June 1, 1988

Discussion of anomalies. Let's consider a pure gauge theory to fix the ideas. This starts with a classical Lagrangian density $\mathcal{L}(A)$ which is a polynomial in the components $A^a_\mu(x)$ and their derivatives. The action is $S(A) = \int_M \mathcal{L}(A)$, where M is space-time or maybe space-imaginary time. For example we have the YM Lagrangian $|F|^2 \cdot vol_M$ and the form $tr(F^2)$ giving the Pontryagin class.

Each of these example Lagrangians is gauge-invariant, although what perhaps is needed is only that the action be gauge invariant. Let's analyze this a bit. Consider an infinitesimal gauge transformation v, i.e. an element of $\Omega^1(M, ad(P))$ ($= \Omega^1(M, End E)$ in the vector bundle case). As the action of $\mathcal{G}$ on $\mathcal{A}$ is $A \cdot g = g^{-1} \cdot (d+A) \cdot g - d = g^{-1}dg + g^{-1}Ag$, we see that the tangent vector at A which is the action of v is $dv + [A,v] = d_A v$. We write

$$\delta_v A = dv + [A,v]$$

for this tangent vector, or variation produced by the inf. gauge transf. v.

Then infinitesimal gauge invariance of the action means

$$0 = \delta_v \int_M \mathcal{L}(A) = \int_M \delta_v \mathcal{L}(A)$$

for all A, v. One way to insure this, and perhaps this is what one means by a local gauge-invariant theory, is for $\delta_v \mathcal{L}(A)$ to be divergence

$$\circledast \quad \delta_v \mathcal{L}(A) = d\,\mathcal{L}'(A,v)$$

where $\mathcal{L}'(A,v)$ is again a polynomial in the $A^a_\mu(x)$, $v^a(x)$ and their derivatives with values in $(n-1)$-forms, $n = \dim M$. One says when $\circledast$ holds that $\mathcal{L}(A)$ is a δ-cocycle modulo d.

Now when it comes to quantizing the gauge theory formally in the sense of constructing the renormalized perturbation theory, one is apparently led to consider expressions

$$\Delta(A; v_1, \ldots, v_k) = \int_M \mathcal{L}(A; v_1, \ldots, v_k)$$

where $\mathcal{L}(A; v_1, \ldots, v_k)$ is Lie algebra cochain with values in n-forms depending on A in a "local" way. Let's write this simply

$$\Delta = \int \mathcal{L}$$

where now instead of $\mathcal{L}$ depending just on A, it also is an alternating multilinear functional on the Lie algebra $\tilde{\mathfrak{g}}$.

Such expressions occur in the perturbation series construction which one is trying to do gauge invariantly. It's analogous to obstruction theory; at each stage one encounters such an expression which turns out to be a cocycle

$$\delta\Delta = \int \delta\mathcal{L} = 0$$

and which one wants to write as a coboundary

$$\Delta = \delta\Delta' = \int \delta\mathcal{L}'$$

If these equalities are to hold "locally", that is, without regard to the global topology of M, this means that $\mathcal{L}$ has the property

$$\delta \mathcal{L} = d\mathcal{L}_1$$

i.e. it is a δ-cocycle modulo $\mathrm{Im}(d)$, and the obstruction vanishes where there are $\mathcal{L}', \mathcal{L}_2$ with

$$\mathcal{L} = \delta \mathcal{L}' + d\mathcal{L}_2$$

i.e. $\mathcal{L}$ is a coboundary mod $\mathrm{Im}(d)$.

Now in the paper of Dubois-Violette, Talon, Viallet one considers only Lie algebra cochains $\mathcal{L}(A;\chi)$ constructed by taking the component 1,2 forms A^a, F^a and 0,1-forms $\chi^a, d\chi^a$ and multiplying them together in $\Omega_M^{\bullet}$. Their universal model is the bigraded differential algebra

$$\Lambda[A^a] \otimes S[F^a] \otimes \Lambda[\chi^a] \otimes S[d\chi^a]$$

where

$$dA = F - A^2$$
$$\delta A = -d\chi - [A,\chi]$$
$$\delta \chi = -\chi^2$$

Now I would like to see if I can use this algebra to produce cyclic cocycles on $\mathrm{End}(E)$ associated to a connection on E. Thus I want to work over $G \times M$, where $G = \mathrm{Aut}(E)$.

Let E be a trivial bundle V over M equipped with the connection $d+A$, and let G be a Lie group of automorphisms of E. Then consider matrix forms on $G \times M$ which are left G-invariant.

Let E be the trivial bundle $\tilde{V}_M$ over M equipped with the connection $d + A$ where A is an $\mathrm{End}(V)$-valued 1-form on M. Let $\mathcal{G} = C^\infty(M, \mathrm{Aut}(V))$ be the group of autos of $\tilde{V}_M$. We then consider the product $\mathcal{G} \times M$ and pull-back $\tilde{V}_M$ to obtain $\tilde{V}_{\mathcal{G}\times M}$. On this trivial bundle we have the pullback connection $\delta + d + A$ and we have the tautological automorphism g. This is a canonical matrix-valued function, namely the evaluation map $\mathcal{G} \times M \longrightarrow \mathrm{Aut}(V)$. Now consider the various matrix forms that we have over $\mathcal{G}\times M$. Our idea is to try to find a geometric model for the BRS algebra of Ω-V, T, + V.

First of all we have the connection 1-form A and the curvature 2-form $F = dA + A^2$. These come by pullback from M. They can be transformed by the canonical automorphism g.

$$g^{-1}(\delta + d + A)g = \delta + d + (g^{-1}\delta g) + (g^{-1}dg + g^{-1}Ag)$$

In addition we have the pull-back of the MC form on $\mathrm{Aut}(V)$:

$$g^{-1}(\delta + d)g = g^{-1}\delta g + g^{-1}dg$$

We can identify $g^{-1}\delta g \in \Omega^{1,0}(\mathcal{G}\times M) \otimes \mathrm{End}(V) = \Omega^1(\mathcal{G}, \Omega^0(M) \otimes \mathrm{End}(V))$ with the MC form of $\mathcal{G}$.

Lets consider a principal G-bundle P with base M equipped with a connection $A_0 \in \Omega^1(P,\mathfrak{g})^G$. Let $\mathcal{G}$ be the group of gauge transformations. We consider the pullback bundle $\mathcal{G} \times P$ over $\mathcal{G} \times M$ and denote by g the tautological automorphism of this pullback. We can form the connection

$$A = \cancel{g^{-1}dg + g^{-1}A_0 g} \quad g^*(A_0)$$

on $\mathcal{G} \times P / \mathcal{G} \times M$.

This is confusing. To simplify suppose P is trivial so that I can write the given connection in P/M as $d + A_0$. Then on $pr_2^*(P) = \mathcal{G} \times P$ we have the pull-back connection $\delta + d + A_0$ which we can transform

$$g^{-1}(\delta + d + A_0)g = \delta + d \underbrace{+ g^{-1}\delta g}_{X} + \underbrace{(g^{-1}dg + g^{-1}A_0 g)}_{A}$$

Thus over $\mathcal{G} \times M$ we have the trivial G-bundle with the connection $\delta + d + X + A$ whose curvature is

$$(\delta + d)(\overset{(1,0)}{X} + \overset{(0,1)}{A}) + (X + A)^2$$

$$= \underbrace{(\delta X + X^2)}_{(2,0)} + \underbrace{(\delta A + dX + [X, A])}_{(1,1)} + \underbrace{(dA + A^2)}_{(0,2)}$$

on one hand. On the other hand it is

$$g^{-1}(dA_0 + A_0^2)g$$

which is of type $(0,2)$. Thus we get

$$\delta X + X^2 = 0$$

$$\delta A + dX + [X, A] = 0$$

which are the BRS algebra relations

So we learn that the principal bundle $pr_2^*(P)$ over $\mathcal{G} \times M$ has a canonical connection whose curvature is of type (0,2). In fact if we use the bundle map

$$\begin{tikzcd} \mathcal{G} \times P \arrow[r, "\mu"] \arrow[d] & P \arrow[d] \\ \mathcal{G} \times M \arrow[r, "pr_2"] & M \end{tikzcd}$$

where μ is the action of $\mathcal{G}$ on P, then the connection is just $\mu^*(d + A_0)$, so it descends.

Repeat: On $pr_2^*(P)$ over $\mathcal{G} \times M$ we have a canonical connection whose curvature is of type (0,2). Hence we will have a map from the BRS bigraded differential algebra to $\Omega^\cdot(pr_2^* P) = \Omega^\cdot(\mathcal{G} \times M)$.

Now I would like to know if there are any applications. I think it's clear that if we take basic forms, then we get a map from the basic BRS algebra to forms on $\mathcal{G} \times M$. Moreover the image should consist of left $\mathcal{G}$-invariant forms.

In fact what I should have are two $\mathcal{G}$-invariant connections on $pr_2^*(P) = \mathcal{G} \times P$ with $\mathcal{G}$ acting diagonally, namely

$$\delta + d + \underbrace{g^{-1}\delta g}_{\chi} + \underbrace{(g^{-1}dg + g^{-1}A_0 g)}_{A} \quad , \quad \delta + d + \underbrace{(g^{-1}dg + g^{-1}A_0 g)}_{A}$$

Thus forms on $\mathcal{G} \times P$ concocted out of A, χ should be $\mathcal{G}$-invariant, and if also G-basic, should be left $\mathcal{G}$-invariants forms on $\mathcal{G}\times M$.

Thus we might be able to use the results about the BRS algebra to construct cyclic cocycles.

However we have to consider the basic subalgebra of the BRS algebra which apparently doesn't have the same cohomology. So there arises the question as to what this might be.

June 2, 1988:

Let's consider a principal G-bundle P over M, and let $\mathcal{G} = \mathrm{Aut}(P)$ be the group of gauge transformation. We are mainly interested in constructing closed forms on $\mathcal{G}\times M$ which are invariant for the left-translation action of $\mathcal{G}$ on $\mathcal{G}\times M$. Thus we are interested in $\Omega(\mathcal{G}\times M)^{\mathcal{G}} = \Omega(\mathcal{G})^{\mathcal{G}}\otimes\Omega(M)$ = double complex of Lie cochains of $\tilde{g} = \mathrm{Lie}(\mathcal{G})$ with values in the complex $\Omega^{\cdot}(M)$.

One way to produce such forms is to consider the pullback bundle $pr_2^*(P) = \mathcal{G}\times P$ which is a principal G-bundle over $\mathcal{G}\times M$. There are two actions of $\mathcal{G}$ on $\mathcal{G}\times P$, namely, the diagonal action

$$g_1(g_2,\xi) = (g_1 g_2, g_1\xi)$$

and the action on the first factor

$$g_1(g_2,\xi) = (g_1 g_2, \xi).$$

Thus there are two ways of considering the bundles $pr_2^*(P)$ as a $\mathcal{G}$-equivariant principal G-bundle over $\mathcal{G} \times M$. Now in fact these two $\mathcal{G}$-equivariant principal G-bundles over $\mathcal{G} \times M$ are isomorphic via the isomorphism

$$\begin{array}{ccc} \mathcal{G} \times P & \longrightarrow & \mathcal{G} \times P \\ (g_2, \xi) & \longmapsto & (g_2, g_2 \xi) \end{array}$$

This intertwines the $\mathcal{G}$-action with trivial action on P with the diagonal $\mathcal{G}$-action.

Now we can identify $\Omega(\mathcal{G} \times M)$ with $\Omega(\mathcal{G} \times P)_{\text{basic}}$. One way to produce $\mathcal{G}$-invariant forms in $\Omega(\mathcal{G} \times P)$ is to consider a connection form for the principal bundle $\mathcal{G} \times P$ which is $\mathcal{G}$-invariant. It doesn't matter which $\mathcal{G}$ action we take since the resulting $\mathcal{G}$-equivariant G-bundles are isomorphic. Thus let's take the simplest, namely where $\mathcal{G}$ acts trivially on P. Then connection forms live in

$$\left(\Omega^1(\mathcal{G} \times P) \otimes \mathfrak{g}\right)^G = \Omega^1(\mathcal{G}) \otimes \Omega^0(P, \mathfrak{g})^G \oplus \Omega^0(\mathcal{G}) \otimes \Omega^1(P, \mathfrak{g})^G$$

and $\mathcal{G}$-invariant connection forms live in

$$\left(\Omega^1(\mathcal{G} \times P)^{\mathcal{G}} \otimes \mathfrak{g}\right)^G = C^1(\tilde{\mathfrak{g}}, \Omega^0(P, \mathfrak{g})^G) \oplus \Omega^1(P, \mathfrak{g})^G$$

From this viewpoint it doesn't look very interesting because one must have a connection A in P, so $A \in \Omega^1(P, \mathfrak{g})^G$ belongs to the set of A such that $\iota_X A = X$ for all $X \in \mathfrak{g}$. Then

One can add to A multiples of the MC form $\chi \in C^1(\tilde{\mathfrak{g}}, \Omega^0(P,\mathfrak{g})^G)$ which is the isomorphism $\tilde{\mathfrak{g}} \xrightarrow{\sim} \Omega^0(P,\mathfrak{g})^G$.

Then A, χ satisfy

$$\delta\chi = -\chi^2 \qquad \delta A = 0$$

so the curvature of $\chi + A$ is

$$(\delta + d)(\chi + A) + (\chi + A^2)$$

$$= d\chi + [A,\chi] + dA + A^2$$

so we recognize that the path of connections $A + t\chi$ is our old $D + t\theta$, whose curvature is $D^2 + t[D,\theta] + (t^2 - t)\chi^2$.

But what's the relation with the BRS alg? In the BRS algebra one has $\chi + A$ such that

$$(\delta + d)(\chi + A) + (\chi + A)^2 = dA + A^2$$

so

$$\delta\chi + \chi^2 = 0$$

$$\delta A + d\chi + [A,\chi] = 0$$

Thus the path of connection $t\chi + A$ starts at $t = 0$ with the curvature

$$(\delta + d)A + A^2 = -d\chi - [A,\chi] + dA + A^2$$

and ends at $t = 1$ with the curvature $dA + A^2$.

Question: Is it possible that, although different as bigraded differential algebras, the associated filtered algebras are isomorphic?

June 3, 1988 (married 27 years)

Let M be a manifold with S^1-action. Then one has the complex of equivariant forms $\Omega_{S^1}(M) = \mathbb{C}[u] \otimes \Omega(M)^{S^1}$ [redacted] with the differential $d + u\iota_X$, where $u = -$ curvature form. Following [redacted] Witten one specializes u to be a number, and then we obtain a $(\mathbb{Z}/2)$-graded complex:

$$\circledast \qquad \Omega^{ev}(M)^{S^1} \underset{\longleftarrow}{\xrightarrow{d+u\iota_X}} \Omega^{odd}(M)^{S^1}$$

which up to isomorphism is indepe[redacted]ndent of u as long as $u \neq 0$.

If M is finite dimensional, then the localization theorem [redacted] says that the cohomology of $\circledast$ for $u \neq 0$ is the cohomology [redacted] (total even $\oplus$ total odd) of M^{S^1}.

Bismut construction: Suppose E a vector bundle with connection ∇ on M, but we do not assume that S^1 acts on E. Then one considers the operator $\nabla + u\iota_X$ on $\Omega(M,E)$. One has

$$(\nabla + u\iota_X)^2 = \nabla^2 + u\nabla_X$$

and the operator $e^{t(u\nabla_X + \nabla^2)}$ on $\Omega(M,E)$ is a kind of translation operator [redacted] in the sense that it is compatible with [redacted] the diffeomorphism $\exp(tuX)$ on M. Suppose $\exp(X) = id$, then [redacted] taking $t = u^{-1}$ we obtain an $\Omega(M)$-linear operator i.e.

$$e^{\nabla_X + u^{-1}\nabla^2} \in \Omega^{ev}(M, \operatorname{End} E)$$

so can take

$$\operatorname{tr}\left(e^{\nabla_X + u^{-1}\nabla^2}\right) \in \Omega^{ev}(M)$$

This is equiv. closed by

$$(d+uc_X)\, tr\left(e^{\nabla_X + u^{-1}\nabla^2}\right) = tr\left[\nabla + u\iota_X, e^{u^{-1}(\nabla+u\iota_X)^2}\right] = 0$$

~~[illegible]~~

Example: Suppose we have a line bundle. Then ∇^2 is a 2-form on M which ~~[illegible]~~ ~~[illegible]~~ can be averaged over the circle action to obtain an invariant 2-form $\overline{\nabla^2}$. e^{∇_X} is the the monodromy operator for the connection around the circle orbits. I think that in this case we have.

$$tr\left(e^{\nabla_X + u^{-1}\nabla^2}\right) = e^{\nabla_X}\, e^{u^{-1}\overline{\nabla^2}}$$

An important point is that when M is infinite dimensional, these Bismut forms ~~[illegible]~~ have infinitely many components. Thus $\Omega^{ev}(M)^{S^1}$ has to be understood as more than $\bigoplus_n \Omega^{2n}(M)^{S^1}$, which means some sort of completion must be defined.

Next let's turn to cyclic theory. Given ~~[illegible]~~ a unital algebra A, (for example $A = a^+$), we form the ~~[illegible]~~ mixed complex which is $A\otimes \bar{A}^{\otimes n}$ in degree n with the two operators b, B. Then we can consider the $(\mathbb{Z}/2)$ graded complex

$$A\otimes\bar{A}^{\otimes ev} \underset{\longleftarrow}{\xrightarrow{b+B}} A\otimes \bar{A}^{\otimes odd}$$

which ~~[illegible]~~ should be a localized version of $\mathbb{C}[u]\otimes(A\otimes\bar{A}^{\otimes *})$ with differential $b+uB$

Now that we have a curvature approach to the Connes homomorphisms we should try to see that the composition

$$\overline{HC}_{2n+1}(A) \longrightarrow I^{n+1}/[I,I^n] \longrightarrow \begin{array}{c} H_1(R,I^n) \\ \cap \\ I^n \otimes_R \Omega^1_R \otimes R \end{array}$$

is zero. (In fact we should check that the cyclic cohomology classes are well-defined first).

Let $\rho : A \to R$ be a linear map which is a homomorphism modulo I in R. Then ρ can be viewed as an elt of $C^1(A,R)$, and the curvature $d\rho + \rho^2 \in C^2(A, I)$. Then $(d\rho+\rho^2)^{n+1} \in C^{2n+2}(A, I^{n+1})$ and we have

$$d\,(d\rho+\rho^2)^{n+1} = -[\rho, (d\rho+\rho^2)^{n+1}]$$

by the Bianchi identity. The commutator on the right is a cochain with values in $[R, I^{n+1}]$ - NO this isn't correct. To be precise putting $\omega = d\rho+\rho^2$ we have

$$[\rho, \omega^n](a_0, \dots, a_{2n}) = \rho(a_0)\,\omega(a_1,a_2)\cdots\omega(a_{2n-1},a_{2n}) - \omega(a_0,a_1)\cdots\omega(a_{2n-2},a_{2n-1})\,\rho(a_{2n}).$$

Now if we apply N we get the cyclic cochain $[\rho(a_0), \omega(a_1,a_2)\cdots\omega(a_{2n-1},a_{2n})]$ + cyclic perms. Thus

$$\text{Image of } N\,(d\rho+\rho^2)^{n+1} \in C^{2n+1}_\lambda(A, I^{n+1}/[R,I^{n+1}])$$

is a cyclic cocycle.

Now the next thing to check is independence of the choice of ρ. Let's consider a family ρ_t of linear maps which are homomorphisms modulo I. Then

$$\partial_t\,(d\rho+\rho^2) = d\dot\rho + [\rho, \dot\rho]$$

has values in I. This means

$$-\mathring{\rho}(a_0,a_1) + \rho(a_0)\mathring{\rho}(a_1) + \mathring{\rho}(a_0)\rho(a_1) \in I$$

that is $\mathring{\rho}$ is a derivation modulo I relative to the homomorphism ρ mod I.

June 4, 1988 Then

$$\partial_t\, (d\rho+\rho^2)^{n+1} = \sum_{i=0}^{n} (d\rho+\rho^2)^i \left(d\mathring{\rho}+[\rho,\mathring{\rho}]\right)(d\rho+\rho^2)^{n-i}$$

$$= \left[d+\rho\, ,\ \sum_{i=0}^{n} (d\rho+\rho^2)^i\,\mathring{\rho}\,(d\rho+\rho^2)^{n-i}\right]$$

If $\mathring{\rho} \in C^1(A,I)$, which means that we aren't varying ρ mod I, then the big sum in the bracket belongs to $C^{2n+1}(A, I^{n+1})$, and so we have

$$* \qquad \partial_t\left\{N(d\rho+\rho^2)^{n+1}\right\} \equiv d\left\{N\sum_{i=0}^{n} (d\rho+\rho^2)^i\,\mathring{\rho}\,(d\rho+\rho^2)^{n-i}\right\}$$

modulo $[R, I^{n+1}]$.

Thus it appears that we obtain a well-defined map

$$HC_{2n+1}(A) \longrightarrow I^{n+1}/[R,I^{n+1}].$$

This is surprising at first, because we expect to land in $I^{n+1}/[I,I^n]$, however the effect of N on $(d\rho+\rho^2)^{n+1}$ is to do also a sum over the cyclic group $\mathbb{Z}/n+1$ acting on these factors. So one is in effect using the splitting of

$$0 \to [R,I^{n+1}]/[I,I^{n+1}] \longrightarrow \underset{\displaystyle (I\otimes_R)^{n+1}}{\underset{\|}{I^{n+1}/[R,I^{n+1}]}} \longrightarrow \underset{\displaystyle (I\otimes_R)^{n+1}_{\sigma}}{\underset{\|}{I^{n+1}/[I,I^{n+1}]}} \to 0$$

which is the unique invariant one under $\mathbb{Z}/n+1$.

If we work mod $[I, I^n]$, then we can simplify $*$ and we get

$$\boxed{\partial_t \{N(d\rho+\rho^2)^{n+1}\} \equiv d\{\overset{n+1}{\cancel{n}} N \mathring{\rho}(d\rho+\rho^2)^n\} \quad \text{mod } [I, I^n]}$$

provided that $\mathring{\rho} \in C^1(A, I)$.

So far we have been essentially with the cyclic cocycle class of degree $2n+1$ associated to a trace defined on $I^{n+1}/[I, I^n]$. However suppose the trace is actually defined on I^n. Then we don't have to restrict $\mathring{\rho}$ to have values in I, that is, we can define the homomorphism $\rho : A \to B/I$.

Let's review. We have

$$\partial_t \; N(d\rho+\rho^2)^{n+1} = N\left[d+\rho, \sum_{i=0}^{n} (d\rho+\rho^2)^i \mathring{\rho} (d\rho+\rho^2)^{n-i}\right]$$

$$= d\Big\{\underbrace{N\sum_{i=0}^{n} (d\rho+\rho^2)^i \mathring{\rho} (d\rho+\rho^2)^{n-i}}_{\in \; C^{2n}_\lambda(A, I^n)}\Big\}$$

in $C^{2n+1}_\lambda(A, I^n/[R, I^n])$, and

$$\partial_t \{N(d\rho+\rho^2)^{n+1}\} = d\{\overset{n+1}{\cancel{n}} N \mathring{\rho}(d\rho+\rho^2)^n\}$$

in $C^{2n+1}_\lambda(A, I^n/[I, I^{n-1}])$. This proves invariance of the class in $HC^{2n+1}(A)$ under deformations of the map $A \to R/I$ provided the trace is defined on $I^n/[I, I^{n-1}]$.

The next thing I want to do is suppose the trace factors through the map

$$j : I^{n+1}/[I, I^n] \longrightarrow H^1(R, I^n) \subset I^n \otimes_R \Omega^1_R \otimes_R$$

$$x_0 \cdots x_n \longmapsto \sum x_{j+1} \cdots x_n x_0 \cdots x_{j-1} \, dx_j$$

and to show that the cyclic class is trivial; But

$$j\, N(\delta\rho+\rho^2)^{n+1} = N\, (n+1)(\delta\rho+\rho^2)^n \underbrace{d_R(\delta\rho+\rho^2)}_{[\delta+\rho,\, d_R\rho]}$$

$$= \delta\{N\,(n+1)(\delta\rho+\rho^2)^n\, d_R\rho\}$$

so this is clear

~~Let's summarize.~~

~~Prop. Let $\rho_t : A \to R$ be a 1-parameter family of linear maps which are homomorphisms modulo I. Then~~

$$\cancel{N(\delta\rho+\rho^2)^n \in C^{2n-1}_\lambda(A, I^n}$$

Let's summarize

Prop. Let $\rho : A \to R$ be a linear map which is a homomorphism modulo I. Then (1)

$$N(\delta\rho+\rho^2)^n \in C^{2n-1}_\lambda(A, I^n/[I, I^{n-1}])$$

is a cyclic cocycle whose class depends only on the homomorphism $A \to R/I$. (2) If j is the composition

$$I^n/[I, I^{n-1}] \longrightarrow H^1(R, I^{n-1}) \subset I^{n-1} \otimes_R \Omega^1_R \otimes_R$$

then $j\, N(\delta\rho+\rho^2)^n \in C^{2n-1}_\lambda(A, I^{n-1} \otimes_R \Omega^1_R \otimes_R)$

is a cyclic coboundary:

$$j\, N(\delta\rho+\rho^2)^n = \delta\{N\, n\, (\delta\rho+\rho^2)^{n-1}\, d_R\rho\}$$

(3) Let ρ_t be a 1-parameter family of linear maps $\rho_t : A \to B$ which are homomorphisms modulo I.

$$\partial_t N(\delta\rho+\rho^2)^n = \delta\{N n \dot{\rho} (d\rho+\rho^2)^{n-1}\}$$

in $C_\lambda^\cdot(A, I^{n-1}/[I,I^{n-2}])$. Consequently if τ is a linear functional defined on $I^{n-1}/[I,I^{n-2}]$, then $\tau N(d\rho+\rho^2)^2$ is a cyclic $(2n-1)$-cocycle on A whose class is invariant under deformations of ρ.

(4) If further ρ_t is constant modulo I, then the above formula holds in $C_\lambda^\cdot(A, I^n/[I,I^{n-1}])$.

Next I would like to try to refine the homotopy assertion in (3) to a statement about the S-operator. Let's try to formulate an approach. Recall that we already have a sort of solution along the following lines. Given $\rho: A \longrightarrow B$ such that $\rho(1)=1$ we can construct the GNS algebra

$$C = A \oplus A\otimes B\otimes A$$

containing A unitally and B non-unitally in the form $B = eCe$, where e is an idempotent in C; further one has $\rho(a) = eae$. Then if we set $F=2e-1$ we know that the Connes cocycle $\mathrm{tr}(F[F,\theta]^{2n})$ is essentially the same as $\mathrm{tr}\{N(d\rho+\rho^2)^{2n}\}$. But we have seen how to link the Connes cocycles for different n using the family

$$\tilde{F}_t = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

The point is that we can see the S-transforms of the earlier Connes cocycles occurring as coefficients in the polynomial

$$(*)\qquad \left(\sqrt{1+t^2}\right)^{2n-1} \operatorname{tr}\left(\tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n-1}\right) = \operatorname{tr}\left(\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -\theta t\\ t\theta & 0\end{pmatrix}^{2n-1}\right)$$

and on the other hand we know the class of $\operatorname{tr}\left(\tilde{\theta}[\tilde{F},\tilde{\theta}]^{2n-1}\right)$ is independent of t.

Let's note that this whole argument is infinitesimal in t, and in fact one really runs into problems with the existence of traces if one tries to use too many powers of t. So I still have the problem of making a rigorous argument with $(*)$. Possible approaches:

1) Deeper understanding of $(*)$ above. There clearly is some resemblance with the various Chern-Simons components. In any case we have seen the necessity of the Chern-Simons forms in establishing "homotopy" properties of S-transforms of cocycles associated to closed currents.

2) Grassmannian. The forms $\operatorname{tr}(e[\theta,e]^{2n})$ are essentially the character forms on the Grassmannian and their homotopy invariance under deformation is standard. What is the meaning of the deformation $\tilde{F}_t$? It's likely it's a standard periodicity map, but why should its infinitesimal behavior near $t=0$ be so significant?

Consider a Hilbert space H and the space of pairs of projectors e, e' in H such that $e-e'$ is compact and $\operatorname{Im} e$ is of infinite dimension and codimension. We know this space has the same homotopy type as the restricted Grassmannian. In effect it fibres over the space of projectors e with $\operatorname{Im} e$ of infinite dim & codim and the fibre over e is the restricted Grassmannian, whereas the base is contractible.

An old problem was to construct a map to a restricted Grassmannian, which is a homotopy equivalence. I found two maps, the second being symmetric in e, e' in some sense. (See

Here's a better construction, now that I have Cayley transform tools at my disposable. We form $H \oplus H$ and let $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the grading and $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as usual. Then we can identify (e, e') with an even projector, namely $\begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix}$, which commutes with F modulo compacts. We can then reduce F with respect to this projector and we obtain the odd self-adjoint contraction on $eH \oplus e'H$

$$A = \begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e' \end{pmatrix} = \begin{pmatrix} 0 & ee' \\ e'e & 0 \end{pmatrix}$$

which is essentially unitary. Then we take the appropriate Cayley transform

$$\left(\sqrt{1-A^2} + iA\right)^2 = g$$

which is a unitary transformation on $eH \oplus e'H$ congruent to -1 modulo compacts. This can be extended by -1 to a unitary on $H \oplus H$ which is congruent to -1 modulo compacts, and which is reversed by ε. Thus it represents a point in the restricted Grassmannian of $H \oplus H$ relative to $-\varepsilon$.

For example suppose $e = e'$. Then A is unitary on $eH \oplus eH$, in fact its $\begin{pmatrix} 0 & id \\ id & 0 \end{pmatrix}$, so $g = -1$ on H and we obtain the subspace $0 \oplus H$. On the other hand suppose $e' = 1 - e$ (this requires we ignore the compactness conditions). Then $A = 0$, so $g = +1$ on $eH \oplus (1-e)H$ and $g = -1$ on the othogonal complement $(1-e)H \oplus eH$. Thus $g\varepsilon = +1$ on $eH \oplus eH$, so this is the subspace to be associated to $(e, 1-e)$.

June 5, 1988

The problem is to link the different cocycles belonging to an extension by the S-operator. We have a partial solution to this problem which uses the GNS construction to reduce the extension cocycles to Connes cocycles, and then to use the formulas linking the latter via S operator. However a direct approach might be more illuminating.

So I propose now to try to translate the formulas we have in the Connes cocycle approach to the extension situation. We start with $A = \mathcal{a}^+$ and a linear map $\rho : \mathcal{a} \to B$ where B is unital, which is a homomorphism modulo the ideal I of B. We might as well suppose $B = T(\mathcal{a})$. Then we form the GNS construction which gives us the algebra $C = A * \mathbb{C}[F]$ such that $B = eCe$ and $\rho(a) = eae$. Thus we represent ρ in the form $\rho(a) = eae$ in a unital algebra C generated by A, e such that $B = eCe$.

Next we "double" C that is, thinking of $C^{\oplus 2}$ as a left A and right C-bimodule, we look at its endomorphisms as right C-module and obtain $M_2(C)$. Recall that A is to act on $C^{\oplus 2}$ via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and we consider the family of involutions

$$\tilde{F}_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

However we have also seen that it suffices for our purposes to use

$$\tilde{F}_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix} F$$

since the two families are conjugate under $\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}$ which commutes with $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$.

Now the basic idea is that from the extension viewpoint we are interested in the $+1$ eigenspace of $\tilde{F}_t$ and the contraction of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ to this eigenspace. The eigenspace varies with t, but we can find a conjugation

$$\tilde{F}_t = g_t \tilde{F}_0 g_t^{-1}$$

which we can use to trivialize the family of $+1$ eigenspaces, at the expense of making the homomorphism $A \longrightarrow M_2(\mathbb{C})$ vary:

$$a \longmapsto g_t^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} g_t$$

Note that $\tilde{F}_0 = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$ has the $+1$ eigenspace $eC \oplus (1-e)C \subset C^{\oplus 2}$ so that this eigenspace is $\cong C$. Thus we will obtain a family of linear maps from A to C:

$$\rho_t(a) = \begin{pmatrix} e & 0 \\ 0 & (1-e) \end{pmatrix} g_t^{-1} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} g_t \begin{pmatrix} e & 0 \\ 0 & (1-e) \end{pmatrix}$$

Notice that everything is canonical, because there is a canonical choice for g_t as we shall show.

$$\tilde{F}_t \tilde{F}_0 = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \qquad \sin\phi = \frac{t}{\sqrt{1+t^2}}$$

Let

$$g_t = \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}$$

Then g_t is inverted by $\varepsilon = \tilde{F}_0$ so

$$g_t \tilde{F}_0 g_t^{-1} = g_t^2 \tilde{F}_0 = \tilde{F}_t \tilde{F}_0 \tilde{F}_0 = \tilde{F}_t$$

Thus we get

$$\rho_t(a) = \begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix} \begin{pmatrix} \cos(\phi/2)a & \cancelto{0}{\sin(\phi/2)} \\ -\sin(\phi/2)a & \cancelto{0}{\cos(\phi/2)} \end{pmatrix} \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}$$

$$\rho_t(a) = \begin{pmatrix} \cos^2(\phi/2)\; eae & -(\sin\cos)(\phi/2)\; ea\bar{e} \\ -(\sin\cos)(\phi/2)\; \bar{e}ae & \sin^2(\phi/2)\; \bar{e}a\bar{e} \end{pmatrix}$$

$$\boxed{\rho_\phi(a) = \frac{1}{2} \begin{pmatrix} (1+\cos\phi)\, eae & (-\sin\phi)\, ea\bar{e} \\ (-\sin\phi)\, \bar{e}ae & (1-\cos\phi)\, \bar{e}a\bar{e} \end{pmatrix}}$$

Notice that at $t = 0$, $\phi = 0$ so

$$\rho_0(a) = \begin{pmatrix} eae & 0 \\ 0 & 0 \end{pmatrix}$$

and at $t = -\infty$, $\phi = \pi/2$ so

$$\rho_{-\infty}(a) = \frac{1}{2} \begin{pmatrix} eae & ea\bar{e} \\ \bar{e}ae & \bar{e}a\bar{e} \end{pmatrix} = \frac{1}{2} a$$

Also $\rho_{+\infty}(a) = \frac{1}{2} FaF$. Actually it's clear that ϕ is a better parameter than t.

So we have this canonical deformation of $a \mapsto \begin{pmatrix} eae & 0 \\ 0 & 0 \end{pmatrix}$. Now we ought to be able to work to second order in ϕ and obtain the desired link between the cocycles associated to ρ. To do our calculations lets redefine t and put (mod ϕ^3

$$-\frac{\sin\phi}{2} = t \qquad \text{or} \qquad \phi = -2t$$

$$\cos\phi = 1 - \frac{\phi^2}{2} = 1 - 2t^2$$

and so

$$\boxed{\rho_t(a) = \begin{pmatrix} (1-t^2)(eae) & te a\bar{e} \\ t\bar{e}ae & t^2\bar{e}a\bar{e} \end{pmatrix} \qquad \text{where } t^3=0}$$

But there is something wrong because we want $\rho_t(a)$ to be a homomorphism modulo some ideal in C. Certainly this ideal should contain $[F, a]$, $a \in A$. But then we have $\rho_t(a) \equiv \begin{pmatrix} (1-t^2)ae & 0 \\ 0 & t^2 a\bar{e} \end{pmatrix}$

which is not a homomorphism.

June 6, 1988

Review. We start with $A = \mathfrak{a}^+$ unital, augmented and with an algebra C generated by A and an involution F. Then we "double" C i.e. we consider $M_2(C)$ with $A \longrightarrow M_2(C)$ given by $a \longmapsto \tilde{a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ and with the family of involutions

$$\tilde{F}_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

We know then that the Connes cocycle $\mathrm{tr}\, \tilde{\theta}[\tilde{F}, \tilde{\theta}]^{2n-1}$ will involve S-transforms of the cocycles $\mathrm{tr}\, \theta[F,\theta]^{2m-1}$ for $m \leq n$.

However we have an explicit conjugation giving $\tilde{F}_t$ from $\tilde{F}_0$:

$$\tilde{F}_t \tilde{F}_0 = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & -tF \\ tF & 1 \end{pmatrix} = \begin{pmatrix} c' & -s'F \\ s'F & c' \end{pmatrix} = g_t^2$$

where $c' = \cos\phi = \frac{1}{\sqrt{1+t^2}}$ $\quad s' = \sin\phi = \frac{t}{\sqrt{1+t^2}}$, and

$$g_t = \begin{pmatrix} c & -sF \\ sF & c \end{pmatrix} \qquad c = \cos(\phi/2) \quad s = \sin(\phi/2).$$

Thus

$$\tilde{F}_t = g_t \tilde{F}_0 g_t^{-1}$$

So up to conjugation which doesn't affect the cocycles $\mathrm{tr}\,(\tilde{\theta}[\tilde{F},\tilde{\theta}])^{2n-1}$, we can suppose

$$\tilde{F} = \tilde{F}_0 = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$$

and

$$\tilde{\theta} = g_t^{-1} \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} g_t = \begin{pmatrix} c & sF \\ -sF & c \end{pmatrix} \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & -sF \\ sF & c \end{pmatrix}$$

$$= \begin{pmatrix} c\theta & 0 \\ -sF\theta & 0 \end{pmatrix} \begin{pmatrix} c & -sF \\ 0 & 0 \end{pmatrix}$$

so

$$\tilde{F} = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} \qquad \tilde{\Theta} = \begin{pmatrix} c^2\Theta & -sc\Theta F \\ -cs F\Theta & s^2 F\Theta F \end{pmatrix}$$

If we do a further conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}$ then we get

$$\boxed{\tilde{F} = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} \qquad \tilde{\Theta} = \begin{pmatrix} c^2 & -sc \\ -cs & s^2 \end{pmatrix} \Theta}$$

Therefore we learn that [redacted] the doubling process consists of passing from F, C, A to $\tilde{F} = \begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$ and $M_2(C)$, $M_2(A)$ but then restricting to $a \longmapsto e_t\, a$, $\quad e_t = \begin{pmatrix} c^2 & -sc \\ -sc & s^2 \end{pmatrix}$.

June 7, 1988

Let's consider $A * \mathbb{C}[F] = (A*A) \tilde{\otimes} \mathbb{C}[F]$ and let's adjoin ε so as to anti-commute with F and commute with A. Then we obtain a family of involutions

$$\tilde{F}_\phi = (\cos\phi) F + (\sin\phi)\varepsilon$$

such that

$$[(\cos\phi)F + (\sin\phi)\varepsilon \,,\, a] = (\cos\phi)[F,a]$$

Since $\tilde{F}_0 = F$ and $\tilde{F}_{\pi/2} = \varepsilon$ one sees that $\mathrm{tr}\, F[F,\theta]^{2n}$ and $\mathrm{tr}(\varepsilon[\varepsilon,\theta]^{2n}) = 0$ are cohomologous. But this is no surprise since the trace is relative to the algebra $((A*A)\tilde{\otimes}\mathbb{C}[F])\tilde{\otimes}\mathbb{C}[\varepsilon]$. In other words if we view $C = (A*\mathbb{C}[F])[\varepsilon]$ as acting on H, then we have $H = H^+ \oplus H^-$ with $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Better $F[F,\theta]^{2n}$ anticommutes with ε so the trace is zero.

Let's try to discuss in general terms Connes approach to the S-operator. His starting point is the equivalence between cyclic n-cocycles on A and closed degree n traces on Ω_A. Thus any cyclic class of degree n is represented by a cochain algebra $\Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots\cdots\cdots$ with a homomorphism $A \to \Omega^0$ and a (super) trace $\tau : \Omega^n \to \mathbb{C}$ with $\tau \circ d = 0$.

Important example: Let us take a superalgebra with an odd derivation of square zero

$$R^+ \underset{d}{\overset{d}{\rightleftarrows}} R^-$$

Then it gives rise to a cochain algebra

$$R^+ \xrightarrow{d} R^- \xrightarrow{d} R^+ \xrightarrow{d} R^- \longrightarrow \cdots$$

Specialize to the case where $d = [F, \;]$ and $F^2 \in$ center of R, e.g. $F^2 = 1$. Check

$$[F,[F,x]] = \underbrace{[[F,F],x]}_{=0} - [F,[F,x]] \Rightarrow [F,[F,x]] = 0$$

Let's see if we can work in connections in some way. Let E be a vector bundle over M with connection ∇. Then we can consider $\Omega^\cdot(M, \operatorname{End} E)$ with the odd derivation $[\nabla, \;]$ whose square is $[\nabla^2, \;]$. Now suppose $E \underset{i}{\overset{i^*}{\rightleftarrows}} \tilde{V}$ such that $\nabla = i^* d i$. Then we have something which might serve as a model for the proper treatment of a Dirac operator.

We consider $\Omega^\cdot(M, \operatorname{End} \tilde{V}) = \Omega^\cdot(M) \otimes \operatorname{End}(V)$ with its differential d, and we try to locate the cochain algebra generated by $\Omega^0(M, \operatorname{End} E) \cong e(\Omega^\cdot(M) \otimes \operatorname{End}(V))e$ where $e = i i^*$. This will be a subalg of the algebra of matrix forms $\Omega^\cdot(M) \otimes \operatorname{End}(V)$ which we can describe in block form. First of all there is

$$i\, \Omega^0(M, \operatorname{End} E)\, i^* = \begin{pmatrix} \Omega^0(M, \operatorname{End} E) & 0 \\ 0 & 0 \end{pmatrix}$$

and there is the image of this under d. Since

$$d(i \theta i^*) = [d, i\theta i^*] = \begin{pmatrix} [\nabla, \theta] & -\theta i^* dj \\ j^* di\, \theta & 0 \end{pmatrix}$$

June 10, 1985

I want to review what I worked out on the plane to Montreal. It concerns the isomorphism of $A*A$ and $\Omega_A^{\cdot}$. Set $qA = A*A$ and $\varepsilon A = A*\mathbb{C}[F] = (A*A)\tilde{\otimes}\mathbb{C}[F]$.

The last identification $A*\mathbb{C}[F] \cong (A*A)\tilde{\otimes}\mathbb{C}[F]$ proceeds as follows. One has a canonical homom. $A \longrightarrow A*\mathbb{C}[F]$, which we denote $a \mapsto a$. Then one can conjugate to get another homomorphism $a \longmapsto FaF$. Thus one gets ~~two~~ homomorphisms and hence a homomorphism $A*A \longrightarrow A*\mathbb{C}[F]$ such that $in_1(a) \longmapsto a$, $in_2(a) \longmapsto FaF$. Next $A*A$ admits an action of $\mathbb{Z}/2$ which interchanges $in_1(a)$ and $in_2(a)$, so one can take the semi-direct product $(A*A)\tilde{\otimes}\mathbb{C}[F]$, where conjugation by F acts as this automorphism. Then we get a homomorphism

$$(A*A)\tilde{\otimes}\mathbb{C}[F] \longrightarrow A*\mathbb{C}[F]$$

which we know is an isomorphism by the universal property. Notice also that we have a natural grading ε on $A*\mathbb{C}[F]$ defined by $\varepsilon(a) = a$, $\varepsilon(F) = -F$. For this grading $\varepsilon(FaF) = FaF$ so that $A*A = \varepsilon$ even part of $A*\mathbb{C}[F]$ and $(A*A)F$ = odd part.

Set $qa = a - FaF \in A*A$. One thinks of qa as a quantized version of da. This is because one has (setting $\bar{a} = FaF$)

$$q(aa') = aa' - \bar{a}\bar{a}' = (a-\bar{a})a' + \bar{a}(a'-\bar{a}')$$

$$= (a-\bar{a})a' + a(a'-\bar{a'}) - (a-\bar{a})(a'-\bar{a'})$$

or

$$\boxed{q(aa') = q(a)a' + aq(a') - q(a)q(a')}$$

Notice that we can change the sign by changing q to $-q$. In fact if we set $q_h(a) = \frac{1}{h}(a-\bar{a})$, then we have the identity

$$\boxed{q_h(aa') = q_h(a)a' + a\,q_h(a') - h\,q_h(a)\,q_h(a')}$$

Eventually I should review the "classical limit" filtration construction $\bigoplus_{p\geq 0} h^p F_p A$ in the context of the Clifford + exterior algebras, Weyl + polynomial algebras, but for the moment let's go over the formulas. The principal point is to set up an isomorphism of qA with Ω_A such that we have the correspondence

$*$ $\qquad a_0\, qa_1 \cdots qa_n \longleftrightarrow a_0\, da_1 \cdots da_n$

We will define an action of $A * \mathbb{C}[F]$ on [redacted] $\Omega_A \otimes \mathbb{C}[F]$ such that action on 1 gives a bijection $A * \mathbb{C}[F] \xrightarrow{\sim} \Omega_A \otimes \mathbb{C}[F]$ and then $A * A \xrightarrow{\sim} \Omega_A$. The A-action is the obvious left multiplication. To define multiplication by F on $\Omega_A \otimes \mathbb{C}[F]$ we need a suitable involution on Ω_A, say denoted $\omega \mapsto \omega^F$, so that we can define $F\cdot(\omega\otimes 1) = \omega^F \otimes F$ and $F\cdot(\omega\otimes F) = \omega^F \otimes 1$.

We want this conjugation on Ω_A to be compatible with the $\mathbb{Z}_2$-action on $A*A$. Now

$$F(a_0 q a_1 \cdots q a_n)F = F a_0 F (-q a_1) \cdots (-q a_n)$$

because the $\mathbb{Z}_2$-action is conjugation by F in $A * \mathbb{C}[F]$. But this equals

$$(-1)^{n+1} \underbrace{(a_0 - F a_0 F)}_{q a_0} q a_1 \cdots q a_n + (-1)^n a_0 q a_1 \cdots q a_n$$

This corresponds to $(-1)^n a_0 da_1 \cdots da_n + (-1)^{n+1} da_0 \cdots da_n$. Thus we want

$$\omega^F = \sigma(1+d)\,\omega$$

where $\sigma(\omega) = (-1)^{\deg \omega}\omega$. Note that

$$\sigma(1+d)\,\sigma(1+d) = (1-d)(1+d) = 1$$

so indeed $\omega \longmapsto \omega^F$ is an action of $\mathbb{Z}/2$ on Ω_A.

June 11, 1988

Recall we obtain a left module structure over $A * \mathbb{C}[F]$ on Ω_A by using the obvious left A-module structure and defining the action of F to be the operator $\sigma(1+hd)$ on Ω_A. Then we have that FaF acts on Ω_A by

$$\sigma(1+hd)\,a\,\sigma(1+hd) = (1-hd)a(1+hd)$$
$$= a - h\,da - h^2\,da\,d$$

and so $g_h a = \frac{1}{h}(a - FaF)$ becomes the operator

$$g_h a \doteq da(1+hd)$$

Then
$$g_h a_1\, g_h a_2 \doteq da_1(1+hd)da_2(1+hd)$$
$$\doteq da_1 da_2 (1-hd)(1+hd) = da_1 da_2$$

and so we have

$$a_0 g_h a_1 \cdots g_h a_n \doteq \begin{cases} a_0 da_1 \cdots da_n & n \text{ even} \\ a_0 da_1 \cdots da_n (1+hd) & n \text{ odd.} \end{cases}$$

We now consider the composition

$$A * A \subset A * \mathbb{C}[F] \xrightarrow{\alpha \mapsto \alpha 1} \Omega_A .$$

One has

$$a_0 g_h a_1 \cdots g_h a_n \longmapsto a_0 da_1 \cdots da_n$$

Prop. This map $A * A \longrightarrow \Omega_A$ is an isomorphism.

Proof. The map is obviously onto. In fact we can define a linear map $\Omega_A \longrightarrow A * A$ by the formula

$$a_0\, da_1 \cdots da_n \longmapsto a_0\, q_h a_1 \cdots q_h a_n = h^{-n}\, a_0\, q a_1 \cdots q a_n$$

This is well defined because $\Omega_A^n = A \otimes \bar{A}^{\otimes n}$ and $q(1) = 0$. It's clear the composition $\Omega_A \longrightarrow A * A \longrightarrow \Omega_A$ is the identity. To finish we note that by virtue of the identity

$$(*) \qquad q_h(a_1 a_2) = q_h a_1\, a_2 + a_1\, q_h a_2 - h\, q_h a_1\, q_h a_2$$

any element of $A * A$ can be written as a linear combination of elements of the form $a_0\, q_h a_1 \cdots q_h a_n$. However on such elements it is clear that the composition $A * A \longrightarrow \Omega_A \longrightarrow A * A$ is the identity.

Rmk: Note that $A * A$ is a left $A * \mathbb{C}[F]$-module in a natural way where F interchanges the two factors. More precisely

$$A * A \xrightarrow{\sim} A * \mathbb{C}[F] \big/ (A * \mathbb{C}[F])(1 - F)$$

So the isomorphism $A * A \xrightarrow{\sim} \Omega_A$ is the unique isomorphism of $A * \mathbb{C}[F]$-modules such that $1 \mapsto 1$.

Next consider the I-adique filtration on $A * A$, where $I = \operatorname{Ker}\{A * A \longrightarrow A\}$ is the smallest ideal modulo which F and A commute. I is generated by $\{qa \mid a \in A\}$. Thus I^n is spanned by elts. of the form

$$a_0\, q a_1\, a_1'\, q a_2\, a_2' \cdots q a_m\, a_m'$$

for $m \geq n$. Using the identity $(*)$ one can move all the a_i''s to the left obtaining terms with more q-factors. Thus I^n is spanned by elts.

of the form $a_0 q a_1 \cdots q a_m$ with $m \geq n$. and so under the the isomorphism $A*A \xrightarrow{\sim} \Omega_A$ it corresponds to $\bigoplus_{m \geq n} \Omega^m_A$. It follows that we have an isomorphism

(1) $$gr^I_m(A*A) \longrightarrow \Omega^m_A$$

On the other hand from the identity ⊛, we see using the universal property of $\Omega^\bullet_A$, that there is a unique algebra homomorphism

(2) $$\Omega^\bullet_A \longrightarrow gr^I_\bullet(A)$$

which is id_A in degree zero and which sends da to $q_h a$. The map (1) sends the image of $a_0 q_h a_1 \cdots q_h a_m$ in gr^I_m to $a_0 da_1 \cdots da_m$ and (2) does the opposite. So we obtain an isomorphism of $\mathbb{Z}$ graded algebras

$$\boxed{gr^I(A*A) \xrightarrow{\sim} \Omega^\bullet_A}$$

Remark: Unlike the case of the Clifford and exterior algebras where we have increasing algebra filtrations, the filtration we used on $A*A$ is an adique filtration. At first sight $A*A$ doesn't appear to have a natural increasing filtration, although perhaps we can find one using our knowledge about $A*\mathbb{C}[F]$. Until we do so, it would seem that there is not much point in the parameter h.

Let's consider $C = A * \mathbb{C}[F]$. Then there is the ideal K in C which is the smallest ideal modulo which A, F commute. We have

$$C/K = A \otimes \mathbb{C}[F] \cong A \times A$$

and there are then two ideals containing K

$$J = C\bar{e}C \qquad \bar{J} = CeC$$

with $K = J\bar{J} + \bar{J}J = J \cap \bar{J}$. In terms of block notation, recall we have

$$J = \begin{pmatrix} eCe\,eC\bar{e}\,\bar{e}Ce & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e} \end{pmatrix},$$

$$\bar{J} = \begin{pmatrix} eCe & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e}\,\bar{e}Ce\,eC\bar{e} \end{pmatrix}$$

The only natural complement to K in C is $eAe \oplus \bar{e}A\bar{e}$ and this is not a subalgebra. So it appears as if there is no natural way to get an increasing algebra filtration on C. Certainly we can't expect to have one complementary to the K-adic filtration otherwise C would be a graded algebra.

Let's look at $B = eCe = T(A)/(1 = 1_A)$. Then we have an increasing algebra filtration with $F_p B = (eAe)^p$. If $I = \mathrm{Ker}\{B \longrightarrow A\}$ then

$$B = F_1 B \oplus I = F_3 B \oplus I^2 = \cdots$$
$$= F_{2n-1} B \oplus I^n$$

In fact we have

$$F_{2n+1}B = eAe \oplus eAe(eA\bar{e}Ae) \oplus \cdots \oplus eAe(eA\bar{e}Ae)^n$$

$$F_2 B = (eAe)^2 = eAe \oplus eA\bar{e}Ae$$

In effect $ea_1e\, ea_2e - ea_1a_2e = -ea_1\bar{e}a_2e$. So

$$F_{2n}B = eAe \oplus eA\bar{e}Ae \oplus \cdots \oplus (eA\bar{e}Ae)^n$$

seems to be true.

June 12, 1988

Let us consider the construction of Connes cocycles. Ungraded case: One starts with a representation of A on a Hilbert space H together with an involution F on H such that $[F,a]$ is Schatten. Then $\mathcal{E}A = A * \mathbb{C}[F]$ acts on H and the ideal K generated by the $[F,A]$ in $A*\mathbb{C}[F]$ acts as Schatten operators, so a trace is defined on K^n for suff large n. Precisely one has a linear fnl on $K^n/[K,K^{n-1}]$.

So a natural question is to describe these traces on K^n. They are linear functionals and hence can be described by linear functionals on Ω_A via the linear isomorphisms

$$K^n = I_\Delta^n \otimes \mathbb{C}[F] \xrightarrow{\sim} \Omega_A^{\geq n} \otimes \mathbb{C}[F]$$

where here $I_\Delta = \mathrm{Ker}\{A*A \to A\}$. Now it is also possible to use the block description of $C = A*\mathbb{C}[F]$. Then

$$K = \begin{pmatrix} eC\bar{e}Ce & eC\bar{e} \\ \bar{e}Ce & \bar{e}CeC\bar{e} \end{pmatrix} \qquad K^2 = \begin{pmatrix} eC\bar{e}Ce & eC\bar{e}CeC\bar{e} \\ \bar{e}CeC\bar{e}Ce & \bar{e}CeC\bar{e} \end{pmatrix}$$

The condition $\tau[F, K^n] = 0$ implies that τ

sees only the diagonal blocks.

Note that $eC\bar{e}Ce$ is the ideal $I = \mathrm{Ker}(B \to A)$ where $B = eCe$. The diagonal blocks of K^{2n-1} and K^{2n} are I^n and $\bar{I}^n$. Thus it appears that τ is specified by traces on I^n and $\bar{I}^n$.

Notice that $\varrho a = a - FaF = F[F,a]$, hence K^n is spanned by $a_0[F,a_1]\cdots[F,a_m]$ and $a_0[F,a_1]\cdots[F,a_m]F$ for $m \geq n$. The possible nonzero traces are

$$tr\left(\frac{a_0 + Fa_0F}{2}[F,a_1]\cdots[F,a_{2m}]\right)$$

$$tr\left(\frac{a_0+Fa_0F}{2}[F,a_1]\cdots[F,a_{2m}]F\right).$$

However this is very confusing, and I don't see how to conveniently describe traces on K^n in terms of linear functionals on Ω^n.

Notice as far as the cyclic cocycles are concerned

$$tr\ F[F,a_1]\cdots[F,a_{2m}] = 2\ tr\ a_1[F,a_2]\cdots[F,a_{2m}]$$

we seem to be interested in part of the trace, namely, just its restriction to exact even forms times F.

Next let's consider the graded case. This time we start with a representation of A in $H^+ \oplus H^-$ commuting with ε together with an odd F $\ni$ $[F,a]$ is Schatten. Then we can identify H^+ with H^- so that $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We then get a representation of $A * A$ on H^+ such that I_Δ is Schatten. Conversely given a repn. of $A*A$ on H such that I_Δ is Schatten, we can for $H \oplus H$ and get A, F, ε. From such a representation we get a trace defined on I_Δ^n for suff. large n.

So a natural question is to describe these traces, and we might try to describe them as linear fcls. on $\Omega^{\geq n}$ using the vector space isomorphism

$$I_\Delta^n \xrightarrow{\sim} \Omega^{\geq n}$$

Now one thing I failed to use in the ungraded case above is the $\mathbb{Z}/2\mathbb{Z}$-action on $A * \mathbb{C}[F]$ given by ε. This gives a $\mathbb{Z}/2\mathbb{Z}$ action on the traces on K^n and we can split up the traces into even and odd according to ε. Similarly we have a $\mathbb{Z}/2$-action on $A*A$ which we can use to study the traces on I_Δ^n.

Idea: Recall that if M is an A-bimodule with a trace, i.e. a linear map $\tau: M \longrightarrow \mathbb{C}$ such that $\tau(am) = \tau(ma)$, then we have a canonical A-bimodule map

$$M \longrightarrow A^*$$

unique such that τ is this map followed by evaluation at 1. We can also use this trace to go from a Hochschild cocycle in $C^p(A, M)$ to a Hochschild cocycle in $C^p(A, A^*)$, i.e. a Hochschild cocycle in the sense of cyclic theory.

Connes entire cyclic cocycles paper.

The goal is to produce a certain trace on εA, an "odd" trace, for these have to do with cyclic cocycles. The method is to invent a convolution algebra made up of "functions" $T(s)$ for $0 < s < \infty$ with values in $\mathcal{L}(H)$. Actually these are extended to distributions on $\mathbb{R}$ with support in $\mathbb{R}_{\geq 0}$.

This algebra can be identified, using the L.T., with an algebra of holomorphic functions. One maps εA to this algebra; this means giving a homomorphism from A and an F. To a one associates the distribution $a\delta_0$, where δ_0 is the Dirac δ at 0; this has L.T. the constant function a. For F one takes the phase of massive Dirac

$$F = \frac{D}{\sqrt{\lambda + D^2}} + \frac{\varepsilon \lambda^{1/2}}{\sqrt{\lambda + D^2}} = \frac{D + \varepsilon\lambda^{1/2}}{\sqrt{\lambda + D^2}}$$

Finally there is a tricky "trace" which takes the coeff. of $\varepsilon\lambda^{1/2}$, and then evaluates the inverse L.T. at $t=1$

June 14, 1988

Conversation with Stora about gauge fixing and ghosts. One starts with a Lagrangian which is degenerate because of gauge symmetry. Call this action S_{inv}. Then one adds a gauge-fixing action S_{gf} which has a standard form involving 3 new fields, the ghost ω, the Lagrange multiplier b, and the anti-ghost $\bar{\omega}$. One extends the symmetry to the new action.

Suppose the initial symmetry is denoted s. One need a gauge condition which is a function whose level sets are slices for the gauge action. Call this gauge condition function f. The gauge fixing action is

$$S_{gf} = \int bf + \bar{\omega}\, s(f)$$

The effect of bf is to give the δ-function $\delta(f)$, when the b integral is done; more generally, as is the case with the Fourier transform + Lagrange multipliers, it can give $\delta(f-c)$ provided one adds a linear term bc to the exponent. The term $\bar{\omega}\, s(f)$ gives the Fadeev-Popov determinant.

It's important to extend the s-action in such a way that $s^2 = 0$. One wants to carry this Slavnov symmetry through all the steps of the renormalized perturbation expansion so as to establish the gauge invariance.

Stora claims that the cohomology of s modulo d is where one finds the real physics. If this cohomology is trivial, then the theory is trivial.

Linear example related to Ray-Singer torsion. Consider the action

$$S_{inv}(\omega^p) = \int |\omega^p|^2$$

on the space of p-forms. This has the symmetry $\omega^p \mapsto \omega^p + d\omega^{p-1}$. Thus the gauge transformations are given by elements of Ω^{p-1} under addition. I guess one writes this symmetry as

$$s\,\omega^p = d\omega^{p-1}$$

Next one takes the gauge condition

$$f(\omega^p) = d^*\omega^p$$

whence

$$S_{gf}^{(\omega^p, \omega^{p-1}, b, \bar{\omega}^{p-1})} = \int b\, d^*\omega^p + \bar{\omega}^{p-1} d d^* \omega^{p-1}$$

One extends the s-action

$$s(\omega^p) = d\omega^{p-1} \quad \text{as before}$$

$$s(\omega^{p-1}) = d\omega^{p-2}$$

$$s(\bar{\omega}^{p-1}) = b \ (+ d^*?) \quad ?$$

$$s(b) = 0$$

The Lagrangian $S_{inv} + S_{gf}$ is still not non-degenerate, so one starts repeating the gauge fixing procedure. If done correctly one should encountered the various Laplaceans on forms of degree $\leq p$ and the Ray-Singer weighted determinant.

One should work with twisted coefficients so that the Laplaceans have no zero modes.

This model is related to light on curved space time (?).

The extra fields separate out, so one gets a propagator for the field ω^p of interest which is independent of the extra fields. This is not the case for non abelian theories.

June 15, 1988

BRS algebra and $H^*(\mathcal{G})$. Recall that the BRS algebra of $\mathfrak{g}$ is a universal bigraded differential algebra having a $\mathfrak{g}$-valued connection form $\chi + A$ whose curvature is of type $(0,2)$:

$$(\delta + d)(\chi + A) + (\chi + A)^2 = \overbrace{(\delta\chi + \chi^2)}^{0} + \overbrace{(\delta A + d\chi + [A,\chi])}^{0} + (dA + A^2)$$

Consider the trivial G-bundle P over M and let $\mathcal{G} = C^\infty(M,G)$ be its group of gauge transformations. We choose a connection $d + A_0$ in M. Now pull P back to the trivial G-bundle $pr_2^*(P)$ over $\mathcal{G} \times M$, and let g denote tautological autom. of $pr_2^*(P)$. The connection $d + A_0$ pulls back to $\delta + d + A_0$. We now consider the connection

$$g^{-1}\cdot(\delta + d + A_0)\cdot g = \delta + d + \underbrace{g^{-1}\delta g}_{\chi} + \underbrace{(g^{-1}dg + g^{-1}A_0 g)}_{A}$$

Its curvature is

$$g^{-1}(\delta + d + A_0)^2 g = g^{-1}(dA_0 + A_0^2)g = dA + A^2$$

of type $(0,2)$. Hence we have a map from the BRS algebra to $\Omega(\mathcal{G} \times M)$.

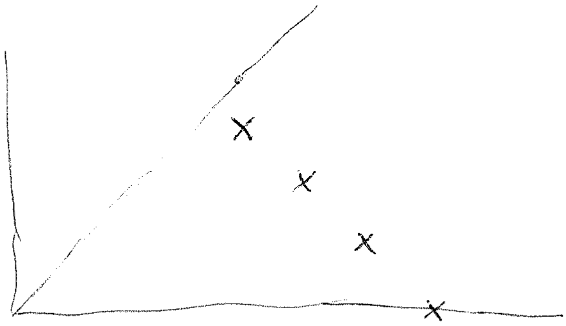
Now we have to understand what cohomology classes, or rather what kind of cohomology, can be obtained from the BRS algebra. At first glance, because the $\delta + d$ cohomology of the BRS algebra is trivial, one can't obtain non-trivial cohomology classes in $H^*(\mathcal{G} \times M)$. But on the other hand one has taken a connection $\delta + d + A_0$ and transformed it by the gauge transformation g, so the linear path between these connections should give odd cohomology classes on $\mathcal{G} \times M$. The resolution of this paradox comes from the fact that A_0 does not come from the BRS algebra.

According to Stora the physics is contained in the δ-cohomology modulo d, and this is non-trivial. Here's how we can use this cohomology to obtain classes in $H^*(\mathcal{G})$. Let ω^p_k be an element of type (p,k) in the BRS algebra such that $\delta \omega^p_k = d\omega^{p+1}_{k-1}$ for some ω^{p+1}_{k-1}, that is ω^p_k is a δ-cocycle mod d. Then if γ_k is a closed k-current on M

$$\delta \int_{\gamma_k} \omega^p_k = \int_{\gamma_k} d\omega^{p+1}_{k-1} = 0$$

so $\int_{\gamma_k} \omega^p_k$ is a closed p-form on $\mathcal{G}$. Here $\int$ map from the BRS algebra to $\Omega(\mathcal{G} \times M)$.

Suppose we try to find examples. First of all we have seen that the subalgebra generated by χ and $d\chi$ (in the BRS algebra) occurs in the construction of cyclic cocycles. This is just the Weil algebra in disguise with a bigrading which is natural from either the holomorphic theory or the contracting homotopy. We have seen that the δ-cohomology modulo d leads to the classes:



In fact recall that the δ-cohomology, or rather the primitive part in the cyclic version, consists of the diagonal classes $\operatorname{tr}(d\theta)^n$ and the classes along the bottom $\operatorname{tr}(d\theta)^{2n-1}$.

What is the δ cohomology for the BRS algebra in general? According to DV, T&V it is $H^*(\mathfrak{g}, S(\mathfrak{g}^*)) = H^*(G) \otimes H^*(BG)$. To be precise the BRS algebra with δ differential is the tensor product of the DGA's

$$\mathbb{C}[\chi^a, F^a] \otimes \mathbb{C}[A^a, \delta A^a]$$

In effect $\delta\chi = -\frac{1}{2}[\chi,\chi]$ $\quad \delta F = -[\chi, F]$ so the first algebra is closed under δ; the second is trivially closed under δ and is contractible. Finally recall that $\delta A = -d\chi - [\chi, A]$, and hence the BRS algebra is freely generated in the following ways

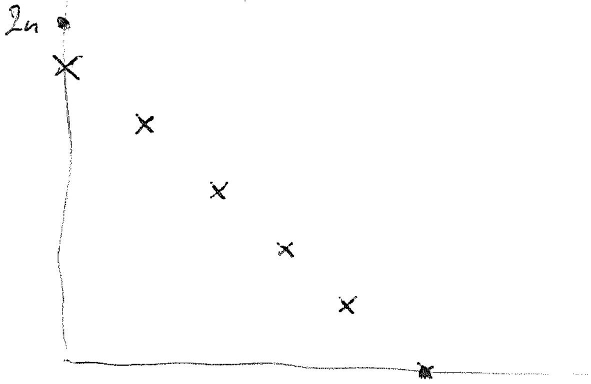
$$\mathbb{C}[\chi^a, d\chi^a, A^a, dA^a] = \mathbb{C}[\chi^a, d\chi^a, A^a, F^a] = \mathbb{C}[\chi^a, \delta A^a, A^a, F^a]$$

replace dA by $F - A^2$ $\qquad$ replace $d\chi$ by $-\delta A - [\chi, A]$

Thus the δ-cohomology is that of $\mathbb{C}[\chi, F]$ which is the complex of Lie cochains on $\mathfrak{g}$ with values in $S(\mathfrak{g}^*)$.

Extrapolating from the situation

we treated before, namely $\mathbb{C}[X, dX]$, the fact that we have δ cohomology (primitive) given by $tr(F)^{even}$ $tr(X)^{odd}$ and trivial d-cohomology should tell us that the δ cohomology modulo d is represented by classes at the odd lattice points:



The ones above the diagonal depend on A and hence are not represented by left invariant forms on $\tilde{\mathcal{G}}$.

This discussion raises many questions.

1). Is there some way to modify the above so that one can understand the δ cohomology modulo d classes as coming from closed odd forms on $\mathcal{G} \times M$, specifically the odd classes associated to $pr_2^*(P)$ and the tautological automorphism g?

2) How should one handle a general principal bundle?

3) Formulas? What formulas for the δ mod d cohomology classes can be obtained using the BRS algebra? In particular for a twisted bundle can one obtain the good cyclic cocycles relative to a connection?

Consider again a connection $d+A_0$ on the trivial G-bundle over M, then consider the connection on the trivial bundle over $\mathcal{G}\times M$

$$g^{-1}(\delta+d+A_0)\,g = \delta+d+\underbrace{g^{-1}\delta g}_{\chi}+\underbrace{(g^{-1}dg+g^{-1}A_0 g)}_{A}$$

Since $(\delta+d)(\chi+A)+(\chi+A)^2 = dA+A^2 = F$ is of type $(0,2)$, there is a homomorphism

$$BRS \longrightarrow \Omega(\mathcal{G}\times M).$$

We saw how the δ mod d cohomology of the BRS algebra integrated over cycles in M gives cohomology on $\mathcal{G}$. Now I propose to describe what one obtains.

Now according to the analysis on the preceding page the primitive δ mod d classes arise by taking $tr(F^n)$ and expressing it as $(d+\delta)(\eta)$ then taking the components of η. One way to do this within the BRS algebra is to use the linear path $t(\chi+A)$ joining A to zero. This gives

$$tr(F^n) = (\delta+d)\;\omega_{2n-1}(\chi+A,0)$$

where ω_{2n-1} is the standard difference form. In the present case the curvature is

$$(\delta+d)\,t(\chi+A)+t^2(\chi+A)^2 = tF+(t^2-t)(\chi+A)^2$$

so

$$\omega_{2n-1}(\chi+A,0) = \int_0^1 dt\; n\, tr\left\{(\chi+A)\left(tF+(t^2-t)(\chi+A)^2\right)^{n-1}\right\}$$

Another possibility is to use the broken path from 0 to X to $X+A$.

But notice once we map into $\Omega(\mathcal{G}\times M)$ we have available $\omega_{2n-1}(A_0, 0)$ which satisfies
$$(\delta+d)\,\omega_{2n-1}(A_0,0) = \operatorname{tr} F_0^n = \operatorname{tr} F^n$$

Thus $\omega_{2n-1}(X+A,0) - \omega_{2n-1}(A_0,0)$ is closed on $\mathcal{G}\times M$ and it represents the odd $(2n-1)$dim class belonging to the odd autom g over $\mathcal{G}\times M$. Thus I learn that the BRS classes are really not going to give us anything new beyond what I understood before. The simplest way to produce the cohomology of $\mathcal{G}$ is to use the closed form
$$\omega_{2n-1}(X+A, A_0)$$

June 20, 1988

Let's recall that we obtain a left $B = eCe = T(A)/(1 - \rho(1_A))$ - module structure on $\Omega_A^{\cdot}$ by associating to $a \in A$ the operator

$$\rho(a) = a + da\, d$$

and moreover that acting on 1, i.e.

$$\rho(a_0) K(a_1, a_2) \cdots K(a_{2n-1}, a_{2n}) \longmapsto a_0\, da_1 \cdots da_{2n}$$

gives a linear isomorphism of B with Ω_A^{ev}. Here

$$K(a_1, a_2) = \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$$

operates as

$$\left.\begin{aligned} &(a_1 + da_1 d)(a_2 + da_2 d) \\ &- a_1 a_2 - (da_1 a_2 + a_1 da_2) d \end{aligned}\right\} = da_1 da_2.$$

Let's compute the B-product in terms of this linear isomorphism. Given even forms

$$\omega = a_0\, da_1 \cdots da_{2m} \qquad \eta = a_0'\, da_1' \cdots da_{2n}'$$

these become the elements of B which are the operators

$$\begin{array}{cc} (a_0 + da_0 d)\, da_1 \cdots da_{2m} & (a_0' + da_0' d)\, da_1' \cdots da_{2n}' \\ \| & \| \\ \omega + d\omega\, d & \eta + d\eta\, d \end{array}$$

Thus if we denote left B-multiplication on $\Omega_A^{\cdot}$ by ω by $\omega *$, it's clear we have

$$\omega * = \omega + d\omega\, d$$

and hence

$$\boxed{\omega * \eta = \omega\eta + d\omega\, d\eta \qquad \begin{cases} \omega \in \Omega_A^{ev} \\ \eta \in \Omega_A^{\cdot} \end{cases}}$$

Thus a trace on B is a linear functional on Ω_A^{ev} such that

$$\boxed{\tau(\omega\eta + d\omega\, d\eta) = \tau(\eta\omega + d\eta\, d\omega)}$$

for all even forms η, ω. Similarly for linear functionals on $I^n/[I, I^{n-1}]$.

Next we would like to go from a trace on B (or I^n) to a trace on C (or a suitable power of K). Recall

$$C = \begin{pmatrix} eCe & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e} \end{pmatrix} \qquad B = eCe$$

A trace on C vanishes on the off-diagonal blocks, and so reduces to a pair of traces, one τ on eCe and the other $\bar{\tau}$ on $\bar{e}C\bar{e}$, since

$$\left[\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right] = \begin{pmatrix} XY & 0 \\ 0 & -YX \end{pmatrix}$$

these are linked by the condition

$$\tau(ez_1\bar{e}z_2e) - \bar{\tau}(\bar{e}z_2ez_1\bar{e}) = 0$$

for $z_1, z_2 \in C$.

The goal next is to show that a trace on B can be extended uniquely to a trace defined on CeC. We first check the formula

$$CeC \xleftarrow{\sim} Ce \otimes_{eCe} eC$$

and similar things.

We adopt the Morita viewpoint. We

have functors

$$X \longmapsto Ce \otimes_{eCe} X$$

$$\mathrm{Mod}(eCe) \rightleftarrows \mathrm{Mod}(C)$$

$$eC \otimes_C Y = eY \longleftarrow\!\!\shortmid\ Y$$

Thus eC is the eCe, C bimodule giving rise to the exact functor $M \mapsto eM$, and Ce is the C, eCe bimodule giving rise to $X \mapsto Ce \otimes_{eCe} X$. Then the composite $X \mapsto Ce \otimes_{eCe} eX$ is represented by the C-bimodule $Ce \otimes_{eCe} eC$. This is then idempotent:

$$(Ce \otimes_{eCe} \underbrace{eC) \otimes_C (Ce}_{= eCe} \otimes_{eCe} eC) = Ce \otimes_{eCe} eC,$$

and so it is the "universal cover" of the ideal CeC.

To show that the map

$$Ce \otimes_{eCe} eC \longrightarrow CeC$$

is an isomorphism it will be enough to show after applying e and $\bar{e}$ to both sides it gives an isomorphism. There are four possibilites, but since

$$\begin{array}{ccc} eC \otimes_C Ce \otimes_{eCe} eC & \longrightarrow & eC \otimes_C CeC \\ \shortparallel & & \shortparallel \\ eCe \otimes_{eCe} eC & \xrightarrow{\sim} & eCeC = eC \end{array}$$

one only has to worry about the $\bar{e}, \bar{e}$ case. Thus we want to show that

$$\bar{e}Ce \otimes_{eCe} eC\bar{e} \longrightarrow \bar{e}CeC\bar{e}$$

is an isomorphism. Now recall that we showed

$$eC\bar{e} = eA\bar{e} \otimes \bar{e}Ce = eCe \otimes eA\bar{e}$$

and we established additive descriptions

$$eC\bar{e} = eA\bar{e} + eAe\, eA\bar{e}\, \bar{e}Ae + eAe\,(eA\bar{e}Ae)^2 + \cdots$$

which give additive isomorphisms

$$eC\bar{e} \xrightarrow{\sim} \Omega_A^{odd}$$

Also

$$\bar{e}CeC\bar{e} = \bar{I} = \bar{e}A\bar{e}(\bar{e}AeA\bar{e}) + \bar{e}A\bar{e}(\bar{e}AeA\bar{e})^2 + \cdots$$

$$\xrightarrow{\sim} \Omega_A^{even, \geq 2}.$$

Thus

$$\bar{e}Ce \otimes_{eCe} eC\bar{e} \xrightarrow{\sim} \bar{e}Ce \otimes eA\bar{e}$$

$$\xrightarrow{\sim} \Omega_A^{odd} \otimes eA\bar{e} \xrightarrow{\sim} \Omega_A^{ev \geq 2}$$

which should do the job.

Summarizing we have

$$\boxed{\begin{aligned} \bar{J} = CeC &\xleftarrow{\sim} Ce \otimes_{eCe} eC \\ \bar{e}CeC\bar{e} &\xleftarrow{\sim} \bar{e}Ce \otimes_{eCe} \bar{e}Ce \end{aligned}}$$

The main question is whether we have an isomorphism

$$I^n/[I, I^{n-1}] \xrightarrow{\sim} K^{2n}/[K, K^{2n-1}]$$

There is a canonical map since $I \subset K^2$ and since $[K, K^{2n-1}]$ contains $[K^2, K^{2n-2}]$. Let's check that this map is onto. Recall

$$K = \begin{pmatrix} eC\bar{e}Ce & eC\bar{e} \\ \bar{e}Ce & \bar{e}CeC\bar{e} \end{pmatrix} \qquad K^2 = \begin{pmatrix} eC\bar{e}Ce & eC\bar{e}CeC\bar{e} \\ \bar{e}CeC\bar{e}Ce & \bar{e}CeC\bar{e} \end{pmatrix}$$

and
$$K^3 = \begin{pmatrix} eC\bar{e}CeC\bar{e}Ce & eC\bar{e}CeC\bar{e} \\ \bar{e}CeC\bar{e}Ce & \bar{e}CeC\bar{e}CeC\bar{e} \end{pmatrix}$$

and in general

$$K^{2n-1} = \begin{pmatrix} e(C\bar{e}Ce)^n & eC\bar{e}(CeC\bar{e})^{n-1} \\ \bar{e}Ce(C\bar{e}Ce)^{n-1} & \bar{e}(CeC\bar{e})^n \end{pmatrix}$$

$$K^{2n} = \begin{pmatrix} e(C\bar{e}Ce)^n & eC\bar{e}(CeC\bar{e})^n \\ \bar{e}Ce(C\bar{e}Ce)^n & \bar{e}(CeC\bar{e})^n \end{pmatrix}$$

The diagonal blocks of K^{2n} are respectively I^n and $\bar{I}^n$. We we divide by $[F, K^{2n}]$ we kill the off-diagonal blocks. Next take

$$X = e z_1 \bar{e} \qquad Y = \bar{e} z_2 e (z_3 \bar{e} z_4 e) \cdots (z_{2n-1} \bar{e} z_{2n} e)$$

I want to prove $I^n/[I,I^{n-1}] \xrightarrow{\sim} K^{2n}/[K,K^{2n-1}]$. It seems worthwhile to find a concrete approach like Connes' which uses linear isomorphisms with diffl forms. One reason is that this representation is suitable for ~~[illegible]~~ describing the I-adic filtration. We want a block description of C i.e. an isomorphism

$$\begin{pmatrix} eCe & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e} \end{pmatrix} \simeq \begin{pmatrix} \Omega^{ev} & \Omega^{odd} \\ \Omega^{odd} & \Omega^{ev} \end{pmatrix}$$

together with a formula for the multiplication in C relative to this isomorphism.

Let's recall the canonical linear isomorphism

$$\circledast \qquad eCe \simeq \Omega^{ev}$$

This is defined in such a way that

$$\rho(a_0)\, K(a_1,a_2)\cdots K(a_{2n-1},a_{2n}) \longleftrightarrow a_0\, da_1 \cdots da_{2n}$$

~~And that this~~ To define it we use $eCe = T(A)/1 = 1_A$ and define a left eCe module structure on Ω^{ev} by letting eae act as $a + dad$. Then

$$K(a_1,a_2) = ea_1ea_2e - ea_1a_2e = -ea_1\bar{e}a_2e$$
$$\longmapsto da_1da_2 \quad (\text{left mult. by})$$

so $\rho(a_0)K(a_1,a_2)\cdots K(a_{2n-1},a_{2n}) \longmapsto \omega + d\omega d$

where $\omega = a_0 da_1 \cdots da_{2n}$. The isomorphism $\circledast$ is obtained by ~~letting~~ applying an element of eCe to

$1 \in \Omega_A^{ev}$. Relative to this isomorphism the multiplication in eCe is

$$\omega * \eta = \omega\eta + d\omega\, d\eta$$

since $(\omega + d\omega\, d)(\eta + d\eta\, d) = \omega\eta + d\omega\, d\eta + d\omega\,\eta\, d + \omega\, d\eta\, d$

$$= (\omega * \eta) + d(\omega * \eta)\, d$$

Let's now consider the first column of C:

$$Ce = \begin{pmatrix} eCe \\ \bar{e}Ce \end{pmatrix}$$

We have a map (alg. homom.) $C \to \mathrm{End}_{eCe^\circ}(Ce)$ compatible with the $\mathbb{Z}/2$ grading. The idea is to combine the isom $eCe \simeq \Omega^{ev}$ with an isomorphism $\bar{e}Ce \simeq \Omega^{odd}$ and compute the C-action. We have

$$\bar{e}Ce = \bar{e}Ae + \bar{e}A\bar{e}C\bar{e}Ae = \bar{e}C\bar{e}Ae$$

$$= \bar{e}C\bar{e} \cdot \bar{e}Ae$$

and this is isomorphic to $\Omega_A^{ev} \otimes \bar{A} = \Omega_A^{odd}$. It clear that we want $\bar{e}C\bar{e}$ to act on Ω_A^{odd} so that left multiplication by $\bar{e}a\bar{e}$ is $a + (da)d$ and hence we want $\bar{e}C\bar{e} \simeq \Omega_A^{ev}$ to act on Ω_A^{odd} by the same formula:

$$\omega * \eta = \omega\eta + d\omega\, d\eta$$

Now we need to understand the operators associated to $\bar{e}ae$ and $ea\bar{e}$. It's reasonable to expect $\bar{e}ae$ to carry $1 \in \Omega^0$ to $da \in \Omega^1$. Let's

Try the formula

$$\eta * \omega = \eta\omega - d\eta\, d\omega \qquad \begin{matrix}\eta\in\Omega^{odd}\\ \omega\in\Omega^{ev}\end{matrix}$$

and see if this is compatible with an identification $\bar{e}Ce = \Omega_A^{odd}$.

$$\begin{aligned}
\omega_1 * (\eta * \omega) &= \omega_1 * (\eta\omega - d\eta\, d\omega) \\
&= \omega_1\eta\omega - \omega_1\, d\eta\, d\omega + d\omega_1\, d(\eta\omega) \\
&= \omega_1\eta\omega - \underset{①}{\omega_1\, d\eta\, d\omega} + \underset{②}{d\omega_1\, d\eta\, \omega} - \underset{③}{d\omega_1\, \eta\, d\omega} \\
(\omega_1 * \eta) * \omega &= (\omega_1\eta + d\omega_1\, d\eta) * \omega \\
&= \omega_1\eta\omega + d\omega_1\, d\eta\,\omega - d(\omega_1\eta)\, d\omega \\
&= \omega_1\eta\omega + \underset{②}{d\omega_1\, d\eta\, \omega} - \underset{③}{d\omega_1\, \eta\, d\omega} - \underset{①}{\omega_1\, d\eta\, d\omega}
\end{aligned}$$

Suppose we associate to a differential form ω the operator

$$\eta \longmapsto \omega * \eta = \omega\eta + (-1)^{\deg\omega} d\omega\, d\eta$$

Here ω and η can be of arbitrary degree.

Then

$$\begin{aligned}
\omega * (\xi * \eta) &= \omega * \left(\xi\eta + (-1)^{\deg\xi} d\xi\, d\eta\right) \\
&= \omega\xi\eta + (-1)^{\deg\xi}\omega\, d\xi\, d\eta + (-1)^{\deg\omega} d\omega\, d(\xi\eta) \\
&= \omega\xi\eta + (-1)^{\deg\xi}\omega\, d\xi\, d\eta + (-1)^{\deg\omega} d\omega\, d\xi\, \eta + (-1)^{\deg\omega+\deg\xi} d\omega\, \xi\, d\eta \\
&= \left(\omega\xi + (-1)^{\deg\omega} d\omega\, d\xi\right)\eta + (-1)^{\deg\xi+\deg\omega}\left(d\omega\,\xi + (-1)^{\deg\omega}\omega\, d\xi\right) d\eta \\
&= (\omega * \xi)\eta + (-1)^{\deg(\xi*\omega)} d(\omega*\xi)\, d\eta = (\omega*\xi)*\eta
\end{aligned}$$

Thus we have shown that

$$\boxed{\omega * \eta = \omega\eta + (-1)^{\deg \omega}\, d\omega\, d\eta}$$

defines an associative product on Ω_A.

Question: Do we get the algebra $A * A$?

If so we need to find two homomorphisms of A into this algebra.

Consider the map $a \longmapsto a + \lambda da$. Then

$$(a_1 + \lambda da_1) * (a_2 + \lambda da_2) = a_1 * a_2 + \lambda da_1 * a_2 + \lambda a_1 * da_2 + \lambda^2 da_1 * da_2$$

$$= a_1 a_2 + da_1 da_2 + \lambda\, da_1\, a_2 + \lambda\, a_1 da_2 + \lambda^2 da_1 da_2$$

$$= a_1 a_2 + \lambda\, d(a_1 a_2) + (1+\lambda^2)(da_1 da_2)$$

Consequently we get homomorphisms for $\lambda = \pm i$. Moreover if we let $F = \sigma$ on Ω_A, then

$$\sigma(a + i da)\sigma = a - ida$$

and so it's clear that we have the algebra $A * A$. We have

$$qa \longmapsto (a + ida) - (a - ida) = 2ida$$

June 23, 1988

From Itzykson, here are examples that might lead to an understanding of BRS. The general idea is that ghosts + BRS are tools needed to describe constrained systems.

1) Particle driven by noise. Here one considers an equation of motion such as

$$m\ddot{x} = f(\omega, t)$$

where $f(\omega, \cdot)$, $\omega \in$ prob. space, is a random function. One is interested in ~~some~~ the average value of some functional $F(x(t))$, where $x(t)$ is the solution of the equation of motion. Thus $x(t)$ and hence $F(x(t))$ is a fn. of ω and we integrate it over the prob. space. (This is classical?)

To treat as a constrained system one works in (path space) × (prob. space) and treats the equation of motion as a constraint, i.e. one uses

$$\delta(m\ddot{x} - f_\omega(t))$$

There's also a Jacobian factor leading to ghosts. (It seems this gives a quantum mechanical ~~[illegible]~~ treatment? ~~[illegible]~~ Yes.)

2) Motion on a sphere. Finite dimensional constraint situations were treated by BRS formalism in a paper of Fradkin. Historical notes: Ghosts first appeared (in the form of negative energy states) in a paper of Feynmann. The Fadeev-Popov paper is recommended although it's concisely written.

3) Gauge theory. Although mathematicians

would think the [redacted] problem was to pass to a quotient, physicists treat it as a constraint problem, because they choose a gauge-fixing

June 24, 1988

We are trying to find an additive description of

$$C = \begin{pmatrix} eCe & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e} \end{pmatrix}$$

which is convenient for understanding the filtration K^n. Thus we want an isomorphism

$$Ce = \begin{pmatrix} eCe \\ \bar{e}Ce \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \Omega^{ev} \\ \Omega^{odd} \end{pmatrix} = \Omega$$

and similarly for the other column $C\bar{e}$. As usual we obtain this by defining a left-module structure on Ω such that acting on 1 gives the isomorphism.

$$Ce = C/C\bar{e} \xrightarrow{\sim} \Omega$$

$$z \longmapsto z \cdot 1$$

~~[illegible]~~ Clearly $F \in C$ should act as σ on Ω, so the [redacted] problem is to find the action of A.

When we also consider the second column $C\bar{e}$ we see that we want to construct a C-action on $\Omega \oplus \Omega$ such that $F = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$ and such that [redacted] $\Omega \oplus \Omega$ becomes the free C-module with generator $1 \oplus 1$. ~~[illegible]~~ If we define

$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\Omega \oplus \Omega$, then ε induces the autom ε on C such that
$\varepsilon(a) = a$ and $\varepsilon(F) = -F$ provided A acts the same on both factors of $\Omega \oplus \Omega$.

We have seen that $\sigma(1+hd)$, h constant is an involution on Ω, so one way to obtain an action of A on Ω is

$$a \longmapsto \sigma(1+hd)\, a\, \sigma(1+hd) = (1-hd)\, a\, (1+hd)$$
$$= a - hda - h^2 da\, d$$

The simplest choice is $h = -1$. So let us define the A-action on $\Omega \oplus \Omega$ to be

$$a \longmapsto a + da - dad$$

Then
$$a = \frac{a + FaF}{2} + \frac{a - FaF}{2}$$
$$\longmapsto a - dad + da \qquad \text{as operators}$$

because $F(a + da - dad)F = \sigma(a + da - dad)\sigma = a - da - dad$.

The next step will be to compute the map $Ce \longrightarrow \Omega$, $z \longmapsto z \cdot 1$ defined by this C-action. Here F is acting as σ so e is the projection on Ω^{ev} and $\bar{e}$ is the projection on Ω^{odd}.

$$\frac{a - FaF}{2} = \bar{e}ae + ea\bar{e} \longmapsto da$$

$$(\bar{e}a_1 e + ea_1\bar{e})(\bar{e}a_2 e + ea_2\bar{e}) \longmapsto da_1 da_2$$
$$\shortparallel$$
$$\bar{e}a_1\bar{e}a_2 e + \bar{e}a_1 e a_2 e$$

Also we have

$$\frac{a + FaF}{2} = ea\bar{e} + \bar{e}ae \longrightarrow a - dad$$

Remark 1: Recall that to construct the Connes cocycles we have started with the flat "connection form" $\theta \in C^1(A, \overset{C}{\cancel{\operatorname{End} H}})$ and then split it into parts commuting and anticommuting with F

$$\rho = \frac{\theta + F\theta F}{2} \qquad \alpha = \frac{\theta - F\theta\theta}{2} = \bar{e}\theta e + e\theta\bar{e}$$

Here we are replacing $\overset{C}{\cancel{\operatorname{End} H}}$ with operators on Ω so that $\theta \in C^1(A, Op(\Omega))$ is the algebra homom. $a \longmapsto a + da - dad$ and ρ is the linear map $a \longmapsto a - dad$ so that the curvature is

$$-\alpha^2 = -\bar{e}\theta e\theta\bar{e} - e\theta\bar{e}\theta e = -d\theta d\theta.$$

(except that we are not certain about the last sign. In any case the curvature is the 2 cochain with values in $Op(\Omega)$ given by

$$(\delta\rho + \rho^2)(a_1, a_2) = \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$$

$$= (a_1 - da_1 d)(a_2 - da_2 d) - a_1 a_2 + d(a_1 a_2) d$$

$$= -\, da_1 da_2$$

Remark 2: The group of operators on Ω generated by $1 + hd = \exp(hd)$ and σ is a kind of infinite dihedral group $\mathbb{R} \rtimes \mathbb{Z}/2$

Now we are in a position to discuss the formula for the linear isomorphism $C \longrightarrow \Omega \oplus \Omega$. Recall the additive decomposition

$$eCe = eAe + eAe(A\bar{e}Ae) + eAe(A\bar{e}Ae)(A\bar{e}Ae) + \cdots$$

$$\bar{e}Ce = \bar{e}A\bar{e}Ae + \bar{e}A\bar{e}AeA\bar{e}Ae + \cdots$$

Take a typical generating element of eCe namely

$$ea_0e\,a_1\bar{e}\,a_2e \cdots\cdots a_{2n-1}\bar{e}\,a_{2n}e$$

This is realized by the operator

$$(a_0 - da_0d)\,da_1 \cdots da_{2n} \oplus 0 \qquad \text{on } \Omega\oplus\Omega$$

and so we have

$$ea_0e\,a_1\bar{e}\,a_2e \cdots a_{2n}e \longmapsto a_0\,da_1 \cdots da_{2n} \oplus 0 \in \Omega\oplus\Omega$$

Similarly a typical generating ellment of $\bar{e}Ce$ namely

$$\bar{e}a_0\bar{e}\,a_1e\,a_2\bar{e} \cdots a_{2n+1}e$$

acts as the operator

$$(a_0 - da_0d)\,da_1 \cdots da_{2n+1} \oplus 0 \quad \text{on} \quad \Omega\oplus\Omega$$

so we get

$$\bar{e}a_0\bar{e}a_1e \cdots a_{2n+1}e \longmapsto a_0\,da_1 \cdots da_{2n+1} \oplus 0 \in \Omega\oplus\Omega$$

So we see the correspondence between C and $\Omega\oplus\Omega$. Under this correspondence the subalg $A * A$ fixed by ε corresponds to $\Delta\Omega \subset \Omega\oplus\Omega$.

Our next step will be to find a formula

for the multiplication in C as a twisted multiplication in $\Omega \oplus \Omega$. Let's begin with the subalgebra fixed by ε. Given (ω, ω) where $\omega = a_0 da_1 \ldots da_n$ we know it corresponds to

$$\begin{pmatrix} e a_0 e & 0 \\ 0 & \bar{e} a_0 \bar{e} \end{pmatrix} \begin{pmatrix} 0 & e a_1 \bar{e} \\ \bar{e} a_1 e & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & e a_n \bar{e} \\ \bar{e} a_n e & 0 \end{pmatrix}$$

which in turn acts as the operator

$$(a_0 - da_0 d) da_1 \ldots da_n = \omega + d \circ (d\omega)$$
$$= \omega + (-1)^{\deg \omega + 1} d\omega\, d$$

Thus

$$(\omega, \omega) * (\eta_1, \eta_2) = (\omega * \eta_1, \omega * \eta_2)$$

where

$$\omega * \eta = \omega\eta - (-1)^{\deg \omega} d\omega\, d\eta .$$

Next we have to adjoin F which acts as $\sigma \oplus (-\sigma)$ on $\Omega \oplus \Omega$.

Recapitulate: We are trying to define an action of C on the set of matrices over Ω

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where α, δ are even forms and where β, γ are odd forms. We want

$$F = \text{left mult by} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that F acts as σ on the first column and $-\sigma$ on the second. We also want

$$a = \text{left mult. by the operator} \begin{pmatrix} a - dad & da \\ da & a - dad \end{pmatrix}$$

which means

$$a \longmapsto \begin{pmatrix} a & da \\ da & a \end{pmatrix}^{*}$$

$*$ means matrix mult. relative to the $*$ product.

Thus it seems we should simply start with the subalgebra of $M_2(\Omega)$, where Ω is equipped with the $*$ product

$$\omega * \eta = \omega\eta - (-1)^{\deg \omega} d\omega d\eta,$$

consisting of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ with $\alpha, \delta \in \Omega^{ev}$, $\beta, \gamma \in \Omega^{odd}$.

We take $F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and note that

$$a \longmapsto \begin{pmatrix} a & da \\ da & a \end{pmatrix}$$

is a homomorphism. Thus we get a map $C \longrightarrow$ this algebra, which should be an isom.

Let's call this algebra $\tilde{C}$ and define ε to conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The centralizer of C is the subalgebra of $\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ and it can be identified with Ω equipped with the $*$ product. This is the model for $A * A$. Note also that if we adjoin $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\tilde{C}$ we get $M_2(\Omega, *)$

Next I want to understand traces defined on a power K^{2n} of $C = \tilde{C}$ Now K^{2n} consists $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\deg \alpha, \deg \delta \geq 2n$ and $\deg \gamma, \deg \beta \geq 2n+1$.

Here is the definitive picture of the relations between $A*A$ and Ω_A.

Let

$$C = A * \mathbb{C}[F] = (A*A) \tilde{\otimes} \mathbb{C}[F]$$

If $a \in A$, let's write

$$a = \underbrace{\frac{a+FaF}{2}}_{a^+} + \underbrace{\frac{a-FaF}{2}}_{a^-}$$

in C. I claim any element of C is a linear combination of elements of the form

$$a_0^+ a_1^- \cdots a_n^- \qquad \text{or} \qquad a_0^+ a_1^- \cdots a_n^- F$$

for different $n \geq 0$. To see this we have to show the subspace of C spanned by these elements is closed under left mult. by F and any $a \in A$. But

$$F(a_0^+ a_1^- \cdots a_n^-) = (-1)^n a_0^+ a_1^- \cdots a_n^- F$$

$$a a_0^+ = a(a_0 - a_0^-) = (aa_0)^+ + (aa_0)^- - a^+ a_0^- - a^- a_0^-$$

$$\boxed{a(a_0^+ a_1^- \cdots a_n^-) = \left[(aa_0)^+ + (aa_0)^- - a^+ a_0^- - a^- a_0^-\right] a_1^- \cdots a_n^-}$$

Another approach: a superalgebra is simply an algebra equipped with an action of $\mathbb{Z}/2$. $A * A$ is a superalgebra equipped with a morphism of algebras $A \longrightarrow A*A$, namely $a \longmapsto in_1(a)$; this is a universal map from A to the underlying algebra of a superalgebra. Let

$$a = a^+ + a^-$$

be the decomposition of $a = in_1(a)$ in $A*A$. Then the claim is that $A*A$ is spanned by elements of the form $a_0^+ a_1^- \cdots a_n^-$. The ~~abs~~space spanned by these elements is stable under the $\mathbb{Z}/2$-action clearly and closed under left multiplication by elements of A hence it [redacted] must be all of $A*A$.

Next map Ω_A to $A*A$ by

$$\Phi : a_0 da_1 \cdots da_n \longmapsto a_0^+ a_1^- \cdots a_n^-$$

and note that this map is onto by the above discussion. Also

$$a\,\Phi(a_0 da_1 \cdots da_n) = a(a_0^+ a_1^- \cdots a_n^-)$$

$$= \left((aa_0)^+ + (aa_0)^- - a^+a_0^- - a^-a_0^-\right) a_1^- \cdots a_n^-$$

$$= \Phi\left(\left(aa_0 + d(aa_0) - a\,da_0 - da\,da_0\right) da_1 \cdots da_n\right)$$

$$= \Phi\left((a + da - da\,d)(a_0 da_1 \cdots da_n)\right)$$

Next we define an action of $A*A$ on Ω_A. Let $\sigma(\omega) = (-1)^{\deg \omega}\,\omega$ be the usual $\mathbb{Z}/2$ grading on $\Omega = \Omega_A$, so that $\mathrm{End}(\Omega)$ becomes a superalgebra. We note that $\sigma(1-d) = (1+d)\sigma$ is an involution

$$(1+d)\sigma(1+d)\sigma = (1+d)(1-d) = 1$$

hence [redacted] the map $A \longrightarrow \mathrm{End}(\Omega)$ given by

$$a \longmapsto \sigma(1-d)\,a\,\sigma(1-d) = (1+d)\,a\,(1-d)$$

$$= a + da - da\,d$$

is an alg. homomorphism. This algebra morphism extends uniquely to a superalgebra morphism $A*A \longrightarrow \mathrm{End}(\Omega)$, defining a left $A*A$-module structure on Ω. We have

$$a^+ \longmapsto a - da\,d \qquad\qquad a^- \longmapsto da$$

hence

$$a_0^+ a_1^- \cdots a_n^- \longmapsto (a_0 - da_0 d)\, da_1 \cdots da_n$$

Now let $\Psi : A * A \longrightarrow \Omega$ be the $A*A$ module map with $\Psi(1) = 1$. Thus

$$\Phi(a_0^+ a_1^- \cdots a_n^-) = (a_0 - da_0 d)\, da_1 \cdots da_n (1)$$

$$= a_0\, da_1 \cdots da_n .$$

Clearly Φ and Ψ are inverses.

Let

$$\omega * \eta = \Phi(\Phi^{-1}\omega \cdot \Phi^{-1}\eta) = \Phi^{-1}(\omega)\cdot \eta$$

be the product in $A*A$ transported to Ω by the isomorphism Φ. If $\omega = a_0\, da_1 \cdots da_n$ we have

$$\Phi^{-1}(\omega)\cdot\eta = (a_0 - da_0 d)\, da_1 \cdots da_n\, \eta$$

$$= \omega\eta - (-1)^n\, d\omega\, d\eta .$$

Hence we have proved

<u>Proposition:</u> The correspondence

$$a_0^+ a_1^- \cdots a_n^- \longleftrightarrow a_0\, da_1 \cdots da_n$$

gives a vector space isomorphism between $A*A$ and Ω_A relative to which the product in $A*A$ becomes the twisted product on Ω_A given by

$$\boxed{\omega * \eta = \omega\eta - (-1)^{\deg\omega}\, d\omega\, d\eta}$$

Further, the $\mathbb{Z}/2$-action on $A*A$ becomes the involution $\sigma(\omega) = (-1)^{\deg\omega}\omega$, on Ω_A. If $K = \operatorname{Ker}\{A*A \longrightarrow A\}$, then K^n corresponds to $\Omega_A^{\geq n}$.

Next we consider cyclic cocycles. Suppose τ is a linear functional on $K^n/[K, K^{n-1}]$. Then we obtain cyclic $2k$-cocycles on A for $2k+1 \geq n$ given by

$$\varphi(a_0, \ldots, a_{2k}) = \tau(\bar{a_0} \cdots \bar{a_{2k}})$$

To see this is a cyclic cocycle, we first observe it satisfies the cyclic symmetry condition. Secondly if we use the isomorphism $A * A \simeq \Omega$, then τ becomes a trace on $\Omega_A^{\geq n}$ for the $*$-product, and we have

$$\varphi(a_0, \ldots, a_{2k}) = \tau(da_0 \ldots da_{2k})$$
$$= \tau d(a_0 da_1 \ldots da_{2k})$$

This will be a Hochschild $2k$-cocycle provided $\tau d : \Omega^{2k} \longrightarrow \mathbb{C}$ satisfies

$$\begin{array}{ccc} \tau d(a\omega) & \overset{?}{=} & \tau d(\omega a) \\ \| & & \| \\ \tau(da\,\omega + a\,d\omega) & & \tau(d\omega\, a + \omega\, da) \end{array}$$

for all $a \in A$, $\omega \in \Omega^{2k}$. But this follows because $\omega * \eta = \omega\eta$ when either ω or η is closed, and because τ is a trace for the $*$ product. To be more precise this argument requires τ to be a linear functional on

$$K^{2k+1}/[K, K^{2k}]$$

so it seems to work.

Next we observe that the cocycle φ uses only [redacted] τ on $\Omega^{\text{odd}, \geq n}$. Thus we get the same cyclic cocycles from the trace $\frac{1}{2}(\tau - \tau\sigma)$ which is supported on $\Omega^{\text{odd}, \geq n}$.

Interesting point. Let us consider a homomorphism $A \longrightarrow R$ where R is a superalgebra. Suppose τ is a trace on R, but do not suppose that the $\mathbb{Z}/2$ grading of R is given by an involution in R. Then we obtain even cocycles on A as follows.

We consider the homomorphism $A \longrightarrow R$ as a connection form $\theta \in C^1(A, R)$ which is flat. We split the connection $\delta + \theta$ into parts

$$\delta + \theta = \underbrace{(\delta + \theta^+)}_{\nabla} + \theta^-$$

which are even and odd. Then the usual argument shows that $[\nabla, \theta^-] = 0$

$$\delta \operatorname{tr} (\theta^-)^m = \operatorname{tr} [\nabla, (\theta^-)^m] = 0$$

Now $\operatorname{tr}(\theta^-)^m = 0$ for m even > 0, however we can't conclude $\operatorname{tr}(\theta^-)^{2k+1} = 0$ as before because we don't have an F in R giving the $\mathbb{Z}/2$ grading. Thus we have potentially nontrivial cocycles $\operatorname{tr}(\theta^-)^{2k+1}$.

Thus we can give a θ-type proof that $\tau(a_0^- \cdots a_{2k}^-)$ is a cyclic cocycle.

Next we consider the ungraded case. Here we start with a trace τ on $\tilde{K}^n/[\tilde{R}, \tilde{K}^{n-1}]$ where $C/\tilde{K} = A \otimes \mathbb{C}[F]$ or $\tilde{K} = K \tilde{\otimes} \mathbb{C}[F] \subset (A * A) \tilde{\otimes} \mathbb{C}[F] = C$.

Then we use τ to define odd cocycles

$$\varphi(a_0, \ldots, a_{2k-1}) = \tau(F\bar{a}_0 \cdots \bar{a}_{2k-1})$$

Let ε be the grading operator on C with $\varepsilon(a) = a$, $\varepsilon(F) = -F$. Then

$$\tau\varepsilon\left[F\bar{a}_0 \cdots \bar{a}_{2k-1}\right) = -\tau(F\bar{a}_0 \cdots \bar{a}_{2k-1})$$

so that we get the same cocycle φ from the trace $\frac{1}{2}(\tau - \tau\varepsilon)$ on C which is odd relative to the grading ε.

Let's now try to describe traces τ on C with $\tau\varepsilon = -\tau$. If we use $C = (A*A) + (A*A)F$ then $\tau\varepsilon = -\tau$ means τ is supported on $(A*A)F$, and so τ is determined from the linear fnl. λ on $A*A$ given by

$$\lambda(\xi) = \tau(F\xi).$$

Then $\lambda(F\xi F) = \lambda(\xi)$ so λ is supported on the even part of $A*A$. Let's use the isom. $A*A = \Omega$. Then we have a linear functional λ on Ω which is supported on Ω_A^{ev}. From the fact that τ is a trace we conclude that λ is an even supertrace on Ω supported on Ω^{ev}.

<u>Lemma</u>: Let R be a superalgebra.

1) There's an equivalence between odd traces and odd supertraces on R. Here odd means supported on R^-.

2) There's an equivalence between even supertraces on R and odd traces on $R \hat{\otimes} \mathbb{C}[F]$

Proof: 1) To verify a trace or supertrace identity one considers elements appearing to be homogeneous. Supertraces and traces τ differ only when applied to a pair of odd elements x, y of R. In this case xy is even so $\tau(xy) = \tau(yx) = 0$.

2). In general $A \hat{\otimes} B / [\,,\,]_{super} = A/[\,,\,]_s \otimes B/[\,,\,]_s$ hence there's an equivalence between supertraces on R on $R \hat{\otimes} \mathbb{C}[F] = \tilde{R}$, but with parity reversal, i.e. even supertraces on R correspond to odd supertraces on $\tilde{R}$. **But odd supertraces and odd traces on $\tilde{R}$ are equivalent** by 1). ■ QED.

At this point I have reached an understanding of Connes theorem identifying odd traces on $A * A$ and $C = (A*A) \hat{\otimes} \mathbb{C}[F]$ with respectively odd and even supertraces, on Ω_A equipped with the $*$ product.

Actually there appears to be an error in the ungraded case. First of all if we take the tensor product in the sense of superalgebras of $A*A$ and $\mathbb{C}[F] = C_1$, then we do get $C = A * \mathbb{C}[F]$. as algebras, but not as superalgebras; because the $\mathbb{Z}/2$ grading is wrong. However since the cyclic cocycle φ depends upon τ applied to $F a_0 \ldots a_{2k-1}$ which has odd degree for either the total degree or the F-degree, the total degree might also work.

?

In any case let's try to describe the odd traces on C for the F-grading. Thus τ is a trace on

$$C = A*A \oplus (A*A)F \qquad \text{which vanishes on } A*A.$$

Look at $\lambda(\xi) = \tau(\xi F)$ for $\xi \in A*A$.
Then $\lambda(\xi) = 0$ if ξ is odd i.e. $F\xi F = -\xi$.
Moreover

$$\lambda(\xi_1\xi_2) = \tau(\xi_1\xi_2 F) = (-1)^{\deg \xi_2}\tau(\xi_1 F \xi_2)$$
$$= (-1)^{\deg \xi_2}\lambda(\xi_2\xi_1).$$

Thus we have

$$\lambda(\xi_1\xi_2) = (-1)^{\deg \xi_1 \deg \xi_2}\lambda(\xi_2\xi_1)$$

since this is true when ξ_1, ξ_2 have the same parity, and since both sides are zero where they have different parity. Thus λ is an even supertrace on $A*A$.

Conversely suppose λ is an even supertrace on $A*A$. Define τ on C by

$$\tau(\xi) = 0 \qquad \xi \in A*A$$
$$\tau(\xi F) = \lambda(\xi)$$

We have to check that τ is a trace on C. The only thing to check is that

$$\tau(\xi_1\xi_2 F) \stackrel{?}{=} \tau(\xi_2 F \xi_1)$$

where ξ_1, ξ_2 have the same parity. This becomes

$$\lambda(\xi_1\xi_2) \stackrel{?}{=} (-1)^{\deg \xi_1}\lambda(\xi_2\xi_1)$$

which is OK.