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A gauge equivalent for superconnections

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Let's recall some earlier work on cyclic cocycles and left-invariant differential forms on groups of gauge transformations. Let \( M, E, D \) be a manifold, vector bundle, a connection, and \( \mathcal{G} = \text{Aut}(E) \). We consider \( \text{pr}_2^*(E) = \mathcal{G} \times E \), where \( \text{pr}_2 : \mathcal{G} \times M \to M \), and we identify 
\[
\Omega^n(D \times M, \text{End}(\text{pr}_2^*E)) = \Omega^n(\mathcal{G}, \Omega^*(M, \text{End} E))
\]

On \( \text{pr}_2^*(E) \) we have the full-back of \( D \) which we denote \( \delta + D \) where \( \delta = d_g \). We also have a natural automorphism \( \tilde{g} \) of \( \text{pr}_2^*(E) \), and we transform this connection via \( \tilde{g} \):
\[
\tilde{g}^{-1}(\delta + D)g = \delta + \tilde{g}^{-1}(dg) + D + \tilde{g}^{-1}(Dg)
\]
where \( \Theta = \text{the Maurer-Cartan form on } \mathcal{G} \).

To get left-invariant forms on \( \mathcal{G} \), we can use the family of connections \( \delta + D + t\Theta \); the curvature is
\[
(\delta + D + t\Theta)^2 = D^2 + tD(\Theta) + (t^2 - t)\Theta^2
\]
For \( t = 0,1 \) this is flat in the \( \mathcal{G} \)-direction.

To simplify suppose \( E = V, \quad D = d \). Then the transgression formula becomes
\[
(\delta + d) \int_0^1 \text{tr} \left( e^{t d\Theta} (t^2 - t)\Theta^2 \right) dt = \text{tr} \left( e^{d\Theta} - 1 \right)
\]

or better taking a component we see
\[
\eta = \int_{2n+1}^1 \text{tr} \left( \Theta (d\Theta + (t^2 - t)\Theta)^n \right) dt \in \left[ \Omega^*(\mathcal{G}), \Omega^*(M) \right]_{2n+1}
\]
satisfies
\[
\frac{1}{n!} (\delta + d) \eta = \frac{1}{(n+1)!} \text{tr} (d\Theta)^{n+1}
\]
In particular
\[ \Theta(d\Theta)^n \in C^{n+1}(\Omega^n, \Omega^{n+1}/\Omega^n) \]
will be killed by \( \partial \), i.e., it is a cyclic \( n \)-coboundary with values in \( \Omega^n/\Omega^{n-1} \).

We can make this all algebraic as follows. Given \( A \) we can consider the double complex
\[ C_\cdot^n(A, \Omega^\cdot) \]
of Hochschild cochains on \( A \) with values in \( \Omega^\cdot \) equipped with the zero bimodule structure. There is a natural trace on this algebra which values in the double complex
\[ C_\cdot^n(A, \Omega^\cdot/[\Omega^\cdot, \Omega^\cdot]) \]
with a shift in degree.

We have \( \Theta \in C^1(A, \Omega^\cdot) \) given by the identity map \( A \to A \), and it satisfies
\[ \delta \Theta = -\Theta^2 \]
so we can use the above formalism, and we obtain from the components of
\[ \int \Theta + \left(t \Theta + t^2 \Theta^2\right) dt \]
various cyclic cochains on \( A \) with values in \( \Omega^\cdot/[\Omega^\cdot, \Omega^\cdot] \). Moreover if we divide out by the image of \( d \Omega^\cdot \), we get cyclic cocycles.
The goal is now to see if we can achieve an understanding of the $\delta$-operator. I want to obtain it from a non-unital homomorphism $a \mapsto a e_j$ as Connes does.

Let's begin with a simple example. Let $M$ be a manifold, let $A = C^\infty(M)$, let $\delta$ be a closed current of dim $p$ with compact support. Then we obtain a cyclic $p$-co-cycle on $A$ given by

$$\varphi(a_0, \ldots, a_p) = \int a_0 da_1 \ldots da_p.$$ 

Moreover we have cyclic $p$-co-cycles in the matrix algebras $M_n(A)$ given by

$$\varphi(x_0, \ldots, x_p) = \int tr(x_0 dx_1 \ldots dx_p).$$

Let's identify $M_n(A)$ with $\text{End}(\tilde{V})$ where $\tilde{V}$ is the trivial vector bundle with fibre $V = C^\infty$. Let $E$ be a vector bundle which is a direct summand of $\tilde{V}$:

$$E \xrightarrow{i} \tilde{V} \xleftarrow{i^*} \quad i^* i = id_E.$$ 

Then we have a (non-unital) homomorphism

$$\text{End}(E) \to \text{End}(\tilde{V})$$

and so we obtain a cyclic $p$-co-cycle on $\text{End}(E)$. On the other hand we have a (unital) homomorphism

$$A \to \text{End}(E)$$

which is
an isomorphism when $E$ is a line bundle. Thus we obtain another cyclic $p$-cocycle on $A$.

The composition

$$A \rightarrow \text{End}(E) \rightarrow \text{End}(\mathcal{V}) = \mathcal{M}_n(A)$$

is $a \rightarrow a e$ where $e = 1_{\mathcal{V}^*}$, and so the new cyclic cocycle is

$$\psi(a_0, \ldots, a_p) = \int \text{tr} \left( a e \circ d(a_0) \circ \cdots \circ d(a_p) \right)$$

Comes method of defining the $S$-operator in cyclic cocycles is to form the cochain algebra

$$\mathcal{O}_A^+ \otimes \mathcal{O}_C[e]$$

and use the universal property of $\mathcal{O}_A^+$ to extend the homomorphism

$$A^+ \rightarrow A^+ \otimes C[e]$$

$$a \rightarrow a e$$

to a map of cochain algebras

$$\mathcal{O}_A^+ \rightarrow \mathcal{O}_A^+ \otimes \mathcal{O}_C[e]$$

This gives a map of complexes

$$\mathcal{O}_A^+ /[\cdot, \cdot] \rightarrow \left( \mathcal{O}_A^+ /[\cdot, \cdot] \right) \otimes \left( \mathcal{O}_C[e]/[\cdot, \cdot] \right)$$

and calculation shows

$$\mathcal{O}_C[e]/[\cdot, \cdot]$$

has basis $e (de)^i$ for $i \geq 0$. 


Returning to (2) we see that it is a sum of $2p$ terms and in each we can move all the $a_i$'s to the left and the $e_i$'s to the right. If $p = 2$ we have

$$
\text{tr} \ a_0 (da_1 e + a_1 de)(da_2 e + a_2 de) = a_0 da_1 da_2 \text{tr}(e) + a_0 da_1 a_2 \text{tr}(ede) + a_0 a_1 da_2 \text{tr}(cde e) + a_0 a_1 a_2 \text{tr}(edede)
$$

Thus we find that

$$
\int \text{tr} \left\{ a_0 e \, d(a_1 e) \, d(a_2 e) \right\} = \left( \int a_0 da_1 da_2 \right) \text{tr} + \int a_0 a_1 a_2 \text{tr}(ede^2).
$$

The $j$-th term in general can be described as $S_j$ applied to the cyclic $p-2j$ cocycle associated to the current $\text{tr}(c^{(de)^2j}) \wedge \gamma$.

**General question**: We've seen how to associate to a closed current $\gamma$ in $M$ and a vector bundle $E$ which is expressed as a direct summand of a trivial bundle a cyclic cocycle on $\text{End}(E)$. Can we write this cyclic cocycle in terms of the induced connection on $E$? The induced connection is

$$
\nabla = i^* d i
$$

and

$$
\nabla^2 = i^* d i \wedge i^* d i = -i^* d j \wedge i^* d i = -i^* [d_j] \wedge [d_i].
$$
So if \( \alpha_0, \alpha_1, \alpha_2 \in \text{End}(E) \)

\[
\text{tr}(\pi L^*_x \ast [d, i \alpha_1 \ast] [d, i \alpha_2 \ast]) = \text{tr} (\alpha_0 i^* [d, i \alpha_1 \ast] (i i^* + j j^*) [d, i \alpha_2 \ast] i)
\]

Now \( i^* [d, i \alpha_1 \ast] i = i^* d i \alpha x - \alpha i^* d i = [\nabla_0 \alpha] \)

and

\[
[d, i \alpha_1 \ast] j^* [d, i \alpha_2 \ast] = \alpha_1 [\nabla_0 \alpha] j^* [d, i] \alpha_2 \ast
\]

\[
- i^* [d, i] j^*
\]

\( = \alpha_1 \nabla^2 \alpha_2 \ast \)

So the 2-cocycle is

\[
\text{tr} (\alpha_0 ([\nabla \alpha_1] [\nabla \alpha_2] + \alpha_1 \nabla^2 \alpha_2))
\]

And the 3-cocycle is

\[
\text{tr} (\alpha_0 i^* [d, i \alpha_1 i^*] (i i^* + j j^*) [d, i \alpha_2 i^*] (i i^* + j j^*) [d, i \alpha_3 i^*] i)
\]

\[
= \text{tr} (\alpha_0 [\nabla \alpha_1] [\nabla \alpha_2] [\nabla \alpha_3] + \alpha_0 [\nabla \alpha_1] \alpha_2 \nabla^2 \alpha_3
\]

\[
+ \alpha_0 \alpha_1 \nabla^2 \alpha_2 [\nabla \alpha_3]
\]

In the case of the n-cocycle there are potentially \( 2^{n-1} \) terms however one can't choose \( j j^* \) consecutively.

Thus the 4-cocycle should be

\[
\text{tr} \left( \alpha_0 [\nabla \alpha_1] [\nabla \alpha_2] [\nabla \alpha_3] [\nabla \alpha_4] + \alpha_0 \alpha_1 \nabla^2 \alpha_2 [\nabla \alpha_3] [\nabla \alpha_4]
\right.
\]

\[
+ \alpha_0 [\nabla \alpha_1] \alpha_2 \nabla^2 \alpha_3 [\nabla \alpha_4] + \alpha_0 [\nabla \alpha_1] [\nabla \alpha_2] \alpha_3 \nabla^2 \alpha_4
\]

\[
+ \alpha_0 \alpha_1 \nabla^2 \alpha_2 \alpha_3 \nabla^2 \alpha_4
\]
Consider a vector bundle which is a direct summand of a trivial bundle

$$\tilde{V} = E \oplus E' \quad E \xrightarrow{i} \tilde{V} \xrightarrow{d^*} E'$$

Suppose we have a cyclic cocycle on $C^\infty(M)$. Then by the trace it gives a cyclic cocycle on $\text{End}(\tilde{V})$ and by restriction a cyclic cocycle on $\text{End}(E)$.

To be specific take $\varphi(a_0, \cdots, a_p) = \int a_0 \, da_1 \cdots da_p$.

More generally we should take

$$\varphi(a_0, \cdots, a_p) = a_0 \, da_1 \cdots da_p \mod d \Omega^{p-1}$$

or using Maurer-Cartan form notation

$$\varphi = \Theta(d\Theta)^p \mod d \Omega^{p-1}.$$ 

On $\text{End}(\tilde{V})$ this becomes

$$\varphi = \text{tr} \Theta(d\Theta)^p \mod d \Omega^{p-1}$$

where $\text{tr}$ denotes $\text{tr}_V$.

Now restrict to $\text{End}(E)$. Let's use block notation relative to the decomposition $E \oplus E' = \tilde{V}$.

Let $\Theta$ now be the canonical element in $C'(\text{End} E, \text{End} E)$.

The restriction of $\varphi$ is then

$$\text{tr}_V(i \Theta \iota^* \cdot d(i \Theta \iota)^p)$$

Now

$$i \Theta \iota^* = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} \quad - \Theta(i \iota^* d)$$

$$d(i \Theta \iota^*) = [d, i \Theta \iota^*] = \begin{pmatrix} i \iota^*[d, i \Theta \iota^*] & i \iota^*[d, i \Theta \iota^*] \\ j \iota^*[d, i \Theta \iota^*] & j \iota^*[d, i \Theta \iota^*] \end{pmatrix}$$

$$i \iota^*[d, i \Theta \iota^*] = (i \iota^* d) \Theta - \Theta(i \iota^* d) = [\nabla, \Theta] = \nabla(\Theta)$$
Thus we are dealing with the cocycle
\[ \text{tr}_V \left\{ (0 \ 0) \begin{bmatrix} [\nabla, \Theta] & \Theta(i^*d_j) \end{bmatrix}^p \right\} \]

This is a sum of terms which can be described as follows. Think in terms of two state system with states \( E, E' \) where \( E' \) has "zero energy".

Actually we should probably calculate the resolvent. Recall the identity
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}
\]
The natural generating function for the family of cocycles \( \Theta \) is
\[ \text{tr}_V \left\{ (0 \ 0) \begin{bmatrix} a-[\nabla, \Theta] & \Theta(i^*d_j) \end{bmatrix}^{-1} \right\} \]
\[
\begin{aligned}
= & \text{tr}_E \left( \Theta \begin{pmatrix} 1 \\ -[\nabla, \Theta] \end{pmatrix} \begin{pmatrix} \lambda & \frac{1}{\lambda} \Theta \nabla^2 \Theta \\ \lambda^{-1} \Theta(i^*d_j) \frac{1}{\lambda} (-i^*d_j) \Theta \end{pmatrix} \right) \\
= & \text{tr}_E \left( \Theta \frac{1}{\lambda - [\nabla, \Theta] - \frac{1}{\lambda} \Theta \nabla^2 \Theta} \right)
\end{aligned}
\]

Alternatively the \( p \)-th cocycle is the \( \text{tr}(\Theta H) \) where \( H \) is the sum of all monomials in \([\nabla, \Theta]\) and \( \Theta \nabla^2 \Theta \) of degree \( p \), where \([\nabla, \Theta]\) has degree 1 and \( \Theta \nabla^2 \Theta \) has degree 2.

If we further restrict to \( C^\infty(M) \subset \text{End}(E) \), then \([\nabla, \Theta] \mapsto d\Theta\), \( \Theta \nabla^2 \Theta \mapsto \Theta^2 \nabla^2 \) and we get the formula for the \( p \)-th cocycle:
Our next project should be the link between the $S$-operator and the different cyclic cocycles attached to an operator. Let's consider the odd case. We have an algebra $A$ acting in a Hilbert space $H$ and an involution $F$ on $H$ such that $[F,F_0]$ belongs to an ideal $I$ such that a trace is defined on $I$. Then we have the cyclic cocycles

$$
\psi(q_0, \ldots, q_{2n-1}) = \text{tr} (F[F_0, q_0] \cdots [F, q_{2n-1}])
$$

$$
= 2 \text{tr} (q_0 [F_0 q_1] \cdots [F, q_{2n-1}])
$$
defined for $n$ large enough.

What is needed is a way to link $S_{2n-1}$ and $S_{2n+1}$. This requires an understanding of $S_{2n-1}$. According to Cremers what we do is to enlarge $A$ by adjoining an idempotent commuting with $A$, and then you find a suitable cyclic cocycle of degree $2n+1$ on the enlarged algebra. This gives $S_{2n-1}$. 
Let $\mathcal{R}$ be a superalgebra: $\mathcal{R} = \mathcal{R}^+ \oplus \mathcal{R}^-$ and let $\mathcal{F}$ be an odd involution in $\mathcal{R}$. Then $[\mathcal{F}, \mathcal{F}]$ is an odd derivation of $\mathcal{D}$ whose square is zero, since

$$
[F, [F, x]] = F(Fx - xF) + (Fx - xF)F \\
= [F^2, x] = 0
$$

$$
[F, F, x] = F(Fx + xF) - (Fx + xF)F \\
= [F^2, x] = 0
$$

(Actually the argument should be for $\mathcal{D}$ of odd degree)

$$
[D, [D, x]] = [D, D, x] - [D, [D, x]]
$$

so

$$
[D, [D, x]] = \frac{1}{2} [D, D^2, x] = [D^3, x].
$$

Thus we see that

$$
\mathcal{R}^+ \xrightarrow{[\mathcal{F}, \mathcal{F}]} \mathcal{R}^- \xleftarrow{[\mathcal{F}, \mathcal{F}]} \mathcal{R}^+
$$

is a super version of a DGA. Notice that such a super version can be expanded to become either a chain or cochain algebra

$$
\mathcal{R}^+ \rightarrow \mathcal{R}^- \rightarrow \mathcal{R}^+ \rightarrow \ldots
$$

$$
\mathcal{R}^+ \leftarrow \mathcal{R}^- \leftarrow \mathcal{R}^+ \leftarrow \ldots
$$

Principle to be followed (perhaps): Use super arguments with adic filtrations.
Consider next an algebra $R$ with involution $F$, ungraded. Then we can form the superalgebra $R \otimes C_1 = R \oplus R\sigma$ with the odd involution $\sigma F$. This gives the super DGA:

$$
\begin{array}{c}
R \\ \downarrow \quad [F, \cdot] \\
R \quad \quad [F, \cdot] +
\end{array}
$$

Example. Let $C = \varepsilon A = A \times C[F] = (A \times A) \times \mathbb{Z}/2$. This is a superalgebra where $\varepsilon = 1$ on $A$ and $\varepsilon = -1$ on $F$. In this case

$$
C^+ = A \times A \\
C^- = (A \times A) F
$$

Note that $C^+ = A \times A$ is a superalgebra and that $C$ is the result of adjoining the grading on $C^+$ to $C^+$. This grading on $A \times A$ is compatible with the axiom

$$
\text{gr}(A \times A) = \Omega_A
$$

In the ungraded case we look at an algebra $A$ and an involution $F$ acting on a space $H$. Thus $C = A \times C[F]$ acts on $H$. In the graded case we ask that there be an grading $\varepsilon$ on $H$ compatible with the grading on $C$. That is, we want $A$ to be of even degree and $F$ of odd degree with respect to their action on $H$. What this means is that we have an action of

$$
\hat{C} \otimes C[\varepsilon] = C^+ \otimes C_2
$$

i.e. $H = H^+ \oplus H^+$ with $\varepsilon = (1, 0)$, $F = (0, 1)$

and where we have two actions of $A$ on $H^+$.
It might be useful to try to describe our program. I think the main goal is to understand the Connes cocycles without being tied to $\Omega^A$.

Let's now look at Connes cocycles. We start with $A, F$ acting on $H$ with $[F, \theta]$ in some Schatten ideal so that

$$\text{tr}(F[F, \theta]^{2n}) = 2\text{tr}(\theta[F, \theta]^{2n-1})$$

is defined. We claim this is a cyclic $2n$-cocycle.

There are two proofs at least. Let's start with my proof using the connection and curvature formalisms.

Let $\widetilde{W} \oplus \widetilde{W}' \sim H$ be the $t+1$ eigenspaces of $F$ and $i_+^*(\eta) \in F^*$ as usual. Over $G = \text{GL}_n(A)$ we have maps of a bundles

$$\tilde{W}^n \xleftarrow{\pi^*g^{-1}} \tilde{H}^n \xrightarrow{\pi^*g^{-1}} \tilde{W}'^n$$

$$\tilde{W}^n \xrightarrow{g_i} \tilde{H}^n \xrightarrow{g_j} \tilde{W}'^n$$

where $g$ is the topological automorphism of $\tilde{H}^n$ over $G$.

Thus on $\tilde{W}^n \oplus \tilde{W}'^n$ we have the connection

$$d + (i^*\Theta_i \oplus j^*\Theta_j)$$

connection form $\eta$

$$\Theta = g^{-1}dg$$

$\in C'(A, \text{End}(H))$

with curvature

$$\omega = dg + \eta^2 = (i^*\Theta_i)^2 - i^*\Theta^2_i) + (j^*\Theta_j)^2 - j^*\Theta^2_j).$$
Now \((i^* \Omega^i)^2 - i^* \Omega^2_i = i^* \theta(i i^* - i) \theta i\)

\[= -i^* \theta j^* \theta i\]

Also
\[
\frac{i}{2} \{ F, \theta \} = [c, \theta] = \begin{pmatrix} 0 & i^* \theta j \\ -j^* \theta i & 0 \end{pmatrix}
\]

so we see the curvature of the direct sum \(\tilde{\mathbb{W}}^n \oplus \tilde{\mathbb{W}}^n\) is
\[\omega = \left(\frac{1}{2} \{ F, \theta \} \right)^2\]

We have the \textit{Bianchi identity}
\[d \omega + [\eta, \omega] = 0.\]

Using this one sees in the usual way that
\[\text{tr} (F \omega^n) \quad (\text{or} \quad \text{tr} (c \omega^n) \quad \text{or} \quad \text{tr} ((i-c) \omega^n))\]
are all closed. This gives
\[\delta \text{tr} (F \{ F, \theta \}^{2n}) = 0\]

Next let's go over Connes' proof. He considers the \textit{DGA} defined by the algebra \text{End}(H) and the involution \(F:\)
\[
\text{End}(H) \xrightarrow{\{ F, \_ \}} \text{End}(H) \xrightarrow{\text{EF, } I_+} \cdots \cdots \cdots \]

Inside there is the \textit{DG subalgebra}
\[
\text{End}(H) \rightarrow I \rightarrow I^2 \rightarrow \cdots \cdots \]

where \(I\) is the Schatten ideal containing \([F, \theta].\)

In large odd degrees the trace on \(I^{2n-1}\) is
a closed supertrace on this DG subalgebra. By his theory one gets a cyclic $(2n-1)$ cocycle defined by
\[ \text{tr} \left( \Theta [F, \Theta]^{2n-1} \right) \]

Finally there is a direct method of calculating, the simplest being to compute directly
\[ 8 \text{tr} \left( F [F, \Theta]^{2n} \right) \]
using $8\Theta = -\Theta^2$ and see that the various terms cancel.

The next thing to look at is the graded case. Here we suppose that there is a grading $H = H^+ \oplus H^-$ given on $H$ such that $F$ is of odd degree. Then $\text{End}(H)$ is a superalgebra with an odd involution, and we have the DG algebra
\[ \text{End}^+(H) \xrightarrow{[F, \_]} \text{End}^-(H) \xrightarrow{[F, \_]} \ldots \]
We have the DG subalgebra
\[ \text{End}^+(H) \rightarrow (I^-) \rightarrow (I^2)^+ \rightarrow \ldots \]
and in large even degrees the supertrace $\text{tr}$ on $I^{2n}$ is a closed supertrace on this DG subalgebra. This gives by Connes theory $2n$-cocycles
\[ \text{tr} (\Theta [F, \Theta]^{2n}) \]
The question is now whether there is a natural connection curvature construction of this cocycle.
We have to understand the relation between Chern cocycles and the S-operator.
Basic to his construction is the fact that to a superalgebra \( R = R^+ \oplus R^- \) with involution belongs a cochain algebra
\[
R^+ \xrightarrow{[F, \cdot]} R^- \xrightarrow{[F, \cdot]} R^+ \rightarrow \ldots
\]

**Example:** Take \( R = \text{End}(\mathbb{C}^2) \) with \( e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Then the above cochain algebra should be the non-commutative DR algebra of
\( r^+ = \mathbb{C} \oplus \mathbb{C}e \), \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \).
Recall that
\[
\Omega^n_{\mathbb{C}^2} = \mathbb{C} \otimes \mathbb{C}^n \quad \text{so if} \quad A = \mathbb{C}[e]
\]
\[
\Omega^n_A = \mathbb{C}[e] \otimes \mathbb{C}^n
\]
Also,
\[
[F, e] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
\[
[F, e]^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]
so it's clear that the canonical map
\[
\Omega^n_{\mathbb{C}[e]} \xrightarrow{\cdot} (R^+ \xrightarrow{[F, \cdot]} R^- \xrightarrow{[F, \cdot]} R^+ \rightarrow \ldots)
\]
is an isomorphism.
The next step is take the tensor product with this example.
Let us now consider the $S$-operator in my setting, by which I mean we consider left-invariant differential forms on $G = GL(A)$. Thus I have a $\mathbb{A}$ acting on $V$ and an $F$ on $V$ and I construct a direct embedding

\[ W \xrightarrow{i^{-1} g \cdot} V \]

$W = +1$ eigenspace of $F$ of trivial bundles over $G$. Then I use the trace on $W$ to obtain Chern character forms on $G$.

I'd like to describe the situation by saying that I have a nice description of certain kinds of cyclic cocycles using the connection and curvature formalism. In particular the cocycles associated to extensions are of this type. (However I'm still stuck on the cocycles on $C^\infty(M)$ or more generally $\text{End}(E)$. In this situation I haven't really understood the good way to construct these cocycles via transgression. Hence I have nothing better than Connes' approach using cochain algebras and the formula $\int \theta(\partial)$.)

So let us start with cocycles in the form I like and try to find their $S$-transforms. The idea geometrically goes as follows. We have a cocycle on $A$ which we are viewing as a character form on $G$ of a bundle $W$ with connection. We now lift $W$ up to $G \times S^2$ and tensor with the Hopf line bundle. Then
we get character forms over $G \times S^2$
which can be integrated to give invariant forms on $G$. Unfortunately this doesn't
raise the degree of the form, so we don't have the right operation: ?

It appears that we have to use the "cycle" (fundamental class) on $S^2$, and this
has to be multiplied with the "cycle" represented by $F$. 
Our goal is to understand Connes' claim showing the different cyclic cocycles attached to a Fredholm module are linked by the S-operator.

Let's work in the graded case $H = H^+ \oplus H^-$ with\[ \varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]. Let $R = \text{End}(H)$. We have the DG algebra with traces:

\[ R^+ \xrightarrow{[F, \cdot]} R^- \xrightarrow{[F, \cdot]} R^+ \]

\[ U \quad U \quad U \]

\[ R^+ \xrightarrow{\text{tr}_\varepsilon} I \xrightarrow{\text{tr}_\varepsilon} I^2 \xrightarrow{\text{tr}_\varepsilon} I^3 \xrightarrow{\text{tr}_\varepsilon} I^4 \]

\[ C \quad C \quad C \]

and our problem is to show that the different cyclic cocycles attached to these traces are linked by $S$.

The simplest case is to relate $\text{tr}_\varepsilon \Theta^3$ and $\text{tr}_\varepsilon \Theta[F, \Theta]^2$.

We have:

\[ \Theta[F, \Theta]^2 = \Theta(FOF - \Theta F)^2 = \Theta(FOFOF - FOF^2 + \Theta^2 + \Theta FOF) \]

\[ = \Theta F O F O F - \Theta F O F^2 + \Theta^2 F O F - \Theta^3 \]

\[ \equiv 3 \Theta^2 F O F - \Theta^3 \quad (\equiv \text{means same for } \text{tr}_\varepsilon) \]

But

\[ -8 \Theta(FOF) = \Theta^2 F O F - \Theta F O F^2 = 2 \Theta^2 F O F \]

consequently $\text{tr}_\varepsilon (\Theta^2 F O F)$ is a cyclic coboundary and

\[ \text{tr}_\varepsilon \Theta[F, \Theta]^2 \sim - \text{tr}_\varepsilon \Theta^3 \]
This calculation raises an interesting point, namely, we saw before in the geometric case of a manifold \( M \) and cycle \( R \) that the \( S \) transform of the cyclic cocycle attached to \( R \) can be realized exactly by lifting functions on \( M \) to \( M \times S^2 \), then multiplying by the idempotent \( e \) describing the Hopf bundle, and then using the cycle \( [e] \times [S^2] \).

Suppose that a similar procedure is applied to the cocycles \( \text{tr}_e(\Theta[F,0]^2) \), or really to the DGA with traces

\[
\begin{array}{c}
R^+ \\ R^- \\ R^+
\end{array} \xrightarrow{[F,0]} \begin{array}{c}
R^+ \\ R^- \\ R^+
\end{array}
\]

\[
\begin{array}{c}
t_r \varepsilon \\ t_r \varepsilon
\end{array}
\]

Call this cochain algebra \( R^* \). The idea is that we form

\[
R^* \otimes \Omega^*_{\mathcal{O}[e]}
\]

and take the tensor product of the traces \( \rho_{2n} \xrightarrow{t_r \varepsilon} \mathbb{C} \) and \( \Omega^2_{\mathcal{O}[e]} \xrightarrow{t_r \varepsilon} \mathbb{C} \). Clearly we then obtain the \((2n+2)\)-cocycle \( S_{2n} \).

We see now that, contrary to our initial impression, the \( S \)-transforms are completely analogous in the two cases. The issue is why in the "super" example the cocycles \( T_{2n+2} \) and \( S_{2n} \) are cohomologous.

However, let's consider some general
nonsense connected with cyclic cocycles on \( C^\infty(M) \). We have a means to construct lots of cyclic cocycles as follows. First of all we can pull-back \( f^*: C^\infty(M) \to C^\infty(N) \) with respect to a map \( f: M \to N \). Then we have the possibility of a non-unital homomorphism

\[
C^\infty(M) \to \text{End}(E) \to \mathbb{M}_n(C^\infty(M))
\]

corresponding to a direct summand \( E \subset C^n \).

Finally we have the basic Connes cocycle \( \int_{O(d\theta)^n} \) attached to a closed current. To what extent can one prove directly without using Connes' determination of \( HC^*(C^\infty(M)) \) that cocycles constructed geometrically in the way described can be expressed in terms of \( S \)-transforms of cocycles attached to closed currents on \( M \)?

This question suggests a program, namely to completely describe \( HC^*(C^\infty(M)) \), rather a candidate for this cyclic homology, by directly defining the \( S \)-operator and manipulating it's necessary to prove that \( S \) applied to the cyclic cocycle attached to a closed current which is a boundary is a cyclic coboundary. I guess I would like geometric arguments where possible.
Typical question. Let's consider $E \xrightarrow{\pi} V$ over $M$ and the corresponding cocycle $\int \tau (\Omega \circ \delta \Omega)^n$. We know how to write this as

$$\int \sum_{p} \text{Sp}(\Omega \circ \delta \Omega)^{n-2p} \cdot \tau (\delta \Omega)^{2p}$$

We carry this out on the level of formulas, but the question is whether there is a geometric way to proceed. The idea here is that $e$ defines a map $M \to Gr(V)$ and pulling back via

$$M \times Gr(V) \xrightarrow{p_1} M$$

and integrating over $V \times \text{(suitable cycles in } Gr(V))$ gives $\text{Sp} \int \ldots$.

It seems better to observe that certain kind of data gives rise to a cyclic cocycle in $C_0(M)$. The data is a triple consisting of a manifold $f: N \to M$ mapping to $M$, a vector bundle $E$ with connection on $N$ and a cycle $\mathfrak{e}$ (closed current) $f$ on $N$. We can replace the bundle with connection by an idempotent matrix, or perhaps a map $N \to Gr(V)$. Thus we have a correspondence of some sort with the Grassmannian. In fact we have simply a

closed current on $M \times Gr(V)$. 

\[ \text{(Equation)} \]
Let's check this. We are given a closed current \( \gamma \) in \( M \times \text{Gr}(V) \). Then we obtain an \( n \)-dimensional cyclic cocycle on \( C^\infty(M) \) as follows: We have (non-unital) algebra homomorphisms

\[
C^\infty(M) \longrightarrow C^\infty(M \times \text{Gr}(V)) \longrightarrow C^\infty(M \times \mathbb{O}(V)) \otimes \text{End}(V)
\]

\[
f \longmapsto f \circ \text{pr}_1 \longmapsto (f \circ \text{pr}_1)(\text{c} \circ \text{pr}_2)
\]

and the cyclic cocycle on the latter given by \( \int \text{tr}_V \theta(\Omega)^n \). Thus in \( C^\infty(M) \) we have the cocycle

\[
\int \text{tr}_V \left[ \theta(\text{c}(\Omega)^n) \right]
\]

\[
= \sum_{p > 0} \int \text{pr}_1^* s^p \left\{ \theta(\Omega)^{n-2p} \right\} \circ \text{pr}_1^* \text{tr}_V \left( \text{c}(\Omega)^{2p} \right)
\]

\[
= \sum_{p > 0} s^p \left\{ \theta(\Omega)^{n-2p} \right\} \cup \text{pr}_1^* \left[ \text{tr}(\text{c}(\Omega)^{2p}) \right] \cup V
\]

Thus this cyclic cocycle can be reconstructed from the one on \( M \) obtained by "intersecting" \( \gamma \) with the character forms on the Grassmannians and then pushing down to \( M \).

However I still need a proof that if the closed \( n \)-current \( \gamma \) is a boundary, say \( \gamma = \partial(\nu) \), then the cyclic cocycles

\[
s^p \left\{ \theta(\Omega)^{n-2p} \right\} \cup V
\]

for \( p > 0 \) are cyclic coboundaries.
Here's a proof based on earlier works. We work in the double cochain algebra \( C^*(A, \Omega A) \). This has the total differential \( S + d \), where \( d \) is in \( C^0(A, \Omega A) \) involves the sign \((-1)^p\) times the effect of \( d \) on \( \Omega A \). Recall the formula

\[
(S + d) \int dt (n+1) \ln \left\{ \Theta \left( t d\Theta + (k^2-t)\Theta^2 \right)^n \right\} = \text{tr} \left( d\Theta \right)^{n+1}
\]

as well as the fact, which I think I proved, that in the diagram

\[
\begin{array}{ccc}
\bullet & \xrightarrow{d} & \bullet \\
\downarrow & & \downarrow \\
\bullet & \xrightarrow{S} & \bullet \\
\end{array}
\]

the \( S \) operator transforms a term at a cross into the terms 2 steps to the right (up to some factor).

If all this is true, then the identity \( \otimes \) says that (up to numerical factors)

\[
(S + d) \left\{ \text{tr} \left( \Theta d\Theta^n \right) + S \text{tr} \left( \Theta d\Theta^n \right)^{n-1} + S^2 \text{tr} \left( \Theta d\Theta^n \right)^{n-2} + \cdots \right\} = \text{tr} \left( d\Theta \right)^n
\]

Thus

\[
\int \text{tr} \left( \Theta d\Theta^n \right) \eta = \int d \eta \text{tr} \left( \Theta d\Theta^n \right)
\]

\[
= -\int S \text{tr} \left( \Theta d\Theta \right)^{n+1} \eta = -S \int \text{tr} \left( \Theta d\Theta \right)^{n} \eta
\]
May 11, 1988

Let us consider finite-dimensional algebras. Consider $R = \text{End}(V)$ where $V$ is a finite-dimensional vector space. In order to get something interesting, I guess we must assume $V$ is $\mathbb{Z}_2$-graded: $V = V^+ \oplus V^-$. Suppose $F$ is an odd involution, whence we can suppose $V^+ = V^-$ and $F = -I$. Then we have even cyclic cocycles on $R^+$ defined by

$$\tau_{2n} = \text{tr}(\varepsilon F [F, \theta]^{2n+1}) = 2 \text{tr}(\varepsilon \theta [F, \theta]^{2n})$$

I would like to understand why these cocycles are related by Connes $S$-operator, and in particular why their cyclic cohomology classes are independent of $F$. ($\tau_0 = 2 \text{tr}(\varepsilon \theta)$ is ind. of $F$).

I propose to use the dilation concepts together with homotopy. Homotopy means that the class of $\tau_{2n}$ doesn't change as $F$ is deformed. I think this is fairly clear from the left-invariant form interpretation of cyclic cochain. In any case a direct proof should be possible.

Consider the ungraded situation $A$, $H$, $F = 2e - 1$. I recall that if we work in the $D$-algebra $\mathcal{O}(A, \text{End}(H)\rtimes)$, we have

- connection form $\mathcal{F} = e\theta e$
- curvature $\omega = d\mathcal{F} + \mathcal{F}^2 = e\theta e\theta e - e\theta^2 e = e\theta(e - 1)\theta e = e[\theta, e]^2$
and the \((2n-1)\)-coycle is
\[
\text{tr}(\omega^n) = \text{tr}(e[\theta, e]^{2n})
\]

Now I would like to show that an infinitesimal change in \(e\) changes this cocycle by a cyclic coboundary. Let \(\dot{e} = \frac{d}{dt} e(t)\big|_{t=0}\) where \(e(t)\) is a path of projectors with \(e(0) = e\).

Then \(\dot{\omega} = d\dot{\mu} + \dot{\mu}^* + \dot{\mu} = [d + \mu^*] \) and
\[
\text{tr}(\omega^n) = n \text{tr}(\omega \omega^{n-1}) = n \text{tr}(d + \mu^*) \omega^{n-1}
\]
Thus the cyclic cochain of which
\[
\text{tr}(\omega^n) = \text{tr}(e[\theta, e]^{2n})
\]
is to be the coboundary is
\[
h \text{tr}((\dot{e} h e + e \dot{h} e) e[\theta, e]^{2n-2})
\]

It might be useful to write this in terms of \(F\) for generalization to the graded case. We have

connection form
\[
j = \frac{E+1}{2} \Theta + \frac{E+1}{2} \frac{1-F}{2} \Theta - \frac{1-F}{2}
\]

\[
= \frac{\Theta + \Theta F F}{2}
\]

curvature
\[
\omega = \frac{(\Theta + \Theta F F)^2}{4} - \frac{\Theta^2 + \Theta F^2 F}{2}
\]

\[
= \frac{\Theta^2 + \Theta F F + F F \Theta + F \Theta^2 F - 2 \Theta^2 - 2 F \Theta^2 F}{4}
\]

\[
= \frac{1}{4} (F \Theta F \Theta - F \Theta^2 F - \Theta^2 + \Theta F F) = \frac{1}{4} [F \Theta]^2
\]
We are interested in the form
\[ \frac{1}{2} \text{tr} (Fw^n) = \text{tr} (e^{\mathcal{L}_c} w^n) \]

(Notice: \( \frac{1}{2} \text{tr} (F \mathcal{L}_c w^n) = \frac{1}{2} \text{tr} (F \mathcal{L}_c w^n) = \text{tr} (e^{\mathcal{L}_c} w^n) \))

since \( \text{tr} (e^{\mathcal{L}_c} w^n) = 0 \); this is true for any matrix 1-form.

Now
\[ \text{tr} (Fw^n) = \text{tr} (F \omega^n) + \text{tr} (F \sum_{i=0}^{n-1} \omega^i \omega^{n-i-1}) \]

because \( F \) anti-commutes with \( F \)
and \( \omega \) commutes with \( F \)

\[ = n \text{tr} (F \omega w^{n-1}) \]
\[ = n \text{tr} (F \mathcal{L} \omega w^{n-1}) \]
\[ = n \text{tr} (\mathcal{L} \omega w^{n-1}) \]
\[ = n \text{tr} (\mathcal{L} \omega w^{n-1}) \]

\[ = d \text{tr} (\mathcal{L} \omega w^{n-1}) \]
May 12, 1988

Let's check the transgression calculation from yesterday.

Connection

\[ j = e \theta e + (1 - e) \Theta (1 - e) \]

\[ = \Theta + F \Theta F = \Theta - \frac{1}{2} F [F, \Theta] \]

Curvature

\[ \omega = \frac{1}{4} [F, \Theta]^2 = [\Theta, e]^2 \]. The identity to check is

\[ \partial_t \text{tr} (F w^n) = \partial_t \text{tr} (F \dot{\omega}^{n-1}) \]

\[ = 2^{-2n} \text{tr} (F [F, \Theta]^2) \partial_t \text{tr} \left( \frac{F \dot{F} \Theta F \dot{F} \Theta}{2} \right) \frac{1}{2^{2n-2}} [F, \Theta]^{2n-2} \]

\[ \partial_t \text{tr} (F [F, \Theta]^2) = \partial_t 2n \text{tr} \left( (F \Theta + \Theta F) [F, \Theta]^{2n-2} \right) \]

Compute RHS. We have from the Bianchi identity

\[ d \omega^n = -j \omega^{n+1} j = -\Theta \omega^n + \omega^n \Theta \]

because \( \frac{1}{2} F [F, \Theta] \) commutes with \( \omega \). In fact

\[ d [F, \Theta]^m = \sum_{i=0}^{m-1} (-1)^i [F, \Theta]^i [F, \Theta^2] [F, \Theta]^{m-1-i} \]

\[ = -\Theta [F, \Theta]^m - (-1)^{m-1} [F, \Theta]^m \Theta \]

\[ = -\Theta [F, \Theta]^m + (-1)^m [F, \Theta]^m \Theta = -[\Theta, [F, \Theta]^m] \]

Thus, the RHS is

\[ 2n \text{tr} \left\{ (-F \Theta^2 - \Theta^2 F) [F, \Theta]^{2n-2} \Theta (F \Theta + \Theta F) (-\Theta [F, \Theta]^{2n-2} + [F, \Theta]^{2n-2}) \right\} \]

\[ = 2n \text{tr} \left\{ (-F \Theta^2 - \Theta^2 F + \Theta^2 F + F \Theta F + \Theta F + \Theta \dot{F}) [F, \Theta]^{2n-2} \right\} \]

The LHS is

\[ \partial_t \text{tr} (F [F, \Theta]^2) = \text{tr} \left( \dot{F} [F, \Theta]^2 \right) + \text{tr} \left( F \sum_{i=0}^{2n-1} [F, \Theta]^i \Theta [F, \Theta]^{2n-1-i} \right) \]
\[ = 2n \text{tr} \{ F [\hat{\Theta}, \Theta] [F, \Theta]^{2n-1} \} \]

To see these are the same note that \( \hat{\Theta} \) is an arbitrary operator anti-commuting with \( F \). This suggests writing both sides as \( \text{tr} (\hat{\Theta} X) \) where \( X \) anti-commutes with \( F \). On the left

\[
\text{tr} (2 \hat{\Theta} F [F, \Theta]^{2n-2}) = -2 \text{tr} (\hat{\Theta} [F, \Theta]^{2n-2} \Theta)
\]

\[
= -2 \text{tr} \left( \hat{\Theta} \left[ F, \Theta [F, \Theta]^{2n-2} \Theta \right] \right)
\times \text{as above}
\]

\[
= \text{tr} (F \hat{\Theta} [F, \Theta]^{2n-1} \Theta + \Theta [F, \Theta]^{2n-2} \Theta + \Theta [F, \Theta]^{2n-1}) = 0 \text{ as } F \text{ anti-commutes with } [F, \Theta]
\]

\[
= \text{tr} \left( FF \Theta [F, \Theta]^{2n-1} - \Theta FF [F, \Theta]^{2n-1} \right)
\]

\[
= \text{tr} \left( F(\Theta - \hat{\Theta}) [F, \Theta]^{2n-1} \right)
\]

so it works.

---

Our next project should be to find a suitable graded version.

But first we can simplify the above computation. Go back to the cochain

\[
\text{tr} (\hat{\Theta} F + \Theta F) [F, \Theta]^{2n-2}
\]

and apply the principle of writing it in the form \( \text{tr} (\hat{\Theta} X) \) where \( X \) anti-commutes with \( F \).

\[
\text{tr} \left( \Theta [F, \Theta]^{2n-2} + [F, \Theta]^{2n-2} \Theta \right)
\]

\[
= \frac{1}{2} \text{tr} \left( FF [F, \Theta [F, \Theta]^{2n-2} + [F, \Theta]^{2n-2} \Theta] \right)
\]

\[
= \text{tr} \left( FF [F, \Theta]^{2n-1} \right)
\]
Then
\[
\frac{d}{dt} \text{tr}(\mathcal{F}[F,\Theta]^{2n-1}) = \text{tr}(\mathcal{F}(-\Theta[F,\Theta]^{2n-2} - [F,\Theta]^{2n-1}))
\]
\[
= \text{tr}(\mathcal{F}(F\Theta - \Theta F)[F,\Theta]^{2n-1})
\]

so the moral is that there is a simpler transgression formula, namely
\[
\frac{d}{dt} \text{tr}(\mathcal{F}[F,\Theta]^{2n}) = d \ln \text{tr}(\mathcal{F}[F,\Theta]^{2n-1})
\]

It now begins to appear that the connection-curvature formalism you have been pushing is perhaps not fundamental.

So let us now turn to the graded case. Here one has $A, H, \varepsilon, F$, and $H = H^+ \oplus H^-$ where $H^+ = H^-$ and $F = (\sigma F)$ rel. to this identification.

Effectively this means we have two $A$-module structures on $H^+$, i.e. we are dealing with $A \times A$ as emphasized by Cuntz.

A central problem will be to link the even cocycles of Connes with my even cocycles.

It would be nice to continue with the connection-curvature formalism in this graded case. The problem one encounters at the start is how to make sense of the even cocycles
\[
\text{tr}(\varepsilon F[F,\Theta]^{2n+1}) = 2\text{tr}(\varepsilon \Theta[F,\Theta]^{2n})
\]
These are not the usual functions of the curvature $\frac{1}{4} [F,\Theta]^2$. Recall that we have connection $\mathcal{F} +$ curvature forms
\[
\mathcal{F} = \Theta - \frac{1}{2} F[F,\Theta]
\]
\[
\Theta = \frac{1}{4} [F,\Theta]^2
\]
\[
\text{diag part of } \Theta \text{ rel } F
\]
I think we should emphasize the fact that the crossed cocycles use the off-diagonal part of $\Theta$, namely
\[ \frac{1}{2} F[F, \Theta] = \Theta - F \Theta F, \]
whose square is the curvature except for sign.

Let's write
\[ \alpha = \frac{\Theta - F \Theta F}{2}, \quad \beta = \frac{\Theta + F \Theta F}{2} \]
then
\[ \alpha^2 = \frac{1}{4} F[F, \Theta] F[F, \Theta] = -\frac{1}{4} [F, \Theta]^2 = -\omega \]
\[ \alpha + \beta = \Theta \]

so
\[ d\Theta = d\alpha + d\beta \]
\[ -\Theta^2 \quad \omega - \beta^2 \]
\[ \alpha \]
\[ d\alpha = -\Theta^2 - \omega + \beta^2 \]
\[ = -\alpha^2 - \omega + \beta^2 = \omega - \beta^2 - \omega \]

so
\[ d\alpha + \beta \alpha + \alpha \beta = 0 \]
\[ \alpha \]
\[ [d + \beta, \alpha] = 0 \]
This is a better "Bianchi identity" than $[d + \beta, \omega] = 0$.

Check:
\[ \alpha = \begin{pmatrix} 0 & i*\Theta_j \\ j*\Theta_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} i*\Theta_i & 0 \\ 0 & j*\Theta_j \end{pmatrix} \]
\[ -d i*\Theta_j = +i*\Theta^2 \beta = (i*\Theta_i)(i*\Theta_j) + (i*\Theta_j)(j*\Theta_j) \]
\[ -d j*\Theta_i = j*\Theta^2 \alpha = (j*\Theta_i)(i*\Theta_i) + (j*\Theta_i)(j*\Theta_i) \]
\[ d\beta + \beta \alpha + \alpha \beta = \begin{pmatrix} 0 & i*\Theta_i i*\Theta_i + i*\Theta_i j*\Theta_j \\ j*\Theta_i j*\Theta_i + i*\Theta_i i*\Theta_i & 0 \end{pmatrix} \]
Here is a more intrinsic description of the formalism. Consider a vector bundle with a flat connection $D$ and an involution $F$. Then we have the induced connections $eD e$, $(1-e)D(1-e)$ on the two eigenbundles of $F$, and we can form the direct sum connection

$$\nabla = eD e + (1-e)D(1-e) = \frac{D + FDF}{2}$$

and define $\alpha$ to be the difference operator 1-form:

$$\alpha = D - \nabla, \quad D = \nabla + \alpha$$

We note that $[\nabla, F] = 0$ and that $F$ anti-commutes with $\alpha$. Then flatness implies

$$0 = D^2 = \nabla^2 + [\nabla, \alpha] + \alpha^2.$$

Taking the parts commuting + anti-commuting with $F$ we obtain

$$\begin{cases} \nabla^2 = -\alpha^2 \\ [\nabla, \alpha] = 0. \end{cases}$$

Now suppose we have a 1-parameter family of $F$'s. Then we work over $\mathbb{R} \times M$ with the flat connection $\tilde{D} = dt \partial_t + D$ and the involution $F$ on $pr_2^*(E)$ over $\mathbb{R} \times M$ corresponding to the family of involutions on $E$,

$$\tilde{\nabla} = \frac{1}{2}(\tilde{D} + F\tilde{D} F) \otimes$$

$$\tilde{\alpha} = \frac{1}{2}(\tilde{D} - F\tilde{D} F) = \frac{1}{2}(dt \partial_t + D - F(dt \partial_t + D) F)
= \alpha - \frac{1}{2} dt FF$$
Then
\[ \text{tr} \left( F \dot{\alpha}^{2n} \right) = \text{tr} \left( F \left( \alpha - \frac{1}{2} dt F F \right)^{2n} \right) \]
\[ = \text{tr} \left( F \alpha^{2n} \right) + \sum_{i=0}^{2n-1} \text{tr} \left( F \alpha^{i} \left( -\frac{1}{2} dt F F \right) \alpha^{2n-1-i} \right) \]
\[ = \text{tr} \left( F \alpha^{2n} \right) - \frac{1}{2} (2n) dt \text{tr} \left( F \alpha^{2n-1} \right) \]

This is a closed form over \( R \times M \) and so we should obtain the transgression formula
\[ \partial_t \text{tr} \left( F \alpha^{2n} \right) = -d \left\{ n \text{tr} \left( F \alpha^{2n-1} \right) \right\} \]

Let's check that this agrees with the formula on pages 59-61:
\[ \alpha = \frac{1}{2} F [F, \Theta] \quad \text{tr} \left( F \alpha^{2n} \right) = \frac{1}{2^{2n}} (-1)^{n} \text{tr} \left( F [F, \Theta]^{2n} \right) \]
\[ -n \text{tr} \left( F \alpha^{2n-1} \right) = (-1)^{\frac{n}{2}} (-1)^{n-1} \text{tr} \left( F [F, \Theta]^{2n-1} \right) \]

So now we have a formalism of the connection curvature type which explains the odd Breen cocycles in the ungraded case.

Next we should check that the graded case works in essentially the same manner.

In the graded case we have \( M, E, D \), flat \( F \) and an \( \varepsilon \) on \( E \) such that \( D \) is even and \( F \) is odd relative to \( \varepsilon \). Then we have that
\[ \alpha = \frac{1}{2} (D - FDF) \quad \text{is even wrt } \varepsilon \]
\[ D = D - \alpha \quad \text{is even wrt } \varepsilon \]

Recall also that \( \nabla F = F \nabla \) and \( F \alpha + \alpha F = 0 \).
It turns out to be possible to write down lots of closed forms using traces and these involutions, however, most of them are zero. For example in the ungraded case we have
\[\text{tr} (\alpha^{2n}) = 0\]
\[\text{tr} (\alpha^{2n+1}) = 0\]
\[\text{tr} (F \alpha^{2n+1}) = 0\]
so the only possibility is \[\text{tr} (F \alpha^{2n})\] for any matrix 1-form as \(\alpha^{2n+1}\) anti-commutes with \(F\) for the same reason.

In the graded case we have
\[\text{tr} (\varepsilon \alpha^{2n}) = 0\]
\[\text{tr} (F \alpha^{2n}) = 0\] (anti-commute with \(\varepsilon\))
\[\text{tr} (\varepsilon F \alpha^{2n}) = 0\]
\[\text{tr} (\varepsilon F \alpha^{2n+1}) = 0\]
\[\varepsilon \text{tr} (\varepsilon F \alpha^{2n}) = -\text{tr} (\alpha \varepsilon F \alpha^{2n}) = -\text{tr} (\varepsilon F \alpha^{2n+1})\]
\[\therefore \text{tr} (\varepsilon F \alpha^{2n+1}) = 0\].

Thus the only possibility is \[\text{tr} (\varepsilon \alpha^{2n+1})\]. This is closed by the usual argument
\[d \text{tr} (\varepsilon \alpha^{2n+1}) = \text{tr} \left[ \nabla, \varepsilon \alpha^{2n+1} \right] = 0\]
since \([\nabla, \varepsilon] = [\nabla, \alpha] = 0\). Note that \([\nabla, \varepsilon] = [\nabla, \alpha] = [\nabla, \pi] +\]

Finally, we look at the homotopy property. We set \(\tilde{D} = dt \pi + D\) on \(pr_2^*(E)\) over \(\mathbb{R} \times M\), and consider a family \(F\), i.e. an involution on \(pr_2^*(E)\). Then we consider
\[ \tilde{\alpha} = \frac{1}{2} (\tilde{D} - \tilde{F} \tilde{D} \tilde{F}) = -\frac{1}{2} \tilde{F} \tilde{D} \tilde{F} \]

\[ = \alpha - \frac{1}{2} \int dt \tilde{F} \]

\[ \text{tr}(\varepsilon \tilde{\alpha}^2) = \text{tr} \varepsilon (\tilde{\alpha}^{2n+1} + \sum_{i=0}^{2n} \alpha^i \varepsilon (-\frac{1}{2} \int dt \tilde{F} \tilde{D} \tilde{F}) \alpha^{2n-i}) \]

\[ = \text{tr} \varepsilon \tilde{\alpha}^{2n+1} - \sum_{i=0}^{2n} \text{tr} \alpha^i \varepsilon (-\frac{1}{2} \int dt \tilde{F} \tilde{D} \tilde{F}) \alpha^{2n-i} \]

\[ = \text{tr} \varepsilon \tilde{\alpha}^{2n+1} - \frac{1}{2} \frac{dt}{dt} \tilde{F} \text{tr} \varepsilon \tilde{F} \tilde{D} \tilde{F} \alpha^{2n} \]

Consequently, we have

\[ \partial_t \text{tr} (\varepsilon \tilde{\alpha}^{2n+1}) = - \alpha \left\{ (n+\frac{1}{2}) \text{tr} (\varepsilon \tilde{F} \tilde{D} \tilde{F} \alpha^{2n}) \right\} \]

I would like to check this by a standard Bianchi identity proof. This means expressing \( \tilde{\alpha} \) in the form \([\nabla, \tilde{\alpha}]\). We have

\[ \tilde{\alpha} = \frac{1}{2} (\tilde{D} - \tilde{F} \tilde{D} \tilde{F}) \quad \dot{\alpha} = (-\frac{1}{2})(\tilde{F} \tilde{D} \tilde{F} + \tilde{F} \tilde{D} \tilde{F}) \]

\[ (-2) \tilde{\alpha} = \tilde{F} \tilde{D} \tilde{F} + \tilde{F} \tilde{D} \tilde{F} \]

\[ = \tilde{F} \nabla \tilde{F} + \tilde{D} \tilde{F} \tilde{F} + \tilde{F} \nabla \tilde{F} + \tilde{F} \tilde{D} \tilde{F} \]

\[ = -\dot{\tilde{F}} \alpha - \tilde{D} \alpha \tilde{F} \]

\[ = -(\tilde{F} \tilde{D} \tilde{F} + \tilde{D} \tilde{F} \tilde{F} + \tilde{F} \nabla \tilde{F} + \tilde{F} \nabla \tilde{F}) \alpha - \alpha (\tilde{F} \tilde{D} \tilde{F}) \]

\[ = \left[ \nabla - \alpha, \tilde{F} \tilde{D} \tilde{F} \right] \]

\[ \alpha = \left[ \nabla - \alpha, -\frac{1}{2} \tilde{F} \tilde{D} \tilde{F} \right] \]

Thus

\[ \partial_t \text{tr} (\varepsilon \tilde{\alpha}^{2n+1}) = \sum_{i=0}^{2n} \text{tr} (\varepsilon \alpha^i \left[ \nabla - \alpha, -\frac{1}{2} \tilde{F} \tilde{D} \tilde{F} \right] \alpha^{2n-i}) \]
continued from previous page. It is necessary to be careful because
\[ [\alpha, \alpha^2] = 2\alpha^2 \]
\[ \partial_t \text{tr} (e^{\alpha 2^n+i}) = \sum_{i=0}^{2n} \text{tr} \left( e^{\alpha i} \cdot \alpha^{2^n-i} \right) \]
\[ \left[ \text{tr} - \frac{i}{2} \text{tr} \right] \]
\[ = \sum_{i=0}^{2n} \text{tr} \left( e^{\alpha} \left[ \alpha - \frac{i}{2} \text{tr} \right] \alpha^{2^n-i} \right) \]
\[ = (2n+1) \text{tr} \left( [\alpha, \alpha^2] \right) \]
\[ = d \left( 2n+1 \right) \text{tr} \left( e^{\alpha \cdot \alpha^2} \right) \]

**Ungraded case**
\[ \partial_t \text{tr} \left( F \alpha^{2^n} \right) = \text{tr} \left( F \alpha^{2^n} \right) + \sum_{i=0}^{2n-1} \text{tr} \left( F e^{\alpha i} \left[ \alpha - \frac{i}{2} \text{tr} \right] \alpha^{2^n-i} \right) \]
\[ = \sum_{i=0}^{2n-1} \text{tr} \left( \left[ \alpha, \alpha^2 \right] \alpha^{2^n-i} \right) \left( -i(i-1)(2^n-i) \right) \]
\[ = 2n \text{tr} \left( \left[ \alpha, \alpha^2 \right] \alpha^{2^n-1} \right) \]
\[ = 2n \text{tr} \left( \left[ \alpha, \alpha^2 \right] \alpha^{2^n-1} \right) \]
\[ = d 2n \text{tr} \left( \beta \alpha^{2^n-1} \right) \]
\[ = d 2n \text{tr} \left( F \beta \alpha^{2^n-1} \right) \]

So it all works but just barely.

Now let’s continue with transgression. We consider a geometric structure where \( A = C^\infty(M) \) acts on \( H = L^2(M, S) \), and on \( H \) one has an involution modulo some Schatten ideal defined. I want to study the equivalence of various cocycles obtained by dilating.
The other example to keep in mind is the case of the finite-diml algebra \( \text{End}^+(V) \). The point should be to see the equivalence of the different cocycles.

Let's try to describe the goal in the geometric case. We have \( A \) acting on a Hilbert space \( H \) and an involution modulo compacts \( \gamma \) on \( H \) such that \([\gamma] \) is self-adjoint compacts. We can lift this involution to a contraction and dilate to obtain an involution on a larger Hilbert space. Using this involution one obtains (under suitable hypotheses) cyclic cocycles. The goal is to show that these cocycles are cohomologous, better yet, their cyclic cohomology classes are independent of the choice of contraction and dilation.
Let's consider the dilation process abstractly. We consider a vector
space \( V \) with an endomorphism \( T \) and
wish to find \( V \xrightarrow{\iota} V_1 \) and an
involution \( F \) on \( V_1 \) such that \( T = \iota^* F \iota \).
(In the C*-setting one supposes that \( T \) is a
self-adjoint contraction) and
one wants \( \iota \) to be an isometric embedding
with adjoint \( \iota^* \) and \( F \) to be unitary.)

This is a special case of the GNS construction.
Consider \( A = C[\mathbb{Z}/2] \) and form the universal
algebra + map \( \varphi : A \rightarrow B \) such that \( \varphi(1) = 1 \).
Thus if \( A = C \oplus CF \), then \( B = C[T] \) where \( T = \varphi(\pi) \).
Then \( C = A \oplus A \otimes B \otimes A = A \ast C[e] \simeq C[\text{dihedral group } \langle F, \pi \rangle] \), acts on \( V_1 \).

Recall that \( B = eCe \) where \( e = \frac{\pi + 1}{2} \).

Thus we have a non-unital homomorphism
\[ B = eCe \rightarrow C \]
and a \( B \)-module \( V \) and we are looking for
a \( C \)-module \( V_1 \). We should probably break
this non-unital homomorphism into 2 steps
\[ B = eCe \xrightarrow{\text{adjoining \( B^+ \)}} eCe \oplus eCe \rightarrow C \]

since \( B \) is unital we have \( B^+ = C \times B \) so
\[ \text{Mod}(B^+) = \text{Mod}(B) \times \text{Mod}(C) \]

In other words a non-unital \( B \)-module (equivalently
a unital \( B^+ \)-module) is uniquely the direct sum of a (unital) \( B \)-module and a vector space with zero \( B \)-multiplication.

Thus our problem is given a (unital) \( B \)-module \( V \), we wish to find a (unital) \( C \)-module such that \( V \) is the unital part of \( V \) restricted to \( B \).

Next let's look carefully at the Morita type relations between \( \text{Mod}(eCe) \) and \( \text{Mod}(C) \). We have functors

\[
\text{Mod}(eCe) \leftrightarrow \text{Mod}(C)
\]

\[
X \rightarrow eCe \otimes eCe \rightarrow X
\]

\[
eY = eC \otimes_C Y \rightarrow Y
\]

It's clear that the composition \( \Rightarrow \) is the identity since

\[
eCe = eCe \oplus C(-e)
\]

(\text{use } C = Ce \oplus C(-e)). The other composition is the identity when

\[
Ce \otimes eCe \rightarrow C.
\]

This certainly isn't the case when \( CeC < C \) which is typical for the \( C \)'s I am looking at.

\[
\text{Past work: } Ce \otimes eCe eC \rightarrow CeC
\]

\[
\tilde{I} \rightarrow \tilde{I}
\]

is the universal bimodule map \( u(\tilde{I}y) = \tilde{I}u(y) \). See p 688, 673-674. So \( \tilde{I} = I \) in the case \( C = A \star \mathbb{C}[\mathbb{Z}/2] \).
Notice that the functor
\[ Y \mapsto eC \otimes_C Y = eY \]
simply takes the \( C \)-module \( Y \) restricts the \( C \)-multiplication to the non-unital subalgebra \( eC e \) and then takes the unital part. Thus if we start with \( V \) over \( eC e \), there is an obvious choice for \( V_1 = eY \), namely \( V_1 = C e \otimes eC e V \). This should actually amount to doubling \( V \).

\[ B = C[T] \quad T = eFe \]
\[ C = C[F] \times C[\varepsilon] \quad \varepsilon = \frac{1 + \varepsilon}{2} \]
\[ C e \otimes e C e \oplus (1 - e) C e = B \oplus (1 - e) Fe, B \]
\[ C e \otimes e C e V = (e) V \oplus ((1 - e) Fe) V \leftrightarrow \mathbb{V}^{\oplus 2} \]
\[ F e \sigma = e F e \sigma + (1 - e) F e \sigma \leftrightarrow \begin{pmatrix} T_0 \\ v \end{pmatrix} \]
\[ F (1 - e) F e \sigma = e \sigma - F e F e \sigma \]
\[ = e \sigma - e \frac{F e T \sigma}{T^2} - (1 - e) F e T \sigma \leftrightarrow \begin{pmatrix} \frac{v - T^2 \sigma}{T} \\ T_0 \end{pmatrix} \]

So
\[ F = \begin{pmatrix} T & 1 - T^2 \\ 1 & -T \end{pmatrix} \quad \mathbb{V}^{\oplus 2} \]

Eigenvalues of \( F \)
\[ \begin{pmatrix} T & 1 - T^2 \\ 1 & -T \end{pmatrix} \begin{pmatrix} T + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} T + 1 \\ T + 1 - T \end{pmatrix} = \begin{pmatrix} T + 1 \\ 1 \end{pmatrix} \]
\[ \begin{pmatrix} T & 1 - T^2 \\ 1 & -T \end{pmatrix} \begin{pmatrix} T - 1 \\ 1 \end{pmatrix} = \begin{pmatrix} T^2 - T + 1 - T^2 \\ T - 1 - T \end{pmatrix} = (-1) \begin{pmatrix} T - 1 \\ 1 \end{pmatrix} \]
Now that I have carried the dilation process algebra, what do I do next?

Let's go back over the transgression problem with a view toward seeing whether our new ideas about superconnection forms are relevant.

Situation: $M$ odd Riemannian compact manifold, $S$ Clifford bundle on $M$, $D_0$ Dirac type operator on $H_0 = L^2(M, S)$. Let $E$ be a vector bundle with inner product over $M$, let $G = \text{Aut}(E)$, $\mathcal{A} = \text{connections on } E$. Given a connection $A$ on $E$, then we have an associated Dirac $D_A$ on $L^2(M, S \otimes E)$. The map $A_1 \mapsto D_A$ is $G$-equivariant. Therefore, there is a canonical $G$-equivariant family of Dirac operators on $M$ parametrized by $A$.

The index of this family is a $K$-cohomology class on $A_g \sim B \frac{g}{2}$. My goal is to apply the families index theorem to this situation. Really what I want to do is to describe the Chern character of the family index by a superconnection form.

For a superconnection form we need a bundle, connection, and endomorphism $X$. From our new viewpoint the connection should come from an embedding into a trivial bundle.

Let's not worry about $B \frac{g}{2}$ and instead work with a principal $H$-bundle $P$ over $Y$. 
Our Hilbert bundle over $Y$ is

$$\mathcal{P} \times^G H \quad H = L^2(M, S \otimes E)$$

and we want a skew-adjoint endomorphism $\mathcal{D}$ on it, which means that we have an equivariant map from $\mathcal{P}$ to Dirac operators on $H$. We also need an embedding of $\mathcal{P} \times^G H$ into a trivial Hilbert bundle, i.e., an equivariant map from $\mathcal{P}$ to embeddings of $H$ in a fixed Hilbert space $H'$.

It seems the way to do this is to suppose we have an equivariant map from $\mathcal{P}$ to the embeddings of $E$ into $\tilde{V}$ over $M$. Then we automatically get an embedding of $\mathcal{P} \times^G H$ into the trivial bundle over $Y$ with fibre $H' = L^2(M, S \otimes \tilde{V})$. Moreover, given $i : E \to \tilde{V}$, then $i^*(\mathcal{D} \otimes 1_V)i$ is just the Dirac operator $\mathcal{D}_A$ on $L^2(M, S \otimes E)$ where $A$ is the connection $i^*d\tilde{i}^*i$.

Thus we reach the following situation: We have a Hilbert space $H' = L^2(M, S \otimes \tilde{V})$ with an unbounded operator $\mathcal{D}_0 = i^*d\tilde{i}^*i$. We consider a subbundle $\mathcal{H}$ of $\tilde{H}_Y$. Let $e$ be the endomorphism of $\tilde{H}_Y$ given by the projection on $\mathcal{H}$. Then we consider the superconnection $e(d + \mathcal{D}_0)e$ on $\mathcal{H}$. 
We consider a Dirac operator \( D_0 \) on \( H_0 = L^2(M, S) \) where \( M \) is odd-dimensional. Let 

\[ e_y \in M_n(C^\infty(M)) \]

be a family of projectors over \( M \) parametrized by \( Y \); equivalently let 

\[ e \in M_n(C^\infty(Y \times M)) \].

Then we have a family of Dirac operators on \( M \) parametrized by \( Y \) given by 

\[ D_y = e D_0 e \quad \text{on} \quad H_y = e H_0 e \]

Moreover we have the Grassmannian connection \( \nabla \) on the Hilbert bundle \( H \) over \( Y \). Thus we have a superconnection on \( Y \) given by 

\[ \nabla = e dy e \quad \text{and} \quad X e = e D_0 e \]

in other words one has taken the "constant" superconnection \( d + D_0 \sigma \) on \( H_0 \) over \( Y \) and reduced it by the idempotent \( e \).

We have now a geometric way to look at the superconnection character forms as follows. We take the Cayley transform of \( X \). This gives a unitary operator on the bundle \( H \) and we extend it by \( j = -1 \) on the complementary bundle \( (1-e) H_0 \). This gives a map \( j \) from \( Y \) to the unitary group of \( H_0 \).

Moreover it's a map to the space of unitaries congruent to \(-1\) modulo some Schatten ideal. Universal.

There are certain superconnection character forms on this space of unitaries, and pulling these back gives the forms for \( \nabla + X \sigma \) over \( Y \).
The question arises as to whether (or why) this map
\[ g : \mathcal{Y} \to -U^\infty(\mathcal{H}_0) \]
represents the index of the family of Dirac operators.

Now Atiyah-Singer define the index of the family as follows. One converts the family \( \{D_y\} \) to a family of self-adjoint contractions with essential spectrum \( \{\pm 1\} \)

\[ T_y = \frac{1}{i} D_y \frac{1}{\sqrt{1 + D_y^2}}. \]

By K"{u}nneth's theorem the Hilbert bundle is trivial and we obtain a map
\[ \mathcal{Y} \to F_1 \] - space of self-adj. contractions with essential spec. \( \{\pm 1\} \)

They prove \( F_1 \) is a classifying space for \( K_{odd} \), so this map represents an element of \( K_{odd}(\mathcal{Y}) \), which they define to be the index.

Now it is easy to see that if we use their homotopy equivalence of \( F_1 \) with the space of unitaries \( = -1 \mod 2 \), call this \( -U^\infty \), then our map \( \mathcal{Y} \to -U^\infty \) and there index map define the same element of \( K_{odd}(\mathcal{Y}) \). One uses the map \( [-1, 1] \to \text{unit circle} \)

\[ T \mapsto (\sqrt{1-T^2} + iT)^2 \]
to get the homotopy equivalence \( F_1 \to -U^\infty \), and if \( T = \frac{D}{\sqrt{1 + D^2}} \), then this gives the Cayley transform of \( D \).
Thus if we appeal to the AS defn of the index of the family, we see our map represents the index. Consequently our superconnection form represent the character of the index.

However, it appears that something important is missing. We started with $\phi_0$ an $H_0 = L^2(M, S)$, and this operator represents an old $K^0$-homology class of $M$. Our map which assigns to $\phi$ over $Y \times M$ the family $e \otimes \phi$ of operators over $Y$ is the cap (or slant) product map

$$K^0(Y \times M) \rightarrow K'(Y)$$

with this $K$-homology class. Better, it's the integral in $K$-theory over the fundamental class of $M$.

At the moment I am very confused about foundations. Maybe the way to handle things is to use Kasparov theory, which is the accepted foundation for the constructions we are using.

Kasparov theory takes $C^0(M), L^2(M, S), \phi_0$ and replaces $\phi_0$ by $\phi_0$ of order zero

$$F = \frac{i \phi_0}{1 - \phi_0^2}$$

and then $C^0(M)$ can be replaced by $C(M)$. I like to think of this geometrically as giving a splitting of $H_0$ modulo compacta.

Now suppose we are given $\phi$ over $Y \times M$. 

Geometrically we consider the Hilbert bundle $H_0^n$ over $Y$ and use the idempotent $e$ to define a subbundle $H = e H_0^n$. Now we have to use $F_0$ on $H_0^n$ to obtain an $F$ on $H$. The idea is that $e$ and $F_0$ commute modulo compacts, and so there is a well-defined splitting of $H$ modulo compacts. A specific choice of $F$ is $F = e F_0 e$. 

We now reach the following problem. The operator $D_0$ on $H_0^n$ defines a splitting of $H_0^n$ modulo compacts represented by the contraction $F_0 = \frac{i e D_0}{\sqrt{1 - (e D_0 e)^2}}$. The operator $e y D_0 e y$ on $e y H_0^n$ defines a splitting of $e y H_0^n$ modulo compacts represented by the contraction $F_y = \frac{i e y D_0 e y}{\sqrt{1 - (e y D_0 e y)^2}}$.

Is it true that $e y F_0 e y$ and $F_y$ are congruent modulo compacts?

This should follow just by looking at the symbols. These are both $400$'s of order zero with the same symbol.

But I would like a better version of this, and I hope that one might get an
explicit link between these two using the massive Dirac operator. The idea here is that the massive Dirac:

\[
\begin{pmatrix}
\partial_0 & m \\
m & -\partial_0
\end{pmatrix} = \gamma^1 \partial_0 + m \gamma^1
\]

when reduced by \((1, 0)\) always gives \(\partial_0\), but as \(m\) becomes big its spectrum has a big gap.
Recall the formalism used in constructing Connes cocycles. One has a flat connection $D$ and also a splitting $F$. Then we have:

$$\nabla = \frac{1}{2} (D + FDF) \quad \alpha = \frac{1}{2} (D - FDF)$$

So $D = \nabla + \alpha$ with $\nabla$ commuting and $\alpha$ anticommuting with $F$. Then

$$0 = D^2 = \nabla^2 + \alpha^2 + [\nabla, \alpha]$$

where $[\nabla, \alpha] = 0$ and $\nabla^2 = -\alpha^2$.

The relevant closed differential forms are

- for $(F \omega^{2n})$ ungraded case
- for $(\varepsilon \omega^{2n+1})$ graded case

To get cyclic cocycles, we apply this to a representation of $G$ as $H$, where $H$ is equipped with a splitting $F_0$. We consider the trivial bundle $H$, with connection $D = d$ and splitting $F = g F_0 g^{-1}$, where $g$ is the tautological automorphism of $H$.

Equivalently, we use $g$ as a gauge transform. We have:

$$D = g^{-1} dg = d + \Theta$$

$$F = F_0$$

whence

$$\alpha = \frac{1}{2} (D - F_0 DF_0) = \frac{1}{2} (\Theta - F_0 F) = \frac{1}{2} F [F, \Theta].$$

Now we want to generalize this.
The problem with the superconnection forms are the supersymmetric forms are $\frac{1}{2} \sqrt{g} + X^2$ connected by the connection $+ \partial X \theta$.

The action is path is as follows:

$$\frac{1}{2} \sqrt{g} + X^2 = 0$$

$$d + \theta + X^2$$

Looking at the formula from the viewpoint of the superconnection, the idea is to construct a cyclic operator $X$, the idea to extend into the superconnection operator $X$, as $X_2$.

The first idea is to look at the formula as a smoothing type operator $d + \theta + X^2$. The second idea is to look at the formula as an $e^X$.
Now that we understand convex cycles in a fashion, let us try to prove his theorem that the consecutive forms are related by the S-operator. From his viewpoint one takes the initial situation \( A, H, F \) and tensors it with a standard 2-diml "cycle" given by \( A = \mathbb{C} \langle e \rangle, \ H = \mathbb{C}^2, \ F = (\begin{array}{c} 1 \\ 0 \end{array}) \).

Tensoring means we consider \( \tilde{H} = \mathbb{C}^2 \otimes H \) with the involution roughly \( \tilde{F} : \ v \otimes F + v' \otimes 1 \).

To get an involution one has to pick a "mixing angle":
\[
\tilde{F} = \cos(\phi) \ e \otimes F + \sin(\phi) \ v' \otimes 1
\]
\[
= \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & -\cos \phi \end{pmatrix}
\]

A acts on \( \tilde{H} \) via \( \phi e = (\begin{array}{c} e \\ 0 \end{array}) \). Thus
\[
\tilde{\Theta} = (\begin{array}{cc} \Theta & 0 \\ 0 & 0 \end{array})
\]

and
\[
[\tilde{F}, \tilde{\Theta}] = \begin{pmatrix} \cos \phi [F, \Theta] & -\sin \phi \Theta \\ \sin \phi \Theta & 0 \end{pmatrix}
\]

We consider \( \text{tr} (\tilde{F} [\tilde{F}, \tilde{\Theta}]^{2n}) \). If \( \phi = 0 \), we get \( \text{tr}(F[F, \Theta]^{2n}) \). It looks like the term 2nd order in \( \phi \) is \( S \text{tr}(F[F, \Theta]^{2n-2}) \).
Start again and put \( t = \tan(\phi) \) so that
\[
\tilde{F} = \cos \phi \begin{pmatrix} F & t \\ t & -F \end{pmatrix}
\]

We consider in the ungraded case
\[
\operatorname{tr} \tilde{F}^{2n} = 2 \operatorname{tr} \tilde{E}^{2n-1} \quad \text{for } n > 1
\]
and in the graded case
\[
\operatorname{tr} \tilde{F}^{2n+1} = 2 \operatorname{tr} \tilde{E}^{2n} \quad \text{for } n > 0
\]
To handle both we consider the generating fun.

\[
\sum_{n > 0} \frac{1}{\lambda^{n+1}} \tilde{E}^{2n} = \sum_{n > 0} \frac{(\cos \phi)^n}{\lambda^{n+1}} \begin{pmatrix} 0 & 0 \\ 0 & (t \theta & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} \frac{1}{1 - \frac{\cos \phi}{\lambda} \begin{pmatrix} F \theta & -t \theta \\ t \theta & 0 \end{pmatrix}}
\]

\[
= \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} \begin{pmatrix} \lambda & \begin{pmatrix} \cos \phi [F, \theta] & -\sin \phi \theta \\ \sin \phi \theta & 0 \end{pmatrix} \\ \begin{pmatrix} \sin \phi \theta & 0 \end{pmatrix} \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ (\frac{\sin \phi}{\lambda}) \theta & 1 \end{pmatrix} \begin{pmatrix} (\lambda - \cos \phi [F, \theta] + \frac{1}{\lambda} \sin^2 \phi \theta^2)^{-1} & 0 \\ 0 & 1 - \frac{\sin \phi}{\lambda} \theta \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
\sum_{n > 0} \frac{1}{\lambda^{n+1}} \operatorname{tr}(\tilde{E}^{2n}) = \operatorname{tr} \begin{pmatrix} \theta & 0 \\ \lambda - \cos \phi [F, \theta] + \frac{1}{\lambda} \sin^2 \phi \theta^2 \\ 0 & \lambda^{-1} \end{pmatrix}
\]

and similarly with \( \operatorname{tr}(E^{2n}) \) in the graded case.
From the homotopy property established earlier, we know that the cohomology class of $tr(\tilde{\Theta} [F, \Theta]^{2n})$ is independent of $\phi$. The above formula turns out not to be very useful because there are two terms depending on $\phi$.

Instead, we should look at

$$tr \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \left[ \left( \begin{array}{cc} t & F \\ -F & t \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right]^{2n} \right\} = (1 + t^2)^n \ tr(\tilde{\Theta} [F, \Theta]^{2n})$$

The cohomology class of the cocycle in the right is $(1 + t^2)^n$ class of $tr(\Theta [F, \Theta]^{2n})$. The coefficient of $t^{2k}$ on the left should be $(a^2) \Sigma \ tr(\Theta [F, \Theta]^{2n-2k})$ yielding the desired relations.

Let's check this out using the generating functions

$$\sum \ tr \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \left[ \left( \begin{array}{cc} F & \Theta \\ \Theta & 0 \end{array} \right) \right]^{2n} \right\} \frac{1}{\lambda^{n+1}} \ \text{where} \quad a = \lambda - [F, \Theta] \ \ \ b = \Theta t \ \ \ c = -t \Theta \ \ \ d = \lambda$$

$$= tr \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \frac{1}{\lambda - \left( \begin{array}{cc} F & \Theta \\ \Theta & 0 \end{array} \right) t \Theta} \right\}$$

$$= tr \left\{ \Theta \ \frac{1}{\lambda - [F, \Theta] - \Theta t \frac{1}{\lambda} (-t \Theta)} \right\}$$

$$= tr \left\{ \Theta \ \frac{1}{\lambda - [F, \Theta] + \frac{t^2}{\lambda} \Theta^2} \right\}$$

Now if we compare this with $\sum_{i=1}^7$ we see that the coefficient of $t^2$ in this is $-S$ applied to $tr \Theta [F, \Theta]^{2n-2}$, and so this $S$-transform...
Consider a flat connection on the trivial bundle. Then we have two flat connections $d + \Theta$ on the same bundle. The difference of the character forms for these two connection vanishes for two reasons, flatness and the general fact that the cohomology class of the character form is independent of the connection. This leads to odd forms.

The usual way of constructing the odd forms is to join the connections by a linear path

$$D_t = d + t\Theta$$

which gives

$$0 = \text{tr}(e^{uD_t^2}) - \text{tr}(e^{uD_0^2}) = d\int_0^1 dt \text{tr}(\Theta e^{u(t^2 - t)\Theta^2})$$

The coefficient of $u^{k+1}$ in the integral is

$$\text{tr}(\Theta^{2k+1}) \int_0^1 \frac{1}{k!} (t^2 - t)^k dt$$

$$= \frac{1}{k!} (-1)^k \int_0^1 t^k (1-t)^k dt = \frac{(-1)^k}{k!} \frac{\Gamma(k+1) \Gamma(k+1)}{\Gamma(2k+2)} = \frac{(-1)^k k!}{(2k+1)!}$$

Thus

$$\frac{(-1)^k k!}{(2k+1)!} \text{tr}(\Theta^{2k+1})$$

are the odd forms.
Another way to obtain odd forms is to use superconnections. More generally, given two connection \( D_1, D_0 \) on the same vector bundle \( E, \) we form the super vector bundle \( E \oplus E \) with the connection \( D = D_1 \oplus D_0 \) and the odd endomorphism \( (\sigma, 0) \). This gives the superconnection

\[
\begin{pmatrix}
D_1 & 0 \\
0 & D_0
\end{pmatrix}
+ t
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\sigma
\]

depending on \( t \). For \( t=0 \) we get the difference of the character forms for \( D_1 \) and \( D_0 \), and for \( t=\infty \) the superconnection character forms are zero. Thus the homotopy from \( t=0 \) to \( t=\infty \) gives an odd form whose differential is \( \text{tr}(e^{-\sigma D_1^2}) - \text{tr}(e^{-\sigma D_0^2}) \).

So consider this construction when \( D_0 = \sigma \) and \( D_1 = \sigma + \theta \). We have the family of superconnections

\[
\tilde{D}_t = \sigma + \begin{pmatrix}
\theta & 0 \\
0 & 0
\end{pmatrix}
+ t
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\sigma
\]

with curvatures

\[
- t^2 + t \left( \begin{pmatrix}
\theta & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\sigma
+ \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\sigma
\begin{pmatrix}
\theta & 0 \\
0 & 0
\end{pmatrix}
\right)
\]

\[
\left[
\begin{pmatrix}
\theta & 0 \\
0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\right]
\sigma
= - \theta \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\sigma
\]

\[
\tilde{D}_t^2 = - t^2 - t \theta \sigma \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

\[
e^{-t \tilde{D}_t^2} = e^{-ut^2} e^{-ut \theta \sigma \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}}
\]

Now \( (\theta \sigma)^2 = - \theta^2 \) so the second exponential
will give cosine and sine series.

$$e^{u \tilde{D}_t^2} = e^{-ut^2} \begin{pmatrix} \cos(u \theta) & -\sin(u \theta) \\ -\sin(u \theta) & \cos(u \theta) \end{pmatrix}$$

Our transgression formula is then

$$0 = \text{tr}_e(e^{u \tilde{D}_t^2}) - \text{tr}_e(e^{u \tilde{D}_0^2}) = -d \int_0^\infty \text{tr}_e(u t^{01}) e^{u \tilde{D}_t^2} dt$$

and the integral on the right is the negative of $$[\because \text{tr}(1^{01}) = -<(01)>]$$

$$\int_0^\infty \text{tr}(u t^{01}) e^{-ut^2} \begin{pmatrix} \cos(u \theta) & -\sin(u \theta) \\ -\sin(u \theta) & \cos(u \theta) \end{pmatrix}$$

$$= \int_0^\infty dt \ e^{-ut^2} 2 \sin(u \theta) \sum_{k=0}^\infty (-1)^k (u \theta)^{2k+1} \frac{1}{(2k+1)!}$$

$$= \sum_{k=0}^\infty u^{2k+2} \text{tr}(\theta^{2k+1}) \int_0^\infty dt \frac{(-1)^k t^{2k+1} e^{-ut^2}}{(2k+1)!}$$

$$= \int_0^\infty dt \ \frac{(-1)^k}{(2k+1)!} \ t^k e^{-ut} = \frac{(-1)^k}{(2k+1)!} \Gamma(k+1) \ u^{-k-1}$$

$$= \sum_{k=0}^\infty \frac{u^{k+1} \text{tr}(\theta^{2k+1}) (-1)^k k!}{(2k+1)!}$$

Now the next step will be to go back and do the analogue of the formula

$$\text{tr} (e^{u \theta - 1}) = (d+1) \int_0^\infty dt \ \text{tr} \{ u \theta e^{u \theta + u(t^2-t)\theta} \}$$
The setting is as follows. We have the trivial bundle over $\mathbb{R} \times M$ with the two connections $\delta + d$ and $\delta + d + \Theta$ where $d = dM$, $\delta = dy$. If we use the linear path $D_t = \delta + d + t\Theta$, then

$$D_t^2 = td\Theta + (t^2 - t)\Theta^2$$

as $d\Theta = -\Theta^2$

and we obtain the above transgression formula. Instead we want to look at the family of superconnections

$$\tilde{D}_t = \delta + d + \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\tilde{D}_t^2 = \begin{pmatrix} d\Theta & 0 \\ 0 & 0 \end{pmatrix} - t\Theta\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^2$$

which will give the formula

$$-\text{tr}(e^{-\delta d\Theta}) - 1 = (\delta + d)\int_0^t \text{tr}_\sigma \left\{ e^{(t - s)\delta d\Theta} (\sigma)^{-2} e^{-\tilde{D}_s^2} \right\}$$

Calculation: Suppose we apply superconnection methods to derive a transgression form like the Chern-Simons form which is based on the path of connections $d + tA$ with curvature $tdA + t^2A^2$. Then we form the family of superconnection

$$\tilde{D} = \delta + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with curvature

$$\tilde{D}^2 = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix} + (-t\Theta\sigma)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - t^2$$

The transgression formula is
\[ \text{tr} \left( e^{uF} - 1 \right) = -\int_0^\infty dt \, \text{tr} \left( e^{u(0, 1)} e^{u\sigma^2} \right) \]

Now we divide by \( n \) and use the L.T.

\[ \int_0^\infty e^{-\lambda u} \, \frac{e^{uF} - 1}{u} \, du = \sum_{k \geq 1} \frac{F^k}{k!} \int_0^\infty e^{-\lambda u} u^k \, du \]

\[ \frac{(k-1)!}{\lambda^k} \]

\[ \int_0^\infty e^{-\lambda u} \, \text{tr} \left( \frac{e^{uF} - 1}{u} \right) \, du = \sum_{k \geq 1} \frac{1}{k \lambda^k} \, \text{tr} \left( F^k \right) \]

\[ \int_0^\infty e^{-\lambda u} \, du \int_0^\infty dt \, \text{tr} \left\{ \sigma \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \sigma e^{u\sigma^2} \right\} \]

\[ = \int_0^\infty dt \, \text{tr} \left\{ \sigma \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \sigma \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \sigma \right\} \]

\[ = \left( \begin{array}{cc} \lambda+t^2-F & tA\sigma \\ tA\sigma & \lambda+t^2 \end{array} \right)^{-1} \]

Recall
\[ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \left( \begin{array}{cc} 1 & 0 \\ -a & 1 \end{array} \right) \left( \begin{array}{cc} a-bd^{-1}c & 0 \\ 0 & a^{-1} \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \]

\[ \text{tr} \sigma \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)^{-1} = \text{tr} \left( \sigma \left( \begin{array}{c} 0 \\ a-bd^{-1}c \end{array} \right) \frac{1}{a-bd^{-1}c} \right) + \text{tr} \sigma \left( \frac{tA\sigma}{\lambda+t^2} \right) \left( \begin{array}{c} 0 \\ -bd^{-1} \end{array} \right) \]

\[ a-bd^{-1}c = \lambda+t^2-F-tA\sigma \frac{1}{\lambda+t^2} tA\sigma \]

\[ = \lambda+t^2-F + \frac{t^2}{\lambda+t^2} A^2 = \lambda+t^2 - dA - \frac{A^2}{\lambda+t^2} \]
The previous calculation shows quite generally that if we go between a flat connection and an arbitrary connection by either the linear path or the superconnection, then we get the same transgression forms. In particular we see in the case of $S + d + \Theta$ and $S + d$ that the transgression forms over three pages are the same, i.e.
Let's discuss some general issues. We still have to organize lectures at Montreal. I propose to start with cyclic cocycles on $\mathcal{C}^\infty(M)$. Exhibit Connes cocycles associated to closed currents on $M$. Exhibit others obtained by vector bundles with connections. This gives some feeling for the $S$-operator. One seems to get formulas for the $S$-iterates of the cocycles associated to closed currents, although it is not clear that there is a basic operator $S$ on cocycles.

A problem is to show that the $S$-transforms of a closed current on $M$ depend up to coboundaries only on the homology class of the current.

to go in outline

cyclic cocycles on $\text{End}(E)$ assoc to connection 824-26

cyclic cocycles on $\mathcal{C}^\infty(M)$ 840

Connes cocycles assoc. to flat connection with $F$ 849-53

Derivation of a transgression form using superconnection methods leads to usual Chern-Simons 873-75

Proof that different Connes cocycles are related by the $S$-operator 869
May 19, 1988

BRS-geometry. Atiyah has a possible interpretation of Witten's Lagrangian for topological QFT, which uses equivariant differential forms for G-manifolds. It is necessary to make precise Witten's formulas in the finite dimensional context.

On one hand this should lead to an understanding of the BRS formalism, and on the other hand it might have applications to G-manifolds where the action is not free.

Let then $G$ be a compact connected Lie group acting on a manifold $M$. We suppose the action is free. In this case $M$ is a principal $G$-bundle over $M/G$. If we choose a connection $\Theta \in \Omega^1(M, g)^G$ in this principal bundle then we get a Chern-Weil homomorphism

\[ \{ (\mathfrak{g}^*) \otimes \Omega(M) \} \xrightarrow{\Theta} \{ W(\mathfrak{g}) \otimes \Omega(M) \}_{\text{basic}} \rightarrow \Omega(M/G) \]

from equivariant forms on $M$ to forms on $M/G$.

In the Witten approach one supposes a metric given on $M$ which is $G$-invariant. We can then define a $\mathfrak{g}^*$ valued 1-form $\Phi \in \Omega^1(M, \mathfrak{g}^*)$ which is $G$-invariant, in the following way.

Given $X \in \mathfrak{g}$, then inner product with the vector field on $M$ associated to $X$ by the $\mathfrak{g}$-action is a 1-form. Put another way the metric transforms the vector field to a 1-form. If $g \in \Gamma(S^2 \mathfrak{g}^*)$ is the
metric tensor, then the 1-form is the contraction \( \chi \cdot \theta \in \Gamma(T^*) \). Thus we get a map \( \phi : \Omega^1(M) \rightarrow \Omega^1(M) \) which is \( G \)-invariant, whence a 1-form \( \tilde{\theta} \in \Omega^1(M, g^*) \).

Note that \( \tilde{\theta} \) can be defined even when the action isn't free. Thus if we can write a formula for the Chern-Weil homomorphism in terms of \( \tilde{\theta} \), it might have applications in the non-free case.

Returning to our free situation, let's ask what an invariant metric on \( M \) looks like. First of all such a metric determines a connection in the principal bundle \( M \) over \( M/G \); the horizontal space for the connection is the orthogonal complement of the vertical space. Secondly, the metric restricted to the horizontal subbundle of TM descends to a metric on \( M/G \). Thirdly, at each point of \( M \) the metric restricted to the vertical tangent space gives an inner product on \( g_f \). Thus we get an equivariant map \( s \) from \( M \) to the positive cone in \( S^2(g_f^*) \). Let's denote this by

\[
\tilde{s} \in \Omega^0(M, S^2(g_f^*))^G
\]

Conversely, given a metric on \( M/G \), a connection \( \theta \in \Omega^1(M, g) \) in \( M \) over \( M/G \), and an equivariant map \( s \) from \( M \) to inner products on \( g_f \), we get an invariant metric on \( M \).

Notice that \( s \) is defined in the case \( \tilde{s} \) a non-free action, but its values are only positive semidefinite. Also, relative to the
we have the formula
\[ \tilde{\Theta} = \Theta + \phi \]

To fix the ideas we consider the simple case where \( \phi \) is constant. In other words we fix a invariant inner product on \( \mathfrak{g} \), and we use this to define the metric in the vertical direction. This inner product identifies \( \mathfrak{g} \) and \( \mathfrak{g}^* \), and then relative to this identification we have \( \Theta = \tilde{\Theta} \). Let's fix an orthonormal basis \( X_a \) for \( \mathfrak{g} \), and then the connection has the components \( \Theta^a \in \Omega^1(M) \), i.e. \( \Theta = X_a \Theta^a \) and the curvature is \( \Omega = X_a \Omega^a \), where
\[ \Omega^a = \pi^a a^a + \frac{1}{2} f^a_{bc} \Theta^b \Theta^c \]
\[ [X_b, X_c] = f^a_{bc} X_a \]

Now we are after a formula for the Chern-Weil homomorphism \( \otimes \) on \( \Omega \).

What one wants to do is to start with an equivariant function \( \alpha(\omega) \) from \( \mathfrak{g} \) to forms on \( M \) and then produce a basic form on \( M \). The first stage is to produce from \( \alpha \in \mathcal{S}(\mathfrak{g}^*) \otimes \Omega(M) \) an element of \( \mathcal{W}(\mathfrak{g}) \otimes \Omega(M) \) which is horizontal. I recall the formula from my paper with Mathai
\[ \left( \prod_{a=1}^n (1 - \Theta^a_{a}) \right) \alpha(\mathfrak{g}) = \prod_{a=1}^n (1 - \Theta^a_{a}) \alpha(\mathfrak{g}) \]
where $\hat{\Theta}^a$, $\hat{\Omega}^a$ refer to the generators of $W(g)$. The next stage is to substitute $\hat{\Theta}^a \to \Theta^a$, $\hat{\Omega}^a \to \Omega^a$.

Now according to Atiyah, Witten has a formula for doing this which uses a double integration over $\Omega$. My guess is this double integral is essentially two Fourier transforms which by the Fourier inversion formula amounts to the substitution $\Theta \to \Omega$.

The problem is to figure out Witten's formula which should give the map

* $\{S(g^{*}) \otimes \Omega(M)\}^G \to \Omega(M)_{\text{basic}}$

Take the simplest case where $G = S^1$ and the metric on $M$ is such that all orbits have same length, i.e. $M$ is a unit circle bundle in a complex line bundle with inner product over $M/G$. In this case we have the Atiyah-Bott model

$$ \{S(g^{*}) \otimes \Omega(M)\}^G = S(g^{*}) \otimes \Omega(M)^G $$

$$ = C[u] \otimes \Omega(M)^G $$

and the map above sends $u$ to the curvature $d\Theta$ and $\alpha \in \Omega(M)^G$ to

$$ \alpha \to \Theta \circ \alpha $$

Now it seems to me that there is no way to adjoint variables to $\Omega(M)$ and manipulate algebraically (fermiion integration + integrating out the extra variables) that will give $\alpha$ from $\alpha$. 
Thus we must it seems pull back $\alpha$ via the map $M \times G \to M$.

This leads me to suspect that in general one needs a similar action map

$$M \times \text{exp} \to M \times G \to M$$

and that also, as Atyah said, one must do some sort of limiting process that concentrates the measure near the origin of $G$, where the exponential map has least distortion. (Vergne-Berline?)

Maybe it's not so involved. What we have to understand is the operation which take a form $\alpha \in \Omega(M)$ and projects it onto the horizontal forms. This is the operation

$$\alpha \mapsto \left( \prod_{a=1}^{n} i_{a} \Theta \right) \alpha = \prod_{a=1}^{n} \left( 1 - \Theta i_{a} \right) \alpha$$

and it's an algebra homomorphism. To see this note we have an algebra isomorphism

$$\Omega(M) \cong \Omega(M)_{h} \otimes \Lambda\Theta^{*}$$

and we are just picking out the degree zero component, where degree refers to the $\Theta$ in $\Lambda\Theta^{*}$.

For example what is the horizontal part of $d\Theta$? It's clearly the curvature, since one knows that the curvature is horizontal, and since the horizontal projection of $\Theta$ is zero.

Consider the map

$$\Omega(M) \to \Omega(M \times G) = \Omega(M) \otimes \Omega(G) \to \Omega(M) \otimes \Lambda\Theta^{*}$$

induced by action $M \times G \to M$

and induced by evaluating a form on $G$ at the identity.
This is a ring homomorphism in fact a map of cochain algebras by construction. Let $\lambda^x \in \Omega^x$ be a dual basis to $X \in \mathcal{D}$. This map should be:

$$x \mapsto \sum \lambda^x \cdot \lambda$$

essentially. Next we have a homomorphism of algebras:

$$\Lambda \Omega^x \longrightarrow \Omega \mathcal{D}\Omega^x$$

which is not compatible with differentials.

Note that $\Omega \mathcal{D}\Omega$ doesn't respect differentials.
May 20, 1988

Let $M$ be a free $G$-manifold. We are trying to obtain the formula which Witten seems to have produced for the map (Chern-Weil homomorphism)

$$\Omega_G(M) = \{S(g^*) \otimes \Omega(M)\}^G \to \Omega(M/G)$$

associated to a connection in $M$ over $M/G$. The idea to explore is the following. Suppose we consider the $G$-map

$$\mu: M \times G \to M, \quad (m, g) \mapsto mg$$

where $M \times G$ is the trivial principal $G$-bundle over $M$ (i.e., $G$ acts trivially on $M$ and by right multiplication on $G$). We have a commutative square

$$\begin{array}{ccc}
\{S(g^*) \otimes \Omega(M)\}^G & \to & \Omega(M) \\
\uparrow & & \uparrow \\
\{S(g^*) \otimes \Omega(M)\}^G & \to & \Omega(M/G)
\end{array}$$

so we are reduced to finding a formula for the Chern-Weil homomorphism associated to the trivial $G$-bundle $M \times G$, but for the connection which is the pullback of the connection in $M$ over $M/G$ by the map $\mu$.

Note

$$S(g^*) \otimes \Omega(M \times G) = S(g^*) \otimes \Omega(G) \otimes \Omega(M)$$
\[ \{ S(g^*) \otimes \Omega(M) \}^G \cong S(g^*) \otimes \Lambda(g^*) \otimes \Omega(M) \]

Actually the Chern-Weil map
\[ \{ S(g^*) \otimes \Omega(M) \}^G \rightarrow \Omega(M/G) = \Omega(M)_{bas} \]

can be described as follows. The connection gives an isomorphism
\[ \Omega(M) \cong \Omega(g^*) \otimes \Omega(M)_{hor} \]

and then the Chern-Weil map results by taking G-invariants for the map
\[ S(g^*) \otimes \Omega(M) \cong S(g^*) \otimes \Lambda(g^*) \otimes \Omega(M)_{hor} \xrightarrow{\alpha} \Omega(M)_{hor} \]

where \( \alpha \) in \( S^1(g^*) \) is the curvature and \( \alpha \) or \( \Lambda(g^*) \)

is zero.

Consider the Chern-Weil map
\[ \{ S(g^*) \otimes \Omega(P) \}^G \rightarrow \Omega(B) \]

in the case of a principal G-bundle \( P/B \) equipped with connection and section. The section gives an isom. \( P = B \times G \), and so the equivariant forms

is isom. to
\[ \{ S(g^*) \otimes \Omega(G) \}^G \otimes \Omega(B) \]

Thus one needs a map \( W(g) \rightarrow \Omega(B) \). But the difference of the two connections on \( P \), or better just
the pull-back of the connection form via
the section gives us a collection \( \eta^a \) of
1-forms on \( \Omega(\mathcal{B}) \). Since \( \Omega(\mathcal{B}) \) is the free
cochain algebra generated by the \( \eta^a \), we have
then a map \( \Omega(\mathcal{B}) \rightarrow \Omega(\mathcal{B}) \).

The question is whether there is a kind of
integral formula for this sort of map.

May 21, 1988:

Here seems to be an example of gauge-fixing.

Let \( P \) be a principal \( G \)-bundle over \( \mathcal{B} \)
and let \( V \) be a representation of \( G \).
If \( s : \mathcal{B} \rightarrow P \) is a section (choice of gauge),
then we have an isomorphism \( \mathcal{B} \times G \rightarrow P \) and so

\[
(\Omega(P) \otimes V)^G = \Omega(B) \otimes (\Omega(G) \otimes V)^G = \Omega(B) \otimes \Lambda^* V
\]

Notice that \( (\Omega(G) \otimes V)^G = \Lambda^* V \) is the complex
of Lie algebra cochains on \( G \) with values in the
\( \mathfrak{g} \)-module \( V \), denoted \( C(\mathfrak{g}, V) \). It is the
free module over \( C(\mathfrak{g}) = \Lambda^* \mathfrak{g} = \Omega(\mathfrak{g})^G \) generated
by \( V \).

To describe the differential, let choose a basis
\( X_a \) for \( \mathfrak{g} \), \( [X_a, X_b] = f_{ab}^c X_c \) and let \( X^a \) denote
the dual basis for \( \Lambda^* \mathfrak{g} \). Then \( C(\mathfrak{g}) = \Omega(\mathfrak{g})^G \) is
the exterior algebra on the \( X^a \) with differential
given by

\[
dx = -X_a \eta^a
= -\frac{1}{2} [X_a, X_b]
\]

i.e.

\[
dx^a = -\frac{1}{2} f_{bc}^a X^b X^c
\]
The differential on $\mathit{C}(g, V) = \mathit{C}(g) \otimes V$ is determined by requiring a $\mathit{G}$ module over $\mathit{C}(g)$ such that

$$
d\nu = \theta^a(X_a \nu)$$

Let's check this formula. We are using the isomorphism $(\Omega(G) \otimes V)^G \rightarrow \Lambda g^* \otimes V$ given by restricting a form to the tangent spaces at the identity. Given $\nu \in V$, the function on $G$ corresponding to it under $(\Omega(G) \otimes V)^G = V$ is the function $f_\nu : \xi \mapsto \xi^{-1} \nu$. In effect we must check the invariance

$$(g \cdot f_\nu)(x) = g f_\nu(g x)$$

$$= g(xg)^{-1} \nu$$

$$= f_\nu(x).$$

What is $df_\nu$, or rather the image of $df_\nu \in (\Omega(G) \otimes V)^G$ under the isomorphism to $g^* \otimes V$?

Unfortunately I have got left + right mixed up. The Maurer-Cartan form $\mathcal{X} = g^{-1} dg$ is a left-invariant differential form, or rather the components $\mathcal{X}^a$ are invariant under left multiplication.

The way to rectify this is to use the map $\mathbf{B} \times G \rightarrow P$, $(x g) \mapsto s(x) g^{-1}$, so that right $G$-multiplication corresponds to lift $G$-mult. by $g^{-1}$ on $G$. Then $\Omega(G) \mathcal{X}$ is the complex of left-invariant forms on $G$ and it is the exterior alg. on the $\mathcal{X}^a$ with $d\mathcal{X}^a = -\frac{1}{2} f_{bc}^a \mathcal{X}^b \mathcal{X}^c$. 

Now consider \((\Omega(G) \otimes V)^G\) where 
\(G\) acts on itself by left multiplication.

Given \(v \in V\), let \(f_v : G \to V\) be the function \(f_v(g) = gv\). This is invariant:

\[(g, f_v)(x) = g(f_v(g^{-1}x)) = g g^{-1}x v = x v = f_v(x)\].

Its differential is

\[(df_v)(g) = f_v(g + dg) - f_v(g) = dg v = g(g^{-1}dg)v = g x^a x_a v\]

Then \(d : V \to \Omega^* \otimes V\) is

\[\Omega(G) \otimes V^G \to (\Omega(G) \otimes V)^G\]

\[d v = x^a x_a v\]

i.e. \((d v)(x) = X v\)

Check

\[0 = d^2 v = d (x^a x_a v) = dx^a x_a v - x^a x^b X_b X_a v\]

\[= dx^c x_c v - x^a x^b X_b X_a v\]

\[= dx^c x_c v + \frac{1}{2} x^a x^b [X_a, X_b] v\]

\[= \left(dx^c + \frac{1}{2} x^a x^b f_{ab}\right) x_c v = 0\]

Let's now apply this to the case of the equivariant forms on \(G\). This means we are considering the \(G\)-cochain algebra

\[\left(\Omega(G) \otimes \Omega(G)\right)^G\]

where \(G\) acts by left multiplication, and where the differential is \(d = d - \omega^a X_a\).
What is \( i_x \) in \( \Omega(G)^G = \Lambda g^* \)? It is interior product with respect to infinitesimal left multiplication by \((1-eX)\).

Hence \( i_e \cdot X^b = -\delta^b_a \). (According to our conventions \( G \) acts to the right and then \( iX, LX \) are defined by the transpose of multiplying by \( e^T X \).)

Thus in the algebra \( (S(G^*) \otimes \Lambda G)^G \) we have

\[
dX^a = d_e X^a - \omega^a_b X^b = -\frac{1}{2} X^b X^c f^a_{bc} + \omega^a
\]

which means we have the Weil algebra.
May 22, 1988

Go back to the beginning. We start with a principal $G$-bundle $M \rightarrow M/G$ with a connection form $\omega$. There is then a Chern-Weil homomorphism

\[ \{ S(g^*) \otimes \Omega(M) \} \rightarrow \Omega(M/G) = \Omega(M)_{bas} \]

which we know is given by the formula

\[ \alpha(\omega) \rightarrow e^{-\theta^g \iota_{\omega}} : \alpha(\omega). \]

Our problem is to discover the "integral formula" for the Chern-Weil map which Witten seems to have. Here "integral formula" means that we adjoin variables which may be fermionic, that is, we lift up to some (super?) manifold over $M$; then we multiply by a kernel and integrate over the fibre relative to some other map to $M$.

Let's study the operator $e^{-\theta^g \iota_{\omega}}$.

The bosonic analogue is

\[ \frac{t x^i \partial_i}{\alpha} : e^{t x^i \partial_i} : \]

acting on polynomials. Except only part of the variables occur. So consider polynomials in $x, y$ and the operator $e^{t x \partial_x}$. We have

\[ e^{t x \partial_x} : f(x, y) = \sum_{n \geq 0} \frac{t^n x^n}{n!} \partial^x f(x, y) \]

\[ = f((1 + t)x, y) \]

so that if $t = -1$ we get the projection.
\[ f(x,y) \rightarrow f(0,y). \] To what extent can we find a nice "integral formula" for this operator? It is a substitution operator so the Schwartz kernel should be a \( S \)-function supported on the graph of the substitution.

We have

\[
\begin{align*}
&\int e^{tx} \partial_x^x f(x,y) = \left[ e^{u \partial_x} f(x,y) \right]_{u=tx} \\
&= \left[ f(x+u,y) \right]_{u=tx} \\
&= \int \frac{dx}{2\pi} e^{itx} \int du e^{-iu} f(x+u,y)
\end{align*}
\]

I guess what we learn from this calculation is that to get \( e^{-\Theta \partial_x} \), where \( \Theta \in \mathbb{R}(\mathbb{M}) \), we first consider the translation operator

\[ e^{x \lambda} \lambda \]

where \( \lambda \) are auxiliary Grassmann variables and then we substitute \( \lambda \rightarrow -\Theta \). This substitution can be done by introducing a new set of Grassman variables \( \mu_a \) and then double integrating

\[
\int D\mu_a \int DA e^{(\Theta \partial_x + \lambda) \mu_a}
\]
Here seems to be the logical structure, or maybe it is better to say the "yoga" or way of thinking. One starts with the complex of equivariant forms: \( \{W(\mathfrak{g}) \otimes \Omega(M)\}_{bas} \).

Here we have adjoined to \( \Omega(M) \) extra fermionic and bosonic variables, then divided out by the supergroup \( G + \mathfrak{g} \). Because the \( \mathfrak{g} \) action is free we can divide out by it and we obtain the small model:

\[
\left\{ S(\mathfrak{g}^*) \otimes \Omega(M) \right\}^G
\]

in which only the new bosonic variables remain.

Next we want to divide out by the \( G \) in some sense, so we pull back to \( M \times G \). This introduces a new set of bosonic variables and fermionic variables, but the fermionic ones disappear on taking the quotient:

\[
\left\{ S(\mathfrak{g}^*) \otimes \Omega(M \times G) \right\}^G = \left\{ S(\mathfrak{g}^*) \otimes \Omega(G) \right\}^G \otimes \Omega(M) \]

\( W(\mathfrak{g}) \)

Thus we now have an algebra freely generated by \( \Omega(M) \) and a set of bosonic and fermionic variables. Finally these have to be specialized to curvature and connection forms in \( \Omega(M) \).

This can be done by introducing a new set of Fourier transform bosonic + fermionic variables.

Another point: Instead of lifting back to \( M \times G \) one can think of...
choosing (fixing) a gauge) a section of $M$ over $M/G$. Until one does this one can't speak about the connection forms on $M/G$.

Now I think we have the correct yoga or philosophy, and the next stage is to work out formulas. This means "working in a gauge".

It's probably a good idea to ponder what I might end up with. We will start with $x = (x^i \mapsto x^i(\omega)) \in (S(L^*) \otimes \Omega(M))^G$. If a gauge is fixed we have an isomorphism

$$(S(L^*) \otimes \Omega(M))^G = W(L) \otimes \Omega(M/G)$$

and so $x$ becomes a polynomial in the universal connection + curvature $(\xi^a, \omega^a)$ with coefficients from $\Omega(M/G)$, specifically

$$e^{-\xi^a \eta_a} \omega(\omega) \mid_{\text{cross section } s(M/G)}$$

$$\equiv (s^* (e^{-\xi^a \eta_a}))(\omega)$$

Now introduce the dual variables to $\xi^a$ and $\omega^a$, call them $\lambda^a$ and $\eta^a$. Then the evaluation formula is something like

$$\int d\eta^a \int D\lambda^a \delta[\xi^a - \lambda^a] (\Theta^a - \xi^a) \lambda_a + (\Omega^a - \omega^a) \eta^a s^* (e^{-\lambda^a \eta_a}(\omega))$$
The idea would be to write the exponential in a really nice form as Atiyah suggested. It is tempting to identify \( \mathcal{W}(g) = S(g^*) \otimes N(g^*) \) with the polynomial differential forms on \( g^* \), and in some sense by using Fourier transform you are doing this. But there are problems with the grading.

Review Atiyah's idea. One starts with the connection form \( \Theta = X_a \Theta^a \) on \( M \), and one interprets \( X_a \) as linear functions \( \eta_a \) on \( g^* \). This gives the invariant 1-form \( \Theta^a \eta_a \) on \( M \times g \) of type \((1,0)\). An invariant 1-form can be viewed as an equivariant 1-form. The differential in \((S(g^*) \otimes \Omega(M \times g))_G\) is

\[
(d - \omega^a c_a)(\Theta^a \eta_a) = (d\Theta^a - \omega^a) \eta_a - \Theta^a d\eta_a
\]

We want something like

\[
(\Omega^a - \omega^a) \eta_a \pm (\Theta^a - K^a) \eta_a
\]
May 24, 1988

Consider the double cochain algebra $C^\bullet(A, S^\bullet)$ with the differentials $\delta, d$ and the canonical element $\Theta \in C^1(A, \Omega^0_A)$.

We wish to understand the universal relations that can be generated in the double complex, which is essentially the commutator quotient of the above, namely

$$C^\bullet_A(A, S^\bullet) / [\Omega^\bullet_A, \Omega^\bullet_A]$$

For this purpose we note the relation $\delta \Theta = -\Theta^2$, so that $(\delta + d + \Theta)^2 = d\Theta$. Let's change notation and consider the universal DG algebra generated by elements $A, F$ of degrees 1, 2 respectively such that $F = dA + A^2$. We can bidegraded this algebra by calling $A$ of degree $(1, 0)$ and $F$ of degree $(1, 1)$, since its the tensor algebra generated by $A, F$. We can also define a horizontal differential $\delta$ by $\delta A = -A^2$ and $\delta F = [F, A]$ and a vertical differential $d$ by $dA = F$ and $dF = 0$. Then we compute

$$\delta^2 A = -\delta(A^2) = -[(-A^2)A, A] - A(-A^2)] = 0$$

$$\delta^2 F = \delta [F, A] = [F, [A, A]] + [F, -A^2] = (FA - AF)A + A(FA - AF) - FA^2 + A^2F = 0$$

$$\delta d + d\delta A = \delta F + d(-A^2) = [F, A] - \text{Cayley} \ FA + AF = 0$$

$$\delta d + d\delta F = d[F, A] = [F, F] = 0.$$
What we learn then is that the universal Chern-Simons algebra \( C(A,F) \) with \( \tilde{\pi} = F - A^2 \) is in fact a bigraded diff. algebra, i.e. a double cochain algebra. We are interested in the double complex obtained by taking the commutator quotients.

Proof. The vertical cohomology of \( C(A,F)/[\cdot,\cdot] \), that is with respect to \( d \), is trivial. Hence the total cohomology is also trivial.

Proof. As far as \( d \) is concerned \( C(A,F) \) with \( d \) is the tensor algebra on the complex \( 0 \rightarrow CA \xrightarrow{d} CF \rightarrow 0 \). The commutator quotient complex is the direct sum of the symmetric cyclic powers of this complex and so the cohomology is trivial. The second assertion is clear from the spectral sequence which incidentally sits in a sector.

\[
\begin{array}{c}
F^2 \\
F \quad AF, FA \\
0 \quad A \quad A^2
\end{array}
\]

An interesting question is what one can say about the \( S \)-cohomology, and also because this is what Dubois-Violette, Tan, Viallet (Comm. Math. Phys 102 (1985)) say gives the "anomalous" terms, what can one says about the \( S \)-cohomology model, that is after dividing by \( \text{Im} \, d \).

Our next step will be to get a conjecture.
about the $S$-cohomology from the D-V, T, V results. Let's review these results.

Let $g$ be a Lie algebra; they define the BRS algebra of $g$ as a kind of bigraded generalization of the Weil algebra. It is a double cochain algebra $\mathcal{B}$ equipped with a universal "connection form" $\omega \in \mathcal{B}^{0,1}$ such that the curvature is of type $(0,2)$ say:

$$(\delta + \partial) \omega + \omega^2 \in \mathcal{B}^{0,2}.$$ 

One writes $\omega = \chi + A \in \mathcal{B}^{1,0} \oplus \mathcal{B}^{0,1}$, and then has relations

$$8\chi + \chi^2 = 0,$$

$$d\chi + 8A + [\chi, A] = 0.$$ 

The universal algebra $\mathcal{B}$ is generated by the components of the fields $\chi, A$, $q = d\chi$, $F = dA + A^2$.

There are two interesting subalgebras of $\mathcal{B}$. The first is generated by $A$, and is stable under $\delta$. This is the Weil algebra. The second is generated by the components of $\chi$ and $q = d\chi$ and is stable under both $\delta, \partial$. In fact this second algebra is really quite analogous to the non-commutative algebra I have been studying (the Chern-Simons algebra).

This suggests I should change notation immediately to $\chi, q$ instead of $A, F$.

We a map from the non-commutative algebra...
to matrices over the commutative algebra \( \text{gen. by } x^a, \varphi^a \), when we take \( g^e_N \). Further we have the trace map from the non-commutative algebra to the commutative algebra.

The commutative algebra is generated by the components of \( X = x^a x^a \), \( \varphi = x^a \varphi^a \) where the differential is

\[
\delta X = -X^2 \\
\delta \varphi = [\varphi, X]
\]

Therefore one has the complex of Lie algebra cochains \( \delta g \) with values in \( S(g^*) \). In the reductive case, we know

\[
H^*(g, S(g^*)) = H^*(g) \otimes S(g^*)^G
\]

Now I want to take \( g^e_N \) for large \( N \) and take the primitive part. It seems that we get exactly one generator in each degree.

Thus the conjecture should be that the \( S \)-homology of the commutator quotient of the non-commutative Chern-Simons algebra contains a single generator in each degree. In fact they seem to be

\[
\text{tr } (x \text{ odd}) \quad + \quad \text{tr } (\varphi) \quad \text{ for any}
\]

Check: Observe that

\[
\text{tr } (\partial^{2n+1}) \quad \text{tr } (d\varphi^n)
\]

are cyclic cocycles, i.e., killed by \( \Delta \).

Let's see if we can prove our conjecture.
about the $\mathbb{C}$-cohomology of
\[ C \langle x, \varphi \rangle / \{ \} \]
where $\delta x = -x^2$, $\delta \varphi = [\varphi, x]$. This
splits up into rows where the $g$th row
has $\varphi$ occurring $g - 1$ times interspersed with
powers of $x$. I propose to use the Goodwillie
trick of labelling the $\varphi$'s in order. This
should give a $g$-fold cyclic covering complex
where the monomials all begin with $\varphi$. So
we should first look at $g = 1$.

In this case we have

\[
\delta (\varphi x^j) = -\varphi x^j + \varphi x^{i+1} (1 + \sum_{i=1}^j (-1)^i)
\]

\[
= \varphi x^{j+1} \left( 1 + \sum_{i=1}^j (-1)^i \right)
\]

so the conjecture works for $g = 1$. Try $g = 2$.

\[
\delta \{ \varphi x^k, \varphi x^l \} = \varphi \left( 1 + \frac{(-1)^k}{2} x^{k+1} \right) \varphi x^l
\]

\[
+ (-1)^k \varphi x^k \varphi x^l
\]

\[
+ (-1)^{k+1} \frac{\varphi x^{k+1}}{1} (1 - 1 + \ldots + (-1)^{l+1})
\]

\[
= \varphi \left( 1 - \frac{(-1)^k}{2} \right) x^{k+1} \varphi x^l + (-1)^k \varphi x^k \varphi \left( \frac{1 + \sum_{i=1}^l (-1)^i + (-1)^{l+1}}{1 - (-1)^l} \right)
\]
Thus it seems to work

Next we should look at the $d$-cohomology modulo $d$ in the sense of Anomaly theory. This means that we pick a row of the commutator quotient complex and divide out by the image of $d$ from the lower row. Since we know the $d$-cohomology, the vertical cohomology is trivial, we ought therefore to be able to reduce things to our computation of the $d$-cohomology.

This seems clear. Let's go back to our notation $X \to \Theta$, $q \to d\Theta$ and our diagram

\[
\begin{array}{cccc}
    d\Theta^3 & \times & & \\
    d\Theta^2 & \times & \times & \\
    d\Theta & \times & \times & \times \\
    \times & \times & \times & \times \\
\end{array}
\]

Notice that in the universal situation, but note $C^\bullet(X, \Omega^\bullet_{\mathbb{C}/\mathbb{C}})$, the columns are acyclic.

In the universal case then, we see by diagram chasing that at each of the crosses there is $d$-cohomology modulo $d$.

Now we ought to see if we can produce Chern-Simons formulas for the relevant cyclic cochains, now that we know about the Goodwillie picture for the rows.
Let's see if we can explain how the BRS obj B of DTVT is related to anomalies. An "anomalous term" is of the form
\[ \Delta = \int Q \]
where the integral is over space-time, and Q is a polynomial in the fields + their derivatives, which satisfies a "consistency equation"
\[ \delta \Delta = 0 \]

However, solutions of this equation of the form \[ \Delta = \delta \Delta' \] with \[ \Delta' = \int Q' \] are considered trivial.

We get solutions of the consistency equation from Q such that
\[ \delta Q = dQ' \]
for some Q', that is, if Q is a 1-cocycle mod d.

If
\[ Q = \delta L' + dL \]
then
\[ \int Q = \int \delta L' = \delta \int L' \]
is a trivial solution. Thus anomalous terms arise from the Lie algebra cohomology \[ H^1 \]
with values in forms depending on the fields modulo \[ Im d \] on these forms.

Now consider a pure gauge theory which means that one considers functionals of a variable gauge field A, e.g. YM. The sort of Q's to consider are polynomials in the 1-forms A^a and their differentials dA^a together with the components A^a of
a variable infinitesimal gauge transformation \( X = (X^a) \) and their differentials \( dX^a \). A mystery is why the \( X^a \) should be treated as the component of the MC form of the group of gauge transformations. Thus why do the sort of terms
\[
\Delta = \int Q \text{ considered have the form where } Q \text{ is a polynomial in the fermionic variables } A^a, X^a \text{ and the bosonic variables } dA^a, dX^a? \] This must have something to do with the quantum theory.

The BRST algebra has in addition to \( A, X, dA, dX \) also a contractible piece involving the second Faddeev-Popov ghost \( \bar{\psi} \) and the gauge fixing Lagrange multiplier \( \gamma \) satisfying \( S\bar{\psi} = \gamma \).
May 25, 1988:

Chern-Simons forms. These are the components of

$$\int_0^1 dt \, \text{tr} \left( \Theta \exp \left( t d\Theta + (t^2-t)\Theta^2 \right) \right)$$

Let us look at the form of degree \((p,g)\), so that there are \(g\) \(d\Theta\)'s and the \(\Theta\)-degree is \(p\). I find it easier to think in terms of the number of \(\Theta^2\) used; call this \(k\). Then \(p = \Theta\)-degree \(= 1 + g + 2k\) and the total degree is \(p + g = 1 + 2g + 2k\). We are after therefore the component of

$$\int_0^1 dt \, \text{tr} \left( \Theta \left( \frac{1}{(n-1)!} \right) (t d\Theta + (t^2-t)\Theta^2)^{n-1} \right)$$

of degree \(g\) in \(d\Theta\). Thus we get

$$\text{tr} \left( \Theta \left( \frac{\text{monomials of degree } g \text{ in } d\Theta} \right) \right) \text{non-commutative and } k \text{ in } \Theta^2$$

times the numerical factor

$$\frac{1}{(n-1)!} \int_0^1 dt \, t^g (t^2-t)^k = \frac{(-1)^k}{(g+k)!} \int_0^1 t^{g+k+1} - (1-t)^{(k+1)} \, dt$$

$$= \frac{(-1)^k}{(g+k)!} \frac{\Gamma(g+k+1) \Gamma(k+1)}{\Gamma(g+2k+2)} = \frac{(-1)^k k!}{(g+2k+1)!}$$

so we learn that if we set

$$\zeta_{g,k} = \text{tr} \left\{ \Theta \left( \text{sum of the monomials of degree } g,k \text{ in } d\Theta, \Theta^2 \right) \right\}$$

then \(\zeta_{g,k} \) and \(d\zeta_{g-1,k+1}\) coincide up to numerical factors.
For example

\[ \text{tr} \, \Theta(d\Theta)^{n} = \text{const} \cdot \text{tr}(\Theta \, \text{Sym}(d\Theta^{2}, d\Theta^{2})) \]

It would be nice to see this directly.

May 26, 1988

There seems to be hope that the Chern-Simons transgression forms and their splitting into homogeneous components might explain most of cyclic formalism. We learned yesterday that in the usual formula

\[ \text{tr} \left( \frac{d\Theta^{n}}{n!} \right) = \int_{0}^{1} dt \, \text{tr} \, \Theta \left( \frac{4d\Theta + (t^2 - t)\Theta}{(n-1)!} \right) \]

all the complexity due to the factors of \( t \)

and the integration disappears if one takes the homogeneous components for the begetting. Let's now do the combinatorics as simply as possible.

Thus we consider the expression

\[ \text{tr} \, \Theta (d\Theta + \Theta^{2})^{n} \]

in the commutator quotient of \( \langle \Theta, d\Theta \rangle \)

which is a bigraded differential algebra with \( 8\Theta = -\Theta^{2} \), \( 8d\Theta = [d\Theta, \Theta] \). We can write

\[ \text{tr} \, \Theta (d\Theta + \Theta^{2})^{n} = \text{tr} \Theta (d\Theta)^{n} + \text{tr} \Theta^{2n+1} + \text{tr} \Theta^{2n+1} \]

and where \( f_{n-k,k} \) involves \( d\Theta \) \( n-k \) times and \( \Theta^{2} \) \( k \)-times.
Notice that
\[ \delta (d\theta) = [d\theta, \theta] \quad \implies [\delta + \theta, d\theta + \theta^2] = 0 \]
\[ d (\theta^2) = [d\theta, \theta] \quad \implies [d + \theta, d\theta + \theta^2] = 0 \]
and so

\[ \delta \text{ tr} (\theta (d\theta + \theta^2)^n) = \text{ tr} [\delta + \theta, \theta (d\theta + \theta^2)^n] \]
\[ = \text{ tr} [\delta + \theta, \theta (d\theta + \theta^2)^n] = \text{ tr} (\theta^2 (d\theta + \theta^2)^n) \]
\[ d \text{ tr} (\theta (d\theta + \theta^2)^n) = \text{ tr} [d + \theta, \theta (d\theta + \theta^2)^n] \]
\[ = \text{ tr} [d + \theta, \theta (d\theta + \theta^2)^n] = \text{ tr} (d\theta (d\theta + \theta^2)^n) \]

Now we want to identify the components up to numerical factors.

Let’s consider two non-commuting variables \( A, B \) and form in \( \mathfrak{C} \langle A, B \rangle / [\cdot, \cdot] \) \((A, B \text{ both even})\)

\[ \text{ tr} (A + B)^{n+1} = \text{ tr} (A^{n+1}) + (n+1) \text{ tr}(A^nB) + \ldots + \text{ tr}(B^{n+1}) \]

We propose to calculate the term of degree \( k \) in \( A \) and \( B = n+1-k \) \( \in \mathbb{N} \) in two ways; this is for \( 0 < k < n+1 \).

The term of bidegree \((k, p)\) is the sum of traces of monomials of degree \((k, p)\) in \((A, B)\) and their \((k+p)\) of these monomials, one for each and \(k\) subset of \(\{1, \ldots, n+1\}\). The traces are the same for monomials differing by a cyclic permutation; this is the cyclic group of order \(n+1\).

Thus we get a sum over orbits, and the contribution of an orbit is the trace of one of its representative monomials times the size of the orbit. So if we
divide by the order $n+1$ of the group we get a sum over the orbits where the contribution of an orbit is the trace of a representative monomial in the orbit divided by the order of the stabilizer.

But now let us restrict to those monomials which begin with $A$. This set of monomials is in 1-1 correspondence with the set of monomials of degree $(k,p)$, There is also action of $\mathbb{Z}/k$ on this set which cyclically moves the “blocks” $AB^t$. So the sum of the traces of the monomials of degree $(k,p)$ which begin with $A$ divided by $k$ equals the sum over the orbits where the contribution from an orbit is the trace of a representative term divided by the order of the stabilizer. But now we apply the equivalence of categories used in our proof of Green's theorem and we see

$$\frac{1}{n+1} \text{tr} \left( \text{sum of monomials of degree } (k,p) \text{ in } A,B \right) = \frac{1}{k} \text{tr} \left( A \left( \text{sum of monomials of degree } (k-1,p) \text{ in } A,B \right) \right)$$

Reversing the roles of $A, B$ we see this also equals

$$\frac{1}{p} \text{tr} \left( B \left( \text{sum of monomials of degree } k(p-1) \text{ in } A,B \right) \right)$$

We use this when $A = \Theta^2$, $B = d\Theta$. We have

$$\frac{1}{k+1} \text{tr} \left( \Theta \left( \text{mon. in } d\Theta, \Theta^2 \text{ of degree } n-k,k \right) \right) = \frac{1}{k+1} \text{tr} \left( \Theta^2 \left( \text{mon. in } d\Theta, \Theta^2 \text{ of degree } n-k,k \right) \right)$$

$$\frac{1}{(n-k)+k+1} \text{tr} \left( \text{mon. in } d\Theta, \Theta^2 \text{ of degree } n-k,k+1 \right)$$
\[
\frac{1}{n-k} \quad \text{tr} \quad \Theta \quad (\text{min in } d^\Theta, \Theta^2) \quad \text{degree } n-k-1, k+1
\]

Now that we understand the Chern-Simons we want to apply them to understand cyclic formalism.

Let's review the Connes cocycles. We suppose \( A \) acts on \( H \) and there is given an involution \( F \) of \( H \). We then do calculations in \( \text{End}(H) \), but perhaps one should work in \( A \times C[F] \). This is so far ungraded. In the graded case we suppose \( H = H^+ \otimes H^- \) \( \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) such that \( A \) is even and \( F \) is odd with respect to \( \varepsilon \). The universal algebra would be \((A \times C[F]) \otimes C[\varepsilon]\) which is isomorphic to \((A \times A) \otimes M_2(C)\).

I have two ways to understand the Connes cocycles. First of all there is his method where one forms the DG algebra

\[
R^+ \xrightarrow{[FJ]} R^- \xrightarrow{[FJ]^*} R^+ \xrightarrow{[FJ]^*} \ldots
\]

where \( \text{End}(H) \) in the ungraded case and in the graded case \( \text{End}^\varepsilon(H) \) relative to \( \varepsilon \).

In the ungraded case a trace defined on a suitable ideal of \( R \) will give odd degree traces on the above DGA. Then we can form the cyclic cocycles \( \text{tr} (\Theta[F], \Theta)^{2n-1} \).
In the graded case one has \( R^+ = \text{End}^+(H) \), or some ideal say, and this gives closed even degree traces on the DGA. Thus one obtains even cyclic cocycles \( \text{tr}(\Theta[F, \Theta]^{2n}) \).

In addition to Lemma we have a vector bundle with a flat connection \( D \) and a splitting given by an involution \( F \). Then we can write \( D = \nabla + \alpha \) as the sum of a connection \( \nabla = \frac{1}{2}(D + FDF) \) commuting with \( F \) and a 1-form \( \alpha = \frac{1}{2}(D - FDF) \) anti-commuting with \( F \). From the flatness (or even the weaker assertion that \( D^2 \) commutes with \( F \)) we find that

\[
D^2 = \nabla^2 + \alpha^2 + [[\nabla, \alpha]]
\]

commutes with \( F \) anti-commutes with \( F \)

so that \([[\nabla, \alpha]] = 0 \). This implies by the usual argument that \( \text{tr}(F^{2n}) \) is closed.

In the graded case one considers the form \( \text{tr}(\Theta[F, \Theta]^{2n+1}) \) which is closed.

To obtain cyclic cocycles one works in the algebra \( C^*(A, \text{End} H) \) with

\[
D = \delta + \Theta,
\]

Then

\[
\alpha = \frac{1}{2}((\delta + \Theta) \Theta - F(\delta + \Theta)F) = \frac{1}{2}(\Theta - FGF)
\]

\[
= \frac{1}{2}F[F, \Theta]
\]

and

\[
\text{tr}(F^{2n}) = (-1)^n 2^{-2n} \text{tr}(F[F, \Theta]^{2n}) = (-1)^n 2^{-2n} 2 \text{tr}(\Theta[F, \Theta]^{2n-1})
\]

\[
\text{tr}(\Theta[F, \Theta]^{2n+1}) = 2^{-2n-1}(-1)^n \text{tr}(\Theta[F, \Theta]^{2n}) = (-1)^n 2^{-2n-1} 2 \text{tr}(\Theta[F, \Theta]^{2n})
\]
Now the problem I want to get to the bottom of is to relate these different cocycles by the S-operator.

As usual there is the basic problem of what the S-operator is.

The goal now is to bring to bear all we know about the formalism so far. We know that we ought to be able to replace \( \text{End}(H) \) by \( A \times C[F] \) in the ungraded cases.

But we also did find a proof that the different cocycles are S-related using the homotopy property established in my formalism. Let's go over this.
I want to discuss the $S$-operation for the odd cocycles belonging to an extension. My idea is to use the homotopy invariance in roughly the following way. Let $T_{2n-1}$ be the cocycle's associated to $\Theta, F$. Then by doubling one achieves another pair

$$\tilde{\Theta} = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{F} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}$$

and it seems possible to relate $skT_{2k}$ for $\Theta, F$ to the coefficient of $t^{2k}$ in $T_{2k+2}$ for $\tilde{\Theta}, \tilde{F}$. Then the homotopy invariance should show $skT_{2k}$ and $T_{2k+2}$ are cohomologous.

I would now like to see what this sort of construction amounts to when I work with the idempotent $e = \frac{1+F}{2}$. The idea is roughly that what we are doing is to deform the idempotent $e$, and in general such a deformation is described by a map from $\text{Im}(e)$ to $\text{Im}(1-e)$. If we stay within the given space $H$, then $eH$ and $(1-e)H$ are unrelated, so there is nothing we can do, i.e. there is no deformation singled out. On the other hand if we double $H$, then there is the copy of $eH$ in the 2nd factor.

So we should just add $eH$ to $H$ perhaps.
May 28, 1988

The problem is still to relate the cocycles belonging to an extension via the $S$ operator. To do this, I propose to start from $A, H, F$ then double $H$ and use

$$
\tilde{\Theta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{F} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix}
$$

Notice that

$$
\tilde{F}^{t} \tilde{F} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & tF \\ tF & -F \end{pmatrix} = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix} \otimes F
$$

should work as well, since $\tilde{F}$ and $\tilde{F}^{t}$ are conjugate by $\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix}$ which commutes with $\tilde{\Theta}$.

We have

$$
\begin{pmatrix} (F & t) \\ (t & -F) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} [F, \Theta] & -\Theta t \\ t \Theta & 0 \end{pmatrix}
$$

On the right side we have a polynomial in $t$ and $t^{2}$ whose coefficients are $S$-transforms of the cocycles $tr \Theta [F, \Theta]^{2m-1}$. On the left we see $tr \tilde{\Theta} [\tilde{F}, \tilde{\Theta}]^{2m-1}$ whose cohomology class is independent of $t$.

The problem with this proof is that one must work infinitesimally in $t$. Thus the "ideal of $\Theta$" must contain $t\Theta$. 

Let \( C = A \otimes C[\mathbb{F}] = (A \otimes A) \otimes C[\mathbb{F}] \).

This is \( \mathbb{Z}_2 \)-graded where \( A \) is even and \( F \) is odd. We want to set up an additive isomorphism of \( A \otimes A \) with \( \Omega A \). This isomorphism is constructed by defining a module structure on \( \Omega A \) over \( A \otimes A \), and then acting on 1.

Another feature of the isomorphism is that \( g(a) = a - FaF \) should correspond to \( da \). In fact we want

\[
\begin{align*}
\ast & : a_0 da_1 \ldots da_n & \mapsto a_0 g(a_1) \ldots g(a_n) \\
& & = a_0 F[F[a_1] \ldots F[a_n]]
\end{align*}
\]

The simplest way to proceed is to define a \( C = (A \otimes A) \otimes C[\mathbb{F}] \)-module structure on \( \Omega A \otimes C[\mathbb{F}] \). For this we have to have an \( A \)-module structure - this is just left multiplication of \( A \) on \( \Omega A \), and an \( F \)-operator. We have to move the \( F \) through \( \Omega A \), and so must define conjugation by \( F \) on \( \Omega A \). But \( F \) anti-commutes with \( g(a) = a - FaF \), so

\[
F(a_0 g(a_1) \ldots g(a_n))F = (-1)^n F_{a_0} F g(a_1) \ldots g(a_n)
\]

\[
= (-1)^n \left( a_0 g(a_1) \ldots g(a_n) + (F_{a_0} F - a_0) g(a_1) \ldots g(a_n) \right)
\]

Thus if we want \( \ast \) we must define

\[
F(a_0 da_1 \ldots da_n)F = (-1)^n (a_0 da_1 \ldots da_n - da_0 \ldots da_n)
\]

\[
= \sigma(1 + a_0 da_1 \ldots da_n)
\]
where \( \sigma(\omega) = (-1)^\omega \omega \).

Let's check \( \sigma(1+d) \) as \( \Omega^*_A \) is an involution:

\[
\sigma(1+d) \sigma(1+d) = (1-d)(1+d) = 1.
\]

So now we can make \( \Omega^*_A \) into a left \( C \)-module by making \( F \) act as \( \sigma(1+d) \) and \( A \) by the obvious left multiplication. One then has

\[
F a F = \sigma(1+d) a \sigma(1+d)
= (1-d) a (1+d) = a - (da) - (da) d
\]

and \( g(a) = a - Fa F = da + (da) d \). Thus

\[
g(a) \text{ acting on a closed form is multiplication by } da.
\]

Thus

\[
a_0 g(a_1) \cdots g(a_n) \text{ acting at } 1 = q_0 da_1 \cdots da_n.
\]

Notice that

\[
F \ast 1 = \sigma(1+d) 1 = 1
\]

so that

\[
\frac{C}{C(1-F)} \overset{\sim}{\longrightarrow} \Omega^*_A
\]

Thus

\[
C \overset{\sim}{\longrightarrow} \Omega^*_A \otimes C^e
\]

\[
c e \longmapsto e \otimes 1
\]

However this is not the expected isomorphism since

\[
ea e \longmapsto \frac{1 + F a}{2} = \frac{1}{2} (a + \sigma(1+d) a)
= \frac{1}{2} (a + a - da) = a - \frac{1}{2} da
\]
Idea: We are looking at the family \( F_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix} \) with the \( A \)-module structure \( \tilde{\Theta} = (\Theta \ 0) \). Then we form the cocycles \( \tilde{\Theta} \in (\mathbb{F}_2, \tilde{\Theta}^2 \mathcal{E}) \). It might be interesting to have, instead of \( \tilde{\Theta} \) fixed and \( F \) varying, for \( F \) to be fixed and \( \tilde{\Theta} \) to vary. Now we have seen that we can replace \( F_t \) above by

\[
F_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} 1 & t \\ t & -1 \end{pmatrix} \otimes F
\]

where \( \nu_t = \begin{pmatrix} \cos \phi/2 & -\sin \phi/2 \\ \sin \phi/2 & \cos \phi/2 \end{pmatrix} \), \( \tan \phi = t \)

It follows that

\[
[F_t, \tilde{\Theta}] = [\nu_t (0 \ 1 \ 1 \ 0)^{-1} \otimes F, (0 \ 0) \otimes \Theta]
\]

\[
\nu_t^{-1} [F_t, \tilde{\Theta}] \nu_t = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \begin{pmatrix} \cos^2 & -\cos \sin \\ -\sin \cos & \sin^2 \end{pmatrix}
\]

\[
= \begin{pmatrix} \cos^2 & -\cos \sin \\ \sin \cos & -\sin^2 \end{pmatrix} F \Theta - \begin{pmatrix} \cos^2 & \cos \sin \\ -\sin \cos & -\sin^2 \end{pmatrix} \Theta F
\]

\[
= \begin{pmatrix} \cos^2 [F, \Theta] & -\sin \cos [F, \Theta]_+ \\ \sin \cos [F, \Theta]_+ & -\sin^2 [F, \Theta] \end{pmatrix}
\]
In other words we have made things more complicated. 

Thus let's return to the family 

\[ F_t = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix} \]

\[ [F_t, \tilde{\Theta}] = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} [F, \Theta] & -\Theta t \\ t\Theta & 0 \end{pmatrix} \]

Our goal should now be to completely understand the proof of the $S$-equivalence of the different cocycles.

Question: If $t = 0$ then 

\[ \text{tr} \left( F_t [F_t, \tilde{\Theta}]^{2n} \right) = \text{tr} \left( F[F, \Theta]^{2n} \right) \]

and if $t = \infty$, then $F_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

\[ \text{tr} \left( F_\infty [F_\infty, \tilde{\Theta}]^{2n} \right) = \text{tr} \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Theta & 0 \\ 0 & -\Theta \end{pmatrix}^{2n} \right) = 0 \]

Is the "transgression" form equal to the Chern-Simons form in this case?
In studying the deformation $F_\varepsilon = \frac{1}{\sqrt{1+t^2}} (F + \varepsilon)$, it has become clear that it should be useful to work with $F$'s such that $F^2$ is scalar, or more generally $F^2$ commutes with $A$.

Let's prove that $\text{tr} F[F_\varepsilon, \theta]^{2n}$ is a cyclic cocycle assuming only that $[F^2, \theta] = 0$. This implies $0 = [F^2, \theta] = F[F, \theta] + [F, \theta] F$

hence $[F, \theta]$ anti-commutes with $F$. We have

$$[\delta + \theta, F[F, \theta]] = 0. \quad \text{Thus}$$

$$\delta \text{ tr} (F[F, \theta]^{2n}) = \text{tr} \delta [F[F, \theta]^{2n}] = -\text{tr} ([F, \theta]^{2n+1}) = 0$$

because $[F, \theta]$ anti-commutes with $F$.

Alternative elementary proof.

$$\delta \text{ tr} (F[F, \theta]^{2n}) = \sum_{i=0}^{2n-1} (-1)^i \text{tr} \left\{ F[F, \theta]^i [F, \theta]^{2n-i} \right\}$$

$$= \text{tr} (-F\theta [F, \theta]^{2n+1} - (-1)^{2n-1} F[F, \theta]^{2n+1} \theta)$$

$$= -\text{tr} ([F, \theta]^{2n+1}) = 0 \quad \text{Same reason.}$$

Let's check it also works in the graded case.

$$\delta \text{ tr} (\varepsilon F[F, \theta]^{2n+1}) = \text{tr} (\delta + \theta, \varepsilon F[F, \theta]^{2n+1})$$

$$= \text{tr} (\varepsilon [F, \theta]^{2n+2}) = 0 \quad \text{as}$$

$$\varepsilon [F, \theta]^{2n+1} \text{ anti-commutes with } F.$$
Notice that

\[ [F^2 \theta] = F[F, \theta] + [F, \theta] F \]

so we have \( F \) anti-commutes with \([F, \theta]\) if \( F^2 \) commutes with \( \theta \). This means that the hypothesis that \( F^2 \) commutes with \( \theta \) is essential to use the method employed.

I would now like to derive homotopy properties for these cocycles. As a first step let us consider the problem of generalizing the construction of forms associated to a flat bundle with splitting.

Previously we considered a bundle with flat connection \( D \) and splitting \( F \) satisfying \( F^2 = 1 \); the example was \( D = \delta + \delta \) where \( \delta(F) = 0 \). Now we relax the condition \( F^2 = 1 \), but require \([D, F^2] = 0\) and also that \( F \) be invertible. Then we can define

\[ \nabla = \frac{1}{2} (D + F^{-1} DF) \]
\[ \alpha = \frac{1}{2} (D - F^{-1} DF) = -\frac{1}{2} F^{-1} [D, F] \]

so that \( \nabla \) commutes with \( F \) and \( \alpha \) anti-commutes with \( F \). It follows by the usual argument that \( m \)-cohomology

\[ \operatorname{tr}(F^m \otimes \alpha^{2n}) \]

are closed forms. For \( m \) even they are zero because \( \operatorname{tr}(F^m \otimes \alpha^{2n}) = \operatorname{tr}(\alpha F^m \otimes \alpha^{2n-1}) = -\operatorname{tr}(F^m \otimes \alpha^{2n-1}) \).

I think I might as well assume that \( F^2 \) is a scalar, more precisely that \( \operatorname{tr}(F^2 \alpha) = F^2 \alpha \).

Then the form \( \alpha \) with \( m = 1 \) essentially

\[ \operatorname{tr}(F \otimes \alpha) \]
gives the others. There might be some interesting play with the hypothesis that $F$ be invertible.

Next we consider a family of $[F^2_t]$ such that $[D, F^2_t] = 0$ for all $t$. In order to prove homotopy invariance we need to construct a similar setup over our manifold $xR$. Normally we take $\delta_c = dtD_c + D$, but

$$[dtD_c + D, F^2_t] = dt(FF_t + FF_t)$$

Unfortunately, if $F^2$ is a scalar endomorphism, then there is no connection on the bundle relative to which $F^2$ is constant.

Thus we run into a problem with our approach based on connections $D$ and endomorphisms $F$ such that $[D,F^2] = 0$. Actually flatness is not needed; it suffices for $D^2F = 0$.

This implies

$$[D, [D,F]] = D[D,F] + [D,F]D$$

$$= [D^2,F] = 0$$

and so

$$d \quad tr \quad (F^{2k+1}_c [D,F]^{2n}) = tr \quad (D,F^{2k+1}_c [D,F]^{2n+1})$$

$$= tr \quad (F^{2k+1}_c [D,F]^{2n+1}) = 0$$

It's clear from the preceding that one eventually has to find a way to weaken the condition $[F^2_a] = 0$, e.g., the Kasparov condition $a(F^2-1) \in \text{ideal}$. 
Let's return to $F_t = \frac{1}{1+t^2} (F \cdot t - F)$ and review the proof that the cyclic cohomology class of $tr(F[F, \theta]^{2n})$ is independent of $t$. Recall the proof (p 859) with some simplifications.

We have $D$ fixed but $F$ varying with respect to $t$. Thus

$$
\partial_t \tr(F^{2n}) = \tr(F^2 \cdot 2n) + \sum_{i=0}^{2n-1} \tr(F \cdot \alpha^i \cdot \alpha^{2n-1-i})
$$

$$
= 2n \tr(F^2 \cdot \alpha^{2n-1})
$$

The point is that the latter depends only on the part of $\alpha$ commuting with $F$. We have

$$
\alpha = \frac{1}{2} (D - FDF)
$$

$$
\alpha^* = \frac{1}{2} (\hat{F}DF + FDF)
$$

$$
= (-\frac{1}{2}) \left( \frac{\hat{F}DF + FDF}{\left[ \hat{F}, \theta \right]} \right)
$$

with $\alpha$ commuting with $F$ and $\alpha^*$ anti-commuting with $F$.

Thus

$$
\partial_t \tr(F^{2n}) = 2n \tr(F\left[ \nabla, -\frac{1}{2} \hat{F} \theta \right] \alpha^{2n-1})
$$

$$
= d \left[ 2n \tr\left\{ (-\frac{1}{2}) \alpha^{2n-1} \right\} \right]
$$

Let's substitute $\alpha = \frac{1}{2} F[F, \theta]$ and $\alpha^{2n-1} = \frac{(-1)^{n-1}}{2^{2n-1}} F[F, \theta]^{2n-1}$

$$
\alpha^{2n} = \frac{(-1)^n}{2^{2n}} [F, \theta]^{2n}
$$

$$
\partial_t \left( \frac{(-1)^n}{2^{2n}} \tr(F[F, \theta]^{2n}) \right) = d \left[ 2n \tr\left\{ \hat{F} F[F, \theta]^{2n-1} \right\} \right]
$$

$$
\partial_t \left\{ 2 \tr(F[F, \theta]^{2n-1}) \right\} = d \left[ 2n \tr\left\{ \hat{F} F[F, \theta]^{2n-1} \right\} \right]
$$
Actually this can be simplified further because

\[ \text{tr} \left( \tilde{F} \left( \tilde{F} - \Theta \right) \left( \tilde{F}, \tilde{\Theta} \right)^{2n-2} \right) = 2 \text{tr} \left( \tilde{F} \Theta \left( \tilde{F}, \tilde{\Theta} \right)^{2n-2} \right) \]

but we don't use this because the \( \Theta \) does not have a parity with respect to \( F \).

Now apply this formula to \( \tilde{\Theta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) and

\[ F = \begin{pmatrix} \frac{1}{\sqrt{1+t^2}} (F \ t) \\ t - F \end{pmatrix} \]

\[ \left[ \tilde{F}, \tilde{\Theta} \right] = \frac{1}{\sqrt{t \Theta}} \begin{pmatrix} [F, \Theta] - \Theta t \\ t \Theta - 0 \end{pmatrix} \]

Note that \( \text{tr} \left( \tilde{F} \left[ \tilde{F}, \tilde{\Theta} \right]^{2n-1} \right) \) depends only on the part of \( \tilde{F} \) anti-commuting with \( \tilde{F} \). Thus we get

The formula

\[ \partial_t \left[ 2 \text{tr} \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \left[ [F, \Theta] - \Theta t \right]^{2n-1} \right) \left( \frac{1}{(1+t^2)} \right)^{2n-1} \right] = \]

\[ d \ln \text{tr} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \left[ \begin{pmatrix} F \ t \\ t - F \end{pmatrix} \right] \left[ \begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix} \right] \]

Actually it is probably clearer if we use

\[ \tilde{F} \tilde{F} = \frac{1}{2} \left[ \tilde{F}, \tilde{F} \right] = \frac{1}{2} \left[ \partial_t \sqrt{t \ t - F}, \sqrt{t \ t - F} \right] \]

\[ = \frac{1}{2 (1+t^2)} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} F \ t \\ t - F \end{pmatrix} \right] = \frac{1}{(1+t^2)} \begin{pmatrix} 0 & -F \\ F & 0 \end{pmatrix} \]

Now

\[ (1+t^2) \left( \frac{1}{\sqrt{1+t^2}} \right)^{2n-1} = (1+t^2) \left( \partial_t - \frac{2n-1}{2} \frac{2t}{1+t^2} \right) \]

\[ = (1+t^2) \partial_t - (2n-1) t \]

So we obtain the formula.
\[
(1+t^2) \frac{\partial}{\partial t} \Theta + (2n-1)t \left(2 \text{tr} \left\{ \left[ \begin{array}{c, c}
\emptyset & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right\} - 2t \text{tr} \left\{ \left[ \begin{array}{c, c}
0 & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right\} \right)
\]

\[
= d \cdot 2n \cdot \text{tr} \left\{ \left( \begin{array}{c, c}
0 & -\Theta \\
\Theta & 0 \\
\end{array} \right) \left[ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right]^{2n-1} \right\}
\]

Notice that both sides are odd polynomials in \( t \). Let's look at the coefficient of \( t \) on both sides. On the right we have

\[-(2n-1)2 \text{tr} \Theta \left[ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right]^{2n-1} + 4 \sum (-1)^k \text{tr} \left\{ \Theta \text{Sym} \left( \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right) \right\} \]

and on the left we have \( d \cdot 2n \) of coefficient of \( t \) in

\[\text{tr} \left\{ \left( \begin{array}{c, c}
0 & -\Theta \\
\Theta & 0 \\
\end{array} \right) \left[ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right]^{2n-1} \right\} \]

\[\text{tr} \left\{ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right\}^{2n-1} \]

\[\text{tr} \left\{ \Theta \text{Sym} \left( \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right) \right\} \]

2n-1 times

\[= \text{tr} \left( F \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right)^{2n-2} (\Theta) - F \Theta \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta^{2n-2} \]

\[+ O(t^3)\]

\[= (-2t) \text{tr} \left( F \Theta \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right)^{2n-2} \]

so it would seem we have the identity

\[(2n-1) \text{tr} \Theta \left[ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right]^{2n-1} - 2 \text{tr} \left\{ \Theta \text{Sym} \left( \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right) \right\} \]

\[= d \cdot 2n \cdot \text{tr} \left( F \Theta \left[ \left[ \begin{array}{c, c}
\Theta & 0 \\
0 & 0 \\
\end{array} \right] \Theta \right]^{2n-2} \right)\]

Let's now work out the formulas in the graded case:

\[\frac{\partial}{\partial t} \text{tr} (\epsilon \alpha^{2n+1}) = \sum_{i=0}^{2n} \text{tr} (\epsilon \alpha^i \times \alpha^{2n-i}) = (2n+1) \text{tr} (\epsilon \alpha^{2n})\]
\[ \dot{x} = (-\frac{1}{2})(\underbrace{[\varepsilon, F F^2]}_{\text{anti-corr}} - \underbrace{[x, F F^2]}_{\text{corr with } F}) \]

\[ \partial_t \, tr(\varepsilon x^{2n+1}) = (2n+1)(-\frac{1}{2}) \, tr(\varepsilon \, [\varepsilon, F F^2] x^{2n}) \]
\[ = d \, (2n+1)(-\frac{1}{2}) \, tr(\varepsilon F x^{2n}) \]

\[ \partial_t \, tr(\varepsilon F[F, \theta]^{2n+1}) \left(\frac{-1}{2^{2n+1}}\right) = d (2n+1)(-\frac{1}{2}) \, tr(\varepsilon F F[F, \theta]^{2n}) \left(\frac{-1}{2^{2n}}\right) \]
\[ \partial_t \, tr(\varepsilon F[F, \theta]^{2n+1}) = d (2n+1) \, tr(\varepsilon F F[F, \theta]^{2n}) \]

Now apply to \( \tilde{\Theta} = (0 \circ \theta) \), \( F = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} F & t \\ t & -F \end{pmatrix} \)
\( \tilde{\varepsilon} = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \)

\[ \partial_t \, 2 \, tr(\tilde{\varepsilon} \tilde{\Theta} [F, \tilde{\Theta}]^{2n}) = d (2n+1) \, tr(\tilde{\varepsilon} \frac{1}{2} [\tilde{\varepsilon}, [F, \tilde{\Theta}]] [F, \tilde{\Theta}]^{2n}) \]

\[ \partial_t \, \frac{1}{(1+t^2)^n} \, 2 \, tr(\tilde{\varepsilon} [F, \tilde{\Theta}] [F, \tilde{\Theta}]^{2n}) = d (2n+1) \, tr(\tilde{\varepsilon} \frac{1}{2} [F, \tilde{\Theta}] [F, \tilde{\Theta}]^{2n}) \times \left( \begin{pmatrix} [F, \tilde{\Theta}] \theta & -\theta \\ \theta & 0 \end{pmatrix}^{2n} - \varepsilon \begin{pmatrix} 0 & I \\ I & -F \end{pmatrix} \right) \]

So

\[ \left(1+t^2 \right) \partial_t - 2nt \right) \, 2 \, tr\left( \tilde{\varepsilon} [F, \tilde{\Theta}] [F, \tilde{\Theta}]^{2n} \right) \]
\[ = d (2n+1) \, tr\left( \tilde{\varepsilon} \frac{1}{2} [F, \tilde{\Theta}] [F, \tilde{\Theta}]^{2n} \right) \times \left( \begin{pmatrix} [F, \tilde{\Theta}] \theta & -\theta \\ \theta & 0 \end{pmatrix}^{2n} - \varepsilon \begin{pmatrix} 0 & I \\ I & -F \end{pmatrix} \right) \]

Now we look at the coefficients of \( \dot{t} \). On the left:

\[-4n \, tr(\varepsilon F[F, \theta]^{2n}) - 4 \, tr(\varepsilon \Theta \, \text{Sym}( [F, \theta], \theta^2) \]

On the right:

\[ d (2n+1) \] applied to

\[ tr\{ \varepsilon (-F) \theta [F, \theta]^{2n-1} + (-\varepsilon) F [F, \theta] (\theta \Theta) \} \]
\[ = -tr\{ \varepsilon F \Theta [F, \theta]^{2n-1} + \varepsilon \Theta F [F, \theta]^{2n-1} \} \]
so it seems we get the identity

\[ 2^n \text{tr}(\varepsilon \Theta[F, \Theta]^{2n}) - 2n \text{tr} \{ \varepsilon \Theta \text{Sym}([F, \Theta], \Theta)^{2n-2} \} \]

\[ = d (2n+1) \text{tr}(\varepsilon F \Theta [F, \Theta]^{2n-1}) \]

which checks for \( n = 1 \).
Here's a mystery. In practice when we deal with the S-operator, we go from a known cocycle of degree n and produce another cocycle of degree n-2 which turns out to be S of a cocycle of degree n-2. We still don't have an operation on cyclic cocycles, although Connes does this by means of non-commutative differential forms.

Let's review the geometric picture of the S-operation for cocycles on $C^\infty(M)$. We have seen that we obtain a cyclic cocycle from a n-dimensional correspondence from $M$ to a Grassmannian. This means a closed n-dim current $\gamma$ on $M \times Gr(V)$.

To obtain $S$ of this cocycle one multiplies by $S^2$ as follows. One has the current $8 \times [S^2]$ of dim (n+2) on $M \times Gr(V) \times S^2$ and one pulls forward under the Bott map $Gr(V) \times S^2 \to Gr(V(\mathbb{C}))$.

Connes has an operator analogue of this which goes as follows. Suppose the algebra $A$ acts on $H$ preserving a grading $\epsilon$ and that $F$ is an odd involution on $H$. Form

$$\tilde{\epsilon} = \epsilon \otimes \epsilon_c \quad \epsilon_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{\Theta} = \Theta \otimes e \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\tilde{F} = \frac{1}{\sqrt{1 + t^2}} (F \otimes 1 + t \epsilon \otimes F_c) \quad F_c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

I forgot to say that the cyclic cocycle of interest is $\text{tr}(\tilde{\epsilon} \Theta [F, \Theta]^{2n})$; this is formally...
analogous to \( \int tr(\theta \otimes \delta^k) \) if \( \delta \) has dim \( k \). The analogue of the geometric construction of multiplying by the 2-sphere is to consider \( tr(\tilde{\varepsilon} \otimes [F, \tilde{\varepsilon}]^{2n+2}) \).

We have, on writing \( H \otimes \mathbb{C}^2 \cong H \oplus H \),

\[
[F, \tilde{\varepsilon}] = [F \otimes 1 + t \varepsilon \otimes F_e, \theta \otimes e] \frac{1}{\sqrt{1 + t^2}}
\]

\[
= ([F, \theta] \otimes e + t \varepsilon \otimes [F_e, e]) \frac{1}{\sqrt{1 + t^2}}
\]

\[
[F_e, e] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
\therefore [F, \tilde{\varepsilon}] = \frac{1}{\sqrt{1 + t^2}} \begin{bmatrix} [F, \theta] & -t \varepsilon \theta \\ t \varepsilon \theta & 0 \end{bmatrix}
\]

and so

\[
tr(\tilde{\varepsilon} \otimes [F, \tilde{\varepsilon}]^{2n}) = (1 + t^2)^{-n} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [F, \theta] & -t \varepsilon \theta \\ t \varepsilon \theta & 0 \end{bmatrix}^{2n}
\]

\[
= (1 + t^2)^{-n} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} [F, \theta] & -\theta \\ \theta & 0 \end{bmatrix}^{2n}
\]

The reason we can get rid of the \( \varepsilon \) is because we can conjugate by \( (1 0 0 \varepsilon) \). This is exactly the formula we used on p. 921 in the graded case. (See also next page)

Ungraded case: One can view an ungraded \((H, F)\) as a graded pair \((C_1 \otimes H, \varepsilon \otimes F)\). More precisely suppose given a graded \( C_1 \)-module \( H' \) \( H^+ \oplus H^- \) with an odd \( F' \) commuting with \( \varepsilon \) strictly. Then we can identify \( H^+ = H^- \) so that \( \varepsilon \) becomes \( (0 1) \), and then \( F' \) becomes \( (0 F) = \varepsilon \otimes F \).
Actually it looks like in the
graded case the best thing to do
is to form
\[ \tilde{H} = C^2 \otimes H, \quad \tilde{\varepsilon} = \varepsilon_C \otimes \varepsilon, \quad \tilde{\Theta} = \Theta \otimes \Theta. \]
\[ \tilde{F} = \frac{1}{\sqrt{1+t^2}} \left( \varepsilon_C \otimes F + t F_C \otimes I \right) \]

Then
\[ [\tilde{F}, \tilde{\Theta}] = \frac{1}{1+t^2} \left( \varepsilon \otimes [F, \Theta] + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \Theta \right) \]
\[ = \frac{1}{\sqrt{1+t^2}} \begin{pmatrix} [F, \Theta] & -t \Theta \\ t \Theta & 0 \end{pmatrix} \]

and
\[ \text{tr} \varepsilon \tilde{\Theta} [F, \Theta]^{2n} = \frac{1}{(1+t^2)^n} \text{tr} \left( \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \Theta \otimes \Theta \right) \left( [F, \Theta] - t \Theta \right)^{2n} \]

When we do the ungraded case, then?
Consider a vector bundle $E$ over $M$ with a connection $D$. We have various ways to produce cyclic cocycles on $\text{End} E$ using a closed current $M$, for example the fundamental class if $M$ is compact and oriented.

Let's review the construction. We think of cyclic cochains as left-invariant forms on a group of gauge transformations $G = \text{Aut}(E^*)$, and we work with differential forms on $G \times M$. The bundle $\mathfrak{g}(E^*)^*$ over $G \times M$ has the connection the inverse image of $D$ which we denote $S + D$. To this we can add the Maurer-Cartan form $\Theta$. Then we get a connection $S + D + \Theta$ with curvature

$$(S + D + \Theta)^2 = D^2 + [D, \Theta]$$

which is of type $(0,2) + (1,1)$ over $G \times M$, hence is flat in the $G$-direction.

Now we have two ways of producing odd forms. Better, when a connection is flat along the leaves of a foliation, one has the Bott observation that the characteristic forms are of a certain filtration in the DR complex, and that they vanish in certain cases. In the above situation one has two connections $S + D$ and $S + D + \Theta$ which are partially flat in the $G$-direction. Thus powers of the curvature of degree $> \dim M$ are zero. Thus we have two...
reasons for the characteristic classes of \( \hat{P}^*_2(\mathcal{E}^{\oplus n}) \) on \( \mathcal{E} \times M \) of certain degrees to be zero. The difference of these two reasons is an odd form on \( \mathcal{E} \times M \) which is closed.

We have two ways to construct this odd form. The first uses the linear path \((\delta + D + t\Theta)\) between the two connections and leads to the formula

\[
(S + d) \left( \int \frac{dt}{u^{1/2}(D + t^2 \Theta)} \right) = \text{tr} \left( e^{u(D + t\Theta)} - e^{-uD^2} \right)
\]

The second uses the superconnection

\[
\delta + D + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

whose curvature is

\[
-\frac{t^2}{2} + D^2 + \begin{pmatrix} [D, \Theta] & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Theta
\]

\[
= -\frac{t^2}{2} + D^2 + \begin{pmatrix} [D, \Theta] & t\Theta \\ t\Theta & 0 \end{pmatrix}
\]

and leads to the formula
\[
\text{tr} \left( e^{u(D^2 + [D,\Theta])} \right) - \text{tr} \left( e^{uD^2} \right) \\
= (d+1) \int_0^\infty dt \text{ tr} \left\{ u(0,1) e^{-t^2 + D^2 + (d\Theta + [D,\Theta]) \tau + \Theta} \right\}
\]

However, there is another way to obtain odd forms, namely, one can use an embedding \( i : E \to \tilde{V} \) such that \( i^* di = D \) and construct the odd forms in the case of the trivial bundle \( \tilde{V} \) and then restrict from \( \text{Aut}(\tilde{V}) \) to \( G = \text{Aut}(E^\Theta) \). Thus we start with the odd form

\[
\int_0^\infty dt \text{ tr} \left\{ u(0,1) e^{-t^2 + (d\Theta + [D,\Theta]) \tau + \Theta} \right\}
\]

which we know coincides with the super connection versions, p. 873-875

\[
\int_0^\infty dt \text{ tr} \left\{ u(0,1) e^{-t^2 + (d\Theta + [D,\Theta]) \tau + \Theta} \right\}
\]

Then we restrict this to \( i^* \) and use

\[
d(i^* \Theta) = \begin{pmatrix} [D,\Theta] & -\Theta i^* di^* \\ (i^* di)^* & 0 \end{pmatrix}
\]

Now we have seen the effect of \( i^* \) to take \( \text{tr} \Theta(e^\Theta)^n \) into \( \text{tr} \left( [D,\Theta] \Theta - \Theta i^* di^* \right)^n \)

which is

\[
\text{tr} \Theta \left( \Theta \Theta \Theta \text{ of degree } n \text{ in } \Theta \right)
\]

and also in \( D \)
Now we can also do this substituting to a typical component of the Chern--Gaussians form. Let's take
\[ \text{tr } \Theta (\text{words in } d\Theta \text{ and } \Theta^2) \]
of degree \( n-k \), \( \Theta^k \).

Then it is clear that when we substitute \( \Theta \rightarrow d(i\Theta^*) \) the \( d\Theta \) changes to \([D,\Theta]\)
and when we have consecutive \( d\Theta \)'s in a word we are allowed to produce extra terms by \( d\Theta^2 \rightarrow \Theta d\Theta \).

We notice that the sort of cyclic cochains obtained sit in the sector of the double complex \( C^p_q (\mathcal{A}, \mathcal{M}) \) where \( p > q \). If we start with \( \text{tr } \Theta \circ S(d\Theta^{n-k}(\Theta^2)^k) \) we have \( p = 1 + n-k + 2k = 1 + n + k \) and \( q = n-k \), so \( p-q = 1 + 2k > 1 \). Now the extra terms produced by allowing \( d(\Theta^*) \) to become \( \Theta d\Theta \) have the same \( p, q \).

Thus we conclude that the odd form
\[
\int_{0}^{1} dt \text{tr} \{ u \Theta \circ e^{u(d(i\Theta^*))} \}
\]
when pulled back via \( \Theta \rightarrow i\Theta^* \) is probably not equal to either of the odd forms produced on p. 927 or 928. Let's see if this checks for the total differential, i.e. let's compare
\[
\text{tr} (e^{u(D^2 + [D,\Theta])} - e^{uD^2}) \quad \text{and} \quad \text{tr} (e^{i\Theta^* dx} - 1)
\]
since \( d(i\Theta^*) = (\Theta^* dx, -\Theta^* dy) \) the latter
is a polynomial in \( \Theta D^2 \Theta \) and \([D,\Theta]\), whereas
the former is a polynomial in $D^2$ and $[D; \theta]$.

The conclusion perhaps is that the former two schemes based on some method of joining $S+D$ and $S+D+\theta$ are "wrong."

Recall

\[
\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{a-bd^2-c}
\]

Similarly

\[
\begin{pmatrix} 0 & 1 \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{d-ca^{-1}b}
\]

and so

\[
\text{tr} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) = \text{tr} \left( \frac{1}{a-bd^2-c} \right) + \text{tr} \left( \frac{1}{d-ca^{-1}b} \right)
\]