October 5, 1988

Let e, e' be idempotents over R which are congruent modulo an ideal I. Let τ be a trace on I^n.

The pair (e,e') determines an element of $K_0(I)$. (This is not immediately obvious. One has to manipulate to arrange that the idempotent over R/I have its entries in $\mathbb{C}$. However it follows from Milnor's excision property, or really his patching theory, that the relative K_0 group for $I \to R \to R/I$ depends only on I.)

On the other hand $K_0(I^n) \twoheadrightarrow K_0(I)$, so the element $[e]-[e']$ in $K_0(I)$ can be lifted to $K_0(I^n)$ and then paired with the trace τ.

Let's carry this out. We want to modify e' by conjugation so that it becomes congruent to e mod I^n. We find $1-\mu$ which intertwines e and e' and is such that $\mu \equiv 0$ mod I. A first guess is $1-\mu = ee' + (1-e)(1-e')$ which satisfies

$$e(1-\mu) = ee' = (1-\mu)e'$$

Next $1-\mu$ is invertible modulo I^n, but it can't be lifted to an invertible over R. However the Whitney sum of $1-\mu$ and a "parametrix" can be lifted. We have the invertible matrix

$$\begin{pmatrix} 1-\mu & -\mu^n \\ \mu^n & 1+\cdots+\mu^{2n-1} \end{pmatrix} \cdot \begin{pmatrix} 1+\cdots+\mu^{2n-1} & \mu^n \\ -\mu^n & 1-\mu \end{pmatrix} = 1$$

and we conjugate $\begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix}$ by it to obtain the idempotent

$$\tilde{e}' = \begin{pmatrix} (1-\mu)e'(1+\cdots+\mu^{2n-1}) & (1-\mu)e'\mu^n \\ \mu^n e'(1+\cdots+\mu^{2n-1}) & \mu^n e'\mu^n \end{pmatrix}$$

$$= \begin{pmatrix} e(1-\mu^{2n}) & e(1-\mu^n)\mu^n \\ \mu^n e'(1+\cdots+\mu^{2n-1}) & \mu^n e'\mu^n \end{pmatrix}$$

which is congruent to $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$ mod I^n. Then the pairing with τ is

$$\tau\left(\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} - \tilde{e}'\right) = \tau(e\mu^{2n} - \mu^n e'\mu^n)$$

$$= \tau((e-e')\mu^{2n})$$

provided $\tau([I^n, I^n]) = 0$.

Finally we need μ. Put $\varepsilon = 2e'-1$, $F = 2e-1$

Then $1-\mu = ee' + (1-e)(1-e') = \left(\frac{1+F}{2}\right)\left(\frac{1+\varepsilon}{2}\right) + \left(\frac{1-F}{2}\right)\left(\frac{1-\varepsilon}{2}\right)$

$$= \frac{1+F\varepsilon}{2} = \frac{1+g}{2} \qquad g = F\varepsilon$$

and $\mu = 1 - \frac{1+g}{2} = \frac{1-g}{2}$

$$\mu^2 = -g\underbrace{\frac{(1-g)(1-g^{-1})}{4}}_{\text{commutes with } F,\varepsilon} \qquad \begin{matrix} (F-\varepsilon)g & = & g^{-1}(F-\varepsilon) \\ \| & & \| \\ (F-\varepsilon)F\varepsilon & & \varepsilon F(F-\varepsilon) \end{matrix}$$

so

$$\tau((e-e')\mu^{2n}) = \tau\left(\frac{F-\varepsilon}{2}(-g)^n\left(\frac{(1-g)}{2}\frac{(1-g^{-1})}{2}\right)^n\right)$$

Now

$$\frac{(1-g)(1-g^{-1})}{4} = \frac{(1-F\varepsilon)(1-\varepsilon F)}{4} = \frac{1}{4}(2-F\varepsilon-\varepsilon F)$$

$$= \frac{1}{4}(F-\varepsilon)^2 = \left(\frac{F-\varepsilon}{2}\right)^2.$$

Also $\quad \frac{F-\varepsilon}{2}F = -\varepsilon\frac{F-\varepsilon}{2}$. $\quad$ Thus

$$\tau((e-e')\mu^{2n}) = \tau\left((-F\varepsilon)^n\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\right)$$

$$= \tau\left((-F\varepsilon)^{n-1}F\underbrace{\varepsilon}_{-\varepsilon}\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\right)$$

$$= \tau\left((-F\varepsilon)^{n-1}F\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\varepsilon\right)$$

$$= \tau\left((-\varepsilon)(F\varepsilon)^{n-2}F\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\right)$$

$$= \tau\left((-\varepsilon F)^{n-1}\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\right) \qquad \text{etc.}$$

$$\tau((e-e')\mu^{2n}) = \tau\left(\left(\frac{F-\varepsilon}{2}\right)^{2n+1}\right) = \tau\left((e-e')^{2n+1}\right).$$

Recall also the calculation

$$\tau\left((F-\varepsilon)^{2n+1}\right) = \tau\left(F(F-\varepsilon)^{2n}\right) - \tau\left(\underbrace{\varepsilon(F-\varepsilon)}_{F}(F-\varepsilon)^{2n-1}\right)$$

$$= 2\tau\left(F(F-\varepsilon)^{2n}\right)$$

$$= 2\tau\left((F-\varepsilon)^{2n-1} - F\underbrace{\varepsilon(F-\varepsilon)^{2n-1}}_{-F}\right)$$

$$= 4\tau\left((F-\varepsilon)^{2n-1}\right)$$

Let's now return to the problem under discussion which is to link Connes approach to even cyclic cocycles based on $A*A$ with my approach based on Chern-Simons forms and the algebra $B = T(A)/(1=\rho(1_A)) = $ even subalgebra of $A*A$.

In my approach I show how to associate to a trace on B/I^{n+1} a cyclic $2n$-cocycle on A. In Connes approach he associates to a trace on J^{2n+1} a cyclic $2n$-cocycle on A. Here $J = \mathrm{Ker}\{A*A \to A\}$.

Since B sits inside $A*A$ and I sits inside J^2, one would like to find a correspondence between traces on B/I^{n+1} and on J^{2n+1} which is compatible with the construction of cyclic cocycles.

There is an obvious difficulty with the constants which one sees upon pairing with K-classes. If e is an idempotent over A then its pairing with τ on J^{2n+1} is as we have seen

$$\tau\left((e-\bar{e})^{2n+1}\right)$$

where $a \mapsto a$, $a \mapsto \bar{a}$ are the two canonical maps $A \to A*A$.

On the other hand suppose we are given a trace τ on B/I^{n+1} and an idempotent e over $B/I = A$. To pair τ and e we lift e to $\rho(e)$ in B and substitute into a polynomial $f(x)$ which is $\equiv 0 \mod x^{n+1}$ and $\equiv 1$ modulo $(X-1)^{n+1}$. Such a polynomial represents the unique idempotent in

$$\mathbb{C}[X]/(X^2-X)^{n+1} \xrightarrow{\sim} \mathbb{C}[X]/(X^{n+1}) \times \mathbb{C}[X]/(X-1)^{n+1}$$

which is zero in the first factor and 1 in the second. Thus $f(\rho(e))$ will be an idempotent in B/I^{n+1} lifting e in A. Then we apply τ to $f(\rho(e))$.

The polynomial $f(X)$ is unique if we require its degree to be $\leq 2n+1$, and it is given by

$$f(X) = \int_0^X t^n(1-t)^n \, dt \Big/ \int_0^1 t^n(1-t)^n \, dt$$

since $f'(X)$ clearly vanishes to nth order at both $X = 0, 1$. We have also

$$f(X) = \int_0^1 (tX)^n(1-tX)^n X \, dt \cdot \beta(n+1, n+1)^{-1}$$

$$= \int_0^1 X(tX - t^2X^2)^n \, dt \cdot \beta(n+1, n+1)^{-1}$$

Thus we have the formula

$$\langle [\tau], [e] \rangle = \tau\left(\int_0^1 \rho(e)\left(t\rho(e) - t^2\rho(e)^2\right)^n dt\right)\left(\frac{(2n+1)!}{n!\,n!}\right)$$

Now let's recall the formulas for the cyclic $2n$-cocycle attached to τ on B/I^{n+1}. It is

$$\boxed{\varphi_\tau^{(2n)} = \operatorname{trace} \int_0^1 \rho \frac{(t\delta\rho + t^2\rho^2)^n}{n!} \, dt}$$

where trace means we apply τ to the values of the cochain and $\wedge$ to the arguments.

Let's apply $\varphi_\tau^{(2n)}$ to $(\overbrace{e, \ldots, e}^{2n+1})$. We get, with the good sign conventions $(\delta\rho)(a_1, a_2) = \rho(a_1 a_2)$, $\rho^2(a_1, a_2) = -\rho(a_1)\rho(a_2)$,

$$\varphi_\tau^{(2n)}(e,\dots,\overset{2n+1}{e}) = (2n+1)\,\tau\left(\int_0^1 \rho(e)\frac{\left(t\rho(e)-t^2\rho(e)^2\right)^n}{n!}\,dt\right)$$

Hence we conclude

$$\boxed{\langle[\tau],[e]\rangle = \varphi_\tau^{(2n)}(e,\dots,\overset{2n+1}{e})\,\frac{(2n)!}{n!}}$$

which checks with our earlier formula for the class in $HC_{2n}(A)$ corresponding ~~to~~ $[e]$ being represented by $\frac{(2n)!}{n!}\,e^{\otimes(2n+1)} \in A_\lambda^{\otimes(2n+1)}$

Next let us consider a trace τ defined on J^{2n+1} and define the associated cyclic $2n$-cocycle with the appropriate constants. We have two homomorphisms $a \mapsto a, \bar{a}$ from A to $A * A$ whence two flat connection forms

$$\theta, \bar\theta \in C^1(A, A*A)$$

We use the linear path

$$D_t = \delta + \bar\theta + t(\theta - \bar\theta)$$

with curvature

$$D_t^2 = t\underbrace{[\delta+\bar\theta,\theta-\bar\theta]}_{-\theta^2+\bar\theta^2+[\bar\theta,\theta]-2\bar\theta^2 \;=\; -(\theta-\bar\theta)^2} + t^2(\theta-\bar\theta)^2$$

$$= (t^2-t)(\theta-\bar\theta)^2.$$

This path leads to the closed form

$$\operatorname{trace}\int_0^1 \dot{D}_t\frac{(D_t^2)^n}{n!}\,dt = \operatorname{trace}(\theta-\bar\theta)^{2n+1}\int_0^1\frac{(t^2-t)^n}{n!}\,dt$$

$$\boxed{\varphi_\tau^{(2n)} = (-1)^n\frac{n!}{(2n+1)!}\operatorname{trace}(\theta-\bar\theta)^{2n+1}}$$

Applying the cochain $(\theta-\bar{\theta})^{2n+1}$ to $(a_0, \dots, a_{2n})$ gives

$$\underbrace{(-1)^{\frac{(2n)(2n+1)}{2}}}\ (a_0-\bar{a}_0)\cdots(a_{2n}-\bar{a}_{2n})$$

$$(-1)^{2n+2n-1+\cdots+1} \qquad = (-1)^n$$

Thus

$$\boxed{\varphi_\tau^{(2n)}(a_0, \dots, a_{2n}) = \frac{n!}{(2n)!}\, tr\,(a_0-\bar{a}_0)\cdots(a_{2n}-\bar{a}_{2n})}$$

where the $2n+1$ is removed because of applying N.

Now we have to take up the main problem of relating traces on J^{2n+1} and B/I^{n+1}

October 6, 1988

Let A be unital and consider the DGA of (bar) cochains with values in the Cuntz algebras $A*A$ or $(A*A)\otimes \mathbb{C}[F] = \varepsilon A$. We have two algebra homomorphism $a \mapsto a$, $a \mapsto \bar{a} = FaF$, whence we have 1-cochains θ, $\bar{\theta} = F\theta F \in C^1(A, A*A)$, which are "flat".

It might be better to say that we have the flat connection $\delta + \theta$ over $C^\cdot(A, A*A)$ and the automorphism of order 2 given by F conjugating. Then we decomposition the connection

$$\delta + \theta = \underbrace{\delta + \frac{\theta + \bar{\theta}}{2}}_{\nabla} + \underbrace{\frac{\theta - \bar{\theta}}{2}}_{\alpha}$$

into parts even and odd with respect to the involution. As usual

$$[\nabla, \alpha] = 0 \qquad \nabla^2 + \alpha^2 = 0$$

Let τ be a trace on εA (resp. on $A*A$). Then combining τ with N we obtain a trace on $C^\cdot(A, \varepsilon A)$ (resp. $C^\cdot(A, A*A)$), and we obtain cyclic cocycles

$$tr(F\alpha^{2n}) \qquad \text{resp.} \qquad tr(\alpha^{2n+1}).$$

Now these cyclic cocycles for different n are linked by Connes S-operator, and to understand this relation we have been led to a family of Hochschild cochains which appear to be more basic than the cyclic cocycles. Let's review this in the ungraded case first and then work out the graded version.

We start by lifting the cyclic cocycle $tr(F\alpha^{2n})$ to the bar cochain $\tau(F\alpha^{2n})$. Then with $\rho = \frac{\theta + \bar\theta}{2}$ so $\nabla = \delta + \rho$ we have

$$0 = [\nabla, \alpha] = \delta\alpha + [\rho, \alpha],$$

so

$$\delta\tau(F\alpha^{2n}) = \tau(F(-\rho\alpha^{2n} + \alpha^{2n}\rho))$$
$$= \beta\,\tau(F\alpha^{2n}\partial\rho)$$

On the other hand

$$\partial\tau(F\alpha^{2n+2}) = \tau(\underbrace{F\sum_{i=0}^{2n+1}\alpha^i\,\partial\alpha\,\alpha^{2n+1-i}}_{(-1)^{i-1}})$$
$$= (2n+2)\,\tau(F\alpha^{2n+1}\partial\alpha)$$

$$\delta\tau(F\alpha^{2n}\partial\rho) = \tau(F((-\rho\alpha^{2n} + \alpha^{2n}\rho)\partial\rho + \alpha^{2n}\partial\delta\rho))$$
$$= \tau F(\alpha^{2n}(\underbrace{\partial\delta\rho + \rho\partial\rho + \partial\rho\,\rho}_{\partial(\delta\rho + \rho^2) = \partial(-\alpha^2)}))$$
$$= -\tau(F(\alpha^{2n}(\partial\alpha\,\alpha + \alpha\,\partial\alpha))) = -2\tau(F(\alpha^{2n+1}\partial\alpha))$$

Now take the "graded case". We lift $tr(\alpha^{2n-1})$ to the bar cochain $\tau(\alpha^{2n-1})$

$$\delta\tau(\alpha^{2n-1}) = \tau(-\rho\alpha^{2n-1} - \alpha^{2n-1}\rho)$$
$$= -\beta\tau(\alpha^{2n-1}\partial\rho)$$

$$\partial\tau(\alpha^{2n+1}) = \tau\sum_{i=0}^{2n}\alpha^i\,\partial\alpha\,\alpha^{2n-i} = (2n+1)\tau(\alpha^{2n}\partial\alpha)$$

$$\delta\tau(\alpha^{2n-1}\partial\rho) = \tau\big((-\rho\alpha^{2n-1}-\alpha^{2n-1}\rho)\partial\rho - \alpha^{2n-1}\partial\delta\rho\big)$$

$$= -\tau\big(\alpha^{2n-1}(\partial\rho\rho+\rho\partial\rho+\partial\delta\rho)\big)$$

$$= \tau\big(\alpha^{2n-1}(\partial\alpha\cdot\alpha+\alpha\cdot\partial\alpha)\big)$$

$$= 2\tau(\alpha^{2n}\partial\alpha)$$

I think the above proof becomes uniform in the two cases, ~~[illegible]~~ if one uses a supertrace τ on $A * A$.

$$\delta\tau(\alpha^n) = \tau(-[\rho,\alpha^n])$$

$$= \tau(-\rho\alpha^n+(-1)^n\alpha^n\rho)$$

$$= (-1)^n\,\beta\,\tau(\alpha^n\partial\rho)$$

$$\partial\tau(\alpha^{n+2}) = \tau\Big(\sum_0^{n+1}\alpha^i\,\partial\alpha\,\alpha^{n+1-i}\Big)$$

$$= \sum_{i=0}^{n+1}\tau\Big(\alpha^{n+1-i}\alpha^i\,\partial\alpha\,\underbrace{(-1)^{(i+1)(n+1-i)}}_{\text{cochain sign}}\underbrace{(-1)^{(i+1)(n+1-i)}}_{\substack{A*A\\ \text{sign}}}\Big)$$

$$= (n+1)\,\tau(\alpha^{n+1}\partial\alpha)$$

~~[illegible]~~

$$\delta\tau(\alpha^n\partial\rho) = \tau\big((-\rho\alpha^n+(-1)^n\alpha^n\rho)\partial\rho \overset{(-1)^n}{+} \alpha^n\partial\delta\rho\big)$$

$$= \underbrace{\tau(-\rho\alpha^n\partial\rho)}_{\tau\big(-\alpha^n\partial\rho\rho\underbrace{(-1)^{n+1}}_{\substack{\text{cochain}\\ \text{sign}}}\underbrace{(1)}_{\substack{A*A\\ \text{sign}}}\big)\;=\;(-1)^n\tau(\alpha^n\partial\rho\rho)} + (-1)^n\tau\big(\alpha^n(\rho\partial\rho+\partial\delta\rho)\big)$$

$$= (-1)^n\tau(\alpha^n\partial(-\alpha^2)) = (-1)^{n+1}\tau\big(\alpha^n(\partial\alpha\,\alpha+\alpha\,\partial\alpha)\big)$$

$$= (-1)^{n+2}2\tau(\alpha^n\partial\alpha)$$

This seems OK but it needs to be carefully checked on the cochain level. Let's first check that the linear functionals $\omega \mapsto \tau(F\omega)$, $\omega \mapsto \tau(\omega)$ in the ungraded and graded cases respectively give supertraces on $A*A$.

τ is a trace on $(A*A)\tilde{\otimes}\mathbb{C}[F]$. Then $\tau(\omega') = 0$ for any $\omega' \in$ ______ anti-comm. with F, in particular if $\omega' = \omega$ or ωF with $\omega \in (A*A)^-$. Thus $\omega \mapsto \tau(F\omega)$, $\omega \in A*A$ is supported on $(A*A)^+$. To check the supertrace identity let $\omega_1, \omega_2 \in (A*A)^+$, i.e. they commute with F. Then

$$\tau(F\omega_1\omega_2) = \tau(\omega_1 F\omega_2) = \tau(F\omega_2\omega_1)$$

If ω_1, ω_2 both anti-commute with F, then

$$\tau(F\omega_1\omega_2) = -\tau(\omega_1 F\omega_2) = -\tau(F\omega_2\omega_1)$$

Thus we have a supertrace which is in fact even, i.e. supported on $(A*A)^+$.

Next let τ be a trace on $A*A$ which is odd, i.e. supported on $(A*A)^-$. Then automatically τ is a supertrace on $A*A$, since the difference between traces and supertraces occurs with the product of two odd elements, and τ applied to such a product is zero.

Maybe the simplest way to remember this is as follows:

$$(A*A)\tilde{\otimes}\mathbb{C}[F] = (A*A)\hat{\otimes}C_1 \qquad \text{as superalgebras.}$$

A trace on this algebra is supported in $(A*A)^+\otimes C_1$; if in addition we require it to be odd, i.e. supported in $(A*A)^- \oplus (A*A)^+F$, then it is supported in $(A*A)^+F$.

Better: An odd trace on a superalgebra is the same as an odd supertrace. Thus an odd trace on $(A*A)\hat{\otimes}C_1$ is the same as an odd supertrace which is the same as an even supertrace on $A*A$.

Similarly an odd trace on $A*A$ is the same thing as an ~~odd~~ supertrace. Thus the two cases together are [redacted]

$$\text{odd traces on}\left\{\begin{array}{lcl} (A*A)\hat{\otimes}C_1 & \longleftrightarrow & \text{even} \\ A*A & \longleftrightarrow & \text{odd}\end{array}\right\}\text{supertraces on } A*A.$$

The motivation for the left side comes from the positive cocycle idea, namely, reconstructing the Hilbert space.

Now let's go back to the actual cochains. Let τ be a supertrace defined on $A*A$ of a given parity, say [redacted] odd to begin with. Then we define a sequence of Hochschild cochains which are normalized

$$\psi_{2n}(a_0,\dots,a_{2n}) = \tau\{\rho(a_0)\alpha(a_1)\dots\alpha(a_{2n})\}$$

Notice that $\psi_{2n}(1,a_1,\dots,a_{2n}) = \tau\{\alpha(a_1)\dots\alpha(a_{2n})\}$ is invariant under cyclic permutations with sign because τ is a supertrace. Thus

$$(B\psi_{2n})(a_1,\dots,a_{2n}) = \sum_{i=1}^{2n}(-1)^{i-1}\psi_{2n}(1,a_i,\dots,a_{2n},a_1,\dots,a_{i-1})$$

$$= 2n\,\psi_{2n}(1,a_1,\dots,a_{2n})$$

Next we want to compute $b\psi_{2n}$ for this we need some identities

$$\alpha(a_1 a_2) = \frac{a_1 a_2 - \bar a_1 \bar a_2}{2} = \frac{(a_1 - \bar a_1)}{2}\frac{(a_2 + \bar a_2)}{2} + \frac{(a_1 + \bar a_1)}{2}\frac{(a_2 - \bar a_2)}{2}$$
$$= \alpha(a_1)\rho(a_2) + \rho(a_1)\alpha(a_2)$$

$$\rho(a_1 a_2) - \rho(a_1)\rho(a_2) = \frac{a_1 a_2 + \bar a_1 \bar a_2}{2} - \frac{(a_1 + \bar a_1)(a_2 + \bar a_2)}{2 \cdot 2}$$
$$= \frac{a_1 a_2 + \bar a_1 \bar a_2 - a_1 \bar a_2 - \bar a_1 a_2}{4} = \frac{(a_1 - \bar a_1)}{2}\frac{(a_2 - \bar a_2)}{2}$$
$$= \alpha(a_1)\alpha(a_2)$$

Now

$$(b\psi_{2n})(a_0, \ldots, a_{2n+1}) = \tau\{\rho(a_0 a_1)\alpha(a_2)\cdots\alpha(a_{2n+1})\}$$
$$- \tau\{\rho(a_0)[\alpha(a_1)\cancel{\rho(a_2)} + \rho(a_1)\alpha(a_2)]\alpha(a_3)\cdots\}$$
$$+ \tau\{\rho(a_0)\alpha(a_1)[\alpha(a_2)\cancel{\rho(a_3)} + \cancel{\rho(a_2)\alpha(a_3)}]\alpha(a_4)\cdots\}$$
$$\cdots\cdots$$
$$+ \tau\{\rho(a_0)\alpha(a_1)\cdots\alpha(a_{2n-1})[\alpha(a_{2n})\rho(a_{2n+1}) + \cancel{\rho(a_{2n})\alpha(a_{2n+1})}]\}$$
$$- \tau\{\rho(a_{2n+1}a_0)\alpha(a_1)\cdots\alpha(a_{2n})\}$$
$$= \tau\{\underbrace{[\rho(a_0 a_1) - \rho(a_0)\rho(a_1)]}_{\alpha(a_0)\alpha(a_1)}\alpha(a_2)\cdots\alpha(a_{2n+1})\}$$
$$- \tau\{\underbrace{[\rho(a_{2n+1}a_0) - \rho(a_{2n+1})\rho(a_0)]}_{\alpha(a_{2n+1})\alpha(a_0)}\alpha(a_1)\cdots\alpha(a_{2n})\}$$
$$= 2\tau\{\alpha(a_0)\cdots\alpha(a_{2n+1})\}$$
$$= 2\psi_{2n+2}(1, a_0, \ldots, a_{2n+1})$$

This same calculation could have been done for

$$\boxed{\begin{aligned} \psi_n(a_0, \ldots, a_n) &= \tau\{\rho(a_0)\alpha(a_1)\cdots\alpha(a_n)\} \\ (B\psi_n)(a_1, \ldots, a_n) &= n\psi_n(1, a_1, \ldots, a_n) \\ (b\psi_n)(a_0, \ldots, a_{n+1}) &= 2\psi_{n+2}(1, a_0, \ldots, a_{n+1}) \end{aligned}}$$

Conversely, suppose one is given a family of cochains $\psi_n(a_0, \ldots, a_n)$ satisfying the properties

1) $\psi_n(a_0, a_1, \ldots, a_n) = 0$ if $a_i = 1$ some $i \geq 1$

2) $\psi_n(1, a_1, \ldots, a_n)$ is λ-invariant, equivalently

$$B\psi_n([redacted]\, a_1, \ldots, a_n) = n\,\psi_n(1, a_1, \ldots, a_n).$$

3) $b\psi_n(a_0, \ldots, a_{n+1}) = 2\,\psi_{n+2}(1, a_0, \ldots, a_{n+1}).$

Do we actually have a supertrace on $A * A$. The point is that one has an isomorphism

$$\bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n} \xrightarrow{\ \sim\ } A * A$$

$$(a_0, a_1, \ldots, a_n) \longmapsto \rho(a_0)\,\alpha(a_1) \cdots \alpha(a_n)$$

so that τ is completely determined by the cochains $\psi_n(a_0, \ldots, a_n)$. In fact we can say that a linear functional on $A * A$ is equivalent to a sequence of normalized cochains $\{\psi_n(a_0, a_1, \ldots, a_n), n \geq 0\}$

October 7, 1988

Try to obtain the Jaffe-Lesniewski cocycles, using present familiarity with Hochschild cochains. On skew-adjoint ops. we have

$$\bar{g} = \frac{1+X}{1-X} = -1 + \frac{2}{1-X}$$

$$\omega = \bar{g}^{-1} d\bar{g} = \frac{2}{1+X} dX \frac{1}{1-X}$$

$$d\omega = -\omega^2 \qquad d(\omega^2) = 0.$$

$$d\{\mathrm{tr}(\omega)^{2n+1}\} = -\mathrm{tr}(\omega^{2n+2}) = 0$$

So now fix X_0 and consider $g \longmapsto g X_0 g^{-1}$, which maps unitaries to skew-adj. ops. Then

$$dX = g[g^{-1}dg\, X_0 - X_0 g^{-1} dg] g^{-1} = g[\theta, X_0] g^{-1}$$

so

$$\omega = \frac{2}{1+X} dX \frac{1}{1-X} = g\left\{\frac{2}{1+X_0}[\theta, X_0]\frac{1}{1-X_0}\right\} g^{-1}$$

$$d(g^{-1}\omega g) = -\underbrace{g^{-1}dg}_{\theta}\, g^{-1}\omega g - g^{-1}\omega^2 g - g^{-1}\omega\, dg$$

$$d(g^{-1}\omega g) + [\theta, g^{-1}\omega g] + (g^{-1}\omega g)^2 = 0.$$

Thus if we put

$$\boxed{\eta = \frac{2}{1+X_0}[\theta, X_0]\frac{1}{1-X_0}}$$

we have

$$\boxed{[\delta + \theta, \eta] + \eta^2 = 0}$$

$\delta = d$ on group

also

$$\boxed{\eta^2 = \frac{2}{1+X_0}[\theta, X_0]\frac{1}{1-X_0^2}[\theta, X_0]\frac{1}{1-X} \quad \text{satisfies} \quad [\delta + \theta, \eta^2] = 0}$$

And ~~as~~ above we have that

$$\mathrm{tr}\left(\eta^{2n+1}\right) = \mathrm{tr}\left(\frac{2}{1-X_0^2}[\theta, X_0]\right)^{2n+1}$$

is closed and hence it gives a cyclic cocycle. Unfortunately it is of the wrong parity, and moreover when $X_0 = iF$ we get const $\mathrm{tr}\left([F,\theta]^{2n+1}\right) = 0$, and $[F,\theta]^{2n+1}$ anti-commutes with F.

Return ~~to~~ the problem of the link between even ~~cyclic~~ classes defined via Chern-Simons or via $A * A$.

<u>Observation</u>: Let τ be a trace defined on all of $A * \mathbb{C}[F]$, so that we have the sequence of cyclic (odd) cocycles $\mathrm{tr}_\tau(F\alpha^{2n})$. Then these are cyclic coboundaries. Now you know this is the case by Chern-Simons. In effect these are character forms, actually the difference of the character forms, for the two eigenbundles of F with respect to the connections $\delta + \rho$, and we have the deformation $\delta + t\rho$.

The observation is that Connes formalism also gives this result. In effect $\mathrm{tr}_\tau(F\alpha^{2n})$ is essentially the cyclic cochain $B\psi_{2n}$ and the other cochains ψ_{2i} $i = n-1, \dots, 1, 0$ ~~constitute~~ in the double complex a way of writing $B\psi_{2n}$ as $(b+B)\chi$. By the exactness of the rows we can replace χ by a Hochschild $n-2$-cochain alone.

So we guess, maybe conjecture that the standard contracting homotopies for the rows applied to $\psi_0, \psi_2, \dots, \psi_{2n-2}$ should give the Chern-Simons form.

October 12, 1988

Let's concentrate on what appears to be [redacted] the main obstacle to a good understanding of cyclic formalism. I think there is a single obstruction whose elimination will lead to the solution of several other problems. These are the following.

1) To find the good (i.e. geometric or analytic) interpretation of Hochschild cohomology. At the moment we have an interpretation of the cochain algebra $C^\cdot(A,R)$ in terms of left-$\mathcal{G}$-invariant forms on [redacted] $\mathcal{G}=GL_n(A)$ with values in $M_n(R)$, which are invariant under $GL_n(\mathbb{C})$ conjugation. (In other words, whereas $C_\lambda(A)$ is linked to $\mathbb{C}$-valued left-$\mathcal{G}$-invariant forms, the cochain algebra $C^\cdot(A,R)$ is linked to matrix-valued forms, which are rather natural when dealing with connections.)

Thus $C^\cdot(A,A)$ is linked to the complex of Lie algebra cochains with values in the adjoint representation $C^\cdot(\tilde{\mathfrak{g}},\tilde{\mathfrak{g}})$. But we are interested in $C^\cdot(A,A^*)$, which suggests that we want the Lie algebra cohomology with coefficients in the coadjoint representation $C^\cdot(\tilde{\mathfrak{g}},\tilde{\mathfrak{g}}^*)$. (I forgot to mention that [redacted] when M is an A-bimodule, $C^\cdot(A,M)$ with the differential $\delta+ad(\theta)$ is the Hochschild cochain complex. Notice that we now have a nice interpretation of $C^\cdot(\mathfrak{g},M)$, the Lie algebra cochain complex. It is the complex of M-valued left-invariant forms on G with the differential $d=\delta+\rho$ [redacted] (θ), where $\rho:\mathfrak{g}\to End(M)$ is the action of $\mathfrak{g}$.)

2) <u>To find a good construction for cyclic cohomology classes associated to Dirac operators</u>

3) <u>To establish a transgression link between superconnection forms and cyclic cocycles.</u>

The problems 2) + 3) are related, because it is very likely that the cyclic classes attached to a Dirac operator are linked by transgression to the superconnection form associated to the family of convex combinations of the gauge transforms of the Dirac operator.

In order to study these questions, let adopt the Hilbert space picture of Connes. In this setup, one has a $*$-algebra A acting on a Hilbert space H, and one has an unbounded self-adjoint operator D. One assumes that $B = \frac{D}{\sqrt{1+D^2}}$ is a quasi-involution commuting with A modulo compacts. (<u>Quasi-involution</u> $=$ self-adjoint operator B such that $-1 \leq B \leq 1$ and the essential spectrum of B is exactly $\{\pm 1\}$.)

Let's then fix the Hilbert space H and a non-trivial involution η in the Calkin algebra H. Let $\mathcal{J}$ be the space of quasi-involutions on H, and $\mathcal{I}(\mathcal{Q})$ the space of nontrivial involutions in $\mathcal{Q}$. By Atiyah-Singer one has homotopy equivalence

$$\mathcal{J} \longrightarrow \mathcal{I}(\mathcal{Q})$$

because the fibres are convex, hence contractible.

One also has a homotopy equivalence

$$\mathcal{J} \longrightarrow \mathcal{U}^\infty(H,-1) \qquad B \mapsto (\sqrt{1-B^2}+iB)^2$$

unitaries congruent to -1 mod $\mathcal{L}^\infty(H)$ = compacts.

by a quasi-fibring type arguments using the fact that the fibres are essentially Grassmannians of Hilbert space, and ~~hence~~ they are contractible by Kuiper's thm.

One has the picture

$$\begin{array}{ccccc}
\text{restricted Grass}\ \mathrm{Grass}(H,\eta) & \longrightarrow & \mathcal{J}_\eta \sim pt & & \\
\downarrow & & \downarrow & & \\
pt \sim \mathrm{Grass}(H) & \longrightarrow & \mathcal{J} & \xrightarrow{\text{heq}} & \mathcal{U} = \mathcal{U}^\infty(H,-1) \\
 & & \downarrow{\scriptstyle \text{heq}} & & \\
 & & \mathcal{D}(\mathcal{Q}) & &
\end{array}$$

By homotopy theory, one has a map

$$\begin{array}{ccc}
\mathrm{Grass}(H,\eta) & \longrightarrow & \Omega\mathcal{U} \\
\wr & & \\
\mathbb{Z}\times B\mathcal{U} & &
\end{array}$$

which is the Bott periodicity map.

Now the important thing here is the space $\mathcal{J}_\eta$ which one may view as the abstract analogue of the space of a connections. It is convex and contractible like $\mathcal{A}$. On $\mathcal{J}_\eta$ the restricted unitary group ~~[illegible]~~ $\mathcal{U}(H)_\eta$ acts, just as the group of gauge transformations $\mathcal{G}$ acts on the space of connections. Moreover the map

$$\underset{\text{conn.}}{A} \longmapsto \frac{1}{i}\not{D}_A = D \longmapsto \frac{D}{\sqrt{1+D^2}}$$

really maps $\mathcal{A}$ to $\mathcal{J}_\eta$ equivariantly with respect to the $\mathcal{G}$-action.

Next we have the C.T. map, which will take any Dirac operator X to the unitary $\frac{1+X}{1-X}$ in $U^\infty(H,-1)$. So we have the following setup. We have the contractible space $\mathcal{A}$ on which $\mathcal{G}$ operates, and we have an equivariant map

$$\mathcal{A} \longrightarrow U^\infty(H,-1)$$

where $\mathcal{G}$ acts on the latter by conjugation. On the latter space we have conjugation invariant closed forms which represent the Chern character classes.

Summarizing we have the group of gauge transformations acting the space of connections and on the space $\mathcal{U} = U^\infty(H,-1)$ representing odd K-theory, and we have an equivariant map $\mathcal{A} \to \mathcal{U}$ given by the C.T. of the Dirac operator associated to a connection. Now do we want to accomplish?

First look at the topology. First of all we know there is a canonical odd K-coh. class on $\mathcal{U}$ and an even K-class on $\mathcal{G}$ which are linked by transgression in some sense. We see this as follows. Take a point $A \in \mathcal{A}$; actually we want to consider the image $B \in \mathcal{J}_\eta$. This point B can be deformed to an involution F. The corresponding composition

$$(*) \qquad \mathcal{G} \xrightarrow{*B} \mathcal{J}_\eta \xrightarrow{C.T.} \mathcal{U}$$

is then deformed to the constant map with value -1. On the other hand $\mathcal{J}_\eta$ is contractible giving a second reason for $(*)$ to be null-homotopic. These two null-homotopies lead to a map $\Sigma_i G \longrightarrow \mathcal{U}$ giving rise to an even K-class on G.

Note in this construction the deformation of B to F is where one usual assume D invertible, or makes it so by dilation.

Taking the Chern character leads to even cohomology classes on G. Our goal is to represent these by left-G-invariant differential forms.

Now as we deform B to F the family of maps $G \longrightarrow \mathcal{J}\eta \longrightarrow \mathcal{U}$ are G-equivariant, since G acts by conjugation on $\mathcal{J}\eta$ and $\mathcal{U}$. Thus our problem reduces to the contractibility of $\mathcal{J}\eta$, or say $\mathcal{A}$. $\mathcal{A}$ is contractible but not equivariantly contractible.

October 13, 1988

Let $\mathcal{G}$ be the group of gauge transfs. acting on $\mathcal{A}$ the space of connections. [redacted] By associating to a connection the Cayley transform of the corresponding Dirac operator, we obtain a map $\mathcal{A} \longrightarrow U^\infty(H,-1) = \{\text{unitaries} \equiv -1 \bmod \mathcal{L}^\infty\}$, which is $\mathcal{G}$-equivariant, where $\mathcal{G}$ acts by conjugation on the latter space. Yesterday I described how this leads to a map defined up to homotopy

$$\Sigma \mathcal{G} \longrightarrow U,$$

[redacted] [redacted] which leads to even cohomology classes on $\mathcal{G}$. The idea is to pick a point in $\mathcal{A}$ and look at the composition

$$\mathcal{G} \longrightarrow \mathcal{A} \longrightarrow U^\infty(H,-1)$$

This map is null-homotopic [redacted] for two reasons - because $\mathcal{A}$ is contractible and because [redacted] we can pick a point not in the spectrum of the operator and deform the Cayley transform to -1. This last step looks non-canonical and doesn't work in the graded case, but such defects can probably be eliminated by dilating, i.e. doubling $H \mapsto H \oplus H$ with trivial $\mathcal{G}$-action on the second factor.

Generalization: Set $U = U^\infty(H,-1)$, $\tilde{U} = U(H)$. Then our $\mathcal{G}$-equivariant map provides a map

$$\begin{array}{ccc} P\mathcal{G} \times^{\mathcal{G}} \mathcal{A} & \longrightarrow & P\tilde{U} \times^{\tilde{U}} U \\ \downarrow \wr \ \text{because } \mathcal{A} \text{ contractible} & & \uparrow \wr \ \text{because } \tilde{U}, B\tilde{U} \text{ are contractible by Kuiper's thm} \\ B\mathcal{G} & & U \end{array}$$

[redacted] I feel that the use of Kuiper's thm can be avoided by dilating. In any case, the point to emphasize is that on the topological level at least <u>the odd cohomology classes on $\mathcal{U}$ extend to equivariant cohomology classes for the conjugation action by $\tilde{\mathcal{U}}$.</u>

This leads to the problem of whether they can be represented by equivariant diffl. forms under suitable Schatten conditions.

If this is the case, then we obtain equivariant differential forms on $\mathcal{A}$ for the action of $\mathcal{G}$ which represent the Chern character classes of the index of the equivariant family of operators.

Notice that this is fairly canonical — the only choice will be the universal equivariant forms for $\mathcal{U}$ with the action of $\tilde{\mathcal{U}}$. Thus we will have sidestepped the problems encountered before with putting a connection on the bundle $P\mathcal{G}\times^{\mathcal{G}} E$ over $B\mathcal{G}\times M$. ?

Let's consider $G = U_n$ and the conjugation action of G on itself. The equivariant cohomology for this action is the cohomology of $PG\times^G(G)_{conj}$ which classifies bundles equipped with automorphism. One has a cartesian square

$$\begin{tikzcd}
PG\times^G(G)_{conj} \arrow[r] \arrow[d] & (PG\times PG)\times^{G\times G}(G)_{l,r} \sim BG \arrow[d] \arrow[dr, phantom] & \\
BG \arrow[r, "\Delta"] & BG\times BG &
\end{tikzcd}$$

with $BG \xrightarrow{\Delta} BG\times BG$.

which shows incidently that
$PG \times^G (G)_{conj} \sim (BG)^{S^1}$ (free loop space).
This leads to a diagram in cohomology

$$\begin{tikzcd}
H^*(PG \times^G G) & H^*(BG) \arrow[l] \\
H^*(BG) \arrow[u] & H^*(BG \times BG) \arrow[l] \arrow[u, "\Delta"'] \\
H^*(\mathrm{Cone}(PG \times^G G \to BG)) \arrow[u] & H^*(\mathrm{Cone}(BG \xrightarrow{\Delta} BG^2)) \arrow[l] \arrow[u] \\
H^{*-1}(PG \times^G G) \arrow[u] & H^{*-1}(BG) = 0 \quad \text{if } * \text{ even} \arrow[l] \arrow[u] \\
H^{*-1}(BG) = 0 \arrow[u] & {} \arrow[u] \\
{} \arrow[u] &
\end{tikzcd}$$

which shows that the even classes $pr_1^*\alpha - pr_2^*\alpha$ on $BG \times BG$ will determine odd classes in $PG \times^G G$. If we take the generators for $H^*(BG)$, say the even character classes, then we get odd classes in $H^*_G(G)$ which restrict to the primitive generators for $H^*(G)$.

This shows that the odd character classes on $H^*(G)$ extend in a canonical to equivariant classes for the conjugation action.

Notice that $P(G\times G) \times^{(G\times G)} (G)_{l,r}$ classifies pairs of bundles together with an isomorphism between them. Equivariant forms for the left-right action of $G\times G$ on G should give a universal model for

the forms on a manifold associated
to a single vector bundle with a
pair of connections.

Equivariant forms for the conjugation action of G on itself should give a universal model for the forms on a manifold which can be constructed from a vector bundle equipped with connection and automorphism. The map $PG\times^G G \to P(G\times G)\times^{(G\times G)} G$ on the equivariant form level should associate to (E, D, g) the bundle E with the two connections D, $g^{-1}Dg$.

Let's consider the class α in $H^*(BG) = S(\mathfrak{g}^*)^G$ which associates to (E, D) the DR class of $\mathrm{tr}\,\frac{(D^2)^n}{n!}$. We know two ways of associating to a bundle with two connections, a difference (E, D_1, D_0), a form whose d is $\mathrm{tr}(D_1^{2n}/n!) - \mathrm{tr}(D_0^{2n}/n!)$. The first is based on the path of connections

$$D_t = (1-t)D_0 + tD_1$$

$$D_t^2 = (1-t)D_0^2 + tD_1^2 + (t^2-t)(D_1-D_0)^2$$

and the second is based on the family of superconnections

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sigma$$

with curvature

$$\begin{pmatrix} D_1^2 & 0 \\ 0 & D_0^2 \end{pmatrix} + t\begin{pmatrix} 0 & D_0-D_1 \\ D_0-D_1 & 0 \end{pmatrix}\sigma$$

Such a difference form becomes closed upon restricting from $(G\times G, G_{lr})$ to (G, G_{conj})

and it represents the odd class of degree $2n-1$ associated to (E, D, g).

So we have at least two ways to construct odd equivariant forms for the conjugation action which [redacted] extend the primitive generators $\mathrm{tr}(g^{-1}dg)^{\mathrm{odd}}$ on the level of cohomology.

Now we want to actually construct explicitly, and as simply as possible, equivariant forms extending $\mathrm{tr}(g^{-1}dg)^{\mathrm{odd}}$.

Let's begin by describing the equivariant forms.

October 14, 1988

Let G compact act on M and let $\Omega_G(M) = \{W(\mathfrak{g}) \otimes \Omega(M)\}_{G\text{-basic}}$ be the DGalgebra of equivariant forms. This algebra maps to $\Omega(P \times^G M)$ for any principal G-bundle P/B equipped with connection.

Let E be a G-equivariant vector bundle over M and let D be a G-invariant connection in E. Then there is a natural connection in the $P\times^G E$ over $P\times^G M$ associated to D and to the connection given in P.

For example, suppose M is a point, whence E is a representation [redacted] of G on a vector space W. The representation gives a Lie homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(W)$, and the connection in $P \times^G W$ over BG is obtained from the operator $d + \rho(\theta)$ on $\Omega(P) \otimes W$. (In other words a connection θ in a principal bundle induces connections in associated fibre bundles). To see this one must show $d + \rho(\theta)$ induces an operator on the basic subspace. But for $X \in \mathfrak{g}$

$$[\iota_X, d + \rho(\theta)] = \mathcal{L}_X + \rho(X) = \begin{array}{c}\text{total Lie}\\ \text{derivative}\\ \mathcal{L}_X^{tot} \text{ on } \Omega(P) \otimes W.\end{array}$$

From this follows that if ξ is basic, then so is $(d + \rho(\theta))\xi$. Note also that

$$(d + \rho(\theta))^2 = [d, \rho(\theta)] + \tfrac{1}{2}[\rho(\theta), \rho(\theta)]$$

$$= \rho(d\theta + \tfrac{1}{2}[\theta,\theta]) \qquad \rho \text{ Lie hom.}$$

so the curvature of $d + \rho(\theta)$ is the image of the curvature of θ under ρ.

Consider (M,E) acted on by G and an invariant connection ∇ on E. Let Q be the principal bundle of E, i.e. the space of isos. of W with the fibres of E. Then $G \times U(W)$ acts on Q.

If $\pi : Q \longrightarrow M$, then $\pi^* E = \tilde{W}$ over Q. If we lift ∇ via it becomes $\nabla = d + \alpha$ where $\alpha \in \Omega^1(Q, \mathrm{End}\, W)^G = \Omega^1(G)^G \otimes \mathrm{End}\, W$. Next we take the product with P. ~~or better, we tensor $W(g)$ with $\Omega(Q)$. We have $d+\alpha$ acting on $\Omega(Q) \otimes W$ and we want to modify it so that~~ This gives the connection $d + \alpha$ operating on $\Omega(P) \otimes \Omega(Q) \otimes W$. Now we want to modify so that it induces a map on G-basic elements. If $X \in g$, then we have

$\mathcal{L}_X$ on $\Omega(P \times Q) \otimes W$ no G action

$$[\iota_X, d + \alpha] = [\iota_X, d] + [\iota_X, \alpha]$$

and we want to remove the last term. So consider a basis X_a for g and put $\theta = \theta^a X_a \in \Omega^1(P) \otimes g$. Then

$$\left[\iota_X,\ d + \alpha - \theta^a \iota_{X_a} \alpha\right] = \mathcal{L}_X$$

so we get a connection which descends to the quotient $P \times^G Q$.

Let $M = pt$, $E = V$, $G = U(V)$. Then $Q = \mathrm{Isom}(W, V)$. $\pi^*(E) = \tilde{V} \xleftarrow[g]{\sim} \tilde{W}$. On $\tilde{V}$ $\pi^*(\nabla)$ is d, but on $\tilde{W}$ it is $g^{-1} \cdot d \cdot g = g + g^{-1} dg$, so $\alpha = g^{-1} dg \in \Omega^1(Q) \otimes \mathrm{End}(W)$. ?

Let's return to the example of $G = U(V)$ acting on $M = U(V)$ by conjugation, and the G-equivariant bundle $\tilde{V}$ over $U(V)$, where G acts by the standard representation. We have the invariant connection d on $\tilde{V}$ which we modify to $d+\theta$, so it descends to basics. We also have a canonical autom. of $\tilde{V}$ over M

$$g \in \{\Omega^0(M) \otimes \mathrm{End}(V)\}^G$$

So we can transform our connection to

$$g^{-1}\cdot(d+\theta)\cdot g = d + g^{-1}dg + g^{-1}\theta g$$

Curvatures

$$\omega = (d+\theta)^2 = d\theta + \theta^2$$

$$\left(g^{-1}(d+\theta)g\right)^2 = g^{-1}\omega g$$

The difference of these connections is

$$\circledast \qquad g^{-1}[d+\theta, g] = g^{-1}dg + g^{-1}[\theta, g] \in \Omega^1(P\times M)\otimes \mathrm{End}\, V$$

Let's check that $\circledast$ is horizontal. First we have because g is G-invariant

$$\mathcal{L}_X g + [X, g] = 0$$

so

$$\iota_X dg = \mathcal{L}_X g = -[X, g]. \qquad \text{Thus}$$

$$\iota_X\left(g^{-1}dg + g^{-1}[\theta, g]\right) = -g^{-1}[X,g] + g^{-1}[X,g] = 0.$$

Thus in $\Omega^\bullet(P\times M)\otimes \mathrm{End}(V)$ we have the basic elements ω, $g^{-1}\omega g$, $g^{-1}[d+\theta, g]$. When we actually pass to the small equivariant form model $\{S(\mathfrak{g}^*)\otimes\Omega(G)\otimes \mathrm{End}(V)\}^G$, θ goes to zero so we obtain the two curvatures $\omega, g^{-1}\omega g$, and $g^{-1}dg$ = the difference of the two connections.

The goal now should be to obtain simple equivariant extensions of the odd character forms $tr\,(g^{-1}dg)^{odd}$. These should be simple expressions involving the trace of a polynomial in g, g^{-1}, ω, dg.

Conclusions Oct 15-17.

All the business about equivariant forms for $U(H)$ acting by conjugation on $U^{\infty}(H,-1)$ only leads to representing the character of the index of the canonical family of Dirac operators over $\mathcal{A}$ with $\mathcal{G}$ acting as an equivariant form in $\Omega_{\mathcal{G}}(\mathcal{A})$. It doesn't solve the transgression problem.

The space of quasi-involutions on H is not a manifold, so one has to be careful about discussing differential forms on it. So there is an a priori problem in carrying out your Chicago program, namely, to find the character forms on $B\mathcal{G}$.

Let A be a (unital) algebra, & let's define a <u>nonunital A-bimodule algebra</u> to be an A-bimodule R equipped with a product map

$$R \otimes_A R \longrightarrow R$$

which is a bimodule map and which is associative. Thus R is a nonunital algebra and an A-bimodule and these two structures are compatible. Example: Take an alg homomorphism $u: A \longrightarrow S$ and take R to be an ideal in S. In fact any nonunital A-bimodule algebra occurs in this way, since we can take S to be the semi-direct product $A \oplus R$.

Suppose the nonunital A-bimodule algebra R happens to be unital as a algebra, that is, there's an element $1 \in R$ such that $1x = x1 = x$ for all $x \in R$. Denote the bimodule multiplications by $a \cdot x$ and $x \cdot a$. Then we have

$$1 \cdot a = (1 \cdot a) 1 = 1(a \cdot 1) = a \cdot 1$$

where we use that the product in R is a map $R \otimes_A R \longrightarrow R$. Let's check that $a \longmapsto u(a) = 1 \cdot a = a \cdot 1$ is an algebra homom.

$$u(a)u(a') = (1 \cdot a)(a' \cdot 1) = 1(a \cdot (a' \cdot 1))$$
$$= 1(aa' \cdot 1) = u(aa')$$

Also

$$u(a)x = (1 \cdot a)x = 1(a \cdot x) = a \cdot x$$
$$x u(a) = x(a \cdot 1) = (x \cdot a)1 = x \cdot a$$

Thus we conclude that <u>a unital A-bimodule algebra R is the same as an algebra equipped with an alg hom $A \to R$</u>

Question: What form does multiplier theory take for nonunital bimodule algebras?

Let's return to A acting on the Hilbert space H, where A is a $*$-algebra. Then $M_n(A)$ acts on H^n. We have a $*$-homomorphism $M_n(A) \longrightarrow \mathcal{L}(H^n) = M_n(\mathcal{L}(H))$ which is just tensoring with $M_n(\mathbb{C})$ the given $*$-homomorphism $A \longrightarrow \mathcal{L}(H)$. The group $U_n(A)$ of unitary elements in $M_n(A)$ then acts on $\mathcal{L}(H^n) = M_n(\mathcal{L}(H))$ in four ways - left mult., right mult, conjugation, trivially.

Actually it's useful to consider the situation where A is a nonunital algebra. In this case $\tilde{A}$ acts on H so we have a unital algebra homomorphism $M_n(\tilde{A}) \longrightarrow M_n(\mathcal{L}(H))$ and we consider the group

$$U_n(A) \overset{def}{=} \operatorname{Ker}\{U_n(\tilde{A}) \longrightarrow U_n(\mathbb{C})\}$$

Actually I would like to start with a nonunital algebra A and a bimodule M over it. Then $M_n(M) = \operatorname{End}(V) \otimes M$ is a $M_n(A) = \operatorname{End}(V) \otimes A$ - bimodule. ??

An important problem is to understand Hochschild cohomology. Let $\mathcal{a}$ be a nonunital algebra and let M be an $\mathcal{a}$-bimodule, i.e. vector space with associative commuting left + right multiplication by elements of $\mathcal{a}$. Then there are three other $\mathcal{a}$-bimodule structures on M, namely by replacing either the left or right multiplication or both by zero mult.

In other words M is a ordinary unital $A \otimes A^{op}$ module where $A = \tilde{a}$, and then the other bimodule structures result from the map $A \to k \to A$.

Now consider the spaces $C^p(a, M)$ of cochains. There are corresponding to the four a-bimodule structures four differentials one can define

$$\delta, \quad \delta + l(\theta), \quad \delta - r(\theta), \quad \delta + l(\theta) - r(\theta) = \delta + ad(\theta).$$

Here $\theta \in C^1(a, a)$ is the canonical element satisfying $\delta(\theta) = -\theta^2$, and $l(\theta), r(\theta)$ stand for left and right multiplication by θ with signs in the case of $r(\theta)$:

$$\text{if } \omega \in C^p(a, M) \qquad (\delta - r(\theta))(\omega) = \delta\omega - (-1)^p \omega\theta$$

Check: $$(\delta - r(\theta))^2(\omega) = \delta\left(\delta\omega + (-1)^{p+1}\omega\theta\right) + (-1)^{p+2}\left(\delta\omega + (-1)^{p+1}\omega\theta\right)\theta$$

$$= \delta^2(\omega) + (-1)^{p+1}\underbrace{\delta(\omega\theta)}_{(\delta\omega)\theta + (-1)^p \omega(-\theta^2)} + (-1)^p(\delta\omega)\theta + (-1)\,\omega\theta^2$$

$$= 0$$

One sees that really the basic differential is $\delta + ad(\theta)$, and the others are obtained by changing the a-bimodule structure on M

The nice thing to do would be to start with a bimodule M over a unital A, then equip $C^\bullet(A, M)$ with $\delta + ad(\theta)$, and ~~then~~ explain in a good fashion why the normalized cochains $C_N^\bullet(A, M)$ form a subcomplex with the same homology.

Next let's review Lie algebra cochains and invariant theory. Let M be a bimodule over the (unital) algebra A. Then

$$M_n(M) = M_n(k) \otimes M$$

is a bimodule over $M_n(A)$, so it can be viewed as a module over the underlying Lie algebra of $M_n(A)$ which is $\mathfrak{gl}_n(A)$. In fact this can be done in four ways. We shall be interested in the "adjoint" action

$$\rho(X)Z = [X, Z] = XZ - ZX \qquad X \in M_n(A), Y \in M_n(M)$$

Now we consider the complex

$$C^{\cdot}(\mathfrak{gl}_n(A), M_n(M))^{\mathfrak{gl}_n(k)}$$

of cochains on $\mathfrak{gl}_n(A)$ with values in the module $M_n(M)$ which are invariant under the conjugation action of $\mathfrak{gl}_n(k)$. Actually it might be better (from the viewpoint of unital-nonunital confusion) to consider the complex of relative cochains

$$C^{\cdot}(\mathfrak{gl}_n(A), \mathfrak{gl}_n(k); M_n(M))^{\bullet}$$

$$= \mathrm{Hom}\left(\Lambda^{\cdot}(\mathfrak{gl}_n \otimes \bar{A}), \mathfrak{gl}_n \otimes M\right)^{\mathfrak{gl}_n}$$

It's necessary in order to obtain $\mathfrak{gl}_n$ invariants that we use the conjugation i.e. adjoint action of $\mathfrak{gl}_n$ on itself. Thus we want $\mathfrak{gl}_n(A)$ to acts by the adjoint action on $\mathfrak{gl}_n \otimes M$.

Now apply invariant theory to

$$\mathrm{Hom}\Big(\underset{(V \otimes V^*)^{\otimes p}}{\mathfrak{g}^{\otimes p}} \otimes \bar{A}^{\otimes p}, \underbrace{\mathfrak{g}}_{V \otimes V^*} \otimes M\Big)^{\mathfrak{g}}$$

which gives a sum over ways of contracting; these are indexed by Σ_{p+1}, and then acted on by Σ_p.

Pay attention to the contraction path starting from the V in $\mathfrak{g} \otimes M$ and ending with the V^* there. This gives rise to cochains $\mathrm{Hom}(\bar{A}^{\otimes \bullet}, M)$. It's pretty clear that we get a decomposition

$$C^\bullet(\mathfrak{gl}_n(A), \mathfrak{gl}_n(k); M_n(M))$$

$$= C_N^\bullet(A, M) \otimes C^\bullet(\mathfrak{gl}_n(A), \mathfrak{gl}_n(k); k)$$

which should explain the normalized Hochschild cochain complex $C_N^\bullet(A, M)$ as a primitive piece of the left-invariant forms over $\mathcal{G}/G = U_n(A)/U_n$ with coefficients in a certain equivariant bundle with flat connection.

Let G be a Lie group, let it act on itself by left multiplication, and let E be an equivariant vector bundle over G. Then E is canonically isomorphic to the trivial bundle V, where G acts trivially on V, and V is the fibre over the identity. An invariant connection on E is of the form $\nabla = d + \theta$ where

$$\theta \in \{\Omega^1(G) \otimes \mathrm{End}(V)\}^G = \mathfrak{g}^* \otimes \mathrm{End}(V)$$

This connection is flat iff θ interpreted as a map $\mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a Lie algebra homomorphism and [redacted] In effect this map is $X \longmapsto \iota_X\theta$ and

$\iota_Y \iota_X (d\theta + \theta^2) = X$ [redacted] $\cancel{\iota_Y(\theta)} - \cancel{Y\iota_X(\theta)} - \iota_{[X,Y]}\theta$

$$+ \iota_X\theta\,\iota_Y\theta - \iota_Y\theta\,\iota_X\theta$$

which shows the curvature gives the deviation of $X \mapsto \iota_X\theta$ from being a Lie homomorphism. Thus we have:

<u>Prop.</u> If E is an equivariant vector bundle over G for the left translation action, then an invariant flat connection on E is the same as a representation of the Lie algebra on the fibre over the identity.

<u>Example</u>: Suppose we start with a representation of [redacted] G on the vector space V. This gives an equivariant bundle over a point and we let E be the pull-back to G. Thus $E = \tilde{V}$ with a nontrivial G-action and E has the invariant connection d. To find the Lie alg. repn in this example we use the [redacted] isomorphism

$$\begin{aligned} G \times V &\longrightarrow G \times V = E \\ (g, v) &\longmapsto (g, gv) \end{aligned}$$

[redacted] of E with $\tilde{V}$, where now G acts trivially on V. Let's denote this isomorphism $\rho(\tilde{g})$, $\tilde{g}$ being the tautological map from G to itself and ρ being the representation on V. Thus $\rho(\tilde{g})$ is a gauge transformation which transforms d to the connection

$$\rho(\tilde{g})^{-1} \cdot d \cdot \rho(\tilde{g}) \quad = \quad d + \rho(\underbrace{\tilde{g}^{-1} d\tilde{g}}_{\text{Maurer-Cartan form of } G.})$$

Thus the representation of $\mathfrak{g}$ in this case is just the infinitesimal representation associated to the representation ρ of G on V.

If G is simply-connected, then any flat connection on a bundle over G corresponds to a trivialization. Hence by reversing the preceding discussion, we see there's a 1-1 correspondence between representations of G & $\mathfrak{g}$, as is well-known.

Let's consider next a representation of G on V together with an involution F on V. Then we get [redacted] equivariant maps

$$\begin{array}{ccccc} G & \longrightarrow & \mathrm{Grass}(V) & \longrightarrow & pt \\ g & \longmapsto & gFg^{-1} & & \end{array}$$

and we can pullback the sub and quotient bundles over the Grassmannian as well as $\tilde{V}$ (which is obtained by pulling back from a point.) The normal geometric picture is that we have over G a splitting of the bundles $\tilde{V}$ defined by a varying family of F's. Write this

$$\tilde{V} = E \oplus E^{\perp}$$

Moreover there are induced connections on $E, E^{\perp}$ obtained from d on $\tilde{V}$ which have curvatures which are 2-forms in $\Omega^2(G, \mathrm{End}\, E)$, etc.

This typical geometric picture is the way to view things over the Grassmannian. However we can transform by the tautological gauge transformation $\tilde{g}$ as we have seen in the above example. This makes all the variation over G disappear, and one has the trivial bundle $\tilde{V}$ with trivial G action on V, and a constant involution F so the subbundles $E, E^\perp$ are constant inside $\tilde{V}$. All the variation is contained in the connection on $\tilde{V}$ which becomes $d + \theta$, $\theta = \rho(\tilde{g}^{-1} d\tilde{g}) \in \Omega^1(G)^G \otimes \mathrm{End}(V)$

Next let's turn to the (A, H, F) situation. We take the Lie group + representation to be $\mathcal{G} = U_n(A)$ acting on H^n. By the previous discussion the place we do curvature calculations is in the algebra of differential forms on $\mathcal{G}$ with values in operators on H^n, but where the differential is $d + \mathrm{ad}(\theta)$, θ being the Maurer–Cartan form. Thus we are using an isomorphism

$$\Omega^{\cdot}(\mathcal{G}, \mathrm{End}(H^n)) \simeq \Omega^{\cdot}(\mathcal{G}) \hat{\otimes} \mathrm{End}(H^n)$$

non trivial action on H^n, but trivial connection d

Ordinary "matrix" forms but with twisted differential $d + \mathrm{ad}(\theta)$.

and the advantage is that when we take left-invariant forms for the $\mathcal{G}$-action, we see the differentials more clearly.

~~Thus perhaps the lesson learned is that if Λ is a representation of G, then the complex of left-invariant forms on G with values in the vector bundle $\tilde{V}$ with non-trivial G-action (and~~

Thus perhaps the lesson learned is the following. Given a $\mathfrak{g}$-module V there is a complex of cochains $C^\bullet(\mathfrak{g}, V)$ which yields Lie algebra cohomology.

Geometrically one obtains this complex by associating to the repn. of $\mathfrak{g}$ on V the corresponding equivariant bundle with flat connection over G, and then taking the complex of left-invariant forms with values in this equivariant bundle, and with differential obtained from the flat connection.

Let's consider a subgroup H of G and an equivariant vector bundle E over G for the left mult action equipped with a flat connection. As we have seen above E is equivalent to a representation of $\mathfrak{g} = \mathrm{Lie}(G)$ on $V = E_{\{e\}}$. Let's restrict this representation to $\mathfrak{h}$ and suppose given a representation of H on V whose infinitesimal rep is this restriction. Then I claim everything descends down to G/H.

To understand this, let's first consider the case the Lie algebra representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ comes from a repn. $\tilde{\rho}$ of G on V, and that the repn. of H is the restriction of $\tilde{\rho}$ to H. Then we have seen that $E = G \times V$ with connection d and diagonal G-action. This obviously descends to $G/H \times V$ with the connection d and the diagonal G-action. Moreover we have an isomorphism

$$G \times^H V \xrightarrow{\sim} G/H \times V$$
$$(g, v) \longmapsto (gH, gv)$$

better, we have isomorphisms.

$$\begin{tikzcd} G \times V \arrow[r, "\sim"] \arrow[r, "(g,v) \mapsto (g,gv)"'] \arrow[d, "\text{left } G \text{ action}"'] & G \times V \arrow[d, "\text{diag. } G \text{ action}"] \\ G \times^H V \arrow[r] & G/H \times V \end{tikzcd}$$

So we conclude that at least in this case the bundle $E = G \times V$, with G acting only on the left and with the flat conn. $d + \rho(\theta)$, descends to give the equivariant bundle $G \times^H V$ over G/H with some flat connection.

What I want to see is why $d + \rho(\theta)$ descends to give a connection on $G \times^H V$. Recall

$$\Omega(G/H,\ G \times^H V) = \Omega(G, V)_{H\text{-basic}}$$

so we have to see that the operator $d + \rho(\theta)$ on $\Omega(G, V)$ preserves the subspace of H-basic elements. We check the operator $\rho(\theta)$ is H-invariant where H acts of G by right-mult and on V by the given repn. This uses the identity

$$g^* \theta = Ad(g^{-1})\theta$$

for a connection form. Thus

$$\begin{pmatrix} \text{right mult} \\ \text{by } h \end{pmatrix}^* \rho(\theta) = \rho\left(Ad(h^{-1})\theta\right)$$
$$= \hat{\rho}(h)^{-1} \rho(\theta)\, \hat{\rho}(h)$$

where we have used that $\rho : \mathfrak{g} \longrightarrow \operatorname{End} V$ is compatible with the H-action, since it comes from a repn. of G.

Next we check that $d + \rho(\theta)$ preserves H-basic elements. Let $\omega \in \Omega(G, V)$ satisfy $i_v(\omega) = 0$ for $v \in \mathfrak{h}$ and also be H invariant $h^*(\omega) = \hat{\rho}(h)^{-1}(\omega)$.

Then $(d+\rho(\Theta))\omega$ is clearly H invariant and for $v \in \mathfrak{h}$

$$\iota_v\,(d+\rho(\Theta))\,\omega = \mathcal{L}_v\omega + \rho(v)\,\omega = 0$$

The last equation is the infinitesimal version of H-invariance $\hat{\rho}(h)\,h^*(\omega) = \omega$.

So we seem to be able to prove the following extension of this calculation.

Prop: Let $\rho: \mathfrak{g} \to \mathrm{End}(V)$ be a Lie algebra representation, let $\hat{\rho}: H \to \mathrm{Aut}(V)$ be a Lie group representation and assume 1) $\rho|\mathfrak{h} =$ infinitesimal rep of $\hat{\rho}$, 2) ρ is compatible with the H-action given by the restriction of the adjoint repn. of G to H. Then the flat connection $d+\rho(\Theta)$ on $\tilde{V} = G \times V$ (G trivial on V) induces a flat invariant connection on $G \times^H V$ over G/H.

Presumably the complex of G-invariant forms on G/H with values in $G \times^H V$ is the relative Lie cochain complex $C^\cdot(\mathfrak{g}, H; V) = \{\Lambda(\mathfrak{g}/\mathfrak{h})^* \otimes V\}^H$

Example: $H =$ discrete subgp Γ. Then assuming that the given Lie repn. $\rho: \mathfrak{g} \to \mathrm{End}(V)$ is compatible with Γ action, we get a flat bundle $G \times^\Gamma V$ over G/Γ.

Let's try to apply the above discussion to the analytical situation of a Fredholm module (A, H, F). Here $\mathcal{G} = U_n(A)$ acts on H^n so we have a flat invariant connection $d+\Theta$ on the trivial bundle $\tilde{H}^n$ over $\mathcal{G}$, where $\mathcal{G}$ acts trivially on H^n. We have the splitting F^n of this bundle which is invariant, infact constant, and the two eigenbundles for this splitting are constant however their connections are not flat.

We consider the subgroup $G = U_n$ of $\mathcal{G}$ which leaves the splitting F^n invariant. Let's recall the diagram

$$\begin{tikzcd}
\mathcal{G} \times V \arrow[r] \arrow[d] & \mathcal{G} \times V \arrow[d] & (g,v) \mapsto (g, gv) \\
\mathcal{G} \times^{G} V \arrow[r] & \mathcal{G}/G \times V &
\end{tikzcd}$$

This diagram shows that because the connection d descends to on the right, the connection $d+\theta$ on $\mathcal{G} \times V$ descends to a flat connection on the bundle $\mathcal{G} \times^G V$ over $\mathcal{G}/G$.

It's the operators, or really the DG algebra of endomorphism-valued forms for the bundle $\mathcal{G} \times^G V$ that I want to be able to handle. The DGA of $\mathcal{G}$-invariant forms is the same as the DGA of relative Lie alg cochains

$$C^{\bullet}(\tilde{\mathfrak{g}}, \mathfrak{g} ; \mathcal{L}(H^n))$$

which contains the DGA of Hochschild cochains $C^{\bullet}_{N}(A, \mathcal{L}(H))$.

October 21, 1988

Recall that a supertrace τ on $A*A$ yields a sequence of normalized Hochschild cochains

$$\varphi_n(a_0, a_1, \ldots, a_n) = \tau(a_0^+ a_1^- \cdots a_n^-)$$

linked by b, B operators:

$$b\varphi_{n-1} = 2\tau(a_0^- \cdots a_n^-)$$

$$B\varphi_{n+1} = (n+1)\tau(a_0^- \cdots a_n^-)$$

Here $a = a^+ + a^-$ is the even odd splitting of a in the superalgebra.

Now the natural question which arises is whether these Hochschild cochains have a natural Lie interpretation. A related question is whether these cochains viewed as lying in $C_N(A, A^*)$ arise from cochains with values in an A-bimodule M with trace. Recall A^* is the universal A-bimodule with trace, in the sense that given an A-bimodule with $\tau : M \to k$ vanishing on $[A, M]$, there is a unique bimod map $M \xrightarrow{u} A^*$ such that $\tau = (\text{eval at } 1) \circ u$. In fact u is [redacted] just $u(m)(a) = \tau(am)$.

First suppose τ is odd. Odd traces are the same as odd supertraces on a superalgebra, so τ is a trace on $A*A$ considered as an A bimodule in the usual way. We have the ~~coef~~ cochain

$$\{(\theta^-)^n(a_1, \ldots, a_n) = a_1^- \cdots a_n^-\} \in C^n(A, A*A)$$

and so the resulting cochain in $C^n(A, A^*)$ is

[redacted]

$$(a_0, a_1, \dots, a_n) \longmapsto \tau(a_0 a_1^- \cdots a_n^-)$$

If n is odd, which one can suppose since otherwise $\varphi_n = 0$, then we have

$$\tau(a_0 a_1^- \cdots a_n^-) = \tau(a_0^+ a_1^- \cdots a_n^-) = \varphi_n(a_0, \dots, a_n)$$

On the other hand suppose τ is even and consider $A * A$ as an A-bimodule using multiplication by a on left and by $\tilde{a}$ on the right. Then the supertrace τ is in fact a trace on this bimodule

$$\left.\begin{aligned} \tau(\omega \tilde{a}) &= \tau(\omega^+ a^+ + \omega^-(-a^-)) \\ \tau(a\omega) &= \tau(a^+\omega^+ + a^-\omega^-) \end{aligned}\right)$$

The cochain in $C^n(A, A^*)$ resulting from $(\theta^-)^n \in C^n(A, A*A)$ is, supposing n even,

$$\tau(a_0 a_1^- \cdots a_n^-) = \tau(a_0^+ a_1^- \cdots a_n^-) = \varphi_n(a_0, \dots, a_n)$$

Similar results hold when τ is defined only on J^m for some m.

The moral of the preceding calculation appears to be that the occurrence of Hochschild cochains is naturally associated to the [redacted] existence of traces in bimodules. What still needs explaining is the function of the [redacted] operator valued cochain [redacted] $(\theta^-)^n$, since it is not a cocycle with values in the bimodule.

Consider A acting on H and a skew-adjoint operator X satisfying a suitable compactness condition. This condition is part of the definition of unbounded Fredholm module, and I would like to understand what it ought to be.

Proceeding as in the case of a Fredholm module (A, H, F), it seems reasonable to look for a universal algebra like the Cuntz-Zekri algebra $\varepsilon A = A * \mathbb{C}[F]$. Thus we want the bounded operators on H which can be constructed from A, X. And we would like ultimately to explain the Jaffe-Lesniewski-Osterwalder entire cyclic cocycle, we means in some ~~sense~~ that we want to combine $[X,a]$ operator with time evolution e^{tX^2}. Following Connes, this means forming some sort of convolution algebra, and passing to Laplace transform, some sort of algebra made out of $\frac{1}{\lambda - X^2}$, $[X,a]$ where λ is a indeterminate.

As a first step one should understand the index pairing with K-theory. In the ungraded case this means restricting to A being generated by a unitary, say $A = C^\infty(S^1)$. A first guess might be that since we are interested in things like $\frac{1}{\lambda - X^2}$ it might be possible to replace X by its C.T. $g = \frac{1+X}{1-X}$, which we know via the holomorphic functional calculus will generate the resolvent operators $\frac{1}{t - X}$ for different $t \notin i\mathbb{R}$.

Therefore one is led to consider trying to attach an index to a pair of unitary operators u, g on H satisfying some compactness condition. For example this condition might be that $[u,g]$ is compact. In effect

$$\left[u, \frac{1+X}{1-X}\right] = \left[u, -1+\frac{2}{1-X}\right] = \frac{1}{1-X}\,[u,X]\,\frac{1}{1-X}$$

and in the case of Dirac operators this is an operator of degree -1.

However to give two unitaries commuting modulo compacts gives an extension of $C(\mathbb{T}^2)$ by compact operators. Such extensions are classified by odd K-homology of the torus $\mathbb{T}^2$, which should be $\mathbb{Z}\oplus\mathbb{Z}$ and linked to $H_1(\mathbb{T}^2, \mathbb{Z})$. Now this extension should be trivial because we have already said that the two unitaries in the Calkin algebra have index zero since they come from unitaries on H.

So we reach a puzzle as how or whether one can assign an index to two unitaries which commute modulo compacts.

Except that I've forgotten that $g \equiv -1$ mod compacts. I've also forgotten my previous discussion of this index using spectral flow, that is, using a path to join X and uXu^{-1}

Let's review the link between the Weil algebra and Narasimhan-Ramanan theorem viewpoints. We concentrate on "primitive" endomorphism valued forms.

We start with a vector bundle E equipped with connection ∇ over M. In the Weil algebra approach we suppose E trivialized $E = \tilde{W}$, say by passing to the principal bundle. Then $\nabla = d + \rho$ where $\rho \in \Omega^1(M) \otimes \operatorname{End}(W)$. We then have a DGA map $\mathbb{C}\langle \rho, \omega \rangle \longrightarrow \Omega(M) \otimes \operatorname{End}(W)$, where $\omega = d\rho + \rho^2$, from which we obtain the character forms.

In the NR approach we find a [redacted] split embedding $E \oplus E^\perp = \tilde{V}$ of E in a trivial bundle such that ∇ is the induced connection from d. Then we have a DGA map $\mathbb{C}\langle e, de \rangle \longrightarrow \Omega(M) \otimes \operatorname{End}(V)$. $\mathbb{C}\langle e, de \rangle$ is the DGA generated by an involution F in degree 0.

To link these picture we suppose both $E, E^\perp$ trivialized $E = \tilde{W}$, $E^\perp = \tilde{W'}$. Then on $\widetilde{W+W'}$ we have not only d, F with $dF = 0$ but the flat connection $d + \theta$ corresponding to d on $\tilde{V}$. Thus we get the universal DGA.

$$\mathbb{C}\langle F, \theta \rangle \qquad dF = 0 \quad , \quad d\theta = -\theta^2$$

mapping to $\Omega(M) \otimes \operatorname{End}(W \oplus W')$. This DGA is the crossproduct of $\mathbb{C}[F]$ with

$$\mathbb{C}\langle \theta \rangle * \mathbb{C}\langle \tilde{\theta} \rangle \qquad d\theta + \theta^2 = d\tilde{\theta} + \tilde{\theta}^2 = 0.$$

[redacted]

It seems that $\mathbb{C}\langle \theta, d\theta = -\theta^2 \rangle * \mathbb{C}[F, dF = 0\rangle$ contains $\mathbb{C}\langle \rho, \omega = d\rho + \rho^2 \rangle$, $\mathbb{C}\langle F, dF \rangle$ (became $[d+\theta, F]$) as (nonunital) subalgebras. [redacted] This is analogous to $T_r(A)$ being a noncentral subalgebra of $A * \mathbb{C}[F]$. I don't see any deeper relations.

The next project will be the study of equivariant forms on U_n for the conjugation action. These yield characteristic forms for bundles equipped with connection ∇ and automorphism g. I have found three ways to construct odd character forms associated to (∇, g).

① using the path $(1-t)\nabla + t\, g\nabla g^{-1}$

② using the superconnection $\begin{pmatrix} \nabla & 0 \\ 0 & g\nabla g^{-1} \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\sigma$

or equivalently $\nabla + t\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$

③ superconnection forms derived via the CT representation $g = \frac{1+X}{1-X}$. This method emphasizes the eigenvalue -1 unlike the first two.

Since I am ultimately interested in ∞-dim'l applications, where g is the CT of a Dirac operator, I will now explore ③ in more detail.

Let's start then with (E, ∇, g) over M. We apply NR to represent ∇ as $i^* d\, i$, where $i : E \to \tilde{V}$ is an isometric embedding. Then we extend g to $\tilde{V}$ by using -1 on $E^\perp$. This extension gives a map $\tilde{g} : M \longrightarrow U(V)$ and the superconnection forms associated to (E, ∇, g) are obtained by pulling back universal forms defined on $U(V)$.

The point to emphasize is that this process reduces the general case of a bundle with connection to the case where the bundle + connection are trivial. It therefore appears that we have a natural way within the superconnection theory to obtain the equivariant odd character forms relative to the conjugation action from the odd character forms relative to the trivial action.

Another important point is that we obtain a map from M to a unitary group ~~starting~~ from (E, ∇, g). It's not yet clear how unique the map is, that is, the arbitrariness in the NR embedding. However we do know the forms obtained back on M don't depend on this choice. So it ought to be possible to give a formal algebraic treatment.

Suppose then $\nabla = i^* d i$, where $i : E \longrightarrow \tilde{V}$ is an isometric embedding with complement $j : E^\perp \longrightarrow \tilde{V}$. We have

$$d = \begin{pmatrix} i^* d i & i^* d j \\ j^* d i & j^* d j \end{pmatrix} \qquad \tilde{g} = \begin{pmatrix} g & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{g}^{-1}[d, \tilde{g}] = \begin{pmatrix} g^{-1} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} [\nabla, g] & -(g+1)\, i^* d j \\ (j^* d i)(g+1) & 0 \end{pmatrix}$$

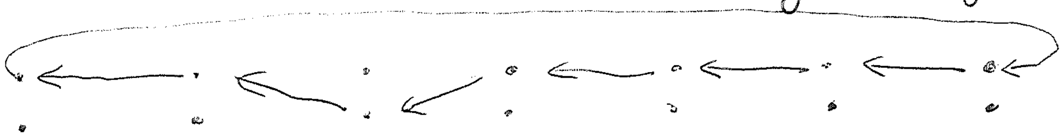
$$= \begin{pmatrix} g^{-1}[\nabla, g] & -(g^{-1}+1)\, i^* d j \\ (-j^* d i)(g+1) & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} \frac{1}{1+X}[\nabla, X]\frac{1}{1-X} & \frac{1}{1+X}(-i^* d j) \\ (-j^* d i)\frac{1}{1-X} & 0 \end{pmatrix}$$

When we take $\operatorname{tr}\left(\tilde{g}^{-1}[d, \tilde{g}]\right)^{odd}$ we get, up to a constant the leading terms

$$\operatorname{tr}\left(\frac{1}{1-X^2}[\nabla,X]\right)^{odd}$$

but then there are correction terms involving $\nabla^2 = -i^* d j j^* d i$. One can label the various terms by diagrams



where each dip contributes a curvature ∇^2 contribution. One ought to be able to describe the terms using the same symmetrized products I used in studying Chern-Simons forms.

Also instead of starting with the invariant character forms $\operatorname{tr}(g^{-1}dg)^{odd}$, one could work with the superconnection forms, which appear in resolvent form (under L.T.) as

$$\operatorname{tr}\left(\left(\frac{1}{\lambda-X^2}\,dX\right)^{odd}\frac{1}{\lambda-X^2}\right)$$

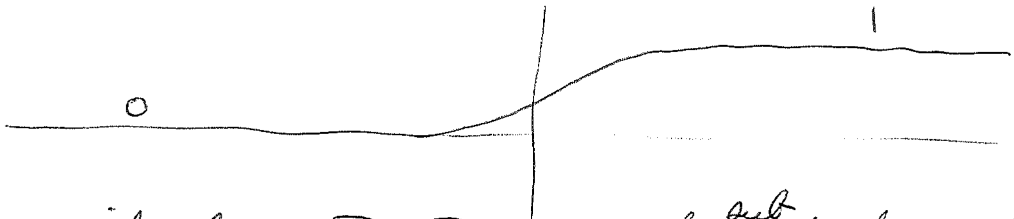
In the superconnection version the passage to equivariant forms just means the curvature goes from $X^2+dX\sigma$ to $X^2+\nabla^2+[\nabla,X]\sigma$.

Now let's consider the index associated to say a Dirac operator X and a gauge transformation u. Here the idea is that the linear path $(1-t)X + t\,u^{-1}Xu$ is a loop modulo gauge transformation, so since we have an equivariant 1-form we can integrate it over this loop.

October 27, 1988

The main observaton today is that, as Getzler explained, one can understand the JLO big cocycle via looking at the cochain algebra $C(A, \mathcal{L}(H))[\sigma]$ and using the superconnection $\delta + \theta + X\sigma$.

An earlier idea was that although a Dirac operator $\not{D}$ doesn't have a canonical F attached to it (I'm thinking of the ungraded case to fix the ideas) it would seem that by using a partition $1 = \rho + (1-\rho)$ where ρ looks like



that one gets ideals I, J in the subalgebra of $\mathcal{L}(H)$ generated by $A, \not{D}$. Thus one would expect cyclic cocycles (classes at least) of the "Tate type", see letter to Kassel.

October 28, 1988

Let's return to our key lemma for establishing the S-relation between cyclic cocycles. We have a DGA R and we form the double complex.

$$\longrightarrow R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{\beta} R \xrightarrow{\partial}$$

Recall that $\Omega^1_R \otimes_R$ is a vector space with the following universal property: There is a map

$$R \otimes R \longrightarrow \Omega^1_R \otimes_R \qquad x \otimes y \longmapsto (x;y)$$

satisfying

$$(x;yz) = (xy;z) + (zx;y)$$

Here $(x;y)$ = class module $[R,\Omega^1_R]$ of $x\,\partial y$. Then $\Omega^1_R \otimes_R$ is the ~~quotient~~ of $R \otimes R$ by this relation.

Then the calculation in the lemma goes starting from $\rho \in R^1$, $\gamma = d\rho + \rho^2$.

$$\begin{cases} d\gamma^n = -\rho\gamma^n + \gamma^n\rho \\ \beta(\gamma^n;\rho) = \gamma^n\rho - \rho\gamma^n \\ d(\gamma^n;\rho) = (-\rho\gamma^n + \gamma^n\rho;\rho) + (\gamma^n;\overset{d\rho}{\gamma - \rho^2}) \\ \qquad = (\cancel{-\rho\gamma^n} + \cancel{\gamma^n\rho;\rho}) + (\gamma^n;\gamma) + \cancel{(\rho\gamma^n;\rho)} - \cancel{(\gamma^n\rho;\rho)} \\ \qquad = (\gamma^n;\gamma) \\ \partial(\gamma^{n+1}) = (n+1)(\gamma^n;\gamma) \end{cases}$$

Here observe that $\partial(\rho^2) = \partial\rho\,\rho + \rho\,\partial\rho$

Now this is such an important link between the different component forms in the Chern character that one wonders if it has already a geometric interpretation in the case of curvature of vector bundles.

The problem is whether there is a geometric analogue of $\Omega^1_R \otimes_R$.

In the geometric context R is the DGA of matrix-valued forms. Now matrices don't affect the complex $\Omega^1_R \otimes_R \xrightarrow{b} R$ much, so let's look at the DGA of forms. This is free commutative essentially a poly ring $\otimes$ ext. alg, so it's $\Omega^1_R \otimes_R$ is the R module of Kahler differentials so it should be ~~the~~ "free" R-module generated by two copies of $\Gamma(T^*)$. So one is led to look at forms ~~with~~ values in $T^* \oplus T^*$. (Any link with Witten-Taubes-elliptic stuff?)

The universal R above is the free DGA with one generator ρ of degree 1. Thus $\Omega^1_R \otimes_R$ looks like a free left R-module generated by $\partial\rho$, $\partial d\rho$.

Idea: If I am serious about $\Omega^1_R \otimes_R$ for the algebra of forms ~~then~~ I should be looking at the cohomology of the free loop space.

November 3, 1988

<u>Proposition</u>. Let $R = T(V)$. Then the sequence

$$R^{\otimes 4} \xrightarrow{b'} R^{\otimes 3} \xrightarrow{p} R \otimes V \otimes R \longrightarrow 0$$

where $b'(x,y,z,w) = (xy,z,w) - (x,yz,w) + (x,y,zw)$

and

$$p(x, v_1 \cdots v_n, y) = \sum_{i=1}^{n} (xv_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n y)$$

is exact.

<u>Proof</u>: b', p are maps of R-bimodules, so it's enough to check that on applying $\operatorname{Hom}_{R\otimes R^{op}}(?, M)$ we get a left exact sequence. However a homomorphism of bimodules from $R^{\otimes 3}$ to M vanishing on $b'R^{\otimes 4}$ is the same thing as a derivation of R with values in M. Thus the result follows from the fact that any linear map $V \xrightarrow{u} M$ extends uniquely to a derivation D given by

$$D(v_1 \cdots v_n) = \sum v_1 \cdots v_{i-1} Dv_i \, v_{i+1} \cdots v_n .$$

More concrete proof. Define

$$s : R \otimes V \otimes R \longrightarrow R^{\otimes 3}$$
$$h : R^{\otimes 3} \longrightarrow R^{\otimes 4}$$

to be the bimodule maps such that

$$s(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$$

$$h(1 \otimes v_1 \cdots v_n \otimes 1) = 1 \otimes v_1 \cdots v_n \otimes 1 \otimes 1 - \sum_{i=1}^{n} v_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_n \otimes 1$$

where the empty product is 1 and empty sum is 0.

Check the identities

$$pb' = 0, \quad ps = id, \quad b'h + sp = id$$

Do the last one

$$b'h(1, v_1 \cdots v_n, 1) = b' \left\{ \begin{array}{l} (1, v_1 \cdots v_n, 1, 1) \\ -\sum_{i=1}^{n} (v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n, 1) \end{array} \right.$$

$$= (v_1 \cdots v_n, 1, 1) - \cancel{(1, v_1 \cdots v_n, 1)} + \cancel{(1, v_1 \cdots v_n, 1)}$$

$$+ \sum_{1}^{n} -(v_1 \cdots v_i, v_{i+1} \cdots v_n, 1) + (v_1 \cdots v_{i-1}, v_i \cdots v_n, 1)$$

$$- (v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$$

$$= \cancel{(v_1 \cdots v_n, 1, 1)} - \cancel{(v_1 \cdots v_n, 1, 1)} + (1, v_1 \cdots v_n, 1)$$

$$- \sum_{1}^{n} (v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$$

$$= (id - sp)(1, v_1 \cdots v_n, 1)$$

Cor. Exact sequence

$$R^{\otimes 3} \xrightarrow{b} R^{\otimes 2} \xrightarrow{\bar{p}} R \otimes V \longrightarrow 0$$

$$\bar{p}(x, v_1 \cdots v_n) = \sum_{i=1}^{n} (v_{i+1} \cdots v_n x v_1 \cdots v_{i-1}, v_i)$$

with contracting homotopy

$$\bar{s}(x, v) = x \otimes v$$

$$\bar{h}(x, v_1 \cdots v_n) = (x, v_1 \cdots v_n, 1) - \sum_{i=1}^{n} (x v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$$

Check: $b\bar{h}(x, v_1 \cdots v_n) = (x v_1 \cdots v_n, 1)$

$$+ \sum_{1}^{n} -(x v_1 \cdots v_i, v_{i+1} \cdots v_n) + (x v_1 \cdots v_{i-1}, v_i \cdots v_n)$$

$$- (v_{i+1} \cdots v_n x v_1 \cdots v_{i-1}, v_i)$$

$$= (id - \bar{s}\bar{p})(x, v_1 \cdots v_n)$$

Now we want to apply this to the bar construction of an augmented algebra $\tilde{A} = k \oplus A$.

But before doing so, I should record how the formula for h was found in case I wonder about this in the future. Consider the standard resolution

$$\begin{tikzcd}[column sep=small]
 & \Omega^2_R \otimes_{\bar{R}} R \arrow[d, equal] & & \Omega^1_R \otimes R \arrow[d, equal] & & & \\
{} \arrow[r, "b'"] & R \otimes \bar{R}^{\otimes 2} \otimes R \arrow[rr, "b'"] \arrow[dr, two heads, "p"] & & R \otimes \bar{R} \otimes R \arrow[r, "b'"] \arrow[dr, two heads, "p"] & R \otimes R \arrow[r] & R \arrow[r] & 0 \\
 & & \Omega^2_R \arrow[ur, hook, "i"] & & \Omega^1_R \arrow[u, hook, "i"'] & &
\end{tikzcd}$$

Notice that

$$\begin{array}{ccc}
b'(x\,dy\,dz \otimes w) & \longleftrightarrow & b'(x, y, z, w) \\
\parallel & & \parallel \\
\begin{array}{l} xy\,dz \otimes w \\ -\, x\, d(yz) \otimes w \\ +\, x\, dy \otimes zw \end{array} & \longleftrightarrow & \begin{array}{l} (xy, z, w) - (x, yz, w) \\ +\, (x, y, zw) \end{array} \\
\parallel & & \\
-\, x\,dy\,(z \otimes 1 - 1 \otimes z)\, w & &
\end{array}$$

In general

$$b'(x_0\, dx_1 \cdots dx_n \otimes y) = (-1)^{n-1} x_0\, dx_1 \cdots dx_{n-1} (x_n \otimes y - 1 \otimes x_n y)$$

So we have

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2_R \arrow[r, "i"] & \Omega^1_R \otimes R \arrow[r, "p"', "\text{right mult.}"] \arrow[l, bend left, "g"] & \Omega^1_R \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

$$s\, d(v_1 \cdots v_n) = \sum v_1 \cdots v_{i-1}\, dv_i \otimes v_{i+1} \cdots v_n$$

$$d(v_1 \cdots v_n) \otimes 1 = \sum v_1 \cdots v_{i-1}\, dv_i\, v_{i+1} \cdots v_n \otimes 1$$

$$sp(d(v_1 \cdots v_n) \otimes 1) = \sum v_1 \cdots v_{i-1}\, dv_i \otimes v_{i+1} \cdots v_n$$

so if we put

$$q(d(v_1 \cdots v_n) \otimes 1) = -\sum_{i=1}^{n} v_1 \cdots v_{i-1}\, dv_i\, d(v_{i+1} \cdots v_n)$$

then $\quad \iota q + ps = \mathrm{id}.$

Then we took h ~~to~~ lift q up ~~into~~ $\Omega^2_R \otimes R$ namely

$$h(d(v_1 \cdots v_n) \otimes 1) = -\sum v_1 \cdots v_{i-1}\, dv_i\, d(v_{i+1} \cdots v_n) \otimes 1.$$

The thing to do if one wants formulas in higher degrees is to define the homotopy operator to be compatible with form multiplication on the left. Thus we want the lifting

$$\begin{tikzcd} \Omega^2_R \otimes R \arrow[r] & \Omega^2_R \arrow[l, bend left, "s"] \end{tikzcd}$$

given by $\quad s(x\,dy\,dz) = x\,dy\; s(dz)$

~~so~~

$$sq(d(v_1 \cdots v_n) \otimes 1) = -\sum_{1 \le i \le j \le n} v_1 \cdots v_{i-1}\, dv_i\, v_{i+1} \cdots v_{j-1}\, dv_j \otimes v_{j+1} \cdots v_n$$

<u>Remark</u>: It's not clear whether an explicit R-bimodule contracting homotopy for

$$\longrightarrow R \otimes \bar{R}^{\otimes 2} \otimes R \longrightarrow R \otimes \bar{R} \otimes R \longrightarrow \Omega^1_R \longrightarrow 0$$

when $R = T(V)$ has any real use. It possibly might be related to Goodwillie's derivation theory.

Let us now consider the free coalgebra situation. Let $C = T(A)$ where A is a vector space. C is a coalgebra with

$$\Delta(a_1, \dots, a_n) = \sum_{i=0}^{n} (a_1, \dots, a_i) \otimes (a_{i+1}, \dots, a_n)$$

$$\eta(a_1, \dots, a_m) = \begin{cases} 0 & n > 0 \\ 1 & n = 0 \end{cases}$$

where the empty symbol $(a_1, \dots, a_n)$ $n=0$ is $\underline{1}$.

For any coalgebra we have the dual of the standard resolution. It is an exact sequence of C-bicomodules

$$0 \longrightarrow C \xrightarrow{\Delta} C \otimes C \xrightarrow{\Delta \otimes 1 - 1 \otimes \Delta} C^{\otimes 3} \longrightarrow C^{\otimes 4}$$

and a contracting homotopy as a complex of right C comodules is $\eta \otimes 1^{\otimes n} : C^{\otimes(n+1)} \longrightarrow C^{\otimes n}$. Put

$$\Lambda = \operatorname{Im}\left\{ C^{\otimes 2} \xrightarrow{\Delta \otimes 1 - 1 \otimes \Delta} C^{\otimes 3} \right\}$$

so that we have exact sequences

$$0 \longrightarrow C \xrightarrow{\Delta} C \otimes C \xrightarrow{p} \Lambda \longrightarrow 0$$

$$0 \longrightarrow \Lambda \xrightarrow{i} C^{\otimes 3} \xrightarrow{\Delta \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \Delta} C^{\otimes 4}$$

Formally Λ is the bicomodule of differentials, i.e. any coderivation $M \xrightarrow{D} C$ is obtained from a unique bicomodule map $M \longrightarrow \Lambda$. The canonical coderivation is

$$\Lambda \longrightarrow C^{\otimes 3} \xrightarrow{\eta \otimes 1 \otimes \eta} C.$$

Now suppose $C = T(A)$ as above. Then we wish to show that we have

$$\begin{tikzcd} & C^{\otimes 3} \arrow[dr, "1\otimes\tau\otimes 1"] & \\ \Lambda \arrow[ur, "i"] \arrow[rr, "\sim"] & & C\otimes A\otimes C \end{tikzcd}$$

where $\tau : C \to A$ is the projection on the tensors of degree 1. Thus Λ is the free bicomodule cogenerated by A, which is equivalent to the assertion that any linear map from a bicomodule M to A extends to a unique coderivation $M \to C$.

We can prove this assertion by interpreting coderivations as coalgebra homomorphisms $C \oplus M \to C$, where $C\oplus M$ denotes the semi-direct product coalgebra. Then the universal property of the tensor coalgebra allows one to extend linear maps $M \to A$ to coderivations $C\oplus M \to C$.

However we want to find formulas for the canonical maps p, i in terms of this isomorphism. Thus we want explicit maps

$$C\otimes C \xrightarrow{p} C\otimes A\otimes C \xrightarrow{i} C^{\otimes 3}$$

and perhaps these can be used to prove the isomorphism $\Lambda \xrightarrow{\sim} C\otimes A\otimes C$.

Since $ip = (\Delta\otimes 1 - 1\otimes\Delta) : C^{\otimes 2} \to C^{\otimes 3}$ and $1\otimes\tau\otimes 1 : C^{\otimes 3} \to C\otimes A\otimes C$ retracts $C^{\otimes 3}$ onto $\Lambda = C\otimes A\otimes C$, it is easy to find a formula for p. Namely

$$(\Delta\otimes 1 - 1\otimes\Delta)(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)$$

$$= \sum_{0\le i\le p} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_p)\otimes(a_{p+1},\dots,a_n) - \sum_{p\le j\le n}(a_1,\dots,a_p)\otimes \overbrace{}^{(a_{p+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n)}$$

and applying $1 \otimes \tau \otimes 1$ gives

$$\boxed{\begin{aligned} p\{(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)\} &= \\ &(a_1,\dots,a_{p-1})\otimes a_p \otimes (a_{p+1},\dots,a_n) \\ &- (a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n) \end{aligned}}$$

where the first term is 0 if $p=0$ and the last term is zero if $p=n$.

It's clear that we have an exact sequence

$$0 \longrightarrow C \xrightarrow{\Delta} C\otimes C \xrightarrow{p} C\otimes A\otimes C \longrightarrow 0$$

In effect, in degree n in A, this is just

$$0 \longrightarrow A^{\otimes n} \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix}} \bigoplus_0^n A^{\otimes n} \xrightarrow{\begin{pmatrix}1 & & \\ -1 & 1 & \\ & -1 & \ddots\end{pmatrix}} \bigoplus_1^n A^{\otimes n} \longrightarrow 0$$

which one can easily see is exact. p onto is all that's needed

To find a formula for i, recall that it is the C-bicomodule map extending $C\otimes A\otimes C \xrightarrow{i} C^{\otimes 3} \xrightarrow{\eta\otimes 1\otimes\eta} C$ which is supposed to be the canonical coderivation. Thus it's dual to $R \longrightarrow R\otimes V\otimes R$, $v_1\cdots v_n \longmapsto \sum v_1\cdots v_{i-1}\otimes v_i\otimes v_{i+1}\cdots v_n$

So we should have

$$(\eta\otimes 1\otimes\eta)i : C\otimes A\otimes C \longrightarrow C \quad \text{is}$$

$$(a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n) \longmapsto (a_1,\dots,a_n)$$

and hence

$$\boxed{\begin{aligned} &i\{(a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n)\} \\ &= \sum_{0\le i\le p}\ \sum_{p+1\le j\le n} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_p,a_{p+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n) \end{aligned}}$$

Note that i is injective, because $(1\otimes\tau\otimes 1)\, i = \text{identity}$.

Next we have

$$ip\{(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)\}$$

$$= i\left\{ \begin{array}{l} (a_1,\dots,a_{p-1})\otimes a_p \otimes (a_{p+1},\dots,a_n) \\ \quad -(a_1,\dots,a_p)\otimes a_{p+1}\otimes(a_{p+2},\dots,a_n) \end{array}\right\}$$

$$= \sum_{\substack{0\le i\le p-1 \\ p\le j\le n}} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n)$$

$$- \sum_{\substack{0\le i\le p \\ p+1\le j\le n}} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n)$$

$$= -\sum_{p\le j\le n} (a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n)$$

$$+ \sum_{0\le i\le p} (a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_p)\otimes(a_{p+1},\dots,a_n)$$

$$= (\Delta\otimes 1 - 1\otimes\Delta)\{(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_n)\}$$

Thus since p is ~~onto~~ and i is injective we see that p, i give the canonical factorization of the map $\Delta\otimes 1 - 1\otimes\Delta : C^{\otimes 2}\longrightarrow C^{\otimes 3}$. Thus we get the exact sequence of C-bicomodules

$$0\longrightarrow C\otimes A\otimes C\xrightarrow{\ i\ } C^{\otimes 3}\longrightarrow C^{\otimes 4}$$

Here's is the algebra variant of the preceding argument:

$$\begin{tikzcd} R^{\otimes 4} \arrow[r, "b'"] & R^{\otimes 3} \arrow[r, "b'"] \arrow[dr, "p"'] & R^{\otimes 2} \arrow[r, "b'"] & R \\ & & R\otimes V \otimes R \arrow[u, "i"'] & \end{tikzcd} \qquad \text{exact}$$

$$p(x, v_1 \cdots v_n, y) = \sum_{i=1}^{n} (x v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$$

$$i(x, v, y) = (xv, y) - (x, vy)$$

Then

$$\iota p(x, v_1 \cdots v_n, y) = \sum_{i=1}^{n} (x v_1 \cdots v_i, v_{i+1} \cdots v_n y) - (x v_1 \cdots v_{i-1}, v_i \cdots v_n y)$$

$$= (x v_1 \cdots v_n, y) - (x, v_1 \cdots v_n y) = b'(x, v_1 \cdots v_n, y)$$

p is onto since $p(x, v, y) = (x, v, y)$

i injective because if $g : R^{\otimes 2} \longrightarrow R\otimes V \otimes R$ is defined by $g(x, v_1 \cdots v_n) = \sum_{i=1}^{n} (x v_1 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$

then

$$g i(x, v_0, v_1 \cdots v_n) = g(x v_0, v_1 \cdots v_n) - g(x, v_0 \cdots v_n)$$

$$= \sum_{i=1}^{n} (x v_0 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n) - \sum_{i=0} (x v_0 \cdots v_{i-1}, v_i, v_{i+1} \cdots v_n)$$

$$= -(x, v_0, v_1 \cdots v_n)$$

Also if one verifies $pb' = 0$ then p surjective $\Rightarrow$ i injective by diagram chasing

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So now we have the exact sequence of C bicomodules

$$0 \longrightarrow C \otimes A \otimes C \xrightarrow{\ i\ } C^{\otimes 3} \xrightarrow{(b')} C^{\otimes 4}$$

and we wish to take cotensor product $\boxtimes^{C\otimes C^{op}} C$.

In general given a C-bicomodule M we can define

$$M \boxtimes^{C} = \mathrm{Ker}\left\{ M \underset{\sigma\Delta_r}{\overset{\Delta_\ell}{\rightrightarrows}} C \otimes M \right\}$$

and in the case of a free bicomodule [redacted] $C \otimes V \otimes C$ we can identify this with $C \otimes V$, the isomorphism being given by the map

$$C \otimes V \xrightarrow{\Delta \otimes 1} C \otimes C \otimes V \xrightarrow{\sigma^{-1}} C \otimes V \otimes C$$

where σ denotes the forward shift cyclic permutation.

We want to work out the maps in the induced exact sequence

$$0 \longrightarrow C \otimes A \longrightarrow C^{\otimes 2} \xrightarrow{\text{[redacted]}} C^{\otimes 3}$$

obtained by applying $? \boxtimes^C$ to $\ast$.

$$\begin{tikzcd}
C \otimes A \otimes C \arrow[r, "i"] & C^{\otimes 3} \\
C \otimes A \arrow[u, "\sigma^{-1}(\Delta \otimes 1)"] \arrow[r, "?"] & C^{\otimes 2} \arrow[u, "\sigma^{-1}(\Delta \otimes 1)"']
\end{tikzcd}$$

So start with $(a_1, \ldots, a_p) \otimes \alpha \in C \otimes A$. Then $\sigma^{-1}(\Delta \otimes 1)$ takes this to

$$\sum_{i=0}^{p} (a_{i+1}, \ldots, a_p) \otimes \alpha \otimes (a_1, \ldots, a_i)$$

and i takes this to

$$\sum_{0 \le i \le p} \sum_{\substack{0 \le j \le i \\ i \le k \le p}} (a_{i+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_p, \alpha, a_1, \ldots, a_j) \otimes (a_{j+1}, \ldots, a_i)$$

which we recognize as $\sigma^{-1}(\Delta \otimes 1)$ applied to

$$\sum_{0 \le j \le k \le p} (a_{j+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_p, \alpha, a_1, \ldots, a_j)$$

So we should have an exact sequence

$$0 \longrightarrow C \otimes A \xrightarrow{\ \tilde{\imath}\ } C^{\otimes 2} \xrightarrow{\ b\ } C^{\otimes 3}$$

where

$$\boxed{\tilde{\imath}\big((a_1, \ldots, a_p) \otimes \alpha\big) = \sum_{0 \le j \le k \le p} (a_{j+1}, \ldots, a_k) \otimes (a_{k+1}, \ldots, a_p, \alpha, a_1, \ldots, a_j)}$$

This should follow by left-exactness of the functor $M \longmapsto M \boxtimes C$. However it would perhaps be desirable to give a direct proof, as short as possible.

The reason we want the exactness is so that in the case where C is the bar construction on $\tilde{A}$, we can conclude that the image of $\tilde{\imath}$ is a subcomplex.

Actually I think that one really ought to be able to give a loop type proof.

November 4, 1988

Here seems to be a simple ~~[illegible]~~ construction of the needed exact sequence

$$* \qquad 0 \longrightarrow C \otimes A \xrightarrow{\iota} C^{\otimes 2} \longrightarrow C^{\otimes 3}$$

by duality. We first do it for A a finite dimensional vector space and then take the limit. We let $V = A^*$, $R = T(V)$ and use the exact sequence

$$** \qquad R^{\otimes 3} \xrightarrow{b} R^{\otimes 2} \xrightarrow{p} R \otimes V \longrightarrow 0$$

where $p(x \otimes v_1 \cdots v_n) = \sum_{i=1}^{n} v_{i+1} \cdots v_n x v_1 \cdots v_{i-1} \otimes v_i$

The dual of $**$ as a graded vector space, the grading being the tensor degree in V will then give $*$.

We now calculate $\iota = p^t$.

$$p(v_1 \cdots v_m \otimes v_{m+1} \cdots v_n) = \sum_{j=0}^{n-m-1} v_{n-j+1} \cdots v_n v_1 \cdots v_{n-j-1} \otimes v_{n-j}$$

$$\langle (a_1, \dots, a_{n-1}) \otimes a_n, \; p(v_1 \cdots v_m \otimes v_{m+1} \cdots v_n) \rangle$$

$$= \sum_{0 \le j < n-m} \langle a_1, v_{n-j+1} \rangle \cdots \langle a_j, v_n \rangle \langle a_{j+1}, v_1 \rangle \cdots \langle a_{n-1}, v_{n-j-1} \rangle \langle a_n, v_{n-j} \rangle$$

$$= \Big\langle \sum_{0 \le j < n-m} (a_{j+1}, \dots, a_{j+m}) \otimes (a_{j+m}, \dots, a_n, a_1, \dots, a_j), \; v_1 \otimes v_2 \cdots v_m \otimes v_{m+1} \cdots v_n \Big\rangle$$

and now put $k = j+m$ to obtain

$$\boxed{\iota\{(a_1, \dots, a_{n-1}) \otimes a_n\} = \sum_{0 \le j \le k < n} (a_{j+1}, \dots, a_k) \otimes (a_{k+1}, \dots, a_n, \dots, a_j)}$$

I want to exhibit a contracting homotopy to show that the sequence

$$0 \longrightarrow C\otimes A \xrightarrow{\ i\ } C\otimes \bar C \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta - \sigma^{-1}(\Delta\otimes 1)} C\otimes \bar C^{\otimes 2}$$

is exact. Let $s : C\otimes\bar C \longrightarrow C\otimes A$ be $1\otimes\tau$:

$$s\{(a_1,\dots,a_m)\otimes(a_{m+1},\dots,a_n)\} = \begin{cases} 0 & m+1<n \\ (a_1,\dots,a_{n-1})\otimes a_n & m+1=n \end{cases}$$

Clearly $si = id$. Next define $h : C\otimes\bar C^{\otimes 2} \to C\otimes\bar C$ to correspond under duality to

$$h : R\otimes\bar R \longrightarrow R\otimes\bar R^{\otimes 2}, \qquad h(x, v_1\cdots v_n) = \sum_1^n v_{i+1}\cdots v_n x v_1\cdots v_{i-1}\otimes v_i$$

namely

$$\boxed{\begin{aligned} &h\{(a_1,\dots,a_p)\otimes(a_{p+1},\dots,a_q)\otimes(a_{q+1},\dots,a_n)\} \qquad p<q,\ q<n \\ &= \begin{cases} 0 & \text{if } p<q-1 \\ -\displaystyle\sum_{0\le k\le p}(a_1,\dots,a_k)\otimes(a_{k+1},\dots,a_n) & \text{if } p=q-1 \end{cases}\end{aligned}}$$

~~That~~ Let's verify that $h(\Delta\otimes 1 - 1\otimes\Delta + \sigma^{-1}(\Delta\otimes 1)) = id - is$

$$\big(\Delta\otimes 1 - 1\otimes\Delta + \sigma^{-1}(\Delta\otimes 1)\big)\{(a_1,\dots,a_m)\otimes(a_{m+1},\dots,a_n)\} \qquad m<n$$

$$= \sum_{0\le i\le m}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_m)\otimes(a_{m+1},\dots,a_n)$$

$$-\sum_{m\le j\le n}{}'(a_1,\dots,a_m)\otimes(a_{m+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n)$$

$$+\sum_{0\le i\le m}(a_{i+1},\dots,a_m)\otimes(a_{m+1},\dots,a_n)\otimes(a_1,\dots,a_i)$$

Notice that the $i=m$ term in the first sum and the $j=m$ term in the second ~~[illegible]~~ cancel, and similarly the $j=n$ term in the second cancels the $i=0$ term in the third, so this does lie in $C\otimes\bar C^{\otimes 2}$

Now apply h. Suppose $m<n-1$, so there's no contribution from the last sum. We obtain

$$-\sum_{0\le k\le m-1}' (a_1,\dots,a_k)\otimes(a_{k+1},\dots,a_m,\dots,a_n)$$

$$+\sum_{0\le k\le m}' (a_1,\dots,a_k)\otimes(a_{k+1},\dots,\dots,a_n)$$

$$=(a_1,\dots,a_m)\otimes(a_{m+1},\dots,a_n).$$ Notice that S kills $(a_1,\dots,a_m)\otimes(a_{m+1},\dots,a_n)$ when $m<n-1$. Thus we do get $1 = h(\text{"}b\text{"}) + iS$ on elements with $m<n-1$.

Next suppose $m=n-1$. Then we get

$$-\sum_{0\le k\le n-2}(a_1,\dots,a_k)\otimes(a_{k+1},\dots,a_n)$$

$$-\sum_{0<i\le j\le n-1}(a_{i+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n,a_1,\dots,a_i)$$

$$=(a_1,\dots,a_{n-1})\otimes a_n-\underbrace{\sum_{0\le i\le j\le n-1}'(a_{i+1},\dots,a_j)\otimes(a_{j+1},\dots,a_n,a_1,\dots,a_i)}_{i\{(a_1,\dots,a_{n-1})\otimes a_n\}}$$

So it works, but is very messy

In the applications we take $C = T(A[1])$ where A is an algebra, so that C is the bar construction on $\bar{A}$. C is DGA with $d = b'$.

The formula for i then involves signs. It is the map

$$C \otimes A[1] \xrightarrow{i} C \otimes \bar{C}$$

given by the formula

$$\boxed{i\{(a_1,\dots,a_n)\otimes\alpha\} = \sum_{0\le j\le k\le n} (-1)^{jn} (a_{j+1},\dots,a_k)\otimes(a_{k+1},\dots,a_n,\alpha,a_1,\dots,a_j)}$$

Since the sequence

$$0 \longrightarrow C\otimes A[1] \xrightarrow{i} C\otimes\bar{C} \xrightarrow{\text{"}b\text{"}} C\otimes\bar{C}^{\otimes 2}$$

is exact and "b" is a map of complexes where C has the differential $d = b'$, it follows that there is an induced differential on $C\otimes A[1]$. ~~The~~ We now work this out.

Start with $(a_1,\dots,a_n)\otimes\alpha$, apply i using the above formula, and then apply $d\otimes 1 + 1\otimes d$. This gives

$$\sum_{0\le j\le k\le n} (-1)^{jn} \left\{ \begin{aligned} &b'(a_{j+1},\dots,a_k)\otimes(a_{k+1},\dots,a_n,\alpha,a_1,\dots,a_j) \\ &+(-1)^{k-j}(a_{j+1},\dots,a_k)\otimes b'(a_{k+1},\dots,\alpha,a_1,\dots,a_j) \end{aligned} \right\}$$

To find out what element of $C\otimes A[1]$ this comes from we apply s. The only contributions are for $j=0$, $k=n$ from the top and $(j=0, k=n-1)$ $(j=1, k=n)$ from the bottom:

$$b'(a_1,\dots,a_n)\otimes\alpha + (-1)^{n-1}(a_1,\dots,a_{n-1})\otimes a_n\alpha$$
$$+ (-1)^n(-1)^{n-1}(a_2,\dots,a_n)\otimes\alpha a_1$$

$$= -\left\{ \begin{aligned} &(a_2,\dots,a_n)\otimes\alpha a_1 + \sum_{i=1}^{n-1}(-1)^i(a_1,\dots,a_i a_{i+1},\dots,a_n)\otimes\alpha \\ &+(-1)^n(a_1,\dots,a_{n-1})\otimes a_n\alpha \end{aligned} \right\}$$

So let us now use the flip map

$$A[1]\otimes C \xrightarrow{\sim} C\otimes A[1]$$

$$\alpha\otimes(a_1,\dots,a_n) \longmapsto (-1)^n(a_1,\dots,a_n)\otimes\alpha$$

and we find that

$$d(\alpha\otimes(a_1,\dots,a_n)) \longmapsto (-1)^n(-1)\left\{\begin{aligned}&(a_2,\dots,a_n)\otimes\alpha a_1\\ &+\sum_{i=1}^{n-1}(-1)^i(a_1,\dots,a_ia_{i+1},\dots a_n)\otimes\alpha\\ &+(-1)^n(a_1,\dots,a_{n-1})\otimes a_n\alpha\end{aligned}\right\}$$

so

$$d(a_0,a_1,\dots,a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0,\dots,a_ia_{i+1},\dots,a_n) + (-1)^n(a_na_0,a_1,\dots,a_{n-1})$$

Therefore we have proved the following.

<u>Theorem</u>: There is an exact sequence of complexes

$$0 \longrightarrow A[1]\otimes C \xrightarrow{\tilde{i}} C\otimes\bar{C} \xrightarrow{\Delta\otimes 1 - 1\otimes\Delta - \sigma^{-1}(\Delta\otimes 1)} C\otimes\bar{C}^{\otimes 2}$$

where $\tilde{i}(a_0,\dots,a_n) = \sum_{0\le j\le k\le n}(-1)^{(j+1)n}(a_{j+1},\dots,a_k)\otimes(a_{k+1},\dots,a_n,a_0,\dots,a_j)$

and where $A[1]\otimes C$ is equipped the differential b C is given the differential b'.

<u>Cons. 1)</u> $b^2=0$ (because $\tilde{i}$ injective)

<u>Cons. 2)</u> Applying $\eta\otimes 1: C\otimes\bar{C}\longrightarrow\bar{C}$, we see that

$$(a_0,\dots,a_n)\longmapsto\sum_{0\le j\le n}(-1)^{(j+1)n}(a_{j+1},\dots,a_n,a_0,\dots,a_j)$$

is a map of complexes from ~~the~~ Hochschild complex to $\bar{C}$. Thus we get the ~~identity~~ $Nb = b'N$

Next consider the map

$$\bar{C} \xrightarrow{\Delta - \sigma\Delta} C \otimes \bar{C}$$

Since its composition with $\Delta\otimes 1 - 1\otimes\Delta + \sigma^{-1}(\Delta\otimes 1)$ is zero this map must factor through $i : C\otimes A[1] \longrightarrow C\otimes\bar{C}$. To find the map $\bar{C} \longrightarrow C\otimes A[1]$ we apply $1\otimes\tau : C\otimes\bar{C} \to C\otimes A[1]$

$$(1\otimes\tau)(\Delta - \sigma\Delta)(a_1,\dots,a_n)$$

$$= (1\otimes\tau)\sum_{0\le i\le n}(a_1,\dots,a_i)\otimes(a_{i+1},\dots,a_n)$$

$$= (1\otimes\tau)\sum_{0\le i\le n}(-1)^{i(n-i)}(a_{i+1},\dots,a_n)\otimes(a_1,\dots,a_i)$$

$$= (a_1,\dots,a_{n-1})\otimes a_n + (-1)^{n-1}(a_2,\dots,a_n)\otimes a_1$$

Under the flip isomorphism $C\otimes A[1] \simeq A[1]\otimes C$ we find the map

$$(a_1,\dots,a_n) \longmapsto (-1)^{n-1}(a_n,a_1,\dots,a_{n-1}) - (a_1,a_2,\dots,a_n)$$

which is the negative of $1-t$. Thus we see that $1-t : \bar{C} \longrightarrow A[1]\otimes C$ is a map of complexes, i.e. $b(1-t) = (1-t)b'$.

November 9, 1988

Let's consider an unbounded Fredholm module situation (A, H, X). We are going to work in the algebra of cochains on A with values operators on H, and ultimately we want some kind of convergence condition on the cochains.

To fix the ideas let $L = \mathcal{L}(H)$ and consider $C(A,L) = \mathrm{Hom}_k(\mathcal{B}, L)$, where $\mathcal{B}$ is the bar construction on $\tilde{A}$. Adjoin an elt σ to $C(A,L)$ satisfying $\sigma^2 = 1$, $\sigma f = (-1)^{\deg f} f\sigma$. Then on $C(A,L)[\sigma]$ we have the derivation

$$(1) \qquad \delta + \mathrm{ad}(\theta + X\sigma)$$

where $\theta \in C^1(A,L)$ is the cochain giving the homomorphism $A \longrightarrow L$. Consider (1) as a superconnection; ~~[illegible]~~ really one should be thinking of it as bracketing with respect to the superconnection $\delta + \theta + X\sigma$. The square of (1) is

$$\left(\delta + \mathrm{ad}(\theta + X\sigma)\right)^2 = \mathrm{ad}\left(X^2 + [\theta, X\sigma]\right)$$

so the curvature is $K = X^2 + [\theta, X\sigma]$. Now we can form

$$e^{uK} \in C(A,L)[\sigma]$$

and (formally) we have the identity

$$(*) \qquad \left(\delta + \mathrm{ad}(\theta + X\sigma)\right) e^{uK} = 0.$$

The existence of e^{uK} should follow from the perturbation expansion, since $X^2 \leq 0$. Here as usual $\mathrm{Re}(u) > 0$. It should also be true that e^{uK} is a cochain whose values are trace class operators, assuming θ-summability. If this is OK, then we can form the supertrace τ on $L[\sigma]$ appropriate to the parity of X (graded or ungraded) and look at the following cochains

$$\tau(e^{uK}) \in C^{\cdot}(A) \qquad \text{bar cochain}$$

$$\tau(\partial\theta\, e^{uK}) \in C^{\cdot}(A, A^*) \qquad \text{cyclic bar cochain.}$$

These are linked by the following calculations:

$$\begin{aligned}
\delta\tau(e^{uK}) &= \tau(\delta e^{uK}) \\
&= -\tau([\theta + X\sigma, e^{uK}]) \qquad \text{from } (*). \\
&= -\tau([\theta, e^{uK}]) \\
&= -\beta\, \tau(\partial\theta e^{uK})
\end{aligned}$$

$$\begin{aligned}
\delta\tau(\partial\theta e^{uK}) &= \tau(\cancel{\partial(-\theta^2)e^{uK}} + \partial\theta[\theta + X\sigma, e^{uK}]) \\
&= \tau(\partial\theta\,[X\sigma, e^{uK}]) \\
&= \tau(\partial[\theta, X\sigma]\, e^{uK}) = \tau(\partial(X^2 + [\theta, X\sigma])e^{uK}) \\
&= \tau(\partial K\, e^{uK})
\end{aligned}$$

$$\partial\tau(e^{uK}) = u\,\tau(\partial K e^{uK})$$

Now I would like to go over these formulas using homogeneous components to check that things works.

The component of degree n in e^{uK} is

$$\int_{u_0 + \cdots + u_n = u} \cdots \int e^{u_0 X^2} [\theta, X\sigma] e^{u_1 X^2} \cdots [\theta, X\sigma] e^{u_n X^2}\, du_1 \cdots du_n$$

and to study it we use the L.T. which gives

$$\left(\frac{1}{\lambda - X^2}[\theta, X\sigma]\right)^n \frac{1}{\lambda - X^2}$$

Set $G = \dfrac{1}{\lambda - X^2}$. We have

$$[\delta + \theta, X\sigma] = [\theta, X\sigma]$$

$$\therefore [\delta+\theta, [\theta, X\sigma]] = 0.$$

$$[\delta+\theta, G] = G[\delta+\theta, X^2]G$$
$$= G[[\theta, X\sigma], X\sigma]G$$

Thus

$$\delta \left(G[\theta, X\sigma]\right)^n G + [\theta, \left(G[\theta, X\sigma]\right)^n G]$$

$$= [\delta+\theta, \left(G[\theta, X\sigma]\right)^n G]$$

$$= \sum_{i=0}^{n-1} \left(G[\theta, X\sigma]\right)^i \underbrace{[\delta+\theta, G]}_{G[[\theta, X\sigma], X\sigma]G} [\theta, X\sigma]\left(G[\theta, X\sigma]\right)^{n-1-i} G$$

$$+ \left(G[\theta, X\sigma]\right)^n G[[\theta, X\sigma], X\sigma]G$$

$$= \sum_{i=0}^{n} \left(G[\theta, X\sigma]\right)^i G[[\theta, X\sigma], X\sigma]\left(G[\theta, X\sigma]\right)^{n-i} G$$

$$= [\left(G[\theta, X\sigma]\right)^{n+1} G, X\sigma]$$

i.e.

$$\boxed{[\delta+\theta, \left(G[\theta, X\sigma]\right)^n G] + [X\sigma, \left(G[\theta, X\sigma]\right)^{n+1} G] = 0}$$

This is just the component version of the identity

$$[\delta + \theta + X\sigma, f(K)] = 0 \qquad \text{where } f(x) = \frac{1}{\lambda - x}$$

Applying the supertrace τ gives

$$\delta\ \tau\left\{\left(\frac{1}{\lambda-X^2}[\theta,X\sigma]\right)^n\frac{1}{\lambda-X^2}\right\} = -\beta\ \tau\left\{\partial\theta\left(\frac{1}{\lambda-X^2}[\theta,X\sigma]\right)^n\frac{1}{\lambda-X^2}\right\}$$

Now for the second identity

$$\frac{1}{u}\,\partial\tau(e^{uK}) = \tau(\partial K\, e^{uK}) = \delta\tau(\partial\theta\, e^{uK})$$

we have

$$\left.\begin{aligned}
&\delta\tau\{\partial\theta\,(G[\theta,X\sigma])^n G\}\\
&=\tau\left\{\partial(-\theta^2)\cancel{(G[\theta,X\sigma])^n}G + \partial\theta\left\{[\theta,\cancel{(G[\theta,X\sigma])^n G}]\right.\right.\\
&\qquad\qquad\left.\left. + [X\sigma,(G[\theta,X\sigma])^{n+1}G\right\}\right\}\\
&= \tau\{\ \partial\theta\,[X\sigma,(G[\theta,X\sigma])^{n+1}G]\}\\
&= \tau\{\ \partial([\theta,X\sigma])\,(G[\theta,X\sigma])^{n+1}G\}.
\end{aligned}\right\} \ast\ast$$

Now let's recall that the L.T. of $\frac{e^{uK}}{u}$ is roughly $-\log(\lambda-K)$. A precise statement is that

$$\int_0^\infty \frac{1}{u}\,\tau\left(e^{u(X^2+[\theta,X\sigma])} - e^{uX^2}\right)e^{-\lambda u}\,du$$

$$= -\tau\log\left(\frac{\lambda-X^2-[\theta,X\sigma]}{\lambda-X^2}\right)$$

$$= \sum_{k\geq 1}\frac{1}{k}\,\tau\left(\frac{1}{\lambda-X^2}[\theta,X\sigma]\right)^k$$

So the kth component of $\frac{1}{u}\tau(e^{uK})$ should have L.T. $\frac{1}{k}\tau((G[\theta,X\sigma])^k)$. Now we have

$$\partial\,\frac{1}{n+2}\,\tau\left((G[\theta,X\sigma])^{n+2}\right) = \tau\left(\partial([\theta,X\sigma])\,G[\theta,X\sigma]^{n+1}G\right)$$

which agrees with $\ast\ast$ above.

The question arises whether there is a deeper meaning to the preceding calculations.

Let's review the interpretation of the cochain $tr\left(\frac{1}{\lambda-X^2}[\theta,X]\right)^{odd}$ in the ungraded case; take $\lambda=1$.

We consider $G = U_n(A)$ with $Lie(G)_c = gl_n(A)$ acting on $H^{\oplus n}$. By associating to $g \in G$ the operator gXg^{-1} and then taking its C.T., we get a map from G to unitary operators on $H^{\oplus n}$ congruent to -1 modulo some Schatten ideal. Then one can pull back the odd character forms on this unitary group and obtain left-invariant closed forms on G. The corresponding cyclic cocycles are the cyclic cocycles

$$tr\left(\frac{1}{1-X^2}[\theta,X]\right)^{odd}$$

~~Recall that the G-bundle over G~~

~~Recall that the representation of G on $H^{\oplus n}$ furnishes a G-vector bundle for the left trans. action, namely $H^{\oplus n}$~~

Recall that the G-vector bundle over a point given by the representation on $H^{\oplus n}$ lifts back over G to an equivariant bundle with flat invariant connection, namely $H^{\oplus n}$ over G with the connection d; here G acts on itself via left translation and on $H^{\oplus n}$ using the representation. There's an equivariant family of operators on the fibres given $g \mapsto gXg^{-1}$. By a tautological gauge transformation one gets an isomorphic equivariant bundle with flat connection and family, namely $\tilde{H}^n$ with $d+\theta$ and the constant family X.

Recall also that the left-invariant character forms obtained on G have natural refinements to equivariant forms for G acting on the space of bounded perturbations of X.

Now come the mysteries. ~~[illegible]~~ ~~The~~ One has a geometric interpretation of the cyclic cocycles $tr\left(\frac{1}{\lambda - X^2}[\theta,X]\right)^{odd}$ as Chern character forms. But the above calculation deals with the bar + cyclic bar cochains.

November 11, 1988

Consider the simplest case of the big cochains associated to a bundle + connection and a cycle.

Review the setting. One has a connection ∇ on E, whence a derivation $ad(\nabla)$ on $\Omega(M, \mathrm{End}\, E)$, and an elt ∇^2 of degree 2 such that $(ad\,\nabla)^2 = ad(\nabla^2)$. One has $\tau = tr_E : \Omega(M, \mathrm{End}\, E) \longrightarrow \Omega(E)$ satisfying

$$d\tau = \tau \circ ad(\nabla).$$

Put $A = \Omega^0(M, \mathrm{End}\, E)$ and let $\theta \in C^1(A, \Omega^0(M, \mathrm{End}\, E))$ be the identity map. Form the bar & cyclic bar cochains

$$\tau\left(e^{\nabla^2 + [\nabla,\theta]}\right) \in C(A; \Omega(M))$$

$$\tau\left(\partial\theta\, e^{\nabla^2 + [\nabla,\theta]}\right) \in C(A, A^*; \Omega(M))$$

One has then the formulas

$$(\delta + d)\,\tau\left(e^{\nabla^2 + [\nabla,\theta]}\right) = \beta\left\{\tau\left(\partial\theta\, e^{\nabla^2 + [\nabla,\theta]}\right)\right\}$$

$$(\delta + d)\,\tau\left(\partial\theta\, e^{\nabla^2 + [\nabla,\theta]}\right) = \partial\left\{\tau\left(e^{\nabla^2 + [\nabla,\theta]}\right)\right\}$$

Integrating over a k-cycle z gives a map

$$\tau = \int_z tr_E : \Omega(M, \mathrm{End}\, E) \longrightarrow \Omega(M) \longrightarrow \mathbb{C}$$

and one gets the same formulas except the d disappears.

Now take $E = \tilde{\mathbb{C}}$, $\nabla = d$. Then

$$\tau\left(e^{\nabla^2 + [\nabla,\theta]}\right) = \int_z e^{d\theta} = \int_z \frac{(d\theta)^k}{k!} = \int_z d\left(\frac{\theta (d\theta)^{k-1}}{k!}\right)$$

$$= 0$$

and

$$\tau\left(\partial\theta\, e^{\nabla^2+[\nabla,\theta]}\right) = \int_Z \partial\theta \frac{(d\theta)^k}{k!}$$

So the whole family of cochains (the "big" cocycle) reduces to the single cyclic bar cochain

$$\int_Z \partial\theta \frac{(d\theta)^k}{k!} \in C^k(A, A^*)$$

This is killed by δ, so it's a ~~[illegible]~~ cyclic bar (or Hochschild) cocycle, and it's killed by β, so it's a cyclic cochain.

The moral is this. One has an incredibly simple "big" cocycle attached to a closed current in M. By diagram chasing one can associate to it lots of cyclic cocycles.

IDEA: Compare the cyclic bar cochains attached to a vector bundle with connection $\mathrm{tr}_E\left(\partial\theta\, e^{\nabla^2+[\nabla,\theta]}\right)$ with the Bismut forms on the free loop space.