March 27, 1988

We have seen how to make

\[ C = A \oplus A \otimes B \otimes A \]

into an algebra given a linear map \( p : A \to B \)
such that \( p(1) = 1 \). There is a (unital)

homomorphism \( A \to C \) and a map \( \nu : B \to C \),
given by \( \nu(b) = 1 \otimes b \otimes 1 \), which satisfies

1) \( \\nu(b_1) \cdot \nu(b_2) = \nu(b_1 p(a) b_2) \)

This implies \( \nu(1) \) is an idempotent and that \( \nu \)
is a non-unital homomorphism.

Suppose that \( R \) is an algebra and that
we are given a homomorphism \( A \to R \) and
a map \( \nu : B \to R \) such that 1) is satisfied.
Then we have a unique homomorphism

\[ C \to R \]

\[ a + a \otimes b \otimes a' \mapsto a + a \nu(b) a' \]

Check:

\[ (x_1 \otimes y \otimes x_1') (x_2 \otimes y \otimes x_2') = (x_1 \nu(b_1) x_2') \otimes (x_2 \nu(b_2) x_2') \]

\[ (x_1 \otimes y \otimes x_1') (x_2 \otimes y \otimes x_2') = x_1 \nu(b_1 p(a) x_2') \otimes x_2' \]

Next let's examine 1) more closely. It

implies

2) \[ \nu(1) \cdot \nu(1) = \nu(p(a)) \]

and conversely if 2) hold, then

\[ \nu(b_1) \cdot \nu(b_2) = \nu(b_1) \nu(1) \cdot \nu(1) \cdot \nu(b_2) = \nu(b_1 p(a) b_2) \]

\[ = \nu(b_1) \nu(p(a)) \nu(b_2) = \nu(b_1 p(a) b_2) \]
So we can conclude that the algebra \( C = A \oplus A \otimes B \otimes A \) is universal among algebras \( R \) equipped with \( A \to R \) a homomorphism and \( v : B \to R \) satisfying 2).

Now let us take \( B \) to be the universal algebra equipped with a linear map \( f : A \to B \) such that \( f(1) = 1 \). Thus

\[
B = T(A) / T(A)(1 - 1_A) T(A) \approx T(L)
\]

where \( L \) is a subspace of \( A \) complementary to \( C \).

(Note that one can be slightly more intrinsic by saying that \( B \) has an increasing filtration \( F_p B = f(A)^p \) and that \( \text{gr}(B) = T(A) \).)

Let us now consider an algebra \( R \) equipped with a homom. \( A \to R \) and consider those maps \( v : B \to R \) satisfying 2). I claim that \( v \) is completely determined by the idempotent \( v(1) \) in \( R \) and that there is a 1-1 correspondence between the maps \( v \) satisfying 2) and idempotents in \( R \). In effect, given \( e \) in \( R \), the map \( a \mapsto eae \) is linear from \( A \) to the unital algebra \( e e e \) and satisfies \( f(1) = e \), hence there is a unique homomorphism \( B \to e e e \) carrying \( f \) to \( f' \). Then \( v : B \to R \) satisfies 2) and idempotents

\[
v(1) v(1) = eae = f'(a) = v(f(a)).
\]

Thus we see that when \( f : A \to B \) is universal, the algebra \( C = A \oplus A \otimes B \otimes A \) is just the free product \( A \ast C[\mathbb{Z}/2] \).
What this means is that all of the calculations I have been doing in the algebra $C$ for a general $\psi: A \to B$ are just the images of calculations in the free product $A \star C[2/2]$. This free product has been studied by Connes + Kunita.

March 28, 1988

I would like next to use what I have learned about the Steinspring game to construct the odd cyclic cocycles attached to an extension of $A$. Thus I start with an extension and choose a lifting

$$0 \to I \to R \overset{\psi}{\to} A \to 0$$

and I want a cyclic $(2n-1)$-cocycle on $A$ with values in $I^n/[I, I^n]$. By naturality I can suppose $R$ is the universal algebra $B = T(A)/(1 - 1_A)$ and that $I$ is the ideal generated by $\{ \psi(a_1) \psi(a_2) - \psi(a_1 a_2)^2 \}$ (better: the kernel of $B \to A$). I want to do the calculations in the Steinspring algebra

$$C = A \oplus A \otimes B \otimes A = A \star C[2/2]$$

I know that if $K$ is an ideal in $C$ containing $[F_a]$ for all $a \in A$, and if $\tau$ is a linear functional defined on $K^{2n}/[K, K^{2n-1}]$, then

$$\psi(a_0, \ldots, a_{2n-1}) = \tau(F[F_{a_0}] \cdots [F_{a_{2n-1}}])$$

is a cyclic cocycle.

I should go over this and see exactly what is needed, i.e. what are the minimal assumptions on
First of all we need $K$ to be a bimodule over $A$ and $\alpha \mapsto [F, \alpha]$ to be a derivation of $A$ with values in $K$.

Tate structure: Let $I_1, I_2$ be ideals in $R$ such that $R = I_1 + I_2$, and let

$$\tau : (I_1 \cap I_2) / [I_1, I_2] \to \mathbb{C}$$

be a linear functional. Then we can define an element of $\text{HC}^0(R)$ as follows.

We have the bimodule extension

$$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to R \to 0$$

and the class of this extension is an element of $\text{HC}^0(R, I_1 \cap I_2)$.

If $\tau$ is a linear map $I_1 \cap I_2 / [R, I_1 \cap I_2] \to \mathbb{C}$, then we would have a bimodule map

$$I_1 \cap I_2 \to R^*$$

and hence an element in $\text{HC}^0(R, R^*) = (\text{HC}^0(R, R))^*$

i.e. a Hochschild cohomology class. In dimension 1 we have from the Cohn's exact sequence

$$0 \to \text{HC}^0(R) \to \text{H}^1(R, R^*) \to \text{HC}^0(R) \xrightarrow{S}$$

so the real point is to see why the extra condition that $\tau([I_1, I_2]) = 0$ implies the Hochschild class is cyclic.

At this point we compute: Write $1 = \alpha + (1-\alpha)$ where $\alpha \in I_1$ and $1-\alpha \in I_2$. Then
the 1-cocycle describing the extension \( \eta \) is the derivation \( \eta \mapsto [r, (\alpha, 1-\alpha)] = ([\alpha, \alpha], -[\alpha, \alpha]) \) with values in \( I_1 \cap I_2 \). The corresponding Hochschild 1-cocycle is
\[
\varphi(r_0, r_1) = \tau(r_0 [r_1, \alpha])
\]
Changing \( \alpha \) by \( \beta \in I_1 \cap I_2 \) changes \( \varphi \) by
\[
\tau(r_0 [r_1, \beta]) = \tau(r_0 \beta r_1 - r_0 \beta r_1) = \tau([r_0, r_1] \beta)
\]
which is \( b \lambda \) where \( \lambda(\alpha) = \tau(r_0 \beta) \).
Note that cyclic and Hochschild 1-cocycles coincide. The issue now is why \( \varphi \) is cyclic, i.e., why
\[
\varphi(1, r) = \tau([\alpha \alpha]) = 0.
\]
Note that this is a trace on \( R \):
\[
0 = (b \varphi)(1, r_1, r_2) = \varphi(r_1, r_2) - \varphi(1, r_1 r_2) + \varphi(r_2, r_1)
\]
so \( \varphi(1, r_1 r_2) \) is symmetric in \( r_1, r_2 \).
So finally if we know \( \tau([I_1, I_2]) = 0 \), then we have
\[
\tau([\alpha \alpha]) = \tau([\alpha \alpha, \alpha] + [\alpha (1-\alpha), \alpha]) = 0.
\]
Actually this follows without the assumption \( \tau([I_1, I_2]) = 0 \).
Remark: From $I_1 + I_2 = R$ we can deduce

\[ I_2 \otimes I_1 + I_1 \otimes I_2 = I_1 \cap I_2. \]

In effect, let $I = \alpha + (1-\alpha)$ as above. If $r \in I_1 \cap I_2$, then

\[ r = r \alpha + r (1-\alpha) \]

\[ I_2 I_1 \quad I_1 I_2 \]

This suggests the following. Note that

\[ [\beta \lambda] = r \alpha - \alpha r = \alpha (1-\alpha) r \alpha - \alpha r (1-\alpha) \]

so we can define a map

\[ R \longrightarrow I_2 \otimes_R I_1 \oplus I_1 \otimes_R I_2 \]

\[ r \longmapsto (\alpha (1-\alpha) r \alpha, - \alpha r (1-\alpha)) \]

Unfortunately, I can't see if this is a derivation.

So we see how Tate's structure leads to a cyclic 1-dimensional class. I'd like to generalize this construction so as to obtain a Hochschild type construction for cyclic classes.
March 29, 1988

Let \( R = I_1 + I_2 \), where \( I_1, I_2 \) are ideals, and suppose \( e \) is an idempotent with \( eeI_1, 1-e \in I_2 \). We learned yesterday that

\[
I_1 \cap I_2 = I_1I_2 + I_2I_1
\]

because if \( x \in I_1 \cap I_2 \), then

\[
x = xe + x(1-e) \in I_1I_2 + I_2I_1.
\]

This means that

\[
I_1 \cap I_2 / [R, I_1 \cap I_2] \quad \text{and} \quad I_1 \cap I_2 / [I_1, I_2]
\]

are quotients of \( I_1 \otimes_R I_2 \oplus I_2 \otimes_R I_1 \), so the question arises as to whether we can refine the maps

\[
H_1(R, R) \rightarrow (I_1 \cap I_2) \otimes_R \quad \text{HC}_1(R) \rightarrow I_1 \cap I_2 / [I_1, I_2]
\]

discussed yesterday. In the same way that the Connes homomorphism \( HC_{2n-1}(A) \rightarrow I^n / [I_2, I_2^n] \)

is refined by a map \( HC_{2n-1}(A) \rightarrow (I \otimes_R)^n \).

Consider the derivation with values in \( I_1 \cap I_2 \):

\[
x \mapsto [x, e] = (1-e)x - xe(1-e)
\]

We can lift this map into \( I_1 \otimes_R I_2 \oplus I_2 \otimes_R I_1 \) by

\[
x \mapsto ((1-e)\otimes xe, -e\otimes(1-e))
\]

Let \( f(x) = (1-e)\otimes xe = (1-e)x \otimes e \)

Then \( f(x) \otimes e = f((1-e)x e) \). This shows we have nice liftings

\[
(1-e) \otimes e \rightarrow I_1 \otimes_R I_1, \quad e \otimes (-e) \rightarrow I_1 \otimes_R I_2.
\]
So if a derivation?

\[ f(xy) = (1-e) \otimes xy e = (1-e) \otimes [xy, e] + (1-e) \otimes [x] \otimes y \]

\[ xf(y) + f(x)y = x(1-e) \otimes y e + (1-e) \otimes x e y \]

\[ f(xy) = xf(y) + f(x)y + [x, e] \otimes [y, e] \]

Similarly set \( f'(x) = e \otimes x(1-e) \). Then

\[ f'(xy) = e \otimes xy(1-e) = [e, xy] \otimes (1-e) \]

\[ = [e, x] \otimes y(1-e) + x [e, y] \otimes (1-e) \]

\[ xf''(y) + f'(x) y = x e \otimes y(1-e) + [x, e] \otimes (1-e) y \]

\[ f'(xy) = xf'(y) + f'(x)y + [x, e] \otimes [y, e] \]

It appears that I missed something yesterday, namely, assume \( \tau \) is a linear ful. on \( (I_1 \cap I_2) \otimes_R \), then

\[ \tau([x, e]) = \tau((1-e)xe - ex(1-e)) \]

\[ = \tau((1-e)xe) - \tau(ex(1-e)) \]

\[ = \tau(e(x(1-e)x)) - \tau(ex(1-e)e) = 0 \]

This works even if \( e \) is replaced by any \( x \in I_1 \), \( 1-x \in I_2 \). (See below)

More generally let \( x \) us consider elements
\[ \bar{z} = xey \in I_1, \ \eta = y'(1-e)x' \in I_2. \] Then modulo \([R, I_1 \cap I_2]\) we have
\[ \bar{z} \eta = xeyy'(1-e)x' = x[e_yy'](1-e)x' \]
\[ \equiv (1-e)x'x[e_yy'] \]
\[ = (1-e)x'xeyy' + (1-e)x'xyy'ee \]
\[ \equiv y'(1-e)x'xey + e(1-e)x'xyy'ee = \eta \bar{z} \]

This shows in the case \(I_1 = R(eR, I_2 = R((1-e)e)\) that
\([I_1, I_2] = [R, I_1 \cap I_2]\).

Suppose we now try to define a cyclic cocycle with values in \(I_2 \otimes_R I_1 \otimes_R\) by
\[ \varphi(r_0, r_1) = r_0(1-e) \otimes r_1 e - r_1(1-e) \otimes r_0 e. \]
\[ = r_0 f(r_1) - r_1 f(r_0). \]
\[ \varphi(r_0, r_1, r_2) = r_0 r_1 f(r_2) - r_2 f(r_0, r_1) \]
\[ + \varphi(r_1, r_2, r_0) \]
\[ + \varphi(r_2, r_0, r_1) \]
\[ + \mbox{cyc.} \]

\[ (b\varphi)(r_0, r_1, r_2) = r_0 r_1 f(r_2) - \quad ? \]

Let us start again with an algebra \(R\) and an idempotent in it \(e\). Let \(I_1 = ReR\) and \(I_2 = R((1-e)e)\). Then
\[ I_1 = (eRe, eR(1-e)) \]
\[ I_2 = (eR(1-e)Re, eR((1-e)e)) \]
\[ I_1 I_2 = \begin{pmatrix} \text{eR}(1-e)\text{Re} & \text{eR}(1-e) \\ (1-e)\text{Re} \text{R}(1-e) & (1-e)\text{Re} \text{R}(1-e) \end{pmatrix} \]

\[ I_2 I_1 = \begin{pmatrix} \text{eR}(1-e)\text{Re} & \text{eR}(1-e)\text{Re} \text{R}(1-e) \\ (1-e)\text{Re} & (1-e)\text{Re} \text{R}(1-e) \end{pmatrix} \]

This formulas follow taking the block decomposition and using the identities \( \text{Re} \text{Re} = \text{Re} \), \( \text{eR} \text{eR} = \text{eR} \) and similarly for \( 1-e \). Thus \( (1-e)I_1 e = (1-e)\text{Re} \text{Re} = (1-e)\text{Re} \).

The mystery to be solved is the following.

Given ideals \( I_1, I_2 \) in \( R \) such that \( I_1 + I_2 = R \), the bi-module extension

\[ 0 \rightarrow I_1 \cap I_2 \rightarrow I_1 \oplus I_2 \rightarrow R \rightarrow 0 \]

determines a Hochschild class:

\[ H_1(R, R) \rightarrow (I_1 \cap I_2) \otimes_R \]

which turns out to be cyclic, i.e., to factor through \( H_1(R) \). This is false for a general bi-module extension, e.g., if we take

\[ 0 \rightarrow \Omega^1_R \rightarrow R \otimes_R R \rightarrow R \rightarrow 0 \]

then we get the map

\[ H_1(R, R) \rightarrow \Omega^1_R \otimes_R . \]

The composition with \( H_0(R) \rightarrow H_1(R, R) \) is the map \( S: H_0(R) \rightarrow \Omega^1_R \otimes_R \) which is usually non-zero.
So the problem is to learn how this ideal structure leads to a cyclic cocycle. Actually I should first check carefully in the general case that if \( a \in I_1, 1-a \in I_2 \), then \([r, a] \in \mathbb{R}[I_1, I_2].\) But

\[
[r, a] = (1-a)r - a[r, 1-a] \\
= (1-a)r_1 x + (1-a)r_1(1-a) - a[r, (1-a)^2] - (1-a)x[r, 1-a] \\
\equiv a x(1-a)r + a(1-a)^2 r - a[r, (1-a)^2] - (1-a)^2 x r \\
\equiv a^2 (1-a)r - (1-a)^2 x r \equiv 0.
\]

So our problem is to work out a bimodule style construction of the cyclic cocycle. Let's try the formula

\[
\psi(r_0, r_1) \equiv T \left( FR_0 FR_1 \right) \left[ I_1, I_2 \right] \left( \frac{1}{4} \right)
\]

We have to understand where the action is taking place. Let's figure out what we need to get a cyclic cocycle. First of all, we look at

\[
\frac{1}{2} [F, r_0] = [e, r_0] = er_0 (-e) - (1-e)r_0 e
\]

This is the sum of the two off-diagonal parts. A first idea would be to work with the off-diagonal blocks \( eR(1-e) \) and \((1-e)Re \) as bimodules for \( eRe \) and \((1-e)Re \).

This raises the question as to what extent \( R \) is Morita equivalent to \( eRe \) or maybe \( eRe \oplus (1-e)Re \)? It is not Morita equivalent to the direct sum, e.g. take \( R = M_2(\mathbb{C}) \), but I have the feeling of being in the presence of a partition of unity.
Let's now try to find the best target space for the 1-cocycle
\[ \varphi(x,y) = tr(F[\frac{1}{2} F_x \frac{1}{2} F_y]) \]
\[ tr F[e x][e y] = tr \begin{pmatrix} 1 & 0 \\ 0 & (1-e)x e & 0 \end{pmatrix} \begin{pmatrix} 0 & ey(1-e) \\ (1-e)x e & 0 \end{pmatrix} = (1-e)x e y(1-e) - ey(1-e)y \]
To simplify the writing put \( e = 1-e \). Let's try first putting
\[ \varphi(x,y) = (\overline{e} x e \otimes ey \bar{e} \bar{x}) - (\bar{e} x e \otimes \overline{e} y \bar{e} \bar{x}) \]
\[ \in \overline{e} Re \otimes e \overline{e} \bar{e} \bar{x} \otimes \overline{e} Re \otimes e \overline{e} \bar{e} \bar{x} \]
We want \( \varphi(x,y) = - \varphi(y,x) \), this means we want to add the two factors after interchanging the factors of the second summand. Thus we try
\[ \varphi(x,y) = (\overline{e} x e \otimes ey \bar{e} \bar{x}) - (\bar{e} x e \otimes \overline{e} y \bar{e} \bar{x}) \]
\[ \in \overline{e} Re \otimes e \overline{e} \bar{e} \bar{x} \otimes \overline{e} Re \otimes e \overline{e} \bar{e} \bar{x} \]
Let's see if this is a cyclic cocycle
\[ \varphi(x,y,\bar{e}) = (\overline{e} x e \otimes ez \bar{e} \bar{x}) - (ez \bar{e} \otimes ey \bar{e} \bar{x}) \]
\[ = [\overline{e} x e y + \overline{e} x e y \bar{e}] \otimes ez \bar{e} \bar{x} - ez \bar{e} \otimes [\overline{e} x e y + \overline{e} x e y \bar{e}] \]
\[ - \varphi(xy,\bar{e}) = -\overline{e} x e \otimes ey \bar{e} \bar{x} + \overline{e} y e \otimes ez \bar{e} \bar{x} \]
\[ = [\overline{e} x e y + \overline{e} x e y \bar{e}] \otimes ez \bar{e} \bar{x} - [\overline{e} y e + \overline{e} y e \bar{e}] \otimes ez \bar{e} \bar{x} \]
\[ \varphi(x,y) = \overline{e} x e \otimes ey \bar{e} \bar{x} - ey \otimes ez \bar{e} \bar{x} \]
\[ = [\overline{e} x e + ez \bar{e} e] \otimes ey \bar{e} \bar{x} - ey \otimes [\overline{e} x e + ez \bar{e} e] \]

Thus it works and we have a cyclic cocycle on \( R \) with values in \( \overline{e} Re \otimes e Re \otimes e Re \otimes e Re \).
March 30, 1988

Yesterday, I considered an algebra $R$ with idempotent $e$, and I constructed a cyclic 1-cycle with values in

1) \[(1-e)R \otimes_{eR} eR(1-e) \otimes_{eR} (1-e)R(1-e)\]

We can rewrite this space as follows. First note that $eR$ is a projective $R$-module and that for any $R$-module $M$ we have

\[eR \otimes_R M \cong eM\]

and similarly for $(1-e)R$. Thus we have

\[eR \otimes_R R(1-e) = eR(1-e)\]

\[(1-e)R \otimes_R R = (1-e)R\]

whence 1) becomes isomorphic to

2) \[(R \otimes_{eR} eR) \otimes_R (R(1-e) \otimes_{(1-e)R} (1-e)R) \otimes_R\]

We have surjections

3) \[R \otimes_{eR} eR \longrightarrow \bigodot \bigodot \ ReR = I_1\]

\[R \otimes_{eR} eR \longrightarrow \bigodot \bigodot \ R \bigodot \bigodot R = I_2\]

so 2) maps onto $I_1 \otimes_{R} I_2 \otimes_{R}$. Thus we obtain a cyclic 1-cycle with values in $I_1 \otimes_{R} I_2 \otimes_{R}$.

Next, let's examine the surjection 3) which in block form appears

\[
\begin{pmatrix}
(eRe) & (eRe \ eR(1-e)) \\
((1-e)Re) & (eR(1-e))
\end{pmatrix}
\]
Thus we see that the kernel of $3)$ is the same as the kernel of

$$(1-e)Re \otimes eR \xrightarrow{eRe} (1-e)ReR(1-e)$$

Recall that we encountered this map before in the study of the algebra $C = A + A(Bi^*A)$, see pp. 653–654. There we wanted to extend a trace defined on the ideal $I = eR(1-e)Re = eI_2e$ in $eRe$, which vanished on $[eRe, I]$, to a trace on $K = I_1 \cap I_2$ vanishing on $[I_1, I_2]$.

Conclude: The good things to work with are the bimodules

$$\tilde{I}_1 = Re \otimes eRe$$
$$\tilde{I}_2 = R(1-e) \otimes (1-e)R(1-e)$$

instead of the ideals $I_1 = ReR$, $I_2 = R(1-e)R$. The standard cyclic $1$-cocycle has values in the space $\tilde{I}_1 \otimes_R \tilde{I}_2 \otimes_R$

Questions: $\tilde{I}_1 = I_1$ in the universal case $R = A \times C[\mathbb{Z}/2]$? Higher cocycles? Morita aspects of these bimodules?

Let's consider the Morita aspects, especially the links between the categories of modules over the rings $Re$, $eRe$, and $(1-e)R(1-e)$ defined by the various bimodules. We have functors:

$$i^* = eRe$$
$$j^* = eR$$

Diagram:

$$\text{Mod}(eRe) \xrightarrow{i^* = eRe} \text{Mod}(R) \xrightarrow{j^* = eR} \text{Mod}(eRe)$$

$\tilde{j} = eR(1-e)$
We have
\[ i^* i = id \quad f^* f = id \]
\[ i^* r^* = (R e \otimes e R) \otimes_R ? = i_1 \otimes_R ? \]
\[ f^* f^* = (R e \otimes e R) \otimes_R ? = i_2 \otimes_R ? \]

The value group for the cyclic 1-cocycle is the fixed point space for any of the functors
\[ i^* f^* i, \quad f^* i^* f, \quad i^* (i^* f), \quad f^* (f^* i) \]

\[ \text{Observation:} \quad \tilde{i}_1 = R e \otimes e e R \text{ is an } R\text{-bimodule with a bimodule map}\]
\[ \begin{array}{c}
R e \otimes e e R \\
\xrightarrow{\partial}
\end{array} \rightarrow R \]

I claim this forms a DG algebra, i.e.
\[ \partial(\xi) \eta = \xi \partial(\eta) \]

Check: Let \( \xi = r_1 e \otimes e r_2 \), \( \eta = r_3 e \otimes e r_4 \).
Then
\[ \partial(\xi) \eta = r_1 e r_2 (r_3 e \otimes e r_4) = r_1 e (e r_2 r_3 e) \otimes e r_4 \]
\[ \xi \partial(\eta) = (r_1 e \otimes e r_2) e r_4 = r_1 e (e r_2 e r_3 e) e r_4 \]
and these are equal in \( \tilde{i}_1 \).

Now we can form the free product
\[ (\tilde{i}_1 \rightarrow R) \star_R (\tilde{i}_2 \rightarrow R) \]
of DGA's. This gives a DGA which in degree $n$ will be

$$
\left( \tilde{I}_1 \otimes R \tilde{I}_2 \otimes R \tilde{I}_3 \otimes R \cdots \right) \oplus \left( \tilde{I}_2 \otimes R \tilde{I}_1 \otimes R \tilde{I}_3 \otimes R \cdots \right)
$$

$n$-factors $n$-factors.

Recall that $(\tilde{I}_1)^2 = 0$ in the DGA $(\tilde{I}_1 \rightarrow R)$. This implies that for $\tilde{e}_1, \tilde{e}_3, \ldots \in \tilde{I}_1$, and $\tilde{e}_2, \tilde{e}_4, \ldots \in \tilde{I}_2$

$$
\partial(\tilde{e}_1, \tilde{e}_2, \ldots, \tilde{e}_n) = (\partial(\tilde{e}_1) \tilde{e}_2, \ldots, \tilde{e}_n) + (-1)^{n-1}(\tilde{e}_1, \ldots, \tilde{e}_{n-1} \partial(\tilde{e}_n))
$$

By our assumption that $\tilde{I}_1 + \tilde{I}_2 = R$ it follows that $H_0$ of this amalgamated product is zero, and hence the DGA has zero homology. So it gives a bimodule resolution of $R$:

$$
\cdots \rightarrow \tilde{I}_1 \otimes \tilde{I}_2 \otimes \tilde{I}_3 \otimes R \rightarrow \tilde{I}_1 \otimes \tilde{I}_2 \rightarrow R \rightarrow 0
$$

Suppose $\tilde{I}$ is an ideal in $R$ such that $I^2 = I$, for example $I = ReR$ where $e$ is an idempotent. Does there exist a universal $R$-bimodule extension

$$
0 \rightarrow M \rightarrow \tilde{I} \otimes \tilde{I}_1 \otimes \tilde{I}_2 \otimes \tilde{I}_3 \otimes R \rightarrow I \rightarrow 0
$$

such that $u(x) y = x u(y)$ for all $x, y \in \tilde{I}$?

Following Prüfer, we might ask whether such extensions form a cofibrant category. Notice that

$$
I \cdot M = u(\tilde{I}) M = \tilde{I} \cdot u(M) = 0
$$

$$
M \cdot I = M u(\tilde{I}) = u(M) \tilde{I} = 0
$$
so that $M$ is an $R/I$-bimodule.

Next consider a pushout situation

$$
0 \rightarrow M \rightarrow E \xrightarrow{u} I \rightarrow 0
$$

$$
0 \rightarrow M' \rightarrow E' \xrightarrow{u'} I \rightarrow 0
$$

Let $x', y' \in E'$. Then $x' = x + m'$, $y' = y + m'_2$

for $m' \in M'$ and $x, y \in E$.

$$
u'(x')y' = u(x)(y + m'_2) = u(x)y$$

as $Im'_2 = 0$

$$
x'v'(y') = (x + m')(u(y) = xu(y)$$

as $m'_2I = 0$

Thus $E'$ is in the same class of extensions. Next note that two maps $E \rightarrow E'$ inducing the same map $M \rightarrow M'$ differ by an element of

$$
\text{Hom}_{R \otimes R^0}(I, M') = \text{Hom}_{R/I \otimes R/I^0}(I/I^2, M')
$$

As $I/I^2 = 0$, there is at most one map of extensions consistent with a given $R/I$-bimodule map $M \rightarrow M'$.

Next we analyze an extension of the above type by choosing a linear section $s$ of $u$.

$$
0 \rightarrow M \rightarrow E \xrightarrow{u} I \rightarrow 0.
$$

We then get a bilinear map $I \times I \rightarrow M$ given by

$$
f(x, y) = xs(y) - s(xy)
$$

For $r \in R$ we have

$$
f(x, yr) = xrs(y) - s(xry)
$$

$$
f(xr, y) = xs(ry) - s(xry)
$$

$$
f(xry) - f(x, ry) = x \frac{rs(y) - s(ry)}{I. M} = 0
$$

Note

$$
xs(y) = u(s(x))s(y) = s(x)u(s(y)) = s(x)y.
$$
where \( \varphi(x,y) = x s(y) \). \( \therefore \varphi = i f + s b' \)

In other words define \( \varphi \) by this formula. Then

\[
\varphi(xr, y) = x r s(y) \\
\varphi(x, ry) = x s(ry)
\]

and these are equal as \( x e I \) and \( rs(y) - s(ry) \in M \).

Also \( \varphi(x, y) = x s(y) \) shows \( \varphi(rx, y) = r \varphi(x, y) \), whereas \( \varphi(x, y) = s(x) y \) shows \( \varphi(x, yr) = \varphi(x, y)r \). Thus \( \varphi \) is a bimodule map.

Let's check that the extension defined by \( I \otimes_R I \)

is in our class:

\[
(x \otimes y) \cdot b'(z \otimes w) = x \otimes y z \otimes w \\
b'(x \otimes y) \cdot (z \otimes w) = x y z \otimes w
\]

these are in \( I \otimes_R I \).

Next note that changing \( s \) by \( h: I \rightarrow M \)

changes \( \varphi(x, y) \) by \( x h(y) - h(xy) = -h(xy) \), so the map \( K Rb' \rightarrow M \) in deduced by \( \varphi \) is independent of the choice of \( s \).

Actually you seem to be proceeding stupidly.

The point is given \( E \xrightarrow{i} I \) with \( u(x)y = x u(y) \)

for all \( x, y \in E \), one can define a unique bimodule map

\[
I \otimes_R I \xrightarrow{\varphi} E
\]

over \( I \), namely
set \( \varphi(xy, y) = x \tilde{y} \) or \( \tilde{x}y \).

where \( u(y) = y \) or \( u(x) = x \). Uniqueness follows because two choices differ by a bimodule map \( I \otimes_R I \to M \) which must be zero as \( I^2 = I \) and \( IM = 0 \).

Conclusion: If \( I \) is an ideal in \( R \) such that \( I^2 = I \), then

\[
I \otimes_R I \xrightarrow{b'} I
\]

is the universal bimodule extension \( E \) of \( I \) such that the map \( E \xrightarrow{\varphi} I \) satisfies \( \varphi(x, y) = xu(y) \).

Let's examine Wodzicki's homologically unital condition in a slightly more general situation. Let \( I \) be an ideal in \( R \); Wodzicki takes \( R = I^+ = C \otimes I \). Then his HU. condition bears the analogue

1) \( \text{Tor}^R_1(R/I, R/I) = 0 \)

Since

\[
\text{Tor}^R_1(R/I, R/I) = I/I^2
\]

\[
\text{Tor}^R_2(R/I, R/I) = \ker\{I \otimes_R I \to I^2\}
\]

\[
\text{Tor}^R_{q+2}(R/I, R/I) = \text{Tor}^R_q(I, I) \quad q > 0
\]

we see 1) is equivalent to

1') \( \begin{cases} I \otimes_R I \xrightarrow{\mu} I \\ \text{Tor}^R_+(I, I) = 0 \end{cases} \)

or 1") \( I \otimes_R I \xrightarrow{\mu} I \)

If \( I \) has this property, then \( I \otimes_R \cdots \otimes_R I \xrightarrow{\mu} I \), and

\[
(I \otimes_R \cdots \otimes_R I)^n \to I \otimes_R \cdots \otimes_R I
\]
One can speculate then about the spectral sequences associated to the extension. Assuming the cyclic group actions to be trivial, we find the spectral sequences have the Hochschild $H_x(R, I)$ or $H_x(R, R/I)$ in the columns. This suggests there should be a long exact sequence related $HC_x(R, I)$ to $H_x(R, I)$. This would be reasonable provided

$$H_x(R, R/I) = H_x(R/I, R/I)$$

which I ought to be able to check.

Multiply's algebra. Let $I$ be a non-unital algebra. We want to embed $I$ as an ideal in a larger algebra called $R$. Then each $r$ will determine a left multiplication $\lambda_r$ on $I$ and a right multiplication $\rho_r$. $\lambda_r$ commutes with $\rho_x$ for $x \in I$ and $\rho_r$ commutes with $\lambda_x$ for $x \in I$. Thus we have a homomorphism

$$R \rightarrow \text{End}_0(I) \times \text{End}_0(I)$$

$$r \mapsto (\lambda_r, \rho_r)$$

The first candidate is to take $M(I) = I$, that is, to consist of pairs $(u, v)$ of operators on $I$ satisfying

1) \[ u(xy) = u(x)y, \quad x, y \in I \]

\[ v(xy) = x v(y) \]

Now the condition that $u, v$ should commute turns out to be too strong a requirement as we saw when $I^2 = 0$. However if $I^2 = I$, then any $u \in \text{End}_0(I)$ commutes with any $v \in \text{End}_0(I)$ since

\[ u(v(xy)) = u(xv(y)) = u(x)v(y) \]

\[ v(u(xy)) = v(u(x)y) = u(x)v(y) \]
so 1) implies the commutativity of left + right multipliers when $I^2 = I$.

Next we map $I$ to $\text{End}_R(I) \times \text{End}_I(I)$ by $x \mapsto (p_x, s_x)$; this is a homomorphism with the product in the latter being $(u, v)(u', v') = (uu', vv')$.

We want the image to be an ideal:

$$(u, v)(p_x, s_x) = (u p_x, s_x)$$

$$(u, v)(y) = u(y) = u(x)y = s(x)v(y)$$

$$(s_x, v)(y) = v(y)x$$

At this point we need a further condition, which we can find by thinking of $u, v$ as left and right multiplication by $x$. Thus

$$v(y)x = yx = yu(x)$$

and so we add the condition

2) $v(y)x = yu(x)$ for $x, y \in I$

to define the multiplier algebra $M(I)$. Check:

$$(v', v)(y) = v'(v(y))x = v(y)u(x) = yu(x)$$

Then

$$(s_x, v)(y) = v(y)x = yu(x) = s(x)v(y)$$

so

$$(u, v)(p_x, s_x) = (u, v)(x)$$

showing the image of $I$ in $M(I)$ is a left ideal. Similarly it is a right ideal:

$$(p_x, s_x)(y) = p(x)y = s(x)v(y)$$

$$(v, s_x)(y) = v(y)x = yv(x) = s(x)v(y)$$

1) $$(p_x, s_x)(u, v) = (p_x, s_x)(u, v)$$
March 31, 1988

Let's consider two ideals $I, J$ in $R$ such that $I + J = R$. Then $I$ can form the DGA

$$(I \rightarrow R) \ast (J \rightarrow R)$$

In degree $n$, it is the direct sum

$$I \otimes_R J \otimes_R I \otimes_R \cdots \otimes_R J \otimes_R I \otimes_R J \otimes_R \cdots$$

with $n$ factors in each tensor product, and so it appears

1) \[ \cdots \rightarrow I \otimes_R J \otimes_R I \xrightarrow{d} I \otimes J \xrightarrow{d} R \rightarrow 0 \]

The differential is

$$d(\chi_1, \ldots, \chi_n) = (\chi_1 \chi_2, \ldots, \chi_n) \oplus (-1)^{n-1}(\chi_1 \cdots \chi_{n-1}, \chi_n)$$

where $\chi_1, \ldots, \chi_n$ is a sequence of elements alternately from $I$ and $J$. (Note $I \cdot I = J \cdot J$ in these DGA's)

We've seen this DGA is acyclic because the homology is a unital algebra in which $1 = 0$. Specifically if $f = \chi \oplus (1-\chi) \in I \oplus J$ is an elt. with $df = f$, then we obtain a homotopy operator which is an $R$-module morphism

$$h(\omega) = f \cdot \omega$$

Check: \[ d(h(\omega)) = d(f \cdot \omega) = df \cdot \omega \oplus f \cdot dw = \omega - h(d\omega) \]

Unfortunately $h^2$ is multiplication by

$$f^2 = \chi \otimes (1-\chi) \oplus (1-\chi) \otimes \chi \neq 0$$
so we can't use our E formulas without modification.

In order to use the resolution 1) to obtain Hochschild cohomology classes we need to find linear functionals \( \tau \) on the complex 1) divided by its commutators with \( R \) such that \( \tau d = 0 \).

In degree 1 we have to find a linear functional on \( (I/[R,I]) \oplus (J/[R,J]) \) vanishing on the images of pairs

\[ xy \oplus (-yx) \quad x \in I, \; y \in J. \]

But actually by right exactness of \( \otimes_R \) one sees the only possibilities in degree 1 are traces on \( R \).

In degree 3 we want a linear functional on \( I \otimes_R J \otimes_R I \otimes_R I \oplus J \otimes_R I \otimes_R J \otimes_R J \) which vanishes on

\[
d(x_1, y_1, x_2, y_2) = \varepsilon(1)(x_1 y_1 x_2 y_2) \oplus (x_1 y_1 x_2, y_2) \]

\[
d(y_1, x_1, y_2, x_2) = \varepsilon(1)(y_1 x_1 y_2 x_2) \oplus (y_1 x_1 y_2, x_2) \]

The problem is to find a group to which both factors \( I \otimes_J I \otimes_R J \otimes_R I \otimes_R I \) and \( J \otimes_R I \otimes_R J \otimes_R J \) can be mapped. Otherwise you are stuck with

\[
\frac{I \otimes_J I \otimes_R J \otimes_R I}{I \otimes_J I \otimes_R (J \otimes_R (I \otimes_J I)) \otimes_R I} \]

and similarly with \( J, I \) interchanged. It's
not clear whether this gives anything interesting.

So let's consider the case where the degree is even. We then have to divide

\[(I \otimes_R J \otimes_R)^n \oplus (J \otimes_R I \otimes_R)^n\]

by two relations:

\[d(y_0, x_1, y_1, \ldots, x_n, y_n) = (y_0, x_1, y_1, \ldots, y_n, x_n) \oplus (y_0, \ldots, x_n, y_n)\]

\[d(x_0, y_1, x_1, \ldots, y_n, x_n) = (x_0, y_1, x_1, \ldots, x_n, y_n) \oplus (x_0, x_1, y_1, x_1, \ldots, y_n, x_n)\]

If we identify \((y_0, x_1, \ldots, y_n, x_n) \equiv (x_1, \ldots, x_n, y_1)\), then the first relation becomes

\[\equiv (y_0, x_1, y_1, \ldots, x_n, y_n) - (x_0, y_1, \ldots, y_n, x_n, y_0)\]

which is zero in \((I \otimes_R J \otimes_R)^n\). The second relation becomes

\[\equiv (x_0, y_1, x_1, \ldots, y_n, x_n) - (x_1, \ldots, x_{n-1}, y_n, x_n, x_0, y_1)\]

\[\equiv (x_n, x_0, y_1, x_1, \ldots, y_n) - (x_1, y_2, \ldots, x_{n-1}, y_n, x_n, x_0, y_1)\]

and so for this to vanish we must divide out by 0 on \((I \otimes_R J \otimes_R)^n\).

**Conclusion:** There exists a Hochschild cocycle class of degree 2n-1 with values in

\[(I \otimes_R J \otimes_R)^n\]

**Alternative method:** Let us consider the DGA 1) and take its commutator for quotient. I claim this kills all odd degrees.
except degree 1. In effect, consider $I \otimes_R (J \otimes_R I \otimes_R)^n$. A typical element 

$$(x_0, y_1, x_1, \ldots, y_n, x_n)$$

here is the bracket

$$x_0 \cdot (y_1, x_1, \ldots, y_n, x_n) - (y_1, \ldots, x_n) \cdot x_0$$

as the latter term is zero.

Consider even degrees. First take degree 2. Then we must divide out $I \otimes_R J \otimes_R \oplus J \otimes_R I \otimes_R$ by elements $x \cdot y + y \cdot x = (x, y) + (y, x)$.

This leads to the quotient $I \otimes_R J \otimes_R$.

In general to get the commutator quotient one divide out with brackets where one of the elements comes from a set of generators for the algebra. Thus we have to divide out by brackets coming from $R, I, J$. This means that we work with cyclic tensor products $I \otimes_R J \otimes_R \ldots \otimes_R R$, and we have to be able to move the factors around with the appropriate signs.

**Conclusion:** Let $R = (I \to R)^R \otimes (J \to R)$. Then $R/[[R, R]]$ is

$$
\begin{array}{cccccc}
I \otimes_R J \otimes_R & \rightarrow & O & \rightarrow & I \otimes_R J \otimes_R & \rightarrow & I/[[R, R]] \oplus J/[[R, R]] & \rightarrow & R/[[R, R]]
\end{array}
$$

Now let us apply the functor $CC$ to the DGA $R$. This will give us a double chain complex with acyclic rows since $R$ is a complex is acyclic. Hence the columns of positive degree gives a resolution of the cyclic complex $CC(R)$. Looking at the bottom edge
of this double complex gives us canonical maps
\[ \mathcal{C}_{2n}^2(R) \longrightarrow (I \otimes J \otimes R)^n \quad n > 1 \]
which are the edge homomorphisms of the associated spectral sequence
\[ \begin{array}{c}
\begin{array}{c}
\downarrow \\
R^2 \leftarrow R^2 \otimes (I \otimes J) \\
\downarrow \\
R^1 \leftarrow R \otimes (I \otimes J) \\
\downarrow \\
R \leftarrow I \otimes J \\
\end{array}
\end{array} \]
It's clear that the first column is \((I \otimes J) \otimes R\)

It follows by diagram chasing that
\[ \mathcal{C}_{1}^1(R) \longrightarrow I \otimes J \otimes R \longrightarrow (I \otimes J) \otimes R \longrightarrow R \otimes R \longrightarrow \]
\[\mathcal{C}_{1}^2(R) \quad (xy) \otimes (yx) \]

is exact. An interesting question is how much one is seeing of \(\mathcal{C}_{1}^1(R)\) in \(I \otimes J \otimes R\). Note we have
\[ I \otimes J \otimes R \longrightarrow (I \otimes J) \otimes R \]
\[ \downarrow \\
(\mathcal{C}_{1}^1(R) \otimes R) \quad (\otimes -1) \rightarrow (I \otimes J) \otimes R \]
\[ \text{commutes} \]
\[ \mathcal{C}_{1}^1(R/\mathcal{C}_{1}^1(R)) \quad \mathcal{C}_{1}^1(R/\mathcal{C}_{1}^1(R)) \oplus \mathcal{C}_{1}^1(R/\mathcal{C}_{1}^1(R)) \]
so it would seem that \(\otimes\) is not consistent with the Eilenberg homomorphism for \(R/\mathcal{C}_{1}^1(R)\).
Variants: We can replace the ideals $I$ in $R$ by a bimodule map $u: \tilde{I} \rightarrow R$

satisfying $u(xy) = xu(y)$ for $x, y \in \tilde{I}$, because this gives us a DGA:

$\cdots \rightarrow 0 \rightarrow \tilde{I} \rightarrow R$

An example of such an $\tilde{I}$ is

$$\tilde{I} = I \otimes_{R} \cdots \otimes_{R} I$$

with $u$ the multiplication map. Note that if $\xi = x_1 \otimes \cdots \otimes x_n \quad \eta = y_1 \otimes \cdots \otimes y_n$, then

$$u(\xi) \eta = x_1 \cdots x_n y_1 \otimes \cdots \otimes y_n$$

$$= x_1 \otimes x_2 \cdots x_n y_1 y_2 \otimes \cdots$$

$$= x_1 \otimes x_2 \otimes x_3 \cdots x_n y_1 y_2 y_3 \otimes \cdots$$

$$\cdots$$

$$= x_1 \otimes x_2 \otimes \cdots \otimes x_n y_1 y_2 \cdots y_n$$

$$= \xi \otimes u(\eta)$$

Notice if $I + J = R$, then $I^n + J^k = R$ for $n, k \geq 0$, since

$$R = (I + J)^{2n-1} \subseteq I^n + J^n$$

(Better: $1 = (x + 1 - x)^{2n-1} = \sum_{k+l=2n-1} x^k (1-x)^l$, and one of $k, l$ must be $\geq n$)

So we can start with $I, J$ replace them by $I \otimes_{R} \cdots \otimes_{R} I$, $J \otimes_{R} \cdots \otimes_{R} J$ and proceed to define

$$HC_{2n-1}(R) \rightarrow (I \otimes_{R} \cdots \otimes_{R} I \otimes_{R} J \otimes_{R} \cdots \otimes_{R} J)^n$$
Discussion: We have found another way to construct cyclic cocycles using the cyclic complex of DGA's. But we still don't have any better understanding of what the cyclic complex \( \text{CC}(A) \) means!

However there are two important features of the present example worth thinking about. First of all instead of a DGA resolution of the algebra \( R \) one actually uses an acyclic DGA with \( R \) in degree zero. A similar thing was done with the DGA \( R \to R \)

and leads to maps

\[ \text{HC}_{2n}(R) \to \text{HC}_0(R) \]

which ought to be the iterated S-homomorphisms.

A second feature is the fact that our cyclic cocycles come from traces on the DGA.

Let's consider this carefully. Suppose we have a DGA \( R \).

\[ \cdots \to R_3 \to R_2 \to R_1 \to R_0 \]

which resolves an algebra \( A \). Then we can pass to non-unital rings and replace it by an acyclic DGA where \( R_0 \) is the ideal \( I = \text{Ker}(R_0 \to A) \). It appears that we have two acyclic complexes. On one hand we have

\[ \text{CC}(- \to R_2 \to R_1 \to I) \]

whose rows are acyclic + which gives a resolution of \( \text{CC}(I) \). On the other hand we have

\[ \text{CC}(- \to R_2 \to R_1 \to R_0) \]
which gives a resolution of \[ CC(A) \].

As far as traces on the \( DGA \) is concerned these are already different because
\[ R_n/[I,R_n] \neq R_n/[R_0,R_n] \).

It seems that the good setting is to assume
\( R_\bullet \) is unital and resolves \( A \). Then we can think of \[ CC(R_\bullet) \] as resolving either \( CC(A) \) or
think of its positive degree columns as we can think of its positive degree columns as
resolving the relative complex \( \text{Ker}\{CC(R_\bullet) \to CC(A)\} \).

April 1, 1988

Let \( I \) be an ideal in \( R \) such that \( I^2 = I \).

Let \( M \xrightarrow{\mu} I \) be a bimodule map such that
\( u(x)y = u(y)x \). The kernel \( K = \text{Ker}(\mu) \) is then killed by \( I \). Since \( I^2 = I \), we conclude that
\( K \otimes_R I = I \otimes_R K = 0 \). Hence

\[
\begin{array}{ccc}
0 & \longrightarrow & M \otimes_R I \\
& \downarrow \text{mult} & \text{id} \\
& M & \longrightarrow \\
\end{array}
\]

Thus \( \varphi(x \otimes y) = \bar{x}y \) where \( \bar{x} \in u^{-1}(x) \) is a bimodule map from \( I \otimes_R I \) to \( M \). Note
\( \bar{x}y = \bar{x}u(y) = u(x)y = xy \).

Next note that \( I \otimes_R I \xrightarrow{\text{mult}} I \) is one of our extensions. Moreover
\[
\begin{array}{ccc}
(I \otimes_R I) \otimes_R I & \xrightarrow{\mu \otimes 1} & I \otimes_R I \\
& \downarrow \text{id} & \mu \otimes \text{id} \\
I \otimes_R I & \rightarrow & I \otimes_R I \\
\end{array}
\]

commutes: \( xy \otimes z = x \otimes yz \).
It's still not very clear. Suppose I start with a bimodule surjection $M \twoheadrightarrow I$ in our class. I claim there is a unique map $\varphi : I \otimes_R I \rightarrow M$ compatible with the maps to $I$. Existence:

\[
\begin{align*}
M \otimes_R I & \twoheadrightarrow I \otimes_R I \\
\downarrow & \\
0 & \rightarrow K & M & \rightarrow I & \rightarrow 0
\end{align*}
\]

Uniqueness: Since $I(I \otimes_R I) = I \otimes_R I$ and $IK = KI = 0$, there can be no bimodule maps $I \otimes_R I \rightarrow K$.

This proves the universal character of $I \otimes_R I$, and on the other hand the above diagram shows that the map $\varphi : I \otimes_R I \rightarrow M$ is an isomorphism $\Rightarrow M \otimes_R I \cong M$ (or similarly $I \otimes_R M \cong M$).

Next consider $I = eR$ where $e$ is an idempotent. We take $M \twoheadrightarrow I$ to be $\tilde{I} = eR \otimes_R eR \overset{\mu}{\longrightarrow} eRe$

Check:

$\mu(re_1 \otimes_r r_2) (re_3 \otimes_r e_4) = r_1 e_2 e_3 e_4$ (1)

$(re_1 \otimes_r r_2) \mu(r_3 e_4) = r_1 e_2 e_3 e_4$ (2)

Then $\tilde{I} \otimes_R I = eR \otimes_R eR \otimes_R eR \overset{R}{\longrightarrow} eReR = eR$

$= \tilde{I}$

where we have used that $eR \otimes_R M = eM$ for any $R$-module $M$. Thus we conclude that

\[
\begin{align*}
Re \otimes_R Re & = \text{universal extension} \quad eR \otimes_R eR
\end{align*}
\]
Review: Let $R$ be an algebra, let $e$ be an idempotent in $R$. Then Connes has defined certain canonical odd degree cyclic cocycles by means of the formula

$$
\varphi(\tau_{n}^* \cdot x_{2n-1}) = \tau_{n}(F[e, R]^{n} - [F, R_{2n-1}])
$$

Here $\tau$ is a kind of trace-like map. For example if $K$ is an ideal in $R$ containing $[F, R]$ for all $r \in R$, then $\tau$ can be the canonical map

$$
K^{2n} \to K^{2n}/[K, K^{2n-1}]
$$

A first problem is to find the natural range space for Connes cocycle. I suspect that it is

$$
1) \quad (1-e)_{Re} \otimes_{eRe} eR(1-e) \otimes_{(1-e)R(1-e)} R^{n}
$$

and I checked this for $n = 1$.

On the other hand by considering the DGA

$$(\tilde{I} \to R) \otimes_{R} (\tilde{J} \to R)$$

where $\tilde{I} = Re \otimes_{eRe} eR$, $\tilde{J} = R(1-e) \otimes_{(1-e)R(1-e)} (1-e)R$ I have been able to produce a cyclic $(2n-1)$-cocycle on $R$ with values in

$$
1) \quad (\tilde{I} \otimes_{R} \tilde{J} \otimes_{R})^{n}
$$

But 1) and 1) are isomorphic.

So the problem is to see whether my cocycle class is represented by the
Connes formula with values in the space \( \mathfrak{l} \Rightarrow \mathfrak{l}' \). In particular, this means showing the latter multilinear functional is a cyclic cocycle.

This problem might be significant because it links Connes approach, which involves traces on DG cochain algebras, with mine which uses traces and more generally cyclic cocycles on DG chain algebras.

First, we must understand \( n = 1 \) thoroughly.

Review the problem. We have \( R, e \) and we have my cyclic classes

\[
HC_{2n-1}(R) \longrightarrow (\mathcal{I}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_R)^n
\]

On the other hand, we have actual cyclic cocycles defined by Connes where targets are roughly the same. Connes's method can be described as follows. He has a derivation \( \tau \) with values in a bimodule \( M \) and a linear functional \( \mathcal{T} \) on \( (M \otimes_R)^{2n-1} \).

Then \( \mathcal{T}(a_0, \ldots, a_{2n-1}) \) is a Hochschild cocycle. If it happens that \( \tau(a_0, \ldots, a_{2n-1}) = 0 \), then one has a cyclic cocycle.

This is Connes basic method and one can try to see if by some chance it produces a cyclic 1-cocycle with values in \( \mathcal{T}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{T}_{\mathcal{O}} \). The obvious thing is to start with the derivation \( r \mapsto \frac{1}{2}[F, r] \). We want to find the bimodule \( M \).
Let $R = \mathbb{Z}/2 \oplus R^-$ be the $\mathbb{Z}/2$-graded of elements commuting and anti-commuting with $F$. We have in block notation

$$[\frac{1}{2}F, r] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} r = \begin{pmatrix} 0 & e r(1-e) \\ -(1-e) r e & 0 \end{pmatrix}$$

so $[\frac{1}{2}F, r] \in R^-$. The obvious candidate for the bimodule $M$ is

$$M = R \otimes_{R^+ \otimes R} R$$

$$= R \otimes_{R^+} e \bar{e} \otimes e \bar{e} R \oplus R \otimes_{e \bar{e} \bar{e}} e \bar{e} e \otimes e \bar{e} R$$

$$= R \otimes_{e \bar{e} \bar{e}} e \bar{e} \otimes e \bar{e} R \oplus R \otimes_{e \bar{e} \bar{e}} e \bar{e} e \otimes e \bar{e}$$

$$= \tilde{I} \otimes \tilde{J} \oplus \tilde{J} \otimes \tilde{I}$$

so the question is whether $r \mapsto [\frac{1}{2}F, r]$ is a derivation with values in $M$. Look at the component in $\tilde{I} \otimes \tilde{J}$, that is

$$e \otimes e r(1-e) \otimes (1-e) \in R \otimes_{e \bar{e} \bar{e}} e \bar{e} \otimes e \bar{e} R$$

From p. 667 we know this map

$$f : R \rightarrow R \otimes_{e \bar{e} \bar{e}} e \bar{e} \otimes e \bar{e} \otimes e \bar{e}$$

given by

$$f(r) = e \otimes e r(1-e) \otimes (1-e) = e \otimes e \bar{e} \otimes \bar{e}$$

is not a derivation. Let's review the calculation:

$$f(r s) = e \otimes [e \bar{e} s + e \bar{e} \bar{e} \bar{e}] \otimes \bar{e}$$

$$r f(s) = re \otimes e \bar{e} s \otimes \bar{e}$$

$$f(r) s = e \otimes e r \bar{e} \otimes e \bar{e}$$
Then
\[ rf(r) + fr(s) - f(rs) = \overline{e}re \otimes es \overline{e} \otimes \overline{e} + e \otimes ere \otimes \overline{e}se \]

If I push this into R by multiplication I get
\[ \overline{e}res \overline{e} + ere \overline{e}se \]
\[ = \overline{e} \left[r, e]\overline{[s, e]} + e \left[r, e]\overline{[s, e]} \right] \]
\[ = -(\overline{e} + e)\left[r, e]\overline{[s, e]} = -[r, e]\overline{[s, e]} \]

On the other hand we could try the second component of \([1/2, r]\) in \(\overline{J} \otimes R \overline{I}\), that is
\[ \overline{e} \otimes ere \otimes e \in \overline{R} \otimes \overline{e}re \otimes er \]
\[ g(r) \]

Then we have
\[ rg(s) + g(r)s - g(rs) = \left(ere \otimes es \otimes e \right) + \left(e \otimes ere \otimes es \overline{e} \right) \]

In order to continue we would have to find some bimodule quotient of
\[ M = \overline{I} \otimes \overline{J} \oplus \overline{J} \otimes \overline{I} \]
in which the sum of 1) + 2) becomes zero.

A first thing to try is to replace \(\overline{I}\) by I and \(\overline{J}\) by J. As on p. 667 this leads to
1) \[ = -[r, e] \otimes [s, e] \in I \otimes J \]
2) \[ = + [r, e] \otimes [s, e] \in J \otimes I \]

Thus we can take the R-bimodule of \(I \otimes J \oplus J \otimes I\) generated by the elements
\[ [r, e] \otimes [s, e] \oplus [r, e] \otimes [s, e] \]
Similarly I can divide $M$ itself by the $R$-bimodule generated by the direct sum of the elements $1 + 2$. I don't see what all this means.

The conclusion seems to be the following: Given an algebra $R$ and an idempotent $e$ in $R$, there is a canonical cyclic cocycle on $R$ with values in

$$\overline{I \otimes R} \cong eRe \otimes eRe \otimes eRe$$

However it doesn't seem to be possible to obtain this cocycle by Cézanne's methods, namely, by means of a derivation $s : R \to M$ and a suitable trace map on the latter.

The next step to take is to see what can be done with the 2nd formula

$$\varphi(a_0, \ldots, a_{2n-1}) = \frac{1}{2^n} \sum \left( F \left[ F_j a_0 \right] \cdots \left[ F_j a_{2n-1} \right] \right)$$

This should be a well-defined cyclic cocycle with values in $\left( K \otimes_R \overline{I} \right)^{2n}$, where $K$ is the ideal $I \cap J = IJ + JI$ generated by the elements $\left[ F_j a \right]$ as $a \in R$. However there is a map

$$\left( K \otimes_R \overline{I} \right)^{2n} \rightarrow \left( I \otimes_R J \otimes_R \right)^n$$

and the natural question is whether we can obtain an explicit cyclic cocycle with values in $\left( \overline{I \otimes R} \otimes \overline{J \otimes R} \right)^n$. 
April 2, 1988

Let $R$ be an algebra, $e$ an idempotent in $R$, $F = 2e - 1$, and let $\delta: R \to M$ be a derivation where $M$ is a bimodule. Assume that

$$F\delta(a) = -\delta(a)F \quad \forall a \in A$$

and set

1. \(\varphi(a_0, \ldots, a_{2n-1}) = F\delta(a_0) \otimes \delta(a_1) \otimes \cdots \otimes \delta(a_{2n-1}) \in (M \otimes R)^{2n} \)

**Claim:** $\varphi$ is a cyclic $(2n-1)$-coycle.

**Careful check for $n = 1$:**

\[(b\varphi)(a_0, a_1, a_2) = (F\delta(a_0 a_1), \delta(a_2)) = \begin{cases} 
1 & (F\delta(a_0) a_1, \delta(a_2)) + (F\delta(a_0), \delta(a_1) a_2) \\
2 & (F\delta(a_0), \delta(a_1) a_2) - (F\delta(a_0), \delta(a_1) a_2) \\
3 & (F\delta(a_0), \delta(a_1) a_2) + (F\delta(a_0), \delta(a_1) a_2)
\end{cases}\]

2. \((Fa_0 \delta a_1, \delta a_2) = (\delta a_2, Fa_0 \delta a_1) = (\delta a_2, Fa_0, a_1) = -(F\delta a_2, a_0, a_1)\)

3. \((F\delta a_0, \delta a_1 a_2) = -(\delta a_0 F, \delta a_1 a_2) = -(\delta a_0, F\delta a_1 a_2) = (\delta a_0, \delta a_1 Fa_2) = (Fa_0 \delta a_1, \delta a_2)\)

**Cyclic property:** \((F\delta a_1, \delta a_0) = -(\delta a_1, F\delta a_0) = -(\delta a_1, F\delta a_0) = -(Fa_0, a_1)\)

**General case:**

\[(b\varphi)(a_0, \ldots, a_{2n}) = (F(\delta a_0 + \ldots + \delta a_0), \delta a_2) \cdots \delta a_{2n})
+ \cdots + (-1)^n(F\delta a_0, \ldots, \delta a_{2n})
+ (-1)^{2n-1}(F\delta a_0, \ldots, \delta a_{2n-1}, \delta a_{2n})
+ (-1)^{2n}(F(\delta a_0 a_0 + \ldots + \delta a_{2n-1}, \delta a_{2n}))\]

Also,

\((F\delta a_{2n-1}, \delta a_0, \ldots, \delta a_{2n-2}) = -(\delta a_{2n-1}, F\delta a_0, \ldots, \delta a_{2n-2}) = -(F\delta a_0, \delta a_1, \ldots, \delta a_{2n-1})\)
Here's a simpler version without the $F$. Suppose $\delta: R \to M$ is a derivation and set
\[ \varphi(a_0, \ldots, a_n) = (\delta a_0, \ldots, \delta a_n) \in (M \otimes_R)^{n+1} \]

Claim: $\varphi$ is a cyclic $n$-cocycle for all $n$.

To see this we can of course calculate, however here's a general argument:

\[\begin{array}{ccc}
\Omega^{n+1}_R & \xrightarrow{d} & M \otimes_R \cdots \otimes_R M \\
\Omega^n_R / \Sigma [a_i^{n+i} \cdots] & \xrightarrow{d} & \Omega^{n+1}_R / \Sigma [a_i^{n+i} \cdots] \\
\xrightarrow{\varphi} & & \varphi(a_0, \ldots, a_n) \\
\end{array}\]

Thus $\varphi$ is a closed trace on $\Omega^*_R$ of degree $n$, and so it is a cyclic cocycle.

The problem with these cocycles $\varphi$ is that they come from Hochschild cocycles of one higher degree, which means that the classes are killed by the $S$ operator

\[\begin{array}{ccc}
H_{n+2}(A) & \xrightarrow{S} & H_n(A) \\
\end{array}\]

Example: $n=0$. $\varphi(a) = \delta(a)$. This is a $0$-cocycle
\[\tau(\delta a_0, a_1 + a_1 \delta a_0) - \tau(\delta a_1, a_0 + a_1 \delta a_0) = 0\]
but it's also killed by $S$.

Remark: $d_d$, does not induce a map on $\Omega^*_R \otimes_R$, which is why given a normalized Hochschild cocycle $\varphi(a_0, \ldots, a_n)$, it's not true that $\varphi(1, a_0, \ldots, a_n)$ is a cyclic cocycle, or even a Hochschild cocycle for $n \geq 2$. 
Consider a DG chain algebra

\[ \cdots \rightarrow R_2 \rightarrow R_1 \rightarrow R_0 \]

Let

\[
K_n = \begin{cases} 
R_n & n \geq p+1 \\
\alpha R_{p+1} & n = p \\
0 & n < p 
\end{cases}
\]

Then \(K_n\) is DG ideal in \(R_n\). It's obviously closed under \(\alpha\). Also the differentials are \(R_0\)-bimodule homomorphisms, so \(\alpha R_{p+1}\) is an \(R_0\)-bimodule, and \(K_n\) is an \(R_0\)-bimodule complex. Finally \(K_n\) is closed under multiplication by \(R_0 > 0\).

Variant: Replace \(\alpha R_{p+1}\) by \(\text{Ker}(R_0 \rightarrow R_{p-1})\).

Note that \(H_*(R_n/K_n) = H_*(R_0)/H_{p}R_0(K_0)\).

Let's use this observation in the case of the DG algebra

\[
(\mathcal{I} \rightarrow R) \times (\mathcal{J} \rightarrow R)
\]

where we obtain a DG algebra

\[
\begin{array}{c}
\rightarrow 0 \\
\rightarrow \mathcal{I} \oplus \mathcal{J} \\
\rightarrow R
\end{array}
\]

which is acyclic assuming \(I + J = R\). Take the commutator quotient.

Thus if \(d : \mathcal{I} \oplus \mathcal{J} \rightarrow \mathcal{I} \oplus \mathcal{J}\) is \(z \mapsto (z, -z)\) then we have the product rules

\[
d(x * y) = dx \cdot y - x dy = (\mathcal{I} \oplus \mathcal{J} \oplus \mathcal{I} \oplus \mathcal{J})
\]

Thus if \(d : \mathcal{I} \oplus \mathcal{J} \rightarrow \mathcal{I} \oplus \mathcal{J}\) is \(z \mapsto (z, -z)\) then we have the product rules
\[ x \ast y = -xy \in I \ast J \\
y \ast x = -yx \in I \ast J \]

Thus the bracket \( x \ast y + y \ast x \) is \(-xy + yx\) and we find that the commutator quotient complex is

\[
I \ast J / [R, I \ast J] + [I, J] \rightarrow I / [R, I] \oplus J / [R, J] \rightarrow R / [R, R] \rightarrow 0
\]

(Recall \([R, I \ast J] \subset [I, J]\) and that these are equal in the case \(I = R \ast R, J = R(-e)R\), see p.668; possibly they are equal in general by generalization of the argument on p.670.)

**Key Problem:** We have acyclic DC algebra starting with \(R\) and with a trace:

\[
\begin{array}{cccccc}
0 & \rightarrow & I \ast J & \rightarrow & I \oplus J & \rightarrow & R & \rightarrow & 0 \\
& \downarrow & \tau & & & & \downarrow & & \\
& & I \ast J / [I, J] & & & & & &
\end{array}
\]

We know this leads to a map \(HC_1(R) \rightarrow I \ast J / [I, J]\)

Can one explain this cyclic cohomology class in Connes' terms? By this I mean:

- using a cochain algebra.
Discussion: One can prove a lot about cyclic homology using the cyclic complex applied to DG chain algebras. For example, your paper is self-contained and proves directly without using the Connes exact sequence that the cyclic homology of a free algebra is the same as that of the field $\mathbb{C}$. (This follows from the exact sequence

$$
0 \to HC_{2n}(A) \to HC_0(R/I^{n+1}) \to H_1(R, R/I^n) \to HC_{2n-1}(A) \to 0
$$

when $R$ is free, together with the explicit calculation of $d'$ when $I = 0$. Thus this calculation uses the cyclic complex of $R \leftarrow R$.)

However, I still don't understand the cyclic complex on a fundamental level. It arises naturally out of the Lie algebra homology of matrices, so the cyclic complex is a natural object.

A key principle should be that cyclic cocycles are higher traces. There seem to be many interpretations of "higher trace." Here is a list:

1) Closed trace on a DG chain algebra resolution of $A$ or an acyclic DG chain algebra with a homomorphism $A \to R_0$.

2) If $A = R/I$, then trace on $R/I^{n+1}$ or on $I^{n+1}$ vanishing on $[I, I^n]$. 

3) Closed trace on a DG cochain algebra $\Omega$ with a homomorphism $A \to \Omega$. 

Now let us consider a free DG chain algebra resolution of $A$:

$$\ldots \to R_2 \to R_1 \to R_0 \to A \to 0$$

Let's apply the reduced cyclic complex functor to it: $\text{CC}(\mathbb{R})$. This will give a double complex which resolves $\text{CC}(A)$ in the row direction. If one ignores the horizontal differential one has the reduced cyclic complex of a free graded algebra. It should be true that the reduced cyclic homology of a free super algebra is trivial. Assuming this to be the case, the columns of $\text{CC}(\mathbb{R})$ are acyclic except at the bottom, and so we obtain an isomorphism

$$\text{HC}_\ast(A) = H_\ast(\mathbb{R}/[\mathbb{R},\mathbb{R}])$$

Thus we see that traces on DG chain algebra resolutions of $A$ gives all the cyclic cohomology of $A$.

How useful is this representation for cyclic cohomology classes?

For example, given a chain algebra resolution $\mathbb{R}$ of $A$ and a closed trace on it, can I pair this with $K_0(A)$ (or $K_1(A)$ depending on the parity)? Is there some way to see the $S$-operator?

The $S$-operator can perhaps be seen in Comes fashion, namely, by using the tensor product of DG algebras. Thus if $\mathbb{R} \to A$ is
a chain algebra resolution, and similarly for \( R' \to A' \), then by the
Kunneth formula we know that \( R \otimes R' \)
is a chain algebra resolution of \( A \otimes A' \).
We also have
\[
R \otimes R' / [R \otimes R', R \otimes R'] = R / [R, R] \otimes R' / [R, R']
\]
and so closed traces on \( R \) and \( R' \) combine
to get a closed trace on \( R \otimes R' \). Thus if
we take \( A' = C \) we go from the cyclic
class on \( A \) of degree \( n \) represented by \( i \) in \( R \)
to the cyclic class on \( A \otimes C[A] \)
of degree \( n + 2 \) represented by \( ? \).

So I have run right into the unital-
unital problem.

Here's a program: First describe the
fundamental cyclic \( 2 \)-dimensional cohomology of \( C[e] = \)
out of means of a □ chain algebra resolution. Then the
cup product should give us a way
to go from \( R(A) \) to \( R(A \otimes C[e]) \).
Working in the unital category you would need to have a unital map \( A \to A \times A \).
Unfortunately there is only one \( \eta \) map around, the diagonal, whereas we need something like the Connes map
\[ A \to A \otimes C[e] \]
Problem: Suppose $R \to A$ is a chain algebra resolution, that $i$ is a closed trace on $R$ of even degree, and that $e$ is an idempotent in $A$. Then we would like to pair the cyclic class on $A$ represented by $(R,e)$ with the class of $e$ in $K_0 A$.

The obvious way to proceed is to take a free chain algebra resolution $F_\cdot$ of $C[e]$, and to extend the homomorphism $C[e] \to A$ to a chain algebra map $F_\cdot \to R$. Then there's an induced map

$$H_\ast(F_\cdot / [F,F]) \to H_\ast(R_\cdot / [R,R,R]).$$

which ought to be independent of the extension (by standard homotopical algebra?). Now the former space is isomorphic to $HC(C[e])$, so there are canonical generators in the even degrees. These can be combined with $i$ to give numbers.

To understand the situation we might as well start with a chain algebra resolution $R_\cdot$ of $C[e]$. We want to produce canonical even classes in $H_\ast(R_\cdot / [R,R,R])$; let's consider $n = 2$.

$$
\begin{array}{c}
\tilde{H}_3(R_3 / [R_0,R_3] + [R,R,R]) \\
\downarrow \\
R_2 / [R_0,R_2] + [R,R,R] \\
\downarrow \\
R_2 / dR_3 + [R_0,R_2] + [R,R,R] \\
\downarrow \\
H_2(R_\cdot / [R,R,R])
\end{array}
$$
Therefore the group $H_2(R/[R, R])$ depends on the truncated chain algebra resolution

\[ \begin{array}{c}
0 \rightarrow R_2/dR_3 \rightarrow R_1 \rightarrow R_0 \rightarrow A \rightarrow 0
\end{array} \]

Let's now start constructing a free chain algebra resolution of $C[e] = C[Z/2Z]$.

We take $R_0 = C[x]$ with $x \mapsto F = 2e-1$ in $C[e]$.

The kernel of $R_0 \rightarrow C[e]$ is generated by $x^2-1$, so we can take $R_1 = C[x] \otimes C[x] \cong C[x] \otimes C[x]$ to be the free $R_0$-bimodule generated by an element $y$ such that $d(y) = x^2-1$.

If we were to continue to construct a free chain algebra resolution, then $R_2$ would be the direct sum of

\[ R_1 \otimes_{R_0} R_1 = C[x] \otimes C[x] \otimes C[x] \cong C[x] \otimes 3 \]

with a free $R_0$ bimodule need to kill extra relations in $R_1$. We will however stop the process in degree 2 and let $R_2 = \text{Ker}[R_1 \rightarrow R_0]$.

Then we have

\[ \begin{array}{c}
R_1 \xrightarrow{d} C[x](x^2-1) \rightarrow R_0
\end{array} \]

\[ \downarrow \quad \downarrow \quad \text{mult.} \quad \downarrow \]

\[ C[x] \otimes C[x] \rightarrow C[x] \]

so $R_2 = C[x] \ast C[x] \cong C[x] \otimes 2$, where $u$ is such that $du = xy - yx$. Thus our resolution appears

\[ \begin{array}{c}
0 \rightarrow R_2 \xrightarrow{d} R_1 \rightarrow R_0 \rightarrow C[e] \rightarrow 0
\end{array} \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \begin{array}{c}
o 
\rightarrow C[x] \ast C[x] \xrightarrow{d} C[x] \ast C[x] \xrightarrow{d} C[x] \rightarrow C[e] \rightarrow 0
\end{array} \]

$du = [x, y]$ \quad $dy = x^2-1$.  
The product map \( R_1 \times R_1 \to R_2 \) is determined by the requirement that \( d \) be a derivation. Thus if we take \( f_1 y g_1, f_2 y g_2 \in R_1 \) we want to have

\[
d(f_1 y g_1 \cdot f_2 y g_2) = f_1(x^2 + 1) f_2 y g_2 - f_1 y g_1 f_2 (x^2 + 1) g_2
\]

\[= f_1 \left[ (x^2 + 1) f_2 y, g_2 \right] g_2 \]

But

\[
[x^n, y] = \sum_{i=0}^{n-1} x^i [x, y] x^{n-1-i}
\]

\[= d \left( \sum_{i=0}^{n-1} x^i u x^{n-1-i} \right) \]

so this determines the product.

We now want to determine the trace spaces

\[
0 \longrightarrow R_2/[R_0, R_2] + [R_1, R_1] \longrightarrow R_1/[R_0, R_1] \longrightarrow R_0/[R_0, R_0]
\]

We have \( R_2/[R_0, R_2] = C[x, y] \) and the trace space in degree 2 is the interesting one.

We have \( R_2 \otimes R_0 = C[x, y] \wedge \otimes \)

and the trace space in degree 2 is the cokernel of the bracket map which is just symmetrized multiplication

\[
(f_1 y g_1, f_2 y g_2) \mapsto f_1 y g_1 f_2 y g_2 + f_2 y g_2 f_1 y g_1
\]

\[= (g_2 f_1) y (g_1 f_2) y + (g_1 f_2) y (g_2 f_1) y \]
Thus we are after the cokernel of
\[(a, b) \quad \mapsto \quad a \otimes b = \mathbb{C}[x] u\]

\[(f g, g y) \quad \mapsto \quad f g y y + g y f y\]

\[d (f g y y) = f (x^2 - 1) y y - f y g x^2 - 1 = f [(x^2 - 1) g, y]\]
\[d (g y f y) = g [(x^2 - 1) f, y]\]

Take \(f = x^m, \quad g = x^n, \) then we have
\[x^m [(x^2 - 1) x^n, y] + x^n [(x^2 - 1) x^m, y]\]
\[= x^m \{ [(x^2 + n), y] - [(x^n, y)] \} + x^n \{ [(x^2 + m), y] - [(x^m, y)] \}\]

But \( [(x^2 + n), y] = d (\sum x^i u x^{n+1-i}) \), hence
\[[x^m, x^n] = \sum_{i+j=n+2} x^{m+i} u x^{n+1-i} - \sum_{i+j=n} x^{m+i} u x^{n-1-i}\]
\[+ \sum_{c+j=m+2} x^{n+i} u x^{m+1-i} - \sum_{c+j=m} x^{n+i} u x^{m-1-i}\]

and the image of this module \([R_0, R_2]\) is
\[(n+2) x^{m+n+1} - n x^{m+n-1} + (m+2) x^{n+m+1} - m x^{n+m-1}) u\]
\[= (n+m+4) x^{m+n+1} - (n+m) x^{m+n-1}) u\]

This just depends on \(m+n\), so we want the quotient space of \(\mathbb{C}[x]\) by the polynomials
\[(n+m+4) x^{n+1} - n x^{n-1} = \{ (x^2 \partial_x - \partial_x) + 4x \} x^n\]
\[= [(x^2 - 1) \partial_x + 4x] x^n\]

Conclude \(R_2/[R_0, R_2] + [R_1, R_1] = \text{Coker} \{ \mathbb{C}[x] \xrightarrow{(x^2 - 1) \partial_x + 4x} \mathbb{C}[x] \}\)
It's clear this differential operator is injective on polynomials, since its kernel is spanned by
\[- \int \frac{4x}{x^2-1} \, dx = e^{-2 \log (x^2-1)} = \frac{1}{(x^2-1)^2}\]

However, if we use the pairing \( \int f'g \, dx \), then the image of \((x^2-1)\partial_x + 4x\) is \(\perp\) to the kernel of \(-\partial_x (x^2-1) + 4x\). This kernel is spanned by \((x^2-1), x\) so
\[(x^2-1)\partial_x + 4x \mathbb{C}[x] \subseteq \{ f \mid \int f (x^2-1) \, dx = 0 \}.
\]

In fact, we know the image of \((x^2-1)\partial_x + 4x\) on polynomials contains a polynomial of each degree \(\geq 0\), namely \((n+4)x^{n+1} - nx^{n-1}\). So we conclude
\[
\mathbb{C}[x]/((x^2-1)\partial_x + 4x) \mathbb{C}[x] \cong \mathbb{C}
\]
\[
\frac{1}{1} \leftarrow \mathbb{C}
\]

Conclude \(R_2/[(R_0, R_2) + (R_1, R_1)] \cong \mathbb{C}\) and we have identified the inverse isomorphism
\[
\tau : f \mapsto \int_{-1}^{1} f(x) (x^2-1) \, dx
\]
up to a scalar factor.

The next question is whether there is a much smaller chain algebra resolution of \(\mathbb{C}[x]\) which still supports the canonical 2-dim trace class.

One thing we've learned is how to calculate the pairing of \(R_0 \otimes (R_0 \otimes T)\) of dim 2 with an e.e.a. We lift \(2e-1\) to \(x \in R_0\) and \(x^2-1\) to \(y \in R_1\), and \([xy]\) to \(u \in R_2\); then the pairing is \(\mathbb{C}u\).
April 4, 1988

Here's a smaller version of the length 2 chain algebra resolution of \( C[Li^2] \). We start with the trace \( \tau : R_2 \to \mathbb{C} \),

\[
\tau(f u g) = \int f g(x^2 - 1) dx,
\]

and we look for the smallest \( R_0 \)-bimodule which supports it. This is the image of \( R_2 \to R_0^* \), and in the present case it is \( C[x] u \), where \( [x, u] = 0 \). Then we want to divide \( R_1 \) out by the \( R_0 \)-bimodule generated by \( d[x,u] = [x, [x,y]] \). Thus we end up with

\[
\begin{align*}
\overline{R_2} & \longrightarrow \overline{R_1} \longrightarrow R_0 \\
0 & \longrightarrow C[x] u \longrightarrow C[x] y \oplus C[x] h \longrightarrow C[x] \longrightarrow C[f] \longrightarrow 0
\end{align*}
\]

\( d u = h \), \( d y = x^2 - 1 \), \( [x, u] = [x, h] = 0 \), \( [x, y] = h \). The product \( \overline{R_1} \times \overline{R_1} \to \overline{R_2} \) is found as follows

\[
d(f_1 y g_1, f_2 y g_2) = f_1 (x^2 - 1) g_1 f_2 g_2 - f_1 y g_1 f_2 (x^2 - 1) g_2
\]

\[
= f_1 [(x^2 - 1) g_1 f_2, y] g_2 = f_1 (x^2 - 1) g_1 f_2 g_2 h
\]

\[
\implies f_1 y g_1 f_2 g_2 = f_1 (x^2 - 1) g_1 f_2 g_2 h
\]

The trace \( \tau(\cdot) \) on \( \overline{R_2} \) must vanish on

\[
f ((x^2 - 1) g)' + g ((x^2 - 1)f)'
\]

and it's clear that

\[
\tau(f) = \int f(x) (x^2 - 1) dx
\]

has this property.
It's more or less clear that this lengthy chain algebra resolution of $C(2,2)$ is the smallest one which supports the trace $t$.

In effect, because the linear functionals $t^*x^*$ are all independent, $R_2$ couldn't be any smaller, hence also $R_0$ couldn't be any smaller, etc.

At this point we conclude that chain algebra resolutions are not likely to be very useful in cyclic theory. The free ones grow too fast. And one misses the nice periodicity, so we must return to the GNS-Stormspring circle of ideas.
Consider the algebra
\[ R = A \otimes C[\mathbb{Z}/2] = (A \otimes A) \otimes C[\mathbb{Z}/2] \]
which is freely generated by \( A \) and the involution \( F \). Then we have seen that \( R \) admits a description as the GNS algebra
\[ R = A \otimes A \otimes B \otimes A \]
where \( f : A \to B \) is universal i.e.
\[ B = T(A)/(1 = f(1_A)) . \]

Also relative to the idempotent \( e = \frac{F+1}{2} \)
we have the block description
\[ R = \begin{pmatrix}
    eRe & eR(1-e) \\
    (1-e)Re & (1-e)R(1-e)
\end{pmatrix} \]
\[ eRe = B \]

But notice that \( R \) has a unique automorphism
which is the identity on \( A \) and changes \( F \) to \(-F\).
This shows that
\[ a \mapsto (1-e)a(1-e) \in (1-e)R(1-e) \]
is also a universal linear map \( \alpha : 1 \to 1 \) and hence
also that
\[ (1-e)R(1-e) \cong T(A)/(1 = f(1_A)) \]
Let $C$ be an algebra containing the algebra $A$ and the idempotent $e$. Then

1) $AeCeA \cdot AeCeA = AeCeCeA \subseteq AeCeA$

and so $A + AeCeA$ is a unital subalgebra of $C$.

Assume $C$ is generated by $A, e$. Then we conclude

2) $C = A + AeCeA$

and similarly

2') $C = A + AeCeA$.

Now

1) shows that $AeCeA$ is an ideal in $C$, and it has to be the ideal generated by $e$. Let's denote this ideal by $I$:

$$I = CeC = AeCeA$$

and similarly set

$$J = C\bar{e}C = A\bar{e}CeA.$$ 

Next we look at the blocks relative to $e$ of $C$ and these ideals. We have

$$\bar{e}Ie \subseteq \bar{e}Ce = \bar{e}Cee \subseteq \bar{e}Ie$$

hence the off-diagonal blocks coincide

$$\bar{e}Ie = \bar{e}Je = \bar{e}Ce$$

$$e\bar{e} = eJ\bar{e} = eC\bar{e}$$

Also we have similarly

$$eIe = eCe$$

$$eJ\bar{e} = eC\bar{e}$$
so what's interesting are the ideals $eJ_e \leq eC_e$ and $eI_e$ in $eC_e$.

We have

$$eC_e = eAe + eAeC_eAe$$

$\leq eC_e \cdot eAe \leq eC_e$

$$eC_e = eAe + eAeC_eAe$$

$\leq eAe \cdot eC_e \leq eC_e$

Thus

$$eC_e = eAe \cdot eC_e = eC_e \cdot eAe$$

Consider now the ideal $eJ_e$ in $eC_e$. We have

$$eJ_e = eC_eC_e = eAe \cdot eC_e \cdot eAe$$

$$eI_e = eC_eC_e = eAe \cdot eC_e \cdot eAe$$

and

$$eC_e = eAe + eAe \cdot eC_e \cdot eAe$$

$$eC_e = eAe + eAe \cdot eC_e \cdot eAe$$

Since $C = A + I = A + J$. (In fact we have

$$C = eAe + I = eAe + J.$$ )

Note that

$$(eJ_e)^2 = eAe \cdot eC_e \cdot eAe \cdot eAe \cdot eC_e \cdot eAe$$

$$= eAe \cdot eAe \cdot eC_e \cdot eAe \cdot eAe = (eAe \cdot eAe)^2 \cdot eC_e$$
Let's start again.

\[ A \varepsilon C e A \cdot A e C e A \subseteq A e (C e A e e) e A \]
\[ \subseteq A e C e A \]

Thus, \( A + A e C e A \) is a unital subalgebra of \( C \) containing \( A, e \) so if these generate \( C \) we have
\[ C = A + A e C e A \]
also
\[ C = A + A e C e A \]
and
\[ J = C e C = A e C e A, \quad J = C e C = A e C e A \]

\[ e C e = e (A + A e C e A) e \]
\[ = e A e + e A e C e A e \subseteq e A e . e C e \]
\[ \subseteq e C e \]

\[ \therefore e C e = e A e . e C e \]

Also
\[ e C e = e (A + A e C e A) e \]
\[ = e A e + e A e C e A e \subseteq e C e . e A e \]
\[ \subseteq e C e \]

\[ \therefore e C e = e C e . e A e = e A e . e C e \]

Next
\[ e J e = e A e C e A e \]
\[ = e A e . e C e . e A e \]
\[ = e C e . e A e . e A e \geq e A e . e A e . e C e \]

\[ (e J e)^n = (e A e . e A e)^n e C e \]
Thus
\[ eCe = e(A + \frac{J}{A e C e A})e = e(A + J)e \]
\[ = eAe + (eA e \cdot eA e) eCe \]
\[ = eAe + eA e \cdot eA e \cdot eA e + (eA e \cdot eA e)^2 eCe \]
\[ = eAe + eA e \cdot eA e + eA e \cdot eA e \cdot eA e + (eA e \cdot eA e)^2 eCe \]
\[ = eAe + (eA e \cdot eA e) + (eA e \cdot eA e)^2 + eJe (eA e \cdot eA e)^2 \]

Also we have
\[ eCe = eAe + eJe \]
\[ eJe = eCe \cdot eA e \cdot eA e \]
\[ = eAe (eA e \cdot eA e) + eJe (eA e \cdot eA e) \]
\[ = eAe (eA e \cdot eA e) + eAe (eA e \cdot eA e)^2 + eJe (eA e \cdot eA e)^2 \]

from which we see that
\[ eCe = eAe + eAe (eA e \cdot eA e) + \cdots + eAe (eA e \cdot eA e)^n \]
\[ + eJe (eA e \cdot eA e)^n \]

Better approach
\[ eCe = eAe + eJe = eAe + eCe(eA e \cdot eA e) \]

Now iterate this equation to get
\[ eCe = eAe + eAe(eA e \cdot eA e) + \cdots + eAe(eA e \cdot eA e)^n + \frac{eCe(eA e \cdot eA e)^{n+1}}{(eJe)^{n+1}} \]

Similarly we have
\[ eC\bar{e} = eCe \cdot eA \bar{e} = eAe \cdot eA \bar{e} + eCe \cdot eA \bar{e} \cdot eA \bar{e} \cdot eA \bar{e} \]
which leads to
\[ eC\bar{e} = eAe \cdot e\bar{A}e + eAe \cdot e\bar{A}e (eAe \cdot e\bar{A}e) + \ldots \]
\[ + eAe \cdot e\bar{A}e (eAe \cdot e\bar{A}e)^2 \]
\[ + \frac{eC\bar{e} \cdot e\bar{A}e (eAe \cdot e\bar{A}e)^{n+1}}{(eJe)^{n+1} \cdot e\bar{A}e} \]

In fact we can say the following.

\[ eC\bar{e} = \sum_{k>0} eAe (e\bar{A}e \cdot eAe)^k \]

\[ (eJe)^n = \sum_{k>n} eAe (e\bar{A}e \cdot eAe)^k = eC\bar{e} (e\bar{A}e \cdot eAe)^n \]

\[ eC\bar{e} = eAe \cdot \sum_{k>0} (e\bar{A}e \cdot eAe)^k \cdot e\bar{A}e \]

Question: \( eA\bar{e} \subset eC\bar{e} \). Is there a decomposition for \( eC\bar{e} \) starting with \( eA\bar{e} + \ldots \)?

In other words, you would like there to be an \( eC\bar{e} \oplus (eC\bar{e})^c = \) submodule of \( eC\bar{e} \) complementary to \( eA\bar{e} \).

Problem: Make the link with Connes–Cuntz theory.

---

Let's try to understand the even–odd, Connes homomorphism. I recall the idea that one of degree 2n should arise from the odd one of degree 2n+1:

\[ HC_{2n+1}(A) \to 0 \oplus I^{n+1}/[I,I^n] \]
as follows. Representing this by a cyclic $2n+1$ cocycle $\varphi$, one should be able to write the image of $\varphi$ in $R/[R,R]$ as $b\varphi$, with $\varphi : A_{\mathbb{A}}^{2n+1} \rightarrow R/[R,R]$ a cyclic $2n$-cochain. Then $\bar{\varphi} : A_{\mathbb{A}}^{2n+1} \rightarrow H^0(R/R)$ is a cyclic $2n$-cocycle.

The real problem appears then to assume one has a trace $\tau$ on $\mathbb{C}$ and then to show that Connes cocycle

$$\tau([F,F,g_0] - [F,F,g_0])$$

is a coboundary. But recall that this formula can be interpreted as being obtained from the invariant character form $\tau_1(F(\tilde{F})^{2n+2})$ on the Grassmannian pulled back via the map $g \mapsto gFg^{-1} = F$.

So we reach the problem of writing the pullback of the invariant character form on $\mathbb{C}(U)$ to $U(U)$ as a coboundary.

Let $W = +1$ eigenspace of $F$ on $V$. We will show $\tau_1(F(\tilde{F})^{2n})$ becomes a coboundary when lifted to the Stiefel manifold of embeddings $W \hookrightarrow V$. This uses the standard Chern-Weil ideas.

Think of the Stiefel manifold of consisting of iso.$\,$embeddings $i : W \rightarrow V$, then the subbundle is iso.$\,$to the trivial bundle $\tilde{W}$ with the connection $i^*d. i = d + A$

$$A = i^*d(i)$$

Use the deformation $d + tA$, $0 \leq t \leq 1$.

$$(d(tA + d + tA)^2 = dt A + tdA + t^2 A^2$$

and so

$$\tau_1((dA + A^2)^n = \int_0^1 \tau_1(dt A + tdA + t^2 A^2)^n$$
by standard Chern-Simons calculations.

Now pull-back to $\mathbb{U}(V)$ by $g \mapsto g_0^* W \rightarrow V$

where $i_0$ is the inclusion of $W$ in $V$. We have

$$A = i_0^* g_0^{-1} dg i_0 = \rho(\Theta)$$

where $\Theta$ is the Maurer-Cartan form on $\mathbb{U}(V)$, and $\rho(r) = i_0^* r i_0$ for any operator $r$. We have with this notation

$$dA = -\rho(\Theta^2)$$

so the curvature is

$$dA + A^2 = \rho(\Theta)^2 - \rho(\Theta^2).$$

At this point, at least on a formal level we begin to see justification for feeling $\rho$ should be viewed as a connection with curvature $\rho(a_1) \rho(a_2) = \rho(a_1 a_2)$

Thus it appears that the transgression form we want is

$$\int_0^1 \text{tr} \left( dt \rho(\Theta) + t^2 \rho(\Theta)^2 - t \rho(\Theta^2) \right)^n$$

so for $n = 1$ we obtain

$$\text{tr} \rho(\Theta)$$

and for $n = 2$ we obtain

$$\int_0^1 2 dt \text{tr} \left\{ \rho(\Theta) \left[ (t^2 + t) \rho(\Theta)^2 + t (\rho(\Theta)^2 - \rho(\Theta^2)) \right] \right\}$$

$$= \text{tr} (\rho(\Theta)^3) 2 \left( \frac{1}{3} - \frac{1}{2} \right) + \text{tr} (\rho(\Theta) (\rho(\Theta)^2 - \rho(\Theta^2)))$$
\[ = \text{tr} \left\{ \rho(\Theta)(\rho(\Theta)^2 - \rho(\Theta^3)) \right\} - \frac{1}{3} \text{tr} \left\{ \rho(\Theta)^3 \right\} \]

This is the formula on p.591, essentially.

April 7, 1988

Chern-Simons algebra. We work with the non-commutative polynomials in two variables \( A, F \) of degrees 1, 2 respectively with differential such that

\[ dA + A^2 = F \quad d(dA) = 0 \]

(hence \( dF = (dA)A - A(dA) = FA - AF \)).

Thus we have a cochain algebra. Algebraically it is the tensor algebra with the generators \( A, F \) and it has a differential of degree +1. We can also describe it as the free cochain algebra with the single generator \( A \) of degree +1. Thus it's the tensor algebra on the complex with the generator \( A \) and \( dA \). It follows that the commutator quotient algebra has trivial homology.

In fact if \( V \) is the complex \( GA \xrightarrow{d} CdA \),
then the algebra is \( T(V) \) and the commutator quotient is \( \bigoplus_{n \geq 0} V_{\sigma} \).

But the Kunneth formula tells us that each of the complexes \( V_{\sigma} \) has trivial homology. In fact if we introduce the derivation \( h \) of \( T(V) \) of degree -1 (relative to the grading of \( V \)) such that \( hA = A \) and \( hA = 0 \), then one has \( dh + hd = 0 \) on \( T_n(V) \) and only \( V_{\sigma} \).

It seems that this homology operator has to be the "Chern-Simons" one, namely the one associated to the action \( A \mapsto tA \).
April 8, 1988

Consider the cochain algebra \( C(A,dA) \) where \( A \) has degree 1. If \( V \) is the complex

\[
\begin{array}{cccc}
0 & \xrightarrow{d} & C & \xrightarrow{dA} & 0 & \xrightarrow{\cdots}
\end{array}
\]

then \( C(A,dA) = T(V) \). Let \( \Theta_t \) denote the automorphism of this cochain algebra such that

\[
\Theta_t(A) = tA
\]

Then because we have a 1-parameter group of automorphisms given by \( x \mapsto \Theta_1 \), we obtain a derivation

\[
X = \frac{d}{dt} \Theta_1 |_{t=0} = \frac{d}{dt} \Theta_t |_{t=1}
\]

This derivation is multiplication by \( n \) on \( V \otimes n \) and commutes with \( \Theta_t \).

Let \( h \) be the degree -1 derivation on \( T(V) \) such that

\[
h(A) = 0 \quad h(dA) = A
\]

Then

\[
[h, X] = X
\]

because both are derivations agreeing on \( V \).

Let \( \eta \in T(V) \) be closed. Then

\[
\triangledown_t (\Theta_t \eta) = X \Theta_t \eta = dh \Theta_t \eta
\]

and so if \( \eta \) has no constant term

\[
\eta = \int_0^1 \frac{dt}{t} h \Theta_t \eta
\]

Let's take

\[
\eta = tr(F^n) = tr \left( (dA + A^2)^n \right)
\]

Then

\[
\Theta_t \eta = tr \left( (tdA + t^2A^2)^n \right)
\]

\[
h \Theta_t \eta = n tr \left( tA \left( (tdA + t^2A^2)^n \right) \right)
\]
and so the \( (F^n) \) is of
\[
\int_0^1 \text{det} \cdot \text{tr} \left\{ A(t^2A + t^2A^2)^{n-1} \right\}
\]
which is the Chern-Simons form.

Let's go over the program. I think at this point I understand the even-dimensional \( \text{C}^* \)-homomorphism on the level of formulas. I still might want to show that the Chern-Simons formula is consistent with diagram-chasing. Still I will find it hard to show compatibility with the \( S \)-operator.

Let's start again with the algebra \( C \) obtained by adjoining an idempotent \( e \) to \( A \). Apparently \( C \) contains all the information relative to the cyclic homology of \( A \).

I propose to use \( C \) with all its structure to prove the theorems at the end of my paper on extensions. To be more specific we have

\[
C = A \times C[z/2], \quad J = C e C, \quad \overline{T} = C e C
\]

\[
B = e C e, \quad s(a) = e a e, \quad s : A \to B.
\]

By universal arguments we know that
\[
B = T(A)/(I - s(\bar{1})) \cong T(\bar{A})
\]

It should be possible to deduce from this or similar arguments that there are canonical isomorphisms
\[
gr^I(B) = \Omega_A^{\text{even}}, \quad gr^I(B) = \Omega_A^{\text{odd}}
\]

\[
gr^I(e C e) = \Omega_A^{\text{even}}, \quad gr^I(e C e) = \Omega_A^{\text{odd}}
\]
Now the idea will be as follows. I have explicitly constructed cyclic cocycles
\[ \overline{C}_{2n+1}(\mathcal{E}) \rightarrow I^{n+1}/[I,I^n] \]

\[ (x_0, \ldots, x_{2n+1}) \mapsto e[x_0,e] \cdots e[x_{2n+1},e]e \]
which ought to induce an injection
\[ \overline{H}_{2n+1}(A) \rightarrow I^{n+1}/[I,I^n]. \]
So it might be possible to see explicitly the exact sequence
\[ 0 \rightarrow \overline{H}_{2n+1}(A) \rightarrow I^{n+1}/[I,I^n] \xrightarrow{\delta} I^n \otimes \Omega^1_B \otimes_B \rightarrow H_{2n}(A) \rightarrow 0 \]
\[ \rightarrow I^n \otimes \overline{A} \]
by means of these calculations within \( C \).
Having to refereee Wagoner's paper it seems desirable to learn some of the background concerning subshifts of finite type, Markov partitions, Cantor-Krieger C*-algebras, and so forth. Earlier work of this sort was done June 85 pp 49-59 when I looked at David Fried's paper on Ruelle's zeta functions.

Let $A$ be a zero-one matrix, say $A = (A(x,y))$ where $x,y$ run over the finite state space $I$. Then the subshift of finite type associated to $A$ is the subspace $X_A$ of $I^\mathbb{Z}$ consisting of sequences $(x_n)$ such that $A(x_n, x_{n+1}) = 1$ for all $n$. Then $X_A$ is a compact totally disconnected metric space with the shift automorphism. It's customary to assume each row and column of $A$ is non-zero.

The Cantor-Krieger algebra $\mathcal{O}_A$ is the C*-algebra freely generated by partial isometries $s_i$, $i \in I$, (this means $s_i^* s_i$ and $s_i s_i^*$ are projectors) such that the range projectors $s_i s_i^*$ are mutually orthogonal and give the domain projectors by the rule

$$s_i^* s_i = \sum_j A(i,j) s_j s_j^*$$

The simplest case is when all $A(i,j) = 1$ whence we have the C*-algebra $\mathcal{O}_n$ for $n = \text{card}(I)$, which contains $n$ isometric embeddings $s_i \cdots s_n$ whose images decompose the Hilbert space orthogonally (assuming we have a representation. Thus

$$s_i^* s_i = 1 = \sum_j s_j s_j^*$$

for all $i$, orthogonal idempotents.
It's clear that in $K_0(C_A)$ one has $722 \equiv 1 \mod n$. Cuntz has shown $K_0(C_A) = \mathbb{Z}/(n-1)\mathbb{Z}$.

Let's now consider $A = (1, 1)$ and consider the half-infinite shift space $\omega$ consisting of all sequences $(x_0, x_1, \ldots)$ where $x_i \in \{0, 1\}$. Then the backwards shift $\sigma$ maps $\omega$ 2 to 1 onto itself. For any measurable subset $S \subseteq \omega$, the measure of $\sigma^{-S}$ is twice that of $S$. Here use the Bernoulli measure on $\omega$.

$\omega$ splits into $\{0\} \times \omega$, $\{1\} \times \omega$, each of which can be identified with $\omega$. Thus one sees how to define maps $s_0, s_1$ on functions on $\omega$ which are embeddings with complementary images. Given $f(x_0, \ldots)$ one puts

\[
(s_0 f)(x_0, x_1, \ldots) = X^0_0 (x_0) f(x_1, x_2, \ldots) \\
(s_1 f)(x_0, x_1, \ldots) = X^1_1 (x_0) f(x_1, x_2, \ldots)
\]

In this case $P_0 = s_0 s_0^*$ projects onto $f \equiv f(\ldots) = 0$ and $P_1 = s_1 s_1^*$ projects onto $f$ supported in $\{1\} \times \omega$.

Let's try to handle the general case by the same method. Again we want $P_i$ to project onto $f(x_0, \ldots)$ which are supported where $x_0 = i$. Then the $P_i$'s give an orthogonal decomposition of functions on $\omega$. 

Next let $Q_i$ project onto functions supported where $A_i(x_0) = 0$, i.e., where we can extend the sequence to $x_{i-1} = i$, or equivalently, where the sequence is the backwards shift of a sequence starting with $i$. It's then clear that

\[
Q_i = \sum A_i(x_0) P_j
\]

Maybe one can see that the shift sets up an equivalence between $Q_i, P_i$. It's clear the back-
wants shift maps the support of \( P_i \) bijectively onto the support of \( Q_i \).

To feel completely happy I would like to find a nice invariant measure on \( X_A \) on the half-sequence space \( S \) and to check that I really get a Hilbert space representation of \( Q_A \).

Let's remark that because the entries of \( A \) are \( \geq 0 \), the Frobenius theorem tells us that (at least under non-degeneracy conditions) there are unique non-negative left and right eigenvectors

\[
\sum_y A(x, y) \nu(y) = \lambda \nu(x)
\]

\[
\sum_x \mu(x) A(x, y) = \lambda \mu(x).
\]

Thus setting \( p(x, y) = \frac{1}{\lambda} A(x, y) \), we get a consistent family of measures on \( I^n \) given by

\[
\mu(x_1) p(x_1, x_2) \cdots p(x_{n-1}, x_n) \nu(x_n) = \text{measure of } (x_1, \ldots, x_n)
\]

Now let's return to Wagner's paper, which is based on Williams' paper. The problem is to understand when two \( 0,1 \) matrices give the same "dynamical system," i.e., when \( (x_A, f_A) \approx (x_B, f_B) \). Sufficient for this is for there to be an elementary string shift equivalence, i.e., \( (0,1) \) matrices \( R, S \) (of appropriate sizes) such that

\[
RS = A \quad \text{and} \quad SR = B
\]

Williams showed that any equivalence is a finite composition of elementary string shift equivalences. The idea is to define something
called a Markov partition for a dynamical system \((X, \sigma, (\mathcal{A}_i))\). This is a partition of \(X\) into open sets \(\mathcal{A}_i\) with certain properties. Any Markov partition determines a \(0,1\) matrix: \(A(i,j) = 1 \iff U_i \cap \sigma^{-1}(U_j) \neq \emptyset\), and one has a canonical isomorphism
\[
(X, \sigma) \cong (X, \mathcal{A})
\]

preserving the tautological Markov partition on the left to the given one on the right.

Thus Markov partitions are something like lattices that go into buildings. We should probably develop this analogy.

Let’s start again. Let’s begin with a dynamical system \((X, \sigma)\) consisting of a compact totally disconnected space and an automorphism. We assume it admits Markov partitions; these are partitions into open sets \(\{U_i\}_{i \in I}\) such that if \(A\) is the \(0,1\) matrix \(A(i,j) = 1 \iff U_i \cap \sigma^{-1}(U_j) \neq \emptyset\), then one gets a unique isomorphism
\[
(X, \sigma) \cong (X, \mathcal{A})
\]
carrying the standard Markov partition \(U_i = \{(x_0) | x_0 = i\}\) on the left to the given one on the right.

In the analogy we think of \((X, \sigma)\) as a f.d. vector space \(V\) over a local field \(F\) and a Markov partition as being a lattice \(L\) in \(V\).

Once we have a Markov partition then we can act by the automorphisms and produce lots more. Now there’s a concept of when two Markov partitions are close which is analogous to having two lattices one included in the other and the quotient killed by the maximal ideal.

If \(U, V\) are close then the associated
0-1 matrices $A$, $B$ are related by an elementary strong shift equivalence. This is part of Williams' proof, the rest being (according to Wagner) the connectivity of the space of Markov partitions.

Wagner's definition of "close" precisely:

$U \rightarrow V$ means that

\[
\begin{align*}
U & \sqsubseteq V < U \cap \sigma^{-1}(U) & U \rightarrow U \cap V \\
U & \sqsubset V < V \cap \sigma(V) & V \rightarrow U \cap V
\end{align*}
\]

where $\sqsubseteq$ means refines and $\sqsubset$ denotes the intersection of partitions = coarsest common refinement.

Lots of technical difficulties seem to arise because of the fact that $U$, $V$ are related by refinement. I suspect it might be possible to simplify things a lot.
April 9, 1988

Let's return to $C = A \ast C[2/2]$. We let $\alpha \mapsto \tilde{\alpha}$ denote the automorphism of $C$ such that $\tilde{\alpha} = \alpha$ and $\tilde{\alpha} = -\alpha$. Thus $\tilde{\alpha} = \alpha$.

Let $B = eCe$, $I = e\tilde{e}e = eC\tilde{e}C$.

(Review previous calculations. We have that $J = AeCeA$, $\tilde{J} = A\tilde{e}C\tilde{e}A$ are ideals in $C = A + AeCeA = A + A\tilde{e}C\tilde{e}A$. Logically, the subalgebra $AeCeA$ of $C$ is closed under left multiplication by $A, e$, and also right multiplication by $A, e$; hence it's an ideal in $C$.

Clearly $CeC = AeCeA$. Then $A + AeCeA$ is a subalgebra containing $A, e$ hence equal to $C$.)

$$C = A + \overline{AeCeA} = CeC$$

and similarly with $\alpha$'s. Then

$$eCe\tilde{e} = eA\tilde{e} + eAeCeA\tilde{e} = eCeA\tilde{e}$$

$$= eA\tilde{e} + eA\tilde{e}C\tilde{e}A\tilde{e} = eA\tilde{e}C\tilde{e}$$

giving

$$eCe = eCe\cdot eA\tilde{e} = eA\tilde{e} \cdot eCe$$

and similarly with $e, \tilde{e}$ reversed. Next

$$eCe = eAe + eA\tilde{e}C\tilde{e}Ae$$

$$eA\tilde{e} \cdot eCe \cdot eAe = eCe \cdot eA\tilde{e} \cdot eAe$$

$$\vdots$$

$$eCe = eAe + eCe \cdot eA\tilde{e} \cdot eAe$$

So iterating gives

$$eCe = eAe + eAe \cdot eA\tilde{e} \cdot eAe + eAe \cdot (eA\tilde{e} \cdot eAe)^2 + \ldots + eAe \cdot (eA\tilde{e} \cdot eAe)^n + I^{n+1}$$
Now I want to identify $e e A e$ with $\Omega_A$. How to do this is clear from the formula

\[ e e A e = e A e + e A e \cdot e A e + \cdots \]

There's a natural map of $\Omega_A$ to this because $\Omega_A \cong A \otimes A^*$. Now we know from our universal property arguments that

\[ e e A e = T(A)/(1 = p_A(1)). \]

We use this to define a representation (left module structure) of $e e A e$ on $\Omega_A$. All we have to do is to associate to $a \in A$ an operator on $\Omega_A$ such that $1 \mapsto \text{id}$. To see what the operator should be look at $\otimes$ and describe left mul.

by $e A e$.

\[ e a e \cdot e a e = e a e e + e [a a e e - e a a e] e \]

\[ e [a e] a e = e [a e] [a e] \]

better:

\[ = e a e e + e a (e^{-1} a) e \]

\[ = e (a a e) e - e a e \cdot e e a e \]

Thus we define the operator $p(a)$ on $\Omega_A$ by the formula

\[ p(a) a_0 a_1 \ldots a_n = a a_0 a_1 \ldots a_n + da_0 d a_1 \ldots d a_n \]

In other words

\[ p(a) = a + (da) d \]

Note that
\[
\rho(a_0) \rho(a_1) - \rho(a_0 a_1) = (a_0 + (da_0) d)(a_1 + (da_1) d)
- (a_0 a_1 + d(a_0) a_1) = \\
= a_0 a_1 + da_0 \cdot d \cdot a_1 + a_0 (da_1) d + (da_0) d (da_1) d
- a_0 a_1 - (da_0) a_1 d - a_0 (da_1) d = 0
\]

Thus
\[
\rho(a_0) \rho(a_1) - \rho(a_0 a_1) = da_0 \cdot da_1.
\]

Next let's try to set up a reasonable program. We are interested in the "universal" extension:

\[
0 \longrightarrow I \longrightarrow B \overset{\rho}{\longrightarrow} A \longrightarrow 0
\]

\[
T(A)/(1 = f)(\omega)
\]

Working inside \( C \) it appears that canonical representatives for Connes homomorphisms. The goal will be to prove the exact sequences you establish at the end of your paper. What this amounts to is a proof in the spirit of Connes where he relates non-comm. DR and cyclic homology.

The method I would like to use to produce Connes homomorphisms is by a Chern-Weil process. Thus I want the cyclic cochains on \( A \) to come from lift-invariant differential forms on \( GL_n(A) \) associated to connections and curvature.

Let's take \( n = 1 \). We have a map

\[
A^x \longrightarrow C^x \longrightarrow \text{Grass}
\]

Let's go over this business again.
Consider a vector space $V$ with an idempotent $e$, all supposed compatible with some inner product. Better, we can forget this but we must replace $GL(V)$ by the space of idempotents $\mathcal{E}(V)$. Over $\mathcal{E}(V)$ we have the subbundle embedded as a direct summand of the trivial bundle $\tilde{V}$, so it has a canonical connection. The principal bundle of the canonical subbundle can be identified with the space of embeddings $W = \text{Im}(e)$ as a direct factor of $V$ (this means we give $i: W \to V$ and $i^*: V \to W$ such that $i^* i = \text{id}$).

Now we pull this all back to $GL(V)$.

Let

$$W \xrightarrow{i_0} V \xrightarrow{i^*} W$$

be the inclusion and projection onto $W = \text{Im} e$. Over $GL(V)$ we have the bundle $g \mapsto gW$ of $\tilde{V}$ with its natural trivialization. Thus over $GL(V)$ we have the bundle $\tilde{W}$ embedded into $\tilde{V}$ via the map $g \mapsto \tilde{W}$. Thus we have the connection form

$$\alpha = i^* g^{-1} dg i_0$$

which is a left-invariant 1-form on $GL(V)$ with values in $\text{End}(W)$.

Here is the problem to be solved. In order
to apply the Chern--Simons transgression method I want to work in a cochain algebra, for example, the DR complex of left-invariant differential forms on with values in $\text{End}(W)$. However I want the final answer to be cyclic cochains on $\text{End}(V)$ not just Lie algebra cochains. Thus I need some version of the $L^A$ forms, which means looking at the $n$-fold direct sum of the situation and asking for $\text{GL}_n(\mathbb{C})$ invariants. But our connection form has values then in $\text{End}(W^\otimes n) = \text{End}(W) \otimes \otimes_n \mathbb{C}$. This isn’t very clear. However, maybe we should try to proceed formally. Thus let’s suppose we have the algebra $A$ acting on the vector space $V$ and the idempotent $e$ on $V$ with image $W$. I then have $\Lambda: A \rightarrow \text{End}(W)$; this is somehow a 1-form with values in operators. The next question is what should be $\Lambda^2$ and $d\Lambda$. I think they ought to be the bilinear functions $a_1, a_2 \rightarrow \Lambda(a_1) \Lambda(a_2)$ and $-\Lambda(a_1, a_2)$. So a natural question is whether these rules lead to a cochain algebra.
Consider a group $G$ acting on a vector space $V$. Let $W = \text{Im}(e)$, let $i_0 : W \hookrightarrow V$ be the inclusion and $i_0^* : V \to W$ the projection. Let $G = A^\times$. Over $G$ we have an embedding of the trivial bundle $\tilde{W}$ as a direct factor of $\tilde{V}$ given by the map

$$\tilde{W} \xrightarrow{i_0} \tilde{V} \xrightarrow{i_0^*} \tilde{W}$$

where $g$ here stands for the action of $G$ over $G$ (induced by the action $G = A^\times \to \text{GL}(V)$). This gives the connection form

$$i_0^* g^{-1} dg i_0 \in \Omega^1(G, \text{End}(\tilde{W}))$$

which is left-invariant, from which we can construct characteristic forms and Chern--Simons forms by the Chern--Weil methods, provided we have a trace defined.

Thus except for the trace question we work in the algebra of left-invariant forms

$$\Omega^\ast(G, \text{End}(\tilde{W}))^G = \text{Hom}_c(\Lambda^\ast g, \text{End}(\tilde{W}))$$

which are the same as Lie algebra cochains on the Lie algebra $g$ with values in the algebra $\text{End}(\tilde{W})$ considered as a trivial $g$--module. Thus the complex of Lie cochains

$$C^\ast(g, \text{End}(\tilde{W}))$$

is a cochain algebra. This is clear at least when $W$ is finite-dimensional as one just has the matrix algebra on $C^\ast(g)$. 
More generally it ought to be the case that on the Lie algebra cochain complexes one has cup products
\[ C^*(g, M) \times C^*(g, N) \longrightarrow C^*(g, M \otimes N) \]
for \( M, N \) \( g \)-modules with usual associativity etc. properties.

So much for the Lie algebra cochains; let's consider next the algebra cochains which should be simpler (both the formulæ for \( d \) and \( \sigma \) are simpler).

Thus we consider \( \text{End}(W) \) as a trivial \( A \)-module which means we forget that \( A \) is unital, and so we work in the complex of normalized Hochschild cochains on \( A^+ \) with values in \( \text{End}(W) \):

\[ C^p_N(A^+, \text{End}W) = \text{Hom}(A^{op}, \text{End}W) \]

This will form a cochain algebra with product
\[ (\psi \chi)(a_1, \ldots, a_p, a_{p+1}, \ldots) = \psi(a_1, \ldots, a_p) \chi(a_{p+1}, \ldots) \]
\[ (b \psi) (a_1, \ldots, a_m) = \sum_{I} b(I) \phi(\ldots, q_{i}, q_{i+1}, \ldots) \]

Thus we have found a cochain algebra in which if \( \alpha : A \rightarrow \text{End}(W) \) is a 1-cochain, then we have

\[ (\alpha^2)(a_0, a_1) = \alpha(a_0) \alpha(a_1) \]
\[ (d \alpha)(a_0, a_1) = - \alpha(a_0 a_1) \]
\[ (d \alpha + \alpha^2)(a_0, a_1) = \alpha(a_0) \alpha(a_1) - \alpha(a_0 a_1) \]
Recall that in the situation of interest $B$ is $\text{End}(W)$ essentially. Thus we are led to the cochain algebra $C^*_N(A^+, B)$.

Unfortunately, there is going to be some confusion of the unit - monoidal sort, but for the moment let’s continue and start applying the Chern - Simons - Weil algebra.

Actually we are above all interested in the case where $B = T(A)/(I = f(A))$, where we take the DG subalgebra generated by $\chi = f: A \to B$. What we have to do is to apply the Chern - Simons game. This means we have just a cochain algebra map

$$C\langle \chi, d\chi \rangle \longrightarrow C^*_N(A^+, B)$$

What is important is the filtration, i.e. the powers of the ideal $I = \text{Ker}(B \to A)$. This is why we want the curvature $\chi^2 + d\chi$. Thus there’s probably a natural filtration on $C\langle \chi, d\chi \rangle$ which should be described.

We need to study the $I$ - adic filtration on $R = C\langle \chi, d\chi \rangle$ where $I$ is the ideal generated by the "curvature" $\beta = \chi^2 + d\chi$. First we note that

$$d\beta = d\chi \cdot \chi - \chi \cdot d\chi = [\beta, \chi]$$

hence the ideal $I$ is stable under $d$. This implies that the associated graded algebra...
$\text{gr} \mathcal{I}(R)$ is a cochain algebra.

We can easily determine $\text{gr} \mathcal{I}(R)$ as an algebra. Note that as a graded algebra $R^*$ is the non-commutative polynomial ring on $\alpha$ and $\beta$ with $\deg \alpha = 1$ and $\deg \beta = 2$. Consider the grading giving the $\beta$-degree. Then we have natural representatives for $I^n / I^{n+1}$ in this grading, namely those polynomial which are linear combinations of monomials in $\alpha, \beta$ with exactly $n \beta$'s occurring.

Thus $\text{gr} \mathcal{I}(R) = R^*$ is a free algebra generated by the image of $\alpha$ in $\text{gr} \mathcal{I}_0(R) = R/I$ and the image of $\beta$ in $\text{gr} \mathcal{I}(R) = I/I^2$.

Thus

$$\text{gr} \mathcal{I}(C<\alpha, d\alpha>) = C<\alpha, \beta>$$

where

$$d \alpha = -\alpha^2$$

$$d \beta = [\beta, \alpha)$$

What is the cohomology $H^*(\text{gr} \mathcal{I}(R), d)$?

$$R'/I' = C[\alpha], \quad d\alpha = -\alpha^2$$

so

$$H^*(R'/I') = C.$$  In effect when you apply $d$ to $\alpha^n$ you get an alternating sum of the terms $-\alpha^{n+1}$ and there are $n$ terms; thus you get $d\alpha^n = 0$ for even and $d\alpha^n = -\alpha^{n+1}$ for odd.

Maybe it's not so interesting what the cohomology of $\text{gr} \mathcal{I}(R)$ is, since we ultimately
want to pass to commutator quotients.

Let's return to basic principles. We have the cochain algebra $C^*_N(A^+, B)$ and the element $x = f: A \to B$. Recall that in the universal case $B = T(A)/(1=\rho(1))$, we have that the 2-cochain

$$(dx + x^2)(a_0, a_1) = \rho(a_0)f(a_1) - f(a_0a_1)$$

has values in the ideal $I = \ker (B \to A)$. Actually this would be true for any extension $0 \to I \to B \xrightarrow{\beta} A \to 0$ and $f$ a lifting. Therefore the homomorphism

$$\mathcal{C}(x, dx) \to C^*_N(A^+, B)$$

is compatible with the filtration where the ideal $J$ on the left is generated by $\beta$ and the ideal $J_0$ on the right is generated by $C^2_N(A^+, I)$.

Now our goal is to prove the Cunnes homomorphisms associated to the extension $B/I = A$, and so it is a very good idea to see if this works in the expected way. The issue is again the shift in degree. Thus from the curvature $dx + x^2$ in $C^2_N(A^+, I)$ we expect a cyclic 1-cocycle on $A$ with values in $I/[B, I]$.

Let's see what we need to construct a trace in the algebra $C^*(A, B) = C^*_N(A^+, B)$.

Is it possible to describe the commutator quotient?
If $A$ is finite-dimensional, then it looks as if $C^\cdot(A, B)$ is generated by $C^0(A, B) = B$ and $C^1(A, B) = \text{Hom}_B(A, B)$. In fact, we have the commutative square

$$C^2(A, B) \otimes_B C^1(A, B) \hookrightarrow B \otimes A^* \otimes A^*$$

$$(a_1 \mapsto b_1 \lambda_1(a_1) \otimes (a_2 \mapsto b_2 \lambda_2(a_2)) \hookrightarrow (b_1 \otimes \lambda_1) \otimes (b_2 \otimes \lambda_2)$$

$$\downarrow$$

$$(a_1, a_2) \mapsto b_1 b_2 \lambda_1(a_1) \lambda_2(a_2) \hookrightarrow b_1 b_2 \otimes \lambda_1 \otimes \lambda_2$$

where the horizontal arrows are isomorphisms in the finite-dimensional case, and similarly for cochains of higher degree. Thus $C^\cdot(A, B)$ is essentially the tensor algebra of the $B$-bimodule $C^1(A, B)$ over $B$.

Recall that the commutator quotient of $T_B(M) = \bigoplus_{n \geq 0} M \otimes_B \cdots \otimes_B M$ is $\bigoplus_{n \geq 0} (M \otimes_B)^n$. In the present situation we view $T_B(M)$ as a superalgebra with $M$ of odd degree and so the commutator quotient is

$$T_B(M)/[T_B(M), T_B(M)] = \bigoplus_{n \geq 0} (M \otimes_B)^n$$

In the case of interest $M = B \otimes A^*$ essentially...
We are, therefore, that it should be possible
to define a trace map
\[(*) \quad C^0(A, B) \xrightarrow{\sim} \text{Hom}(C^0(A), B/[B,B])\]
compatible with differentials and shifting degree
by 1. Moreover, this should be best possible,
therefore, if it should be an isomorphism between the
commutator quotient of the algebra on the left
with the complex on the right, when \( A \) is finite-
dimensional.

We need the formula for this trace map
\[(*) \text{ in dimension 1 it must be the map}\]
\[C^1(A, B) = \text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(A, B/[B,B])\]
induced by the universal trace \( B \rightarrow B/[B,B] \). In
higher dimensions, it must be cyclic \( A \)-symmetrizing,
up to a constant factor. So I try
on \( C^2(A, B) \) the formula
\[(\tau \phi)(a_1, a_2) = \phi(a_1, a_2) - \phi(a_2, a_1)\]
Then
\[\tau (d\phi)(a_1, a_2) = (d\phi)(a_1, a_2) - (d\phi)(a_2, a_1)\]
\[= -\phi(a_1 a_2) + \phi(a_2 a_1)\]
\[(b \tau \phi)(a_1, a_2) = (\tau \phi)(a_1, a_2) - (\tau \phi)(a_2, a_1)\]
\[= \phi(a_1 a_2) - \phi(a_2 a_1)\]
On $C^3(A, B)$ let's try
\[(τφ)(a_1, a_2, a_3) = φ(a_1, a_2, a_3) + φ(a_2, a_3, a_1) + φ(a_3, a_1, a_2)\]

Then
\[τdφ(a_1, a_2, a_3) = (dφ)(a_1, a_2, a_3) + cyc.\]
\[= -φ(a_1, a_2, a_3) - φ(a_2, a_3, a_1) - φ(a_3, a_1, a_2) + φ(a_1, a_2, a_3) + φ(a_2, a_3, a_1) + φ(a_3, a_1, a_2)\]
\[= 0\]
\[(bτφ)(a_1, a_2, a_3) = (τφ)(a_1, a_2, a_3) - (τφ)(a_2, a_3, a_1) + (τφ)(a_3, a_1, a_2)\]
\[= φ(a_1, a_2, a_3) - φ(a_1, a_2, a_3) + φ(a_2, a_3, a_1) - φ(a_2, a_3, a_1) - φ(a_3, a_1, a_2) + φ(a_3, a_1, a_2)\]

Thus it appears as if we should have $T = λ$-symmetrization:

\[(τφ)(a_1, ..., a_n) = φ(a_1, ..., a_n) + (-1)^{n-1} φ(a_2, ..., a_n, a_1) + ... + (-1)^{n-1} φ(a_n, a_1, a_2, ..., a_{n-1})\]

in which case we have $Td = -bT$.

But this should all be clear because we have $d = -b'$ on $C^*(A, B)$, and because of the identity $Nb' = bN$ on cochains.
April 11, 1988

I need some details for the Kassel letter.

Let $C = A \otimes A \otimes B \otimes A$ be the GNS algebra associated to $\phi$. $C$ contains the idempotent

$e = 1 \otimes 1 \otimes 1$ and

$eae = (1, 1, 1)a(1, 1, 1) = (1, 1, 1)(a, 1, 1) = (1, \phi(a), 1)$

$e(a, b, a')e = (1, 1)(a, b, a')(1, 1) = (1, 1)(a, b, \phi(a'), 1)$

$= (1, \phi(a)b, \phi(a'), 1)$.

so that $eCe = 1 \otimes B \otimes 1 = B$. We have

$AeCeA = A \otimes B \otimes A$

is an ideal in $C$. Thus

$AeCeA = CeC$

Let $\tilde{e} = 1 - e$. Is it true that $AeC\tilde{e}A$

is an ideal in $C$? We have to check that

it's closed under multiplication by $B = eCe$.

$eCe \cdot AeC\tilde{e}A \subseteq eCeC\tilde{e}A$

$e(A + AeCeA)\tilde{e} = eA\tilde{e} + \frac{eAeCeAe\tilde{e}}{eCe}$

$= eCe eA\tilde{e}$

It doesn't seem to work.