1 - 72 Aug 6 - Sept. 2, 1982 1-72

ground energy as a det. 1-21
Graeme's notes on ground energy 21-22

parametrices for $\bar{\partial}$: 28-42
flat Green's fn. 31

anomaly + higher order G fns. 43-53

$\langle 0|S|0\rangle$ as a determinant 54-72

August 6, 1982

The general question is [redacted] how to interpret the ground energy of an independent fermion gas as a determinant.

Let's suppose the single particles are described by a Hamiltonian operator H acting on V. First suppose V finite-diml. Extend H to a derivation $\hat{H}$ on $\Lambda(V)$. Assume H doesn't have 0 for an eigenvalue. Then I know the ground energy of $\hat{H}$ is the sum of the negative eigenvalues of H, and the ground state is non-degenerate. Call this ground energy

$$E(\hat{H}) = \sum_{\lambda<0} \lambda .$$

We have the variational formula

$$\delta E(\hat{H}) = \mathrm{Tr}\,(P_- \delta H) \qquad P_- = \text{proj. on } H<0.$$

Next relate this to the operator $\frac{d}{dt} + H$ on $L^2(\mathbb{R}, V)$. The inverse of $\frac{d}{dt}+H$ is given by

$$G(t,t') = \langle t | \left(\tfrac{d}{dt} + H\right)^{-1} | t' \rangle$$

$$= e^{-(t-t')H} \begin{cases} P_+ & t > t' \\ -P_- & t < t' . \end{cases}$$

Hence formally we have

$$\delta \log \det\left(\frac{d}{dt} + H\right) = \mathrm{Tr}\left\{\left(\frac{d}{dt}+H\right)^{-1} \delta H\right\}$$

$$= \int dt \ \mathrm{tr}\,\{G(t,t)\,\delta H\} .$$

We have to make sense of $G(t,t)$ in some way. Let's do this regularization by taking

$$G(t^-, t) = -P_- .$$

Then $$\delta \log \det\left(\frac{d}{dt}+H\right) = -\left(\int dt\right) \operatorname{tr}(P_- \delta H)$$

Since the factor $\int dt$ is infinite, it's clear that the determinant is not defined, but rather one wants to work over a finite interval of length T and divide by T and let $T \to \infty$.

Frequency picture: $\quad f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega)$.

Then $\frac{d}{dt} \longmapsto -i\omega$.

$$\operatorname{Tr}\left\{\left(\frac{d}{dt}+H\right)^{-1}\delta H\right\} = \operatorname{Tr}\left\{(-i\omega+H)^{-1}\delta H\right\}.$$

Here δH and $(-i\omega+H)^{-1}$ are multiplication ops. on $L^2(\frac{d\omega}{2\pi}; V)$, and multiplication operators don't have a traces. The best thing one has is an average. In formulas

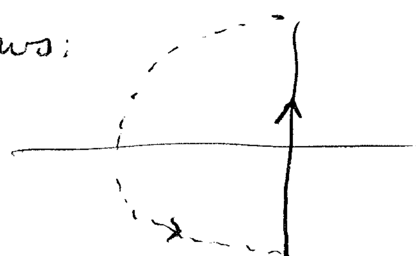
$$\operatorname{Tr}\left\{(-i\omega+H)^{-1}\delta H\right\} = \int d\omega\, d\omega' \operatorname{tr}\left\{(-i\omega+H)^{-1}\delta(\omega-\omega') \times \delta H\, \delta(\omega'-\omega)\right\}$$

$$= \int d\omega \operatorname{tr}\left(\frac{1}{-i\omega+H}\delta H\right) \cdot \delta(0)$$

$$= 2\pi\,\delta(0) \cdot \int \frac{i\,d\omega}{2\pi i} \operatorname{tr}\left(\frac{-1}{i\omega - H}\delta H\right)$$

$$= -2\pi\,\delta(0) \cdot \int_{-i\infty}^{i\infty} \frac{d\lambda}{2\pi i} \operatorname{tr}\left(\frac{1}{\lambda - H}\delta H\right)$$

Same problems occur: The contour integral must be closed off as follows:



so as to get the operator P_-. This is the same as choosing the regularization $G(t^-, t)$. Similarly we have the infinity $2\pi\delta(0) = \int dt$.

Possibility: $\det\left(\frac{d}{dt} + H\right)$ makes sense as a von-Neumann-algebra determinant.

One way to work over a finite interval of length T and let $T \to \infty$ is to use the temperature formalism. In this case we work with

$$e^{-\beta F(\hat{H})} = \mathrm{Tr}\left(e^{-\beta \hat{H}}\right)$$

where $F(\hat{H})$ is the free energy of $\hat{H}$. We have

$$\mathrm{Tr}\left(e^{-\beta \hat{H}}\right) = \det\left(1 + e^{-\beta H}\right) = \prod_\alpha \left(1 + e^{-\beta \varepsilon_\alpha}\right)$$

so that $F(\hat{H}) \to E(\hat{H})$ as $\beta \to \infty$. Variational formula:

$$\delta F(\hat{H}) = -\frac{1}{\beta}\,\delta \log\det\left(1 + e^{-\beta H}\right) = \mathrm{Tr}\left(\frac{e^{-\beta H}}{1+e^{-\beta H}}\,\delta H\right)$$

Here $\frac{e^{-\beta H}}{1+e^{-\beta H}}|\alpha\rangle = n_\alpha |\alpha\rangle$ where $n_\alpha = \frac{e^{-\beta\varepsilon_\alpha}}{1+e^{-\beta\varepsilon_\alpha}}$ is the average no. of particles in the state α.

Now relate the above to the operator $\frac{d}{dt} + H$ on $[0,\beta]$ with anti-periodic bdry conditions. The inverse is given by

$$G(t,t') = e^{-(t-t')H} \begin{cases} \dfrac{1}{1+e^{-\beta H}} & t > t' \\[2ex] \dfrac{-e^{-\beta H}}{1+e^{-\beta H}} & t < t' \end{cases}$$

Hence
$$G(t^-,t) = -\frac{e^{-\beta H}}{1+e^{-\beta H}} .$$

Thus starting from
$$\delta \log\det\left(\frac{d}{dt}+H\right) = \mathrm{Tr}\left\{\left(\frac{d}{dt}+H\right)^{-1}\delta H\right\}$$
$$= \int_0^\beta dt \, \mathrm{tr}\{G(t^-,t)\,\delta H\}$$

we get
$$= -\left(\int_0^\beta dt\right) \underbrace{\mathrm{tr}\left\{\frac{e^{-\beta H}}{1+e^{-\beta H}}\,\delta H\right\}}_{\delta F(H)}.$$

which gives
$$\det\left(\frac{d}{dt}+H \quad \begin{array}{l}\text{on } L^2([0,\beta],V)\\ \text{anti-periodic b.c.}\\ G(t^-,t) \text{ reg.}\end{array}\right) = e^{-\beta F(\hat{H})}.$$

August 7, 1982

The problem is to relate the determinant of $\bar\partial$ operators to the determinants of Dirac operators encountered in physics, e.g. Schwinger's work. So let's begin with the Dirac equation which gives the 1-particle Hamiltonian:

$$i\partial_t \psi = H\psi \qquad H = \alpha \tfrac{1}{i}\partial_x + \beta m$$

We want $(\alpha p + \beta m)^2 = p^2 + m^2$ so $\alpha^2 = \beta^2 = 1$, $\alpha\beta + \beta\alpha = 0$.

$$\left(\tfrac{1}{i}\partial_t + \alpha\tfrac{1}{i}\partial_x + \beta m\right)\psi = 0$$

$$\left(\beta\tfrac{1}{i}\partial_t + \beta\alpha\tfrac{1}{i}\partial_x + m\right)\psi = 0$$

For imaginary time the eqn. is $(\partial_t + H)\psi = 0$, or

$$\left(\beta\partial_t + \beta\alpha\tfrac{1}{i}\partial_x + m\right)\psi = 0$$

so that $\gamma^0 = \beta$ $\gamma^1 = \beta\alpha\frac{1}{i}$ are self-adjoint satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}$. Standard choices are

$$\gamma^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \alpha = \begin{pmatrix} +1 & \\ & -1 \end{pmatrix}$$

so that

$$\tfrac{1}{i}\beta\alpha = \tfrac{1}{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \cancel{\begin{pmatrix} 0 & +i \\ i & 0 \end{pmatrix}}\ \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

Now I am interested in the case $m = 0$, in which case

$$\partial_t + H = \begin{pmatrix} \partial_t + \frac{1}{i}\partial_x & \\ & \partial_t - \frac{1}{i}\partial_x \end{pmatrix}$$

and so the operator $\partial_t + H$ is essentially the direct sum of $\partial_{\bar z}$ and ∂_z. The two helicities.

~~I have seen already that the Green's functions~~
~~are simply expressed in terms of $(\partial_t + H)^{-1}$~~
~~So the conclusion is that the fermion~~

The massless Dirac equation in 1-space dim. is therefore very simple: The Hamiltonian is just

$$H = \begin{pmatrix} +\frac{1}{i}\partial_x & \\ & -\frac{1}{i}\partial_x \end{pmatrix}$$

and if we fix the helicity, then we get

$$H = +\frac{1}{i}\partial_x .$$

At this point I can try to understand the physics of this Hamiltonian and its gauge transforms. For example I can put periodic conditions in x and look at a fermion gas of particles governed by H + gauge potential at a given temperature. I can look at the ground energy shift as I vary the potential. These problems involve regularizing the kernel for $\partial_t + H$ along the diagonal, which seems to involve choices. Somehow the physics makes these choices in a definite way which I should try to understand.

August 8, 1982

I consider the Hamiltonian $H_0 = \frac{1}{i}\partial_x$ acting on functions of x, say periodic with period L. Then I want to form $\hat{H}_0$ on the Fock space. This already involves a choice of an additive constant for the ground energy. Next I want to consider a perturbation $H = \frac{1}{i}(\partial_x + A)$, where A is a function of x, and compute the ground energy shift. To first order it is $\langle 0|\delta\hat{H}|0\rangle$ where

$$\delta\hat{H} = \int \psi^*(x)\,\tfrac{1}{i}A(x)\,\psi(x)\,dx$$

so

$$\langle 0|\,\delta\hat{H}\,|0\rangle = \int dx\ \tfrac{1}{i}A(x)\,\langle 0|\psi^*(x)\psi(x)|0\rangle.$$

But this involves $\langle 0|\psi^*(x)\psi(x)|0\rangle$ which is a diagonal part of a Green's function, and therefore has to be defined by a process of regularization.

Formulas: Orthonormal basis $|k\rangle = \frac{1}{\sqrt{L}}e^{ikx}$ $\quad k \in \frac{2\pi}{L}\mathbb{Z}$.

$$\langle x| = \sum_k \langle x|k\rangle\langle k| \qquad \psi(x) = \frac{1}{\sqrt{L}}\sum_k e^{ikx}\psi_k$$

$$\psi^*(x') = \frac{1}{\sqrt{L}}\sum_{k'} e^{-ik'x'}\psi^*_{k'}$$

Take the ground state $|0\rangle$ in Fock space to be filled with all $|k\rangle$ with $k \le 0$. Then

$$\langle \psi(x)\psi^*(y)\rangle = \frac{1}{L}\sum_k e^{ik(x-y)}\underbrace{\langle \psi_k\psi_k^*\rangle}_{1-n_k} \qquad 1-n_k = \begin{cases} 1 & k>0 \\ 0 & k\le 0\end{cases}$$

$$= \frac{1}{L}\sum_{k>0} e^{ik(x-y)} = \frac{1}{L}\,\frac{e^{i\frac{2\pi}{L}(x-y)}}{1-e^{i\frac{2\pi}{L}(x-y)}}$$

$$= \frac{1}{L}\,\frac{1}{e^{-i\frac{2\pi}{L}(x-y)}-1} \xrightarrow[L\to\infty]{\text{as}} -\frac{1}{2\pi i}\,\frac{1}{x-y+i0^+}$$

We see that there are difficulties with the value of

$$\langle \psi^*(x)\psi(y)\rangle = \frac{1}{L}\sum_{k\le 0} e^{ik(x-y)} = \frac{1}{L}\,\frac{1}{1-e^{-i\frac{2\pi}{L}(x-y)}}$$

$$\xrightarrow[L\to\infty]{\text{as}} \frac{1}{2\pi i}\,\frac{1}{(x-y)-i0^+}$$

at $x = y$.

Review the 1-particle operator (derivation on [redacted] Fock space) corresponding to multiplication by fns. of x.

$$\psi^*(x)\psi(x) = \frac{1}{L}\sum_{k,l}' e^{-ikx}\psi_k^* e^{ilx}\psi_l$$

$$= \frac{1}{L}\sum_q' e^{-iqx} \underbrace{\sum_l \psi_{q+l}^* \psi_l}_{\text{call this } \rho_q}$$

Hence

$$\int_0^L dx\, f(x)\,\psi^*(x)\,\psi(x) = \frac{1}{L}\sum_q'\left(\int_0^L dx\, f(x) e^{-iqx}\right)\rho_q .$$

Recall that [redacted] for $q \neq 0$ the infinite sum in the definition of ρ_q makes sense, so we can regard ρ_q as well-defined for $q \neq 0$. For $q = 0$, however, it is the number operator, and we give the special definition

$$\textcircled{*} \qquad \rho_0 = \sum_{k>0} \psi_k^* \psi_k - \sum_{k\leq 0} \psi_k \psi_k^*$$

so that $\rho_0|0\rangle = 0$, where $|0\rangle$ = state where all $|k\rangle$ with $k \leq 0$ are filled.

Review the commutation relations.

$$[\rho_p, \rho_q] = 0 \quad \text{if} \quad p+q \neq 0$$

For $p>0$, $[\rho_p, \rho_{-p}]$ = − the number of k with $0 \leq k < p$

$$= -p\frac{L}{2\pi}.$$

As a check notice that for $p>0$, $\rho_{-p} = \sum_k \psi_{-p+k}^* \psi_k$ kills the ground state $|0\rangle$ [redacted] in which all $k \leq 0$ are filled. Hence ρ_{-q} is a destruction operator

[redacted] Now we have made sense of the operators ρ_q and hence of the operators

$$\hat{f} = \int dx\, f(x)\,\psi^*(x)\,\psi(x)$$

with f smooth. Actually it might be more relevant to think of Fock space as a representation of the central extension of the abelian Lie algebra of these functions $f(x)$.

The next thing is to perturb $\hat{H}_0$ by an operator $\hat{f}$ and ask for the ground energy shift. What seems to be the case is the following. For each f we get then a number

$$E(H_0 + f) = \text{ground energy of } \hat{H}_0 + \hat{f}$$

where $\hat{f}$ is defined using ⊛. This will be a kind of log determinant for $\frac{d}{dt} + \frac{1}{i}\partial_x + f$, and hence should correspond to a regularization process on the Green's function. (Perhaps we can even handle time-dependent perturbations ultimately.)

August 9, 1982

Let's suppose we have a perturbation $H = H_0 + V$ where H_0 is time independent and $V = V(t)$ has compact support, say inside $[-T, T]$. Let $U(t,t')$ be the propagator for

$$(\partial_t + H)\psi = 0$$

and let $|0\rangle$ be the ground state for H_0. I will be interested in the quantity

$$\langle 0|S|0\rangle = \frac{\langle 0|U(T,-T)|0\rangle}{\langle 0|U_0(T,-T)|0\rangle}$$

which I want to interpret as a determinant in the case of independent fermions. Variational formula:

$$\delta \log \langle 0|S|0\rangle = -\int_{-T}^{T} dt \, \frac{\langle 0|U(T,t)\,\delta H(t)\,U(t,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

Now I have to switch to Fock space. I suppose H, H_0, V are 1-particle operators in Fock space, e.g.

$$(*) \qquad V(t) = \sum_{\beta,\alpha} \psi_\beta^* \, V_{\beta\alpha}(t)\, \psi_\alpha .$$

Then the above integrand is

$$\sum_{\beta,\alpha} \frac{\langle 0|U(T,t)\psi_\beta^* \psi_\alpha U(t,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \, \delta H_{\beta\alpha}(t)$$

But we have the Green's fn.

$$G_{\alpha\beta}(t,t') = \begin{cases} \dfrac{\langle 0|U(T,t)\psi_\alpha U(t,t')\psi_\beta^* U(t',-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} & t>t' \\[2ex] -\dfrac{\langle 0|U(T,t')\psi_\beta^* U(t',t)\psi_\alpha U(t,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} & t<t' \end{cases}$$

which we know is an inverse for $\partial_t + H$.
Combining the above leads to

$$\delta \log \langle 0|S|0\rangle = \int_{-T}^{T} dt \sum_{\alpha,\beta}' G_{\alpha\beta}(t^-, t)\, \delta H_{\beta\alpha}(t)$$

and from this we see that $\langle 0|S|0\rangle$ has an interpretation as a determinant.

Now I want to apply this to the case where the operator H_0 is an extension (to Fock space) of the 1-particle operator $\frac{1}{i}\partial_x$ acting on $L^2(\mathbb{R}/\mathbb{Z}\cdot L)$ and where $V(t)$ comes from a multiplication operator. In this case $V(t)$ does not have the form $(*)$, at least as a finite sum.

August 10, 1982

In the case of a constant perturbation $H = H_0 - \mu$ the partition fn. $\mathrm{Tr}(e^{-\beta\hat{H}})$ should be a determinant of $\frac{d}{dt} + H$ on $[0,\beta]$ with anti-periodic bdry. conditions. I have to define $\hat{H}$ on Fock space which involves the choice of an additive constant. Let's do this by requiring that $\hat{H}|\Phi\rangle = 0$ where Φ is the state where all $k \leq 0$ are filled. Then

$$\hat{H} = \hat{H}_0 - \mu\hat{N} = :\sum (k-\mu)\psi_k^*\psi_k:$$

$$= \sum_{k>0} (k-\mu)\psi_k^*\psi_k - \sum_{k\leq 0}(k-\mu)\psi_k\psi_k^*$$

and

$$\mathrm{Tr}(e^{-\beta\hat{H}}) = \prod_{k>0}\left(1+e^{-\beta(k-\mu)}\right)\prod_{k\leq 0}\left(1+e^{\beta(k-\mu)}\right)$$

Compute variation

$$\delta \log \mathrm{Tr}(e^{-\beta\hat{H}}) = \sum_{k>0}\frac{e^{-\beta(k-\mu)}}{1+e^{-\beta(k-\mu)}}\beta\delta\mu$$

$$+ \sum_{k\leq 0}\frac{e^{\beta(k-\mu)}}{1+e^{\beta(k-\mu)}}(-\beta\delta\mu)$$

Consider $\qquad F(\hat{H}) = -\frac{1}{\beta}\log \mathrm{Tr}(e^{-\beta\hat{H}}) \qquad$ as $\beta \to \infty$.

Suppose $\mu > 0$. The second term has $k-\mu \leq -\mu < 0$ so gives 0 in the limit. The first gives

$$\delta F(\hat{H}) \longrightarrow -\sum_{0<k<\mu}\delta\mu + \begin{cases} -\frac{1}{2}\delta\mu & \text{if } \mu \in \frac{2\pi}{L}\mathbb{Z} \\ 0 & \text{---}\notin\text{---}. \end{cases}$$

Do the limit directly for $F(\hat{H})$

$$F(\hat{H}) = -\frac{1}{\beta}\sum_{k>0}\log(1+e^{-\beta(k-\mu)}) + -\frac{1}{\beta}\sum_{k\leq 0}\cdots$$

For $\mu > 0$

$$F(\hat{H}) \longrightarrow + \sum_{0<k<\mu} (k-\mu) = E(\hat{H})$$

Hence the ground energy is discontinuous in μ, probably because the particle number changes. This suggests that I should stick to the finite β situation.

The problem now is to bring in the elliptic functions. Recall the Jacobi triple product identity

$$\prod_{n\geq 0} (1+q^n t) \prod_{n\geq 1} (1+q^n t^{-1})(1-q^n) = \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} (+t)^n$$

We can use this to [redacted] understand the partition fn. $k \in \frac{2\pi}{L}\mathbb{Z}$. Put

$$q = e^{-\beta\frac{2\pi}{L}} = e^{i(\beta\frac{2\pi i}{L})}$$

$$t = e^{-\beta\mu}$$

Then

$$\mathrm{Tr}(e^{-\beta\hat{H}}) = \prod_{n\geq 0}(1+q^n t) \prod_{n\geq 1}(1+q^n t^{-1}).$$

As a function of μ this vanishes for

$$t = -q^n \quad \text{some } n \in \mathbb{Z}$$

$$-\beta\mu \in i\pi + \mathbb{Z}\left(-\frac{\beta}{L}2\pi\right) + 2\pi i\mathbb{Z}$$

$$\boxed{\mu \in \frac{2\pi}{L}\mathbb{Z} + \frac{2\pi i}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}$$

So our periodicity lattice is $\frac{2\pi}{L}\mathbb{Z} + \frac{2\pi i}{\beta}\mathbb{Z}$, and our Θ function is periodic in the $\frac{2\pi i}{\beta}$ direction.

$$\frac{\delta \log \mathrm{Tr}(e^{-\beta\hat{H}})}{-\beta\,\delta(\mu)} = \sum_{k\leq 0} \frac{e^{\beta(k-\mu)}}{1+e^{\beta(k-\mu)}} - \sum_{k>0} \frac{e^{-\beta(k-\mu)}}{1+e^{-\beta(k-\mu)}}$$

$$\frac{\partial}{\partial\mu} F(\hat{H}) = \sum_{n\geq 0} \frac{q^n t}{1+q^n t} - \sum_{n>0} \frac{q^n t^{-1}}{1+q^n t^{-1}}$$

Use the identity

$$1 - \frac{q^n t^{-1}}{1+q^n t^{-1}} = \frac{1}{1+q^n t^{-1}} = \frac{q^{-n} t}{1+q^{-n} t}$$

and one sees that the above $\frac{\partial}{\partial\mu} F(\hat{H})$ is formally (up to an inf. constant)

$$\sum_{n\in\mathbb{Z}} \frac{q^n t}{1+q^n t}$$

This is going to be a constant times the Weierstrass $\wp(\mu)$ with linear factor added to make it periodic in $\frac{2\pi i}{\beta}$ period.

Calculate the residue as a fn. of μ at $\mu = \frac{\pi i}{\beta}$: One has the term $\frac{e^{-\beta\mu}}{1+e^{-\beta\mu}} = \frac{1}{e^{\beta\mu}+1}$

$$\frac{d}{d\mu}(e^{\beta\mu}+1) = \beta e^{\beta\mu}\Big|_{\mu=\frac{\pi i}{\beta}} = -\beta.$$

Thus

$$\frac{\partial}{\partial\mu} F(\hat{H}) = -\frac{1}{\beta}\Big\{ \sum_{\gamma\in\frac{2\pi}{L}\mathbb{Z}+\frac{2\pi i}{\beta}(\mathbb{Z}+\frac{1}{2})} \Big(\frac{1}{\mu-\gamma} + \frac{1}{\gamma} + \frac{\mu}{\gamma^2}\Big)\Big\}$$

+ linear fn of $\mu, \bar{\mu}$.

Review: I want to interpret $\mathrm{Tr}(e^{-\beta\hat{H}})$ as a determinant of $\partial_t + H$ over $[0,\beta]$ with anti-periodic b.c. I use the variational formula

$$\delta \log\det(\partial_t + H) = \mathrm{Tr}\,(\partial_t+H)^{-1}\delta H$$

$$\delta\log\mathrm{Tr}(e^{-\beta\hat{H}}) = -\beta\,\delta\mu\Big\{\sum_{k\leq 0}\frac{e^{\beta(k-\mu)}}{1+e^{\beta(k-\mu)}} - \sum_{k>0}\frac{e^{-\beta(k-\mu)}}{1+e^{-\beta(k-\mu)}}\Big\}$$

In this case $\delta H = -\delta\mu$ and $-(\partial_t + H)$ has the eigenvalues $\mu - \gamma$ $\quad \gamma \in \frac{2\pi}{L}\mathbb{Z} + \frac{2\pi i}{\beta}(\mathbb{Z}+\frac{1}{2})$

so that

$$-\operatorname{Tr}(\partial_t + H)^{-1} \overset{\text{formally}}{=\!=} \sum_{\gamma} \frac{1}{\mu - \gamma}$$

$$= -\beta\left\{ \sum_{k\le 0} \frac{e^{\beta(k-\mu)}}{1+e^{\beta(k-\mu)}} - \sum_{k>0} \frac{e^{-\beta(k-\mu)}}{1+e^{-\beta(k-\mu)}} \right\}$$

So the last expression is the precise regularized version of $-\operatorname{Tr}(\partial_t + H)^{-1}$. What I want now is to understand this regularization process in terms of the actual Schwarz kernel for $(\partial_t + H)^{-1}$.

Since $\partial_t + \frac{1}{i}\partial_x - \mu$ is a constant coefficient operator the Schwarz kernel of its inverse is a function of $x-x'$, $t-t'$ so I can take $x'=t'=0$. Then as a Fourier transf.

$$G(x,t) = \sum_{\substack{k \in \frac{2\pi}{L}\mathbb{Z} \\ \omega \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})}} \frac{e^{ikx + i\omega t}}{i\omega + k - \mu} \qquad \times \underbrace{\text{const}}_{\frac{1}{\beta L}}$$

so the sum is over $\gamma = k + i\omega \in \frac{2\pi}{L}\mathbb{Z} + \frac{2\pi i}{\beta}(\frac{1}{2}+\mathbb{Z})$.

What is the regularization process?

One idea is to take $t = 0^-$. Thus I need to understand

$$G(t) = \sum_{\omega \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})} \frac{e^{i\omega t}}{i\omega + \varepsilon}$$

This is anti-periodic on $[0,\beta]$ and satisfies $(\partial_t + \varepsilon)G = \beta\delta(t)$

Hence $\quad G(t) = Ae^{-t\varepsilon} \qquad 0<t<\beta$

$$\beta = G(0+) - G(0-) = A + G(\beta-) = A(1+e^{-\beta\varepsilon})$$

$$\therefore \; G(t) = \beta \frac{e^{-t\varepsilon}}{1+e^{-\beta\varepsilon}} \qquad\qquad G(0^-) = -\beta \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$$

$$\therefore \; \sum \frac{e^{i\omega 0^-}}{i\omega + \varepsilon} = -\beta \frac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$$

This doesn't seem to do much. I really need a formula for the Green's function so that I can see what the regularization process is.

August 11, 1982

Calculation of Green's function in the elliptic curve case.

$$(\partial_{\bar z} - w)G = \delta(0) = \frac{1}{vol(\Gamma)} \sum_{\mu \in \Gamma^*} e^{\mu\bar z - \bar\mu z}$$

$$G = \frac{1}{vol\,\Gamma} \sum_{\mu} \frac{e^{\mu\bar z - \bar\mu z}}{\mu - w}$$

But one also has the expression

$$G(z) = \frac{1}{\pi} e^{\frac{w}{m}(m\bar z + lz)} \frac{\sigma(z - \frac{w}{m})}{\sigma(z)\sigma(-\frac{w}{m})} \qquad m = \frac{\pi}{vol\,\Gamma}.$$

I want the Green's fn. for $\partial_t + \frac{1}{i}\partial_x - \mu$:

$$G(x\,t) = \frac{1}{L\beta} \sum_{\substack{k \in \frac{2\pi}{L}\mathbb{Z} \\ \omega \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})}} \frac{e^{ikx + i\omega t}}{i\omega + k - \mu}$$

Let's put $\omega = -\frac{\pi}{\beta} + \lambda$

$$G(x\,t) = e^{-i\frac{\pi}{\beta}t} \underbrace{\frac{1}{L\beta} \sum_{\substack{k \in \frac{2\pi}{L}\mathbb{Z} \\ \lambda \in \frac{2\pi}{\beta}\mathbb{Z}}} \frac{e^{ikx + i\lambda t}}{i\lambda + k - \underbrace{(\mu + i\frac{\pi}{\beta})}_{w}}}_{\text{periodic} \;\therefore\; \text{has form}}$$

$$\text{const.} \times e^{\alpha(m\bar z + lz)} \frac{\sigma(z - \alpha)}{\sigma(z)\sigma(-\alpha)}$$

where α is determined by $\underbrace{\left(\partial_t + \frac{1}{i}\partial_x\right)(\alpha m(x - it))}_{\alpha m(-i + \frac{1}{i})} = w$

$$\alpha = \frac{iw}{2m} = \frac{i}{2m}\left(\mu + i\frac{\pi}{\beta}\right) \qquad m = \frac{\pi}{\beta L}$$

Thus the Green's fn. for $\partial_t + \frac{1}{i}\partial_x - \mu$ periodic in x of period L, anti-per. in t of period β is

$$G(xt) = \text{const}\ e^{-i\frac{\pi}{\beta}t} \cdot e^{\alpha(m\bar{z}+lz)} \frac{\sigma(z-\alpha)}{\sigma(z)\sigma(-\alpha)} \qquad \alpha \text{ as above}$$

Expand near $z=0$:

$$G(xt) = \text{const}\ \frac{1}{z}\left\{1 - i\frac{\pi}{\beta}t + \alpha(m\bar{z}+lz) + \frac{\sigma'(-\alpha)}{\sigma(-\alpha)}z + O(z^2)\right\}$$

Let's go back to old notation

$$G(z) = \text{const}\ \frac{1}{z}F(z)$$

and put $t = \frac{z-\bar{z}}{2i}$. Then

$$F(z) = 1 - \frac{\pi}{2\beta}z + \frac{\pi}{2\beta}\bar{z} + \alpha(lz+m\bar{z}) + \zeta(-\alpha)z + \cdots$$

$$F_b(z) = 1 + \left(\frac{\pi}{2\beta} + \alpha m\right)\bar{z} + Rz + \cdots$$

Here R is a constant, at least I want to try a constant ∂-operator lifting the given $\bar{\partial}$-operator. In general I could try an R depending analytically on μ or α. With the above F_b we get

$$\text{F.P. } G(z) = \text{const}\left\{-\frac{\pi}{2\beta} - R + l\alpha - \zeta(\alpha)\right\}$$

Now I have a candidate for this F.P. which is periodic in μ for the period $\frac{2\pi i}{\beta}$, hence periodic in α with period

$$\Delta\alpha = \frac{i}{2m}\Delta\mu = \frac{i}{2m}\frac{2\pi i}{\beta} = -\frac{\pi}{m\beta} = -\frac{\pi}{\beta\,{}^{\pi}/_{L\beta}} = -L$$

Now I know that $\zeta(\alpha) - l\alpha - m\bar{\alpha}$ is doubly-periodic with the periods $L, i\beta$.

Therefore if I want $\zeta(\alpha) - l\alpha + R(\alpha)$ to have the period L, I must have $\cancel{lL} + mL - \cancel{lL} + R(\alpha+L) - R(\alpha) = 0$, or $R(\alpha+L) - R(\alpha) = mL$. The simplest choice for R seems to be

$$R(\alpha) = -m\alpha$$

August 12, 1982

The problem is to calculate the ground energy of $\hat{H}_0 + \hat{f}$ assuming $\hat{H}_0$ chosen so that its ground energy is zero. Here

$$\hat{H}_0 = \sum :k\,\psi_k^*\psi_k: \;=\; \sum_{k>0} k\psi_k^*\psi_k - \sum_{k\leq 0} k\psi_k\psi_k^*$$

and we use normal ordering relative to the ground state Ω where all $k \leq 0$ are filled. $\hat{f}$ is defined using the same normal ordering. $f(x)$ is periodic of period L so

$$f(x) = \sum_k e^{ikx}\underbrace{\frac{1}{L}\int_0^L e^{-ikx} f(x)\,dx}_{c_k}$$

and then

$$\hat{f} = \sum_k c_k \rho_k .$$

The idea will be to use the boson picture in which case $\hat{f}$ is a linear function of creation and annihilation operators, and then we can calculate the ground energy shift by completing the square. For a simple oscillator

$$\begin{aligned} H &= \omega a^* a + ca + \bar{c}a^* \\ &= \omega\left(a^* + \frac{c}{\omega}\right)\left(a + \frac{\bar{c}}{\omega}\right) - \frac{|c|^2}{\omega} \end{aligned}$$

so the ground energy shift is $-\frac{|c|^2}{\omega}$.

$\rho_k = \sum_\ell \psi_{k+\ell}^*\psi_\ell$ kills Ω for $k<0$, so

$$\rho_{-k} = \text{const}\; a_k \qquad [\rho_{-k}, \rho_k] = \frac{L}{2\pi}k$$

$$\rho_k = \text{const}\; a_k^* \qquad \therefore (\text{const})^2 = \frac{Lk}{2\pi}$$

$$\therefore \quad a_k = \sqrt{\frac{2\pi}{Lk}}\,\rho_{-k} \qquad a_k^* = \sqrt{\frac{2\pi}{Lk}}\,\rho_k \qquad k>0$$

and so $$\hat{f} = c_0\rho_0 + \sum_{k>0}{}' \left(c_k \sqrt{\frac{Lk}{2\pi}}\, a_k^* + \bar{c}_k \sqrt{\frac{Lk}{2\pi}}\, a_k \right).$$

Now $H_0 = \sum_{k>0} k\, a_k^* a_k$ on the $N=0$ piece.

Therefore the ground energy shift on the $N=0$ piece of Fock space is

$$\Delta E = -\sum_{k>0}{}' |c_k|^2 \frac{Lk}{2\pi}\frac{1}{k} = -\frac{L}{2\pi}\sum_{k>0}{}' |c_k|^2 .$$

From $f = \sum' c_k e^{ikx} \Rightarrow \frac{1}{L}\int_0^L |f|^2 dx = \sum_k |c_k|^2.$

So

$$\Delta E = -\frac{L}{4\pi}\left\{ \sum_k{}' |c_k|^2 - |c_0|^2 \right\}$$

$$\boxed{\Delta E = -\frac{1}{4\pi}\int_0^L |f|^2 dx + \frac{L}{4\pi}|c_0|^2}$$

f is real

Take $L = 2\pi$ now.

$$\hat{H}_0 = \sum_{n>0} n\,\psi_n^*\psi_n - \sum_{n\le 0} n\,\psi_n\psi_n^*$$

$$\hat{N} = \rho_0 = \sum_{n>0} \psi_n^*\psi_n - \sum_{n\le 0} \psi_n\psi_n^*$$

on $\Omega = |0,-1,-2,\dots\rangle$ we have $\hat{H}_0 = \rho_0 = 0$

on $|-1,-2,\dots\rangle$ we have $\hat{H}_0 = 0$, $\rho_0 = -1$.

On $|n, n-1, n-2, \dots\rangle$ we have $\rho_0 = n$, $\hat{H}_0 = \frac{n(n+1)}{2}$

and since this is the vacuum state in the $\rho_0 = n$ piece we conclude

$$\boxed{\hat{H}_0 = \sum_{n\ge 1} \rho_n\rho_{-n} + \tfrac{1}{2}(\rho_0^2 + \rho_0)}$$

Ground state energy

Suppose $\varphi : S^1 \to U_n$.

Let $H = L^2(S^1; \mathbb{C}^n) = H_+ \oplus H_-$.

and $\mathcal{H} = \Lambda(H_+ \oplus \bar{H}_-)$

The operator $D_0 = -i\frac{d}{d\theta}$ on H induces a hermitian operator D_0 on $\mathcal{H}$ defined up to an additive constant. Choose D_0 on $\mathcal{H}$ so that $D_0\Omega = 0$.

The loop φ defines a unitary operator U_φ on $\mathcal{H}$, defined up to a scalar multiple.

Consider $D_\varphi = U_\varphi D_0 U_\varphi^{-1}$. This is an operator on $\mathcal{H}$ corresponding to $D_0 + i\varphi'\varphi^{-1}$ on H. Its lowest eigenvalue is 0. Another operator on $\mathcal{H}$ corresponding to $D + i\varphi'\varphi^{-1}$ on H is $\tilde{D}_\varphi = D_0 + :i\varphi'\varphi^{-1}:$.

What is the lowest eigenvalue of $\tilde{D}_\varphi$? In other words, what is $\tilde{D}_\varphi - D_\varphi$?

By assumption $\langle \Omega, \tilde{D}_\varphi \Omega\rangle = 0$, so we want

$$-\langle \Omega, D_\varphi \Omega\rangle = -\langle U_\varphi^{-1}\Omega, D_0 . U_\varphi^{-1}\Omega\rangle = D_0^{(\varphi)} - D_0,$$

where $D_0^{(\varphi)}$ is D_0 normal ordered with respect to $U_\varphi^{-1}H_+ \oplus U_\varphi^{-1}H_-$.

<u>Lemma</u> If $A : H \to H$ is hermitian, and A_0, A_1 are the corresponding operators on $\mathcal{H}$ defined by normal ordering with respect to polarizations J_0, J_1 of H (so that $J_i^2 = 1$),

then $$A_0 - A_1 = \tfrac{1}{2} \operatorname{trace} A(J_0 - J_1).$$

We apply this with $A = D_0 = -i\frac{d}{d\theta}$ and $J_0 = U_\varphi^{-1} J U_\varphi$, $J_1 = J$ = standard polarization. Then

$$D_0^{(\varphi)} - D_0 = \tfrac{1}{2} \operatorname{trace} D_0 \left(U_\varphi^{-1} J U_\varphi - J\right)$$
$$= \tfrac{1}{2} \operatorname{trace}\left\{(D_0 U_\varphi^{-1}).\,[J, U_\varphi]\right\}$$

The standard J is the singular integral operator with kernel $\frac{i}{2\pi} \cot \frac{\theta_1 - \theta_2}{2}$; so $[J, U_\varphi]$ is the integral operator with kernel $-\frac{i}{2\pi} \cot \frac{\theta_1 - \theta_2}{2} \left(\varphi(\theta_1) - \varphi(\theta_2)\right)$. Altogether we must calculate the trace of the operator whose kernel is

$$-\frac{1}{4\pi} \frac{\partial}{\partial \theta_1} \left\{ \cot \frac{\theta_1 - \theta_2}{2} \; \varphi(\theta_1)^{-1} \left(\varphi(\theta_1) - \varphi(\theta_2)\right)\right\}$$

$$= +\frac{1}{8\pi} \operatorname{cosec}^2 \frac{\theta_1 - \theta_2}{2} \cdot \varphi(\theta_1)^{-1}\left(\varphi(\theta_1) - \varphi(\theta_2)\right) + \frac{1}{4\pi} \cot \frac{\theta_1 - \theta_2}{2} \; \varphi(\theta_1)^{-1} \varphi'(\theta_1) \varphi(\theta_1)^{-1} \varphi(\theta_2)$$

$$\sim -\frac{1}{2\pi} \frac{1}{(\theta_1 - \theta_2)^2} \left\{ (\theta_2 - \theta_1)\, \varphi(\theta_1)^{-1} \varphi'(\theta_1) + \tfrac{1}{2} (\theta_2 - \theta_1)^2 \varphi(\theta_1)^{-1} \varphi''(\theta_1) + \ldots \right\}$$
$$- \frac{1}{2\pi} \frac{1}{\theta_1 - \theta_2} \left\{ \varphi(\theta_1)^{-1} \varphi'(\theta_1) + (\theta_2 - \theta_1) \left(\varphi(\theta_1)^{-1} \varphi'(\theta_1)\right)^2 + \ldots \right\}$$

$$\to -\frac{1}{4\pi} \left\{ \varphi(\theta)^{-1} \varphi''(\theta) - 2 \left(\varphi(\theta)^{-1} \varphi'(\theta)\right)^2 \right\} \quad \text{on the diagonal.}$$

$$= -\frac{1}{4\pi} \left\{ \frac{d}{d\theta} \left(\varphi^{-1} \varphi'\right) - \left(\varphi^{-1} \varphi'\right)^2 \right\}.$$

So that finally

$$D_0^{(\varphi)} - D_0 = +\frac{1}{4\pi} \int_0^{2\pi} \operatorname{trace} \left(\varphi^{-1} \varphi'\right)^2 d\theta = -\operatorname{Energy}(\varphi).$$

<u>August 14, 1982</u>

Let's go back to trying to prove a differential form version of GRR, when the fibres are curves. Suppose we have

$$\begin{array}{c} X \\ \downarrow{\scriptstyle \pi} \\ Y \end{array}$$

a map π of complex manifolds which is proper, a submersion, and whose fibres are of dimension 1. Also we have a holom. vector bundle E over X. Then we have the $\bar\partial$ operator along the fibres

$$E \xrightarrow{D} E \otimes T^{0,1}_{X/Y}$$

which commutes with functions on Y. This operator will be used to define a virtual holom. bundle $\pi_!(E)$. I suppose chosen metrics on E and $T^{0,1}_{X/Y}$. Then I get an adjoint operator

$$E \xleftarrow{D^*} E \otimes T^{0,1}_{X/Y}$$

and Laplaceans D^*D, DD^* which are self-adjoint in each fibre over a point of Y.

It seems to be simpler to think of the case of a fixed C^∞ surface & vector bundle and then to vary the metrics and holomorphic structures. This way as we vary over Y we get a varying operator $D_y : V_1 \longrightarrow V_0$ which is Fredholm.

August 15, 1982

Goal: Fix a C^∞ vector bundle E over a Riemann surface M and consider the family of all holom. structures on it. This gives then the picture:

$$\begin{array}{lll} X = & \mathcal{A}\times M & \tilde{E} \\ & \downarrow pr_1 = f & \\ Y = & \mathcal{A} & \end{array}$$

GRR says

$$ch(f_! \tilde{E}) = f_*\big(ch\,\tilde{E}\cdot \underbrace{Todd(X/Y)}_{pr_2^*(\underbrace{Todd(T_M)}_{1+\frac{1}{2}c_1 T_M})}\big)$$

Question: Can $f_*(ch\,\tilde{E}\cdot Todd(X/Y))$ have interesting components of degree ≥ 2?

(I suppose E, M equipped with metrics so that these char. classes are given by differential forms, and f_* is just integrating over M.)

Let's go over the local calculation of $ch\,\tilde{E}$. Locally on M we trivialize E by an orthonormal frame, we choose a local coord. z on M and coordinates t on $\mathcal{A}$. Then the holom. structure on $\tilde{E}$ is given by the $\bar\partial$-operator

$$(\partial_{\bar z} + \alpha)\,d\bar z \quad + \partial_{\bar t}\, d\bar t$$

where α is holomorphic in t. (The notation is lousy - one should think of t-space as a finite-diml subspace of $\mathcal{A}$, [redacted] e.g. $\alpha = \alpha_0 + t_1\alpha_1 + \dots + t_n\alpha_n$). The hermitian connection corresponding to this $\bar\partial$-operator is

$$(\partial_z - \alpha^*)\,dz + (\partial_{\bar z} + \alpha)\,d\bar z + \partial_t\, dt + \partial_{\bar t}\, d\bar t$$

and the curvature is

$$K = \left(\partial_z \alpha + \partial_{\bar z}\bar\alpha + [\alpha, \alpha^*]\right) dz\,d\bar z + (\partial_t \alpha)\,dt\,d\bar z - (\partial_{\bar t}\alpha^*)\,d\bar t\,dz.$$

The character of $\tilde{E}$ is given by the differential form

$$tr\; e^{\frac{i}{2\pi}K} = \sum_n \frac{1}{n!}\left(\frac{i}{2\pi}\right)^n tr(K^n).$$

In this case $K^n = 0$ for $n \geq 3$ and

$$K^2 = \sum_{i,j} \partial_{t_i}\alpha\, \partial_{\bar t_j}\alpha^*\; dt_i\, d\bar t_j\, dz\, d\bar z$$

leading to the curvature of the canonical line bundle $c_1(f_! \tilde{E})$, as we've seen before.

More generally suppose I consider a holom. family of Riemann surfaces with holom. vector bundle E over X and suppose metrics chosen on E and $T_{X/Y}$, so that $f_*(\mathrm{ch}\,E \cdot \mathrm{Todd}(X/Y))$ is given as a differential form.

$$\begin{array}{c} X^{E} \\ \downarrow f \\ Y \end{array}$$

Locally on X I can choose coordinates (t_i, z) where the t_i are coords. on Y. Then the curvature of E, or $T_{X/Y}$ will involve the $(1,1)$ forms

$$dz\, d\bar{z} \qquad dt\, d\bar{z} \qquad d\bar{t}\, dz \qquad dt\, d\bar{t}.$$

If none of the $dt_i\, d\bar{t}_j$ forms occurs, then $f_*(\mathrm{ch}\,E \cdot \mathrm{Todd}(X/Y))$ will involve no terms of degree (p,p) with $p \geq 2$. One can arrange this by tensoring with a suitable line bundle lifted from Y.

August 17, 1982

I am looking at the ~~situation~~ of a family of invert. $\bar\partial$ operators $\begin{matrix} X^{E} \\ \downarrow f \\ Y \end{matrix}$, $D: E \longrightarrow E \otimes T^{0,1}_{X/Y}$ and I suppose metrics chosen, so that I get the function $\zeta(s) = \operatorname{Tr}(D^*D)^{-s}$ defined on Y. The torsion $\tau(D^*D) = \exp(-\zeta'(0))$ gives the norm-squared of the canonical section. Now the problem is that I want to compute the curvature form $\bar\partial\partial \log\tau$ on Y.

Let's discuss carefully the problems involved with the idea of getting a good invariant viewpoint. We have over each point $y \in Y$ a Hilbert space $H_y = L^2(X_y; E_y)$ and an operator $(D^*D)_y$ in H_y and I am looking at the function $\operatorname{Tr}(D^*D)_y^{-s}$. Normally one deals with operators in the same space but here we have a bundle situation.

Simper situation. Suppose E is a vector bundle over X and that A is an endomorphism of E. Then $tr(A)$ is a fn. on X and we can calculate its differential. Choose a connection $\nabla: E \longrightarrow E \otimes T^*_X$. Then

$$d\, tr(A) = tr[\nabla, A]$$

and this right side is clearly independent of the choice of ∇. Similarly when A is invertible

$$d \log\det(A) = tr(A^{-1}[\nabla, A]).$$

(To see this at a point x, let A_0 be an autom. of E $= A$ at x and with $[\nabla, A_0] = 0$ at x. Then we have $d\log\det(A_0) = 0$ at x because A_0 is constant to the first order - actually one can restrict attention to X a curve. So $A_0^{-1}A = I + B$ with $B = 0$ at x, and

$$d\log\det(A) = d\log\det(A_0^{-1}A) = d\, tr(B) = tr([\nabla, B])$$
$$= tr[\nabla, A_0^{-1}A] = tr\, A_0^{-1}[\nabla, A] = tr(A^{-1}[\nabla, A])$$

at x.)

Now let's return to my family of $\bar\partial$ operators, or actually to the Laplacean D^*D on X over Y. I have this Hilbert bundle $H_y = L^2(X_y, E_y)$ and the operator $e^{-t(D^*D)_y}$ in it of trace class, and I want to understand how the trace varies in y.

According to the above I need a connection on this bundle. Does there exist a canonical connection obtained from the inner product?

The following example I think is critical. Take X to be the family of elliptic curves $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$ where $Y = \{\tau \mid \operatorname{Im}\tau > 0\}$, take E = trivial line bundle, and $D^*D = -(\partial_z + \bar w)(\partial_{\bar z} - w)$ where w is a constant chosen so that $-D^*D$ is invertible near the $\tau_0 \in Y$ one is working. In this example, relative to the local coords z, τ the operator D^*D is not changing, however the space $H_\tau = L^2(\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau)$ is changing.

Problem: Given $f: X \to Y$ a holom. family of curves and a holom. v.b. E on X, define $f_! E$ as a virtual holom. vector bundle on Y.

There are several ideas. 1) Use a positive divisor $0 \to \mathcal{O} \to \mathcal{L} \to \mathcal{T} \to 0$ with $\mathcal{T}$ torsion such that $R^1 f_*(E \otimes \mathcal{L}) = 0$ on each fibre. Then $f_! E$ is the difference of $f_*(E \otimes \mathcal{L})$ and $f_*(E \otimes \mathcal{T})$ which are holom. v.b. over Y.

2) Grothendieck's general idea that $f_! E$ is the class of the complex $Rf_*(\mathcal{E})$, $\mathcal{E}$ = sheaf of holom. sections of E, which is a perfect complex of $\mathcal{O}_Y$-modules. What this means on the C^∞ level is that one takes $\mathcal{E}$ replaces it by the Dolbeault complex
$$0 \to E \to E \otimes T_X^{0,1} \to E \otimes T_X^{0,2} \to \cdots$$
and applies f_*.

Use
$$0 \longrightarrow f^*T_Y^{0,1} \longrightarrow T_X^{0,1} \longrightarrow T_{X/Y}^{0,1} \longrightarrow 0$$
$$0 \longrightarrow f^*T_Y^{0,q} \longrightarrow T_X^{0,q} \longrightarrow T_{X/Y}^{0,1} \otimes f^*T_Y^{0,q-1} \longrightarrow 0$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & f_*(E) \arrow[d] & f_*(E)\otimes T_Y^{0,1} \arrow[d] & f_*(E)\otimes T_Y^{0,2} \arrow[d] & \\
0 \arrow[r] & f_*(E) \arrow[r] \arrow[d] & f_*(E\otimes T_X^{0,1}) \arrow[r] \arrow[d] & f_*(E\otimes T_X^{0,2}) \arrow[r] \arrow[d] & \cdots \\
 & 0 & f_*(E\otimes T_{X/Y}^{0,1}) \arrow[d] & f_*(E\otimes T_{X/Y}^{0,1})\otimes T_Y^{0,1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Unfortunately the top row $f_*(E)\otimes T_Y^{0,q} = f_*(E\otimes f^*T_Y^{0,q})$ is not a subcomplex of $f_*(E\otimes T_X^{0,\bullet})$, because one could then think of $f_*(E)$ and $f_*(E\otimes T_{X/Y}^{0,1})$ as being infinite-diml holomorphic v.b. over Y. Actually something like this happens where $X = Y\times M$.

August 18, 1982

I want to go over the nature of a parametrix, or Green's function for a $\bar\partial$-operator on a Riemann surface. This is a local story so I might as well start with $D = \partial_{\bar z}$ on $\mathbb{C}$. The standard parametrix is $\frac{1}{\pi(z-z')}$ relative to the volume $dx\,dy$. By a parametrix I mean a kernel $G(z,z')$ such that DG and GD are the identity + smoothing operators on functions with compact support. Now

$$DG\varphi = \partial_{\bar z}\int dz'\, G(z,z')\varphi(z') = \int dz'\left[\partial_{\bar z}G(z,z')\right]\varphi(z')$$

so that we want $\quad \partial_{\bar z}G(z,z') = \delta(z-z') + \text{smooth kernel}$

and hence $\quad \partial_{\bar z}\left[G(z,z') - \frac{1}{\pi(z-z')}\right] = \text{smooth kernel}.$

As $\partial_{\bar z}$ is elliptic we have $G(z,z') - \frac{1}{\pi(z-z')}$ must be smooth.

From past experience it seems useful to write

$$G(z,z') = \frac{F(z,z')}{\pi(z-z')} \quad \text{with } F \text{ smooth} = 1 \text{ when } z = z'.$$

If I expand F is a Taylor series around $z = z'$:

$$F(z,z') = 1 + a_{z'}(z-z') + b_{z'}(\overline{z-z'}) + \ell_{z'}\frac{(z-z')^2}{2} + m_{z'}|z-z'|^2 + n_{z'}\frac{\overline{z-z'}^{\,2}}{2} + \cdots$$

the condition that $\frac{F(z,z')}{\pi(z-z')} - \frac{1}{\pi(z-z')}$ is smooth amounts simply to requiring the coeff. of $\overline{z-z'}^{\,k}$, $k \geq 1$, vanish

$$b_{z'} = 0, \quad n_{z'} = 0, \quad \text{etc.}$$

For the index thm. I need to know the values on the diagonal of $DG - \mathrm{id}$, $GD - \mathrm{id}$.

$$\left[\partial_{\bar z}G(z,z') - \delta(z-z')\right]_{z=z'} = \partial_{\bar z}\left[\frac{1}{\pi}\left\{a_{z'} + \ell_{z'}\frac{z-z'}{2} + m_{z'}\overline{z-z'} + \cdots\right\}\right]_{z=z'}$$

$$= \frac{1}{\pi}m_z$$

$$\left[-G(z,z')\overleftarrow{\partial}_{\bar z'} - \delta(z-z')\right]_{z=z'} = -\partial_{\bar z'}\left[\frac{F(z,z')}{\pi(z-z')} - \frac{1}{\pi(z-z')}\right]_{z=z'}$$

$$= \frac{-1}{\pi}\partial_{\bar{z}'}\left[a_{z'} + l_{z'}\,\frac{z-z'}{2} + m_{z'}\,\overline{z-z'} + \cdots\right]_{z=z'}$$

$$= -\frac{1}{\pi}\partial_{\bar{z}'}a_{z'} + \frac{1}{\pi}m_{z}$$

Hence we see that the contribution to the index comes from the coeff. $a_{z'}$, which is somehow to be related to the finite part of the Green's function on the diagonal.

Now I want to take a global viewpoint in which I have a $\bar\partial$-operator D on a vector bundle E over a Riemann surface M. A parametrix for D will be a kernel $G(z,z')\,dz' \in \mathrm{Hom}\,(E_{z'}\otimes T^{1,0}_{z'}, E_z)$ with the property that in local holomorphic coordinates

$$G(z,z')\,dz' = \frac{i}{2\pi}\,\frac{dz'}{z-z'} + \text{smooth}$$

(The $\frac{i}{2}$ is needed because $\frac{i}{2}dz'd\bar{z}' = dx'dy'$.)

At this point I've described what a parametrix for $D : E \longrightarrow E\otimes T^{0,1}$ is, and once I have it, I get an integral formula for the index. Now what I want to do is to start with metrics on E, M and construct a parametrix which will yield the index thm. using the Chern differential forms associated to these metrics. My idea is that there is something called a flat Green's fn. belonging to the metrics, or at least to the connections.

Here's how to obtain the flat Green's function. Because we have a connection in the tangent bundle to M we have the notion of a geodesic from z' to a nearby point z, and can use the connection in E to lift this geodesic to parallel translation giving us an isomorphism

$$F(z,z') : E_{z'} \xrightarrow{\sim} E_z$$

Also the connection in the tangent bundle gives an "exponential map" i.e. a local diffeomorphism of $T_{z'}$ with a nbd of z'. Call this exp. map

$$e_{z'} : T_{z'} \longrightarrow M$$

and let $\varphi_{z'}$ be $e_{z'}^{-1}$ defined in a nbd of z' followed by an isomorphism $\mathbb{C} \simeq T_{z'}$. Then

$$\frac{i}{2\pi} \frac{F(z,z')}{\varphi_{z'}(z)} d\varphi_{z'}(z') \in \mathrm{Hom}(E_{z'}, E_z) \otimes T_{z'}^{1,0}$$

is independent of the choice of $\mathbb{C} \simeq T_{z'}$. This expression is defined in a tubular nbd. of the diagonal and then can be extended by 0 by first multiplying by a function $= 1$ near the diagonal and with support in a tubular nbd. of the diagonal.

Questions. ① Should this flat Green's fn. be thought of as a WKB Green's function?

② Can this flat Green's fn. be used to prove RR?

③ What is the relation between the obvious Green's function constructed using orthogonal projection methods and the flat one?

Concerning ③: If D is invertible, then the kernel for D^{-1} is not usually flat – this we know from the anomaly formula. Hence the Hodge-type Green's fn. is not usually flat. In fact the Hodge-type Green's fn, call it G^H, is almost an inverse to D, more precisely

$$DG^H = I - \text{proj on } H^0$$
$$G^H D = I - \text{proj on } H^1.$$

One expects the flat Green's function not to be such a good parametrix.

August 19, 1982

My goal is to understand the flat Green's fn. for a $\bar\partial$ operator over a Riemann surface. The first thing is to show that it actually is suitable for proving the RR thm. For this purpose I will need to know something about the exponential map for a Riemann surface.

More generally consider a Riemannian manifold and Kinetic Energy Lagrangian: $L = \frac{1}{2}\|\dot{q}\|^2$. The geodesics are the trajectories. The action is

$$S(qt, q't') = \frac{d(q,q')^2}{2(t-t')}$$

where $d(q,q')$ is the distance between q and q'. In order to see this recall that the action is the integral over the trajectory going from $q't'$ to qt. In this case one sends a particle along the geodesic from q' to q with speed $\frac{d(q,q')}{\Delta t}$ so that it takes the time Δt. Then

$$S = \int \frac{1}{2}\|\dot{q}\|^2 dt = \frac{1}{2}\left(\frac{d(q,q')}{\Delta t}\right)^2 \Delta t = \frac{d(q,q')^2}{2\Delta t}.$$

Now recall the formula that the initial + final momenta are

$$p' = -\frac{\partial S}{\partial q'} \qquad p = \frac{\partial S}{\partial q}$$

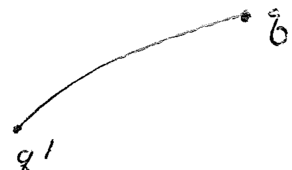
for the trajectory joining $q't'$ to qt. Put $t-t'=1$ and let $S(q,q')$ denote the action which is just $\frac{1}{2}d(q,q')^2$. Then

$$p' = -\frac{\partial S}{\partial q'}(q,q')$$

gives the initial momentum of the trajectory going from q' to q in unit time. In other words the relation $p' \longleftrightarrow q$ expressed by this equation is just the exponential map at the point q'.

August 20, 1982

Yesterday I realized how small geodesics on a Riemannian manifold can be described in terms of the "action" function which is just $\frac{1}{2}d(q,q')^2$. The gradient of this function with respect to q' is the unit tangent vector at q' to the geodesic from q to q'. Hence for q' fixed, the relation:



$$p' = -\partial_{q'}\left\{\tfrac{1}{2}d(q,q')^2\right\}$$

gives the exponential map $p' \longmapsto q$ from the tangent space at q' to M.

Let's now work on finding the flat Green's fn. We have a Riemann surface M with metric $ds^2 = \rho(dx^2+dy^2)$. Hence $\sqrt{\rho}\,dx$, $\sqrt{\rho}\,dy$ is an orthonormal frame in T^*, so

$$\|df\|^2 = \|\partial_x f\,dx + \partial_y f\,dy\|^2 = \frac{1}{\rho}\left[(\partial_x f)^2 + (\partial_y f)^2\right]$$

$$= \frac{4}{\rho}|\partial_z f|^2 \qquad \text{for a real fn. } f$$

The Hamilton-Jacobi equation for $H = \frac{1}{2}\|p\|^2$ is

$$\partial_t S + \frac{1}{2}\|d_q S\|^2 = 0,$$

and so if $S = \frac{u(q)}{t}$, then we get simply

$$\frac{1}{2}\|du\|^2 = u \qquad \text{or} \qquad \|\text{grad}\, u\|^2 = 2u.$$

This has the solution $u = \frac{1}{2}r(q,q')^2$ clearly. Thus

$$\frac{4}{\rho}|\partial_z u|^2 = 2u \qquad \text{or} \qquad |\partial_z u|^2 = \left(\frac{\rho}{2}\right)u$$

is the HJ equation.

In order to ease the calculations, let's remove the factors of 2 and put $v = r(q)^2$ so that the HJ eqn. is

$$|\partial_z v|^2 = \rho\, v.$$

Question: Is v divisible by $|z|^2$ as a C^∞ fn.?

Let try putting $v = z\bar{z} f(z)$. Then

$$\partial_z v = \bar{z} f + z\bar{z}\,\partial_z f = \bar{z}(f + z\partial_z f)$$

and so we ~~solve~~ want to solve

$$(*) \qquad (f + z\partial_z f)(f + \bar{z}\partial_{\bar{z}} f) = \rho f.$$

I know that v is a C^∞ function of z, and hence it has a power series expansion $v = \sum c_{mn} z^m \bar{z}^n$. The question of whether it is divisible by $z\bar{z}$ is the same as whether $c_{mn} = 0$ if $m = 0$ or $n = 0$. Hence it should be possible to prove the divisibility of v by $z\bar{z}$ by exhibiting a power series solution of $(*)$. Start with the series for ρ

$$\rho(z) = \rho_0 + \alpha z + \bar{\alpha}\bar{z} + \frac{\beta}{2} z^2 + \gamma z\bar{z} + \frac{\bar{\beta}}{2}\bar{z}^2 + \cdots$$

where by rescaling we can suppose that $\rho_0 = 1$. Put

$$f(z) = 1 + az + \bar{a}z + \frac{b}{2} z^2 + cz\bar{z} + \frac{\bar{b}}{2}\bar{z}^2 + \cdots$$

Then $$f + z\partial_z f = 1 + 2az + \bar{a}z + \left(\frac{b}{2} + b\right) z^2 + 2c\bar{z} + \frac{\bar{b}}{2}\bar{z}^2 + \cdots$$

and so we see that

$$(f + z\partial_z f)(f + \bar{z}\partial_{\bar{z}} f)\, f^{-1}$$

is a series concocted simply from the coefficients of f. I want to see that there is a unique way to choose the coefficients of f so as to get the series ρ. Assume this has been proved in degrees $< p$. Put $f = \sum c_{mn} z^m \bar{z}^n$. Then the c_{mn} for $m+n < p$ are determined

$$f + z\partial_z f = \sum (c_{mn} + m c_{mn}) z^m \bar{z}^n.$$

The new part in the above product of degree p is

$$\sum_{m+n=p}' c_{mn}(1+m) z^m \bar{z}^n + \sum_{m+n=p} c_{mn}(1+n) z^m \bar{z}^n - \sum_{m+n=p}' c_{mn} z^m \bar{z}^p$$

$$= \sum_{m+n=p} c_{mn}(1+m+n) z^m \bar{z}^n$$

and since $1+m+n \neq 0$, these c_{mn} exist and are unique.

Outline of the proof that $v = r(z)^2$ is divisible by $|z|^2$ as a C^∞ function. v is C^∞ because it comes from the exponential map which know is a local diffeom. So v has a formal power series expansion at $z=0$ satisfying the HJ eqn $\partial_z v \, \partial_{\bar z} v = \rho v$. But this equation has a unique power series solution ~~starting~~ with the 2nd degree terms $\rho_0 z\bar{z}$ (analogous argument to the above) and it has all terms divisible by $z\bar{z}$. So we see the power series of v is divisible by $z\bar{z}$ which in view of the usual Remainder estimates should be enough to guarantee that $v/|z|^2$ is C^∞.

August 21, 1982

More struggle with the flat Green's fn. Take the case of the trivial line bundle in which case the flat Green's function is

$$\frac{\rho(z')\,dz'\,d\bar z'}{-\partial_{\bar z'}\, r(z,z')^2 \;\; d\bar z'}$$

It is clear this is well-defined globally near the diagonal. I still have to see that it is a parametrix for $\bar\partial$.

Yesterday I saw that

$$r(z,z')^2 = |z-z'|^2 f(z,z')$$

where f is smooth and $f(z,z) = \rho(z)$.

$$\begin{aligned} -\partial_{\bar z'}\, r(z,z')^2 &= (z-z') f(z,z') - |z-z'|^2\, \partial_{\bar z'} f(z,z') \\ &= (z-z')\underbrace{\{ f - \overline{(z-z')}\,\partial_{\bar z'} f\}}_{\text{smooth fn } = \rho \text{ on } z=z'.} \end{aligned}$$

Hence we see that the flat Green's function is of the form

$$h(z,z')\frac{dz'}{z-z'}, \qquad \text{where } h \text{ is smooth } = 1 \text{ on } z=z'.$$

Now let's put $z'=0$ to simplify. A true parametrix for $\partial_{\bar z}$ with singularity at $z'=0$ satisfies

$$\pi\, \partial_{\bar z} G(z) = \delta(z) + \text{smooth fn.}$$

$$\pi\, \partial_{\bar z}\left[G(z) - \frac{1}{\pi z}\right] = \text{smooth} \qquad \text{whence by Weyl's lemma}$$

$$\pi G(z) = \frac{1}{z} + \text{smooth}$$

A function like $\dfrac{h(z)}{z} = \dfrac{1 + h_1 z + h_{\bar 1}\bar z + \cdots}{z}$

$$= \frac{1}{z} + h_1 + h_{\bar 1}\frac{\bar z}{z} + h_2\frac{z}{2} + h_{1\bar 1}\bar z + h_{\bar 2}\frac{\bar z^2}{2z} + \cdots$$

will be a parametrix only if $h_{\bar 1}, h_{\bar 2}, \cdots = 0$.

For the purposes of the index thm. we want

$\partial_{\bar z} G - \delta$ to have a value at $z=0$ only, so that it is enough to require h_T and $h_{\bar z} = 0$. (Perhaps even $h_{\bar z}$ could be $\neq 0$ as any symmetric averaging process will assign to $h_{\bar z} \frac{\bar z}{z}$ the value 0.)

So the problem arises as to the exact nature of the singularity of the flat Green's fn.

Let's review the equations. We put

$$r^2(z,z') = |z-z'|^2 f(z,z')$$

$$\begin{cases} \partial_z r^2 = \overline{(z-z')}\left(f + (z-z')\partial_z f\right) \\ \partial_{\bar z} r^2 = (z-z')\left(f + \overline{z-z'}\,\partial_{\bar z} f\right) \end{cases}$$

and want $\partial_z r^2 \, \partial_{\bar z} r^2 = \rho r^2$ or

$$\left[f + (z-z')\partial_z f\right]\left[f + \overline{(z-z')}\,\partial_{\bar z} f\right] = \rho(z) f(z,z')$$

$$1 + (z-z')\partial_z \log f = \frac{\rho(z)}{f + \overline{z-z'}\,\partial_{\bar z} f}$$

$$\frac{1}{z-z'} + \partial_z \log f = \frac{\rho(z)}{\partial_{\bar z} r(z,z')^2}$$

Now interchange z and z' and use that f is symmetric and you get

$$\frac{1}{z-z'} - \partial_{z'} \log f = \frac{\rho(z')}{-\partial_{\bar z'} r(z,z')^2}$$

From this we see that the flat Green's fn. is a parametrix in the strong sense. In fact one sees immediately the desirability of working directly with $\log f$ instead of r^2. $\log f$ might be related to the Green's function for the Laplacean.

This can be developed as follows. In $\mathbb{C}$ the Green's function for $\Delta = 4\partial_{\bar z}\partial_z$ is $\frac{1}{2\pi}\log r = \frac{1}{4\pi}\log r^2$, hence the Green's fns. for $\partial_{\bar z}\partial_z$ is $\frac{1}{\pi}\log r^2$, and so the Green's fn. for $\partial_{\bar z}$ is $\partial \frac{1}{2\pi}\log|z|^2 = \frac{1}{\pi z}$. The obvious parametrix for $-(\bar\partial)^* \bar\partial$ is $\frac{1}{\pi}\log r(z,z')^2 = \frac{1}{\pi}\log|z-z'|^2 + \frac{1}{\pi}\log f(z,z')$, but the order is slightly wrong.

Simpler derivation. We start with the fn. $r(z,z')^2$ intrinsically defined in a nbd. of the diagonal. We also know that it is of the form locally

$$r(z,z')^2 = |z-z'|^2 f(z,z')$$

where f is smooth. Then look at

$$-\partial' \log(r^2) = -\partial_{z'} \log\left(|z-z'|^2 f(z,z')\right) dz'$$

$$= \left(\frac{1}{z-z'} - \partial_{z'} \log f(z,z')\right) dz'$$

which is intrinsically defined in a nbd. of the diagonal, and is clearly a parametrix for $\bar\partial$.

Now that we have the parametrix it should be possible to derive the index thm. Let's review the calculation on p 28+29.

$$G(z,z')dz' = \frac{i}{2\pi}\left\{\frac{1}{z-z'} + a_{z'} + \overset{b_{z'}(z-z') + c_{z'}(\bar z - \bar z') + O((z-z')^2)}{\cancel{l_{z'}\frac{z-z'}{2} + m_{z'}(\overline{z-z'}) + \cdots}}\right\}dz'$$

Then

$$\partial_{\bar z} G(z,z') = \frac{i}{2}\,\delta(z-z') + \frac{i}{2\pi}\, c_{z'} + O(z-z')$$

$$-\partial_{\bar z'} G(z,z') = \frac{i}{2}\,\delta(z-z') - \frac{i}{2\pi}\,\partial_{\bar z'} a_{z'} + \frac{i}{2\pi}\, c_{z'} + O(z-z')$$

Hence the index is given by integrating

$$\frac{i}{2\pi}\,\partial_{\bar z'} a_{z'}\; dz' d\bar z'.$$

In the present case

$$a_{z'} = -\partial_{z'} \log f(z,z')\Big|_{z=z'}$$

Because f is symmetric we have

$$\partial_{z'} f(z',z') = \left[\partial_z f(z,z') + \partial_{z'} f(z,z')\right]_{z=z'}$$

$$= 2\,\partial_{z'} f(z,z')\Big|_{z=z'}$$

hence
$$a_z = -\frac{1}{2}\partial_z \log f(z,\bar z) = -\frac{1}{2}\partial_z \log \rho(z).$$

Now recall that $|dz|^2 = |dx|^2 + |dy|^2 = \frac{2}{\rho}$ and hence the curvature form for the cotangent bundle is
$$\bar\partial\partial \log |dz|^2 = -\partial^2_{\bar z z} \log\left(\frac{2}{\rho}\right) dz d\bar z = \partial^2_{\bar z z} \log(\rho)\, dz d\bar z.$$

Thus
$$\begin{aligned}\partial_{\bar z} a_z \, dz d\bar z &= -\frac{1}{2}\partial^2_{\bar z z} \log \rho(z)\, dz d\bar z \\ &= -\frac{1}{2} \text{ curvature form of } T^{1,0}\end{aligned}$$

[redacted] And so
$$\begin{aligned}\int \frac{i}{2\pi} \partial_{\bar z} a_z \, dz d\bar z &= -\frac{1}{2} \deg K = -\frac{1}{2}(2g-2) \\ &= 1-g\end{aligned}$$

which gives RR.

<u>Remark</u>: ① There is an analogy between constructing a parametrix for $\bar\partial$ and constructing [redacted] the Weierstrass $\wp$-function, or rather a periodic version of it. To do the global construction one must introduce some non-analyticity which is then detected later as a topological characteristic class.

② The above proof of RR using the flat Green's fn. should be related to the one using the ζ-fn. or heat kernel. [redacted] Both seem to use a kind of WKB construction which in the end carries all the information.

August 22, 1982

Yesterday I constructed a parametrix for $\bar\partial$ over a Riemann surface:

$$\frac{i}{2\pi}\Big[\underbrace{-\partial_{z'}\log\big(r(z,z')^2\big)}_{\frac{1}{z-z'}-\partial_{z'}\log v(z,z')}\Big]dz'$$

where $r(z,z')^2 = |z-z'|^2\, v(z,z')$.

There seems to be a close connection between this parametrix and the WKB approximation for the heat kernel e^{-tD^*D}. Here are some formal ideas:

$$\frac{1}{\Gamma(s)}\int_0^\infty e^{-tD^*D}\, t^s\,\frac{dt}{t} = (D^*D)^{-s}$$

Take $s=0$: $$\int_0^\infty e^{-tD^*D}\,dt = (D^*D)^{-1}$$

Let's ~~substitute~~ the WKB version for e^{-tD^*D}

$$\langle z|e^{-tD^*D}|z'\rangle = e^{-\frac{r(z,z')^2}{2t}}\frac{1}{\pi t}\,\theta(z,z')\,[1+O(t)].$$

Maybe we get a parametrix for D^*D by using a cutoff

$$\int_0^\varepsilon e^{-tD^*D}\,dt = \frac{1-e^{-\varepsilon D^*D}}{D^*D}$$

since what we omit is smooth: $e^{-\varepsilon D^*D}(D^*D)^{-1}$.

So
$$\int_0^\varepsilon e^{-\frac{r^2}{2t}}\frac{1}{\pi t}\,dt = \int_{r^2/\varepsilon}^\infty e^{-t/2}\frac{1}{\pi t}\,dt = \int_{r^2/\varepsilon}^1 + \underbrace{\int_1^\infty}_{\text{const.}}$$

$$= -\frac{1}{\pi}\log r^2 + O(1)$$

Hence to a first approximation the parametrix for e^{-tD^*D} yields the parametrix for $(D^*D)^{-1}$.

The parametrix is $\frac{i}{2\pi} G(z,z')\,dz'$ where

$$G(z,z') = \frac{1}{z-z'} - \partial_{z'} \log v(z,z')$$

$$= \frac{1}{z-z'} + a_{z'} + b_{z'}(z-z') + c_{z'}\overline{(z-z')} + O((z-z')^2)$$

I've already seen that $a_z = \frac{-1}{2}\partial_z \log\rho(z)$. Formulas for b_z and c_z might be useful, especially since c_z occurs in the diagonal values of $DP-Id$ and $PD-Id$.

Calculations: First compute $\partial_z \log v(z,z')$ with $z'=0$.

$$r^2 = |z|^2 v = |z|^2 \rho_0 e^g$$

$$\partial_z r^2 = \bar z \rho_0 e^g + |z|^2 \rho_0 e^g \partial_z g = \bar z \rho_0 e^g (1 + z\partial_z g)$$

$$\partial_{\bar z} r^2 = z \rho_0 e^g (1 + \bar z \partial_{\bar z} g)$$

$$\overbrace{\rho\, r^2}^{\rho_0 e^g |z|^2} = \partial_z r^2 \partial_{\bar z} r^2 = |z|^2 \rho_0^2 e^{2g} (1 + z\partial_z g)(1 + \bar z \partial_{\bar z} g)$$

$$\rho = \rho_0 e^g (1 + z\partial_z g)(1 + \bar z \partial_{\bar z} g)$$

$$\log\rho = \log\rho_0 + g + z\partial_z g - \tfrac12 (z\partial_z g)^2 \cdots$$
$$\qquad + \bar z \partial_{\bar z} g - \tfrac12 (\bar z \partial_{\bar z} g)^2 \cdots$$

Put

$$g = g_1 z + g_{\bar 1} \bar z + g_2 \frac{z^2}{2} + g_{1\bar 1} z\bar z + g_{\bar 2} \frac{\bar z^2}{2} + \cdots$$

$$z\partial_z g = g_1 z + 2 g_2 \frac{z^2}{2} + g_{1\bar 1} z \bar z$$

$$-\tfrac12 (z\partial_z g)^2 \qquad -\tfrac12 g_1^2 z^2$$

$$\bar z \partial_{\bar z} g = g_{\bar 1} \bar z + g_{1\bar 1} z\bar z + g_{\bar 2} \bar z^2$$

$$-\tfrac12 (\bar z \partial_{\bar z} g)^2 = -\tfrac12 g_{\bar 1}^2 \bar z^2$$

$$\log\rho = \log\rho_0 + 2 g_1 z + 2 g_{\bar 1} \bar z + (3g_2 - g_1^2)\frac{z^2}{2} + 3 g_{1\bar 1} z\bar z + (3\bar g_2 - \bar g_1^2)\frac{\bar z^2}{2} + \cdots$$

$$g_1 = \tfrac12 (\partial_z \log\rho)_0 \qquad g_2 = \tfrac13\left[(\partial_z^2 \log\rho)_0 + \tfrac14 (\partial_z \log\rho)_0^2\right]$$

$$g_{1\bar 1} = \tfrac13 (\partial^2_{z\bar z} \log\rho)_0$$

I want $\quad \partial_z \log v = \partial_z g = g_1 + g_2 z + g_{1\bar{1}} \bar{z} + \cdots$

and get

$$\partial_z \log v(z,z') = \frac{1}{2}(\partial_z \log\rho)\big|_{z'} + \frac{1}{3}\left[(\partial_z^2 \log\rho)\big|_{z'} + \frac{1}{4}(\partial_z \log\rho)\big|_{z'}^2\right](z-z') + \frac{1}{3}(\partial^2_{z\bar{z}} \log\rho)_{z'}\,\overline{(z-z')} + \cdots$$

Now interchange z, z' and use

$$\frac{1}{2}\partial_z \log\rho = \frac{1}{2}(\partial_z \log\rho)_{z'} + \frac{1}{2}(\partial_z^2 \log\rho)_{z'}(z-z') + \frac{1}{2}(\partial^2_{\bar{z}z} \log\rho)_{z'}\,\overline{(z-z')}$$

and get

$$\partial_{z'} \log v(z,z') = \frac{1}{2}(\partial_z \log\rho)_{z'} + \left[\frac{1}{6}(\partial_z^2 \log\rho)_{z'} - \frac{1}{12}(\partial_z \log\rho)_{z'}^2\right](z-z') + \frac{1}{6}(\partial^2_{z\bar{z}} \log\rho)_{z'}\,\overline{(z-z')}$$

yielding

$$\boxed{\begin{aligned} G(z,z') &= \frac{1}{z-z'} + a_{z'} + b_{z'}(z-z') + c_{z'}\overline{(z-z')} + \cdots\cdots \\ a_{z'} &= -\frac{1}{2}(\partial_z \log\rho)_{z'} \qquad b_z = -\frac{1}{6}(\partial_z^2 \log\rho)_{z'} + \frac{1}{12}(\partial_z \log\rho)_{z'}^2 \\ c_{z'} &= \frac{-1}{6}(\partial^2_{z\bar{z}} \log\rho)_{z'} \end{aligned}}$$

The reason I find this interesting is that

$$\int \frac{i}{2\pi}\, c_{z'}\, dz'\, d\bar{z}' = \frac{1}{6}(2-2g) = \frac{1}{3}(1-g)$$

which is the constant found in the value of $\zeta(0)$.

I still don't understand very well what a parametrix is. The above construction is pretty computational. When I get to the heat kernel the computation becomes even worse. Therefore I want to analyze directly what it means to construct a parametrix. These objects exist locally and one has to combine the local gadgets to obtain a global one.

Let's interchange z, z'. Then the problem is to construct a singular differential form

$$\left\{ \frac{1}{z-z'} + \text{smooth}(z,z') \right\} dz$$

having residue 1 at z' and depending smoothly on z'. This gives a natural fibre space over M, namely to each z' we can associate germs of meromorphic differentials at z' having a simple pole with residue $= 1$ at z'. We can work modulo differential forms vanishing at z'. Then we get a fibre bundle consisting of forms

$$\left\{ \frac{1}{z-z'} + a \right\} dz \qquad \text{mod } \mathfrak{m}_{z'} dz .$$

This is an affine bundle over Ω^1 and hence determines a class in $H^1(\Omega^1)$. This extension

$$0 \longrightarrow \Omega^1 \longrightarrow \quad \longrightarrow \mathcal{O} \longrightarrow 0 \qquad ?$$

has to be non-trivial holomorphically, as we have seen that $\bar\partial a$ gives the topological obstruction.

Try to describe more globally. At a given point z' we have $\mathfrak{m}_{z'}^{-1}$ = germs of functions with at most a simple pole at z' and $\mathfrak{m}_{z'}^{-1} \otimes_{\mathcal{O}_{z'}} \Omega^1_{z'} = \mathfrak{m}_{z'}^{-1} \Omega^1_{z'}$ = germs of differentials with simple pole at most at z'. Since $\mathfrak{m}_{z'}/\mathfrak{m}_{z'}^2 = \Omega^1_{z'}$ it follows that $\mathfrak{m}_{z'}^{-1} \Omega^1_{z'} / \Omega^1_{z'} = \mathcal{O}_{z'}/\mathfrak{m}_{z'} = \mathbb{C}$. Thus

$$gr_q^{\mathfrak{m}_{z'}} \left(\mathfrak{m}_{z'}^{-1} \Omega^1_{z'} \right) = \mathfrak{m}_{z'}^q / \mathfrak{m}_{z'}^{q+1} .$$

As z' varies we get vector bundles over M.

These can be described as follows:

$$gr_q^{\mathcal{I}_\Delta} \left(\mathcal{I}_\Delta^{-1} \otimes pr_2^* \Omega^1_M \right) = \mathcal{I}_\Delta^q / \mathcal{I}_\Delta^{q+1} = \left(\Omega^1_M \right)^{\otimes q}$$

Next project: Anomaly formula for denominator-free Greens functions. Suppose given $D: V_1 \to V_0$ Fredholm whence we get a line

$$\mathcal{L}_D \subset \mathrm{Hom}^{(p)}(\Lambda V_0, \Lambda V_1) \qquad p = \mathrm{Ind}(D)$$

We choose a generator for this line whence we get a linear map $\Lambda V_0 \longrightarrow \Lambda V_1$ of degree p, whose matrix elements will be called denominator-free Green's functions. Use notation $\Lambda(D^{-1})$ for this map, since that is what it is when D is invertible. Fermion integration formula:

$$\int [\mathcal{D}\bar{\psi}\,\mathcal{D}\psi]\; e^{-\bar{\psi}D\psi}\, \psi(x_1)\cdots\psi(x_k)\,\bar{\psi}(y_1)\cdots\bar{\psi}(y_\ell)$$

$$= \langle \psi(x_1)\cdots\psi(x_k) \,|\, \Lambda(D^{-1}) \,|\, \bar{\psi}(y_1)\cdots\bar{\psi}(y_\ell) \rangle .$$

Here $\psi(x_i) \in V_1^*$, $\bar{\psi}(y_j) \in V_0$ and

$$\bar{\psi}D\psi \in V_1^* \otimes V_0$$

$$e^{-\bar{\psi}D\psi} \in \Lambda V_1^* \otimes \Lambda V_0 .$$

The above formula, which is purely formal, does not specify the diagonal matrix elements, and in the case where D is invertible we have a regularization process.

August 23, 1982

Project: Anomaly formula for denominator-free Green's functions. Begin with linear algebra. Suppose given $A : W \longrightarrow V$. Then we can view A as an element of $W^* \otimes V$ which sits inside $\Lambda^2(W^* \oplus V)$ and form

$$e^A \in \Lambda(W^* \oplus V) = (\Lambda W)^* \otimes \Lambda V.$$

Suppose we are given bases of V, W. The natural way to write A as a linear transformation is

$$A = \sum |j\rangle\langle j|A|i\rangle\langle i|$$

where $|i\rangle$ is the [redacted] short notation for $w_i \in W$
$|j\rangle$ ——————————— $v_j \in V$ etc.

[redacted] So $A = \sum v_j a_{ji} w_i^* = \sum a_{ji} v_j w_i^*$ [redacted] $\in V \otimes W^*$. [redacted] I associate to this the element of $\Lambda^2(W^* \oplus V)$

$$A \longmapsto \sum a_{ji} w_i^* \wedge v_j .$$

This is the convention I have to use, namely, the embedding

$$\begin{aligned} W^* \otimes V &\subset \Lambda^2(W^* \oplus V) \\ w^* \otimes v &\longmapsto w^* \wedge v . \end{aligned}$$

Why? Because that

$$\delta \log \int e^A = \frac{\int e^A \delta A}{\int e^A} = \sum \frac{\int e^A w_i^* v_j}{\int e^A} \delta a_{ji} = \operatorname{Tr}(A^{-1}\delta A)$$

and so

$$\boxed{(A^{-1})_{ij} = \frac{\int e^A w_i^* v_j}{\int e^A}}$$

Finally $\sum a_{ji} w_i^* \wedge v_j = - \sum v_j a_{ji} w_i^*$ in the exterior algebra which is why one sees it written

$-\psi^* A \psi$ where ψ destroys
ψ^* creates.

General anomaly picture: Suppose we have an elliptic operator $D: E \to F$ and an automorphism φ of E, F commuting with the symbol of D. We have attached a line $\mathcal{L}_D$ to D and similarly a line $\mathcal{L}_{\varphi D \varphi^{-1}}$ to $\varphi D \varphi^{-1}$, and we have a canonical isomorphism

$$\mathcal{L}_D \simeq \mathcal{L}_{\varphi D \varphi^{-1}}.$$

For example, if D is invertible then both lines are canonically trivial, and the above isomorphism is compatible with the trivializations. On the other hand using analytic torsion, each line has [redacted] a metric and the above canonical isomorphism doesn't preserve the metric, so we get a positive real number attached to φ.

Let's examine this carefully in the invertible case, where we know $\mathcal{L}_D$ has a canonical generator whose norm-squared is the analytic torsion

$$\tau = e^{-\zeta'_{D^*D}(0)}.$$

Replace φ by an infinitesimal autom., i.e. an endom. $\delta\varphi$ of (E,F). Then

$$\delta \log \tau = -\delta \zeta'_{D^*D}(0)$$

and in good cases like $\bar\partial$, [redacted] I know that

$$-\delta\zeta'(0) = \operatorname{Tr}\left((D^*D)^{-s} (D^*D)^{-1} \delta(D^*D) \right) \Big|_{s=0}.$$

$=$ constant term in the $t \downarrow 0$ asymp. exp. for $\operatorname{Tr}\left(e^{-tD^*D} (D^*D)^{-1} \delta(D^*D) \right)$.

$=$ const. term of $\operatorname{Tr}\left(e^{-tD^*D} D^{-1} \delta D \right) + \text{c.c.}$

Now $\delta D = [\delta\varphi, D]$, so ultimately we get an expression for $\delta \log \tau$ [redacted] as an integral of $\delta\varphi$ against a [redacted] quantity depending on D.

For the moment let's work out what happens for a general $\bar\partial$ operator on a Riemann surface, not just an invertible one where I more or less understand what happens. Review first the steps when D is invertible.

$$\mathrm{Tr}\left(e^{-tD^*D}D^{-1}\delta D\right) \xrightarrow{\text{as } t\to 0} \int tr(J\delta D)$$

where J is the finite part of the kernel for D^{-1} on the diagonal constructed using the flat Green's fn.

$$J = G - G_b \quad \text{restricted to } \Delta M.$$

Then when $\delta D = [\delta\varphi, D]$ we get

$$\delta \log \tau = \underbrace{\int tr\left(J[\delta\varphi, D]\right)}_{\int tr\left([D,J]\delta\varphi\right)} + \text{c.c.}$$

and $$[D,J] = -[D,G_b]\Big|\Delta = -(\text{R.R. form for } D)$$

The goal is to understand the anomaly business when D is not invertible. Hence I want to use the line $\mathcal{L}_D \subset \mathrm{Hom}(\Lambda V, \Lambda W)$. Because D is usually only densely-defined it is nice to think of D in terms of its graph

$$\Gamma_D \subset W \times V$$

which is a closed subspace such that the projection

$$pr_2 : \Gamma_D \longrightarrow V$$

is Fredholm. I like to think of $\mathcal{L}_D$ as the line in $\Lambda(W\oplus V)$ belonging to the subspace Γ_D, but this is imprecise. One has

$$\mathcal{L}_D \subset \mathrm{Hom}(\Lambda V, \Lambda W) \supset \Lambda(W\oplus V; V) = \Lambda W \otimes \Lambda V^*$$

I thought that the excellent situation occurred when $\mathcal{L}_D$ is in the ℓ^2-Fock space $\Lambda W \otimes \Lambda V^*$ because then it inherits a metric. But this metric seems to be the wrong one. For example when D is invertible $\mathcal{L}_D$ is spanned by $\Lambda(D^{-1}) : \Lambda V \longrightarrow \Lambda W$ and the good metric on $\mathcal{L}_D$ is given by

$$|\Lambda(D^{-1})|^2 = \det(D^*D)^{-1}.$$

On the other hand the natural inner product on $\mathrm{Hom}(\Lambda V, \Lambda W)$ is $\mathrm{Tr}(A^*B)$. Hence

$$\|\Lambda(D^{-1})\|^2 = \mathrm{Tr}\left(\Lambda(D^{-1})^* \Lambda(D^{-1})\right)$$
$$= \det\left[1 + (DD^*)^{-1}\right].$$

It might be useful to understand these two functions in the situation Graeme works with. [redacted] He considers subspaces Γ of $W \oplus V$ which are ℓ^2-commensurable with V in some sense, and hence which give rise to a line in the Fock space $\Lambda W \otimes \Lambda V^*$. Such subspaces are called polarizations, and each gives then a "vacuum vector" in the Fock space. One should think of these vectors as like the Gaussian functions in the metaplectic representation. By associating to Γ the corresponding line in the Fock space, we get a canonical line bundle $\mathcal{L}$ over the Grassmannian of these polarizations.

Now restrict ~~attention~~ to those Γ such that $pr_2 : \Gamma \xrightarrow{\sim} V$, whence Γ is the graph of a $T : V \longrightarrow W$. This is the fat open set [redacted] in the Grassmannian of Γ of index 0 relative to V. Choose orthonormal bases for V, W such that $Tv_i = \lambda_i w_i$ whence Γ is spanned by $(Tv_i, v_i) = (\lambda_i w_i, v_i)$ and hence the line $\mathcal{L}_\Gamma$ in Fock space corresp. to Γ has the generator $u_\Gamma = \bigwedge_i (v_i + \lambda_i w_i)$.

$\mathcal{L}_\Gamma$ has a unique element u_Γ such that $\langle \Omega | u_\Gamma \rangle = 1$ and one has

$$\|u_\Gamma\|^2 = \prod (1+\lambda_i^2) = \det(1+T^*T)$$

It seems that there is no way to obtain a version of $\det(D^*D)^{-1} = \det(T^*T)$ from the Fock space situation. All I have is this line bundle $\mathcal{L}^*$ over the Grassmannian with a section s which is non-vanishing for the Γ with $pr_2 : \Gamma \simeq V$. The only way to get a determinant function $\det(D)$ is by trivializing the line bundle $\mathcal{L}^*$ over the Γ_D's which occur. For example one can trivialize $\mathcal{L}^*$ by lifting back to the central extension of the restricted unitary gp. In this case one gets a spherical fn. $g \mapsto \langle \Omega | g \Omega \rangle$. The element $g\Omega$ is a specific generator for $\mathcal{L}_{gV}$ belonging to $\Gamma = gV$. Hence $|\langle \Omega | g\Omega \rangle|^2$ depends only on Γ.

$$g\Omega = \frac{u_\Gamma}{\|u_\Gamma\|} \times \text{const of abs. val. } 1$$

$$|\langle \Omega | g\Omega \rangle|^2 = \left| \frac{\langle \Omega | u_\Gamma \rangle}{\|u_\Gamma\|} \right|^2 = \frac{1}{\|u_\Gamma\|^2} = \frac{1}{\det(1+T^*T)}$$

But this still doesn't yield anything like $\det(T^*T)$.

August 24, 1982

$$D: W \longrightarrow V \qquad \mathcal{L}_D \subset \mathrm{Hom}(\Lambda V, \Lambda W)$$

If D invertible, one has formally

$$\frac{\det(D+B)}{\det(D)} = \det(1+D^{-1}B) = \mathrm{tr}(\Lambda D^{-1} \Lambda B).$$

Since ΛD^{-1} generates $\mathcal{L}_D$ one could try to use this formula with ΛD^{-1} replaced by a generator of $\mathcal{L}_D$. Since ΛB is of degree 0, this won't give anything except in the index 0 case.

Nevertheless we can try to prove the above formula for $\bar\partial$ operators. B is a multiplication operator and D^{-1} is given by a Green's fn., so $D^{-1}B$ is represented by a kernel

$$K(z,z') = G(z,z')b(z').$$

$\Lambda^8(D^{-1}B)$ will be represented by the kernel

$$\det\left[G(z_i,z_j')b(z_j')\right] = \det G(z_i,z_j') \cdot \prod_j b(z_j')$$

To take the trace I need to have $z_j' = z_j$ and to make sense of

$$\det\left(G(z_i,z_j)\right).$$

I know what to do about diagonal elements $G(z,z)$. For $q=2$ I get

$$\begin{vmatrix} G(x,x) & G(x,y) \\ G(y,x) & G(y,y) \end{vmatrix}$$

and the main problem concerns integrating

$$G(x,y)G(y,x)\,b(x)b(y) \qquad z=x \quad z'=y.$$

This is an integral of the form $\dfrac{d^2z\,d^2z'}{(z-z')^2} = \dfrac{d^2z\,d^2u}{u^2}$, $u=z'-z$

and in 2 dims $\dfrac{d^2u}{u^2} = \dfrac{r\,dr\,d\theta}{r^2e^{2i\theta}}$ is logarithmically divergent although if we do the θ integral symmetrically there is no problem in getting a good number.

Conclusion: For D invertible we get ("diagonal") Green's fns.

$$G^{(g)}(z_1, \dots, z_g) = \det(G(z_i, z_j))$$ [redacted]

which are well-defined [redacted] distributions on the product M^g such that

$$\frac{\text{"}\det(D+B)\text{"}}{\det D} = \sum_g{}' \int_{M^g} G^{(g)}(z_1, \dots, z_g) \prod b(z_j)\, d^2z_1 \cdots d^2z_g$$

$G^{(g)}$ is a sum of [redacted] terms described by the cycle structure of permutations. The 2 cycles lead[redacted] to the fact the $G^{(g)}$ are distributions and not functions. There is a problem in showing the above [redacted] series converges for any b.

At this point I understand what the diagonal Green's functions are for D invertible. Now the problem arises how to extend this to the general case. [redacted] We choose a generator for $\mathcal{L}_D$ hence get a map from ΛV to ΛW whose matrix elements are what we want.

Consider first the case where $D: W \to V$ is onto and $\dim(\operatorname{Ker} D) = 1$. Then the map $\Lambda V \to \Lambda W$ has degree 1, so we get in particular a map $\Lambda^0 V \to \Lambda^1 W$ defined up to a scalar. This should amount to choosing a generator w for Ker D. Then $\Lambda^1 V \to \Lambda^2 W$ should be $v \mapsto w \wedge D^{-1} v$, or $D^{-1} v \wedge w$ depending on our conventions. Here D^{-1} is a partial inverse for D, but $w \wedge D^{-1} v$ doesn't depend on the choice of D^{-1}. The matrix elements of $v \mapsto w \wedge D^{-1} v$ are

$$\langle z_2, z_1 | w \wedge D^{-1} | z' \rangle = \begin{vmatrix} w(z_1) & G(z_1, z') \\ w(z_2) & G(z_2, z') \end{vmatrix}$$

and the matrix elements of $v_1 \wedge \cdots \wedge v_p \longmapsto w \wedge D^{-1} v_1 \wedge \cdots \wedge D^{-1} v_p$ are

$$\det \begin{pmatrix} w(z_1) & G(z_1, z_1') & \cdots & G(z_1, z_p') \\ \vdots & \vdots & & \vdots \\ w(z_{p+1}) & G(z_{p+1}, z_1') & \cdots & G(z_{p+1}, z_p') \end{pmatrix}.$$

Interesting situation: In the case of $\bar\partial$ operators we know how to trivialize $\mathcal{L}_D$ as D varies over $\mathcal{A}$. Hence, since $\mathcal{L}_D \subset \mathrm{Hom}(\Lambda V, \Lambda W)$ we have attached to a finite-diml subspace $F \subset V$ and f.d. quotient space $W \to Q$ with appropriate dimensions a [redacted] certain holom. fn. on $\mathcal{A}$, which is to be thought of as the determinant of a minor of D. On the other hand these minors are also the denominator-free Green's functions. ∎

August 25, 1982

Associated to any $\bar\partial$ operator $D: W \to V$ is the line $\mathcal{L}_D \subset \mathrm{Hom}(\Lambda V, \Lambda W)$, so if I have a generator of this line, then I get Green's functions which are the matrix elements of the map $\Lambda V \to \Lambda W$. These G-fns. [redacted] are distributions on products of M and do not come provided with a regularization on the diagonal.

Now the theory I am working on constructs a trivialization of $\mathcal{L}_D$ as D varies, and hence the Green's functions obtained from this trivialization will also vary analytically in D. We can [redacted] then differentiate w.r.t. D and this should bring in the regularized diagonal values.

Let's consider this carefully when D is invertible. We can then choose the generator of $\mathcal{L}_D$ given by ΛD^{-1}. In [redacted] this case the Green's fns. are the determinants

$$\det G(z_i, z_j')$$

where G is the kernel for D^{-1}. But now the analytic function $\det(D)$ is constructed so that $(\det D)\cdot \Lambda D^{-1}$ [redacted] extends to a trivialization of $\mathcal{L}$ over all the D of index 0. Thus the distributions on $M^g \times M^g$

$$\det(D) \det G(z_i, z_j')$$

will by extension be defined and analytic in D for all D.

But now we have

$$\delta \det(D) = \mathrm{Tr}\Big[\underbrace{(\det D) G}_{\mathrm{Cof}(D)} \cdot \delta D\Big]$$

so the definition of $\det(D)$ implies a regularization on the diagonal of $\det(D)\,G$. Similarly

$$\delta\big[\det(D)\, G(z_1, z_1')\big] = \delta \det(D) \cdot G(z_1, z_1') + \det(D) \cdot \delta G(z_1, z_1')$$

$$= \det(D)\left[\int dz\, G(z,z)\,\delta D(z)\; G(z_1,z_1') - \int dz\, G(z_1,z)\,\delta D(z)\, G(z,z_1')\right]$$

$$= \underbrace{\int dz\;\; \delta D(z) \begin{vmatrix} G(z,z) & G(z,z_1') \\ G(z_1,z) & G(z_1,z_1') \end{vmatrix} \det(D)}_{\det(D)\; G^{(2)}(z,z_1\,;\, z,z_1')}$$

So what is happening is this. Each time we differentiate a denominator-free Green's function, we get a trace with one of higher order on the diagonal. Hence the regularization procedure needed to define the original trivialization of $\mathcal{L}$ should give a meaning to the other Green's functions when the arguments coincide. There might be a consistency problem.

August 26, 1982

Let's go back to $H_0 = \frac{1}{i}\partial_x$ on $L^2(S^1)$, $S^1 = \mathbb{R}/L\mathbb{Z}$, and calculating the amplitude $\langle 0|S|0\rangle$ for the perturbation $H = \frac{1}{i}\partial_x + f(x,t)$, where f has compact support.

The operator on Fock space belonging to multiplication by $f(x)$ is

$$\hat{f} = \int dx\, f(x)\, \psi^*(x)\psi(x)$$

$$\psi(x) = \sum \langle x|k\rangle \psi_k = \sum \frac{1}{\sqrt{L}} e^{ikx}\psi_k$$

$$\psi(x)^* = \sum \frac{1}{\sqrt{L}} e^{-ik'x}\psi^*_{k'}$$

$$k' = k+q$$

$$\therefore\ \hat{f} = \sum_{k,k'}\left(\frac{1}{L}\int dx\, f(x)\, e^{-i\underbrace{(k'-k)}_{q}x}\right)\psi^*_{k'}\psi_k$$

$$= \sum_q \underbrace{\left(\frac{1}{L}\int dx\, f\, e^{-iqx}\right)}_{f_q} \underbrace{\sum_k \psi^*_{k+q}\psi_k}_{\rho_q} \qquad q \in \frac{2\pi}{L}\mathbb{Z}$$

Now $H_0|k\rangle = k|k\rangle$ and $|0\rangle$ = all $|k\rangle$, $k \le 0$, filled. hence $\rho_q|0\rangle = 0$ for $q<0$; also for $q=0$ by the convention about ρ_0. Thus ρ_{-q} destroys, ρ_q creates for $q>0$, and the non-trivial commutation relation is

$$[\rho_{-q}, \rho_q] = \frac{L}{2\pi} q \qquad q>0$$

In general for a perturbation such as $\hat{H} = \hat{H}_0 + \hat{f}(t)$ we have

$$S\ [redacted] = T\left\{ e^{-\int_{-\infty}^{\infty} dt\, e^{\hat{H}_0 t}\hat{f}(t) e^{-\hat{H}_0 t}} \right\}$$

I need

$$e^{\hat{H}_0 t}\rho_q e^{-\hat{H}_0 t} = e^{+qt}\rho_q .$$

In effect $\hat{H}_0 = :\sum k\psi^*_k\psi_k:$

so

$$[\hat{H}_0, \psi_k] = [k\psi^*_k\psi_k, \psi_k] = -k\{\psi^*_k, \psi_k\}\psi_k = -k\psi_k$$

$$e^{\hat{H}_0 t}\psi_k e^{-\hat{H}_0 t} = e^{-kt}\psi_k \qquad e^{\hat{H}_0 t}\psi^*_k e^{-\hat{H}_0 t} = e^{kt}\psi^*_k$$

$$e^{\hat{H}_0 t} \hat{f}(t) e^{-\hat{H}_0 t} = \sum_q f_q(t)\, e^{qt} \rho_q$$

$$= \sum_{q>0} f_q(t)\, e^{qt} \underset{\text{create}}{\rho_q} + f_{-q}(t)\, e^{-qt} \underset{\text{destroy}}{\rho_{-q}}$$

Recall the formula for a simple oscillator

$$T\left\{ e^{-\int dt\,(f(t) a^* + \tilde{f}(t) a)} \right\} = e^{\int_{t>t'} dt\,dt'\, \tilde{f}(t) f(t')}\, e^{-\left(\int dt f(t) a^*\right)} \times e^{-\left(\int dt \tilde{f}(t)\right) a}$$

Hence in our situation $\langle 0|S|0\rangle$ is exp of

$$\int_{t>t'} dt\,dt' \sum_{q>0} f_{-q}(t)\, e^{-qt} f_q(t')\, e^{qt'} [\rho_{-q}, \rho_q]$$

$$= \int dt\,dt' \sum_q \frac{1}{L} dx\, f(x,t)\, e^{iqx - qt} \frac{1}{L} dx'\, f(x't')\, e^{-iqx' + qt'} \frac{L}{2\pi} q$$

$$= \int_{t>t'} dt\,dt'\,dx\,dx'\; f(xt)\, f(x't') \underbrace{\sum_{q>0} e^{-q[\Delta t - i\Delta x]} \frac{q}{2\pi L}}$$

As $L \to \infty$, $\frac{1}{L}\sum_q \to \int \frac{dq}{2\pi}$ so approaches

$$\int_0^\infty \frac{dq}{(2\pi)^2}\, e^{-q[\Delta t - i\Delta x]}\, q = \frac{1}{(2\pi)^2} \frac{\Gamma(2)}{(\Delta t - i\Delta x)^2}$$

and so the answer is

$$\boxed{\log \langle 0|S|0\rangle = \int_{t>t'} \frac{dt\,dt'\,dx\,dx'}{(2\pi)^2} \frac{f(xt)\, f(x't')}{(\Delta t - i\Delta x)^2}}$$

Why am I interested in the above calculation? This is a situation, where I can see the generating function $Z(J)$ and the S-matrix completely, and where there is some regularization present.

Let's check the answer against previous calculations in the case where f is independent of t. When f is time-independent one lets it act for a long time T and then

$$\langle 0|S|0\rangle \sim e^{-T\Delta E}$$

where ΔE is the ground ~~energy~~ energy shift. Thus

$$\Delta E = -\int_{t>0} \frac{dt\,dx\,dx'}{(2\pi)^2} \frac{f(x)f(x')}{(t-i\Delta x)^2}$$

$$= \int \frac{dx\,dx'}{(2\pi)^2} \frac{f(x)f(x')}{+i\Delta x}$$

Actually $\displaystyle\int_0^\infty dt \frac{-1}{(t-i\Delta x)^2} = \left[\frac{1}{t-i\Delta x}\right]_0^\infty = \frac{-1}{\eta - i\Delta x}$ $\quad \eta = 0^+$.

So by symmetry

$$\Delta E = -\int \frac{dx\,dx'}{(2\pi)^2} f(x)f(x') \underbrace{\left[\frac{1}{\eta - i\Delta x} + \frac{1}{\eta + i\Delta x}\right]\frac{1}{2}}_{\frac{\eta}{\Delta x^2+\eta^2} = \pi\delta(\Delta x)}$$

$$\boxed{\Delta E = -\frac{1}{4\pi}\int dx\, f(x)^2}$$

This agrees with the previous calculation on p. 19.

Now I want to take the viewpoint that $\langle 0|S|0\rangle$ is a determinant for the differential operator $\partial_t + \frac{1}{i}\partial_x + f(x,t)$.

This ~~seems to be~~ is Schwinger's idea, but there seems to be problems with the square $(\Delta t - i\Delta x)^2$ in the formula.

Let's go through Schwinger's derivation.

$$\langle 0|S|0\rangle = \langle 0|e^{\hat{H}_0 T}\, \hat{u}(T,-T)\, e^{\hat{H}_0 T}|0\rangle$$

$$= \frac{\langle 0|\, \hat{u}(T,-T)\,|0\rangle}{\langle 0\,|\hat{u}_0(T,-T)|\,0\rangle}$$

$$\delta \log \langle 0|S|0\rangle = \delta \log \langle 0\,|\,\hat{u}(T,-T)|0\rangle$$

$$= -\int dt\, \frac{\langle 0|\, \hat{u}(T,t)\, \delta\hat{f}(t)\, \hat{u}(t,-T)|0\rangle}{\langle 0\,|\,\hat{u}(T,-T)|0\rangle}$$

$$\delta\hat{f}(t) = \int dx\ \delta f(x,t)\ \psi^*(x)\,\psi(x)$$

$$\therefore\ \delta \log \langle 0|S|0\rangle = -\int dt\,dx\ \delta f(x,t)\ \frac{\langle 0\,|\hat{u}(T,t)\,\psi^*(x)\,\psi(x)\,\hat{u}(t,-T)|0\rangle}{\langle 0|\,\hat{u}(T,-T)|0\rangle}$$

The Green's functions is defined by

$$G(xt,x't') = \langle T[\psi(xt)\,\psi(x't')^*]\rangle$$

$$= \begin{cases} \dfrac{\langle 0|\, \hat{u}(T,t)\psi(x)\, \hat{u}(t,t')\, \psi(x')^*\, u(t,-T)|0\rangle}{\langle 0|\,\hat{u}(T,-T)|0\rangle} & t>t' \\[2ex] -\dfrac{\langle 0|\,\hat{u}(T,t')\psi(x')^*\,\hat{u}(t',t)\,\psi(x)\hat{u}(t,-T)|0\rangle}{\langle 0|\,\hat{u}(T,-T)|0\rangle} & t<t' \end{cases}$$

One has

$$\{\partial_t + \hat{H}(t)\}\ \hat{u}(t,t') = 0$$

hence

$$\partial_t\{\hat{u}(T,t)\,\psi(x)\,\hat{u}(t,t')\} = \hat{u}(T,t)\,[\hat{H}(t),\psi(x)]\,\hat{u}(t,t')$$

and $\hat{H}(t)$, being a 1-particle operator, has the form

$$\hat{H}(t) = \int dy\ \psi^*(y)\left[\tfrac{1}{i}\partial_y + f(y,t)\right]\psi(y)$$

hence

$$[\hat{H}(t),\psi(x)] = -\int dy\ \delta(x-y)\left[\tfrac{1}{i}\partial_y + f(y,t)\right]\psi(y)$$

$$= -\left[\tfrac{1}{i}\partial_x + f(x,t)\right]\psi(x).$$

Therefore it follows that

$$\left[\partial_t + \frac{1}{i}\partial_x + f(x,t)\right] G(xt, x't') = 0 \qquad t \neq t'$$

On the other hand the jump in G as t passes thru t' is $\delta(x-x')$ because $\psi(x)\psi(x')^* + \psi(x')^*\psi(x) = \delta(x-x')$. Thus

$$\left[\partial_t + \frac{1}{i}\partial_x + f(x,t)\right] G(xt, x't') = \delta(x-x')\,\delta(t-t')$$

and so the only remaining info. needed is the boundary conditions.

Let $t \ll 0$ and let t_0 be below t' and the support of f. Then $G(xt, x't')$ is an inner product with a vector independent of xt with

$$\hat{u}(t_0, t)\,\psi(x)\,\hat{u}(t, -T)|0\rangle$$

$$= e^{-\hat{H}_0 t_0} \underbrace{e^{\hat{H}_0 t}\psi(x)\,e^{-\hat{H}_0 t}}_{\frac{1}{\sqrt{L}}\sum_k e^{ikx-kt}\psi_k} \underbrace{e^{\hat{H}_0 T}|0\rangle}_{|0\rangle}$$

hence
$$G(xt, x't') = \text{lin. comb. of } e^{ikx-kt} \text{ for } k \leq 0.$$
$$\text{for } t \ll 0.$$

Analogously since $\langle 0|\,\psi_k = 0$ for $k \leq 0$, one has
$$G(xt, x't') = \text{linear comb. of } e^{ikx-kt} \text{ for } k > 0$$
$$\text{if } t \gg 0.$$

Let's compute G when $f = 0$. Want G fn. for $\partial_t + \frac{1}{i}\partial_x = \frac{2}{i}\,\frac{1}{2}(\partial_x + i\partial_t) = \frac{2}{i}\partial_{\bar{z}}$ if $z = x + it$.

Since $\partial_{\bar{z}}$ has G fn. $\frac{1}{\pi z}$ we get

$$G = \frac{i}{2\pi z} = \frac{i}{2\pi(x+it)} = \boxed{\frac{1}{2\pi(t-ix)}}$$

assuming the bdry conditions are correct. Directly

$$G(xt) = \langle \psi(xt)\psi(0)^* \rangle = \frac{1}{L}\sum_k e^{ikx-kt} \underbrace{\langle \psi_k \psi_k^* \rangle}_{1 \text{ when } k > 0} \qquad t>0$$

$$= \int_0^\infty \frac{dk}{2\pi}\, e^{-k(t-ix)} = \frac{1}{2\pi(t-ix)} \qquad t>0$$

If $t<0$

$$G(xt) = -\langle \psi(0)^* \psi(xt) \rangle = \frac{-1}{L}\sum_k e^{ikx-kt} \underbrace{\langle \psi_k^* \psi_k \rangle}_{1 \quad k \le 0}$$

$$= -\int_{-\infty}^0 \frac{dk}{2\pi}\, e^{ikx-kt} = \frac{1}{2\pi(t-ix)} \qquad t<0$$

Next I want the Green's function for a general f of compact support.

$$\frac{i}{2}\left(\partial_t + \frac{1}{i}\partial_x + f\right)(G) = \frac{i}{2}\delta$$

$$\underbrace{\left(\partial_{\bar z} + \frac{i}{2} f\right)}_{e^{-\varphi}\partial_{\bar z} e^{\varphi}} G(z,z') = \frac{i}{2}\delta(z-z')$$

$$\text{if} \quad \partial_{\bar z}\varphi = \frac{i}{2} f$$

$$\varphi(z) = \int \frac{i}{2\pi}\frac{1}{z-w} f(w)\, d^2w$$

$$\partial_{\bar z} e^{\varphi} G = \frac{i}{2} e^{\varphi(z')}\delta(z-z')$$

$$e^{\varphi} G = \frac{i}{2} e^{\varphi(z')} \frac{1}{\pi(z-z')}$$

$$\therefore G(z,z') = \frac{i}{2\pi}\frac{e^{-\varphi(z)+\varphi(z')}}{z-z'}$$

$$\boxed{G(z,z') = \frac{i}{2\pi}\frac{1}{z-z'}\, e^{\int \frac{i\, d^2w}{2\pi}\left[\frac{1}{w-z} - \frac{1}{w-z'}\right] f(w)}}$$

Next you want to extract a finite part when $z=z'$

$$\frac{1}{w-z} = \frac{1}{w-z'-(z-z')} = \frac{1}{w-z'} + \frac{(z-z')}{(w-z')^2} + \cdots$$

$$G(z,z') = \frac{i}{2\pi}\frac{1}{z-z'}\left\{1 + (z-z')\int \frac{i\,d^2w}{2\pi}\frac{f(w)}{(w-z')^2} + \cdots\right\}$$

$$\therefore \quad \boxed{G(z,z)_{reg} = \left(\frac{i}{2\pi}\right)^2 \int d^2w \, \frac{f(w)}{(w-z)^2}}$$

$$= \frac{1}{(2\pi)^2}\int dx'dt' \, \frac{f(x't')}{(\Delta t - i\Delta x)^2} \qquad \Delta t = t - t'$$

From the formula on p. 55 we have

$$\delta \log \langle 0|S|0\rangle = \int_{t>t'} \frac{dt\,dt'\,dx\,dx'}{(2\pi)^2} \, \frac{\delta f(xt) f(x't') + f(xt)\,\delta f(x't')}{(\Delta t - i\Delta x)^2}$$

$$= \int dt\,dx\,\delta f(xt) \int \frac{dt'dx'}{(2\pi)^2}\,\frac{f(x't')}{(\Delta t - i\Delta x)^2}$$

At this point we have an example of a $\bar\partial$-determinant, namely $\langle 0|S|0\rangle$, a formula for the Green's function of $\frac{2}{i}\partial_{\bar z} + f$, and a way to regularize this Green's function on the diagonal consistent with the determinant. Now there are various things we can do.

1) gauge invariance and trace anomaly.
2) I can rewrite things as energy-momentum integrals.
3) Green's fns. and S-matrix.

For gauge invariance we use

$$\delta \log \langle 0|S|0\rangle = \int d^2z \; \delta f(z)\, J(z)$$

in the case where δf arises from an infinitesimal gauge transf. $\delta f = [\delta\varphi, \frac{2}{i}\partial_{\bar z} + f] = -\frac{2}{i}\partial_{\bar z}(\delta\varphi)$

Now $$J(z) = \left(\frac{i}{2\pi}\right)^2 \int d^2w \, \frac{f(w)}{(z-w)^2}$$ so

$$\delta \log \langle 0|S|0\rangle = \int d^2z \left(-\frac{2}{i}\partial_{\bar z}\,\delta\varphi\right) J(z)$$

$$= \frac{2}{i}\int d^2z \, \delta\varphi \cdot \partial_{\bar z} J(z)$$

$$= \frac{2}{i}\left(\frac{i}{2\pi}\right)^2 \int d^2z \; \delta\varphi(z) \int d^2w \left[\partial_{\bar z}\frac{1}{(z-w)^2}\right] f(w)$$

Hence we must calculate the distribution.

$$\partial_z\left[\partial_{\bar z}\frac{1}{z-w}\right] = \partial_z\left[\pi\,\delta(z-w)\right]$$

$$\partial_{\bar z}\left(\frac{-1}{(z-w)^2}\right) = -\partial_w\left[\pi\,\delta(z-w)\right]$$

hence $$\boxed{\partial_{\bar z}\frac{1}{(z-w)^2} = \pi\,\partial_w \cdot \delta(z-w)}$$

$$\int d^2w \left[\partial_{\bar z}\frac{1}{(z-w)^2}\right] f(w) = \pi\int d^2w \; \partial_w \, \delta(z-w) f(w)$$

$$= \pi\int d^2w \; \delta(z-w)\left(-\partial_w f(w)\right) = -\pi\left(\partial_z f\right)(z).$$

Thus

$$\delta \log \langle 0|S|0\rangle = \frac{2}{i}\left(\frac{i}{2\pi}\right)^2(-\pi)\int d^2z \; \delta\varphi(z)\,\partial_z f(z)$$

$$= \frac{1}{2\pi i}\int d^2z \; \delta\varphi \, \partial_z f$$

and so there is a trace anomaly, and corresponding lack of crude gauge invariance.

Simpler derivation (possibly?)

$$\langle 0|S|0\rangle = \frac{\det\left(\frac{2}{i}\partial_{\bar z} + f\right)}{\det\left(\frac{2}{i}\partial_{\bar z}\right)} = \det\Big(1 + \underbrace{\tfrac{i}{2}(\partial_{\bar z})^{-1} f}_{-K}\Big)$$

K is given by the kernel $\frac{1}{2\pi i}\frac{1}{z-z'}f(z')$ and

$$\log \langle 0|S|0\rangle = \log\det(1-K) = -\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{tr}(K^n)$$

Now $-\frac{1}{2}\,\mathrm{tr}\,K^2 = +\frac{1}{2}\int \frac{d^2z\,d^2z'}{(2\pi i)^2}\,\frac{f(z)f(z')}{(z-z')^2}$, according to the above formulas, is the only non-trivial term in this series. Let's see this directly:

$$\mathrm{tr}\,K^3 = \int \frac{d^2z_1\,d^2z_2\,d^2z_3}{(2\pi i)^3}\,\frac{f(z_2)f(z_3)f(z_1)}{(z_1-z_2)(z_2-z_3)(z_3-z_1)}$$

The numerator is symmetric, the denominator anti-symm. under $z_1 \to z_3$, $z_2 \to z_2$, $z_3 \to z_1$, hence $\mathrm{tr}(K^3)=0$. A similar argument shows that $\mathrm{tr}(K^n)=0$ for n odd, but I don't see why one gets zero for $n=4$.

What matters is the symmetrization of the fn.

$$\frac{1}{(z_1-z_2)\cdots(z_{n-1}-z_n)(z_n-z_1)},$$

call it $\varphi(z_1,\dots,z_n)$. If you multiply φ by the anti-symmetric fn. $\omega = \prod_{i<j}(z_i-z_j)$, then you will get an anti-symmetric polynomial of degree $<$ degree$(\omega) = \frac{n(n-1)}{2}$ and this is impossible. (This is for $n \geq 3$ that $\omega\varphi$ is a polynomial, because then the factors z_i-z_j are distinct. For $n=2$ $\omega\varphi = (z_1-z_2)\frac{-1}{(z_1-z_2)^2}$ is not a poly.)

Therefore we conclude that $\mathrm{tr}(K^n)=0$ $n \geq 3$ and we get the same formula for $\langle 0|S|0\rangle$.

In the non-abelian situation f is a matrix function of z and the numerator is $\mathrm{tr}(f(z_1)\cdots f(z_n))$ which has cyclic symmetry only, so one can't make a similar vanishing assertion.

August 27, 1982

The central point to be clarified is this. By choosing a regularization process I trivialize the line bundle L and hence for each D give a meaning to the fermion integral

$$\int e^{-\bar{\psi} D \psi} \; \psi(x_1) \cdots \psi(x_p) \, \bar{\psi}(y_1) \cdots \bar{\psi}(y_q)$$

as a distribution on $M^p \times M^q$. Now if I differentiate wrt D, then I make sense out of diagonal values of these Green's functions. On the other hand by means of Taylor series, the diagonal Green's functions for one point D_0 determine everything for all other values of D. I would like to check that this all works.

<u>August 28, 1982</u>

Let's take the Schwinger viewpoint towards the Hamiltonian $\frac{1}{i}\partial_x + f(x,t)$ extended to Fock space, where f has compact support. By this I mean that we should work in real time and calculate the S-matrix which should be a unitary operator on Fock space when f is real.

In the present case $e^{i\hat{H}_0 t}\hat{f}(t)e^{-i\hat{H}_0 t}$ is again a multiplication operator. In fact we have

$$e^{iH_0 t} = e^{t\partial_x}$$

and

$$e^{t\partial_x} f_{\text{[redacted]}}\, e^{-t\partial_x} g = e^{t\partial_x} f_{\text{[redacted]}}\, g(\ -t)$$

$$= f_{\text{[redacted]}}(\ +t)\, g$$

so that $\left(e^{t\partial_x} f_{\text{[redacted]}}\, e^{-t\partial_x}\right)(x) = f(x+t)$. Consequently

$$e^{i\hat{H}_0 t}\hat{f}_{\text{[redacted]}}\, e^{-i\hat{H}_0 t} = f(\ +t)^{\wedge}.$$

Thus since these operators commute modulo scalars we have that modulo a scalar c with $|c|=1$.

$$S = T\left\{e^{-i\int dt\, e^{i\hat{H}_0 t}\hat{f}_t e^{-i\hat{H}_0 t}}\right\}$$

$$= c\ e^{-i\int dt\, f_t(\ +t)^{\wedge}} = c\ e^{-i\left[\int dt\, f(x+t,t)\right]^{\wedge}}$$

$$= c'\left[e^{-i\int dt\, f(x+t,t)}\right]^{\wedge}$$

(This last hat represents lifting as an autom. not derivation.)

Thus we conclude that the [redacted] S-matrix on Fock space is [redacted] essentially given by the multiplication operator for the function

$$e^{-i\int dt\, f(x+t,t)}.$$

Next let's check gauge invariance. A gauge transformation consists [redacted] of [redacted] conjugating the Hamiltonian by $e^{i\chi(xt)}$, where χ is real-valued and of compact support. Under this gauge transf. the 1-particle Schroedinger equation becomes

$$0 = e^{-i\chi}\left(\frac{1}{i}\partial_t + \frac{1}{i}\partial_x + f\right)\psi = \left[\frac{1}{i}\partial_t + \frac{1}{i}\partial_x + (\partial_t\chi + \partial_x\chi + f)\right]\left(e^{-i\chi}\psi\right).$$

Thus the gauge transformation changes f to $f + \partial_t\chi + \partial_x\chi$, where χ has compact support. Since

$$\int dt\left[f(x+t) + \underbrace{\partial_t\chi(x+t,t) + \partial_x\chi(x+t,t)}_{\partial_t[\chi(x+t,t)]}\right] = \int dt\ f(x+t,t)$$

it follows that the [redacted] gauge transformation does not affect the 1-particle S-matrix.

Now we should see what happens[redacted] to the vacuum-vacuum amplitude $\langle 0|S|0\rangle$. The calculation goes exactly as on p. 54-55 and yields

$$\log\langle 0|S|0\rangle = -\int_{t>t'} dt\,dt'\,dx\,dx'\ f(xt)f(x't')\ \underbrace{\frac{1}{2\pi L}\sum_{q>0} e^{-iq(\Delta x-\Delta t)}\,q}$$

$$\xrightarrow[L\to\infty]{\text{as}} \frac{1}{(2\pi)^2}\int_0^\infty dq\ e^{-iq(\Delta x-\Delta t+i\eta)}\,q$$

$$= \frac{1}{(2\pi i)^2}\,\frac{1}{(\Delta x-\Delta t+i\eta)^2}$$

$$\log\langle 0|S|0\rangle = -\frac{1}{2}\int dt\,dt'\,dx\,dx'\ f(xt)f(x't')\ \frac{1}{(2\pi i)^2}$$

$$\times\left[\frac{\theta(\Delta t)}{(\Delta x-\Delta t+i\eta)^2} + \frac{\theta(-\Delta t)}{(\Delta x-\Delta t-i\eta)^2}\right]$$

$$= -\frac{1}{2}\int\frac{dt\,dt'\,dx\,dx'}{(2\pi i)^2} f(x+t,t)f(x'+t',t')\left[\frac{\theta(\Delta t)}{(\Delta x+i\eta)^2} + \frac{\theta(-\Delta t)}{(\Delta x-i\eta)^2}\right]$$

Gauge invariance means this expression doesn't change when we make the change

$$f(x+t,t) \longmapsto f(x+t,t) + \partial_t[\chi(x+t,t)]$$

with χ of compact support. This is equivalent to

$$\partial_t \left[\frac{\theta(\Delta t)}{(\Delta x + i\eta)^2} + \frac{\theta(-\Delta t)}{(\Delta x - i\eta)^2} \right] = 0$$

which is certainly true in the open set where $\Delta x \neq 0$ or $\Delta t \neq 0$.

Using $\partial_t \, \theta(\Delta t) = \delta(\Delta t)$ we get

$$\partial_t \left[\frac{\theta(\Delta t)}{(\Delta x+i\eta)^2} + \frac{\theta(-\Delta t)}{(\Delta x - i\eta)^2}\right] = \delta(\Delta t) \left[\frac{1}{(\Delta x+i\eta)^2} - \frac{1}{(\Delta x - i\eta)^2} \right]$$

$$= \delta(\Delta t) \frac{\partial}{\partial x} \left[\underbrace{\frac{-1}{\Delta x + i\eta} + \frac{1}{\Delta x - i\eta}}_{\frac{2i\eta}{(\Delta x)^2+\eta^2} \longrightarrow 2\pi i \, \delta(\Delta x)} \right]$$

Thus

$$\boxed{\partial_t \left[\frac{\theta(\Delta t)}{(\Delta x + i\eta)^2} + \frac{\theta(-\Delta t)}{(\Delta x - i\eta)^2} \right] = 2\pi i \, \delta(\Delta t) \, \delta'(\Delta x)}$$

and so we don't have gauge-invariance. The above is clearly quite consistent with the imaginary time calculation on p.61.

August 29, 1982

Further calculations I could do.
1) Energy-momentum variables
2) S-matrix in imaginary time
3) Schwinger's gauge-invariant determinant
4) Gauge field with non-zero winding number.

Let's consider the problem of the S-matrix in imaginary time. We solve the DE

$$\left(\partial_t + \hat{H}\right)\psi = 0 \qquad H = \frac{1}{i}\partial_x + f$$

in Fock space thereby getting a propagator $\hat{U}(t,t')$ which should be a nice operator for $t \geq t'$ on the Fock space. Then I can try to compare this operator with the free propagator $e^{-\hat{H}_0(t-t')}$.

Good idea: There are 2 determinants. One comes from the fact that $\langle 0|S|0\rangle$ can be interpreted as the determinant of a $\bar{\partial}$-operator. The other comes from the fact that $\langle 0|S|0\rangle$ is the determinant of a corner in a block decomposition; this is Graeme's statement that $\langle 0|S|0\rangle$ is such a determinant.

August 30, 1982

One of the virtues of having worked thru the above example: $\partial_t + \frac{1}{i}\partial_x + f$ with f of compact support, is that it shows we can work over a non-compact Riemann surface provided we deal with compactly supported variations in the $\bar\partial$-operator. The nice thing about the example is that the Green's function for D_0 is so simple.

One possibility now would be to study the case of a small nbd. of a point on a Riemann surface and to allow only changes of the $\bar\partial$-operator within this small nbd. I place myself in an invertible situation and then ask about τ-functions.

Hence it is first necessary to understand what τ-functions are. This requires a review of the loop group representation, etc.

Let's begin with $V = L^2(S^1)$ with its orthonormal basis $|n\rangle = z^n$ $n \in \mathbb{Z}$. We are going to form a Fock space out of V which will then have operators

$$\psi_n = \text{inter. mult. by } \langle n|$$
$$\psi_n^* = \text{ext. mult. by } |n\rangle$$

and then we can introduce the operators

$$\psi(\zeta) = \text{int. mult by } \langle\zeta| = \sum_n \langle\zeta|n\rangle\langle n|$$
$$= \sum_n \zeta^n \psi_n$$

$$\psi(\zeta)^* = \sum_n \zeta^{-n}\psi_n^* .$$

These are formal operators depending on a $\underbrace{\text{complex}}_{\text{non-zero}}$ no. ζ. These are the same kind of gadgets as $\psi(xt) = e^{\hat H_0 t}\psi(x)e^{-\hat H_0 t}$ encountered above. Let's work out the relation:

$$\psi(x) = \frac{1}{\sqrt{L}}\sum{}' e^{ikx}\psi_k \qquad \psi(xt) = \frac{1}{\sqrt{L}}\sum_k{}' e^{ikx}e^{-kt}\psi_k$$

Hence $\psi(xt) = \frac{1}{\sqrt{L}} \sum_k e^{ik(x+it)} \psi_k$

and so I should really think [redacted] now of z as being $e^{i(x+it)}$. Hence $t > 0 \iff |z| < 1$.

$$\langle \psi(\zeta)\psi(\zeta')^* \rangle = \sum_n (\zeta/\zeta')^n \langle \psi_n \psi_n^* \rangle$$

Hence if we choose the vacuum in Fock space to be filled by the z^n with $n \le 0$, we get $\langle \psi_n \psi_n^* \rangle = \begin{cases} 1 & n>0 \\ 0 & n \le 0 \end{cases}$

so
$$\langle \psi(\zeta)\psi(\zeta')^* \rangle = \frac{\zeta/\zeta'}{1-(\zeta/\zeta')}$$

is convergent for $|\zeta| < |\zeta'|$; this corresponds to $t > t'$.

Review next the Japanese factorization formula. The idea here is that we have a subspace W of V which is L^2-commensurable with H^- and which is complementary to H^+. Let Ω, Ω_W be 'the' unit vectors belonging to H^- and W resp.

Because $H^+ = $ Span of z^n, $n > 0$ is complementary to W we know that $(\mathbb{C} + H^+) \cap W$ is 1-dimensional, in fact there is a unique element

$$s \in (1 + H^+) \cap W.$$

Then $s(z) = 1 + a_1 z + a_2 z^2 + \cdots$ is analytic for $|z| < 1$ and we want a formula for it.

Formally $\Omega_W = w_0 \wedge w_1 \wedge \cdots$ where the w_n form a basis for W. Hence

$$\psi(\zeta)\Omega_W = \sum_{n \ge 0}' w_n(\zeta)(-1)^n w_0 \wedge \cdots \wedge \hat{w}_n \wedge \cdots$$

and so for any vector θ in Fock space

$$\zeta \longmapsto \langle \theta | \psi(\zeta)\Omega_W \rangle = \sum_{n \ge 0} w_n(\zeta)(-1)^n \langle \theta | w_0 \wedge \cdots \hat{w}_n \wedge \cdots \rangle$$

is a function [redacted] in the space W. [redacted] On the other hand if we take θ to be the vector belonging to $z^{-1}H^{-}$ = span of $z^{-1}, z^{-2}, z^{-3}, \ldots\ldots$ then

$$\psi(\zeta)^*\theta = \sum \zeta^{-n}\psi_n^*\theta = \sum_{n\geq 0} \zeta^{-n}\underbrace{\psi_n^*\theta}_{\Omega \text{ if } n=0}$$

so $\quad \langle\theta|\psi(\zeta) = \langle\Omega| + \zeta\langle\theta|\psi_1 + \zeta^2\langle\theta|\psi_2 + \cdots$

Thus

$$\frac{\langle\theta|\psi(\zeta)\Omega_W\rangle}{\langle\Omega|\Omega_W\rangle} = 1 + \zeta\langle\theta|\psi_1\Omega_W\rangle + \cdots\cdots$$

$$\in (1+H^+)\cap W$$

is the desired function s. Better:

$$s(\zeta) = \frac{\langle\Omega|\psi_0^*\psi(\zeta)\Omega_W\rangle}{\langle\Omega|\Omega_W\rangle}$$

$$= 1 - \frac{\langle\Omega|\psi(\zeta)\psi_0^*\Omega_W\rangle}{\langle\Omega|\Omega_W\rangle}$$

Baker-Akhieser fn. belonging to W is the unique element of W of the form

$$s(\underline{x}, z) = e^{-\sum_1^\infty x_n z^{-n}}\left\{1+\sum_1^\infty a_n(\underline{x})z^n\right\}$$

Thus you want $\quad e^{\sum x_n z^{-n}} s(\underline{x}, z) \in (1+H^+)\cap e^{\sum x_n z^{-n}} W$

and by the above formula

$$e^{\sum x_n z^{-n}} s(\underline{x}, z) = \frac{\langle\Omega|\psi_0^*\psi(\zeta)\, T_{\underline{x}}\,\Omega_W\rangle}{\langle\Omega|T_{\underline{x}}\Omega_W\rangle}$$

where $T_{\underline{x}}$ is an operator on Fock space corresponding to the function $e^{\sum x_n z^{-n}}$. The denominator is the τ function and the numerator can be written as a τ fn. using the vertex <u>operator formula</u> for $\psi(\zeta)$.

One thing I could try to understand is the relation between the Green's functions and the S-matrix. What could you mean by this? The free Green's fn. is

$$G_0(xt, x't') = \langle T[\psi(xt)\,\psi(x't')^*]\rangle$$

where $\psi(xt) = \frac{1}{\sqrt{L}} \sum_k e^{ikx - kt}\psi_k = \frac{1}{\sqrt{2\pi}} \sum \big(\underbrace{e^{i(x+it)}}_{z}\big)^k \psi_k$

is killed by $\partial_{\bar z} = \frac{i}{2}\left(\partial_t + \frac{1}{i}\partial_x\right)$. Similarly

$$\psi(xt)^* = \frac{1}{\sqrt{L}} \sum e^{-ikx} e^{kt} \psi_k^* = \frac{1}{\sqrt{2\pi}} \sum \left(e^{i(x+it)}\right)^{-k}\psi_k$$

is killed by $\partial_{\bar z}$. Thus the Green's function $G(z,z')$ with interaction will be analytic in both z, z' outside the support of the interaction.

The idea is as follows. Fix z' someway. Then analytically continue $G(z,z')$ from the regions $t \gg 0$ (or $|z| \ll 1$) back to $|z| = 1$. Similarly continue from $|z| \gg 1$ to $|z| = 1$. Then the ratio will be a function on the circle which is the S matrix.

Of course this doesn't work because as a function of z, G either has an analytic continuation or it doesn't, and we know the singularities lie in the support of f, as is clear from the formula $(S^1 \to \mathbb{R})$

$$G(z,z') = \frac{i}{2\pi}\frac{1}{z-z'}\, e^{c\int f(w)\left[\frac{1}{w-z} - \frac{1}{w-z'}\right]d^2w}$$

Nevertheless it should be possible to do the following. Consider the family of $\bar\partial$ operators of the form $\partial_t + \frac{1}{i}\partial_x + f(x,t)$ with f of compact support. Then it should be possible ~~to~~ map this family to the family of function $g(x) \in S^1$, and to cover this map by a map of line bundles. This is clear, and might form a local basis for calculation of determinants.

September 2, 1982

Interesting example: Suppose we have a fermion gas described by $\hat{H}_0 = \sum_k \omega_k \psi_k^* \psi_k$ and we add to it a single fermion state described by the Hamiltonian $\Omega \phi^* \phi$. Thus we get the unperturbed Hamiltonian

$$\hat{H}_0 = \sum_k{}' \omega_k \psi_k^* \psi_k + \Omega \phi^* \phi,$$

to which we add an interaction

$$\hat{H}_{int} = \Big(\sum_k{}' \gamma_k \psi_k^*\Big)\phi + \phi^*\Big(\sum{}' \tilde{\gamma}_k \psi_k\Big).$$

The total Hamiltonian is quadratic, so in principle one should be able to see what happens. The problem is that on the level of the operators $\{\psi_k, \phi\}$ one should have the possibilty of exponential decay, yet somehow this doesn't exist on Fock space.

So let's first understand the Hamiltonian on the 1-particle space which has the basis $|k\rangle$, $|a\rangle$, where ϕ^* = ext. mult by $|a\rangle$, ϕ = int. mult. by $\langle a|$.

Question: There is a close connection between Dirac systems on the line $i\partial_t \psi = \begin{pmatrix} \frac{1}{i}\partial_x & p \\ \bar{p} & -\frac{1}{i}\partial_x \end{pmatrix}\psi$ and wave equations $\partial_t^2 \psi = (\partial_x^2 - q)\psi$ given by factorizing the latter. The natural way to quantize the former uses a fermion Fock space, and for the latter a boson Fock space. Are these two quantizations compatible with the correspondence given by factorization?