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Notes on $\langle 0|S|0\rangle$ for Graeme 73-78

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$$H_0 = \frac{1}{i}\partial_x \quad \text{on } L^2(S^1) \qquad S^1 = \mathbb{R}/L\mathbb{Z}$$

$$H_0|k\rangle = k|k\rangle \qquad \text{where } \langle x|k\rangle = \frac{1}{\sqrt{L}}e^{ikx} \qquad k\in\frac{2\pi}{L}\mathbb{Z}$$

$\mathcal{H}$ = fermion Fock space of $L^2(S^1)$ with vacuum vector $|0\rangle$ corresponding to subspace of $L^2(S^1)$ spanned by $|k\rangle$ with $k\le 0$.

$$\psi(x) = \sum_k \langle x|k\rangle\psi_k = \frac{1}{\sqrt{L}}\sum_k e^{ikx}\psi_k$$

$$\psi(x)^* = \sum_{k'}\frac{1}{\sqrt{L}}e^{-ik'x}\psi_{k'}^*$$

The derivation on Fock space corresponding to multiplication by $f(x)$ is

$$\hat{f} = :\int dx\, f(x)\,\psi(x)^*\psi(x):$$

$$= :\sum_{k,k'}\left(\frac{1}{L}\int dx\, f(x)\,e^{-i(k'-k)x}\right)\psi_{k'}^*\psi_k: \qquad k'=k+q$$

$$= \sum_q \underbrace{\left(\frac{1}{L}\int dx\, f(x)\,e^{-iqx}\right)}_{f_q}\ \underbrace{:\sum_k \psi_{k+q}^*\psi_k:}_{\rho_q}$$

Recall that the ρ_q for $q>0$ are essentially boson creation operators, the only non-trivial commutation relation being

$$[\rho_{-q},\rho_q] = \frac{L}{2\pi}q \qquad q>0.$$

Consider a perturbation $H = \frac{1}{i}\partial_x + f(x,t)$ of H_0, where f has compact support, and extend it to $\mathcal{H}$:

$$\hat{H} = \underset{\displaystyle \| \atop \displaystyle :\sum k\psi_k^*\psi_k:}{\hat{H}_0} + \hat{f}(x,t) = \hat{H}_0 + \sum_q f_q(t)\rho_q$$

The S-matrix (in imaginary time) is

$$S = T\left\{ e^{-\int dt\, e^{\hat{H}_0 t} \hat{f}(t) e^{-\hat{H}_0 t}} \right\}.$$

I need

$$e^{\hat{H}_0 t} \rho_q e^{-\hat{H}_0 t} = e^{qt} \rho_q$$

which follows from

$$e^{\hat{H}_0 t} \begin{Bmatrix} \psi_k \\ \psi_k^* \end{Bmatrix} e^{-\hat{H}_0 t} = \begin{Bmatrix} e^{-kt} \psi_k \\ e^{kt} \psi_k^* \end{Bmatrix}$$

(Check $\quad [\hat{H}_0, \psi_k] = [k\psi_k^* \psi_k, \psi_k] = -k\{\psi_k^*, \psi_k\}\psi_k = -k\psi_k$.)

Hence

$$S = T\left\{ e^{-\int dt \sum_q' f_q(t) e^{qt} \rho_q} \right\}$$

Recall that for a simple oscillator

$$S = T\left\{ e^{\int dt\, (g(t) a^* + \tilde{g}(t) a)} \right\} = e^{\int_{t>t'} dt dt'\, \tilde{g}(t) g(t') [a, a^*]}$$

$$\times\, e^{\left(\int dt\, g(t)\right) a^*} e^{\left(\int dt\, \tilde{g}(t)\right) a}$$

and hence $\langle 0|S|0\rangle = e^{\int_{t>t'} dt dt'\, \tilde{g}(t) g(t')}$. The obvious generalization to a multiple oscillator then gives

$$\log \langle 0|S|0\rangle = \sum_{q>0} \int_{t>t'} dt dt'\, f_{-q}(t) f_q(t') e^{-q(t-t')} [\rho_{-q}, \rho_q]$$

$$= \int_{t>t'} dt dt' \sum_{q>0} \left(\frac{1}{L} \int dx\, f(xt) e^{+iqx} \right)\left(\frac{1}{L} \int dx'\, f(x't') e^{-iqx'} \right) e^{-q(t-t')} \frac{L}{2\pi} q$$

$$= \int_{t>t'} dt dt' dx dx'\, f(xt) f(x't') \sum_{q>0}' e^{iq\Delta x - q\Delta t} q \frac{1}{2\pi L}$$

$\Delta x = x - x'$
$\Delta t = t - t'$

As $L \to \infty$

$$\frac{1}{L} \sum_{\substack{q>0 \\ q \in \frac{2\pi}{L}\mathbb{Z}}} e^{-q(\Delta t - i\Delta x)} q \longrightarrow \int \frac{dq}{2\pi}\, e^{-q(\Delta t - i \Delta x)} q$$

$$= \frac{\Gamma(2)}{2\pi\,(\Delta t - i \Delta x)^2}$$

Thus for the $L \to \infty$ limit

$$\boxed{\begin{aligned} \log \langle 0|S|0\rangle &= \int_{t>t'} dt\, dt'\, dx\, dx' \left(\frac{i}{2\pi}\right)^2 \frac{f(xt)f(x't')}{(\Delta x + i\Delta t)^2} \\ &= \frac{1}{2} \int d^2z\, d^2z' \left(\frac{i}{2\pi}\right)^2 \frac{f(z)f(z')}{(z-z')^2} \end{aligned}}$$

if we put $z = x + it$.

Let's check this against the previous calculation of the ground energy shift when f is time-independent. Turning f on for a long time T gives

$$\langle 0|S|0\rangle \sim e^{-T\Delta E}$$

hence

$$\Delta E = -\int_{t>0} dt\, dx\, dx' \left(\frac{i}{2\pi}\right)^2 \frac{f(x)f(x')}{(\Delta x + it)^2}$$

$$= -\int \frac{dx\,dx'}{(2\pi)^2} f(x)f(x') \underbrace{\int_{t>0} dt\, \frac{+1}{(t - i\Delta x)^2}}_{\left[\frac{-1}{t - i\Delta x}\right]_{0+}^{\infty}}$$

by symmetry

$$= -\int \frac{dx\,dx'\, f(x)f(x')}{(2\pi)^2} \underbrace{\frac{1}{2}\left[\frac{1}{\eta - i\Delta x} + \frac{1}{\eta + i\Delta x}\right]}_{\frac{\eta}{\eta^2 + (\Delta x)^2} = \pi\delta(\Delta x)} \qquad \eta = 0^+$$

$$\therefore \quad \boxed{\Delta E = -\frac{1}{4\pi}\int dx\, f(x)^2}$$

Next I want to identify $\langle 0|S|0\rangle$ with a determinant for the differential operator

$$\partial_t + \frac{1}{i}\partial_x + f(x,t) = \frac{2}{i}\left(\partial_{\bar z} + \frac{i}{2} f\right)$$

The first method uses the formula

$$\delta \log \det(D) = \operatorname{Tr}\left(D^{-1}\delta D\right)$$

where the trace has to be regularized. We need the Green's function for $D = \partial_{\bar z} + \frac{i}{2} f$. To choose φ with

$$e^{\varphi}\partial_{\bar z} e^{-\varphi} = \partial_{\bar z} - \partial_{\bar z}\varphi = \partial_{\bar z} + \frac{i}{2} f$$

$$\partial_{\bar z}\frac{1}{\pi(z-z')} = \delta(z-z')$$

Hence we have $\varphi(z) = \int \frac{d^2z'}{2\pi i}\frac{f(z')}{z-z'}$. Thus

$$\underbrace{D}_{e^{\varphi}\partial_{\bar z} e^{-\varphi}} G(z,z') = \delta(z-z')$$

$$\partial_{\bar z}\, e^{-\varphi(z)} G(z,z') = e^{-\varphi}\delta(z-z') = e^{-\varphi(z')}\delta(z-z')$$

$$e^{-\varphi(z)} G(z,z') = \frac{e^{-\varphi(z')}}{\pi(z-z')}$$

$$\boxed{G(z,z') = \frac{e^{\varphi(z)-\varphi(z')}}{\pi(z-z')} = \frac{1}{\pi(z-z')}\, e^{\int \frac{d^2w}{2\pi i} f(w)\left[\frac{1}{z-w} - \frac{1}{z'-w}\right]}}$$

To regularize the trace we must extract a finite part as $z \to z'$. This we can do by lifting the $\bar\partial$ operator to a connection and subtracting from G a flat Green's function. We have

$$G(z,z') = \frac{1}{\pi(z-z')}\left\{1 + \partial_z\varphi\big|_{z'}\cdot(z-z') + \underbrace{\partial_{\bar z}\varphi\big|_{z'}}_{-\frac{i}{2}f(z')}\overline{(z-z')} + O(|z-z'|^2)\right\}$$

The simplest connection lifting the $\bar\partial$ operator is

$$\nabla = \partial_z\, dz + \left(\partial_{\bar z} + \tfrac{i}{2}f\right)d\bar z$$

and a corresponding flat Green's function has the form

$$G_b(z,z') = \frac{1}{\pi(z-z')}\left\{1 - \tfrac{i}{2}f(z')\,\overline{(z-z')} + O(|z-z'|^2)\right\}.$$

Hence

$$J(z) = F.P.\, G(z,z) = \frac{1}{\pi}\partial_z\varphi$$

$$\boxed{J(z) = \frac{1}{\pi}\int d^2w\left(\frac{i}{2\pi}\right)\frac{f(w)}{(z-w)^2}}$$

On the other hand from p. 75, corresponding to a δf we have

$$\delta\log\langle 0|S|0\rangle = \int d^2z\, d^2z'\left(\frac{i}{2\pi}\right)^2\frac{f(z')\,\delta f(z)}{(z-z')^2}$$

$$= \int d^2z\; J(z)\,\underbrace{\delta\left(\tfrac{i}{2}f(z)\right)}_{\delta D} \qquad \text{if } D = \partial_{\bar z} + \tfrac{i}{2}f$$

Therefore we can identify $\langle 0|S|0\rangle$ with the relative determinant

$$\frac{\det\left(\partial_{\bar z} + \frac{i}{2}f\right)}{\det(\partial_{\bar z})} = \det\left(1 + \partial_{\bar z}^{-1}\tfrac{i}{2}f\right)$$

defined by the above-described regularization process.

The second method for identifying $\langle 0|S|0\rangle$ with the above relative determinant is to use

$$\log\det(1+K) = \sum_1^\infty \frac{(-1)^{n-1}}{n}\,\mathrm{Tr}(K^n)$$

where $K = (\partial_{\bar z})^{-1} \frac{i}{2} f$ has the kernel

$$K = \frac{i}{2\pi} \frac{f(z')}{z - z'} .$$

The regularized trace of K is zero, since the free Green's function $\frac{1}{\pi}\frac{1}{(z-z')}$ has finite part 0. $\mathrm{Tr}(K^2)$ is formally

$$-\int d^2z\, d^2z' \left(\frac{i}{2\pi}\right)^2 \frac{f(z')f(z)}{(z-z')^2}$$

and this has a Cauchy-principal-value interpretation which furnishes the regularization. The operators K^n for $n \geq 3$ are of trace class; one has

$$\begin{aligned}\mathrm{Tr}(K^n) &= \left(\frac{i}{2\pi}\right)^n \int d^2z_1 \cdots d^2z_n \frac{f(z_1)f(z_2)\cdots f(z_n)}{(z_1-z_2)\cdots\cdots(z_n-z_1)} \\ &= \left(\frac{i}{2\pi}\right)^n \int d^2z_1 \cdots d^2z_n \prod f(z_i) \times \left\{ \begin{matrix} \text{the symmetrization} \\ \text{of } \dfrac{1}{(z_1-z_2)\cdots(z_n-z_1)} \end{matrix} \right\}\end{aligned}$$

Call this symmetrization $r(z_1, \ldots, z_n)$. Then $r \cdot \prod_{i<j}(z_i - z_j)$ is a polynomial which is alternating, hence divisible by $\prod_{i<j}(z_i - z_j)$, which implies $r = 0$. Hence $\mathrm{Tr}(K^n) = 0$ for $n \geq 3$, giving

$$\det\left(1 + \partial_{\bar z}^{-1} \tfrac{i}{2} f\right) = e^{-\frac{1}{2}\mathrm{Tr}(K^2)}$$

$$= \langle 0 | S | 0 \rangle .$$

September 4, 1982

Return to the GRR situation where we have a family $f: X \to Y$ of curves, a vector bundle E over X, all this being holomorphic. Put metrics on E and $T_{X/Y}$. The $\bar\partial$ operator along the fibres of f

$$\bar\partial : E \longrightarrow E \otimes T^{0,1}_{X/Y}$$

gives for each $y \in Y$ an operator

$$D_y : \underset{\substack{\| \\ W_y}}{\Gamma(X_y, E)} \longrightarrow \underset{\substack{\| \\ V_y}}{\Gamma(X_y, E \otimes T^{0,1}_{X/Y})}$$

between Hilbert spaces. Hence we can form D_y^* and two Laplaceans. We consider the simple case where D_y is invertible for each y. Then we get a fn. on Y given by

$$y \longmapsto \mathrm{Tr}\left((D_y^* D_y)^{-s}\right)$$

and I want to understand how the analytic torsion $\tau(D_y^* D_y)$ varies as a function of y.

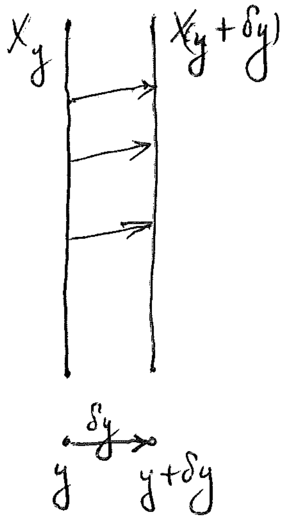
A problem arises in that the operator $\Delta_y = D_y^* D_y$ work on different spaces W_y as y varies, and so we have to do something in order to talk about the change $\delta\Delta$. I want to use the formula

$$-\delta\zeta'(0) = \mathrm{Tr}\left((D^*D)^{-s}(D^*D)^{-1}\delta(D^*D)\right)\Big|_{s=0}$$

$-\delta\zeta'(0)$ refers to the change in $-\zeta'(0)$ corresponding to a tangent vector δy in Y. What I need is some kind of connection in the (pre) Hilbert bundle $\{W_y\}$ over Y.

If E is the trivial bundle, then W_y is the space of C^∞ functions on X_y, so a connection in W gives a way of transporting functions on X_y to fns. on nearby fibers, e.g. a C^∞-trivialization of X/Y does this. To first order such a trivialization gives a vector field on X lying over the tangent vector δy.

Picture:



What I want to see is that such a vector field gives me a way to define $\delta\Delta$, as well a measure of the deformation of the holom. structure on X_y.

September 5, 1982

Situation: $\begin{matrix} X & E \\ \downarrow f & \\ Y & \end{matrix}$ is a family of curves, E holom. v.b./X. Assume E has no cohomology on the fibres, hence the $\bar{\partial}$-operator

$$D_y : W_y \longrightarrow V_y$$

is invertible for each $y \in Y$. Let suppose given a way to extract the diagonal part for the kernel of D_y^{-1}; this will be a (1,0) form $J \in \mathrm{Hom}(E,E) \otimes T^{1,0}_{X/Y}$. I then want to define a fn. $\det(D_y)$ on Y, by the formula

$$\delta \log \det(D_y) = \mathrm{Tr}^{(reg)}(D_y^{-1} \cdot \delta D_y)$$
$$= \int_{X_y} \mathrm{tr}(J_y \cdot \delta D_y).$$

The problem then becomes how to interpret the operator δD_y.

The above is a bit confusing, and I should try to get it clearer. I know that over Y is a canonical line bundle with a canonical section, and that trivializing this line bundle gives me the determinant function $\det(D_y)$ from the canonical section.

Yesterday I decided that in order to define the variation in D or Δ I needed to have a way of lifting vector fields on Y to X. (This is the concept of a connection in the diffeomorphism sense.) One has the exact sequence

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0.$$

A tangent vector at y gives a section of the latter bundle over the fibre X_y. Actually we can look at all these bundles as being holom. over X, and hence holom. bdles when restricted to X_y. Corresponding to $\delta \in T_{Y,y}$ is then a torsor for the holom. bdle $T_{X|Y,y} = T_{X_y}$ which gives an element of $H^1(X_y, T_{X_y})$. This is just the element describing the deformation of X_y in the direction of δy.

Note: Think of a tangent vector at x as an infinitesimal change $x \longrightarrow x + \delta x$, whose effect on functions is $\delta f = f(x+\delta x) - f(x) = \sum_i \partial_{x_i} f(x) \delta x_i = \left(\sum_i \delta x_i \frac{\partial}{\partial x_i} \right) f \Big|_x$.

In $\mathbb{C}$ a tangent vector at a point is given by an operator

$$a \frac{\partial}{\partial z} + \bar{a} \frac{\partial}{\partial \bar{z}} = a \tfrac{1}{2}(\partial_x - i\partial_y) + \bar{a} \tfrac{1}{2}(\partial_x + i\partial_y)$$

$$= \underbrace{\frac{a+\bar{a}}{2}}_{Re(a)} \partial_x + \underbrace{\frac{a-\bar{a}}{2i}}_{Im(a)} \partial_y$$

Hence a displacement $z \longrightarrow z + \delta z$ corresponds to the operator

$$\delta z \frac{\partial}{\partial z} + \delta \bar{z} \frac{\partial}{\partial \bar{z}} .$$

September 6, 1982

Problem: Given a family of Riemann surfaces $X \to Y$ with a metric on $T_{X/Y}$, so that for each $y \in Y$ we get a Laplacean operator Δ_y on $W_y = C^\infty(X_y)$. I want to take the ζ fn. $\mathrm{Tr}(\Delta_y^{-s})$ and make sense out of the formula

$$-\delta\zeta(s) = s\,\mathrm{Tr}\left(\Delta_y^{-s-1}\delta\Delta_y\right).$$

This should be the same as for the heat kernel

$$\delta\,\mathrm{Tr}\left(e^{-t\Delta_y}\right) = -t\,\mathrm{Tr}\left(e^{-t\Delta_y}\,\delta\Delta_y\right).$$

The problem here is that as y changes, Δ_y operates in different spaces W_y.

Go over the linear algebra first. Let W be a vector bundle over a manifold Y and let $A_y : W_y \to W_y$ be an endom. of W. Then $y \mapsto \mathrm{Tr}(A_y)$ is function on Y and so given a tangent vector δy we can differentiate this function with respect to it.

$$\delta\,\mathrm{Tr}(A_y) = \mathrm{Tr}(A_{y+\delta y}) - \mathrm{Tr}(A_y).$$

You want to write this in terms of a δA_y operating on W_y, so you need a way to identify $W_{y+\delta y}$ with W_y. If one asks for this for all δy, that's the same as a connection in W. A connection is a first order op. $\nabla : W \to W \otimes T^*$ and $[\nabla, A] : W \to W \otimes T^*$ is zero-th order, hence it has a trace $\mathrm{Tr}\,[\nabla, A]$ which is a 1-form on Y. One has

$$d\,\mathrm{Tr}(A) = \mathrm{Tr}\,[\nabla, A]$$

for any connection on W.

So now I want to apply this when $W_y = C^\infty(X_y)$ in which case a section of W is a smooth function on X. Given a vector field v on Y if I lift it to a vector field on X, then I get an operator $\nabla_v : W \to W$ satisfying $\nabla_v(f(y)w) = vf \cdot \nabla_v(w)$, so it's clear that a connection in

X/Y, considered as a differentiable fibre bundle, gives a connection in W.

So now I think of a connection in W as being given by an operation of vector fields from Y as vector fields on X. So now let us be given a vector field on Y; call it δ, and think of it as giving an inf. displacement $y \longrightarrow y + \delta y$. Then it gets lifted to X: $\delta : x \longrightarrow x + \delta x$, and precisely it is a first order diff. op. on functions on X. Then $[\delta, \Delta]$ makes sense as a differential operator on X. It should be of order ≤ 2, and linear over $C^\infty(Y)$. This should be the desired operator $\delta\Delta$.

I would like now to make a clarifying calculation. Lets take the case of a family of elliptic curves, say

$$X = \{\operatorname{Im} \tau > 0\} \times \mathbb{C} \Big/ (\tau, z) \sim (\tau, z + m + n\tau) \qquad m, n \in \mathbb{Z}$$

so that $Y = \{\operatorname{Im} \tau > 0\}$, $X_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. The problem is to calculate the change in $-\zeta'(0)$. Now the actual heat kernel for the elliptic curve X_τ is the sum of the usual Euclidean heat kernels translated over the period lattice. In particular the asymptotic expansion of the heat kernel along the diagonal is the same as for the Euclidean kernel, and so doesn't depend on the lattice. Thus the asymptotic expansion of the trace of the heat kernel composed with a local operator such as

$$\operatorname{Tr}(e^{-t\Delta}\, \delta\Delta)$$

will not depend on the lattice. We must have a non-local operator inside to get something interesting, e.g.

$$\operatorname{Tr}(e^{-t\Delta}\, \Delta^{-1}\, \delta\Delta).$$

Let's begin by understanding $\delta\Delta$. What I want to do is to lift the vector field ∂_τ on Y to X. First consider differentials. We have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*T^{1,0}_Y & \longrightarrow & T^{1,0}_X & \longrightarrow & T^{1,0}_{X/Y} & \longrightarrow & 0 \\ & & \Psi & & & & \Psi & & \\ & & d\tau & & & & dz & & \end{array}$$

of holomorphic vector bundles over X. I want to lift dz to a $(1,0)$ form ω on X; call $\tilde{\omega}$ the lift of ω to $Y \times \mathbb{C}$. One has $\tilde{\omega} = dz - \beta(\tau, z)\, d\tau$ and $\tilde{\omega}$ is invariant under $(\tau, z) \longmapsto (\tau, z+m+n\tau)$, i.e.

$$\cancel{dz} + n\,d\tau - \beta(\tau, z+m+n\tau)\,d\tau = \cancel{dz} - \beta(\tau,z)\,d\tau$$

$$\beta(\tau, z+m+n\tau) - \beta(\tau, z) = n$$

The simplest choice for β is $\beta(\tau, z) = \dfrac{\operatorname{Im} z}{\operatorname{Im} \tau}$

so

$$\tilde{\omega} = dz - \frac{\operatorname{Im} z}{\operatorname{Im} \tau}\, d\tau$$

So next consider the exact sequence of holom. v.b.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X/Y} & \longrightarrow & T_X & \longrightarrow & f^*T_Y & \longrightarrow & 0 \\ & & \Psi & & & & \Psi & & \\ & & \partial_z & & & & \partial_\tau & & \end{array}$$

and we want a vector field on X lifting ∂_τ. Lifted to $Y \times \mathbb{C}$ this has the form

$$\partial_\tau + \alpha(\tau, z)\, \partial_z$$

and it must be invariant under $(\tau, z) \mapsto (\tau, z+m+n\tau)$. Consider $(\tau, z) \longmapsto (\tau, z+\tau)$. Then $(\delta\tau, 0) \mapsto (\delta\tau, \delta\tau)$ and $(0, \delta z) \longmapsto (0, \delta z)$, hence we have

$$\partial_\tau \longmapsto \partial_\tau + \partial_z$$
$$\partial_z \longmapsto \partial_z$$

and hence

$$\partial_\tau + \alpha(\tau, z)\, \partial_z \longmapsto \partial_\tau + \partial_z + \alpha(\tau, z-\tau)\, \partial_z .$$

This equals $\partial_\tau + \alpha(\tau, z)\, \partial_z$, hence $\alpha(\tau, z) = \alpha(\tau, z-\tau) + 1$ which has the solution

$$\alpha(\tau, z) = \frac{\operatorname{Im} z}{\operatorname{Im} \tau}$$

Therefore

$$\boxed{\partial_\tau + \frac{\operatorname{Im} z}{\operatorname{Im}\tau}\,\partial_z}$$

is a vector field on X lifting ∂_τ.

Next we need formulas for Δ, $e^{-t\Delta}$. I use the standard metric on $\mathbb{C}$ so that the volume is $d^2z = dx\,dy$. Consider the $\bar\partial$-operator

$$D = (\partial_{\bar z} - w)\,d\bar z .$$

It's adjoint is [redacted]

$$D^*(\alpha\, d\bar z) = -2(\partial_z + \bar w)\alpha$$

and so the Laplacean is given by

$$-\Delta = -D^*D = 2\underbrace{(\partial_z + \bar w)(\partial_{\bar z} - w)}_{e^{w\bar z - \bar w z}\partial_z\partial_{\bar z}\, e^{-w\bar z + \bar w z}}$$

Now $2\,\partial_z\partial_{\bar z} = \frac{1}{2}(\partial_x^2 + \partial_y^2)$ has heat kernel on $\mathbb{C}$

$$\langle z|\, e^{t\partial_z\partial_{\bar z}} |z'\rangle = \frac{1}{2\pi t}\, e^{-\frac{|\Delta z|^2}{2t}}$$

hence

$$\langle z|\, e^{-t\Delta}|z'\rangle = \langle z|\, e^{w\bar z - \bar w z}\, e^{t\,2\partial_z\partial_{\bar z}}\, e^{-w\bar z + \bar w z}|z'\rangle$$

$$\boxed{\langle z|\, e^{-t\Delta}|z'\rangle = \frac{1}{2\pi t}\, e^{-\frac{|\Delta z|^2}{2t} + w\,\Delta\bar z - \bar w\,\Delta z}}$$

on $\mathbb{C}$. To get the heat kernel on $X_\tau = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ we just have to add up these kernels with z' replaced by $z' + \lambda$, $\lambda \in \mathbb{Z} + \mathbb{Z}\tau$.

Next compute $\delta\Delta$

$$\delta\Delta = \left[\cancel{\partial_\tau} + \frac{\operatorname{Im} z}{\operatorname{Im}\tau}\partial_z\,,\ -2(\partial_z + \bar w)(\partial_{\bar z} - w)\right]$$

$$= -2\left\{\underbrace{\left[\frac{\operatorname{Im} z}{\operatorname{Im}\tau}\text{[redacted]},\ \partial_z + \bar w\right]}_{-1/2i\,\operatorname{Im}\tau}(\partial_{\bar z} - w)\partial_z + (\partial_z + \bar w)\underbrace{\left[\frac{\overset{-\bar z/2i}{\operatorname{Im} z}}{\operatorname{Im}\tau},\ \partial_{\bar z} - w\right]}_{1/2i\,\operatorname{Im}\tau}\partial_z\right\}$$

In the preceding it is not clear that we can lift the vector field $\partial_\tau + \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\partial_z$ arbitrarily to sections of the bundle E. Because E is holomorphic and has a metric, it comes with a connection which perhaps we ought to use. The holomorphic structure of E in this elliptic curve family example is given by the $\bar\partial$ operator

$$\partial_{\bar\tau}\, d\bar\tau + (\partial_{\bar z} - w)\, d\bar z .$$

The corresponding connection is

$$\nabla = \partial_\tau d\tau + \partial_{\bar\tau} d\bar\tau + (\partial_z + \bar w)dz + (\partial_{\bar z} - w)d\bar z$$

and hence we should lift $\partial_t + \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\partial_z$ to the operator

$$\partial_t + \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}(\partial_z + \bar w).$$

It's not clear whether this is really necessary, but it certainly simplifies the calculations. So [redacted]

$$\Delta = -2\tilde{\mathcal{D}}\mathcal{D}$$

$$\delta\Delta = -2\left((\delta\tilde{\mathcal{D}})\mathcal{D} + \tilde{\mathcal{D}}\,\delta\mathcal{D}\right)$$

$$\delta\mathcal{D} = \left[\partial_t + \frac{\overset{-\frac{\bar z}{2i}}{\mathrm{Im}(z)}}{\mathrm{Im}(\tau)}\tilde{\mathcal{D}}, \mathcal{D}\right] = \frac{1}{2i\,\mathrm{Im}(\tau)}\tilde{\mathcal{D}}$$

$$\delta\tilde{\mathcal{D}} = \left[\partial_t + \frac{\overset{\frac{z}{2i}}{\mathrm{Im}(z)}}{\mathrm{Im}(\tau)}\tilde{\mathcal{D}}, \underset{\partial_z}{\overset{\partial_{\bar z}}{\tilde{\mathcal{D}}}}\right] = \frac{-1}{2i\,\mathrm{Im}(\tau)}\tilde{\mathcal{D}}$$

$$\mathrm{Tr}(e^{-t\Delta}\Delta^{-1}\delta\Delta) = \underbrace{\mathrm{Tr}(e^{-t\Delta}\,\delta\tilde{\mathcal{D}}\,\tilde{\mathcal{D}}^{-1})}_{-\frac{1}{2i\,\mathrm{Im}(\tau)}\underbrace{\mathrm{Tr}(e^{-t\Delta})}_{}} + \mathrm{Tr}(e^{-t\Delta}\mathcal{D}^{-1}\delta\mathcal{D})$$

has asymptotic expansion to all orders $\sim \frac{\mathrm{vol}(X_\tau)}{2\pi t} = \frac{\mathrm{Im}\,\tau}{2\pi t}$

The second term is $\mathrm{Tr}(e^{-t\Delta}\mathcal{D}^{-1}\tilde{\mathcal{D}})$ essentially and should be more interesting. The operators involved are all translation-invariant so that I only have to worry about the diagonal value at one point (0,0). Furthermore, since I only want the asymptotic exp. I can replace the heat kernel by the Euclidean heat kernel since the other terms decay exponentially. Because of constant coefficients $\mathcal{D}^{-1}\tilde{\mathcal{D}} = \tilde{\mathcal{D}}\mathcal{D}^{-1}$ and so the $\langle 0, 0\rangle$ value is

$$(*) \qquad \int_{X_\tau} d^2z \; \langle 0 | e^{-t\Delta} | z \rangle (\partial_z + \bar{\omega}) \langle z | (\partial_{\bar z} - \omega)^{-1} | 0 \rangle .$$

I already know

$$\langle z | \frac{1}{\partial_{\bar z} - \omega} | 0 \rangle = \frac{1}{\pi} e^{\omega(\bar z + \frac{l}{m} z)} \frac{\sigma(z - \frac{\omega}{m})}{\sigma(z)\sigma(-\frac{\omega}{m})}$$

$$\langle 0 | e^{-t\Delta} | z \rangle = \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t} - \omega\bar z + \bar\omega z} \qquad + \text{ exp. decaying terms in } t.$$

So $(*)$ above has the same asymptotic expansion as

$$\int d^2z \; \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}} \partial_z \underbrace{\left[e^{(\bar\omega + \frac{l}{m}\omega) z} \frac{\sigma(z - \frac{\omega}{m})}{\pi\, \sigma(z)\, \sigma(-\frac{\omega}{m})} \right]}_{\frac{1}{\pi z}\left[1 + a_1 z + a_2 z^2 + \cdots\right]}$$

$$= \int d^2z \; \frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}} \frac{1}{\pi} \left\{ -\frac{1}{z^2} + a_2 + 2a_3 z + \cdots \right]$$

$$= \frac{a_2}{\pi} \qquad \text{Thus} \qquad \boxed{\mathrm{Tr}\left(e^{-t\Delta} \mathcal{D}^{-1} \tilde{\mathcal{D}}\right) \to \frac{a_2 \,\mathrm{Im}\,\tau}{\pi}}$$

Let's calculate a_2 :

$$\log\left(1+a_1 z + a_2 z^2 + \cdots\right) = \left(\bar{\omega} + \frac{l}{m}\omega\right) z + \log \sigma\left(z - \frac{\omega}{m}\right) - \log \underbrace{\frac{\sigma(z)}{z}}_{1+O(z^4)} + c.$$

so
$$\frac{a_1 + 2a_2 z + \cdots}{1 + a_1 z + a_2 z^2 + \cdots} = \bar{\omega} + \frac{l}{m}\omega + \zeta\left(z - \frac{\omega}{m}\right) + O(z^3)$$

$$\left(a_1 + 2a_2 z + \cdots\right)\left(1 - a_1 z + \cdots\right) = \bar{\omega} + \frac{l}{m}\omega + \zeta\left(-\frac{\omega}{m}\right) + \underbrace{\zeta'\left(-\frac{\omega}{m}\right)}_{-\wp\left(\frac{\omega}{m}\right)} z + \cdots$$

$$a_1 + \left(2a_2 - a_1^2\right) z + \cdots =$$

$$a_1 = \bar{\omega} + \frac{l}{m}\omega - \zeta\left(\frac{\omega}{m}\right)$$

$$a_2 = \frac{1}{2}\left[\left(\bar{\omega} + \frac{l}{m}\omega - \zeta\left(\frac{\omega}{m}\right)\right)^2 - \wp\left(\frac{\omega}{m}\right)\right]$$

$$\mathrm{Tr}\left(e^{-t\Delta}\mathcal{D}^{-1}\delta\mathcal{D}\right) = \frac{1}{2i\,\mathrm{Im}\,\tau}\underbrace{\mathrm{Tr}\left(e^{-t\Delta}\mathcal{D}^{-1}\tilde{\mathcal{D}}\right)}_{\frac{a_2\,\mathrm{Im}\,\tau}{\pi}} \sim \frac{a_2}{2\pi i}$$

$$\mathrm{Tr}\left(e^{-t\Delta}\Delta^{-1}\delta\Delta\right) \sim -\frac{1}{4\pi i t} + \frac{a_2}{2\pi i}$$

September 7, 1982

For paper:

$$-\delta\,\mathrm{Tr}(\Delta^{-s}) = s\,\mathrm{Tr}\left(\Delta^{-s-1}\delta\Delta\right)$$

$$= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(e^{-t\Delta}\Delta^{-1}\delta\Delta\right)t^s\frac{dt}{t}$$

Assume $\exists$ asymptotic expansion

$$\mathrm{Tr}\left(e^{-t\Delta}\Delta^{-1}\delta\Delta\right) \sim \frac{a_{-1}}{t} + a_0 + a_1 t + \cdots .$$

Then $-\delta\zeta'(0) = a_0$. Hence if the a_i are local integral quantities, so will be $-\delta\zeta'(0)$.

The problem then becomes one of regularizing $\mathrm{Tr}(\Delta^{-1}\delta\Delta)$. First one has to understand $\delta\Delta$, where $\Delta = D^*D$ and $D:$ [redacted] $W \longrightarrow V$, $W = f_*(C^\infty(E))$ $V = f_*(C^\infty(E\otimes T^{0,1}_{X/Y}))$. Here δ is a vector field on Y and we lift it to differential operators on E, $E\otimes T^{0,1}_{X/Y}$.

$$\delta : \underline{E} \longrightarrow \underline{E\otimes T_Y^*}$$

Recall the exact sequences:

$$(*) \qquad 0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0$$

$$0 \longrightarrow f^*(T_Y^*) \longrightarrow T_X^* \longrightarrow T_{X/Y}^* \longrightarrow 0$$

We want $\delta : \underset{\underset{f_*(E)}{\shortparallel}}{\underline{W}} \longrightarrow \underline{W\otimes T_Y^*} = f_*(\underline{E\otimes f^*T_Y^*})$

which will be given by $\delta : \underline{E} \longrightarrow \underline{E\otimes f^*T_Y}$. Drop —.

A connection on E gives

$$\nabla : E \longrightarrow E\otimes T_X^*$$

and hence [redacted] δ provided [redacted] we split the sequences $(*)$.

It would be nice to list the choices. First of all we have that E is holomorphic on each fibre and that it has a metric. This gives us a $\bar\partial$-operator

$$D : E \longrightarrow E\otimes T^{0,1}_{X/Y}$$

and with the metric a connection along the fibers

$$\nabla : E \longrightarrow E \otimes T^*_{X/Y},$$

as well as Laplaceans, ζ function as a function on Y. Let's keep from using the fact that Y is a complex manifold until required to. We have the canonical line bundle $\mathcal{L}$ on Y with its torsion metric and the canonical section s whose norm^2 is $e^{-\zeta'(0)}$. You want to compute $c_1(\mathcal{L})$ using the GRR formula hence you need a connection on E and on $T_{X/Y}$ in order to get characteristic classes to integrate. So the question arises as to whether connections on $E, T_{X/Y}$ compatible with their metrics and fibrewise holomorphic structure induce a connection on $\mathcal{L}$ compatible with the torsion metric.

The good question: The line $\mathcal{L}$ has a canonical connection, so what is Ds, where s is the canonical section. Because s is a holomorphic section of $\mathcal{L}$ we have $Ds = s\theta$, where θ is a $(1,0)$-form. Also

$$d\left(+e^{-\zeta'(0)}\right) = d|s|^2 = |s|^2(\theta + \bar\theta)$$

hence

$$d(-\zeta'(0)) = \theta + \bar\theta.$$

Hence θ is just the $(1,0)$ part of $d(-\zeta'(0))$. But

$$\delta(-\zeta'(0)) = \overbrace{\mathrm{Tr}}^{\text{constant coeff. of}}\left(e^{-t\Delta}\Delta^{-1}\delta\Delta\right)$$

is a suitable regularization of $\mathrm{Tr}(\Delta^{-1}\delta\Delta)$. Now I believe that δ will be a differential operator both on E and on $E \otimes T^{0,1}_{X/Y}$, and that

$$\delta\Delta = [\delta, D^*D] = [\delta, D^*]D + D^*[\delta, D]$$

$$\mathrm{Tr}^{(r)}(\Delta^{-1}\delta\Delta) = \mathrm{Tr}^{(r)}\left([\delta, D^*](D^*)^{-1}\right) + \mathrm{Tr}^{(r)}\left(D^{-1}[\delta, D]\right)$$

What can we say of a general nature about $\delta D = [\delta, D]$? We can discuss independence of the choice of δ lifting a given vector field on Y. Because of

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0$$

the arbitrariness in δ is a vector field tangent to the fibres, call it η.

$$Tr(e^{-t\Delta}\Delta^{-1}[\eta,\Delta]) = \underbrace{Tr(e^{-t\Delta}\Delta^{-1}\eta\Delta)}_{\text{by Atiyah-Singer-Patodi } Tr(e^{-t\Delta}\eta\Delta\Delta^{-1})} - Tr(e^{-t\Delta}\eta) = 0.$$

However $\quad Tr(e^{-t\Delta}(D^{-1}\eta D - \eta)) \qquad \Delta = D^*D$

$$= Tr(D^{-1}e^{-tDD^*}\eta D) - Tr(e^{-tD^*D}\eta)$$

$$= Tr(e^{-tDD^*}\eta) - Tr(e^{-tD^*D}\eta)$$

and it is not clear that this is zero. In fact when η is an infinitesimal gauge transformation, then $[\eta, D]$ is zero-th order, and

$$Tr(e^{-t\Delta}D^{-1}[\eta,D]) \xrightarrow[t\to 0]{\text{as}} \int tr(J[\eta,D])$$

and this is non-zero in general. So we see that it is necessary to be very clear about $Tr^{(r)}(D^{-1}[\delta,D])$.

I want to understand $[\delta, D]$. The operators are basically pieces of a connection, so the bracket should be some kind of curvature. The holom. structure of E together with the metric give us a connection

$$\nabla : E \longrightarrow E \otimes T_X^*.$$

We also choose a splitting of

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow f^*T_Y \longrightarrow 0$$

or

$$\begin{tikzcd} 0 \arrow[r] & f^*T_Y^* \arrow[r] & T_X^* \arrow[l, bend left, dashed, "\xi"] \arrow[r] & T_{X/Y}^* \arrow[r] & 0 \end{tikzcd}$$

Then

$$E \xrightarrow{\nabla} E \otimes T_X^* \xrightarrow{\xi} E \otimes f^* T_Y$$

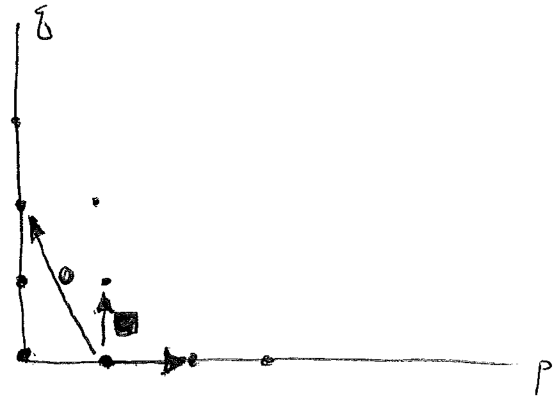
is δ. Maybe it would be useful to set things up without E, but as a differentiable fibre bundle. Foliations? In general how does one see that the de Rham cohomology of the fibres admits a Gauss-Manin connection?

Let write

$$T_X^* = f^* T_Y^* \oplus T_{X/Y}^*$$

$$\Lambda^\cdot T_X^* = \Lambda^\cdot f^* T_Y^* \otimes \Lambda^\cdot T_{X/Y}^* .$$

I want to understand how d looks relative to this decomposition.



$\Lambda^\cdot T_{X/Y}^*$ is a ~~quotient~~ of $\Lambda^\cdot T_X^*$ by the ideal generated by $f^* T_Y^*$. The real question is what does

$$d : T_{X/Y}^* \longrightarrow \Lambda^2 T_X^*$$

look like? Conceivably there is an interesting component

$$(*) \qquad T_{X/Y}^* \longrightarrow f^*(\Lambda^2 T_Y^*)$$

which destroys d having a decomposition $d = d' + d''$. The dual of $(*)$ is a map

$$f^*(\Lambda^2 T_Y) \longrightarrow T_{X/Y}$$

which is obviously the curvature of the connection one has defined by the splitting $f^*T_Y \hookrightarrow T_X$

September 8, 1982:

Go back to the variational formula

$$\mathrm{Tr}^{(r)}\left(\Delta^{-1}[\delta,\Delta]\right) = \mathrm{Tr}^{(r)}\left([\delta,D^*](D^*)^{-1}\right) + \mathrm{Tr}^{(r)}\left(D^{-1}[\delta,D]\right).$$

There is something annoying about the fact that the two terms on the right depend on the choice of δ. I feel that δ can be adjusted locally on X to be very nice. So the question is what is the nicest situation.

The nicest case occurs when $f: X \to Y$ is a product $Y \times M \longrightarrow Y$, hence when only the complex structure on E is changing. Then we define δ using the product structure. We know $[\delta, D]$ is zero-th order.

Let's try to remove all complexities arising from the bundle E by taking it to be the trivial bundle, with some kind of boundary conditions to make D invertible, the boundary conditions being away from where the complex structure is changed. Basically I am thinking of a fixed C^∞-bundle with connection over a Reimann surface of index 0, and I am going to vary the cx. structure on the surface near a point. In the nbd. of where the variations take place I can suppose I have the trivial line bundle with usual metric. The rest of the surface is essentially only providing me with boundary conditions for the Laplacean so as to make it invertible.

So this is a very basic case to consider. Think of working over $\mathbb{C}$ with the trivial line bundle and changing only the holomorphic structure in a small nbd. of 0. Now holom. structure + volume is the same as metric. The arbitrariness of this picture is provided by the diffeos. of the surface.

September 9, 1982

Suppose we have a fixed Riemann surface M and we consider a family [redacted] of holom. v.b. over M. This means over $X = Y \times M$ we have a holom. v.b. E. [redacted] Let us put on M a metric compatible with the holom. structure and an arbitrary metric on E. I want to go over carefully how I compute $-\delta \zeta'(0)$ in this situation. I suppose that E has trivial cohomology.

In the situation I have studied I consider the [redacted] C^∞ bundle with metric of E to be constant as [redacted] we vary over Y. Hence things are described by a changing $\bar\partial$-operator on a fixed C^∞ v.b. over M.

[redacted]

To each complex manifold Y I associate the groupoid of holomorphic vector bundles over $Y \times M$. This gives me a fibred category over the category of complex manifolds Y.

If $M = pt$ then I am getting the category of holom. vector bundles over Y, so I can regard the above as a generalization: [redacted] Over Y I get a new kind of geometric object, namely, a vector bundle over $Y \times M$.

Next [redacted] I can ask about characteristic classes for such a holom. M-bundle over Y. These will be ways to associate to a v.b. E over $Y \times M$ a cohomology class in Y, or maybe, a K-class in Y. For example $(pr_1)_!(E) \in K(Y)$ or what occurs in the RR thm.

[redacted] $(pr_1)_*\big(ch(E) \cdot$ [redacted] $pr_2^*(\text{Todd } M)\big)$.

When $M = pt$, then one finds it convenient to [redacted] consider [redacted] a holom. v.b. together with a hermitian metric. Then one gets a connection and specific differential forms on the base Y representing cohomology characteristic classes.

In the holom. M-bundle situation the analogue of a metric on the bundle appears to be provided by a (Kähler) metric on M and a hermitian metric on E. At least this is enough to make [redacted] $c_1(pr_{1!}E)$ into a definite differential form.

[redacted] Now these M-bundles over Y are local, or better satisfy descent for open coverings. The same with M-bundles equipped with metric; (let's assume the metric on M is fixed). So it makes sense to look for [redacted] local [redacted] models, that is, a covering of the final object in the fibred category. Precisely what I would like to have is a collection of bundles $E_\alpha / Y_\alpha \times M$ such that given any $E/Y\times M$, then locally on Y it is induced from a map $Y \longrightarrow Y_\alpha$ for some α.

[redacted] First forget metrics. If $M = pt$, then any holom. vector bdl. is trivial, hence locally induced from a map $Y \longrightarrow pt$. If M is higher-diml., then locally on Y the bundle E over $Y \times M$ as a C^∞ bundle is a pull-back of a bundle E_0 on M. Then we get a model [redacted] where the base consists of all holomorphic structures on a fixed C^∞ vector bundle E_0 over M.

September 10, 1982

Idea: The trace anomaly should follow from the facts that the curvature of the line bundle is the symplectic 2-form and that the moment [redacted] map is given by the curvature.

One first must get used to thinking in terms of a line bundle with connection instead of simply a closed 2-form.

Let L be a line bundle over M equipped with a connection $\nabla : L \to L \otimes T^*$ whose curvature form is symplectic (i.e. non-degenerate). Given a vector field X on M, one can ask what it means for X to be symplectic in terms of L.

Certainly if X [redacted] is lifted to an action on L and it preserves the connection, then it will be symplectic. However it shouldn't have to preserve the connection in order to be symplectic.

What does a lifting to L look like? X has to be lifted to a vector field on L invariant under the circle action. This is just a connection in the X-direction and hence can be identified with a differential operator of the form $\nabla_X + b$ where b is a function. The condition that this lifting preserve the connection is

$$[\nabla_X + b, \nabla_Y] = \nabla_{[X,Y]} \qquad \text{for all } Y$$

or

$$\underset{\displaystyle i(Y)db}{\underset{\shortparallel}{[\nabla_Y, b]}} = \underbrace{[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}}_{i(Y)i(X)\Omega}$$

or

$$db = i(X)\Omega .$$

Hence we see that symplectic vector fields on X are exactly the ones that lift (modulo H^1, H^0) to L preserving the connection.

So next I want to look at the trace anomaly. Over the space $\mathcal{A}$ of connections ~~we~~ have the line bundle $\mathcal{L}$ with the canonical holomorphic section s and the metric $|s|^2 = e^{-\zeta'(0)}$. Also we have the gauge group acting on $(\mathcal{a}, \mathcal{L})$ preserving the metric and holomorphic structure, hence preserving the connection. So now take an infinitesimal gauge transformation X; this is a skew-hermitian endom. of E.

One thing I forgot to do in the general situation is to point out that if $\nabla_X + f_X$ is the lifted action of X to L, then because of

$$df_X = i(X)\Omega$$

it follows that [redacted] we get control of the moment map. Better: Suppose we have a Lie algebra $\mathfrak{g}$ acting on [redacted] (M, L) preserving the connection. Then we have seen that $X \in \mathfrak{g}$ acts on L via an operator $\nabla_X + f_X$ where $df_X = i(X)\Omega$. Hence $X \mapsto f_X$, $\mathfrak{g} \to C^\infty(M)$ is "the" moment map for the symplectic action of $\mathfrak{g}$ on M.

Now apply this to the Riemann surface situation, where we have that the canonical section s is invariant under the gauge group action:

$$(\nabla_X + f_X)s = 0.$$

On the other hand we have

$$\nabla_X s = s\;\theta(X)$$

(actually $\theta(\bar{X})$ where $\bar{X} = X$ on $\mathcal{a}$.)

where θ is the connection form. Recall that θ is of type $(1,0)$ because s is a holomorphic section, hence $\theta(X)$ [redacted] is $\mathbb{C}$-linear in X. In fact I am pretty [redacted] sure that

$$\theta(X)_D = \int tr(J[X,D]) = \int tr([D,J]X)$$

Hence $\int tr([D,J]X) = \theta(X)_D = -f_X(D)$, and so to finish a proof of the anomaly formula, we need only understand why the curvature of (D) gives the moment map.

So we work at a point ∇ of $\mathcal{a}$, with an infinitesimal gauge transf. $X \in \mathrm{Hom}(E,E)$, and tangent vectors to $\mathcal{a}$ at ∇ given by $B \in \mathrm{Hom}(E,E) \otimes T^*$. The function f_X is supposed to satisfy at [redacted] ∇

$$i(B)\, df_X = i(B)\, i(X)\, \Omega = \Omega(X, B)$$

which amounts to

$$(*) \qquad \underset{\text{mod } O(B^2)}{f_X(\nabla + B) - f_X(\nabla)} = \Omega([X,\nabla], B) = \int \mathrm{tr}\, [X,\nabla] B = \int \mathrm{tr}\big(X [\nabla, B]\big)$$

where here $[\nabla, B] = \nabla B + B \nabla$ because both have degree 1. Hence it seems that

[redacted] $$f_X(\nabla) = \int \mathrm{tr}(X \cdot \nabla^2)$$

so the moment map is given by the curvature.

[redacted] Next we know from Atiyah at the Arbeitstagung that there is a [redacted] small problem with the above at some point, because the anomaly involves also $\frac{1}{2}$ the curvature [redacted] of the tangent bundle.

The difficulty is that one can equally have

$$f_X(\nabla) = \int \mathrm{tr}\big(X \cdot (\nabla^2 + C)\big)$$

with C a [redacted] matrix 2-form, if one requires $(*)$. [redacted]
We have

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \mathbb{C} \arrow[d] \\
 & \text{functions on } \mathcal{a} \arrow[d] \\
\mathfrak{g} \arrow[r] & \text{closed 1-forms on } \mathcal{a} \arrow[d] \\
 & H^1(\mathcal{a}) = 0
\end{tikzcd}$$

and we know that $\mathfrak{g}$ does act on the line bundle, hence the extension of $\mathfrak{g}$ by $\mathbb{C}$ is trivial, but to get an actual moment map one must split the extension. This is unique up to a map $\mathfrak{g} \to \mathbb{C}$ which is given by the term $\int tr(X \cdot C)$.

More precisely the actual $\mathcal{L}$ we have will give us a C [redacted], possibly 0, and we should ask [redacted] about adding a term

$$\varphi(X) = \int tr(X\tilde{C})$$

which is a homomorphism $\mathfrak{g} \to \mathbb{C}$, i.e. vanishes on $[\mathfrak{g},\mathfrak{g}]$. It's clear $\tilde{C}$ [redacted] has to be a scalar 2-form, i.e. in [redacted] $\Lambda^2 T^* \subset End(E) \otimes \Lambda^2 T^*$

Hence the good statement is that the possible moment maps are of the form

$$f_X(\nabla) = \int tr(X \cdot \nabla^2) + \int tr(X) C$$ [redacted]

where C is a scalar 2-form. (We know $\int tr(X \cdot \nabla^2)$ works from Bott + Atiyah) ???

Review: Suppose that we have a line bundle L with connection having non-degenerate curvature over M and a Lie algebra $\mathfrak{g}$ acting on this [redacted]. [redacted] given $X \in \mathfrak{g}$ I let $\bar{X}$ be the associated vector field on M, and the action of X on L is given by an operator

$$\nabla_{\bar{X}} + f_X$$

Then what are the conditions satisfied by [redacted] $f : \mathfrak{g} \to$ fns. on M.

First we want the action to preserve the connection:

$$[\nabla_{\bar{X}} + f_X, \nabla_Y] = \nabla_{[\bar{X},Y]} \qquad \text{for all v.f. } Y \text{ on } M.$$

i.e. $\quad Y f_X = [\nabla_{\bar{X}}, \nabla_Y] - \nabla_{[\bar{X},Y]} = i(Y) i(\bar{X}) \Omega$

i.e. $\quad df_X = i(\bar{X})\Omega$

This says that f is a moment map for the $\mathfrak{g}$-action on M. (Not quite, see below.)

Secondly we want $X \longmapsto \nabla_{\bar{X}} + f_X$ to be compatible with bracket:

$$[\nabla_{\bar{X}} + f_X, \nabla_{\bar{Y}} + f_Y] = \nabla_{[\bar{X},\bar{Y}]} + f_{[X,Y]}$$

In view of the ~~first~~ condition this says

$$[\nabla_{\bar{X}} + f_X, f_Y] = f_{[X,Y]}$$

$$\text{i.e.} \qquad \underbrace{\bar{X} f_Y}_{i(\bar{X})df_Y} = f_{[X,Y]}$$

$$i(\bar{X})df_Y = i(\bar{X})i(\bar{Y})\Omega.$$

The first condition $df_X = i(\bar{X})\Omega$ says that $\bar{X}$ is the Hamilton vector field associated to the function f_X, or that f_X is a Hamiltonian for the symplectic vector field $\bar{X}$. The second condition says that

$$f_{[X,Y]} = \{f_X, f_Y\}$$

i.e. we have a Lie homomorphism for the Poisson bracket. Thus $\{f_X\}$ is a lifting

$$\begin{tikzcd} & 0 \arrow[d] \\ & \mathbb{C} \arrow[d] \\ & C^\infty(M) \arrow[d] \\ \mathfrak{g} \arrow[ur, dashed] \arrow[r] & \text{closed 1-forms on } M \end{tikzcd}$$

Emphasize: A moment map for an action of $\mathfrak{g}$ on a symplectic manifold M is a lifting as above, i.e. a way a choosing a Hamiltonian f_X for $\bar{X}$ for each $X \in \mathfrak{g}$ such that $X \longrightarrow f_X$ is compatible with Poisson bracket.

Now let us **check carefully** the assertion that the curvature is a moment map for the action of the gauge group on the space of connections. There are two things to check. The first is that the function $f_X(\nabla) = \int tr(X\nabla^2)$ generates the vector field $\bar{X}: \nabla \longmapsto [X, \nabla]$.

This means
$$df_X = i(\bar{X})\Omega$$
$$i(B)\,df_X = i(B)\,i(\bar{X})\Omega$$
$$\frac{1}{\varepsilon}\Big[f_X(\nabla+\varepsilon B)-f_X(\nabla)\Big]_{\varepsilon=0} = \Omega(\bar{X},B)\Big|_\nabla$$
$$= \int tr([X,\nabla]B) = \int tr(X[\nabla,B])$$

(Do more carefully:
$$\int tr([X,\nabla]B) = \int tr(X\nabla-\nabla X)B$$
except that it is probably not correct ~~to~~ think of these as operators. Better is to work in $\mathrm{End}(E)\otimes\Lambda T^*$
$$[\nabla, XB] = [\nabla,X]B + X[\nabla,B]$$
and use $\quad tr[\nabla,XB] = d\,tr(XB) \quad$ because $tr: \mathrm{End}(E)\otimes\Lambda T^* \longrightarrow \Lambda T^*$ commutes with differentiation.)

In any case if $\quad f_X(\nabla)=\int tr(X\nabla^2)$, then
$$\frac{1}{\varepsilon}\Big[f_X(\nabla+\varepsilon B)-f_X(\nabla)\Big] = \int tr(X[\nabla,B])$$

Next we have to check that $X\mapsto f_X$ is compatible with Poisson bracket, i.e.
$$\bar{X}f_Y = f_{[X,Y]}$$
$$(\bar{X}f_Y)(\nabla) = \Omega(\bar{Y},\bar{X})(\nabla) = \int tr([Y,\nabla][X,\nabla])$$
$$= \int tr(Y\underbrace{[\nabla,[X,\nabla]]}_{\nabla(\cancel{X\nabla}-\nabla X)+(X\nabla-\cancel{\nabla X})\nabla})$$
$$= \int tr(-Y\nabla^2X + YX\nabla^2) = -\int tr([X,Y]\nabla^2)$$
so it agrees up to the usual sign ambiguity.

September 11, 1982

Summary of yesterday's work.

1) On the moment map. Let L be a line bundle with connection over M such that the curvature is symplectic, and let $\mathfrak{g}$ act on (M,L) preserving the connection. Then the operator on sections of L corresponding to $X \in \mathfrak{g}$ is $\nabla_{\bar{X}} + f_X$, where f is a moment map for the action of $\mathfrak{g}$ on the symplectic manifold M. This means

$$df_X = i(\bar{X})\Omega \qquad \text{hence } f_X \text{ is a Hamiltonian for } \bar{X}$$

$$\bar{X} f_Y = f_{[X,Y]} \qquad \text{hence } X \mapsto f_X \text{ is compatible with Lie and Poisson brackets.}$$

2) Let $\mathcal{G}$ be the gauge gp acting on the space $\mathcal{A}$ of connections on a hermitian v.b. over a Riemann surface. Then the cohomology determinant line bundle $\mathcal{L}$ with its analytic torsion metric is acted on by $\mathcal{G}$. So we have the situation of 1), and so there is a moment map which is obtained by comparing the action in $\mathcal{L}$ with the connection. Since by calculation the curvature is a moment map for the action of $\mathcal{G}$ on $\mathcal{A}$, I know that the moment map for $\mathcal{L}$ is of the form

$$f_X(\nabla) = \int tr(X\nabla^2) + \int tr(X)\cdot\mu$$

where μ is some 2 form on M. In effect two moment maps differ by a Lie homomorphism $\mathfrak{g} \to \mathbb{C}$ which accounts for the 2nd term above.

3) Deduce the trace anomaly: If we use the canonical section s of $\mathcal{L}$, which we know is holom, then the connection is given by $\nabla s = s\theta$ where $\theta = \partial \log |s|^2$.

$$|s|^2 = e^{-\zeta'(0)}$$

$$\delta \log |s|^2 = -\delta\zeta'(0) = Tr_{(r)}(D^{-1}\delta D) + c.c.$$

So we have for any $B \in End(E) \otimes T^{0,1}$ (these are the const. tangent vector fields on $\mathcal{A}$)

$$\nabla_B s\Big|_D = \underbrace{\int tr(J_D B)}_{i(B)\Theta|_D} s$$

The trace anomaly gives the value of this when $B|_D = [X, D]$ is a tangent vector to the gauge gp. orbit through D. One has

$$i([X,D])\Theta_D = \int tr\, J_D [X,D] = \int tr([D,J_D]X)$$

Notationally things are a mess, and very confusing. Look at the end result:

$$(*) \qquad [D, J_D] = -F_D - \mu$$

Here I am using D to describe points of $\mathcal{A}$; before I used ∇. Let's take the Kähler viewpoint: $\mathcal{A}$ is a Kähler manifold hence has a symplectic and a complex side. Since I am dealing with the moment map I want to emphasize the symplectic side. Let's use A to denote a point of $\mathcal{A}$ and d_A'' or D_A the corresponding $\bar\partial$-operator, d_A or ∇_A the corresponding connection.

What kind of quantity is $J = \{J_A\}$? J is a $(1,0)$-form on $\mathcal{A}$ because given a tangent vector $B = B' + B'' \in End(E) \otimes T^*$, $B' = -B''^*$, one gets a number $\int tr(JB) = \int tr(JB'')$ which is $\mathbb{C}$ linear in B'', which is the way the holomorphic ~~structure~~ on $\mathcal{A}$ is defined. J is the same thing as the connection form Θ for $\mathcal{L}, s$.

Ideally one might try to prove $(*)$ by differentiating both sides w.r.t. D.

September 11, 1982 (cont.)

Fix an oriented C^∞ surface (compact) M. Let $\mathcal{H}$ be the space of complex structures on M compatible with the orientation. Recall that a complex structure is a 1-diml subbundle $T^{1,0} \subset T^* = T^*_{\mathbb{R}} \otimes \mathbb{C}$ such that at each point of M the line is in the disk of $\mathbb{P}(T^*) - \mathbb{P}(T^*_{\mathbb{R}})$ which has the correct orientation. (Specifically if $\omega \in T^{1,0}$ then we want $i\,\omega \wedge \bar{\omega} \in \Lambda^2 T^*$ to be positive for the orientation.)

Given a complex structure $T^{1,0} \subset T^*$ we know locally that $T^{1,0}$ is spanned by dz for some function z. Another complex structure

$\widetilde{T^{1,0}}$ [redacted] $\subset T^*$ [redacted] $= T^{1,0} \oplus T^{0,1}$

is given by the graph of a map $T^{1,0} \longrightarrow T^{0,1}$. Hence $\widetilde{T^{1,0}}$ is spanned by $dz + h\,d\bar{z}$ where h is a smooth function which satisfies $|h| < 1$. To see this use

$$i(dz + h\,d\bar{z}) \wedge (d\bar{z} + \bar{h}\,dz) = (1 - |h|^2)\,i\,dz\,d\bar{z}.$$

Thus once we pick a basepoint in $\mathcal{H}$ we can identify $\mathcal{H}$ with the [redacted] sections of the unit disk bundle in the bundle

$$\mathrm{Hom}(T^{1,0}, T^{0,1}) = T \otimes T^{0,1} \qquad T = (T^{1,0})^*$$

whose sections locally are described by $h\,\partial_z \otimes d\bar{z}$. Also the tangent space to $\mathcal{H}$ at the complex structure $T^{1,0}$ can be identified with $\Gamma(T \otimes T^{0,1})$.

Suppose next we choose a volume element on M, locally $\rho \frac{i}{2} dz \wedge d\bar{z}$ with $\rho > 0$. Then we get a hermitian inner product on $\Gamma(T \otimes T^{0,1})$ by

$$\| h\,\partial_z \otimes d\bar{z} \|^2 = \int |h|^2 \rho \tfrac{i}{2} dz\,d\bar{z}.$$

Better description is that $\overline{(T \otimes T^{0,1})} \otimes (T \otimes T^{0,1})$ is canonically the trivial bundle, so that the condition $|h| < 1$ makes intrinsic sense. Thus $T \otimes T^{0,1}$ has an intrinsic hermitian product, so a volume element gives an inner product on sections.

So far we have the metric on the tangent space to a complex structure with the local coordinates dz. It should be possible to give the metric at any point of $\mathcal{H}$ using the coordinates from a basepoint. So fix the basepoint structure and let us consider a given complex structure spanned by $dz + h d\bar{z}$. Consider an infinitesimal change δh and let's compute the norm-squared of δh.

Let $$\begin{pmatrix} dw \\ d\bar{w} \end{pmatrix} = \begin{pmatrix} dz + h d\bar{z} \\ \bar{h} dz + d\bar{z} \end{pmatrix} = \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix}$$

or $$\begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} = \frac{1}{1-|h|^2} \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} dw \\ d\bar{w} \end{pmatrix}$$

The new complex structure is spanned by

$$dz + h d\bar{z} + \delta h d\bar{z} = dw + \delta h \left[\frac{d\bar{w} - \bar{h} dw}{1-|h|^2} \right]$$

$$= \left(1 - \frac{\bar{h}\delta h}{1-|h|^2}\right) dw + \left(\frac{\delta h}{1-|h|^2}\right) d\bar{w}$$

hence is spanned by

$$dw + \frac{\delta h}{1-|h|^2 - \bar{h}\delta h} d\bar{w}.$$

Hence its normed squared is at each point

$$\frac{|\delta h|^2}{(1-|h|^2)^2}$$

and this is to be integrated over M.

The Kähler form is

$$\int \rho \frac{i}{2} dz d\bar{z} \frac{1}{(1-|h|^2)^2} \delta h(z) \delta \bar{h}(z)$$

and this is clearly an infinite dim'l. version of $\sum_j f(w_j) dw_j d\bar{w}_j$ where the coords. are $h \mapsto h(z)$ for each $z \in M$. Hence it is clear the Kähler form is closed.

Let $\mathcal{D}$ be the volume-preserving diffeos. of M and $\mathcal{D}_c$ the orientation-preserving ones. Clearly $\mathcal{D}_c$ acts on $\mathcal{H}$, and $\mathcal{D}$ preserves the Kähler metric. Now we want to compute the moment map which will go from complex structures $\mathcal{H}$, or equivalently Riemann metrics with the given volume, to volume-preserving vector fields, or really the dual of this Lie algebra. A volume-preserving vector field is symplectic, so roughly volume-preserving vector fields are functions, and these are dual to 2 forms. Hence we want a way to assign to a Riemann metric a 2-form, and this should be the curvature.

The first thing we need is how a vector field X acts on $\mathcal{H}$. Hence given a complex structure we want to see a ~~vector field~~ tangent vector produced by X, i.e. a map $T^{1,0} \longrightarrow T^{0,1}$. So take dz act on by X to get $\theta(X)\,dz = d(Xz)$ and project into $T^{0,1}$ to get $\bar\partial(Xz)$. Thus the map is $dz \longmapsto \bar\partial(i(X)dz)$, so if $X = a\partial_z + \bar a \partial_{\bar z}$, then we get the section

$$\bar X : \quad \partial_{\bar z} a \cdot \frac{d}{dz} \otimes d\bar z$$

Now given another tangent vector to $\mathcal{H}$ at our complex structure, say $Y: h\frac{d}{dz}\otimes d\bar z$, we need the symplectic product, which is the imaginary part of the scalar product

$$\Omega(\bar X, Y) = c \cdot \int \left((\partial_{\bar z} a)\bar h - (\partial_z \bar a) h \right) \rho \tfrac{i}{2} dz d\bar z$$

We want to apply this when X is symplectic:

$$\underbrace{i(X)\,\rho \tfrac{i}{2} dz d\bar z}_{\rho \frac{i}{2}(a\,d\bar z - \bar a\, dz)} = d\varphi$$

Put $\rho \frac{i}{2} = \sigma$. Then $\quad -\bar a = \frac{1}{\sigma}\partial_z \varphi \qquad a = \frac{1}{\sigma}\partial_{\bar z}\varphi$

September 12, 1982

Setup: M is a compact oriented surface with a volume Ω [redacted], hence a 2-diml symplectic manifold. $\mathcal{H}$ = complex structures on M, or equivalently the Riemannian metrics with the volume Ω. We know that $\mathcal{H}$ is a Kähler manifold and that the group of [redacted] symplectic diffeomorphisms of M acts on $\mathcal{H}$ preserving its Kähler structure. [redacted] Let G_1 = symplectic diffeos. of M and let $\mathfrak{g}_1$ = symplectic vector fields on M.

Then the Lie algebra $\mathfrak{g}_1$ acts on two symplectic manifolds M and $\mathcal{H}$ and we can ask about the moment map in both cases. First for M, one has

$$0 \longrightarrow \mathbb{C} \longrightarrow C^{\infty}(M) \longrightarrow \mathfrak{g}_1 \longrightarrow H^1_{[redacted]}(M) \longrightarrow 0$$

hence in order to get a moment map, one must first cut $\mathfrak{g}_1$ down to those vector fields given by Hamiltonians, call this $(\mathfrak{g}_1)_h$. It seems likely that the extension of $(\mathfrak{g}_1)_h$ [redacted] given by $C^\infty(M)$ is non-trivial, hence to get a moment map one must use this central extension. Thus [redacted] in order to obtain a moment map for $\mathfrak{g}_1$ acting on M, one replaces $\mathfrak{g}_1$ by $C^\infty(M)$, in which case the map f: $\mathfrak{g} \to C^\infty(M)$ is the identity, and the map $M \to \mathfrak{g}^*$ assigns to any point the Dirac measure of that point. This discussion holds for any symplectic manifolds.

What about the line bundle? Assume that the symplectic form Ω determines an integral class. Then [redacted] there exists a line bundle over M with connection having the curvature [redacted] Ω, and two such line-bundles-with-connection differ by a flat line bundle. Flat line bundles are classified by $H^1(M, S^1) = \mathrm{Hom}(H_1(M,\mathbb{Z}), S^1)$, and infinitesimal changes by elts. in $\mathrm{Hom}(H_1(M,\mathbb{Z}), \mathbb{R}) = H^1(M,\mathbb{R})$. So what emerges is the idea that $\mathfrak{g}_1$ is permuting around the different line-bundles-with-connection belonging to the form Ω, and it is only by [redacted] restricting to $(\mathfrak{g}_1)_h$ and then lifting back to $C^\infty(M)$ that we get an action [redacted] on a line bundle with connection, and hence a moment map.

Question: Is the above related to the idea of different vacua in a case of symmetry breaking?

Next go back to the surface case and the action of $\mathfrak{g}_1$ (= symplectic vector fields on M) on the space $\mathcal{H}$ of complex structures on M (also = metrics on M with the given volume.) Now $\mathcal{H}$ is contractible, hence there is a unique line-bundle-with-connection $\mathcal{L}$ over $\mathcal{H}$ with curvature Ω, and it is unique up to multiplication by an element of S^1. One way to obtain $\mathcal{L}$ is to take the trivial bundle with connection form η where $d\eta = \Omega$. Then sections are functions and

$$\nabla f = (d + \eta) f .$$

~~If X is a vector field with $0 = \theta(X)\Omega = \theta(X)d\eta = d\,i(X)\,d\eta$, then $d\,i(X)$~~

If X preserves Ω, then we have

$$0 = \theta(X)\Omega = d\,i(X)\,\Omega$$

so $i(X)\Omega = d\varphi_X$ for some fn. φ_X. It [redacted] might be possible to produce φ_X from η if η is given with $d\eta = \Omega$. ?

Start again: We have the space $\mathcal{H}$ of complex structures on M, that is, subline bundles $T^{1,0}$ of $T^* = T^*_{\mathbb{R}} \otimes \mathbb{C}$ such that if $\omega \in T^{1,0}$ then $i\,\omega \wedge \bar{\omega} \in \Lambda^2 T^*$ is positive for the orientation. The tangent space to $T^{1,0}$ is, as the tangent bundle to a point in projective space, the homs. from the sub to the quotient bundle:

$$(*) \qquad \Gamma\{\mathrm{Hom}(T^{1,0}, T^{0,1})\}$$

Given a vector field X on M, the corresponding tangent vector to $T^{1,0} \in \mathcal{H}$ is the map whose effect on sections is

$$\Gamma(T^{1,0}) \subset \Gamma(T) \xrightarrow{\theta(X)} \Gamma(T) \xrightarrow{\pi} \Gamma(T^{0,1}).$$

If f is a function, then

$$\begin{aligned} \pi\,\theta(X)(f\,\xi) &= \pi\big[(Xf)\,\xi + f\,\theta(X)\xi\big] \\ &= f\,\pi\,\theta(X)\xi \end{aligned}$$

because $(Xf)\xi \in \Gamma(T^{1,0})$ and π kills it. Thus the map $\xi \longmapsto \pi\,\theta(X)\xi$ is linear over functions and so determines a vector bundle homom. from $T^{1,0}$ to $T^{0,1}$, i.e. an elt. of $(*)$.

Because the definition of the symplectic form on $\mathcal{H}$ uses the volume on M, the above action of X on $\mathcal{H}$ is symplectic when X preserves volume. Because $\mathcal{H}$ is contractible we can lift $\bar{X}$ on $\mathcal{H}$ to a function φ_X:

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \mathbb{C} \arrow[d] \\
 & C^\infty(\mathcal{H}) \arrow[d] \\
X \in \mathfrak{g}_1 \arrow[ur, dotted, "\varphi"] \arrow[r] & \text{symplectic vector fields on } \mathcal{H} \arrow[d] \\
 & 0
\end{tikzcd}$$

September 13, 1982

Some Riemannian geometry. Given a Riem. metric, [redacted] $ds^2 = g_{ij}\,dx^i dx^j$, i.e. a positive-definite quadratic form on $T_{\mathbb{R}}$ such that $(\partial_i|\partial_j) = g_{ij}$. Then we get an induced [redacted] form on $\Lambda^{max} T_{\mathbb{R}}$ given by

$$(\partial_1 \cdots \partial_n | \partial_1 \cdots \partial_n) = \det(g_{ij})$$

and so $\pm \det(g_{ij})\, dx^1 \dots dx^n$ is a unit vector in $\Lambda^{max} T^*_{\mathbb{R}}$. An orientation selects the sign and so gives a volume.

Next consider [redacted] a change in the metric $\delta g_{ij}\, dx^i dx^j$. This is a quadratic form on $T_{\mathbb{R}}$ and so can be represented by a symmetric operator b on $T_{\mathbb{R}}$ for the metric. The formula defining b is

$$\delta g_{ij}\, a^i a^j = \left(\sum a^i \partial_i \,\middle|\, b^j{}_k a^k \partial_j\right).$$

(Better: If $X = a^i \partial_i$, then $\delta g(X) = (X|bX)$ where $bX = |\partial_j\rangle\langle dx^j|b|\partial_k\rangle\langle dx^k|X = \partial_j\, b^j{}_k a^k$) Thus

$$\delta g_{ik} = g_{ij}\, b^j{}_k$$

so $\quad b^j{}_k = g^{ji}\,\delta g_{ik}$, or simply using matrices

$$\boxed{b = g^{-1}\delta g}$$

Notice that b is symmetric for the **quadratic** form $X^t g Y$, because

$$X^t g b Y = X^t \delta g Y = (\delta g X)^t Y = (g^{-1}\delta g X)^t \underbrace{(g^t)}_{g} Y$$

Next consider the case of a surface M_g and let's understand the symplectic structure on the space of metrics with a given volume. We have just seen that the tangent vectors at g can be identified with operators $b : T \to T$ which are symmetric for g and have trace 0. (If I think of g as a map $g : T \longrightarrow T^*$, then b is symmetric for $g \Longleftrightarrow gb : T \longrightarrow T^*$ is symmetric.)

If b_1, b_2 are symmetric for g, then $[b_1, b_2]$ is skew-symmetric, and the space of skew-symmetric operator is 1-dimensional spanned by $J_g =$ ~~unique~~ operator given rotation by $90°$ wrt the orientation. One has

$$J_g = g^{-1/2} J g^{1/2}$$

where $g^{1/2}$ is the positive square root of g, considered as an operator via some fixed basepoint metric.

Thus the symplectic form on the tangent space at g is a multiple of

$$b_1, b_2 \longmapsto \int \operatorname{tr}\left([b_1, b_2] J_g\right)$$

We now want to solve

$$d\varphi_X = i(\bar{X})\Omega$$

where X is a volume-preserving vector field on M, and $\bar{X}$ denotes the vector field $g \longmapsto g + \varepsilon X g$, which corresponds to $b = g^{-1}Xg$. Thus φ_X is to be a function of g such that

$$\varphi_X(g + \varepsilon g b) - \varphi_X(g) = \varepsilon \int \operatorname{tr}\left([b, g^{-1}Xg] J_g\right)$$

for all symmetric $gb : T \to T^*$. (We are ignoring a mult. constant to be determined later)

So we want to find φ_X satisfying

$$\varphi_X(g+\alpha)-\varphi_X(g) = \int tr\left([g^{-1}\alpha, g^{-1}Xg]\,J_g\right) + O(\alpha^2)$$

for all symmetric α. This seems to be a hard equation to integrate, unless the answer is known in advance.

Let's try to do this by brute force. I will suppose that X is supported in a coordinate patch with volume dx^1dx^2. Then the metric g is simply a positive-definite matrix of det 1.

Let's put $ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2$ so that $g = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ with $AC-B^2=1$. ~~Claim~~

$$\cancel{J_g = \begin{pmatrix} -B & A \\ -C & B \end{pmatrix}.}$$

~~In effect~~ $$\cancel{gJ_g = \begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} -B & A \\ -C & B \end{pmatrix} = \begin{pmatrix} & A^2+B^2 \\ -B^2-C^2 & \end{pmatrix}}$$

Set $$g^{1/2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad g^{-1/2} = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}, \quad ac - b^2 = 1$$

Then $$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ab+bc \\ ba+cb & b^2+c^2 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

and $$J_g = g^{-1/2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g^{1/2} = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

$$= \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}\begin{pmatrix} b & c \\ -a & -b \end{pmatrix} = \begin{pmatrix} cb+ba & c^2+b^2 \\ -b^2-a^2 & -bc-ab \end{pmatrix} = \begin{pmatrix} B & C \\ -A & -B \end{pmatrix}$$

Check: $$gJ_g = \begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} B & C \\ -A & -B \end{pmatrix} = \begin{pmatrix} 0 & AC-B^2 \\ B^2-AC & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew-symmetric and SURPRISE!

$$J_g^2 = \begin{pmatrix} B & C \\ -A & -B \end{pmatrix}\begin{pmatrix} B & C \\ -A & -B \end{pmatrix} = \begin{pmatrix} B^2-CA & 0 \\ 0 & -AC+B^2 \end{pmatrix} = -I$$

So we have discovered by calculation that

$$J_g = g^{-1}J = Jg$$

which is the sort of thing one knows from hermitian symmetric space theory. So we can simplify

$$\mathrm{tr}\left([g^{-1}\alpha, g^{-1}X_g]\,J_g\right) = \mathrm{tr}\left(\cancel{g}^{-1}\alpha g^{-1}(X_g)J\cancel{g} - \cancel{g}^{-1}(X_g)g^{-1}\alpha J\cancel{g}\right)$$

$$= \mathrm{tr}\left(\alpha g^{-1}X_g J\right) - \mathrm{tr}\left(J(X_g)g^{-1}\alpha\right)$$

$$= 2\,\mathrm{tr}\left(\alpha g^{-1}X_g J\right)$$

because $\mathrm{tr}(U) + \mathrm{tr}(U^*) = 2\,\mathrm{tr}(U)$ if U is real. Thus our calculations have reduced the symplectic form on the tangent space at g to a multiple of

$$\alpha, \beta \longmapsto \int \mathrm{tr}\left(\alpha\, g^{-1}\beta\, J\right).$$

Now the integration should be manageable because $\int \mathrm{tr}(\alpha\, g^{-1}X_g J)$ is a quadratic function of g.

Suppose we denote the tangent vector α by δg and the tangent vector β by dg. Then

$$\mathrm{tr}\left(\delta g\, g^{-1} dg\, J\right) = \mathrm{tr}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta A & \delta B\\ \delta B & \delta C\end{pmatrix}\begin{pmatrix}C & -B\\ -B & A\end{pmatrix}\begin{pmatrix}dA & dB\\ dB & dC\end{pmatrix}$$

$$= \mathrm{tr}\begin{pmatrix}\delta B & \delta C\\ -\delta A & -\delta B\end{pmatrix}\begin{pmatrix}C\,dA - B\,dB & C\,dB - B\,dC\\ -B\,dA + A\,dB & -B\,dB + A\,dC\end{pmatrix}$$

$$= \delta B(C\,dA - \cancel{B\,dB}) + \delta C(-B\,dA + A\,dB) + (-\delta A)(C\,dB - B\,dC) + (-\delta B)(-\cancel{B\,dB} + A\,dC)$$

$$= -C\,\delta A\,dB \quad - A\,\delta B\,dC \quad - B\,\delta C\,dA$$
$$+ B\,\delta A\,dC \quad + C\,\delta B\,dA \quad + A\,\delta C\,dB$$

Now we have to remember that A, B, C are not independent but are subject to $AC - B^2 = 1$.

Hence

$$(\delta A)C + A\,\delta C - 2B\,\delta B = 0$$

Treat A, B as independent

$$\delta C = 2A^{-1}B\,\delta B - A^{-1}C\,\delta A$$

So the coefficient of δA in $\mathrm{tr}(\delta g\, g^{-1} dg\, J)$ is

[redacted]

[redacted]

$$-C\,dB + B\,dC - B\,dA(-A^{-1}C) + A\,dB(-A^{-1}C)$$

$$= -2C\,dB + B\,\underbrace{dC}_{2A^{-1}B\,dB - A^{-1}C\,dA} + A^{-1}BC\,dA$$

$$= (-2C + 2A^{-1}B^2)\,dB = -2A^{-1}(AC - B^2)\,dB$$

$$= -2A^{-1}dB$$

Similarly the coefficient of δB is

$$-A\,\underbrace{dC}_{(2A^{-1}B\,dB - A^{-1}C\,dA)} + C\,dA - B\,dA(2A^{-1}B) + A\,dB(2A^{-1}B)$$

$$= \cancel{-2B\,dB} + C\,dA + C\,dA - 2A^{-1}B^2\,dA + \cancel{2B\,dB}$$

$$= (2C - 2A^{-1}B^2)\,dA = 2A^{-1}dA$$

Hence

$$\int \mathrm{tr}(\delta g\, g^{-1} dg\, J) = 2\int -A^{-1}\delta A\,dB + A^{-1}\delta B\,dA$$

$$= 2\int A^{-1}(-\delta A\,dB + \delta B\,dA)$$

is the skew form corresponding to two tangent vectors δg, dg

Notice: the above calculation amounts to showing that the only invariant 2-forms on pos. def. $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ of determinant 1 is $A^{-1} dA\, dB$, up to mult. constants. This can be verified directly.

Hence it's clear we haven't made yet any real progress toward why if X is volume-preserving, then the differential equation

$$\varphi_X(g + \delta g) - \varphi_X(g) = \int \mathrm{tr}(\delta g\, g^{-1} X(g)\, J)$$

can be integrated to give a function φ_X on $\{g\}$.

September 14, 1982:

Using A, B as coordinates for pos. definite real 2×2 matrices of determinant 1 is like using the UHP to describe $SL_2(\mathbb{R})/SO_2$. In effect we have

$$SL_2(\mathbb{R})/SO_2 \xrightarrow{\sim} \left\{ \begin{pmatrix} A & B \\ B & C \end{pmatrix} \right\}$$

$$\theta \longmapsto \theta\theta^*$$

wrong should be $\theta \cdot g = (\theta^*)^{-1} g\, \theta^{*-1}$

corresponding to the action $\theta \cdot g = \theta g \theta^*$. So we can lift $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ to $\theta = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as on p113. Then

$$SL_2(\mathbb{R})/SO_2 \longrightarrow UHP$$

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \longmapsto \frac{ai+b}{bi+c} = \frac{(ai+b)(-bi+c)}{b^2+c^2}$$

$$= \frac{(ab+bc)+i}{b^2+c^2} = \frac{B+i}{C}$$

Thus

$$\left\{ \begin{pmatrix} A & B \\ B & C \end{pmatrix} \right\} \xrightarrow{\sim} UHP$$

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \longmapsto \frac{B}{C} + \frac{1}{C} i$$

$$\boxed{\frac{-1}{\frac{B}{C} + \frac{1}{C} i} = \frac{-CB + Ci}{B^2 + 1} \overset{AC}{} = \left(-\frac{B}{A}\right) + \frac{1}{A} i}$$

and the canonical invariant 2-form on the UHP is

$$\frac{dx\,dy}{y^2} = \frac{d\left(\frac{B}{C}\right) \wedge d\left(\frac{1}{C}\right)}{\left(\frac{1}{C}\right)^2} = -\frac{dB \wedge dC}{C} = -\frac{dA \wedge dB}{A}$$

September 14, 1982

Review last night's calculation of the moment map for the action of volume-preserving vector fields on the metrics with this volume Ω on a surface M. Suppose the vector field has support in a coordinate patch and $\Omega = dx^1 dx^2$ there. Metrics will be denoted $A(dx^1)^2 + 2B\,dx^1dx^2 + C(dx^2)^2$ where $AC - B^2 = 1$, and we treat A, B as the independent variables, C as dependent.

We've seen that the canonical 2-form $\tilde{\Omega}$ on the set of metrics is a multiple of $\int A^{-1} dA\, dB$, more precisely

$$\tilde{\Omega}(\delta_1 g, \delta_2 g) = \int A^{-1}(\delta_1 A\, \delta_2 B - \delta_2 A\, \delta_1 B) \cdot \text{volume}$$

Let $X = a_1 \partial_1 + a_2 \partial_2$ be a vector field. We have

$$\theta(X) dx^i = d\,\theta(X) x^i = da^i$$

so

$$\begin{aligned}\theta(X) g = \; & XA\,(dx^1)^2 + 2A\,da^1 dx^1 \\ & 2XB\,dx^1dx^2 + 2B(da^1dx^2 + dx^1da^2) \\ & XC\,(dx^2)^2 + 2C\,da^2dx^2\end{aligned}$$

$$\begin{aligned}= (XA + 2A\,\partial_1 a^1 + 2B\,\partial_1 a^2)(dx^1)^2 + \big(&2XB + 2A\,\partial_2 a^1 \\ & 2B\,\partial_1 a^1 + 2B\,\partial_2 a^2 \\ + (XC + 2B\,\partial_2 a^1 + 2C\,\partial_2 a^2)(dx^2)^2 \qquad & 2C\,\partial_1 a^2\big)\, dx^1dx^2\end{aligned}$$

We want to solve

$$\varphi_X(g + \delta g) - \varphi(g) = \int \Big[A^{-1}\,\delta A\,\big(XB + A\,\partial_2 a^1 + B\,\partial_1 a^1 + B\,\partial_2 a^2 + C\,\partial_1 a^2\big) - A^{-1}\,\delta B\,\big(XA + 2A\,\partial_1 a^1 + 2B\,\partial_1 a^2\big)\Big]\, d^2x$$

when X preserves the volume: $\partial_1 a^1 + \partial_2 a^2 = 0$. Thus

$$\begin{aligned}\frac{\delta \varphi_X}{\delta A(x)} &= A^{-1}(XB + A\,\partial_2 a^1 + C\,\partial_1 a^2) \\ &= A^{-1}(XB) + \partial_2 a^1 + \frac{1+B^2}{A^2}\,\partial_1 a^2\end{aligned}$$

and so

$$\varphi_X = \int \left(\log A\,(XB) + A\,\partial_2 a^1 - \frac{1+B^2}{A}\,\partial_1 a^2 \right) d^2x$$

+ function of B alone.

From

$$\frac{\delta \varphi_X}{\delta B(x)} = -A^{-1}(XA + 2A\,\partial_1 a^1 + 2B\,\partial_1 a^2)$$

$$= -A^{-1}(XA) - 2\,\partial_1 a^1 - 2A^{-1}B\,\partial_1 a^2$$

we see we must add a $-2B\,\partial_1 a^1$ term. Thus

$$\boxed{\varphi_X(g) = \int \left(-A^{-1}(XA)\,B + A\,\partial_2 a^1 - 2B\,\partial_1 a^1 - \frac{1+B^2}{\underbrace{A}_{C}}\,\partial_1 a^2 \right) d^2x}$$

~~Now look at this formula carefully. Quite generally if a moment map exists, then φ_X can be altered by a constant, better $\varphi_X(g)$ can be altered by a function of X alone so that it becomes a linear function of X. It's supposed to be a map $g \to C^\infty(\text{man})$. The above differs from a linear function of X by a non-constant NO the a^i are the coeffs of X.~~

Now notice that relative to the isom.

$$\left\{ \begin{pmatrix} A & B \\ B & C \end{pmatrix} \right\} \simeq UHP \qquad \begin{pmatrix} A & B \\ B & C \end{pmatrix} \mapsto \left(-\frac{B}{A}\right) + \frac{1}{A}\,i$$

one has that the first term in φ_X comes from the form

$$x\,\frac{dy}{y^2} = \left(-\frac{B}{A}\right)\left(-d\,\frac{1}{y}\right) = \frac{B\,dA}{A}$$

(Nicer would be $\frac{dx}{y}$.)

So far I have been working in a coord. patch and assuming X was supported there. One can find an f of compact support with

$$df = i(X)\, dx^1 dx^2 = a_1 dx^2 - a_2 dx^1$$

$$\therefore \quad a_1 = \partial_2 f \qquad a_2 = -\partial_1 f$$

$$\varphi_X = \int \left(-A^{-1}(a^1 \partial_1 A + a^2 \partial_2 A) B + \cdots \right.$$

$$= \int \left\{ \begin{array}{l} a^1 \left[-A^{-1}(\partial_1 A) B \quad \partial_2 A + 2\partial_1 B \right] \\ a^2 \left[-A^{-1}(\partial_2 A) B + \partial_1 C \right] \end{array} \right\}$$

$$= \int f \left\{ \begin{array}{l} \partial_2 (A^{-1} \partial_1 A \; B) + \partial_2^2 A \quad 2\partial_{12}^2 B \\ -\partial_1 (A^{-1} \partial_2 A \; B) + \partial_1^2 C \end{array} \right\}$$

$$\boxed{\varphi_X = \int f \left\{ A^{-1}(\partial_1 A\, \partial_2 B - \partial_2 A\, \partial_1 B) + \partial_2^2 A - 2\partial_{12}^2 B + \partial_1^2 C \right\} d^2 x}$$

September 15, 1982

It is desirable to get the conventions concerning complex structures on an oriented plane and points in the UHP straight. Let $\mathbb{R}^2$ have the usual coordinates x, y and the usual action of $SL(2,\mathbb{R})$. A complex structure on $\mathbb{R}^2$ compatible with the standard orientation can be specified by giving the 1-diml space of linear maps $\mathbb{R}^2 \longrightarrow \mathbb{C}$ which are $\mathbb{C}$-linear for the complex structure. This space of $\mathbb{C}$-linear maps is spanned by a unique map of the form $z = x - \bar{\tau} y$ where $\tau \in$ UHP.

We choose this way instead of $x + \tau y$ so that it is compatible with the action of $SL_2(\mathbb{R})$. Given $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathbb{R}^2$ it acts on a function by $\alpha * f = f \circ \alpha^{-1}$, and so transforms $x - \bar{\tau} y =$ product with $(1 \;\; -\bar{\tau})$ into product with

$$(1 \;\; -\bar{\tau})\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (d + c\bar{\tau} \;\;\; -b - a\bar{\tau}) \sim \left(1 \;\;\; -\overline{\frac{a\tau+b}{c\tau+d}}\right)$$

Now suppose we have a positive-definite quadratic form $Ax^2 + 2Bxy + Cy^2$. Factor into linear factors

$$A(Ax^2 + 2Bxy + Cy^2) = (Ax + By)^2 + (AC - B^2)y^2$$
$$= \left| Ax + (B + i\sqrt{AC - B^2})y \right|^2$$

hence the associated complex structure is given by

$$z = x + \frac{B + i\sqrt{}}{A} y$$

which has $\tau = \dfrac{-B + i\sqrt{AC - B^2}}{A}$. This is equivariant with respect to the action of $SL_2(\mathbb{R})$ (also $GL_2(\mathbb{R})^+$) provided that α acts on $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$ by $(\alpha^{-1})^* \begin{pmatrix} A & B \\ B & C \end{pmatrix} \alpha^{-1}$; see p. 116

Now go back to the case of a surface with metric $A dx^2 + 2B\, dx\, dy + C dy^2$ and the orientation (standard) $dx dy$. Then we know $dx - \bar{\tau} dy$ is a section of $T^{1,0}$ for the associated complex structures. A holomorphic coordinate z is in the form

$$dz = f(dx - \bar{\tau}\, dy)$$

where f has to satisfy

$$0 = d(dz) = df \wedge (dx - \bar\tau dy) + f(-d\bar\tau dy)$$

$$+\partial_y f + \bar\tau \partial_x f + f \partial_x \bar\tau = 0.$$

Notice that any non-zero holom. section of $T^{1,0}$ is in the form dz locally for some coordinate z.

Recall that the curvature of a holom. line bundle with metric is given by

$$\bar\partial \partial \log |s|^2 = -\partial^2_{\bar z z} \log |s|^2 \, dz\, d\bar z$$

where s is a non-zero holom. section. Hence up to a constant the curvature is $\Delta \log |s|^2 \cdot$ volume, where Δ is the Laplacean.

We need to calculate Δ, and so need the metric on T^*. In general if $|a^i \partial_i|^2 = g_{ij} a^i a^j$, then $|b_i dx^i|^2 = g^{ij} b_i b_j$. This is because

$$b_i dx^i \longleftrightarrow b_i g^{ij} \partial_j \xrightarrow{\ |\ |^2\ } b_\ell g^{\ell j} b_m \underbrace{g^{mk} g_{jk}}_{\delta_{mj}}$$

$$= g^{\ell j} b_\ell b_j .$$

So
$$|b_1 dx + b_2 dy|^2 = (b_1 \ b_2) \frac{1}{\det} \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$= \frac{1}{AC - B^2} \left(C b_1^2 - 2B b_1 b_2 + A b_2^2 \right).$$

Put $\sqrt{\rho} = AC - B^2$, so the volume is $\rho\, dx\, dy$.

$$\int |df|^2 \rho\, dx\, dy = \int \frac{1}{\sqrt{\rho}} \left[C(\partial_x f)^2 - 2B\, \partial_x f\, \partial_y f + A(\partial_y f)^2 \right] \sqrt{\rho}\, dx\, dy$$

$$= -\int f \left[\partial_x C \partial_x - \partial_x B \partial_y - \partial_y B \partial_x + \partial_y A \partial_y \right] f \, dx\, dy$$

Hence
$$\boxed{-\Delta = \frac{1}{\sqrt{\rho}}\left[\partial_x C \partial_x - \partial_x B \partial_y - \partial_y B \partial_x + \partial_y A \partial_y\right]}$$

The idea now is to work at a given pt. and choose a section $s = e^h(dx - \bar{\tau}\, dy)$ of $T^{0,1}$ which is holomorphic and such that $\log|s|^2$ is stationary at the point. Then the curvature will be given by the highest order terms of Δ applied to the Hessian of $\log|s|^2$.

Take
$$s = e^h\left[A\,dx + (B + i\sqrt{\rho})\,dy\right].$$

This is a holomorphic section when
$$0 = ds = e^h\, dh\left[A\,dx + (B+i\sqrt{\rho})\,dy\right] + e^h\big(dA\,dx + d(B+i\sqrt{\rho})\,dy\big)$$

or
$$\boxed{-A\partial_y h + (B+i\sqrt{\rho})\partial_x h - \partial_y A + \partial_x(B+i\sqrt{\rho}) = 0}$$

Also
$$\left|A\,dx + (B+i\sqrt{\rho})\,dy\right|^2 = CA^2 - BA(2B) + A\underbrace{(B^2+\rho)}_{AC}$$
$$= 2A\rho$$

so we want
$$|s|^2 = e^{h+\bar{h}}\, 2A\rho$$
to be stationary at $x = y = 0$. Too hard!

September 16, 1982

Review Riemannian geometry. Metric $ds^2 = g_{ij}\,dx^i dx^j$ which means $g(e_i, e_j) = g_{ij}$ where $e_i = \partial_i$ is the state coordinate frame. The Levi-Civita connection

$$\nabla_\mu(e_i) = e_j \Gamma^j_{i\mu}$$

is characterized by the fact that it preserves the metric and has torsion $= 0$. The torsion is

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

so $T=0$ means that $\Gamma^j_{i\mu}$ is symmetric in $i\mu$. Preserving the metric means

$$\partial_k g(e_i, e_j) = g(\nabla_k e_i, e_j) + g(e_i, \nabla_k e_j)$$

$$\partial_k g_{ij} = g_{jl}\Gamma^l_{ik} + g_{il}\Gamma^l_{jk}$$

cyclic permute

$$\partial_i g_{jk} = g_{kl}\Gamma^l_{ji} + g_{jl}\Gamma^l_{ki}$$

$$\partial_j g_{ki} = g_{il}\Gamma^l_{kj} + g_{kl}\Gamma^l_{ij}$$

So

$$g_{il}\Gamma^l_{jk} = \underbrace{\tfrac{1}{2}\left[-\partial_i g_{jk} + \partial_j g_{ki} + \partial_k g_{ij}\right]}$$

Call this Γ_{ijk}

From an invariant viewpoint we are writing down

$$X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$Y\, g(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$$

$$Z\, g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

Using torsion $= 0$ the difference of the above two circled things is $g([X,Y],Z)$, and so it's clear you get a formula for $g(\nabla_X Z, Y)$ the same way as above.

Now that we have the connection we want to get the curvature. The curvature is

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

It is a vector bundle map $T \to T$, skew-symmetric with respect to g, which depends linearly on X, Y over functions. This is the invariant viewpoint, but better for my purposes is to use

$$\nabla : T \longrightarrow T \otimes T^*$$

$$\nabla(e_i) = e_j \Gamma^j_{i\mu} dx^\mu$$

The curvature is then ∇^2.

$$\nabla(\nabla e_i) = e_j\, d(\Gamma^j_{i\mu} dx^\mu) + e_k \Gamma^k_{j\nu} dx^\nu \Gamma^j_{i\mu} dx^\mu$$

$$= e_j R^j_{\ i}$$

where $R^j_{\ i}$ is a matrix of 2-forms. It is skew-symmetric with respect to the metric g which means that the matrix of 2-forms

$$R_{ji} = g_{j\mu} R^\mu_{\ i}$$

is skew-symmetric. ($g(x, Ry) =$
$x^t g R y = - x^t R^t g y = -(Rx)^t g y = - g(Rx, y)$ if gR is skew-symm.)

So to calculate the curvature from the metric let's use the matrix 1-form $\Gamma_{ijk} dx^k = \tilde{\Gamma}$ so that $\cancel{\Gamma_{ijk} = g_{i\mu}\Gamma^\mu_{jk} dx^k}$ the connection form is

$$\theta = \Gamma^i_{jk} dx^k = g^{il}\Gamma_{ljk} dx^k = g^{-1}\tilde{\Gamma}$$

Then $\nabla e_i = e_j \theta^j_{\ i}$ and $R e_i = e_j (d\theta^j_{\ i} + \theta^j_{\ k} \wedge \theta^k_{\ i})$

Hence the skew symmetric R_{ij} will be

$$g(d\theta + \theta\wedge\theta) = g\left[d(g^{-1}\tilde\Gamma) + g^{-1}\tilde\Gamma g^{-1}\tilde\Gamma\right)$$

$$= (g\,dg^{-1})\tilde\Gamma + \text{[redacted]}\; d\tilde\Gamma + \tilde\Gamma g^{-1}\tilde\Gamma$$

$\therefore$

$$\boxed{\begin{array}{l} R_{ij} = -dg\, g^{-1}\tilde\Gamma + d\tilde\Gamma + \tilde\Gamma g^{-1}\tilde\Gamma \\ \tilde\Gamma = \Gamma_{ijk}\,dx^k \qquad \Gamma_{ijk} = \frac{1}{2}\left[-\partial_i g_{jk} + \partial_j g_{ki} + \partial_k g_{ij}\right] \end{array}}$$

Now take $\quad ds^2 = A\,dx^2 + 2B\,dx\,dy + C\,dy^2 \qquad \therefore g = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$$2\Gamma_{111} = -\partial_1 g_{11} + \partial_1 g_{11} + \partial_1 g_{11} = \partial_1 A$$

$$2\Gamma_{\underset{(21)}{112}} = -\partial_1 g_{12} + \partial_1 g_{21} + \partial_2 g_{11} = \partial_2 A$$

$$2\Gamma_{122} = -\partial_1 g_{22} + \partial_2 g_{21} + \partial_2 g_{12} = -\partial_1 C + 2\partial_2 B$$

$$2\Gamma_{211} = -\partial_2 g_{11} + \partial_1 g_{12} + \partial_1 g_{21} = -\partial_2 A + 2\partial_1 B$$

$$2\Gamma_{212} = -\partial_2 g_{12} + \partial_1 g_{22} + \partial_2 g_{21} = \partial_1 C$$

$$2\Gamma_{222} = -\partial_2 g_{22} + \partial_2 g_{22} + \partial_2 g_{22} = \partial_2 C$$

$$\tilde\Gamma = \frac{1}{2}\begin{pmatrix} dA & \partial_2 A\,dx + (-\partial_1 C + 2\partial_2 B)\,dy \\ (-\partial_2 A + 2\partial_1 B)\,dx + \partial_1 C\,dy & dC \end{pmatrix}$$

Notice that $d\tilde\Gamma$ is the only term in R that contributes 2nd degree derivatives. One has

$$d\tilde\Gamma = \frac{1}{2}\begin{pmatrix} 0 & -\partial_2^2 A + 2\partial_1\partial_2 B - \partial_1^2 C \\ \partial_2^2 A - 2\partial_1\partial_2 B + \partial_1^2 C & 0 \end{pmatrix} dx\,dy$$

which shows the[redacted] terms [redacted] involving 2nd derivatives [illegible] in the moment map do occur in the curvature.

Let's continue with this insane waste of time. Observe that

$$\tilde{\Gamma} = \frac{1}{2}\begin{pmatrix} dA & dB \\ dB & dC \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \partial_2 A\,dx - \partial_1 C\,dy - \partial_1 B\,dx + \partial_2 B\,dy \\ -\partial_2 A\,dx + \partial_1 C\,dy + \partial_1 B\,dx - \partial_2 B\,dy & 0 \end{pmatrix}$$

$$= \frac{1}{2}\left[dg + \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]$$

where $\eta = (-\partial_2 A + \partial_1 B)dx + (-\partial_2 B + \partial_1 C)dy$ satisfies

$$d\eta = \left(\partial_2^2 A - 2\partial_{12}^2 B + \partial_1^2 C\right) dx\,dy$$

Now $R = -dg\,g^{-1}\tilde{\Gamma} + d\tilde{\Gamma} + \tilde{\Gamma} g^{-1}\tilde{\Gamma}$

$$= -dg\,g^{-1}\tfrac{1}{2}(dg + \eta J) + \tfrac{1}{2} d\eta J + \tfrac{1}{4}(dg + \eta J)g^{-1}(dg + \eta J)$$

$$= -\tfrac{1}{2} dg\,g^{-1}dg - \tfrac{1}{2} dg\,g^{-1}\eta J + \tfrac{1}{4}\left[dg\,g^{-1}dg + dg\,g^{-1}\eta J + \eta J g^{-1}dg + \eta J g^{-1}\eta J\right]$$

$$= -\tfrac{1}{4} dg\,g^{-1}dg - \tfrac{1}{4} dg\,g^{-1}\eta J + \tfrac{1}{4}\eta J g^{-1} dg + \underbrace{\tfrac{1}{4}\eta J g^{-1}\eta J}_{0 \text{ since } \eta^2 = 0} + \tfrac{1}{2} d\eta J$$

Now $(\eta J g^{-1} dg)^t = -dg\,g^{-1}(-J)\eta = dg\,g^{-1}\eta J$ and R is skew-symm., so there is no cancellation. But if g has determinant $= 1$, then $g^{-1}dg$ has trace zero so $Jg^{-1}dg$ will be symmetric, so we will have $Jg^{-1}dg + dg\,g^{-1}J = 0$.

Hence in the case $\det(g) = 1$ we will have

$$R = -\frac{1}{4} \underbrace{dg\, g^{-1} dg}_{\begin{pmatrix} 0 & AdBdC + BdCdA + CdAdB \\ -AdBdC - BdCdA - CdAdB & 0 \end{pmatrix}} + \frac{1}{2}(d\eta) J$$

Using the relation $C = \dfrac{1+B^2}{A}$ we have

$$dC = \frac{2B}{A} dB - \frac{1+B^2}{A^2} dA$$

so

$$AdBdC + BdCdA + C\,dA\,dB = AdB\left(-\frac{1+B^2}{A^2} dA\right) - \frac{2B^2}{A} dBdA + \frac{1+B^2}{A} dAdB$$

$$= 2A^{-1} dA\, dB$$

So

$$R = \frac{1}{2}\left[A^{-1} dA\, dB + \left(\partial_2^2 A - 2\partial_1\partial_2 B + \partial_1^2 C\right)\overset{dx\,dy}{}\right] J$$

At this point we know the moment map is given by the curvature.

With some more work we can get a complete formula for the Gaussian curvature. We know

$$(J g^{-1} dg)^t = - dg\, g^{-1} J$$

hence $\quad J g^{-1} dg + dg\, g^{-1} J \quad$ is skew-symm.

hence a scalar times J. The scalar must be

$$\frac{1}{2} \operatorname{tr}\left(J g^{-1} dg\, J^{-1} + dg\, g^{-1}\right) = d \log \det(g).$$

Thus we have

$$\boxed{R = -\frac{1}{4} \underbrace{dg\, g^{-1} dg}_{\frac{1}{AC-B^2}\frac{1}{4}(AdBdC + BdCdA + C\,dAdB)\, J} + \left[\frac{1}{4} \overset{(-\partial_2 A + \partial_1 B)dx + (-\partial_2 B + \partial_1 C)dy}{\eta}\, d\log\det(g) + \frac{1}{2} d\eta\right] J}$$

Back to the general theory. Fix the surface M with its volume. Then we have seen that the space $\mathcal{H}$ of complex structures on M has a natural symplectic 2 form, in fact Kähler form. So now the natural question is to produce a holomorphic line bundle over $\mathcal{H}$ with metric having the symplectic form as its curvature. Naturally we should hope to produce this line bundle using $\bar\partial$ operators. In fact the Dirac operators should be natural candidates.

What do I mean by Dirac operator? One has to take a vector bundle intrinsically attached to the surface, e.g. the trivial bundle or powers of the canonical line bundle. Square roots of the canonical line bundle give 2^{2g} different Dirac operators.

The first project is to compute the curvature of the determinant line bundles assuming your conjecture on the GRR thm. Take the case of the trivial bundle. Over each point of $\mathcal{H}$ we get an operator

$$1 \xrightarrow{d} T^* \xrightarrow{p} T^{0,1}.$$

Since E is trivial $ch(E) = 1$, so the important term [redacted] comes from the Todd class of the tangent bundle along the fibres.

$t \mapsto h\, \partial_z \cdot d\bar z$

I take a 1-parameter family of complex structures on M and form $X = \mathbb{C} \times M$. Locally we have coordinates t, z on X and the complex structure on X is defined by requiring $T^{1,0}_X$ to be spanned by [redacted] dt and $dz + h\,d\bar z$. [redacted]

Let's make this clearer. On M we have a initial complex structure for which z is a holom. coordinate. Any other complex structure is given by a line bundle in T^* spanned locally by $dz + h\,d\bar z$. Thus a complex structure on M is the graph of a map $T^{1,0}_M \xrightarrow{h} T^{0,1}_M$

which I describe locally by $h\,\partial_z \otimes d\bar z$, where $|h| < 1$. To give a holomorphic family of complex structures on M parameterized by t means that $h = h(t) : T_M^{1,0} \to T_M^{0,1}$ is a holomorphic function of t with values in a complex vector space.

Thus we are given $h(t,z)$ holomorphic in t and it should be possible to define a complex structure on $X = \mathbb{C} \times M$. The obvious candidate has $T_X^{1,0}$ spanned by dt and $dz + h\,d\bar z$. To see this works one uses the Newlander-Nirenberg thm. Clearly

$$dt,\ \overline{dt},\ dz + h\,d\bar z,\ d\bar z + \bar h\,dz$$

span T_X^* so that $T_X^* = T_X^{1,0} \oplus T_X^{0,1}$. On the other hand we must check that the ideal in ΛT_X^* generated by $T_X^{1,0}$ is closed under d. So

$$d(dt) = 0 \qquad d(dz + h\,d\bar z) = dh\,d\bar z$$

$$dh = \partial_t h\,dt + \cancel{\partial_{\bar t} h\,d\bar t} + \partial_z h\,dz + \partial_{\bar z} h\,d\bar z$$

$$dh\,d\bar z = \partial_t h\,\underbrace{dt}\,d\bar z + \partial_z h\,\underbrace{dz\,d\bar z}_{(dz + h\,d\bar z)\,d\bar z}$$

and so it works.

What is the program? We want the tangent bdl $T_{X/Y}$ along the fibres with its holomorphic structure. So we want the $\bar\partial$ operator $\bar\partial : T_{X/Y} \to T_{X/Y} \otimes T_X^{0,1}$. Then with a metric on $T_{X/Y}$ which will come from the complex structure + volume we should be able to compute the curvature form

September 17, 1982

Ultimately we have to compute the curvature of the tangent bundle along the fibres in a holomorphic family of curves, so we should start with the computation of the curvature of the tangent bundle of a single curve. This is the problem I struggled with the past several days. Perhaps it is possible to simplify the computation of two days ago, by not trying to choose a holomorphic section, and instead using the projection operators.

Let the coordinates on the surface be $z = x+iy$ and the volume $\rho\, dx\, dy$. The complex structure of interest has $dz + h\,d\bar z$ spanning $T^{1,0}$. The metric on $T^{1,0}$ is determined by

$$i\,\alpha \wedge \bar\beta = (\beta|\alpha)\,\rho\, dx dy.$$

For example if $h=0$ we have $\quad i dz\, d\bar z = \frac{2}{\rho}\big(\underbrace{\rho \tfrac{i}{2} dz d\bar z}_{\rho\, dx dy}\big)$

and hence $|dz| = \sqrt{\frac{2}{\rho}}$ which we already know.

Let $\omega = dz + h\,d\bar z$. Then

$$i(\omega \wedge \bar\omega) = i\,(dz + h d\bar z)\wedge(d\bar z + \bar h dz) = i dz d\bar z\,(1-|h|^2)$$

$$= (1-|h|^2)(2/\rho)\,\frac{\rho}{2} i dz d\bar z$$

and so $\quad \boxed{|\omega|^2 = (1-|h|^2)(2/\rho)}$

The $\bar\partial$ operator on $T^{1,0}$ is given by

$$T^{1,0} \xrightarrow{\ d\ } T^{1,1}.$$

So

$$\bar\partial\omega = d(dz + h\,d\bar z) = dh\, d\bar z = \partial_z h\, dz d\bar z$$

$$= \frac{\partial_z h}{1-|h|^2}\,\omega\bar\omega = \omega\otimes\left(\frac{\partial_z h}{1-|h|^2}\,\bar\omega\right) \in T^{1,0}\otimes T^{0,1}$$

I want to lift this operator to a connection $\nabla : T^{1,0} \to T^{1,0}\otimes T^*$

preserving the metric. Suppose

$$\nabla\omega = \omega\otimes(a\omega + b\bar\omega).$$

Then $\boxed{b = \frac{\partial_{\bar z} h}{1-|h|^2}}$ if ∇ is a lift of $\bar\partial$.

For ∇ to preserve the metric means that

$$d(\omega|\omega) = (\nabla\omega|\omega) + (\omega|\nabla\omega)$$

$$d(\omega|\omega) = (\omega|\omega)\,\overline{a\omega+b\bar\omega} + (\omega|\omega)(a\omega+b\bar\omega)$$

$$d\log|\omega|^2 = (a+\bar b)\underbrace{\omega}_{dz+hd\bar z} + (\bar a + b)\underbrace{\bar\omega}_{d\bar z + \bar h dz}$$

Introduce $\partial_\omega, \partial_{\bar\omega}$ the dual base to $\omega, \bar\omega$. Then

$$\partial_\omega = \frac{1}{1-|h|^2}(\partial_z - \bar h\partial_{\bar z}) \qquad \partial_{\bar\omega} = \frac{1}{1-|h|^2}(\partial_{\bar z} - h\partial_z)$$

and

$$\boxed{\begin{aligned} a+\bar b &= \partial_\omega \log|\omega|^2 \\ &= \frac{1}{1-|h|^2}(\partial_z - \bar h\partial_{\bar z})\log\left((1-|h|^2)(2/\rho)\right)\end{aligned}}$$

Finally the curvature is the form

$$d(a\omega+b\bar\omega) = da\cdot\omega + db\cdot\bar\omega + a\underbrace{d\omega}_{b\omega\bar\omega} + b\underbrace{d\bar\omega}_{\bar b\bar\omega\omega}$$

$$= \left[\partial_\omega b - \partial_{\bar\omega} a + ab - b\bar b\right]\omega\bar\omega.$$

This is still very messy.

Let's however continue with the case of a family, because maybe we can work to low order in h around a fibres. Let $X = \underset{Y}{\mathbb{C}}\times M$ with t, z coordinates on X. Take $T_X^{1,0}$ to be spanned by dt, $dz + hd\bar z$ where $h = h(t,z)$ is holomorphic in t. Then $T^{1,0}_{X/Y}$ is spanned by $dz + hd\bar z$ and the metric is given in the same way as above, where in general ρ might be varying wrt t. Next I need the holomorphic structure on $T^{1,0}_{X/Y}$ which is an operator

$$T^{1,0}_{X/Y} \longrightarrow T^{1,0}_{X/Y} \otimes T^{0,1}_{X} .$$

Let's start with the operator

$$T^{1,0}_{X} \longrightarrow T^{1,0}_{X} \otimes T^{0,1}_{X}$$

which comes from the holomorphic structure on bundle $T^{1,0}_X$. This map is induced by d

$$d(dt) = 0 \qquad d(dz + h\,d\bar z) = dh\, d\bar z$$
$$= \partial_t h \, dt\, d\bar z + \partial_z h \, dz\, d\bar z$$

which maps $T^{1,0}_X$ to $T^{2,0}_X \oplus T^{1,1}_X$, and we take the projection into $T^{1,1}_X$. We have to write things not using $dz, d\bar z$ but using $\omega = dz + h\,d\bar z$, $\bar\omega = d\bar z + \bar h\, dz$. So

$$\partial_t h \, dt \underbrace{d\bar z}_{\bar\omega - \bar h dz} + \partial_z h \frac{\omega\bar\omega}{1-|h|^2}$$

so modulo $T^{2,0}_X$ it is just $\partial_t h \, dt\, \bar\omega + \frac{\partial_z h}{1-|h|^2}\omega\bar\omega$.
Since $T^{1,0}_{X/Y}$ is the quotient of $T^{1,0}_X$ by dt, we see that the $\bar\partial$ operator for $T^{1,0}_{X/Y}$ is

$$\bar\partial : T^{1,0}_{X/Y} \longrightarrow T^{1,0}_{X/Y} \otimes T^{0,1}_X$$
$$\omega = dz + h\,d\bar z \longmapsto \omega \otimes b\bar\omega \qquad\qquad b = \frac{\partial_z h}{1-|h|^2}$$

Now we lift $\bar\partial$ to a connection preserving the metric, which is varying wrt. t because the complex structure on M is changing. Suppose

$$\nabla\omega = \omega \otimes \underbrace{(a\omega + b\bar\omega + c\,dt)}_{\Theta}$$

and note that it's the same b and there is no $d\bar t$ term, because ∇ lifts $\bar\partial$. Preserving the metric means

$$d \log|\omega|^2 = \Theta + \bar\Theta$$
$$= (a + \bar b)\omega + (\bar a + b)\bar\omega + c\,dt + \bar c\, d\bar t$$

hence we have

$$a+\bar{b} = \partial_\omega \log|\omega|^2 \qquad c = \partial_t \log|\omega|^2.$$

Finally the curvature is

$$d\theta = d(a\omega + b\bar{\omega}) + dc\,dt.$$

What we want to do is to integrate the Todd class of the tangent bundle of X/Y over the fibres to get a form on Y. We compute the form at the point $t=0$ assuming that $h=0$ at $t=0$, i.e. that our basepoint complex structure coincides with the complex structure ~~[illegible]~~ in the family at $t=0$. We then do low order calculations in t, using that $h = O(t)$.

$$b = \frac{\partial_z h}{1-|h|^2} = \partial_z h\,(1 + O(|h|^2)) = \partial_z h + O(|t|^3)$$

$$a+\bar{b} = \frac{1}{1-|h|^2}\left(\partial_z - \bar{h}\partial_{\bar{z}}\right) \underbrace{\log\,(1-|h|^2)(2/\rho)}_{-\log\rho \;-\; |h|^2 - \frac{1}{2}|h|^4 - \cdots}$$

$$= -\left(\partial_z - \bar{h}\,\partial_{\bar{z}}\right)\log\rho \;-\; \partial_z|h|^2 \;+\; |h|^2(-\partial_z \log\rho) \;+\; O(|t|^3)$$

$$c = \partial_t \log\,(1-|h|^2)(2/\rho) = -\partial_t|h|^2 + O(|t|^3)$$

assuming that ρ is independent of t.

First suppose ρ is constant. Then

$$a = -\partial_z|h|^2 - \partial_{\bar{z}}\bar{h} + \cdots$$
$$b = \partial_z h + \cdots\cdots$$
$$c = -\bar{h}\,\partial_t h + \cdots\cdots$$

so

$$\theta = a\omega + b\bar{\omega} + c\,dt = (-\partial_z|h|^2 - \partial_{\bar{z}}\bar{h})(dz + h\,d\bar{z})$$
$$+\; \partial_z h\,(d\bar{z} + \bar{h}\,dz) - \bar{h}\,\partial_t h\,dt$$
$$= -\partial_{\bar{z}}\bar{h}\,dz + \partial_z h\,d\bar{z} - \bar{h}\,\partial_t h\,dt + O(|t|^2)$$

Then $$d\theta = -\partial^2_{\bar t \bar z}\bar h \, d\bar t\, dz + \partial^2_{tz} h \, d\bar z - \partial_{\bar t}\bar h\, \partial_t h \, d\bar t\, dt + O(|t|)$$

In the general case we have only to calculate a, b, c to first order in h, for then we know $d\theta$ to 0th order.

$$b = \partial_z h \qquad a = -(\partial_z - \bar h \partial_{\bar z}) \log \rho - \partial_{\bar z} \bar h$$

$$c = -\bar h \partial_t h - \partial_t \log \rho$$

$$-\theta = \left[(\partial_z - \bar h \partial_{\bar z}) \log \rho + \partial_{\bar z}\bar h\right](dz + h\, d\bar z) - \partial_z h \, d\bar z + \left[\bar h \partial_t h + \partial_t \log \rho\right] dt$$

$$= \left[\partial_z(\log \rho)\, dz + \partial_t(\log\rho)\, dt\right] + \left[(h\, \partial_z \log \rho)\, d\bar z - (\bar h\, \partial_{\bar z} \log \rho)\, dz\right] + \left[\partial_{\bar z}\bar h\, dz - \partial_z h\, d\bar z - \bar h \partial_t h \, dt\right]$$

<u>Example</u>: Suppose we take an elliptic curve $M = \mathbb{C}/\Gamma$ with constant volume and consider the different complex structures defined by $dz + h\,d\bar z$ with h constant. I can also fix a flat hermitian line bundle, say given by a connection on the trivial line bundle with constant connection form. The above calculations imply that the curvature of the cotangent bundle along the fibres is

$$d\theta = -\partial_t h\, \partial_t h \, d\bar t\, dt$$

where $h = 0$. A fixed flat E has trivial curvature, so pushing the Todd class down gives zero.

Question: Fix a vector bundle E with inner product and connection over a C^∞ surface M. Now consider a holom. family of complex structures and the resulting ~~family of holom.~~ bundles given by E + conn. over this family. The question is whether the bundle is holomorphic

<u>September 18, 1982</u>

Fix a C^∞ compact oriented surface M and vector bundle E over M. We now have two points of view:

a) **Holomorphic:** We consider the space of all holomorphic structures on (M,E) acted on by the group of automorphisms of (M,E). An infinitesimal autom. of (M,E) consists of a vector field on M together with a partial connection on E over this vector field.

A holomorphic structure on M consists of a section of $\mathbb{P}(T^*)$ which lies in the complement of $\mathbb{P}(T^*_{\mathbb{R}})$ with the right orientation. Notation $\mathbb{P}(T^*)_+$. Once a complex structure on M is given, a holom. structure of (M,E) over it consists of a $\bar\partial$-operator $E \longrightarrow E\otimes T^{0,1}$ which is a first order operator with symbol $E\otimes T^* \to E\otimes T^{0,1}$, $id_E \otimes$ (proj. of T^* on $T^{0,1}$).

Tangent space to a complex structure $T^{1,0}$ on M is the space of sections of $Hom(T^{1,0}, T^{0,1})$. The tangent space to the fibre of $SH(M,E) \longrightarrow SH(M)$ at a point given by $\bar\partial : E \longrightarrow E\otimes T^{0,1}$ is $\Gamma\{Hom(E, E\otimes T^{0,1})\}$. Thus we have an exact sequence

$$0 \longrightarrow \Gamma\{Hom(E, E\otimes T^{0,1})\} \longrightarrow T_{SH(M,E)}\big|_{\bar\partial} \longrightarrow \Gamma\{Hom(T^{1,0}, T^{0,1})\} \to 0.$$

I'd like to show that $\sim$ is the space of sections of a vector bundle over M depending on $\bar\partial$. The idea is that $\bar\partial$ is just a section of a fibre bundle over M, and that in general, the tangent space to the space of sections of a fibre bundle at a section, is the space of sections of the normal bundle to the section.

So let's describe $SH(M,E)$ as a space of sections of a fibre bundle over M. First we must give a section of $\mathbb{P}(T^*)$, namely the hol. structure on M, and then look for a diff. op. $E \to E\otimes T^{0,1}$, i.e. a map $J_1(E) \to E\otimes T^{0,1}$. Now $T^{0,1}$ is just $O(1)$ over $\mathbb{P}(T^*)$ restricted to our section, so if $\pi : \mathbb{P}(T^*) \to M$ is the projection ∂ is a map of v.b.

from $\pi^* J_1(E) \longrightarrow \pi^*(E)\otimes\mathcal{O}(1)$ such that

$$\begin{tikzcd} \pi^*(E)\otimes [redacted]\, \pi^*(T^*) \arrow[dr, "\mathrm{id}\otimes\text{canon.}"] \arrow[d, phantom, "\cap"] & \\ \pi^*(J_1(E)) \arrow[r, dashed] & \pi^*(E)\otimes\mathcal{O}(1) \end{tikzcd}$$

commutes. So now it is clear what the fibre bundle is: [redacted]

Over $\mathbb{P}(T^*)$ we have the vector bundle $\mathrm{Hom}(\pi^* J_1 E, \pi^* E\otimes\mathcal{O}(1))$ and inside this a certain coset for $\mathrm{Hom}(\pi^* E, \pi^* E\otimes\mathcal{O}(1))$.

Let's do this [redacted] with coordinates. Fix the holomorphic structure $T^{1,0}$ and the operator $\bar\partial : E \longrightarrow E\otimes T^{0,1}$. Suppose we have a tangent vector δ to $SH(M,E)$ at $\bar\partial$. How can I describe δ in local coordinates. I choose a local coordinate z with dz spanning $T^{1,0}$ and I choose a holomorphic trivialization of E, so that $\bar\partial f = \partial_{\bar z} f\, [redacted]\, d\bar z$. A nearby holomorphic structure $\tilde T^{1,0}$ is spanned by $dz + h\,d\bar z$. [redacted] Suppose we have a $\bar\partial$-operator D belonging to this complex structure $\tilde T^{1,0}$. [redacted] Use

$$\begin{tikzcd} T^{0,1} \arrow[dr, "\sim"] \arrow[d, phantom, "\subset"] & \\ T^* \arrow[r] & \tilde T^{0,1} \end{tikzcd}$$

to identify $\tilde T^{0,1}$ with $T^{0,1}$. Thus I trivialize $\mathcal{O}(1)$ over the part of $\mathbb{P}(T^*)$ [redacted] I called $\mathbb{P}(T^*)_+$. Then the projection $T^* \longrightarrow \tilde T^{0,1}$ can be identified with the map $dz,\ d\bar z \longmapsto -h\,d\bar z,\ d\bar z$ or better the map $i(\partial_{\bar z} - h\,\partial_z)\,d\bar z : T^* \longrightarrow T^{0,1}$. Hence the symbol of a $\bar\partial$ operator for the new holom. structure $dz + h\,d\bar z$ is the same as for the operator

$$(\partial_{\bar z} - h\,\partial_z)\cdot d\bar z : E \longrightarrow E\otimes T^{0,1}$$

Then we can add to this arbitrary lower terms $\beta\, d\bar z$.

Let's go over this more carefully. I start with a $\bar\partial$-operator $E \longrightarrow E\otimes T^{0,1}$ and I want to describe the tangent space to this $\bar\partial$ in $SH(M,E)$. Perhaps I should think of a holom. structure on M, E as a [redacted] quotient-bundle of $J_1(E)$, say $J_1(E) \twoheadrightarrow Q$ with the property that the composition $E\otimes T^* \subset J_1(E) \to Q$ is [redacted] surjective with kernel of the form $E\otimes T^{1,0}$ for some line $T^{1,0} \subset T^*$ in $\mathbb{P}(T^*)^+$. From this point of view a tangent vector should be a map from the subbundle, which is $J_1^h(E)$ h = holom., to the quotient bundle $Q = E\otimes T^{1,0}$ such that

$$\begin{tikzcd} J_1^h(E) \arrow[r, phantom, "\subset"] \arrow[rr, bend right, "\text{tangent vector}"'] & J_1(E) \arrow[r, two heads] & Q = E\otimes T^{0,1} \\ \cup & & \\ E\otimes T^{1,0} \arrow[uurr, "id\otimes h"'] & & \end{tikzcd}$$

This picture has some advantages: a tangent vector is no longer a differential operator but rather an operator [redacted] restricted to holomorphic sections of E whose symbol is $h\,\partial_z\, d\bar z$. So we get

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(E, E\otimes T^{0,1}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(J_1^h(E), E\otimes T^{0,1}) \arrow[r] & \mathrm{Hom}(E\otimes T^{1,0}, E\otimes T^{0,1}) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(E, E\otimes T^{0,1}) \arrow[r] & T_{SH(M,E)}\big|_{\bar\partial} \arrow[r] \arrow[u, phantom, "\cup"] & \mathrm{Hom}(T^{1,0}, T^{0,1}) \arrow[r] \arrow[u, phantom, "\cup"] & 0 \end{tikzcd}$$

If I want to understand the complex structure on this tangent space I can pick a section which is complex linear in h. For example choose a ∂-operator $E \to E\otimes T^{1,0}$, call it d', and then consider the differential operator

$$E \xrightarrow{d'} E\otimes T^{1,0} \xrightarrow{h} E\otimes T^{0,1}$$

restricted to holomorphic sections of E.

Summary: A point of $SH(M,E)$ is a sub-bundle $R \subset J_1(E)$ such that $R \to E$ is surjective with kernel of the form $E \otimes T^{1,0} \subset E \otimes T^*$ for some line $T^{1,0} \subset T^*$ in $\mathbb{P}(T^*)^+$. It follows that $J_1(E)/R \simeq E \otimes T^{0,1}$.

A tangent vector to R in $SH(M,E)$ is a vector bundle homomorphism $R \longrightarrow J_1(E)/R = E \otimes T^{0,1}$ whose restriction to $E \otimes T^{1,0}$ is of the form $id \otimes h$, where $h: T^{1,0} \longrightarrow T^{0,1}$. We have the exact sequence of complex vector spaces

$$0 \longrightarrow \Gamma\{Hom(E, E\otimes T^{0,1})\} \longrightarrow T_{SH(M,E)}\big|_R \longrightarrow \Gamma\{Hom(T^{1,0}, T^{0,1})\} \longrightarrow 0$$

Fix a connection $\nabla: E \longrightarrow E \otimes T^*$. Then given a holomorphic structure $T^{1,0}$ on M we get a holomorphic structure on (M,E) given by the $\bar\partial$-operator

$$E \xrightarrow{\nabla} E \otimes T^* \xrightarrow{id \otimes pr} E \otimes T^{0,1}.$$

Thus we get a section of the map

$$SH(M,E) \longrightarrow SH(M).$$

Given a map $h: T^{1,0} \longrightarrow T^{0,1}$ representing a tangent vector to $SH(M)$ at the point $T^{1,0}$, the image of h under this section is the map

$$R \xrightarrow{\tilde\nabla} E \otimes T^{1,0} \xrightarrow{id \otimes h} E \otimes T^{0,1}$$

which is a complex linear splitting of the above exact sequence. Hence we see that the section is holomorphic.

Since $SH(M,E) \longrightarrow SH(M)$ is a torsor for $\Gamma\{Hom(E, E\otimes T^{0,1})\}$, where now $T^{0,1}$ denotes the quotient line bundle over $\mathbb{P}(T^*)^+$, it follows that $SH(M,E)$ can be identified holomorphically with the bundle over $SH(M)$ with fibre $\Gamma\{Hom(E, E\otimes T^{0,1})\}$.

In addition to the holomorphic viewpoint one also has the

b) metric viewpoint: Here we equip M with a volume and E with a metric, and gauge transformations are autos of (M,E) preserving the volume and metric.

Actually the volume on M + metric on E enable you to form an inner product on $\Gamma(E)$.

It seems one gets a larger gauge group consisting of pairs (f,φ) where f is a diffeom. of M preserving the orientation, and where φ is an isomorphism of bundles $\varphi : f^*E \xrightarrow{\sim} E$ preserving the metric up to the Jacobian factor. More precisely the metric + volume can be replaced by a hermitian inner product on E with values in $\Lambda^2 T^*$. Then we want

$$\begin{tikzcd}
E\otimes\bar{E} \arrow[r, "h"] & \Lambda^2 T^* \\
f^*E\otimes f^*\bar{E} \arrow[u, "\varphi\otimes\bar{\varphi}", "\wr"'] \arrow[r, "f^*(h)"] & f^*(\Lambda^2 T^*) \arrow[u, "\text{canon.}"']
\end{tikzcd}$$

to commute.

September 19, 1982

Program: Yesterday I learned that if I fix a connection on E then I get a holomorphic way to lift holom. structures on M to holom. structures on (M, E). Now I want to understand how to work in the metrics.

So suppose a volume is given on M and a metric is given on E preserved by the connection. Then the volume gives an inner product on $T^{0,1}$ and so we get a Laplacean on E associated to each holom. structure on M.

I want to work around a ~~fixed~~ basepoint holomorphic structure given by dz. Then another holom. structure will be described by $dz + h d\bar{z}$; better $T^{1,0}$ = spanned by $dz + h d\bar{z}$. Let the volume be $\rho \, dx dy = \rho \frac{i}{2} dz d\bar{z}$. The metric on $T^{1,0}$ is then defined by

$$i(dz + h d\bar{z}) \wedge (d\bar{z} + \bar{h} dz) = |dz + h d\bar{z}|^2 \cdot \underbrace{\text{volume}}_{\rho \frac{i}{2} dz d\bar{z}}$$

$$i(1 - |h|^2) \, dz d\bar{z}$$

Hence

$$\boxed{|dz + h d\bar{z}|^2 = (1 - |h|^2)(2/\rho)}$$

Suppose the connection on E is given by

$$\nabla = \nabla_z dz + \nabla_{\bar{z}} d\bar{z}, \qquad \nabla_{\bar{z}} = \partial_{\bar{z}} + \alpha \qquad \nabla_z = \partial_z - \alpha^*,$$

relative to some local orthonormal frame. Then our $\bar{\partial}$-operator is

$$E \xrightarrow{\nabla} E \otimes T^* \longrightarrow E \otimes T^{0,1}$$

where we ~~used~~ use the projection $T^* \to T^{0,1}$ which we know is given by $e(\bar{\omega}) i(\partial_{\bar{\omega}})$, where

$$\partial_{\bar{\omega}} = \frac{1}{1 - |h|^2} \left(\partial_{\bar{z}} - h \partial_z \right)$$

and I put $\omega = dz + h d\bar{z}$, $\bar{\omega} = d\bar{z} + \bar{h} dz$.

Hence the $\bar\partial$-operator is

$$s \longmapsto \underbrace{\frac{1}{1-|h|^2}\left(\nabla_{\bar z} - h\nabla_z\right)}_{\nabla_{\bar\omega}} s \otimes \underbrace{(d\bar z + \bar h\, dz)}_{\bar\omega}$$

In other words

$\nabla = \nabla_\omega$ [redacted] $\omega + \nabla_{\bar\omega}$ [redacted] $\bar\omega$

and the second component is the $\bar\partial$-operator.

Now