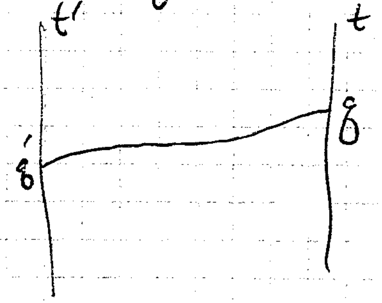
July 18, 1982

WKB approximation again. I still don't understand the determinant term in the heat kernel.

Calc. of variations



$$\delta S = \delta\int L(t,q,\dot q)\,dt = \int\left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta\dot q\right)dt$$

$$= \left[\frac{\partial L}{\partial \dot q}\delta q\right]_{t'}^{t} + \int\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\delta q\,dt$$

Trajectories satisfy $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$. If we let $S(tq)$ be the action $\int L\,dt$ starting at a fixed $t'q'$ and ending at tq, then we have

$$\frac{\partial S}{\partial q} = \frac{\partial L}{\partial \dot q}$$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\dot q = L \qquad H = p\dot q - L$$

so that if we make the definitions $p = \frac{\partial L}{\partial \dot q}$, and change vbles $\dot q \mapsto p$. we get the HJ equation

$$\frac{\partial S}{\partial t} + H\left(t, q, \frac{\partial S}{\partial q}\right) = 0$$

as well as the Hamilton eqns.

$$\dot q = \frac{\partial H}{\partial p} \qquad \dot p = -\frac{\partial H}{\partial q}.$$

The trajectories form a $2n$ diml symplectic manifold which is isomorphic to standard (q,p) space under evaluation at any fixed time. Hence $q' \mapsto q$ is symplectic.

Next I want to work around a given trajectory to the first order. This means to linearize Hamilton's equations $q = q^0 + \delta q$, $p = p^0 + \delta p$ around the given

$$\frac{\sqrt{\det b}}{(2\pi)^{n/2}}\, e^{i\left(\frac{1}{2}aq^2 + bqq' + \frac{c}{2}q'^2\right)}$$

The formula in general for the WKB approx. to the kernel $\langle q|U(t,t')|q'\rangle$ should therefore be

$$\frac{e^{iS(tq,\, t'q')}}{(2\pi)^{n/2}} \quad \det\left(i\,\frac{\partial^2 S}{\partial q\,\partial q'}\right)^{1/2}$$

<u>Flat Green's function.</u>

M compact Riemann surface with metric $ds^2 = \rho\,(dx^2+dy^2)$ given, where $z = x+iy$ is a local coord. We want to construct a parametrix for

$$\bar\partial : T^{0,0} \longrightarrow T^{0,1}, \quad \bar\partial f = (\partial_{\bar z} f)\, d\bar z$$

where parametrix means inverse modulo smoothing operators. For example, locally a parametrix is given by the operator

$$f\,d\bar z \longmapsto \int \frac{i}{2\pi}\,\frac{1}{z-z'}\, f(z')\, dz'\, d\bar z'$$

since

$$\bar\partial \int \frac{i}{2\pi}\,\frac{1}{z-z'}\, f(z')\, dz'\, d\bar z' = d\bar z \int \underbrace{\partial_{\bar z}\frac{1}{\pi(z-z')}}_{\delta(z-z')}\, f(z')\,\underbrace{\tfrac{i}{2}\, dz'\, d\bar z'}_{dx'\,dy'}$$

$$= f(z)\, d\bar z$$

Quite generally a parametrix for $\bar\partial$ is given by a kernel

$$\frac{i}{2\pi}\, G(z,z')\, dz'$$

which is a smooth section of $pr_2^*(T^{1,0})$ over $M\times M - \Delta M$ such that near the diagonal

$$G(z,z') = \frac{1}{z-z'} + \text{smooth}.$$

Here's how to construct a parametrix. Let $r(z,z')$ denote the distance between z and z'; this is well-defined in a nbd. of the diagonal, and $r(z,z')^2$ is a smooth fn.

<u>Lemma:</u> In a local coordinate system one has

$$r(z,z')^2 = |z-z'|^2\, V(z,z')$$

where V is smooth with $V(z,z) = \rho(z)$.

Granting this put $G(z,z') = -\partial_{z'}\left(\log r(z,z')^2\right) dz'$ This is a smooth form defined in a tubular nbd of Δ in $M \times M$ with diagonal deleted. Locally

$$G(z,z')\,dz' = -\partial_{z'}\left[\log(|z-z'|^2 v)\right] dz'$$

$$= \left[\frac{1}{z-z'} - \partial_{z'} \log v\right] dz'$$

so we get a parametrix kernel for $\bar\partial$ defined in a tubular nbd. of Δ in $M \times M$. Extend to the whole of $M \times M$ by multiplying by a suitable fn.

From a parametrix P for D one gets a formula for the index of D as follows. $PD = I - K^0$, $DP = I - K^1$ where K^i is an operator with smooth kernel. Then

$$\mathrm{Ind}(D) = \mathrm{Tr}(K^0) - \mathrm{Tr}(K^1) \qquad \left(= \underset{\text{heuristically}}{\mathrm{Tr}\,[D,P]}\right)$$

where these traces can be evaluated by restricting the kernel to the diagonal and integrating.

Put

$$G(z,z') = \frac{1}{z-z'} + a_{z'} + b_{z'}(z-z') + c_{z'}\overline{(z-z')} + O((z-z')^2)$$

Then DP is represented by the kernel

$$\frac{i}{2\pi}\, d\bar z\ \partial_{\bar z} G(z,z')\, dz' = \frac{i}{2\pi}\, d\bar z\, dz' \left\{ \pi\,\delta(z-z') + c_{z'} + O(z-z') \right\}$$

and PD is the operator

$$f \longmapsto \int \frac{i}{2\pi}\, G(z,z')\,dz'\ \partial_{\bar z'} f(z')\, d\bar z' = \int \frac{i}{2\pi} \underbrace{-\partial_{\bar z'} G(z,z')}_{\pi\delta(z-z') - \partial_{\bar z'} a_{z'} + c_{z'} + O(z-z')} f(z')\, dz'\, d\bar z'$$

Hence the index of D is the integral over M of the form

$$\frac{i}{2\pi}\left(\partial_{\bar z'} a_{z'}\right) dz'\, d\bar z'$$

In the case of the flat parametrix constructed above ©

$$a_{z'} = -\partial_{z'} \log V(z,z')\Big|_{z=z'}$$

But

$$\partial_{z'} \underbrace{\log V(z',z')}_{\log\rho(z')} = \Big[\underbrace{\partial_{z} \log V(z,z')}_{\partial_z \log V(z',z)} + \partial_{z'}\log V(z,z')\Big]_{z=z'}$$

$$= 2\,\partial_{z'}\log V(z,z')\Big]_{z=z'}.$$

Thus

$$a_z = -\frac{1}{2}\partial_z \log\rho$$

and so

$$\mathrm{Ind}\,\bar\partial = \int -\frac{1}{2}\,\frac{i}{2\pi}\underbrace{(\partial_{\bar z}\partial_z \log\rho)\,dz\,d\bar z}_{-\bar\partial\partial\log\rho}$$

$$= \frac{1}{2}\deg \text{ tangent bundle} \quad \text{because}$$

$$|\partial_z|^2 = \frac{1}{4}(\partial_x^2+\partial_y^2) = \frac{\rho}{2}$$

Another possible parametrix would be obtained by piecing together the elementary $\frac{dz'}{z-z'}$ parametrices.

<u>July 30, 1982</u> (1)

In Oxford I made some sense out of denominator-free Green's fns. One has an operator $A : V_1 \to V_0$ which is Fredholm. It gives rise to a line

$$\mathcal{L}_A^* \subset \mathrm{Hom}^{(p)}(\Lambda V_0, \Lambda V_1) \qquad p = \mathrm{Ind}(A).$$

If I choose a generator for $\mathcal{L}_A^*$ then I get a map $\Lambda V_0 \to \Lambda V_1$ whose matrix elements are the Green's functions in question.

Let's now work in the fermion-integration-notation. We have $A \in V_0 \otimes V_1^*$ and hence can write

$$A = \int \tilde{\psi}(x) A(x,y) \psi(y) \qquad (= \tilde{\psi} A \psi \text{ for short})$$

where $\tilde{\psi}(x)$ ~~is a basis for~~ are elements of V_0, $\psi(y)$ ~~is a basis for~~ are elements of V_1^*.
Now a typical Green's fn. is described by an integral of the form

$$G_{rs}(x_1,\dots,x_r, y_1,\dots,y_s) = \int e^{-\tilde{\psi} A \psi} \psi(x_1)\cdots\psi(x_r)\, \tilde{\psi}(y_s)\cdots\tilde{\psi}(y_1)$$

which is a $\mathbb{C}$-number associated to the element $\psi(x_1)\cdots\psi(x_r)\tilde{\psi}(y_s)\cdots\tilde{\psi}(y_1) \in \Lambda^r V_1^* \otimes \Lambda^s V_0$. Hence this G_{rs} is an element of $\Lambda^r V_1 \otimes \Lambda^s V_0^*$.

Thus $\int e^{-\tilde{\psi} A \psi}$ can naturally be interpreted as giving the matrix elements of an element of $\mathrm{Hom}(\Lambda V_0, \Lambda V_1)$.

When A is invertible we can choose the generator of $\mathcal{L}_A^*$ to give the map $1 : \Lambda^0 V_0 \to \Lambda^0 V_1$ in which case we know that the generator is the map $\Lambda(A^{-1}) : \Lambda V_0 \to \Lambda V_1$. On the other hand this choice of generator corresponds to the normalization

$$\int e^{-\tilde{\psi} A \psi} = 1$$

for the fermion integral. The $-$ sign is justified by the formula

$$\frac{\int e^{-\bar{\psi} A \psi} \psi(x) \bar{\psi}(y) \text{ [redacted]}}{\int e^{-\bar{\psi} A \psi}} = \langle x | A^{-1} | y \rangle$$

[redacted] And I know the general formula

$$\int e^{-\bar{\psi} A \psi} \psi(x_1) \cdots \psi(x_r) \bar{\psi}(y_s) \cdots \bar{\psi}(y_1) = \begin{cases} 0 & r \neq s \\ \det \langle x_i | A^{-1} | y_j \rangle & r = s \end{cases}$$

when $\int e^{-\bar{\psi} A \psi} = 1$.

<u>Conclusion:</u> Choosing a generator for $\mathcal{L}_A^* \subset \mathrm{Hom}(\Lambda V_0, \Lambda V_1)$ amounts to choosing a normalization for the fermion integral $\int e^{-\bar{\psi} A \psi}(\cdots\cdots)$, and the fermion integral simply gives the matrix elements of the generator.

At Oxford I noticed [redacted] the similarity of the above fermion integrals with the formula

$$\langle \psi(x_1) \psi(x_2) \psi^*(y_2) \psi^*(y_1) \rangle = \det \langle \psi(x_i) \psi^*(y_j) \rangle$$

encountered when dealing with a [redacted] fermion gas. I want to connect up these formalisms, the idea being [redacted] to relate $\int e^{-\bar{\psi} A \psi}(\cdots\cdot)$, which I have just sort-of-understood, to $\langle \cdots \rangle$ which is the expectation $= \mathrm{tr}\left(e^{-\beta H}(\cdots\cdot)\right) / \mathrm{tr}(e^{-\beta H})$.

Let's now go over the "kinematics" of a fermion gas. One starts with a 1-particle Hilbert space V, and one forms a Clifford algebra out of operators $\psi(x)$, $\psi(x)^*$ for each basis element x of V. ~~notation~~

Then the Clifford module ΛV is the state Hilbert space for the gas. In finite-dims. one has

$$C(V \oplus V^*) = \mathrm{End}(\Lambda V)$$

I am especially interested in operators of degree 0

$$\mathrm{End}^{(0)}(\Lambda V) = \bigoplus_p \Lambda^p(V) \otimes \Lambda^p(V^*) \subset \Lambda(V \oplus V^*)$$

This subspace of $\Lambda(V \oplus V^*)$ is generated by $V \otimes V^* \subset \Lambda^2(V \oplus V^*)$.

Let's begin again with a fermion gas. One has a 1-particle space V and forms the many-particle Fock space ΛV which then has operators ψ_v and ψ_v^* belonging to $v \in V$. What are the Green's functions? These are expectation values

$$\langle \psi(x_1) \cdots \psi(x_n) \psi(y_n)^* \cdots \psi(y_1)^* \rangle$$

where $\psi(x)$ denotes interior multiplication relative to a linear fnl $\langle x|$, etc.

Two examples:

1) Let H be a Hamiltonian on V and extend it to ΛV as a derivation. Suppose $H|k\rangle = \varepsilon_k |k\rangle$ where $|k\rangle$ is an orthonormal basis. If $|x\rangle$ is the standard basis for V, then we have

$$\langle x| = \sum \langle x|k\rangle\langle k| \implies \psi(x) = \sum \langle x|k\rangle \psi_k$$

$$|y\rangle = \sum |k\rangle\langle k|y\rangle \implies \psi(y)^* = \sum \langle k|y\rangle \psi_k^*$$

Take $\langle\ \rangle$ to be the vacuum expectation value, where the vacuum is the state in ΛV where all the $|k\rangle$ with $\varepsilon_k < 0$ are filled. Then

$$G(x,y) = \langle \psi(x)\psi(y)^* \rangle = \sum_{k,k'} \langle x|k\rangle \langle k'|y\rangle \langle \psi_k \psi_{k'}^* \rangle$$

$$= \sum_k \langle x|k\rangle\langle k|y\rangle \underbrace{\langle \psi_k \psi_k^* \rangle}_{\begin{cases} 1 & \varepsilon_k > 0 \\ 0 & \varepsilon_k < 0. \end{cases}}$$

2) Define $\psi(tx) = e^{iHt}\psi(x)e^{-iHt}$ as usual. It is the interior multiplication by the linear fnl $f \longmapsto e^{iHt}\langle x|e^{-iHt}|f\rangle = \sum \langle x|k\rangle e^{-it\varepsilon_k}\langle k|f\rangle$ hence we have

$$\psi_k(t) = e^{iHt}\psi_k e^{-iHt} = e^{-i\varepsilon_k t}\psi_k$$

$$\psi(tx) = \sum_k \langle x|k\rangle e^{-i\varepsilon_k t}\psi_k$$

$$\psi(t'y)^* = \sum_k \langle k|y\rangle e^{i\varepsilon_k t'}\psi_k^*$$

and so

$$\langle \psi(tx)\psi(t'y)^* \rangle = \sum_k \langle x|k\rangle\langle k|y\rangle e^{-i\varepsilon_k(t-t')} \underbrace{\langle \psi_k\psi_k^* \rangle}_{\begin{cases} 1 & \varepsilon_k > 0 \\ 0 & \varepsilon_k < 0 \end{cases}}$$

However the usual time-dependent Green's fns. are

$$\langle T[\psi(tx)\psi(t'x')^*] \rangle.$$

As usual time causes problems.

In the case of a fermion gas the Greens functions are determinants in the 1-particle Green's fn.

$$\langle \psi(x)\psi(y)^* \rangle = \sum_k \langle x|k\rangle\langle k|y\rangle \langle \psi_k \psi_k^* \rangle$$

and $$\langle \psi_k \psi_k^* \rangle = 1 - \underbrace{\langle \psi_k^* \psi_k \rangle}_{n_k} \qquad n_k = \frac{e^{-\beta\varepsilon_k}}{1+e^{-\beta\varepsilon_k}}$$

$$= \frac{1}{1+e^{-\beta\varepsilon_k}} \longrightarrow \begin{cases} 1 & \varepsilon_k > 0 \\ 0 & \varepsilon_k < 0 . \end{cases}$$

$$\therefore \quad \langle \psi(x)\psi(y)^* \rangle = \langle x \,|\, \frac{1}{1+e^{-\beta H}} \,|\, y \rangle .$$

Thus instead of $G(x,y)$ being the kernel of A^{-1}, where $A: V_1 \to V_0$ is $\bar\partial$-operator for example, it is the kernel of $\frac{1}{1+e^{-\beta H}}$ which approaches the projection on the subspace where $H > 0$ as $\beta \to \infty$.

Corresponding to varying the $\bar\partial$-operator is varying the operator H. Thus if we have a perturbation we can ask how the Green's functions change, and also how the ground energy changes, or free energy in the non-zero temperature case. This ground energy is going to be the analogue of the determinant, defined as a function over the family of H, unique up to a multiplicative constant.

August 6, 1982

Review the calculation of $\det(D)$ where $D = \partial_{\bar z} + \alpha$ over an elliptic curve in the rank 1 case. We have

$$e^{-f} D e^{f} = \partial_{\bar z} + \alpha + \partial_{\bar z} f .$$

Choose f such that $\alpha = \alpha_0 - \partial_{\bar z} f$ where α_0 is constant. Then

$$D = e^{f} D_0 e^{-f} \qquad D_0 = \partial_{\bar z} + \alpha_0 .$$

Compute the variation in $\det(D)$ corresp. to δf in f:

$$\delta \log \det(D) = \int tr(J_D\, \delta D)$$

$$\delta D = [\delta f, D]$$

$$\delta \log\det(D) = \int tr\left(J_D\, [\delta f, D]\right)$$

$$= \int tr\left([D, J_D]\, \delta f\right)$$

By the anomaly formula

$[D, J_D] = -$ curvature for the connection defining J_D

So if $D = (\partial_{\bar z} + \alpha) d\bar z$ $\qquad \nabla = \partial_z dz + (\partial_{\bar z} + \alpha) d\bar z$ and

$$\nabla^2 = (\partial_z \alpha)\, dz\, d\bar z .$$

Thus

$$\delta \log \det(D) = -\int \partial_z \alpha\ \delta f\ dz\, d\bar z$$

$$= \int \partial_z \partial_{\bar z} f\ \delta f\ dz\, d\bar z$$

$$= -\int \partial_{\bar z} f\ \delta\, \partial_z f\ dz\, d\bar z$$

$$= -\delta \int \tfrac{1}{2} \partial_{\bar z} f\ \partial_z f\ dz\, d\bar z .$$

So integrating gives

$$\log\det(e^{f} D_0 e^{-f}) = \log\det D_0 - \frac{1}{2}\int \partial_{\bar z} f\, \partial_z f\ dz\, d\bar z$$