March 27, 1988

We have seen how to make

$$C = A \oplus A \otimes B \otimes A$$

into an algebra given a linear map $\rho : A \to B$ such that $\rho(1) = 1$. There is a (unital) homorphism $A \to C$ and a map $v : B \to C$, given by $v(b) = 1 \otimes b \otimes 1$, which satisfies

$$1) \qquad v(b_1)\, a\, v(b_2) = v(b_1 \rho(a) b_2)$$

This implies $v(1)$ is an idempotent and that v is a non-unital homomorphism.

Suppose that R is an algebra and that we are given a homomorphism $A \to R$ and a map $v : B \to R$ such that 1) is satisfied. Then we have a unique homomorphism

$$C \longrightarrow R$$

$$a + \alpha \otimes \beta \otimes \alpha' \longmapsto a + \alpha v(\beta) \alpha'$$

Check:

$$\begin{array}{ccc} (\alpha_1 \otimes \beta_1 \otimes \alpha_1') \cdot (\alpha_2 \otimes \beta_2 \otimes \alpha_2') & & (\alpha_1 v(\beta_1) \alpha_1')(\alpha_2 v(\beta_2) \alpha_2') \\ \| & & \| \\ \alpha_1 \otimes \beta_1 \rho(\alpha_1' \alpha_2) \beta_2 \otimes \alpha_2' & \longmapsto & \alpha_1 v(\beta_1 \rho(\alpha_1' \alpha_2) \beta_2) \alpha_2' \end{array}$$

Next let's examine 1) more closely. It implies

$$2) \qquad \begin{cases} v(b_1) v(b_2) = v(b_1 b_2) \\ v(1)\, a\, v(1) = v(\rho(a)) \end{cases}$$

and conversely if 2) hold, then

$$\begin{aligned} v(b_1)\, a\, v(b_2) &= v(b_1) v(1)\, a\, v(1) v(b_2) \\ &= v(b_1) v(\rho(a)) v(b_2) = v(b_1 \rho(a) b_2). \end{aligned}$$

So we can conclude that the algebra $C = A \oplus A \otimes B \otimes A$ is universal among algebras R equipped with $A \to R$ a homomorphism and $v : B \to R$ satisfying 2).

Now let us take B to be the universal algebra equipped with a linear map $\rho : A \to B$ such that $\rho(1) = 1$. Thus

$$B = T(A) / T(A)(1 - 1_A)T(A)$$
$$\cong T(L)$$

where L is a subspace of A complementary to $\mathbb{C}$. (Note that one can be slightly more intrinsic by saying that B has an increasing filtration $F_p B = \rho(A)^p$ and that $gr(B) = T(\bar{A})$.)

Let us now consider an algebra R equipped with a homom. $A \longrightarrow R$ and consider those maps $v : B \longrightarrow R$ satisfying 2). I claim that v is completely determined by the idempotent $v(1)$ in R and that there is a 1-1 correspondence between the maps v satisfying 2) and idempotents in R. In effect given e in R, the map $a \overset{\rho'}{\longmapsto} eae$ is linear from A to the unital algebra eRe and satisfies $\rho'(1) = e$, hence there is a unique homomorphism $B \xrightarrow{v} eRe$ carrying ρ to ρ'. Then $v : B \to R$ satisfies $v(b_1 b_2) = v(b_1)v(b_2)$ and $v(1)\,a\,v(1) = eae = \rho'(a) = v(\rho(a))$.

Thus we see that when $\rho : A \to B$ is universal the algebra $C = A \oplus A \otimes B \otimes A$ is just the free product $A * \mathbb{C}[\mathbb{Z}/2] = (A * A) \rtimes (\mathbb{Z}/2)$.

What this means is that all of the calculations I have been doing in the algebra C for a general $\rho : A \to B$ are just the images of calculations in the free product $A * \mathbb{C}[\mathbb{Z}/2]$. This free product has been studied by Connes + Kuntz.

March 28, 1988

I would like next to use what I have learned about the Stinespring game to construct the odd cyclic cocycles attached to an extension of A. Thus I start with an extension and choose a lifting

$$0 \to I \longrightarrow R \underset{\rho}{\overset{\leftarrow}{\longrightarrow}} A \longrightarrow 0$$

and I want a cyclic $(2n-1)$-cocycle on A with values in $I^n/[I, I^{n-1}]$.

By naturality I can suppose R is the universal algebra $B = T(A)/(1-1_A)$ and that I is the ideal generated by $\{\rho(a_1)\rho(a_2) - \rho(a_1a_2)\}$, (better: the kernel of $B \to A$). I want to do the calculations in the Steinspring algebra

$$C = A \oplus A \otimes B \otimes A = A * \mathbb{C}[\mathbb{Z}/2]$$

I know that if K is an ideal in C containing $[F, a]$ for all $a \in A$, and if τ is a linear functional defined on $K^{2n}/[K, K^{2n-1}]$, then

$$\varphi(a_0, \dots, a_{2n-1}) = \tau\left(F[F, a_0] \cdots [F, a_{2n-1}]\right)$$

is a cyclic cocycle.

I should go over this and see exactly what is needed, i.e. what are the minimal assumptions on

K, τ? First of all we need K to be a bimodule over A and $a \mapsto [F, a]$ to be a derivation of A with values in K.

Tate structure: Let I_1, I_2 be ideals in R such that $R = I_1 + I_2$, and let

$$\tau : (I_1 \cap I_2)/[I_1, I_2] \longrightarrow \mathbb{C}$$

be a linear functional. Then we can define an element of $HC^1(R)$ as follows.

We have the bimodule extension

$$1) \quad 0 \longrightarrow I_1 \cap I_2 \longrightarrow I_1 \oplus I_2 \longrightarrow R \longrightarrow 0$$

and the class of this extension is an element of

$$H^1(R, I_1 \cap I_2).$$

If τ is a linear map $I_1 \cap I_2/[R, I_1 \cap I_2] \longrightarrow \mathbb{C}$, then we would have a bimodule map

$$I_1 \cap I_2 \longrightarrow R^*$$

and hence an element in

$$H^1(R, R^*) = (H_1(R, R))^*$$

i.e. a Hochschild cohomology class. In dimension 1 we have from the Connes exact sequence

$$0 \longrightarrow HC^1(R) \longrightarrow H^1(R, R^*) \longrightarrow HC^0(R) \xrightarrow{S} \cdots$$

so the real point is to see why the extra condition that $\tau([I_1, I_2]) = 0$ implies the Hochschild class is cyclic.

At this point we compute: Write $1 = \alpha + (1 - \alpha)$ where $\alpha \in I_1$ and $1 - \alpha \in I_2$. Then the 1-cocycle describing the extension 1) is the derivation

$$r \longmapsto [r, (\alpha, 1-\alpha)] = ([r,\alpha], -[r,\alpha])$$

or $\quad r \longmapsto [r,\alpha]$ with values in $I_1 \cap I_2$.

The corresponding Hochschild 1-cocycle is

$$\varphi(r_0, r_1) = \tau(r_0[r_1,\alpha])$$

Changing α by $\beta \in I_1 \cap I_2$ changes φ by

$$\tau(r_0[r_1,\beta]) = \tau(r_0 r_1 \beta - r_0 \beta r_1)$$
$$= \tau([r_0, r_1]\beta)$$

which is $b\lambda$ where $\lambda(r) = \tau(r\beta)$.

Note that cyclic and Hochschild 1-coboundaries coincide. The issue now is why φ is cyclic, i.e. why

$$\varphi(1, r) = \tau([r,\alpha]) = 0.$$

Actually this follows without the assumption $\tau[I_1, I_2] = 0$ see 667

Note that this is a trace on R:

$$0 = (b\varphi)(1, r_1, r_2) = \varphi(r_1, r_2) - \varphi(1, r_1 r_2) + \varphi(r_2, r_1)$$

so $\varphi(1, r_1 r_2)$ is symmetric in r_1, r_2.

So finally if we know $\tau([I_1, I_2]) = 0$, then we have

$$\tau([r,\alpha]) = \tau\Big(\underbrace{[r\alpha, \alpha]}_{\substack{[r\alpha,\ \alpha-1] \\ \cap\quad\ \cap \\ I_1\quad I_2}} + \underset{\substack{\cap \\ I_2}}{[r(1-\alpha)},\ \underset{\substack{\cap \\ I_1}}{\alpha]}\Big) = 0.$$

Remark: From $I_1 + I_2 = R$ we can deduce

$$I_2 \cdot I_1 + I_1 \cdot I_2 = I_1 \cap I_2 .$$

In effect, let $1 = \alpha + (1-\alpha)$ as above. If $r \in I_1 \cap I_2$, then

$$r = \underset{I_2 \cdot I_1}{\underset{\cap\ \ \cap}{r\alpha}} + \underset{I_1 \cdot I_2}{\underset{\cap\ \ \cap}{r(1-\alpha)}}$$

This suggests the following. Note that

$$[r,\alpha] = r\alpha - \alpha r = (1-\alpha)r\alpha - \alpha r(1-\alpha)$$

so we can define a map

$$R \longrightarrow I_2 \otimes_R I_1 \oplus I_1 \otimes_R I_2$$

$$r \longmapsto \big((1-\alpha)\otimes r\alpha,\ -\alpha r \otimes (1-\alpha)\big)$$

Unfortunately I can't see if this is a derivation.

So we see how Tate's structure leads to a cyclic 1-dim'l class. I'd like to generalize this construction so as to obtain a Hochschild type construction for cyclic classes.

March 29, 1988

Let $R = I_1 + I_2$, where I_1, I_2 are ideals, and suppose e is an idempotent with $e \in I_1$, $1-e \in I_2$. We learned yesterday that

$$I_1 \cap I_2 = I_1 I_2 + I_2 I_1$$

because if $x \in I_1 \cap I_2$, then

$$x = xe + x(1-e) \in I_2 I_1 + I_1 I_2$$

This means that

$$I_1 \cap I_2 / [R, I_1 \cap I_2] \quad \text{and} \quad I_1 \cap I_2 / [I_1, I_2]$$

are quotients of $I_1 \otimes_R I_2 \oplus I_2 \otimes_R I_1$. So the question arises as to whether we can refine the maps

$$H_1(R,R) \longrightarrow (I_1 \cap I_2) \otimes_R \qquad HC_1(R) \longrightarrow I_1 \cap I_2 / [I_1, I_2]$$

discussed yesterday in the same way that the Connes homomorphism $HC_{2n-1}(A) \to I^n/[I, I^{n-1}]$ is refined by a map $HC_{2n-1}(A) \to (I \otimes_R)^n_\sigma$.

Consider the derivation with values in $I_1 \cap I_2$:

$$x \longmapsto [x, e] = (1-e)xe - ex(1-e)$$

We can lift this map into $I_1 \otimes_R I_2 \oplus I_2 \otimes_R I_1$ by

$$x \longmapsto ((1-e) \otimes xe, \; -ex \otimes (1-e))$$

Set $f(x) = (1-e) \otimes xe = (1-e)x \otimes e$

Then $f(x) = f((1-e)xe)$. This shows we have nice liftings

$$(1-e)Re \longrightarrow I_2 \otimes_R I_1, \qquad eR(1-e) \longrightarrow I_1 \otimes_R I_2$$

Is f a derivation?

$$f(xy) = (1-e)\otimes xye = (1-e)\otimes[xy,e]$$
$$= (1-e)\otimes x[y,e] + \underbrace{(1-e)\otimes[x,e]y}_{\overset{\parallel}{(1-e)x\otimes ey}}$$

$$xf(y)+f(x)y = x(1-e)\otimes[y,e] + (1-e)\otimes xey$$

$$\therefore\quad f(xy) = xf(y)+f(x)y + [x,e]\otimes[y,e]$$

similarly set $f'(x) = e\otimes x(1-e)$. Then

$$f'(xy) = exy\otimes(1-e) = [e,xy]\otimes(1-e)$$
$$= [e,x]\otimes y(1-e) + \underbrace{x[e,y]\otimes(1-e)}_{xey\otimes(1-e)}$$

$$xf'(y)+f'(x)y = xe\otimes y(1-e) + [e,x]\otimes(1-e)y$$

$$\therefore\quad f'(xy) = xf'(y)+f'(x)y + [x,e]\otimes[y,e]$$

It appears that I missed something yesterday, namely, assume τ is a linear fnl. on $(I_1\cap I_2)\otimes_R$, then

$$\tau([x,e]) = \tau((1-e)xe - ex(1-e))$$
$$= \tau(\underbrace{(1-e)xe}_{\in I_2I_1}\,e) - \tau(e\,\underbrace{ex(1-e)}_{\in I_1I_2})$$
$$= \tau(e(1-e)xe) - \tau(ex(1-e)e) = 0$$

[redacted]

This works even if e is replaced by any $\alpha \ni \alpha\in I_1$, $1-\alpha\in I_2$. (See below) p670

More generally let [redacted] us consider elements

$\xi = xey \in I_1$, $\eta = y'(1-e)x' \in I_2$. Then modulo $[R, I_1 \cap I_2]$ we have

$$\xi\eta = xeyy'(1-e)x' = x\underbrace{[e,yy']}_{\in I_1 \cap I_2}(1-e)x'$$

$$\equiv (1-e)x'x[e,yy']$$

$$= (1-e)x'x\,eyy' + (1-e)x'xyy'ee$$

$$\equiv y'(1-e)x'xey + e(1-e)\cancel{x'xyy'}e = \eta\xi$$

This shows in the case $I_1 = ReR$, $I_2 = R(1-e)R$ that $[I_1, I_2] \subset [R, I_1 \cap I_2]$.

Suppose we now try to define a cyclic cocycle with values in $I_2 \otimes_R I_1 \otimes_R$ by

$$\varphi(r_0, r_1) = r_0(1-e) \otimes r_1 e - r_1(1-e) \otimes r_0 e.$$

$$= r_0 f(r_1) - r_1 f(r_0).$$

$$\left.\begin{array}{l} \varphi(r_0r_1, r_2) \\ +\varphi(r_1r_2, r_0) \\ +\varphi(r_2r_0, r_1) \end{array}\right\} = \left\{\begin{array}{l} r_0r_1f(r_2) - r_2\underbrace{f(r_0r_1)}_{r_0f(r_1)+f(r_0)r_1+[r_0,e]\otimes[r_1,e]} \\ \\ +\text{cyc.} \end{array}\right.$$

$$(b\varphi)(r_0,r_1,r_2) = r_0r_1f(r_2) - \qquad ?$$

Let us start again with an algebra R and an idempotent in it e. Let $I_1 = ReR$ and $I_2 = R(1-e)R$. Then

$$I_1 = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)ReR(1-e) \end{pmatrix}, \quad I_2 = \begin{pmatrix} eR(1-e)Re & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$$

$$I_1 I_2 = \begin{pmatrix} eR(1-e)Re & eR(1-e) \\ (1-e)ReR(1-e)Re & (1-e)ReR(1-e) \end{pmatrix}$$

$$I_2 I_1 = \begin{pmatrix} eR(1-e)Re & eR(1-e)ReR(1-e) \\ (1-e)Re & (1-e)ReR(1-e) \end{pmatrix}$$

This formulas follow taking the block decomposition and using the four identities: $ReRe = Re$, $eReR = eR$ and similarly for $1-e$. Thus $(1-e)I_1 e = (1-e)ReRe$ $= (1-e)Re$.

The mystery to be solved is the following. Given ideals I_1, I_2 in R such that $I_1 + I_2 = R$, the bimodule extension

$$0 \longrightarrow I_1 \cap I_2 \longrightarrow I_1 \oplus I_2 \longrightarrow R \longrightarrow 0$$

determines a Hochschild class:

$$H_1(R,R) \longrightarrow (I_1 \cap I_2) \otimes_R$$

which turns out to be cyclic, i.e. to factor through $HC_1(R)$. This is false for a general bimodule extension, e.g. if we take

$$0 \longrightarrow \Omega^1_R \longrightarrow R \otimes R \longrightarrow R \longrightarrow 0$$

then we get the map

$$H_1(R,R) \hookrightarrow \Omega^1_R \otimes_R .$$

The composition with $HC_0(R) \longrightarrow H_1(R,R)$ is the map $\delta: HC_0(R) \longrightarrow \Omega^1_R \otimes_R$ which is usually non-zero.

So the problem is to learn how this ideal structure leads to a cyclic cocycle. Actually I should first check carefully in the general case that if $\alpha \in I_1$, $1-\alpha \in I_2$, then $[r,\alpha] \in [R, I_1 \cap I_2]$. But

$$
\begin{aligned}
[r,\alpha] &= (1-\alpha) r \alpha - \alpha r (1-\alpha) \\
&= \overbrace{(1-\alpha) r \alpha}\alpha + (1-\alpha) r \overbrace{\alpha(1-\alpha)} - \alpha\overbrace{\alpha r (1-\alpha)} - \overbrace{(1-\alpha)\alpha} r (1-\alpha) \\
&\equiv \overbrace{\alpha(1-\alpha) r}\alpha + \cancel{\alpha(1-\alpha)^2 r} - \alpha r \overbrace{(1-\alpha)\alpha} - \cancel{(1-\alpha)^2 \alpha r} \\
&\equiv \cancel{\alpha^2(1-\alpha) r} - \cancel{(1-\alpha)\alpha^2 r} \equiv 0.
\end{aligned}
$$

So our problem is to work out a bimodule style construction of the cyclic cocycle. Let's try the formula

$$\varphi(r_0, r_1) = \tau\left(F[F,r_0][F,r_1]\right)\frac{1}{4}$$

We have to understand where the action is taking place. Let's figure out what we need to get a cyclic cocycle. First of all, we look at

$$\tfrac{1}{2}[F, r_0] = [e, r_0] = e r_0 (1-e) - (1-e) r_0 e$$

This is the sum of the two off-diagonal parts. A first idea would be to work with the off-diagonal blocks $eR(1-e)$ and $(1-e)Re$ as bimodules for eRe and $(1-e)R(1-e)$.

This raises the question as to what extent R is Morita equivalent to eRe or maybe $eRe \oplus (1-e)R(1-e)$. It is not usually Morita equivalent to the direct sum, e.g. take $\hat{R} = M_2(\mathbb{C})$, but I have the feeling of being in the presence of a partition of unity.

Let's now try to find the best target space for the 1-cocycle

$$\varphi(x,y) = \mathrm{tr}\left(F[\tfrac{1}{2}F,x][\tfrac{1}{2}F,y]\right)$$

$$\mathrm{tr}\, F[e,x][e,y] = \mathrm{tr}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & ex(1-e)\\ -(1-e)xe & 0\end{pmatrix}\begin{pmatrix}0 & ey(1-e)\\ -(1-e)ye & 0\end{pmatrix}$$

$$= (1-e)xey(1-e) - ex(1-e)ye$$

To simplify the writing put $\bar{e} = 1-e$. Let's try first putting

$$\varphi(x,y) = \left(\bar{e}xe \otimes ey\bar{e}\otimes \,,\; -ex\bar{e}\otimes \bar{e}ye\otimes\right)$$

$$\in \bar{e}Re \otimes_{eRe} eR\bar{e} \otimes_{\bar{e}R\bar{e}} \oplus\; eR\bar{e}\otimes_{\bar{e}R\bar{e}} \bar{e}Re \otimes_{eRe}$$

We want $\varphi(x,y) = -\varphi(y,x)$; this means we want to add the two factors after interchanging the factors of the second summand. Thus we try

$$\varphi(x,y) = \left(\bar{e}xe\otimes ey\bar{e}\otimes\right) - \left(\bar{e}ye\otimes ex\bar{e}\otimes\right)$$

$$\in \bar{e}Re \otimes_{eRe} eR\bar{e}\otimes_{\bar{e}R\bar{e}} .$$

Let's see if this is a cyclic cocycle

$$\varphi(xy,z) = \left(\bar{e}xye\otimes ez\bar{e}\otimes\right) - \left(\bar{e}ze\otimes exy\bar{e}\otimes\right)$$

$$= [\bar{e}x\underset{(1)}{e}ye + \bar{e}x\underset{(5)}{\bar{e}}ye]\otimes ez\bar{e}\otimes - \bar{e}ze\otimes[ex\underset{(4)}{e}y\bar{e} + ex\underset{(6)}{\bar{e}}y\bar{e}]\otimes$$

$$-\varphi(x,yz) = -\bar{e}xe\otimes eyz\bar{e}\otimes + \bar{e}yze\otimes ex\bar{e}\otimes$$

$$= -\bar{e}xe\otimes[ey\underset{(1)}{e}z\bar{e} + ey\underset{(3)}{\bar{e}}z\bar{e}]\otimes + [\bar{e}y\underset{(2)}{e}ze + \bar{e}y\underset{(6)}{\bar{e}}ze]\otimes ex\bar{e}\otimes$$

$$\varphi(zx,y) = \bar{e}zxe\otimes ey\bar{e}\otimes - \bar{e}ye\otimes ezx\bar{e}\otimes$$

$$= [\bar{e}z\underset{(4)}{e}xe + \bar{e}z\underset{(3)}{\bar{e}}xe]\otimes ey\bar{e}\otimes - \bar{e}ye\otimes[ez\underset{(2)}{e}x\bar{e} + ez\underset{(5)}{\bar{e}}x\bar{e}]\otimes$$

Thus it works and we have a cyclic cocycle on R with values in $\boxed{\bar{e}Re\otimes_{eRe}\otimes eR\bar{e}\otimes_{\bar{e}R\bar{e}}}$

March 30, 1988

Yesterday I considered an algebra R with idempotent e and I constructed a cyclic 1-cocycle with values in

$$1) \qquad (1-e)Re \otimes_{eRe} eR(1-e) \otimes_{(1-e)R(1-e)}$$

We can rewrite this space as follows. First note that eR is a projective R^0-module and that for any R-module M we have

$$eR \otimes_R M \xrightarrow{\sim} eM$$

and similarly for $(1-e)R$. Thus we have

$$eR \otimes_R R(1-e) = eR(1-e)$$

$$(1-e)R \otimes_R Re = (1-e)Re$$

whence 1) becomes isomorphic to

$$2) \qquad \left(Re \otimes_{eRe} eR\right) \otimes_R \left(R(1-e) \otimes_{(1-e)R(1-e)} (1-e)R\right) \otimes_R$$

We have surjections

$$3) \qquad Re \otimes_{eRe} eR \longrightarrow ReR = I_1$$

$$R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R \longrightarrow R\bar{e}R = I_2$$

so 2) maps onto $I_1 \otimes_R I_2 \otimes_R$. Thus we obtain a cyclic 1-cocycle with values in $I_1 \otimes_R I_2 \otimes_R$

Next let's examine the surjection 3) which in block form appears

$$\begin{pmatrix} eRe \\ (1-e)Re \end{pmatrix} \otimes_{eRe} \begin{pmatrix} eRe & eR(1-e) \end{pmatrix} \longrightarrow \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)ReR(1-e) \end{pmatrix}$$

Thus we see that the kernel of 3) is the same as the kernel of

$$(1-e)Re \otimes_{eRe} eR(1-e) \longrightarrow (1-e)ReR(1-e)$$

Recall that we encountered this map before in the study of the algebra $C = A + A\iota B\iota^* A$, see pp. 653-659. There we wanted to extend a trace defined on the ideal $I = eR(1-e)Re = eI_2e$ in eRe, which vanishes on $[eRe, I]$, to a trace on $K = I_1 \cap I_2$ vanishing on $[I_1, I_2]$.

Conclude: The good things to work with are the bimodules

$$\tilde{I}_1 = Re \otimes_{eRe} eR$$

$$\tilde{I}_2 = R(1-e) \otimes_{(1-e)R(1-e)} (1-e)R$$

instead of the ideals $I_1 = ReR$, $I_2 = R(1-e)R$. The standard cyclic 1-cocycle has values in the space

$$\tilde{I}_1 \otimes_R \tilde{I}_2 \otimes_R$$

Questions: $\tilde{I}_j = I_j$ in the universal case $R = A * \mathbb{C}[\mathbb{Z}/2]$? Higher cocycles? Morita aspects of these bimodules?

Let's consider the Morita aspects, especially the links between the categories of modules over the rings R, eRe, and $(1-e)R(1-e)$ defined by the various bimodules. We have functors

$$\begin{tikzcd}
\mathrm{Mod}(eRe) \arrow[r, shift left, "i = \text{“}Re\text{”}"] \arrow[rr, bend left, "j^* i = (1-e)Re"] & \mathrm{Mod}(R) \arrow[l, shift left, "i^* = \text{“}eR\text{”}"] \arrow[r, shift left, "j^* = \text{“}\bar{e}R\text{”}"] & \mathrm{Mod}(\bar{e}R\bar{e}) \arrow[l, shift left, "j = \text{“}R\bar{e}\text{”}"] \arrow[ll, bend left, "i^* j = eR(1-e)"]
\end{tikzcd}$$

We have

$$i^* i = id \qquad j^* j = id$$

$$i i^* = (Re \otimes_{eRe} eR) \otimes_R ? = \tilde{I}_1 \otimes_R ?$$

$$j j^* = (R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R) \otimes_R ? = \tilde{I}_2 \otimes_R ?$$

The value group for the cyclic 1-cocycle is the fixpoint space for [redacted] any of the functors

$$\iota^* j j^* \iota, \quad j j^* \iota \iota^*, \quad j^* \iota \iota^* j, \quad j^* j \iota \iota^*$$

<u>Observation</u>: $\tilde{I}_1 = Re \otimes_{eRe} eR$ is an R-bimodule with a bimodule map

$$Re \otimes_{eRe} eR \xrightarrow{\partial} R$$

[redacted] I claim this forms a DG algebra, i.e.

$$\partial(\xi)\eta = \xi\partial(\eta)$$

Check: Let $\xi = r_1 e \otimes e r_2$, $\eta = r_3 e \otimes e r_4$.

Then

$$\partial(\xi)\eta = r_1 e r_2 (r_3 e \otimes e r_4) = r_1 e (e r_2 r_3 e) \otimes e r_4$$

$$\xi\partial(\eta) = (r_1 e \otimes e r_2) \cdot r_3 e r_4 = r_1 e \otimes (e r_2 r_3 e) e r_4$$

and these are equal in [redacted] $\tilde{I}_1$.

Now we can form the amalgamated free product

$$(\tilde{I}_1 \longrightarrow R) \underset{R}{*} (\tilde{I}_2 \longrightarrow R)$$

of DGA's. This gives a DGA which in degree n will be

$$\underbrace{\left(\tilde{I}_1 \otimes_R \tilde{I}_2 \otimes_R \tilde{I}_1 \otimes_R \cdots\cdots\right)}_{n\text{-factors}} \oplus \underbrace{\left(\tilde{I}_2 \otimes_R \tilde{I}_1 \otimes_R \tilde{I}_2 \otimes_R \cdots\right)}_{n\text{-factors.}}$$

Recall that $(\tilde{I}_1)^2 = 0$ in the DGA $(\tilde{I}_1 \to R)$. This implies ~~[illegible]~~ that for $\xi_1, \xi_3, \ldots \in \tilde{I}_1$ and $\xi_2, \xi_4, \ldots \in \tilde{I}_2$

$$\partial(\xi_1, \xi_2, \ldots, \xi_n) = (\partial(\xi_1)\xi_2, \ldots, \xi_n)$$
$$+ (-1)^{n-1}(\xi_1, \ldots, \xi_{n-1}, \partial(\xi_n))$$

By our assumption that $I_1 + I_2 = R$ it follows that H_0 of **this** amalgamated product is zero, and hence the DGA has zero homology. So it gives a bimodule resolution of R:

$$\cdots \longrightarrow \tilde{I}_1 \otimes_R \tilde{I}_2 \oplus \tilde{I}_2 \otimes_R \tilde{I}_1 \longrightarrow \tilde{I}_1 \oplus \tilde{I}_2 \longrightarrow R \longrightarrow 0$$

Suppose I is an ideal in R such that $I^2 = I$, for example $I = ReR$ where e is an idempotent. Does there exist a universal R-bimodule extension

$$0 \to M \longrightarrow \tilde{I} \xrightarrow{u} I \longrightarrow 0$$

such that $u(x)y = x\,u(y)$ for all $x, y \in \tilde{I}$?

Following Grothendieck we might ask ~~[illegible]~~ whether such extensions form a cofibred category. Notice that

$$I \cdot M = u(\tilde{I})M = \tilde{I} \cdot u(M) = 0$$
$$M \cdot I = M\,u(\tilde{I}) = u(M)\,\tilde{I} = 0$$

so that M is an R/I-bimodule.

Next consider a pushout situation

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r, "u"] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r] & E' \arrow[r, "u'"] & I \arrow[r] & 0 \end{tikzcd}$$

Let $x', y' \in E'$. Then $x' = x + m_1'$, $y' = y + m_2'$ for $m_i' \in M'$ and $x, y \in E$.

$$u'(x')y' = u(x)(y + m_2') = u(x)y \qquad \text{as } Im_2' = 0$$
$$ \ \ \|$$
$$x'u'(y') = (x + m_1')u(y) = xu(y) \qquad \text{as } m_1' I = 0$$

Thus E' is in the same class of extensions. Next note that two maps $E \rightrightarrows E'$ inducing the same map $M \to M'$ differ by an element of

$$\mathrm{Hom}_{R \otimes R^o}(I, M') = \mathrm{Hom}_{R/I \otimes R/I^o}(I/I^2, M')$$

As $I/I^2 = 0$, there is at most one map of extensions consistent with a given R/I-bimodule map $M \to M'$.

Next we analyze an extension of the above type by choosing a linear section s of u:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \arrow[r, shift right, "u"'] & I \arrow[l, shift right, "s"'] \arrow[r] & 0. \end{tikzcd}$$

We then get a bilinear map $I \times I \to M$ given by

$$f(x,y) = xs(y) - s(xy)$$

Note
$$xs(y) = u(s(x))s(y) = s(x)u(s(y)) = s(x)y.$$

For $r \in R$ we have

$$f(xr, y) = xrs(y) - s(xry)$$
$$f(x, ry) = xs(ry) - s(xry)$$

$$f(xr, y) - f(x, ry) = \underset{\substack{\uparrow \\ I}}{x}\underbrace{[rs(y) - s(ry)]}_{M} = 0$$

Better:

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Ker}\, b') \arrow[r] \arrow[d, dashed] & I \otimes_R I \arrow[r, "b'"] \arrow[dl, "f"'] \arrow[d, "\varphi"] & I \arrow[r] \arrow[dl, "s"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r, "i"'] & E \arrow[r, "u"'] & I \arrow[r] & 0
\end{tikzcd}$$

where $\varphi(x,y) = x s(y)$. $\therefore \varphi = if + sb'$

In other words define φ by this formula. Then

$$\varphi(xr, y) = xrs(y)$$
$$\varphi(x, ry) = xs(ry)$$

and these are equal as $x \in I$ and $rs(y) - s(ry) \in M$. Also $\varphi(x,y) = xs(y)$ shows $\varphi(rx,y) = r\varphi(x,y)$, whereas $\varphi(x,y) = s(x)y$ shows $\varphi(x, yr) = \varphi(x,y)r$. Thus φ is a bimodule map.

Let's check that the extension defined by $I \otimes_R I$ is in our ~~class~~.

$$(x \otimes y) \cdot b'(z \otimes w) = x \otimes yzw$$
$$b'(x \otimes y) \cdot (z \otimes w) = xyz \otimes w$$

these are $=$ in $I \otimes_R I$.

Next note that changing s by $h : I \to M$ changes $\varphi(x,y)$ by $xh(y) - h(xy) = -h(xy)$, so the map $\mathrm{Ker}\, b' \longrightarrow M$ induced by φ is independent of the choice of s.

Actually you seem to be proceeding ~~stupidly~~. The point is given $E \overset{u}{\twoheadrightarrow} I$ with $u(x)y = xu(y)$ for all $x, y \in E$, one can define a unique bimodule map $I \otimes_R I \longrightarrow I$ over I, namely

$$\begin{tikzcd}
I \otimes_R I \arrow[rr] \arrow[dr, "\varphi"'] & & I \\
& E \arrow[ur] &
\end{tikzcd}$$

set
$$\varphi(x \otimes y) = x\tilde{y} \quad \text{or} \quad \tilde{x}y$$
where $u(\tilde{y}) = y$ or $u(\tilde{x}) = x$. Uniqueness follows because two choices different by a bimodule map $I \otimes_R I \longrightarrow M$ which must be zero as $I^2 = I$ and $IM = 0$.

<u>Conclusion</u>: If I is an ideal in R such that $I^2 = I$, then
$$I \otimes_R I \xrightarrow{b'} I$$
is the universal bimodule extension E of I such that the map $E \xrightarrow{u} I$ satisfies $u(x)y = xu(y)$.

Let's examine Wodzicki's homologically unital condition in a slightly more general situation. Let I be an ideal in R; Wodzicki takes $R = I^+ = \mathbb{C} \oplus I$. Then his H.U. condition has the analogue

1) $$\mathrm{Tor}_+^R(R/I, R/I) = 0$$

since
$$\mathrm{Tor}_1^R(R/I, R/I) = I/I^2$$
$$\mathrm{Tor}_2^R(R/I, R/I) = \mathrm{Ker}\{I \otimes_R I \longrightarrow I^2\}$$
$$\mathrm{Tor}_{q+2}^R(R/I, R/I) = \mathrm{Tor}_q^R(I, I) \qquad q > 0$$
we see 1) is equivalent to

1)' $\begin{cases} I \otimes_R I \xrightarrow{\sim} I \\ \mathrm{Tor}_+^R(I, I) = 0. \end{cases}$ or 1)'' $I \overset{\mathbb{L}}{\otimes}_R I \xrightarrow{\text{quis}} I$

If this holds, then
$$I \overset{\mathbb{L}}{\otimes}_R \cdots \overset{n}{\cdots} \overset{\mathbb{L}}{\otimes}_R I \xrightarrow{\sim} I \qquad \text{quis}$$
$$\left(I \overset{\mathbb{L}}{\otimes}_R\right)^n \longrightarrow I \overset{\mathbb{L}}{\otimes}_R$$

One can speculate then about the spectral sequences associated to the extension. Assuming the cyclic group actions to be trivial, we find the spectral sequences to have the Hochschild $H_*(R,I)$ or $H_*(R,R/I)$ in the columns. This suggests there should be a Connes exact sequense related $HC_*(R,I)$ to $H_*(R,I)$. This would be reasenable provided

$$H_*(R,R/I) = H_*(R/I, R/I)$$

which I <u>ought to be able</u> to check.

Multiplier algebra. Let I be a non-unital algebra. We want to embed I as an ideal in a larger algebra call it R. Then each r will determine a left multiplication λ_r on I and a right multiplication ρ_r. λ_r commutes with ρ_x for $x \in I$ and ρ_r commutes with λ_x for $x \in I$. Thus we have a homomorphism

$$R \longrightarrow \underbrace{\operatorname{End}_{I^o}(I) \times \operatorname{End}_I(I)^o}_{} \qquad r \mapsto (\lambda_r, \rho_r)$$

The first candidate is to take $M(I) = \wedge$, that is, to consist of pairs (u,v) of operators on I satisfying

$$1) \quad \boxed{\begin{aligned} u(xy) &= u(x)y \\ v(xy) &= x\,v(y) \end{aligned}} \qquad x,y \in I$$

Now the condition that u,v should commute turns out to be too strong a requirement as we saw when $I^2=0$. However if $I^2 = I$, then any $u \in \operatorname{End}_{I^o}(I)$ commutes with any $v \in \operatorname{End}_I(I)$ since

$$u(v(xy)) = u(x\,v(y)) = u(x)v(y)$$
$$v(u(xy)) = v(u(x)y) = u(x)v(y)$$

so 1) implies the commutativity of left + right multipliers when $I^2 = I$.

Next we map I to $\mathrm{End}_{I^o}(I) \times \mathrm{End}_I(I)^o$ by $x \longmapsto (\rho_x, \lambda_x)$; this is a homomorphism with the product in the latter being $(u,v)(u',v') = (uu', v'v)$.

We want the image to be an ideal.

$$(u,v)(\lambda_x, \rho_x) = (u\lambda_x, \rho_x v)$$

$$(u\lambda_x)(y) = u(xy) = u(x)y = \lambda_{u(x)}(y)$$

$$(\rho_x v)(y) = v(y)x$$

At this point we need a further condition, which we can find by thinking of u, v as left and right multiplication by r. Thus

$$v(y)x = yrx = yu(x)$$

and so we add the condition

2) $\boxed{v(y)x = yu(x)}$ $\quad x, y \in I$

to define the <u>multiplier algebra</u> $M(I)$. Check:

$$(v'v)(y)x = v'(v(y))x = v(y)u'(x) = yu(u'(x))$$

Then

$$(\rho_x v)(y) = v(y)x = yu(x) = \rho_{u(x)}(y)$$

so $\quad (u,v)(\lambda_x, \rho_x) = (\lambda_{u(x)}, \rho_{u(x)})$

showing the image of I in $M(I)$ is a left ideal. Similarly it is a right ideal:

$$(\lambda_x u)(y) = xu(y) = v(x)y = \lambda_{v(x)}$$

$$(v\rho_x)(y) = v(yx) = yv(x) = \rho_{v(x)}(y)$$

$$\therefore \quad (\lambda_x, \rho_x)(u,v) = (\lambda_{v(x)}, \rho_{v(x)})$$

March 31, 1988

Let's consider two ideals I, J in R such that $I + J = R$. Then I can form the DGA

$$(I \longrightarrow R) \underset{R}{*} (J \longrightarrow R)$$

In degree $n > 0$ it is the direct sum

$$I \otimes_R J \otimes_R I \otimes \cdots \quad \oplus \; J \otimes_R I \otimes_R J \otimes_R \cdots$$

with n-factors in each tensor product, and so it appears

$$1) \qquad \cdots \longrightarrow I \otimes_R J \oplus J \otimes_R I \xrightarrow{d} I \oplus J \xrightarrow{d} R \longrightarrow 0$$

The differential is

$$d(x_1, \ldots, x_n) = (x_1 x_2, \ldots, x_n) \oplus (-1)^{n-1}(x_1, \ldots, x_{n-1} x_n)$$

where $x_1, \ldots, x_n$ is a sequence of elements alternately from I and J. (Note $I \cdot I = J \cdot J$ in these DGA's)

We've seen this DGA is acyclic because the homology is a unital algebra in which $1 = 0$. Specifically if $\xi = \alpha \oplus (1-\alpha) \in I \oplus J$ is an elt. with $d\xi = 1$, then we obtain a homotopy operator which is an R^o-module morphism

$$h(\omega) = \xi \cdot \omega$$

Check: $d(h(\omega)) = d(\xi \cdot \omega) = d\xi \cdot \omega - \xi \cdot d\omega$
$= \omega - h(d\omega)$

Unfortunately h^2 is multiplication by

$$\xi^2 = \alpha \otimes (1-\alpha) \oplus (1-\alpha) \otimes \alpha \neq 0$$

so we can't use our formulas without modification.

In order to use the resolution 1) to obtain Hochschild cohomology classes we need to find linear functionals τ on the complex 1) divided by its commutators with R such that $\tau d = 0$.

In degree 1 we have to find a linear functional on $(I/[R,I]) \oplus (J/[R,J]$ vanishing on the images of pairs

$$xy \oplus (-yx) \qquad x \in I, \quad y \in J.$$

But actually by right exactness of $\otimes_R$ one sees the only possibilities in degree 1 are traces on R.

In degree 3 we want a linear fnl. on $I \otimes_R J \otimes_R I \otimes_R \;\oplus\; J \otimes_R I \otimes_R J \otimes_R$ which vanishes on

$$d(x_1, y_1, x_2, y_2) = (-1)(x_1, y_1, x_2 y_2) \oplus (x_1 y_1, x_2, y_2)$$

$$d(y_1, x_1, y_2, x_2) = (y_1 x_1, y_2, x_2) \oplus (-1)(y_1, x_1, y_2 x_2)$$

The problem is to find a group ~~to~~ which both factors $I \otimes_R J \otimes_R I \otimes_R$ and $J \otimes_R I \otimes_R J \otimes_R$ can be mapped. Otherwise you are stuck with

$$I/JI \otimes_R J \otimes_R I/IJ \otimes_R$$

$$\|$$

$$I/JI \otimes_R (J/IJ + JI) \otimes_R I/IJ \otimes_R$$

and similarly with J, I interchanged. It's not clear whether this gives anything interesting.

So let's consider the ~~[illegible]~~ case where the degree n is even. We then have to divide $(I\otimes_R J\otimes_R)^n \oplus (J\otimes_R I\otimes_R)^n$ by two relations:

$x_i \in I$
$y_i \in J$

$$d(y_0, x_1, y_1, \dots, x_n, y_n) = (y_0x_1, \dots, y_n) \oplus (y_0, \dots, x_ny_n)$$

$$d(x_0, y_1, x_1, \dots, y_n, x_n) = (x_0y_1, \dots, y_nx_n) \oplus (x_0y_1, \dots, y_n, x_n)$$

If we identify $(y_1, x_1, \dots, y_n, x_n) \equiv -(x_1, \dots, x_n, y_1)$, then the first relation becomes

$$\equiv (y_0x_1, y_1, \dots, x_n, y_n) - (x_1, y_1, \dots, y_{n-1}, x_ny_n, y_0)$$

which is zero in $(I\otimes_R J\otimes_R)^n$. The second relation becomes

$$\equiv (x_0y_1, x_1, \dots, y_nx_n) - (x_1, \dots, x_{n-1}, y_n, x_n, x_0y_1)$$

$$\quad\;\; \| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$(x_nx_0, y_1, x_1, \dots, y_n) \qquad (x_1, y_2, \dots, x_{n-1}, y_n, x_nx_0, y_1)$$

and so for this to vanish we must divide out by σ on $(I\otimes_R J\otimes_R)^n$.

<u>Conclusion:</u> There exists a Hochschild cocycle class of degree $2n-1$ with values in

$$(I\otimes_R J\otimes_R)^n_\sigma$$

Alternative method: Let us consider the DGA 1) and take its commutator quotient. I claim this kills all odd ~~[illegible]~~ degrees ~~[illegible]~~

~~[illegible] In effect consider~~

except degree 1. In effect, consider $I\otimes_R(J\otimes_R I\otimes_R)^n$. A typical generating element here is the bracket $(x_0, y_1, x_1, \dots, y_n, x_n)$

$$x_0\cdot(y_1, x_1, \dots, y_n, x_n) - (y_1, \dots, x_n)\cdot x_0$$

as the latter term is zero.

Consider even degrees. First take degree 2. Then we must divide out $I\otimes_R J\otimes_R \oplus J\otimes_R I\otimes_R$ by elements $x\cdot y + y\cdot x = (x,y)+(y,x)$; this leads to the ~~quotient~~ $I\otimes_R J\otimes_R$.

In general to get the commutator quotient one divide out with brackets where one of the elements comes from a set of generators for the algebra. Thus we have to divide out by brackets coming from R, I, J. This means that we work with cyclic tensor products $I\otimes_R J\otimes_R \cdots \otimes_R$, $J\otimes_R I\otimes_R \otimes \cdots$ and we have to be able to move the factors around with the appropriate signs.

<u>Conclusion</u>: Let $R_\circ = (I\to R) \underset{R}{*} (J\to R)$. Then $R_\circ/[R_\circ, R_\circ]$ is

$$(I\otimes_R J\otimes_R)^{\otimes 2}_\sigma \longrightarrow 0 \xrightarrow{0} I\otimes_R J\otimes_R \xrightarrow{d} I/[R,I]\oplus J/[R,J] \xrightarrow{d} R/[R,R]$$

Now let us apply the functor CC to the DGA $R_\circ$. This will give us a double chain complex with acyclic rows since $R_\circ$ as a complex is acyclic. Hence the columns of positive degree gives a resolution of the cyclic complex $CC(R)$. Looking at the bottom edge of this double complex gives us then canonical maps

$$\circledast \quad HC_{2n-1}(R) \longrightarrow \left(I \otimes_R J \otimes_R\right)^n_\sigma \qquad n \geq 1$$

which are the edge homomorphisms of the associated spectral sequence

$$\begin{tikzcd}
\downarrow & \downarrow & \\
R^{\otimes 3}_\lambda \arrow[d] & R^{\otimes 2} \otimes (I \oplus J) \arrow[l] \arrow[d] & \\
\Lambda^2 R \arrow[d] & R \otimes (I \oplus J) \arrow[l] \arrow[d] & \arrow[l] \\
R & I \oplus J \arrow[l] & \arrow[l]
\end{tikzcd}$$

It's clear that the first column is $(I \oplus J) \dot{\otimes}_R$

It follows by diagram chasing that

$$\begin{array}{ccccccc}
HC_1(R) \longrightarrow & I \otimes_R J \otimes_R & \longrightarrow & (I \oplus J) \otimes_R & \to & R \otimes_R & \to 0 \\
& x \otimes y & & (xy \oplus -yx) & & &
\end{array}$$

is exact. An interesting question is how much one is seeing of $HC_1(R)$ in $I \otimes_R J \otimes_R$. Note we have

$$\begin{tikzcd}
 & I \otimes_R J \otimes_R \arrow[r] \arrow[d] & (I \oplus J) \otimes_R \arrow[d, equal] \\
HC_1(R) \arrow[ur, dashed, "\text{above } \circledast"] \arrow[dr] & (I \cap J) \otimes_R \arrow[r, "(1 \oplus -1)"] \arrow[dr, phantom, "\text{commutes}"] & (I \oplus J) \otimes_R \\
 & HC_1(R/I \cap J) \arrow[u, "\text{Connes}"] \arrow[r, "1 + (-1)"] & HC_1(R/I) \oplus HC_1(R/J) \arrow[u]
\end{tikzcd}$$

so it would seem that $\circledast$ is not consistent with the Connes homomorphism for $R/I \cap J$.

Variants: We can replace the ideals I in R by a bimodule map

$$u : \tilde{I} \longrightarrow R$$

satisfying $u(x)y = x\,u(y)$ for $x, y \in \tilde{I}$, because this gives us a DGA:

$$\cdots \longrightarrow 0 \longrightarrow \tilde{I} \longrightarrow R$$

An example of such an $\tilde{I}$ is

$$\tilde{I} = I \otimes_R \overset{n}{\cdots} \otimes_R I$$

with u the multiplication map. Note that if $\xi = x_1 \otimes \cdots \otimes x_n$ $\quad \eta = y_1 \otimes \cdots \otimes y_n$, then

$$\begin{aligned} u(\xi)\,\eta &= x_1 \cdots x_n y_1 \otimes \cdots \otimes y_n \\ &= x_1 \otimes x_2 \cdots x_n y_1 y_2 \otimes \cdots \\ &= x_1 \otimes x_2 \otimes x_3 \cdots x_n y_1 y_2 y_3 \otimes \cdots \\ &\;\;\vdots \\ &= x_1 \otimes x_2 \otimes \cdots \otimes x_n y_1 y_2 \cdots y_n \\ &= \xi\, u(\eta) \end{aligned}$$

Notice if $I + J = R$, then $I^n + J^k = R$ for $n, k \geq 0$, since

$$R = (I + J)^{2n-1} \subset I^n + J^n$$

(Better: $1 = (\alpha + 1 - \alpha)^{2n-1} = \sum_{k+l=2n-1} \alpha^k (1-\alpha)^l \binom{2n-1}{k}$ and one of k, l must be $\geq n$.)

So we can start with I, J replace them by $I \otimes_R \overset{k}{\cdots} \otimes_R I$, $J \otimes_R \overset{l}{\cdots} \otimes_R J$ and proceed to define

$$HC_{2n-1}(R) \longrightarrow \left(I \otimes_R \overset{k}{\cdots} \otimes_R I \otimes_R J \otimes_R \overset{l}{\cdots} \otimes_R J \otimes_R R \right)_\sigma^n$$

Discussion: We have found another way to construct cyclic cocycles using the cyclic complex of DGA's. But we still don't have any better understanding of what the cyclic complex $CC(A)$ means!

However there are two important features of the present ~~[illegible]~~ example worth thinking about. First of all instead of a DGA resolution of the algebra R one actually uses an acyclic DGA with R in degree zero. A similar thing was done with the DGA $R \xrightarrow{id} R$ and leads to maps

$$HC_{2n}(R) \longrightarrow HC_0(R)$$

which ought to be the iterated S-homomorphisms.

A second feature is the fact that our cyclic cocycles come from closed traces on the DGA. Let's consider this carefully. Suppose we have a DGA R.

$$\cdots\cdots \longrightarrow R_3 \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0$$

which resolves an algebra A. Then we can pass to non-unital rings and replace it by an acyclic DGA, where R_0 becomes the ideal $I = \operatorname{Ker}\{R_0 \to A\}$. It appears that we have two cyclic complexes. On one hand we have

$$CC(\to R_2 \to R_1 \longrightarrow I)$$

whose rows are acyclic + which gives a resolution of $CC(I)$. On the other hand we have

$$CC(\to R_2 \to R_1 \to R_0)$$

which gives a resolution of $CC(A)$.

As far as traces on the DGA is concerned these are already different because

$$R_n/[I, R_n] \neq R_n/[R_0, R_n].$$

It seems that the good setting is to assume $R_\bullet$ is unital and resolves A. Then we can think of $CC(R_\bullet)$ as resolving either $CC(A)$ or we can think of its positive degree columns as resolving the relative complex $\operatorname{Ker}\{CC(R_\bullet) \to CC(A)\}$.

April 1, 1988

Let I be an ideal in R such that $I^2 = I$. Let $M \overset{u}{\twoheadrightarrow} I$ be a bimodule map such that $u(x)y = xu(y)$. The kernel $K = \operatorname{Ker}(u)$ is then killed by I. Since $I^2 = I$, we conclude that $K \otimes_R I = I \otimes_R K = 0$. Hence

$$\begin{tikzcd} 0 \arrow[r] & M \otimes_R I \arrow[r, "\sim"] \arrow[d, "\text{mult}"] & I \otimes_R I \arrow[r] \arrow[dl, dashed, "\varphi"] & 0 \\ & M & & \end{tikzcd}$$

Thus $\varphi(x \otimes y) = \tilde{x} y$ where $\tilde{x} \in u^{-1}(x)$ is a bimodule map from $I \otimes_R I$ to M. Note

$$\tilde{x} y = \tilde{x}\, u(\tilde{y}) = u(\tilde{x}) \tilde{y} = x \tilde{y}.$$

Next note that $I \otimes_R I \xrightarrow{\mu = \text{mult}} I$ is one of our extensions. Moreover

$$\begin{tikzcd} (I \otimes_R I) \otimes_R I \arrow[r, "\mu \otimes 1"] \arrow[d, "1 \otimes \mu"] & I \otimes_R I \arrow[dl, "\text{id}"] \\ I \otimes_R I & \end{tikzcd}$$

commutes: $\quad xy \otimes z = x \otimes yz$

It's still not very clear. Suppose I start with a bimodule surjection $M \xrightarrow{u} I$ in our class. I claim there is a unique bimodule map $\varphi: I \otimes_R I \longrightarrow M$ compatible with the maps to I. Existence:

$$\begin{tikzcd} & M \otimes_R I \arrow[r, "\sim"] \arrow[d] & I \otimes_R I \arrow[d] & \\ 0 \arrow[r] & K \arrow[r] & M \arrow[r] & I \arrow[r] & 0 \end{tikzcd}$$

Uniqueness: Since $I(I \otimes_R I) = I \otimes_R I$ and $IK = KI = 0$ there can be no nonzero bimodule maps $I \otimes_R I \to K$.

This proves the universal character of $I \otimes_R I$, and on the other hand the above diagram shows that the map $\varphi: I \otimes_R I \longrightarrow M$ is an isomorphism $\Longleftrightarrow$ $M \otimes_R I \xrightarrow{\sim} M$ (or similarly $I \otimes_R M \xrightarrow{\sim} M$).

Next consider $I = ReR$ where e is an idempotent. We take $M \xrightarrow{u} I$ to be

$$\tilde{I} \stackrel{def}{=} Re \otimes_{eRe} eR \xrightarrow{\mu} ReR$$

Check:
$$\mu(r_1 e \otimes e r_2)(r_3 e \otimes e r_4) = r_1 e r_2 r_3 \overset{e^2}{\overset{\shortparallel}{e}} \otimes e r_4$$
$$\shortparallel$$
$$(r_1 e \otimes e r_2)\mu(r_3 e \otimes e r_4) = r_1 e \otimes e r_2 r_3 \underset{e^2}{\underset{\shortparallel}{e}} r_4$$

Then
$$\tilde{I} \otimes_R I = Re \otimes_{eRe} \underbrace{eR \otimes_R ReR}_{= eReR \,=\, eR}$$
$$= \tilde{I}$$

where we have used that $eR \otimes_R M = eM$ for any R-module M. Thus we conclude that

$$\boxed{Re \otimes_{eRe} eR = \text{universal extension } ReR \otimes_R ReR}$$

Review: Let R be an algebra, let e be an idempotent in R. Then Connes has defined certain canonical odd degree cyclic cocycles by means of the formula

$$\varphi(r_0, \ldots, r_{2n-1}) = \tau\left(F[F,r_0]\cdots[F,r_{2n-1}]\right)$$

Here τ is a kind of trace-like map. For example if K is an ideal in R containing $[F,r]$ for all $r \in R$, then τ can be the canonical map

$$K^{2n} \longrightarrow K^{2n}/[K, K^{2n-1}]$$

A first problem is to find the natural range space for Connes cocycle. I suspect that it is

1) $$\left((1-e)Re \otimes_{eRe} eR(1-e) \otimes_{(1-e)R(1-e)}\right)^n_\sigma$$

and I checked this for $n=1$.

On the other hand by considering the DGA

$$(\tilde{I} \longrightarrow R) \underset{R}{*} (\tilde{J} \longrightarrow R)$$

where $\tilde{I} = Re \otimes_{eRe} eR$, $\tilde{J} = R(1-e) \otimes_{(1-e)R(1-e)} (1-e)R$

I have been ~~able~~ to produce a cyclic $(2n-1)$-cocycle class on R with values in

1)' $$\left(\tilde{I} \otimes_R \tilde{J} \otimes_R\right)^n_\sigma.$$

But 1) and 1)' are isomorphic

So the problem is to see whether my cocycle class is represented by the

Connes formula with values in the space $\mathcal{D} \cong \mathcal{D}'$. In particular this means showing the latter multilinear functional is a cyclic cocycle.

This problem might be significant because it links Connes approach, which involves traces on DG cochain algebras, with mine which uses traces and more generally cyclic cocycles on DG chain algebras.

First we must understand $n=1$ thoroughly.

Review the problem. We have R, e and we have my cyclic classes

$$HC_{2n-1}(R) \longrightarrow \left(\tilde{I} \otimes_R \tilde{J} \otimes_R\right)^n_\sigma$$

On the other hand we have actual cyclic cocycles defined by Connes whose targets ~~are~~ roughly the same. Connes's method can be described as follows. He has a derivation δ of R with values in a bimodule M and a linear functional τ on

$$\left(M \otimes_R\right)^{2n-1}$$

Then $\tau(a_0 \delta a_1 \cdots \delta a_{2n-1})$ is a Hochschild cocycle. If it happens that $\tau(\delta a_1 \cdots \delta a_{2n-1}) = 0$, then one has a cyclic cocycle.

This is Connes basic method and one can try to see if by some chance it produces a cyclic 1-cocycle with values in $\tilde{I} \otimes_R \tilde{J} \otimes_R$. The obvious thing is to start with the derivation $r \longmapsto \frac{1}{2}[F, r]$. We want to find ~~the~~ the bimodule M.

Let $R = R^+ \oplus R^-$ be the $\mathbb{Z}/2$ graded of elements commuting and anti-commuting with F. We have in block notation

$$\left[\tfrac{1}{2}F, r\right] = \left[\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, r\right] = \begin{pmatrix} 0 & er(1-e) \\ -(1-e)re & 0\end{pmatrix}$$

so $\left[\tfrac{1}{2}F, r\right] \in R^-$.

The obvious candidate for the bimodule M is

$$M = R \otimes_{R^+} R^- \otimes_{R^+} R$$

$$= R \otimes_{R^+} eR\bar{e} \otimes_{R^+} R \quad \oplus \quad R \otimes_{R^+} \bar{e}Re \otimes_{R^+} R$$

$$= Re \otimes_{eRe} eR\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R \quad \oplus \quad R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}Re \otimes_{eRe} eR$$

$$= \tilde{I} \otimes_R \tilde{J} \quad \oplus \quad \tilde{J} \otimes_R \tilde{I}$$

so the question is whether $r \mapsto \left[\tfrac{1}{2}F, r\right]$ is a derivation with values in M. Look at the component in $\tilde{I} \otimes_R \tilde{J}$, that is

$$e \otimes er(1-e) \otimes (1-e) \in Re \otimes_{eRe} eR\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R$$

From p. 667 we know this map

$f : R \longrightarrow Re \otimes_{eRe} eR\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}R$ given by

$$f(r) = e \otimes er(1-e) \otimes (1-e) = e \otimes er\bar{e} \otimes \bar{e}$$

is not a derivation. Let's review the calculation

$$f(rs) = e \otimes [eres\bar{e} + er\bar{e}s\bar{e}] \otimes \bar{e}$$

$$rf(s) = re \otimes es\bar{e} \otimes \bar{e}$$

$$f(r)s = e \otimes er\bar{e} \otimes \bar{e}s$$

Then

$$1)\qquad rf(s) + f(r)s - f(rs) = \bar{e}re \otimes es\bar{e} \otimes \bar{e} + e \otimes er\bar{e} \otimes \bar{e}se$$

If I push this into R by multiplication I get

$$\bar{e}res\bar{e} + er\bar{e}se$$

$$= \bar{e}[r,e][s,\bar{e}] + e[r,\bar{e}][s,e]$$

$$= -(\bar{e}+e)[r,e][s,e] = -[r,e][s,e]$$

On the other hand we could try the second component of $[\frac{1}{2}F, r]$ in $\tilde{J} \otimes_R \tilde{I}$, that is

$$\underbrace{-\bar{e} \otimes \bar{e}re \otimes e}_{g(r)} \in R\bar{e} \otimes_{\bar{e}R\bar{e}} \bar{e}Re \otimes_{eRe} eR$$

Then we have

$$2)\qquad rg(s) + g(r)s - g(rs) = -\left(\begin{array}{l} er\bar{e} \otimes \bar{e}se \otimes e \\ + \bar{e} \otimes \bar{e}re \otimes es\bar{e} \end{array}\right)$$

In order to continue we would have to find some bimodule ~~quotient~~ of

$$M = \tilde{I} \otimes_R \tilde{J} \oplus \tilde{J} \otimes_R \tilde{I}$$

in which the sum of 1) + 2) becomes zero. A first thing to try is to replace $\tilde{I}$ by I and $\tilde{J}$ by J. As on p. 667 this leads to

$$1) = -[r,e] \otimes [s,e] \in I \otimes_R J$$

$$2) = +[r,e] \otimes [s,e] \in J \otimes_R I$$

Thus we can take the R-bimodule of $I \otimes_R J \oplus J \otimes_R I$ generated by the elements

$$-[r,e] \otimes [s,e] \oplus [r,e] \otimes [s,e]$$

Similarly I can divide M itself by the R-bimodule generated by the direct sum of the elements 1) + 2). I don't see what all this means.

The conclusion seems to be the following: Given an algebra R and an idempotent e in R, there is a canonical cyclic 1-cocycle on R with values in

$$\tilde{I} \otimes_R \tilde{J} \otimes_R = \bar{e}Re \otimes_{eRe} eR\bar{e} \otimes_{\bar{e}R\bar{e}}$$

However it doesn't seem to be possible to obtain this cocycle by Connes' method, namely, by means of a derivation $\delta : R \longrightarrow M$ and a suitable trace map on the latter.

The next step to take is to see what can be done with the 2nd formula

$$\varphi(a_0, \dots, a_{2n-1}) = \frac{1}{2^{2n}} \tau\left(F[F, a_0] \cdots [F, a_{2n-1}]\right)$$

This should be a well-defined cyclic cocycle with values in $(K \otimes_R)^{2n}_\sigma$, where K is the ideal $I \cap J = IJ + JI$ generated by the elements $[F, a]$, $a \in R$. However there is a map

$$(K \otimes_R)^{2n} \longrightarrow (I \otimes_R J \otimes_R)^n$$

and the natural question is whether we can obtain an explicit cyclic $(2n-1)$-cocycle with values in $(\tilde{I} \otimes_R \tilde{J} \otimes_R)^n_\sigma$.

April 2, 1988

Let R be an algebra, e an idempotent in R, $F = 2e-1$, and let $\delta : R \to M$ be a derivation where M is a bimodule. Assume that

$$F\,\delta(a) = -\delta(a)\,F \qquad \forall a \in A$$

and set

$$(1) \qquad \varphi(a_0,\dots,a_{2n-1}) = F\delta(a_0)\otimes\delta(a_1)\otimes\cdots\otimes\delta(a_{2n-1}) \in \left(M^{\otimes_R}\right)^{2n}_{\sigma}$$

Claim: φ is a cyclic $(2n-1)$-cocycle.

Careful check for $n=1$:

$$(b\varphi)(a_0,a_1,a_2) = \left.\begin{aligned} &(F\,\delta(a_0a_1),\ \delta(a_2)) \\ -&(F\delta(a_0),\ \delta(a_1a_2)) \\ +&(F\delta(a_2a_0),\ \delta(a_1)) \end{aligned}\right\} = \left\{\begin{aligned} &\overset{1}{(F\delta(a_0)a_1,\delta(a_2))} + \overset{2}{(Fa_0\delta a_1,\delta a_2)} \\ -&\overset{3}{(F\delta a_0,\delta a_1\cdot a_2)} - \overset{1}{(F\delta a_0, a_1\delta a_2)} \\ +&\overset{2}{(F\cdot\delta(a_2)a_0,\delta a_1)} + \overset{3}{(Fa_2\delta a_0,\delta a_1)} \end{aligned}\right.$$

(2) $\quad (Fa_0\delta a_1,\delta a_2) = (\delta a_2, Fa_0\delta a_1) = (\delta a_2\cdot Fa_0,\delta a_1) = -(F\delta a_2\cdot a_0,\delta a_1)$

(3) $\quad (F\delta a_0,\delta a_1\cdot a_2) = -(\delta a_0F,\ \delta a_1 a_2)$

$$= -(\delta a_0,\ F\delta a_1 a_2) = (\delta a_0,\ \delta a_1 Fa_2)$$

$$= (Fa_2\delta a_0,\ \delta a_1)$$

Cyclic property: $\quad (F\delta a_1,\delta a_0) = -(\delta a_1 F,\delta a_0) = -(\delta a_1, F\delta a_0) = -(F\delta a_0, a_1)$

General case:

$$(b\varphi)(a_0,\dots,a_{2n}) = (F(\cancel{\delta a_0\cdot a_1} + a_0\delta a_1),\ \delta a_2,\dots,\delta a_{2n})$$
$$-(F\delta a_0,\ \cancel{\delta a_1\cdot a_2 + a_1\delta a_2},\ \dots,\ \delta a_{2n})$$

$$+(-1)^{2n-1}(F\delta a_0,\ \dots\dots,\ \delta a_{2n-1}\cdot a_{2n} + \cancel{a_{2n-1}\delta a_{2n}})$$
$$+(-1)^{2n}(F(\delta a_{2n}\cdot a_0 + a_{2n}\cdot\delta a_0),\ \delta a_1,\dots,\delta a_{2n-1})$$

Also

$$(F\delta a_{2n-1},\ \delta a_0,\dots,\delta a_{2n-2}) = -(\delta a_{2n-1}, F\delta a_0,\dots,\delta a_{2n-2}) = -(F\delta a_0,\delta a_1,\dots,\delta a_{2n-1})$$

Here's a simpler version without the F: suppose $\delta: R \to M$ is a derivation and set

$$(2) \quad \varphi(a_0, \dots, a_n) = (\delta a_0, \dots, \delta a_n) \in (M \otimes_R)_\lambda^{n+1}$$

<u>Claim</u>: φ is a cyclic n-cocycle for all n.

To see this we can of course calculate, however here's a general argument:

$$\begin{tikzcd}
 & \Omega_R^{n+1} \arrow[r, "\delta\otimes\cdots\otimes\delta"] \arrow[d] & M \otimes_R \overset{n+1}{\cdots} \otimes_R M \arrow[d] \\
\Omega_R^n / \sum [\Omega^i, \Omega^{n-i}] \arrow[r, "d"] & \Omega_R^{n+1} / \sum [\Omega^i, \Omega^{n+1-i}] \arrow[r] & (M \otimes_R)_\lambda^{n+1}
\end{tikzcd}$$

$$a_0 da_1 \dots da_n \overset{\varphi}{\longmapsto} (\delta a_0, \dots, \delta a_n)$$

Thus φ is a closed trace on $\Omega_R^\cdot$ of degree n, and so it is a cyclic cocycle.

The problem with these cocycles (2) is that they come from Hochschild cocycles of higher degree, which means that the classes are killed by the S operator

$$HC_{n+2}(A) \xrightarrow{S} HC_n(A) \longrightarrow H_{n+1}(A, A)$$

Example: $n = 0$. $\varphi(a) = \tau(\delta a)$. This is a 0-cocycle

$$\tau(\delta a_0 \cdot a_1 + a_0 \delta a_1) - \tau(\delta a_1 \cdot a_0 + a_1 \delta a_0) = 0$$

but it's also killed by S.

Remark: d on $\Omega_R^\cdot$ does not induce a map on $\Omega_R^\cdot \otimes_R$ which is why given a normalized Hochschild cocycle $\varphi(a_0, \dots, a_n)$, it's not true that $\varphi(1, a_0, \dots, a_{n-1})$ is a cyclic cocycle, or even a Hochschild cocycle for $n \geq 2$.

Consider a DG chain algebra

$$\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0$$

Set

$$K_n = \begin{cases} R_n & n \geq p+1 \\ dR_{p+1} & n = p \\ 0 & n < p \end{cases}$$

Then $K.$ is DG ideal in $R.$ It's obviously closed under d. Also the differentials are R_0-bimodule homomorphisms, so dR_{p+1} is an R_0-bimodule, and $K.$ is an R_0-bimodule complex. Finally $K.$ is closed under multiplication by $R_{>0}$.

Variant: Replace dR_{p+1} by $\operatorname{Ker}(R_p \xrightarrow{d} R_{p-1})$.

Note that $H_*(R./K.)$ is $H_*(R.)/H_{>p}(R.)$.

Let's use this observation in the case of the DG algebra

$$(I \to R) \underset{R}{*} (J \longrightarrow R)$$

whence we obtain a DG algebra

$$\to 0 \longrightarrow I \cap J \xrightarrow{d} I \oplus J \xrightarrow{d} R$$

which is acyclic assuming $I + J = R$. Take the commutator quotient. For this I need to have the product explicitly.

$$x \in I, \quad y \in J$$

$$d(x * y) = \underset{\underset{J}{\cap}}{dx} \cdot y \;\oplus\; - x \underset{\underset{I}{\cap}}{dy} = (-xy, xy) \in I \oplus J$$

$$d(y * x) = dy \cdot x - y \cdot dx = (yx, -yx)$$

Thus if $d : I \cap J \longrightarrow I \oplus J$ is $z \longmapsto (z, -z)$ then we have the product rules

$$x * y = -xy \in I \cap J$$
$$y * x = yx \in I \cap J$$

Thus the bracket $x * y + y * x$ is $-xy + yx$ and we find that the commutator quotient complex is

$$I \cap J / [R, I\cap J]_- + [I,J]_- \longrightarrow I/[R,I] \oplus J/[R,J] \longrightarrow R/[R,R] \to 0$$

(Recall $[R, I\cap J] \subset [I,J]$ and that these are equal in the case $I = ReR$, $J = R(1-e)R$, see p668; possibly they are equal in general by generalization of the argument on p.670.)

<u>Key problem:</u> We have acyclic DG algebra ~~starting~~ with R and with a trace:

$$\begin{tikzcd} & & R \arrow[d, "s"] & \\ 0 \arrow[r] & I\cap J \arrow[r] \arrow[d, "\tau"] & I \oplus J \arrow[r] & R \arrow[r] & 0 \\ & I\cap J/[I,J] & & & \end{tikzcd}$$

We know this leads to a map $HC_1(R) \to I\cap J/[I,J]$. Can one explain this cyclic cohomology class in Connes' terms? By this I mean using a cochain algebra.

April 3, 1988

Discussion: One can prove a lot about cyclic homology using the cyclic complex applied to DG chain algebras. For example your paper is self-contained and proves directly without using the Connes exact sequence that the cyclic homology of a free algebra is the same as that of the field $\mathbb{C}$. (This follows from the exact sequence

$$0 \longrightarrow HC_{2n}(A) \rightarrow HC_0(R/I^{n+1}) \xrightarrow{d'} H_1(R, R/I^n)_\sigma \longrightarrow HC_{2n-1}(A) \rightarrow 0$$

when R is free, together with the explicit calculation of d', when $I=0$. Thus this calculation uses the cyclic complex of $R \xleftarrow{1} R$.)

However I still don't understand the cyclic complex on a fundamental level. It arises naturally out of the Lie algebra homology of matrices. So the cyclic complex is a natural object.

A key principle should be that cyclic cocycles are higher traces. There seem to be many interpretations of "higher trace". Here is a list

1) Closed trace on a DG chain algebra resolution of A or an acyclic DG chain algebra $R_\cdot$ with a homomorphism $A \longrightarrow R_0$.

2) If $A = R/I$, then a trace on R/I^{n+1} or on I^{n+1} vanishing on $[I, I^n]$.

3) Closed trace on a DG cochain algebra $\Omega^\cdot$ with a homomorphism $A \longrightarrow \Omega^0$.

Now let us consider a free DG chain algebra resolution of A.

$$\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow A \longrightarrow 0$$

Let's apply the reduced cyclic complex functor to it: $CC(\bar{R}_\cdot)$ This will give a double complex ~~with acyclic rows~~ which resolves $CC(\bar{A})$ in the row direction. If one ignores the horizontal differential one has the reduced cyclic complex of a free graded algebra. It should be true that the reduced cyclic homology of a free superalgebra is trivial. Assuming this to be the case, the columns of $CC_\cdot(\bar{R}_\cdot)$ are acyclic except at the bottom, and so we obtain an isomorphism

$$\bar{H}C_*(A) = H_*\left(\bar{R}_\cdot/[R_\cdot, R_\cdot]\right)$$

Thus we see that traces on DG chain algebra resolutions of A gives all the cyclic cohomology of A.

How useful is this representation for cyclic cohomology classes?

For example given a chain algebra resolution $R_\cdot$ of A and a closed trace on it, can I pair this with $K_0(A)$ (or $K_1(A)$ depending on the parity)? Is there some way to see the S-operator?

The S-operator can perhaps be seen in Connes fashion, namely, by using the tensor product of DG algebras. Thus if $R_\cdot \to A$ is a chain algebra resolution, and similarly for $R'_\cdot \longrightarrow A'$, then by the Kunneth formula we know that $R_\cdot \otimes R'_\cdot$ is a chain algebra resolution of $A \otimes A'$. We also have

$$R_\cdot \otimes R'_\cdot / [R_\cdot \otimes R'_\cdot, R_\cdot \otimes R'_\cdot] = R_\cdot / [R_\cdot, R_\cdot] \otimes R'_\cdot / [R'_\cdot, R'_\cdot]$$

and so closed traces on $R_\cdot$ and $R'_\cdot$ combine to get a closed trace on $R_\cdot \otimes R'_\cdot$. Thus if we take $A' = \mathbb{C}$ we go from the cyclic class on A of degree n represented by τ on R to the cyclic class on $A \otimes \mathbb{C} = A$ of degree $n+2$ represented by ? ?

So I have run right into the unital-nonunital problem.

Here's a program: First describe the fundamental reduced cyclic 2-diml cohomology of $\mathbb{C}[e] = \mathbb{C} \times \mathbb{C}$ by means of a chain algebra resolution. Find the smallest one possible. Then the cup product should give us a way to go from $\overline{HC}^n(A)$ to $\overline{HC}^{n+2}(\underbrace{A \otimes \mathbb{C}[e]}_{A \times A})$. Working in the unital category you would need to have a unital map $A \longrightarrow A \times A$. Unfortunately there is only one map around, the diagonal, whereas we need something like Connes map

$$a \longrightarrow ae \subset a^+ \otimes \mathbb{C}[e].$$

~~Let's start with the algebra $\mathbb{C}[e] = \mathbb{C}[F]$. In general given a chain algebra resolution of A and an idempotent~~

Problem: Suppose $R_\cdot \to A$ is a chain algebra resolution, that τ is a closed trace on R of even degree, and that e is an idempotent in A. Then we would like to pair the cyclic class on A represented by $(R_\cdot, \tau)$ with the class of e in $K_0 A$.

The obvious way to proceed is to take a free chain algebra resolution $F_\cdot$ of $\mathbb{C}[e]$, and to extend the homomorphism $\mathbb{C}[e] \to A$ to a chain algebra map $F_\cdot \to R_\cdot$. Then there's an induced map

$$H_*(F_\cdot/[F_\cdot,F_\cdot]) \longrightarrow H_*(R_\cdot/[R_\cdot,R_\cdot]).$$

which ought to be independent of the extension (by standard homotopical algebra?). Now the former space is isomorphic to $\overline{HC}_*(\mathbb{C}[e])$ so there are canonical generators in the even degrees. These can be combined with τ to give numbers.

To understand the situation we might as well start with a chain algebra resolution $R_\cdot$ of $\mathbb{C}[e]$. We want to produce canonical even classes in $H_*(R_\cdot/[R_\cdot,R_\cdot])$; let's consider $n=2$.

$$\begin{tikzcd}
R_3/[R_0,R_3]+[R_1,R_2] \arrow[r, "d"] & R_2/[R_0,R_2]+[R_1,R_1] \arrow[r] \arrow[d] & R_1/[R_0,R_1] \\
 & R_2/dR_3+[R_0,R_2]+[R_1,R_1] \arrow[ur] & \\
 & H_2(R_\cdot/[R_\cdot,R_\cdot]) \arrow[u] & \\
0 \arrow[ur] & &
\end{tikzcd}$$

Therefore the group $H_2(R_\cdot/[R_\cdot, R_\cdot])$ depends on the truncated chain algebra resolution

$$0 \longrightarrow R_2/dR_3 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow A \longrightarrow 0$$

Let's now start constructing a free chain algebra resolution of $\mathbb{C}[e] = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$. We take $R_0 = \mathbb{C}[x]$ with $x \mapsto F = 2e-1$ in $\mathbb{C}[e]$. The kernel of $R_0 \longrightarrow \mathbb{C}[e]$ is generated by x^2-1, so we can take $R_1 = \mathbb{C}[x]\, y\, \mathbb{C}[x] \cong \mathbb{C}[x] \otimes \mathbb{C}[x]$ to be the free R_0-bimodule generated by an element y such that $d(y) = x^2-1$.

If we were to continue to construct a free chain algebra resolution, then R_2 would be the direct sum of

$$R_1 \otimes_{R_0} R_1 = \mathbb{C}[x]\, y\, \mathbb{C}[x]\, y\, \mathbb{C}[x] \simeq \mathbb{C}[x]^{\otimes 3}$$

with a free R_0 bimodule need to kill ~~extra~~ extra relations in R_1. We will however stop the process in degree 2 and let $R_2 = \mathrm{Ker}\{R_1 \xrightarrow{d} R_0\}$. Then we have

$$\begin{tikzcd} R_1 \arrow[r, two heads, "d"] \arrow[d, phantom, "\wr\|"] & \mathbb{C}[x](x^2-1) \subset R_0 \arrow[d, phantom, "\wr\|"] \\ \mathbb{C}[x]\otimes\mathbb{C}[x] \arrow[r, "\text{mult.}"] & \mathbb{C}[x] \end{tikzcd}$$

so $R_2 \simeq \mathbb{C}[x]\, u\, \mathbb{C}[x] \cong \mathbb{C}[x]^{\otimes 2}$, where u is such that $du = xy - yx$. Thus our resolution appears

$$\begin{tikzcd} 0 \arrow[r] & R_2 \arrow[r, "d"] \arrow[d, equal] & R_1 \arrow[r] \arrow[d, equal] & R_0 \arrow[r] \arrow[d, equal] & \mathbb{C}[e] \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{C}[x]u\mathbb{C}[x] \arrow[r, "d"] & \mathbb{C}[x]y\mathbb{C}[x] \arrow[r, "d"] & \mathbb{C}[x] \arrow[r] & \mathbb{C}[e] \arrow[r] & 0 \end{tikzcd}$$

$$du = [x, y] \qquad dy = x^2-1.$$

The product map $R_1 \times R_1 \to R_2$ is determined by the requirement that d be a derivation. Thus if we take $f_1 y g_1, f_2 y g_2 \in R_1$ we want to have

$$d(f_1 y g_1 \cdot f_2 y g_2) = f_1(x^2-1)g_1 f_2 \, y g_2 - f_1 y g_1 f_2 (x^2-1) g_2$$

$$= f_1 \left[(x^2-1) g_1 f_2 , y\right] g_2$$

But
$$[x^n, y] = \sum_{i=0}^{n-1} x^i [x,y] x^{n-1-i}$$
$$= d\left(\sum_{i=0}^{n-1} x^i u x^{n-1-i}\right)$$

so this determines the product.

We now want to determine the trace spaces

$$\begin{tikzcd}
0 \arrow[r] & R_2/[R_0,R_2]+[R_1,R_1] \arrow[r] \arrow[dr, bend right, "0"'] & R_1/[R_0,R_1] \arrow[r] \arrow[d, equal] & R_0/[R_0,R_0] \arrow[d, equal] \\
& & \mathbb{C}[x]\,y \arrow[r, shift left, "d"] & \mathbb{C}[x] \arrow[l, shift left, "\text{mult by } x^2-1"]
\end{tikzcd}$$

So the trace space in degree 2 is the interesting one. We have

$$R_2/[R_0,R_2] \simeq \mathbb{C}[x]u$$

and the trace space in degree 2 is the cokernel of the bracket map which is just symmetrized multiplication

$$(R_1 \otimes_{R_0} R_1 \otimes_{R_0})_\sigma \longrightarrow R_2 \otimes_{R_0} = \mathbb{C}[x]u$$

$$(f_1 y g_1 , f_2 y g_2) \longmapsto f_1 y g_1 f_2 y g_2 + f_2 y g_2 f_1 y g_1$$

$$\equiv (g_2 f_1) y (g_1 f_2) y + (g_1 f_2) y (g_2 f_1) y$$

Thus we are after the cokernel of

$$\underset{(1)}{(R_1 \otimes_{R_0})^2_\sigma} \longrightarrow R_2 \otimes_{R_0} = \mathbb{C}[x]u$$

$$(fy, gy) \longmapsto fygy + gyfy$$

$$d(fygy) = f(x^2-1)gy - fygx^2-1 = f[(x^2-1)g, y]$$

$$d(gyfy) = \quad = g[(x^2-1)f, y]$$

Take $f = x^m$, $g = x^n$. Then we have

$$x^m[(x^2-1)x^n, y] + x^n[(x^2-1)x^m, y]$$

$$= x^m\{[x^{2+n}, y] - [x^n, y]\} + x^n([x^{2+m}, y] - [x^m, y])$$

But $[x^{2+n}, y] = d\left(\sum_{i=0}^{n+1} x^i u x^{n+1-i}\right)$, hence

$$[x^m y, x^n y] = \sum_{i+j=n+2} x^{m+i} u x^{n+1-i} - \sum_{i+j=n} x^{m+i} u x^{n-1-i}$$

$$+ \sum_{i+j=m+2} x^{n+i} u x^{m+1-i} - \sum_{i+j=m} x^{n+i} u x^{m-1-i}$$

and the image of this modulo $[R_0, R_2]$ is

$$\left((n+2)x^{m+n+1} - nx^{m+n-1} + (m+2)x^{n+m+1} - mx^{n+m-1}\right)u$$

$$= (n+m+4)x^{m+n+1} - (n+m)x^{n+m-1}\big)u$$

This just depends on $n+m$. so we want the quotient space of $\mathbb{C}[x]$ by the polynomials

$$\left((n+4)x^{n+1} - nx^{n-1}\right) = \{(x^2\partial_x - \partial_x) + 4x\}x^n$$

$$= [(x^2-1)\partial_x + 4x]\, x^n$$

Conclude $\quad R_2/[R_0, R_2] + [R_1, R_1] = \mathrm{Coker}\left\{\mathbb{C}[x] \xrightarrow{(x^2-1)\partial_x + 4x} \mathbb{C}[x]\right\}$

It's clear this differential operator is injective on polynomials since its kernel is spanned by

$$e^{-\int \frac{4x}{x^2-1}dx} = e^{-2\log(x^2-1)} = \frac{1}{(x^2-1)^2}$$

However if we use the pairing $\int_{-1}^{1} fg\,dx$, then the image of $(x^2-1)\partial_x + 4x$ is ~~[illegible]~~ $\perp$ to the kernel of $-\partial_x(x^2-1) + 4x$. This kernel is spanned by (x^2-1), so

$$\cancel{\quad}\ \big((x^2-1)\partial_x + 4x\big)\mathbb{C}[x] \subset \Big\{f \;\Big|\; \int_{-1}^{1} f\,(x^2-1)\,dx = 0\Big\}.$$

In fact we know the image of $(x^2-1)\partial_x + 4x$ on polynomials contains a polynomial of each degree >0, namely $(n+4)x^{n+1} - nx^{n-1}$. So we conclude

$$\begin{aligned} \mathbb{C}[x]\big/\big((x^2-1)\partial_x + 4x\big)\mathbb{C}[x] &\xleftarrow{\ \sim\ } \mathbb{C} \\ 1 &\longleftarrow\!\shortmid\ 1 \end{aligned}$$

Conclude $R_2/[R_0,R_2] + (R_1, R_1] \underset{u \,\leftarrow\!\shortmid\, 1}{\xleftarrow{\ \sim\ }} \mathbb{C}$ and we have identified the inverse isomorphism

$$\tau : fug \longmapsto \int_{-1}^{1} fg\,(x^2-1)\,dx$$

up to a scalar factor. ~~[illegible]~~

The next question is whether there is a much smaller chain algebra resolution of $\mathbb{C}[e]$ which still supports the canonical 2 diml trace class.

One thing we've learned is how to calculate the pairing of an (R, τ) of dim 2 with an $e \in A$. We lift $2e-1$ to $x \in R_0$, and x^2-1 to $y \in R_1$, and $[x,y]$ to u in R_2; then the pairing is $\tau(u)$.

Here's a smaller version of the length 2 chain algebra resolution of $\mathbb{C}[\mathbb{Z}/2]$. We start with the trace $\tau : R_2 \longrightarrow \mathbb{C}$,

$$\tau(fug) = \int_{-1}^{1} fg\,(x^2-1)\,dx,$$

and we look for the smallest R_0-bimodule which supports it. This is the image of $R_2 \longrightarrow R_0^*$, and in the present case it is $\mathbb{C}[x]u$ where $[x,u]=0$. Then we want to divide R_1 out by the R_0 bimodule generated by $d[x,u] = [x,[x,y]]$. Thus we end up with

$$\begin{tikzcd}
& \bar{R}_2 \arrow[r] \arrow[d, equal] & \bar{R}_1 \arrow[r] \arrow[d, equal] & R_0 \arrow[d, equal] & & \\
0 \arrow[r] & \mathbb{C}[x]u \arrow[r] & \mathbb{C}[x]y \oplus \mathbb{C}[x]h \arrow[r] & \mathbb{C}[x] \arrow[r] & \mathbb{C}[F] \arrow[r] & 0
\end{tikzcd}$$

$$du = h, \quad dy = x^2-1, \quad [x,u]=[x,h]=0, \quad [x,y]=h$$

The product $\bar{R}_1 \times \bar{R}_1 \longrightarrow \bar{R}_2$ is found as follows

$$d(f_1 y g_1 \cdot f_2 y g_2) = f_1(x^2-1)g_1 f_2\, y g_2 - f_1 y g_1 f_2 (x^2-1) g_2$$

$$= f_1 [(x^2-1)g_1 f_2, y]\, g_2 = f_1((x^2-1)g_1 f_2)' g_2\, h$$

$$\therefore \quad f_1 y g_1 \cdot f_2 y g_2 = f_1((x^2-1)g_1 f_2)' g_2\, u$$

The trace $fu \longmapsto \tau(f)$ on $\bar{R}_2$ must vanish on

$$f((x^2-1)g)' + g((x^2-1)f)'$$

and it's clear that

$$\tau(f) = \int_{-1}^{1} f(x)(x^2-1)\,dx$$

has this property.

It's more or less clear that this length 2 chain algebra resolution of $\mathbb{C}[\mathbb{Z}/2]$ is the smallest one which supports the trace τ. In effect, because the linear functionals $\tau \circ x^n$ are all independent $\bar{R}_2$ couldn't be any smaller hence also R_0 couldn't be any smaller, etc.

At this point we conclude that chain algebra resolutions are not likely to be very useful in cyclic theory. The free ones grow too fast. And one misses the nice periodicity. So we must return to the GNS-Stinespring circle of ideas.

April 5, 1988

Consider the algebra

$$R = A * \mathbb{C}[\mathbb{Z}/2] = (A * A) \tilde{\otimes} \mathbb{C}[\mathbb{Z}/2]$$

which is freely generated by A and the involution F. Then we have seen that R admits a description as the GNS algebra

$$R = A \oplus A \otimes B \otimes A$$

where $\rho : A \longrightarrow B$ is universal i.e.

$$B = T(A) / (1 = \rho(1_A)) .$$

Also relative to the idempotent $e = \frac{F+1}{2}$ we have the block description

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix} \qquad eRe = B$$

But notice that R has a unique automorphism of order 2 which is the identity on A and changes F to $-F$. This shows that

$$a \longmapsto (1-e)a(1-e) \in (1-e)R(1-e)$$

is also a universal linear map $\ni$ $1 \mapsto 1$ and hence also that

$$(1-e)R(1-e) \simeq T(A)/(1 = \rho(1_A))$$

Let C be an algebra containing the algebra A and the idempotent e. Set $\bar{e} = 1-e$. Then

1) $$AeCeA \cdot AeCeA = AeCeCeA \subset AeCeA$$

and so $A + AeCeA$ is a unital subalg of C.

Assume C is generated by A, e. Then we conclude

2) $$C = A + AeCeA$$

and similarly

2)' $$C = A + A\bar{e}C\bar{e}A.$$

Now 1) shows that $AeCeA$ is an ideal in C, and it has to be the ideal generated by e. Let's denote this ideal by I:

$$I = CeC = AeCeA$$

and similarly set

$$J = C\bar{e}C = A\bar{e}C\bar{e}A.$$

Next we look at the blocks relative to e of C and these ideals. We have

$$\bar{e}Ie \subset \bar{e}Ce = \bar{e}Cee \subset \bar{e}Ie$$

hence the off-diagonal blocks coincide

$$\bar{e}Ie = \bar{e}Je = \bar{e}Ce$$

$$eI\bar{e} = eJ\bar{e} = eC\bar{e}$$

Also we have similarly

$$eIe = eCe$$

$$\bar{e}J\bar{e} = \bar{e}C\bar{e}$$

so what's interesting are the ideals $eJe \subset eCe$ and $\bar{e}I\bar{e}$ in $\bar{e}C\bar{e}$.

We have

$$e C\bar{e} = eA\bar{e} + eAeCeA\bar{e}$$
$$\subset eCe \cdot eA\bar{e} \subset eC\bar{e}$$

$$eC\bar{e} = eA\bar{e} + eA\bar{e}C\bar{e}A\bar{e}$$
$$\subset eA\bar{e}\cdot\bar{e}C\bar{e} \subset eC\bar{e}$$

Thus

$$\boxed{\begin{aligned} eC\bar{e} &= eA\bar{e}\cdot\bar{e}C\bar{e} = eCe\cdot eA\bar{e} \\ \bar{e}Ce &= \bar{e}Ae\cdot eCe = \bar{e}C\bar{e}\cdot\bar{e}Ae \end{aligned}}$$

Consider now the ideal eJe ~~in~~ eCe. We have

$$eJe = eC\bar{e}Ce = eA\bar{e}\cdot\bar{e}C\bar{e}\cdot\bar{e}Ae$$
$$\bar{e}I\bar{e} = \bar{e}CeC\bar{e} = \bar{e}Ae\cdot eCe\cdot eA\bar{e}$$

and

$$eCe = eAe + eA\bar{e}\cdot\bar{e}C\bar{e}\cdot\bar{e}Ae$$
$$\bar{e}C\bar{e} = \bar{e}A\bar{e} + \bar{e}Ae\cdot eCe\cdot eA\bar{e}$$

since $C = A + I = A + J$. (In fact we have

$$C = \bar{e}A\bar{e} + I = eAe + J.$$
)

[redacted] Note that

$$(eJe)^2 = eA\bar{e}\cdot\underbrace{\bar{e}C\bar{e}\cdot\bar{e}Ae}_{\bar{e}Ae\cdot eCe}\cdot\underbrace{eA\bar{e}\cdot\bar{e}C\bar{e}}_{eCe\cdot eA\bar{e}}\cdot\bar{e}Ae$$
$$= eA\bar{e}\cdot\bar{e}Ae\cdot eCe\cdot eA\bar{e}\cdot\bar{e}Ae = (eA\bar{e}\,\bar{e}Ae)^2\, eCe$$

Let's start again.

$$AeCeA \cdot AeCeA \subset Ae(CeAeC)eA \subset AeCeA$$

Thus $A + AeCeA$ is a unital subalgebra of C containing A, e so if these generate C we have

$$C = A + AeCeA \qquad \text{also} \qquad C = A + A\bar{e}C\bar{e}A$$

and $J = CeC = AeCeA$, $J = C\bar{e}C = A\bar{e}C\bar{e}A$

$$\begin{aligned} \bar{e}Ce &= \bar{e}(A + AeCeA)e \\ &= \bar{e}Ae + \bar{e}AeCeAe \subseteq \bar{e}Ae \cdot eCe \\ &\subset \bar{e}Ce \end{aligned}$$

$$\therefore \quad \boxed{\bar{e}Ce = \bar{e}Ae \cdot eCe}$$

Also

$$\begin{aligned} \bar{e}Ce &= \bar{e}(A + A\bar{e}C\bar{e}A)e \\ &= \bar{e}Ae + \bar{e}A\bar{e}C\bar{e}Ae \subset \bar{e}C\bar{e} \cdot \bar{e}Ae \\ &\subset \bar{e}Ce \end{aligned}$$

$$\therefore \quad \boxed{\bar{e}Ce = \bar{e}C\bar{e} \cdot \bar{e}Ae = \bar{e}Ae \cdot eCe}$$

Next

$$\begin{aligned} eJe &= eA\bar{e}C\bar{e}Ae \\ &= eA\bar{e} \cdot \bar{e}C\bar{e} \cdot \bar{e}Ae \\ &= eCe \cdot eA\bar{e} \cdot \bar{e}Ae = eA\bar{e} \cdot \bar{e}Ae \cdot eCe \end{aligned}$$

$$(eJe)^n = (eA\bar{e} \cdot \bar{e}Ae)^n eCe$$

Thus

$$eCe = e(A + \overbrace{A\bar{e}C\bar{e}A}^{J})e = e(A+J)e$$

$$= eAe + (eA\bar{e}\cdot\bar{e}Ae)\, eCe$$

$$= eAe + eA\bar{e}\cdot\bar{e}Ae\cdot eAe + \underbrace{(eA\bar{e}\cdot\bar{e}Ae)^2 eCe}_{(eJe)^2}$$

Also we have

$$eCe = eAe + eJe$$

$$eJe = eCe\cdot eA\bar{e}\cdot\bar{e}Ae$$

$$= eAe(eA\bar{e}\cdot\bar{e}Ae) + \underbrace{eJe}_{eAe(eA\bar{e}\cdot\bar{e}Ae) + eJe(eA\bar{e}\cdot\bar{e}Ae)}(eA\bar{e}\bar{e}Ae)$$

$$= eAe(eA\bar{e}\cdot\bar{e}Ae) + eAe(eA\bar{e}\cdot\bar{e}A\bar{e})^2 + eJe(eA\bar{e}\cdot\bar{e}Ae)^2$$

from which we see that

$$eCe = eAe + eAe(eA\bar{e}\cdot\bar{e}Ae) + \cdots + eAe(eA\bar{e}\cdot\bar{e}Ae)^n$$
$$+ eJe(eA\bar{e}\cdot\bar{e}Ae)^n$$

Better approach

$$eCe = eAe + eJe = eAe + \overbrace{eCe(eA\bar{e}\cdot\bar{e}Ae)}.$$

Now iterate this equation to get

$$\boxed{\begin{aligned} eCe = {} & eAe + eAe(eA\bar{e}\cdot\bar{e}Ae) + \cdots\cdots \\ & + eAe(eA\bar{e}\cdot\bar{e}Ae)^n + \underbrace{eCe(eA\bar{e}\cdot\bar{e}Ae)^{n+1}}_{=(eJe)^{n+1}} \end{aligned}}$$

similarly we have

$$eC\bar{e} = eCe\cdot eA\bar{e} = eAe\cdot eA\bar{e} + eCe\, eA\bar{e}\,\bar{e}Ae\, eA\bar{e}$$

which leads to

$$e C \bar{e} = eAe \cdot eA\bar{e} + eAe \cdot eA\bar{e}\,(\bar{e}Ae \cdot eA\bar{e}) + \cdots$$
$$+ \; eAe \cdot eA\bar{e}\,(\bar{e}Ae \cdot eA\bar{e})^{n}$$
$$+ \; \underbrace{eCe \cdot eA\bar{e}\,(\bar{e}Ae \cdot eA\bar{e})^{n+1}}_{(eJe)^{n+1}\, eA\bar{e}}$$

In fact we can say the following.

$$eCe = \sum_{k \geq 0} eAe\,(eA\bar{e} \cdot \bar{e}Ae)^{k}$$

$$(eJe)^{n} = \sum_{k \geq n} eAe\,(eA\bar{e} \cdot \bar{e}Ae)^{k} = eCe\,(eA\bar{e} \cdot \bar{e}Ae)^{n}$$

$$e C \bar{e} = eCe \cdot eA\bar{e}$$
$$= eAe \cdot \sum_{k \geq 0} (eA\bar{e}\, \bar{e}Ae)^{k} \cdot eA\bar{e}$$

Question: $eA\bar{e} \subset eC\bar{e}$. Is there a decomposition for $eC\bar{e}$ starting with $eA\bar{e} + \cdots$?

In other words you would like there to be an $eCe \otimes (\bar{e}C\bar{e})^{\circ}$ - submodule of $eC\bar{e}$ complementary to $eA\bar{e}$.

Problem: Make the link with Connes-Cuntz theory.

Let's try to understand the even-diml. Connes homomorphism. I recall the idea that one of degree $2n$ should arise from the odd one of degree $2n+1$:

$$HC_{2n+1}(A) \longrightarrow I^{n+1}/[I, I^{n}]$$

as follows. Representing this by a cyclic $2n+1$ cocycle φ, one should be able to write the image of φ in $R/[R,R]$ as $b\psi$, with $\psi: A_\lambda^{\otimes(2n+1)} \to R/[R,R]$ a cyclic $2n$-cochain. Then $\bar{\psi}: A_\lambda^{\otimes(2n+1)} \longrightarrow HC_0(R/I^{n+1})$ is a cyclic $2n$-cocycle.

The real problem appears then to assume one has a trace τ on C and then to show that Connes cocycle

$$\tau(F[F,a_0] \cdots [F,a_{2n+1}])$$

is a coboundary. But recall that this formula can be interpreted as being obtained from the invariant character form $tr(\tilde{F}(d\tilde{F})^{2n+2})$ on the Grassmannian pulled back via the map $g \mapsto gFg^{-1} = \tilde{F}$.

So we reach the problem of writing the pull-back of the invariant character form on $Gr(V)$ to $U(V)$ as a coboundary.

Let $W = +1$ eigenspace of F on V. We will show $tr(\tilde{F}(d\tilde{F})^{2n})$ becomes a coboundary when lifted to the Stiefel manifold of isometric embeddings of $W \hookrightarrow V$. This uses the standard Chern-Simons ideas.

Think of the Stiefel manifold of consisting of isom. embeddings $i: W \hookrightarrow V$, then the subbundle is isom. to the trivial bundle $\tilde{W}$ with the connection

$$i^* \cdot d \cdot i = d + A \qquad A = i^* d(i)$$

Use the deformation $d + tA$, $0 \le t \le 1$.

$$(dt\partial_t + d + tA)^2 = dt\,A + t\,dA + t^2A^2$$

and so

$$tr\,(dA+A^2)^n = d\int_0^1 tr\,(dt\,A + t\,dA + t^2A^2)^n$$

by standard Chern-Simons calculations.

Now pull-back to $U(V)$ by $g \mapsto g i_0 : W \to V$ where i_0 is the inclusion of W in V. We have

$$A = i_0^* g^{-1} dg \, i_0 = \rho(\Theta)$$

where Θ is the Maurer-Cartan form on $U(V)$, and where $\rho(r) = i_0^* r i_0$ for any operator r. We have with this notation

$$dA = -\rho(\Theta^2)$$

so the curvature is

$$dA + A^2 = \rho(\Theta)^2 - \rho(\Theta^2).$$

At this point, at least on a formal level we begin to see justification for feeling ρ should be viewed as a connection with curvature

$$\rho(a_1)\rho(a_2) - \rho(a_1 a_2)$$

Thus it appears that the transgression form we want is

$$\int_0^1 tr\left(dt\,\rho(\Theta) + t^2\rho(\Theta)^2 - t\rho(\Theta^2)\right)^n$$

$$= \boxed{\int_0^1 tr\left(dt\,\rho(\Theta) + (t^2-t)\rho(\Theta)^2 + t\left(\rho(\Theta)^2 - \rho(\Theta^2)\right)\right)^n}$$

So for $n=1$ we obtain

$$tr\,\rho(\Theta)$$

and for $n=2$ we obtain

$$\int_0^1 2\,dt\ tr\left\{\rho(\Theta)\left[(t^2-t)\rho(\Theta)^2 + t\left(\rho(\Theta)^2 - \rho(\Theta^2)\right)\right]\right\}$$

$$= tr\left(\rho(\Theta)^3\right) 2\left(\tfrac{1}{3} - \tfrac{1}{2}\right) + tr\left(\rho(\Theta)\left(\rho(\Theta)^2 - \rho(\Theta^2)\right)\right)$$

$$= tr\{\rho(\Theta)(\rho(\Theta)^2 - \rho(\Theta^2))\} - \tfrac{1}{3} tr\{\rho(\Theta)^3\}$$

This is the formula on p.591, essentially.

April 7, 1988

Chern–Simons algebra. We work with the non-commutative polynomials in two variables A, F of degrees 1, 2 respectively with differential such that

$$dA + A^2 = F \qquad d(dA) = 0$$

$$(\text{hence} \quad dF = (dA)A - A(dA) = FA - AF).$$

Thus we have a cochain algebra. Algebraically it is the tensor algebra with the generators A, F and it has a differential of degree $+1$. We can also describe it as the free cochain algebra with the single generator A of degree $+1$. Thus it's the tensor algebra on the complex with the generators A and dA. It follows that the commutator quotient algebra has trivial homology.

In fact if V is the complex $\mathbb{C}A \xrightarrow{d} \mathbb{C}dA$, then the cochain algebra is $T(V)$ and the commutator quotient is $\bigoplus_{n \geq 0} V_\sigma^{\otimes n}$. But the Kuenneth formula tells one that each of the complexes $V_\sigma^{\otimes n}$ has trivial homology. In fact if we introduce the derivation h of $T(V)$ of degree -1 (relative to the grading of V) such that $hdA = A$ and $hA = 0$, then one has $dh + hd = n$ on $T_n(V)$ and only $V_\sigma^{\otimes n}$.

It seems that this homotopy operator has to be the "Chern-Simons" one, namely the one associated to the action $A \mapsto tA$.

April 8, 1988

Consider the free cochain algebra $\mathbb{C}\langle A, dA\rangle$ where A has degree 1. If V is the complex

$$0 \longrightarrow \mathbb{C}A \xrightarrow{d} \mathbb{C}\cdot dA \longrightarrow 0 \longrightarrow \cdots$$

then $\mathbb{C}\langle A, dA\rangle = T(V)$. Let θ_t denote the automorphism of this cochain algebra such that

$$\theta_t(A) = tA$$

Then because we have a 1-parameter group of autos given by $x \mapsto \theta_{e^x}$, we obtain a derivation

$$X = \frac{d}{dx}\theta_{e^x}\Big|_{x=0} = t\frac{d}{dt}\theta_t\Big|_{t=1}$$

This derivation is multiplication by n on $V^{\otimes n}$ and commutes with θ_t.

Let h be the degree -1 derivation [redacted] on $T(V)$ such that

$$h(A) = 0 \qquad h(dA) = A$$

Then

$$[d, h] = X$$

because both are derivations agreeing on V.

Let $\varphi \in T(V)$ be closed. Then

$$t\partial_t(\theta_t\varphi) = X\theta_t\varphi = dh\theta_t\varphi$$

and so if φ has no constant term

$$\varphi = d\int_0^1 \frac{dt}{t} h\theta_t\varphi$$

Let's take

$$\varphi = \operatorname{tr}(F^n) = \operatorname{tr}(dA + A^2)^n$$

Then

$$\theta_t\varphi = \operatorname{tr}(t\,dA + t^2A^2)^n$$

$$h\theta_t\varphi = n \operatorname{tr} tA(t\,dA + t^2A^2)^{n-1}$$

and so $\mathrm{tr}(F^n)$ is d of

$$\int_0^1 dt \cdot n \cdot \mathrm{tr}\left\{A(t\,dA + t^2A^2)^{n-1}\right\}$$

which is the Chern-Simons form.

Let's go over the program. I think at this point I understand the even-dimensional Connes homomorphism on the level of formulas. I still might want to show that the Chern-Simons formula is consistent with diagram-chasing. Still I will find it hard to show compatibility with the S-operator

Let's start again with the algebra C obtained by adjoining an idempotent e to A. Apparently C contains all the information relative to the cyclic homology of A.

I propose to use C with all its structure to prove the theorems at the end of my paper on extensions. To be more specific we have

$$C = A * \mathbb{C}[\mathbb{Z}/2], \quad J = CeC, \quad \bar{J} = C\bar{e}C$$

$$B = eCe, \quad \rho(a) = eae, \quad \rho : A \to B.$$

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[l, dashed, bend right, "\rho"'] & 0 \\ & e\bar{J}e & eCe & & \end{tikzcd}$$

By universal arguments we know that

$$B = T(A)/(1-\rho(1)) \simeq T(\bar{A}).$$

here $\bar{A} = A/\mathbb{C}$ So there's a conflict of notation

It should be possible to deduce from this or similar arguments that there are canonical isomorphisms

$$gr^I(B) = \Omega_A^{\text{even}} = gr^{\bar{I}}(\bar{B})$$

$$gr^I(eC\bar{e}) = \Omega_A^{\text{odd}} = gr^{\bar{I}}(\bar{e}Ce)$$

Now the idea will be as follows. I have explicitly constructed cyclic cocycles

$$\bar{C}C_{2n+1}(\mathbb{C}) \longrightarrow I^{n+1}/[I,I^n]$$

$$(x_0,\dots,x_{2n+1}) \longmapsto e[x_0,e]\cdots[x_{2n+1},e]e$$

which ought to induce an injection

$$\bar{H}C_{2n+1}(A) \hookrightarrow I^{n+1}/[I,I^n].$$

So it might be possible to see explicitly the exact sequence

$$0 \longrightarrow \bar{H}C_{2n+1}(A) \to I^{n+1}/[I,I^n] \xrightarrow{\delta} \underset{\displaystyle \overset{\Vert}{I^n\otimes\bar{A}}}{I^n\otimes_B\Omega^1_B\otimes_B} \longrightarrow HC_{2n}(A) \longrightarrow 0$$

by means of these calculations within C.

Having to referee Wagner's paper it seems desirable to learn some of the background concerning subshifts of finite type, Markov partitions, Cuntz-Krieger C^*-algebras, and so forth. Earlier work of this sort was done June 85 pp 49-59 when I looked at David Fried's paper on Ruelle's zeta functions.

Let A be a zero-one matrix, say $A = (A(x,y))$ where x,y run over the finite state space I. Then the subshift of finite type associated to A is the subspace X_A of $I^{\mathbb{Z}}$ consisting of sequences (x_n) such that $A(x_n, x_{n+1}) = 1$ for all n. Then X_A is a compact totally disconnected metric space with the shift automorphism. It's customary to assume each row and column of A is non-zero.

The Cuntz-Krieger algebra $\mathcal{O}_A$ is the C^*-algebra freely generated by partial isometries s_i, $i \in I$, (this means $s_i^* s_i$, $s_i s_i^*$ are projectors) such that the range projectors $s_i s_i^*$ are mutually orthogonal and give the domain projectors by the rule

$$s_i^* s_i = \sum_j A(i,j)\, s_j s_j^*$$

The simplest case is when all $A(i,j) = 1$ whence we have the C^* algebra $\mathcal{O}_n$ $n = \text{card}(I)$. It contains n isometric embeddings $s_1, \dots, s_n$ whose images decompose the Hilbert space orthogonally (assuming we have a representation. Thus

$$\underset{\text{all } i}{s_i^* s_i = 1} = \sum_j \underbrace{s_j s_j^*}_{\text{orthogonal idempotents}}$$

It's clear that in $K_0(\mathcal{O}_A)$ one has $1 = n$. Cuntz has shown $K_0(\mathcal{O}_A) = \mathbb{Z}/(n-1)\mathbb{Z}$.

Let's now consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and consider the half-infinite shift space Ω consisting of all sequences $(x_0, x_1, \dots)$ where $x_n \in \{0,1\}$. Then the backwards shift σ maps Ω 2 to 1 onto itself. For any measurable subset $S \subset \Omega$, the measure of $\sigma^{-1}S$ is twice that of S. Here we use the Bernoulli measure on Ω.

Ω splits into $\{0\} \times \Omega$, $\{1\} \times \Omega$ each of which can be identified with Ω. Thus one sees how to define maps s_0, s_1 on functions on Ω which are embeddings with complementary images. Given $f(x_0, \dots)$ one puts

$$(s_0 f)(x_0, x_1, \dots) = \chi_{\{0\}}(x_0) f(x_1, x_2, \dots)$$
$$(s_1 f)(x_0, x_1, \dots) = \chi_{\{1\}}(x_0) f(x_1, x_2, \dots)$$

In this case $P_0 = s_0 s_0^*$ projects onto $f \ni f(1, \dots) = 0$ and $P_1 = s_1 s_1^*$ projects onto f supported in $\{1\} \times \Omega$.

Let's try to handle the general case by the same method. Again we want P_i to project onto $f(x_0, \dots)$ which are supported where $x_0 = i$. Then the P_i's given an orthogonal decomposition of functions on Ω. Next let Q_i project onto functions supported where $A(i, x_0) = 1$, i.e. where we can extend the sequence to $x_{-1} = i$, or equivalently where the sequence is the backwards shift of a sequence starting with i. It's then clear that

$$Q_i = \sum A(i,j) P_j$$

Maybe one can see that the shift sets up an equivalence between Q_i, P_i. It's clear the backwards shift maps the support of P_i bijectively onto the support of Q_i

To feel completely happy I would like to find a nice invariant measure on X_A on the half-sequence space Ω and to check that I really get a Hilbert space representation of $\mathcal{O}_A$.

Let's remark that because the entries of A are ≥ 0 the Frobenius thm tells that (at least under non degeneracy conditions) there are unique non-negative left and right eigen~~vectors~~

$$\sum_y A(x,y)\,\nu(y) = \lambda\,\nu(x)$$

$$\sum_x \mu(x)\,A(x,y) = \lambda\,\mu(x).$$

Thus setting $p(x,y) = \frac{1}{\lambda}A(x,y)$, we get a consistent family of measures on I^n given by

$$\mu(x_1)\,p(x_1,x_2)\cdots\cdots p(x_{n-1},x_n)\,\nu(x_n) = \text{measure of } (x_1,\dots,x_n)$$

Now let's return to Wagoner's paper, which is based on Williams paper. The problem is to understand when two $0,1$ matrices give the same "dynamical system", i.e. when $(X_A,\sigma_A) \simeq (X_B,\sigma_B)$. Sufficient for this is for there to be an elementary strong shift equivalence i.e. $(0,1)$ matrices R,S (of appropriate sizes) such that

$$RS = A \qquad \text{and} \qquad SR = B$$

Williams showed that any equivalence is a finite composition of elementary strong shift equivalences.

The idea is to define something

called a Markov partition for a dynamical system (X, σ). This is a partition of X into open sets (U_i) with certain properties. Any Markov partition determines a 0,1 matrix: $A(i,j) = 1 \iff U_i \cap \sigma^{-1}(U_j) \neq \phi$, and one has a canonical isomorphism

$$(X_A, \sigma_A) \xrightarrow{\sim} (X, \sigma)$$

preserving the tautological Markov partition on the left to the given one on the right.

Thus Markov partitions are something like lattices that go into buildings.

We should <u>probably</u> develop this analogy.

Let's start again. Let's begin with a dynamical system (X, σ) consisting of a compact totally disconnected space and an automorphism. We assume it admits Markov partitions; these are partitions into nonempty open sets $\{U_i\}_{i \in I}$ such that if A is the 0-1 matrix $A(i,j) = 1 \iff U_i \cap \sigma^{-1} U_j \neq \phi$, then one gets a unique isomorphism

$$(X_A, \sigma_A) \xrightarrow{\sim} (X, \sigma)$$

carrying the standard Markov partition $U_i = \{(x_n) \mid x_0 = i\}$ on the left to the given one on the right

In the analogy we think of (X, σ) as a f.d. vector space V over a local field F and a Markov partition as being a lattice L in V.

Once we have a Markov partition then we can act by the automorphisms and produce lots more.

Now there's a concept of when two Markov partitions are close which is analogous to having two lattices one included in the other and the quotient killed by the maximal ideal.

If U, V are close then the associated

0-1 matrices A, B are related by an elementary strong shift equivalence.

This is part of Williams' proof, the rest being (according to Wagoner) the connectivity of the space of Markov partitions.

Wagoner's definition of "close" precisely $U \longrightarrow V$ means that

$$V < U \cap \sigma^{-1}(U) \qquad U \xrightarrow{+} U \cap V$$

$$U < V \cap \sigma(V) \qquad V \xrightarrow{-} U \cap V$$

where $<$ means refines and $\cap$ denotes the intersection of partitions = coarsest common refinement.

Lots of technical difficulties seem to arise because of the fact that U, V are related by refinement. I suspect it might be possible to simplify things a lot.

Let's return to $C = A * \mathbb{C}[\mathbb{Z}/2]$. We let $x \longmapsto \tilde{x}$ denote the automorphism of C such that $\tilde{a} = a$ and $\tilde{F} = -F$. Thus $\tilde{e} = 1-e$. Let $B = eCe$, $I = e\tilde{J}e = eC\tilde{e}Ce$.

(Review previous calculations. We have that $J = AeCeA$, $\tilde{J} = A\tilde{e}C\tilde{e}A$ are ideals in $C = A + AeCeA = A + A\tilde{e}C\tilde{e}A$. Logically: the subspace $AeCeA$ of C is closed under left multiplication by A, e and also right multiplication, hence it's an ideal in C. ~~Then (A + AeAeA is~~ Clearly $CeC = AeCeA$. Then $A + AeCeA$ is a subalgebra containing A, e hence equal to C. $\therefore$

$$C = A + \underbrace{AeCeA}_{= CeC}$$

and similarly with $\sim$'s. Then

$$eC\tilde{e} = eA\tilde{e} + eAeCeA\tilde{e} = eCeA\tilde{e}$$
$$= eA\tilde{e} + eA\tilde{e}C\tilde{e}A\tilde{e} = eA\tilde{e}C\tilde{e}$$

giving

$$eC\tilde{e} = eCe \cdot eA\tilde{e} = eA\tilde{e} \cdot \tilde{e}C\tilde{e}$$

and similarly with $e, \tilde{e}$ reversed. Next

$$eCe = eAe + \underbrace{eA\tilde{e}C\tilde{e}Ae}$$

$$eA\tilde{e} \cdot \tilde{e}C\tilde{e} \cdot \tilde{e}Ae = eCe \cdot eA\tilde{e} \cdot \tilde{e}Ae$$

$$\therefore \quad eCe = eAe + \underbrace{eCe \cdot eA\tilde{e} \cdot \tilde{e}Ae}_{I = eC\tilde{e}Ce}$$

so iterating gives

$$eCe = eAe + eAe \cdot eA\tilde{e} \cdot \tilde{e}Ae + eAe \cdot (eA\tilde{e}\tilde{e}Ae)^2 + \cdots$$
$$+ eAe \cdot (eA\tilde{e}\tilde{e}Ae)^n + I^{n+1} \quad)$$

Now I want to identify eCe with Ω_A^{ev}. How to do this is clear from the formula

$$\circledast \quad eCe = eAe + eAe \cdot eA\tilde{e} \cdot \tilde{e}Ae + \cdots$$

There's a natural map of Ω_A^{ev} to this because $\Omega_A^n \xrightarrow{\sim} A \otimes \bar{A}^{\otimes n}$.

Now we know from our universal property arguments that

$$eCe = T(A)/(1 = \rho_A(1)).$$

We use this to define a representation (left module structure) of eCe on Ω_A^{ev}. All we have to do is to associate to $a \in A$ an operator on Ω_A^{ev} such that $1 \mapsto \mathrm{id}$. To see what the operator should be look at $\circledast$ and describe left mult. by eae.

$$eae \cdot ea_1e = eaa_1e + \underbrace{e[aea_1 - eaa_1]e}_{e[a,e]a_1e \; = \; e[a,e][a_1,e]}$$

better:

$$= eaa_1e + ea(e-1)a_1e$$

$$= e(aa_1)e - ea\tilde{e} \cdot \tilde{e}a_1e$$

Thus we define the operator $\rho(a)$ on Ω_A by the formula

$$\rho(a)\, a_0 da_1 \dots da_n = aa_0 da_1 \dots da_n + da_0 da_1 \dots da_n$$

In other words

$$\rho(a) = a + (da) \cdot d$$

Note that

$$\rho(a_0)\rho(a_1) - \rho(a_0a_1) = (a_0 + (da_0)d)(a_1 + (da_1)\cdot d) - (a_0a_1 + d(a_0a_1)\cdot d)$$

$$= \cancel{a_0a_1} + da_0\cdot d\cdot a_1 + a_0\cdot\cancel{(da_1)}\cdot d + \underbrace{(da_0)\cdot d\cdot (da_1)\cdot d}_{=0}$$

$$- \cancel{a_0a_1} - (da_0)\cdot a_1\cdot d - a_0\cdot\cancel{(da_1)}\cdot d$$

Thus

$$\boxed{\rho(a_0)\rho(a_1) - \rho(a_0a_1) = da_0\cdot da_1.}$$

Next let's try to set up a reasonable program. We are interested in the "versal" extension

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & B \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[l, bend right, dashed, "\rho"'] & 0 \\ & & T(A)/(1=\rho(1_A)) & & \end{tikzcd}$$

Working inside C it appears that canonical representatives for Connes homomorphisms. The goal will be to prove the exact sequences you establish at the end of your paper. What this amounts to is a proof in the spirit of Connes where he relates non-comm. DR and cyclic homology.

The method I would like to use to produce Connes homomorphisms is by a Chern-Weil process. Thus I want the cyclic cochains on A to come from left-invariant differential forms on $GL_n(A)$ associated to connections and curvature.

Let's take $n = 1$. We have a map

$$A^\times \longrightarrow C^\times \longrightarrow \text{Grass}$$

Let's go over this business again

Consider a vector space V with an idempotent e, all supposed compatible with some inner product. Better, we can forget this but we must replace $Gr(V)$ by the space of idempotents $\widetilde{Gr}(V)$. Over $\widetilde{Gr}(V)$ we have the canonical subbundle embedded as a direct summand of the trivial bundle $\tilde{V}$, so it has a canonical connection. The principal bundle of the canonical subbundle can be identified with the space of embeddings of $W = \mathrm{Im}(e)$ as a direct factor of V (this means we give $i: W \to V$ and $i^*: V \to W$ such that $i^* i = id.$)

Now we pull ~~this~~ all back to $GL(V)$.

Let

$$W \xrightarrow{i_0} V \xrightarrow{i_0^*} W$$

be the inclusion and projection onto $W = \mathrm{Im}\, e$. Over $GL(V)$ we have the bundle $g \longmapsto gW$ of $\tilde{V}$ with its natural trivialization. Thus over $GL(V)$ we have the bundle $\tilde{W}$ directly embedded into $\tilde{V}$ via the map

$$\tilde{W} \xrightarrow{g i_0} \tilde{V} \xrightarrow{i_0^* g^{-1}} \tilde{W}$$

g here denoting the tautological autom. of $\tilde{V}$ over $GL(V)$. Thus we have the connection form

$$\alpha = i_0^* g^{-1} dg\, i_0$$

which is a left-invariant 1-form on $GL(V)$ with values in $\mathrm{End}(W)$.

Here is the problem to be solved. In order to apply the Chern-Simons transgression method I want to work in a cochain algebra, for example, the DR complex of left-invariant differential forms on $GL(V)$ with values in $End(W)$. However I want the final answer to be cyclic cochains on $End(V)$ not just Lie algebra cochains. Thus I need some version of the LQ thm. which means looking at the n-fold direct sum of this situation and asking for $GL_n(\mathbb{C})$ invariants. But our connection form has values then in $End(W^{\oplus n}) = End(W) \otimes M_n(\mathbb{C})$.

This isn't very clear. However, maybe we should try to proceed formally. Thus let's suppose we have the algebra A acting on the vector space V and the idempotent e on V with image W.

I then have $\alpha : A \xrightarrow{a \mapsto eae} End(W)$; this is somehow a 1-form with values in operators. The next question is what should be α^2 and $d\alpha$. I think they ought to be the bilinear functions

$$a_1, a_2 \longmapsto \alpha(a_1)\alpha(a_2) \quad \text{and} \quad -\alpha(a_1 a_2).$$

So a natural question is whether these rules lead to a cochain algebra.

April 10, 1988

Consider A acting on V and an idempotent e on V. Let $W = \mathrm{Im}(e)$, let $\iota_0 : W \hookrightarrow V$ be the inclusion and $\iota_0^* = e : V \longrightarrow W$ the projection. Let $G = A^\times$. Over G we have an embedding of the trivial bundle $\tilde{W}$ as a direct factor of $\tilde{V}$ given by the maps

$$\tilde{W} \xrightarrow{g\iota_0} \tilde{V} \xrightarrow{\iota_0^* g^{-1}} \tilde{W}$$

where g here stands for tautological autom of $\tilde{V}$ over G (induced by the action $G = A^\times \longrightarrow GL(V)$). This gives the connection form

$$\iota_0^* g^{-1} dg\, \iota_0 \quad \in \Omega^1(G, \mathrm{End}\,\tilde{W})$$

which is left-invariant, from which we can construct character forms and Chern-Simons forms by the Chern-Weil methods, provided we have a trace defined.

Thus except for the trace question we work in the algebra of left-invariant forms

$$\Omega^\cdot(G, \mathrm{End}\,\tilde{W})^G = \mathrm{Hom}_{\mathbb{C}}(\Lambda \mathfrak{g}, \mathrm{End}\, W)$$

which are the same as Lie algebra cochains on the Lie algebra $\mathfrak{g}$ with values in the algebra $\mathrm{End}\, W$ considered as a trivial $\mathfrak{g}$-module. Thus the complex of Lie cochains

$$C^\cdot(\mathfrak{g}, \mathrm{End}(W))$$

is a cochain algebra. This is clear at least when W is finite-dimensional as one just has the matrix algebra on $C^\cdot(\mathfrak{g})$.

More generally it ought to be the case that on the Lie algebra cochain complexes one has cup products

$$C^\cdot(\mathfrak{g}, M) \times C^\cdot(\mathfrak{g}, N) \longrightarrow C^\cdot(\mathfrak{g}, M \otimes N)$$

for M, N $\mathfrak{g}$-modules with usual associativity etc. properties.

So much for the Lie algebra cochains; let's consider next the algebra cochains which should be simpler. (both the formula for d and $\cup$ are simpler.)

Thus we consider End(W) as a trivial A-module which means we forget that A is unital, and so we work in the complex of normalized Hochschild cochains on A^+ with values in End(W):

$$C^p_N(A^+, \operatorname{End} W) = \operatorname{Hom}(A^{\otimes p}, \operatorname{End} W)$$

This will form a cochain algebra with product

$$(\overset{p}{\varphi}\ \overset{q}{\psi})(a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q}) = \varphi(a_1, \ldots, a_p)\,\psi(a_{p+1}, \ldots, a_{p+q})$$

$$(b\varphi)(a_1, \ldots, a_{n+1}) = \sum_{i=1}^{n} (-1)^i \varphi(\ldots, a_i a_{i+1}, \ldots)$$

Thus we have found a cochain algebra in which if $\alpha : A \longrightarrow \operatorname{End}(W)$ is a 1-cochain, then we have

$$(\alpha^2)(a_0, a_1) = \alpha(a_0)\,\alpha(a_1)$$

$$(d\alpha)(a_0, a_1) = -\alpha(a_0 a_1)$$

$$(d\alpha + \alpha^2)(a_0, a_1) = \alpha(a_0)\,\alpha(a_1) - \alpha(a_0 a_1)$$

Recall that in the situation of interest B is $\mathrm{End}(W)$ essentially. Thus we are led to the cochain algebra

$$C_N^\bullet(A^+, B)$$

Unfortunately there is going to be some confusion of the unital - nonunital sort, but for the moment let's continue and start applying the Chern-Simons-Weil algebra.

Actually we are above all interested in the case where $B = T(A)/(1 = \rho(1))$, where we take the DG subalgebra generated by $\alpha = \rho : A \to B$. What we have to do is to apply the Chern-Simons game. This means we have just a cochain algebra map

$$\mathbb{C}\langle \alpha, d\alpha \rangle \longrightarrow C_N^\bullet(A^+, B)$$

What is important is the filtration, i.e. the powers of the ideal $I = \mathrm{Ker}(B \to A)$. This is why we want the curvature $\alpha^2 + d\alpha$. Thus there's probably a natural filtration on $\mathbb{C}\langle \alpha, d\alpha \rangle$ which should be described.

We need to study the I-adic filtration on $R^\bullet = \mathbb{C}\langle \alpha, d\alpha \rangle$ where I is the ideal generated by the "curvature" $\beta = \alpha^2 + d\alpha$. First we note that

$$d\beta = d\alpha . \alpha - \alpha . d\alpha = [\beta, \alpha]$$

hence the ideal $I^\bullet$ is stable under d. This implies that the associated graded algebra

$gr^I(R^\cdot)$ is a cochain algebra.

We can easily determine $gr^I(R^\cdot)$ as an algebra. Note that as a graded algebra $R^\cdot$ is the non-commutative polynomial ring on α and β with deg $\alpha = 1$ and deg $\beta = 2$. Consider the grading giving the β-degree. Then we have natural representatives for I^n/I^{n+1} in this grading, namely those polynomial which are linear combinations of monomials in α, β with exactly n β's occurring. Thus $gr^I(R^\cdot) = R^\cdot$ is a free algebra generated by the image of α in $gr^I_0(R) = R/I$ and the image of β in $gr^I_1(R) = I/I^2$.

Thus

$$gr^I(\mathbb{C}\langle \alpha, d\alpha \rangle) = \mathbb{C}\langle \bar\alpha, \bar\beta \rangle$$

where

$$d\bar\alpha = -\bar\alpha^2$$

$$d\bar\beta = [\bar\beta, \bar\alpha]$$

What is the cohomology $H^\cdot\{gr^I(R), d\}$?

$$R^\cdot/I^\cdot = \mathbb{C}[\bar\alpha], \qquad d\bar\alpha = -\bar\alpha^2$$

so $H^\cdot(R^\cdot/I^\cdot) = \mathbb{C}$. In effect when you apply d to $\bar\alpha^n$ you get an alternating sum of the terms $-\bar\alpha^{n+1}$ and there are n terms; thus you get $d\bar\alpha^n = 0$ for even and $d\bar\alpha^n = -\bar\alpha^{n+1}$ for n odd.

Maybe it's not so interesting what the cohomology of $gr^I(R)$ is, since we ultimately want to pass to commutator quotients.

Let's return to basic principles. We have the cochain algebra $C_N^\cdot(A^+,B)$ and the element $\alpha = \rho: A \to B$. Recall that in the universal case $B = T(A)/(1=\rho(1))$, we have that the 2-cochain

$$(d\alpha + \alpha^2)(a_0, a_1) = \rho(a_0)\rho(a_1) - \rho(a_0 a_1)$$

has values in the ideal $I = \mathrm{Ker}(B \to A)$. Actually this would be true for any extension $0 \to I \to B \overset{\leftarrow \rho}{\to} A \to 0$ and ρ a lifting. Therefore the homomorphism

$$\mathbb{C}\langle \alpha, d\alpha \rangle \longrightarrow C_N^\cdot(A^+,B)$$

is compatible with the J-adic filtration where the ideal J on the left is generated by β and the ideal J on the right is generated by $C_N^2(A^+, I)$.

Now our goal is to prove the Connes homomorphisms associated to the extension $B/I = A$, and so it is a very good idea to see if this works in the expected way. The issue is again the shift in degree. Thus from the curvature $d\alpha + \alpha^2$ in $C_N^2(A^+, I)$ we expect a cyclic 1-cocycle on A with values in $I/[B,I]$.

Let's see what we need to construct a trace on the algebra $C^\cdot(A,B) = C_N^\cdot(A^+,B)$.

Is it possible to describe the commutator quotient?

If A is finite-dimensional, then it looks as if $C^{\cdot}(A,B)$ is generated by $C^0(A,B) = B$ and $C^1(A,B) = \mathrm{Hom}_{\mathbb{C}}(A,B)$. In fact we have the commutative square

$$\begin{tikzcd}
C^1(A,B) \otimes_B C^1(A,B) \arrow[d] & (B\otimes A^*) \otimes_B (B \otimes A^*) \arrow[l] \arrow[d] \\
C^2(A,B) & B \otimes A^* \otimes A^* \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
(a_1 \mapsto b_1\lambda_1(a_1)) \otimes (a_2 \mapsto b_2\lambda_2(a_2)) \arrow[d, mapsto] & (b_1\otimes\lambda_1)\otimes(b_2\otimes\lambda_2) \arrow[l, mapsto] \arrow[d, mapsto] \\
((a_1,a_2) \mapsto b_1 b_2 \ \lambda_1(a_1)\lambda_2(a_2)) & b_1b_2 \otimes \lambda_1 \otimes \lambda_2 \arrow[l, mapsto]
\end{tikzcd}$$

where the horizontal arrows are isomorphisms in the finite-dimensional case, and similarly for cochains of higher degree. Thus $C^{\cdot}(A,B)$ is essentially the tensor algebra of the B-bimodule $C^1(A,B)$ over B.

Recall that the commutator quotient of $T_B(M) = \bigoplus_{n\geq 0} M \otimes_B \overset{n}{\cdots} \otimes_B M$ is $\bigoplus_{n\geq 0} (M\otimes_B)^n_\sigma$. In the present situation we view $T_B(M)$ as a superalg. with M of odd degree and so the commutator quotient is

$$T_B(M)/[\ ,\ ] = \bigoplus_{n\geq 0} (M\otimes_B)^n_{\lambda}$$

In the case of interest $M = B\otimes_{\mathbb{C}} A^*$ essentially so we have that

$$\left(\left(B \otimes_{\mathbb{C}} A^*\right)\otimes_B\right)^n_\lambda = \left(\left(B\otimes_B\right)^n \underset{\mathbb{C}}{\otimes} A^{*\otimes n}\right)_\lambda$$

$$\simeq B/[B,B] \otimes A^{*\otimes n}_\lambda$$

We see therefore that it should be possible to define a trace map

$$(*) \qquad C^\cdot(A,B) \longrightarrow \operatorname{Hom}\left(CC_\cdot(A),\ B/[B,B]\right)$$

compatible with differentials and shifting degrees by 1. Moreover this should be best possible, since it should be an isomorphism between the commutator quotient of the algebra on the left with the complex on the right when A is finite-dimensional.

We need the formula for this trace map $(*)$. In dimension 1 it must be the map

$$C^1(A,B) = \operatorname{Hom}(A,B) \xrightarrow{\ \sim\ } \operatorname{Hom}(A, B/[B,B])$$

induced by the universal trace $B \longrightarrow B/[B,B]$. In higher dimensions it must be cyclic λ-symmetrization up to a constant factor. So I try on $C^2(A,B)$ the formula

$$(\tau\varphi)(a_1,a_2) = \varphi(a_1,a_2) - \varphi(a_2,a_1)$$

Then

$$(\tau d\varphi)(a_1,a_2) = (d\varphi)(a_1,a_2) - (d\varphi)(a_2,a_1)$$

$$= -\varphi(a_1a_2) + \varphi(a_2a_1)$$

$$(b\tau\varphi)(a_1,a_2) = (\tau\varphi)(a_1a_2) - (\tau\varphi)(a_2a_1)$$

$$= \varphi(a_1a_2) - \varphi(a_2a_1)$$

On $C^3(A,B)$ let's try

$$(\tau\varphi)(a_1,a_2,a_3) = \varphi(a_1,a_2,a_3) + \varphi(a_2,a_3,a_1) + \varphi(a_3,a_1,a_2)$$

Then

$$(\tau d\psi)(a_1,a_2,a_3) = (d\psi)(a_1,a_2,a_3) + \text{cyc.}$$

$$= -\psi(a_1a_2,a_3) - \psi(a_2a_3,a_1) - \psi(a_3a_1,a_2)$$
$$+ \psi(a_1,a_2a_3) + \psi(a_2,a_3a_1) + \psi(a_3,a_1a_2)$$

$$(b\tau\psi)(a_1,a_2,a_3) = (\tau\psi)(a_1a_2,a_3) - (\tau\psi)(a_1,a_2a_3) + (\tau\psi)(a_3a_1,a_2)$$

$$= \psi(a_1a_2,a_3) - \psi(a_1,a_2a_3) + \psi(a_3a_1,a_2)$$
$$- \psi(a_3,a_1a_2) + \psi(a_2a_3,a_1) - \psi(a_2,a_3a_1)$$

Thus it appears as if we should have $\tau =$ λ-symmetrization: [redacted]

$$(\tau\varphi)(a_1,\dots,a_n) = \varphi(a_1,\dots,a_n) + (-1)^{n-1}\varphi(a_2,\dots,a_n,a_1) + \dots + (-1)^{(n-1)^2}\varphi(a_n,a_1,\dots,a_{n-1})$$

in which case we have $\boxed{\tau d = -b\tau}$

But this should all be clear because we have $d = -b'$ on $C^\cdot(A,B)$, and because of the identity $Nb' = bN$ on cochains.

April 11, 1988

I need some details for the Kassel letter. Let $C = A \oplus A\otimes B\otimes A$ be the GNS algebra associated to ρ. C contains the idempotent

$e = 1\otimes 1\otimes 1$ and

$$eae = (1,1,1)a(1,1,1) = (1,1,1)\cdot(a,1,1) = (1,\rho(a),1)$$

$$e(a,b,a')e = (1,1,1)(a,b,a')(1,1,1) = (1,1,1)(a,b\rho(a'),1)$$
$$= (1,\rho(a)b\rho(a'),1).$$

so that $eCe = 1\otimes B\otimes 1 \simeq B$. We have

$$AeCeA = A\otimes B\otimes A$$

is an ideal in C. Thus

$$AeCeA = CeC$$

Let $\tilde{e} = 1-e$. Is it true that $A\tilde{e}C\tilde{e}A$ is an ideal in C? We have to check that it's closed under multiplication by $B = eCe$.

$$eCe\cdot A\tilde{e}C\tilde{e}A \subset \underbrace{eC\tilde{e}}\,C\tilde{e}A$$

$$e(A+AeCeA)\tilde{e} = eA\tilde{e} + \underbrace{eAeCe}_{eCe}eA\tilde{e}$$

$$= eCe\,eA\tilde{e}$$

It doesn't seem to work.