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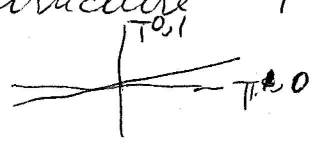
Let's consider varying the holomorphic structure on M. We fix a [redacted] closed smooth orientable [redacted] surface M and a smooth vector bundle E over it. Let $\tilde{a}$ be the space [redacted] of holom. structures on (M,E). Then a point of $\tilde{a}$ consists of a subspace $T^{1,0} \subset T^*_{\mathbb{R}} \otimes \mathbb{C}$ disjoint from $T^{0,1} = \overline{T^{1,0}}$, and a $\bar{\partial}$-operator $D: E \to E \otimes T^{0,1}$. I have forgotten to add the condition that the [redacted]. holomorphic structure on M be compatible with the orientation. Then the holom. structures on M are sections of a fibre bundle with fiber $GL_2(\mathbb{R})^+/\mathbb{C}^* = SL_2(\mathbb{R})/S^1 = UHP$.

Now over $\tilde{a} \times M$ we then have two vector bundles $\tilde{E}$ and $\tilde{E} \otimes \widetilde{T^{0,1}}$, where $\tilde{E} = pr_2^* E$ and $\widetilde{T^{0,1}} \subset pr_2^*(T^*_{\mathbb{R}} \otimes \mathbb{C})$. Also we have the differential operator $\tilde{D}: \tilde{E} \to \tilde{E} \otimes \widetilde{T^{0,1}}$ which is over $\tilde{a}$ (i.e. commutes with fns. on $\tilde{a}$) and elliptic along the fibres of $\tilde{a} \times M \to \tilde{a}$. Hence we can define the index as a virtual bundle over $\tilde{a}$, and we can construct over $\tilde{a}$ the index line bundle $\mathcal{L}$.

(I should think of $T^{0,1}$ as a quotient of $T^*_{\mathbb{R}} \otimes \mathbb{C}$. The [redacted] set of quotients is a projective line $\mathbb{P}(T^*_{\mathbb{R}} \otimes \mathbb{C})$ and $\mathbb{P}(T^*_{\mathbb{R}} \otimes \mathbb{C}) - \mathbb{P}(T^*_{\mathbb{R}})$ consists of two disks, the upper and lower half planes [redacted]. Now it is clear why there is a holom. structure on the set of holom. structures on M.)

Let's suppose we have a [redacted] holomorphic structure $0 \to T^{1,0} \to T^* \otimes \mathbb{C} \to T^{0,1} \to 0$. Then another holom. structure $\widetilde{T^{1,0}}$ will be the graph of a map



from $T^{1,0}$ to $T^{0,1}$

hence $\tilde{T}^{1,0}$ will contain a unique section of the form $dz + h d\bar{z}$ where $|h|^2 < 1$. Hence we can identify the space of holomorphic structures on M with the set of sections

$$h \frac{d\bar{z}}{dz} \quad \text{of} \quad Hom(T^{1,0}, T^{0,1}) = \underbrace{(T^{1,0})^*}_{\text{tangent bundle}} \otimes T^{0,1}$$

such that $\left|h \frac{d\bar{z}}{dz}\right|^2 = |h|^2 < 1$.

Next we want the action of the group of orientation-preserving diffeos. on the space of holomorphic structures. Let $\theta : M \longrightarrow M$ be one such, pick $\overbrace{\text{a point } m \in M}^{\text{local coord. } z \text{ about } a}$, and let w be a local coordinate around θm, both of these coordinates being holom. Then $\theta^* T^{1,0}$ is spanned $\overset{\text{near } m}{\text{by}}$ $\theta^* dw = d(w\theta) = \partial_z(w\theta)\, dz + \partial_{\bar{z}}(w\theta) d\bar{z}$ and so [redacted] $\theta^* T^{1,0}$ is described by $h \frac{d\bar{z}}{dz}$ where $h = \partial_{\bar{z}}(w\theta)/\partial_z(w\theta)$.

Actually [redacted] suppose given a different holom. structure $\tilde{T}^{1,0} \subset T^* \otimes \mathbb{C}$, say spanned locally by $dz + h d\bar{z}$. Then a function f on M is holom. for the new structure when $df = \partial_z f \, dz + \partial_{\bar{z}} f \, d\bar{z}$ is proportional to $dz + h d\bar{z}$, i.e. when

$$\frac{\partial_{\bar{z}} f}{\partial_z f} = h \quad \text{or} \quad (\partial_{\bar{z}} - h \partial_z) f = 0.$$

Let's try to understand what happens when we have a vector field in which case we [redacted] take $w = z$ and then have $z \longmapsto z + \varepsilon(z)$ where ε is infinitesimal. Then

$$h = \frac{\partial_{\bar{z}}(z+\varepsilon)}{\partial_z(z+\varepsilon)} = \frac{\partial_{\bar{z}} \varepsilon}{1 + \partial_z \varepsilon} = \partial_{\bar{z}} \varepsilon.$$

Therefore given a vector field $\varepsilon(z) \frac{\partial}{\partial z}$ on M it transforms the standard holom. structure into

$$\bar{\partial}\left(\varepsilon \frac{\partial}{\partial z}\right) = \partial_{\bar{z}} \varepsilon \cdot \frac{\partial}{\partial z} \otimes d\bar{z}.$$

At this point we can describe nicely a transversal in the space of holomorphic structures on M to the orbit under the vector fields. Namely given a holom. structure γ on M, the tangent space at [redacted] this point γ can be identified with $\Gamma(T \otimes_{\mathbb{C}} T^{0,1})$ where T is the tangent bundle with the given by γ complex structure. The tangent space to the orbit thru γ is the image of

$$\Gamma(T) \xrightarrow{\bar\partial} \Gamma(T \otimes_{\mathbb{C}} T^{0,1})$$

and so the normal space to the [redacted] orbit will be $H^1(T)$ as it should be. [redacted] Now $H^1(T)^* = H^0(T^{1,0} \otimes T^{1,0})$ is the space of holomorphic quadratic differentials, the transpose of the $\bar\partial$ operator above being the $\bar\partial$ operator on $T^{1,0} \otimes T^{1,0}$.

A volume on M combined with the complex structure on T will then give [redacted] an inner product on T allowing one to identify $(\bar\partial)^t$ [redacted] anti-linearly with $(\bar\partial)^*$. [redacted]

Idea of Teichmüller space: Suppose $g > 1$. Then we have $H^0(T) = 0$ [redacted] because degree $(T) = 2 - 2g < 0$. Hence also

$$\dim H^1(T) = -[2 - 2g + 1 - g] = 3g - 3.$$

The fact that $H^0(T) = 0$ means that the vector fields act freely on the holomorphic structures on M. Presumably the connected component of the identity in $\mathrm{Diff}(M)$ acts freely and Teichmüller space is the quotient. But probably one does even better by [redacted] a variational problem. I mean you pick an [redacted] origin and find the point on the orbit closest to the origin.

I want to understand better the $\partial, \bar\partial$ operators belonging to a different holomorphic structure $\widetilde{T}^{1,0}$ = span of $dz + h d\bar z$. We've seen the condition that a fn. be holom. is $(\partial_{\bar z} - h\partial_z) f = 0$. This suggests

$$df = c\left\{(\partial_z - \bar h \partial_{\bar z})f \cdot dz + h d\bar z + (\partial_{\bar z} - h\partial_z) f \cdot d\bar z + \bar h dz\right\}$$

$$= c\{ f_z dz - \cancel{\bar h f_{\bar z} dz} + \cancel{f_z h d\bar z} - |h|^2 f_{\bar z} d\bar z$$

$$+ f_{\bar z} d\bar z - \cancel{h f_z d\bar z} + \cancel{f_{\bar z} \bar h dz} - |h|^2 f_z dz\} \qquad \therefore c = \frac{1}{1-|h|^2}$$

Thus

$$d = \frac{1}{1-|h|^2}\left\{(\partial_z - \bar h \partial_{\bar z})\cdot(dz + h d\bar z) + (\partial_{\bar z} - h\partial_z)\cdot(d\bar z + \bar h dz)\right\}.$$

Now suppose we have a connection form $A = \gamma dz + \alpha d\bar z$ or

$$A = \frac{1}{1-|h|^2}\left\{(\gamma - \bar h \alpha)(dz + h d\bar z) + (\alpha - h\gamma)(d\bar z + \bar h dz)\right\}$$

The new $\bar\partial$-operator $E \longrightarrow E \otimes \widetilde{T}^{0,1}$ is given by $\widetilde{D} = \frac{1}{1-|h|^2}(\mathcal{D} - h\widetilde{\mathcal{D}})\ (d\bar z + h dz)$ where $\mathcal{D} = \partial_{\bar z} + \alpha$, $\widetilde{\mathcal{D}} = \partial_z - \alpha^* = \partial_z + \gamma$.

Next I want the adjoint and the Laplacean. I recall that there is a natural inner product on sections of $E \otimes \widetilde{T}^{0,1}$, once a metric on E is given. Hence to define an adjoint for $\widetilde{D}$, we must give at least a volume on M. But this with the complex structure gives the metric on M. But still, all I have to give is the volume form $\frac{i}{2} g\, dz d\bar z$ and then I should be able to write out the operators $\widetilde{D}^*$ and $\widetilde{D}^* \widetilde{D}$.

Recall that the inner product of two sections $f d\bar z$, $g d\bar z$ of $T^{0,1}$ is defined by

$$(f d\bar z | g d\bar z) = \int \frac{i}{2} \bar f dz \wedge g d\bar z.$$

Extrapolating we get

$$(\tilde{\mathcal{D}} f \,|\, \tilde{\mathcal{D}} \check{g}) = \frac{i}{2}\int \frac{1}{1-|h|^2}\left[(\mathcal{D}-h\tilde{\mathcal{D}})f\right]^*(dz+\bar{h}\,d\bar{z})\,\frac{1}{1-|h|^2}\,(\mathcal{D}-h\tilde{\mathcal{D}})\check{g} \times (d\bar{z}+h\,dz)$$

$$= \int \left[(\mathcal{D}-h\tilde{\mathcal{D}})f\right]^*(\mathcal{D}-h\tilde{\mathcal{D}})\check{g}\,\frac{1}{1-|h|^2}\,\frac{i}{2}\,dz\,d\bar{z}$$

$$= \int f^*\frac{1}{g}\left(-\tilde{\mathcal{D}}+\bar{h}\mathcal{D}\right)\frac{1}{1-|h|^2}\left(\mathcal{D}-h\tilde{\mathcal{D}}\right)\check{g}\;\underbrace{g\frac{i}{2}\,dz\,d\bar{z}}_{\text{volume el.}}$$

Thus

$$\boxed{-\tilde{D}^*\tilde{D} = \frac{1}{g}\left(\tilde{\mathcal{D}}-\bar{h}\mathcal{D}\right)\frac{1}{1-|h|^2}\left(\mathcal{D}-h\tilde{\mathcal{D}}\right)}$$

Don't confuse the ~ over $\tilde{\mathcal{D}}$ with the ~ over $\tilde{D}$; $\tilde{\mathcal{D}} = \partial_{\bar{z}} - \alpha^*$ is fixed independent of h. Now the idea is to suppose that h is infinitesimal, whence

$$-\delta D^*D = -\tilde{D}^*\tilde{D} + D^*D = \cancel{\frac{1}{g}\left[\bar{h}\mathcal{D}^2 - \tilde{\mathcal{D}}h\tilde{\mathcal{D}}\right]} - \frac{1}{g}\left[\mathcal{D}\bar{h}\mathcal{D} + \tilde{\mathcal{D}}h\tilde{\mathcal{D}}\right]$$

which is a 2nd order operator as it must be because the symbol is changing with h.

The general problem is to find the variational formulas for how the torsion of a $\bar\partial$ operator varies with respect to the holomorphic structure on the Riemann surface. I first intend to do the constant coefficient case.

So I start with $M = \mathbb{C}/\Gamma$ and suppose as usual that the volume of M with respect to $\frac{i}{2}\,dz\,d\bar z = dx\,dy$ is π. Next I consider the same torus with the holom. structure such that $T^{1,0}$ is [redacted] spanned by $dz + h\,d\bar z$. Here h is a constant with $|h| < 1$. I consider the [redacted] trivial line bundle over M with the connection

(1) $$e^{w\bar z - \bar w z}\, d\, e^{-w\bar z + \bar w z} \text{ [redacted] } = (\partial_z + \bar w)dz + (\partial_{\bar z} - w)d\bar z$$

I need the d operator in the new holom. structure

$$d = \frac{1}{1-|h|^2}\Big[\big(\partial_{\bar z} - h\partial_z\big)\big(d\bar z + \bar h\,dz\big) + \underbrace{\big(\partial_z - \bar h\partial_{\bar z}\big)}_{\text{new } \partial_z}\big(dz + h\,d\bar z\big)\Big].$$

[redacted] The new Laplacean $\bar\partial^*\bar\partial$, [redacted] according to yesterday's calculations, is

$$-\text{[redacted]}\ \bar\partial^*\bar\partial = 2\big(\partial_z - \bar h\partial_{\bar z}\big)\frac{1}{1-|h|^2}\big(\partial_{\bar z} - h\partial_z\big)$$

and to get $-D^*D$ we apply the gauge transformation (1).

What I want to calculate is the value at $s=0$ of

$$\text{[redacted]}\ \operatorname{Tr}\big((D^*D)^{-s}(D^*D)^{-1}\,\delta(D^*D)\big)$$

where $\delta(D^*D)$ is calculated to first order in h:

$$\delta D^*D = 2\big[\bar h\,\nabla_{\bar z}^2 + h\nabla_z^2\big] \qquad \begin{aligned} \nabla_z &= \partial_z + \bar w \\ \nabla_{\bar z} &= \partial_{\bar z} - w \end{aligned}$$

The value at $s=0$ is the constant term in the asymptotic expansion as $t \downarrow 0$ of

$$\operatorname{Tr}\big[e^{-tD^*D}(D^*D)^{-1}\delta(D^*D)\big]$$

All these operators are [redacted] constant coeff. operators on the trivial bundle, and hence commute with each other. We have $-D^*D = 2\nabla^2_{z\bar z}$. The coeff. of [redacted]^h is

$$\mathrm{Tr}\left[e^{t2\nabla^2_{z\bar z}}\left(-\nabla^2_{z\bar z}\right)^{-1}\nabla_z^2\right] = -\mathrm{Tr}\left[e^{t2\nabla^2_{z\bar z}}\nabla_{\bar z}^{-1}\nabla_z\right].$$

Now

$$\langle z|e^{t2\partial^2_{z\bar z}}|z'\rangle = \sum_\mu \frac{e^{-\frac{|z-z'-\mu|^2}{2t}}}{2\pi t}$$

so

$$\langle z|e^{t2\nabla^2_{z\bar z}}|z'\rangle = \sum_\mu \frac{e^{-\frac{|z-z'-\mu|^2}{2t}}}{2\pi t}\, e^{w(\overline{z-z'-\mu}) - \bar w(z-z'-\mu)}.$$

(The idea is you apply the gauge transf. up on $\mathbb{C}$, then average over Γ.)

Now

$$\langle z|\nabla_{\bar z}^{-1}|z'\rangle = e^{w[(\overline{z-z'})+l(z-z')]}\frac{\sigma(z-z'-w)}{\pi\sigma(z-z')\sigma(-w)}$$

$$-\mathrm{Tr}\left[e^{2t\nabla^2_{z\bar z}}\nabla_z\nabla_{\bar z}^{-1}\right] = -\pi\int_M \underbrace{\langle 0|e^{2t\nabla^2_{z\bar z}}|z\rangle}_{\text{sum over }\Gamma}\underbrace{\nabla_z\langle z|\nabla_{\bar z}^{-1}|0\rangle}_{\text{periodic}}\frac{i}{2}dz\,d\bar z$$

$$= \pi\int_{\mathbb{C}} \frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}\underbrace{e^{-w\bar z+\bar w z}(-\nabla_z)e^{w(\bar z+lz)}}_{(-\partial_z)e^{\bar w z+wlz}}\frac{\sigma(z-w)}{\pi\sigma(z)\sigma(-w)}\frac{i}{2}dz\,d\bar z$$

integrate by part

$$= [redacted]\int_{\mathbb{C}} \frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}\left(-\frac{\bar z}{2t}\right)e^{(\bar w+lw)z}\frac{\sigma(z-w)}{\sigma(z)\sigma(-w)}\frac{i}{2}dz\,d\bar z.$$

Now we let $t\downarrow 0$, and the only part that matters is around $z=0$. Lets write

$$e^{(\bar w+lw)z}\frac{\sigma(z-w)}{\sigma(z)\sigma(-w)} = \frac{1}{z}(1+az+bz^2+\cdots)$$

and then

$$-\frac{1}{2t}\int \frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}\,\bar z\,\frac{1}{z}(1+az+bz^2+\cdots) = -\frac{1}{2t}\,b\underbrace{\int \frac{e^{-\frac{|z|^2}{2t}}}{2\pi t}|z|^2}_{2t} = -b$$

for reasons of symmetry.

Now $\quad \sigma(z) = z(1+O(z^4))$

$$\begin{aligned}\log(1+az+bz^2+\cdots) &= az+bz^2 - \tfrac{1}{2}(az)^2 + O(z^3)\\ &= az + \left(b-\tfrac{a^2}{2}\right)z^2 + \cdots\\ &= (\bar w + lw)z + \log\sigma(z-w) - \log\sigma(-w) + O(z^4)\end{aligned}$$

$$a + 2\left(b-\tfrac{a^2}{2}\right)z + \cdots = \bar w + lw + \zeta(z-w) \implies a = (\bar w + lw - \zeta(w))$$

$$2\left(b-\tfrac{a^2}{2}\right) + \cdots = -\wp(z-w) \implies b = \tfrac{a^2}{2} - \tfrac{1}{2}\wp(w)$$

$$\therefore \quad b = \tfrac{1}{2}\left[(\bar w + lw - \zeta(w))^2 - \wp(w)\right]$$

The final formula is

$$\lim_{t\to 0}\operatorname{Tr}\left[e^{t2\nabla^2_{z\bar z}}(-2\nabla^2_{z\bar z})^{-1}2\nabla_z^2\right] = \tfrac{1}{2}\left[\wp(w) - (\zeta(w) - \bar w - lw)^2\right]$$

Therefore I have the following formulas for a lattice Λ in $\mathbb{C}$

$$\sum_{\mu\in\Lambda}\frac{1}{|w-\mu|^{2s}}\frac{1}{w-\mu} \xrightarrow{s\to 0} \zeta_\Lambda(w) - l_\Lambda w - m_\Lambda \bar w$$

$$\sum_{\mu\in\Lambda}\frac{1}{|w-\mu|^{2s}}\frac{\overline{w-\mu}}{w-\mu} \xrightarrow{s\to 0} \frac{1}{2m_\Lambda}\left[\wp_\Lambda(w) - (\zeta_\Lambda(w) - l_\Lambda w - m_\Lambda\bar w)^2\right]$$

Here we have $\quad \zeta_{t\Lambda}(tw) = \frac{1}{t}\zeta_\Lambda(w) \quad$ hence

$$\boxed{l_{t\Lambda} = \frac{1}{t^2}l_\Lambda \qquad m_{t\Lambda} = \frac{1}{|t|^2}m_\Lambda}$$

In fact I already know that

$$m_\Lambda = \frac{\pi}{\operatorname{vol}(\mathbb{C}/\Lambda)}$$

Next project. We fix $\Lambda \subset \mathbb{C}$ let the complex structures on $\mathbb{C}/\Lambda$ be described by h, and [redacted] the connections by ω as above. Then as h, ω vary we get an analytic torsion, and I want to calculate its variation. The exponential functions on $M = \mathbb{C}/\Lambda$ are $e^{\mu\bar z - \bar\mu z}$ where μ runs over the dual lattice. The Laplacian is (relative to the volume $\frac{i}{2} dz d\bar z$)

$$D^*D = -2\left(\nabla_z - \bar h \nabla_{\bar z}\right)\frac{1}{1-|h|^2}\left(\nabla_{\bar z} - h\nabla_z\right)$$

$$\nabla_{\bar z} = \partial_{\bar z} - \omega \qquad \nabla_z = \partial_z + \bar\omega$$

$\nabla_{\bar z} - h\nabla_z$ has eigenvalues $\mu - \omega - h(-\bar\mu + \bar\omega) = \mu - \omega + h(\overline{\mu - \omega})$

$\nabla_z - \bar h \nabla_{\bar z}$ —————— $-\bar\mu + \bar\omega - \bar h(\mu - \omega) = -(\quad '' \quad)$

hence the eigenvalues of D^*D are $\frac{2}{1-|h|^2}\left|\mu - \omega + h(\overline{\mu-\omega})\right|^2$.

Put [redacted] $\alpha = \omega + h\bar\omega$, $\gamma = \mu + h\bar\mu$ and $\Gamma = \{\gamma \mid \mu \in \Lambda\}$. Then

$$\zeta_{D^*D}(s) = 2^{-s}(1-|h|^2)^s \sum_{\gamma\in\Gamma} \frac{1}{|\gamma - \alpha|^{2s}}$$

Now I know that such a ζ-function vanishes already at $s=0$ for elliptic curves. Hence the first two factors don't contribute to $\zeta'(0)$, and I can ignore them. But this is not a good idea.

$$\cancel{-\frac{\delta\zeta(s)}{s}\Big|_{s=0} = \lim \sum \frac{1}{|\gamma - x|^{2s}} \underbrace{\frac{\delta|\gamma - x|^2}{|\gamma - x|^2}}_{\frac{\delta(x-\gamma)}{x-\gamma} + c.c.}}$$

$$\cancel{\delta(x-\gamma) = \delta\omega + h\,\delta\bar\omega + \delta h\,\bar\omega - \delta h\,\bar\mu}$$

$$-\frac{\delta\zeta(s)}{s}\Big|_{s=0} = \lim_{s\to 0}\sum \frac{1}{|\lambda|^{2s}}\frac{\delta|\lambda|^2}{|\lambda|^2}$$

$$\delta\log|\lambda|^2 = \delta\log\left(\frac{2|\alpha-\gamma|^2}{1-|h|^2}\right) = \frac{\delta(\alpha-\gamma)}{\alpha-\gamma} + \frac{\bar h\,\delta h}{1-|h|^2} + c.c.$$

$$= \frac{1}{\alpha-\gamma}\frac{1}{1-|h|^2}\left[\left(\bar h\,\delta h\,(\omega - \mu + h(\bar\omega - \bar\mu))\right) + (1-|h|^2)\left(\delta h(\bar\omega - \bar\mu) + \delta\omega + h\,\delta\bar\omega\right)\right] + c.c.$$

$$= \frac{1}{\alpha-\gamma}\frac{1}{1-|h|^2}\left[[redacted]\ \bar h\,\delta h\,(\omega - \mu) + (\bar\omega - \bar\mu)\delta h + (1-|h|^2)(\delta\omega + h\,\delta\bar\omega)\right] + c.c.$$

$$= \frac{\delta h}{1-|h|^2}\,\frac{\overline{\alpha-\gamma}}{\alpha-\gamma} + \frac{\delta\omega + h\,\delta\bar{\omega}}{\alpha-\gamma} + \text{c.c.}$$

Thus
$$-\delta\zeta'(0) = \frac{\delta h}{1-|h|^2}\,\frac{1}{2i m_\Gamma}\left[\wp_\Gamma(\alpha) - \left(l_\Gamma\alpha + m_\Gamma\bar{\alpha} - \zeta_\Gamma(\alpha)\right)^2\right]$$
$$+ (\delta\omega + h\,\delta\bar{\omega})\left[\zeta_\Gamma(\alpha) - l_\Gamma\alpha - m_\Gamma\bar{\alpha}\right] + \text{c.c.}$$

The relation $l_{t\Lambda} = \frac{1}{t^2} l_\Lambda$ suggests that l_Λ might be zero, because the only obvious way to construct such a function is as a sum $\sum' \frac{1}{\lambda^k}$, which doesn't converge for $k=2$.

However l_Λ is not an analytic function of Λ. In effect take $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ and let τ vary. Then

$$\zeta_\Lambda(z+1) - \zeta_\Lambda(z) = l + m \qquad m = \frac{\pi}{\operatorname{Im}\tau}$$

The left-side is analytic in Λ and since we know m is not analytic in Λ, it follows that l can't be analytic in Λ. An obvious candidate for l_Λ is

$$\lim_{s\to 0} \sum{}' |\lambda|^{-2s} \frac{1}{\lambda^2}$$

June 14, 1982

Let's try to derive the analytic continuation for series like $\sum |z-\gamma|^{-2s}(z-\gamma)^k\overline{(z-\gamma)}^\ell$.

$$\langle z|e^{t\partial_z\partial_{\bar z}}|z'\rangle = \frac{1}{\pi t}e^{-\frac{|z-z'|^2}{t}} \quad \text{in } \mathbb{C}.$$

$$\langle z|\underbrace{e^{t(\partial_z+\bar\omega)(\partial_{\bar z}-\omega)}}_{e^{\omega\bar z-\bar\omega z}\,e^{t\partial_z\partial_{\bar z}}\,e^{-\omega\bar z+\bar\omega z}}|z'\rangle = \frac{1}{\pi t}e^{-\frac{|z-z'|^2}{t}}e^{\omega(\overline{z-z'})-\bar\omega(z-z')}$$

Thus on $\mathbb{C}/\Gamma$ we have

$$\langle z|e^{t(\partial_z+\bar\omega)(\partial_{\bar z}-\omega)}|z'\rangle = \frac{1}{\pi t}\sum_\gamma e^{-\frac{|z-z'-\gamma|^2}{t}+\omega(\overline{z-z'-\gamma})-\bar\omega(z-z'-\gamma)}$$

$$= \frac{1}{vol(\mathbb{C}/\Gamma)}\sum_\mu e^{-t|\mu-\omega|^2}e^{\mu(\overline{z-z'})-\bar\mu(z-z')}$$

I set $z'=0$ change $t \mapsto 1/t$ $\quad \omega \mapsto J$

$$\sum_\gamma e^{-t|z-\gamma|^2+J(\overline{z-\gamma})-\bar J(z-\gamma)} = \frac{\pi}{vol}\sum_\mu \frac{e^{-\frac{1}{t}|\mu-J|^2+\mu\bar z-\bar\mu z}}{t}$$

$$\boxed{\sum_\gamma |z-\gamma|^{-2s}e^{J(\overline{z-\gamma})-\bar J(z-\gamma)}} = \frac{1}{\Gamma(s)}\int_0^\infty \frac{\pi}{vol}\sum_\mu e^{-\frac{1}{t}|\mu-J|^2+\mu\bar z-\bar\mu z}\,t^{s-1}\frac{dt}{t}$$

To do the analytic continuation one ~~[illegible]~~ first assumes J not a μ period. Then you get

$$\boxed{\frac{\Gamma(1-s)}{\Gamma(s)}\frac{\pi}{vol(\mathbb{C}/\Gamma)}\sum_\mu |\mu-J|^{2s-2}e^{\mu\bar z-\bar\mu z}}$$

How to calculate ℓ, m (after Lang)

Form

$\zeta(z,s) =$ Weierstrass ζ with an s thrown in.

Need
$$\frac{1}{(z-\gamma)|z-\gamma|^{2s}} + \frac{1}{\gamma|\gamma|^{2s}} = \frac{-1}{\gamma|\gamma|^{2s}}\left\{\frac{1}{(1-\frac{z}{\gamma})|1-\frac{z}{\gamma}|^{2s}} - 1\right\}$$
$$= \frac{-1}{\gamma|\gamma|^{2s}}\left\{\left(1+\frac{z}{\gamma}+\cdots\right)\left(1-\frac{z}{\gamma}-\frac{\bar z}{\bar\gamma}+\cdots\right)^{-s} - 1\right\}$$
$$= \frac{-1}{\gamma|\gamma|^{2s}}\left\{\frac{(s+1)z}{\gamma} + \frac{s\bar z}{\bar\gamma} + \cdots\right\}$$

Thus I put
$$\zeta(s,z) = \frac{1}{z|z|^{2s}} + \sum{}'\left(\frac{1}{(z-\gamma)|z-\gamma|^{2s}} + \frac{1}{\gamma|\gamma|^{2s}} + \frac{(s+1)z}{\gamma^2|\gamma|^{2s}} + \frac{s\bar z}{|\gamma|^{2s+2}}\right)$$
and this approaches the usual ζ as $s \to 0$. On the other hand for large s it is a sum of individual convergent series which are known to have analytic continuations. Now we have
$$\zeta(s,z+\lambda) - \zeta(s,z) = \cancel{\sum{}'\frac{1}{\gamma|\gamma|^{2s}}} + (s+1)\lambda\sum{}'\frac{1}{\gamma^2|\gamma|^{2s}} + s\bar\lambda\sum{}'\frac{1}{\gamma^{2s+2}}$$
as $s\to 0$
$$\zeta(z+\lambda) - \zeta(z) \doteq \lambda\sum{}'\frac{1}{\gamma^2|\gamma|^{2s}}\Big|_{s=0} + \bar\lambda\frac{\pi}{\operatorname{vol}(\mathbb{C}/\Gamma)}$$

which proves the formula $\boxed{\ell = \sum{}'\frac{1}{\gamma^2|\gamma|^{2s}}\Big|_{s=0}}$

Go back to the general question of varying the analytic structure on the Riemann surface. Recall that the volume $g\frac{i}{2}dz\,d\bar z$ is fixed and that the connection
$$\nabla = \underbrace{(\partial_z - \alpha^*)}_{\nabla_z}dz + \underbrace{(\partial_{\bar z}+\alpha)}_{\nabla_{\bar z}}d\bar z$$
on the bundle is fixed. Now consider the family of analytic structures given by $(dz + h\,d\bar z) \in T^{1,0}$, whence we have seen that the Laplacean is
$$D^*D = -\frac{1}{g}\left(\nabla_z - \bar h\nabla_{\bar z}\right)\frac{1}{1-|h|^2}\left(\nabla_{\bar z} - h\nabla_z\right)$$
and so
$$\delta(D^*D) = \frac{1}{g}\nabla_z\,\delta h\,\nabla_z + \frac{1}{g}\,\delta\bar h\,\nabla_{\bar z}^2 \qquad \text{around } h=0$$

I am interested in computing

$$\operatorname{Tr}\left(\left(D^*D\right)^{-s}\left(D^*D\right)^{-1}\delta\left(D^*D\right)\right).$$

We should write $\delta'(D^*D) = \frac{1}{g}\nabla_{\bar z}\,\delta h\,\nabla_z$ invariantly. The point is that $\delta h \in \Gamma\{\operatorname{Hom}(T^{1,0}, T^{0,1})\}$, hence $\frac{1}{g}\nabla_{\bar z}\,\delta h\,\nabla_z$ can be interpreted as the composition

$$E \xrightarrow{D'} E\otimes T^{1,0} \xrightarrow{id\otimes\delta h} E\otimes T^{0,1} \xrightarrow{-D^*} E$$

where $D' = \nabla_z\, dz$. Thus the operator whose regularized trace we are after is just

$$E \xrightarrow{D'} E\otimes T^{1,0} \xrightarrow{id\otimes\delta h} E\otimes T^{1,0} \xrightarrow{-D^{-1}} E$$

June 15, 1982

Problem: We have a fixed C^∞ surface + vector bundle (M,E) and we are trying to define $\det(D)$ where D is the $\bar\partial$-operator as we vary the holomorphic structure on (M,E). This means we want to define $\mathrm{Tr}(D^{-1}\delta D)$, where δD is equivalent to a tangent vector at D in the space of holomorphic structures. The problem is to describe in a completely invariant fashion what δD is, and how one is going to regularize the trace.

The first idea is that δD is to be an operator from E ~~to~~ $E\otimes T^{0,1}$ since we need that to define a trace with D^{-1}. However then one must ask about the symbol which should depend only on the change in the complex structure, i.e. a homomorphism $\delta h : T^{1,0} \longrightarrow T^{0,1}$. Then one has

$$\begin{tikzcd} & E\otimes T^{1,0} \arrow[d, "id\otimes\delta h"] \\ E\otimes T^{*} \arrow[ur, dashed, "?"] \arrow[r, "\substack{\text{proposed}\\ \text{symbol}\\ \text{for } \delta D}"'] & E\otimes T^{0,1} \end{tikzcd}$$

and so it seems one has to use the projection $T^*\to T^{1,0}$ which is inherently non-analytic. This is suspicious. [redacted] The problem is perhaps due to the fact that $T^{0,1}$ is varying as D moves around. We have an affine bundle

$$\Gamma(\mathrm{Hom}(E,E)\otimes\mathcal{O}(1)) \longrightarrow \{D\} \longrightarrow \Gamma(\mathbb{P}(T^*)_+).$$

I think of a holomorphic structure on M as a quotient line $T^*\longrightarrow T^{0,1}$ of $T^* = T^*_{\mathbb{R}}\otimes\mathbb{C}$ in the + component. Hence $\mathcal{O}(1)$ is the line bundle which assigns to a holom. structure the space $T^{0,1}$.

Let's look at what happens over a pt. of M. Then we give ourselves lines in T^* at that point, and then a diff'l operator $J_1(E) \longrightarrow E\otimes\mathcal{O}(1)$ with a prescribed symbol. The point is that the above bundle is obtained by taking Γ of a fibre bundle

$$\mathrm{Hom}(E, E\otimes\mathcal{O}(1)) \longrightarrow \{\quad\} \longrightarrow \mathbb{P}(T^*)_+$$

where the middle is the subbundle of $\mathrm{Hom}(J_1(E), E\otimes\mathcal{O}(1))$ inverse image of $id_E \otimes \pi : E\otimes T^* \longrightarrow E\otimes\mathcal{O}(1)$.

Now our question is to [redacted] describe the tangent vectors to the bundle in the middle. Maybe a simple case is to describe tangent vectors to $\mathcal{O}(1)$ over $\mathbb{P}(T^*)_+$. A tangent vector to $\mathbb{P}(T^*)_+$ is an element of the bundle $\cancel{\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}(1))}$ $\mathrm{Hom}(\Lambda^2T^*\otimes\mathcal{O}(-1), \mathcal{O}(1)) \simeq \Lambda^2T\otimes\mathcal{O}(2)$.

In the case $E = \mathcal{O}$ I have an affine space bundle over $\mathbb{P}(T^*)_+$ for $\mathcal{O}(1)$.

June 19, 1982

Atiyah's idea: Consider the canonical line bundle $\mathcal{L}$ over the space $\mathcal{A}$ with its analytic torsion metric. We know this metric is invariant under gauge transf. but not complex gauge transf. So one can consider the variational problem of minimizing the distance [redacted] from [redacted] a complex gauge orbit of $\mathcal{L}$ to the zero section.

Consider first the case of index 0 and the open set of invertible D. Here [redacted] one has the canonical section s which is fixed under the complex-gauge group and so one wants to minimize $|s|^2 =$ anal. torsion of D^*D over a complex gauge orbit. Let's suppose we are at a critical point. The differential of $\log |s|^2$ is $B \longmapsto \mathrm{Tr}\left((D^*D)^{-s} D^{-1} B\right)'_{s=0} = \int_M \mathrm{tr}(JB)$, where J is the heat kernel F.P. of D^{-1} on the diagonal. Tangent vectors coming from complex gauge transformation are of the form $B = [D, X]$ where $X \in \Gamma(\mathrm{End}(E))$. Hence we are at a critical point when for all X

$$0 = \int \mathrm{tr}(J[D,X]) = -\int \mathrm{tr}([D,J]X)$$

i.e. when $[D,J] = 0$. However by the anomaly formula I know that

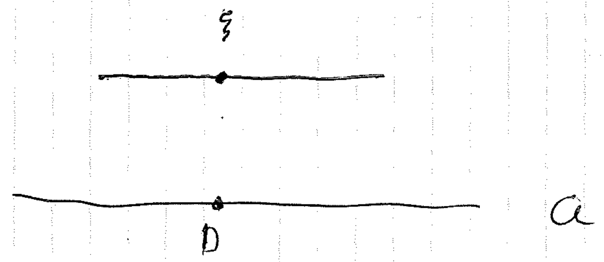
$$[D,J] = \frac{i}{2\pi}\left(\begin{matrix}\text{curvature of } \nabla \\ \text{assoc. to } D\end{matrix} + \frac{1}{2}\begin{matrix}\text{curv. of metric} \\ \text{on } M\end{matrix}\right)$$

It follows that D is critical $\Longleftrightarrow$

$$\begin{matrix}\text{curvature} \\ \text{of } D\end{matrix} = -\frac{1}{2}(\text{curvature of } M)$$

(Check: $\deg E = -\frac{1}{2}(2-2g) \times \text{rank} = (g-1) \times \text{rank}$).

Next I want to work out the ~~situation~~ in general and thereby understand the general version of the anomaly formula. So let us fix a D (or a connection $A \in \mathcal{A}$) and a point ξ of $\mathcal{L}$ over it such that ξ is a critical point for the distance-from-the-zero-section-function on the complex gauge orbit of ξ. (This depends only on D.) Actually without making this last assumption note the following geometry.



$\mathcal{L}$ is a holom. bundle over $\mathcal{A}$ with metric so it has a canonical connection, hence through ξ is a canonical tangent plane transversal to the fibre. On the other hand because of the $\mathcal{G}^c$-action there is a canonical way to assign tangent vectors at ξ to $X \in \Gamma(\mathrm{End}\, E)$, and $X \cdot \xi$ lies over $X \cdot D = [X, D]$. So given X we can compare $X \cdot \xi$ with the lift of $[X, D]$. The difference is a tangent vector to the fibre, which can be identified with a number, thanks to ξ. <u>Hence we get a canonical linear functional on</u> $\Gamma(\mathrm{End}\, E)$ <u>depending</u> on D. It should be possible to express this as an inner product with a section of $\mathrm{End}(E) \otimes T^{1,1}$, like the form $[D, J]$ described above.

However if the two-form were of the form $[D, J]$ then we would have for X a holom. endo. of E

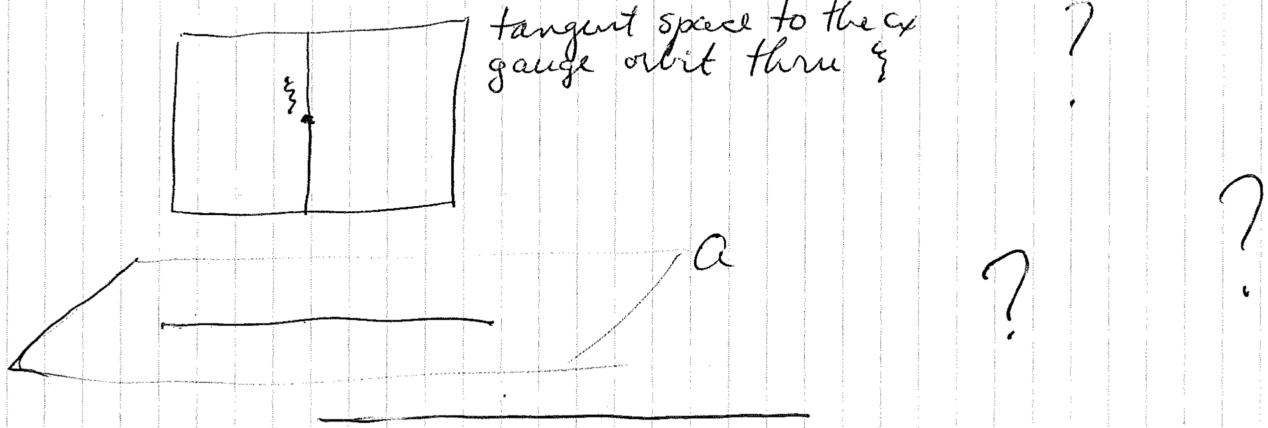
$$\int tr([D,J]\, X) = \int tr(J\, [X, D]) = 0$$

and this would mean $H^0(\text{End}\, E)$ acted trivially on $\lambda(H^0(E))^* \otimes \lambda(H^1(E))$, which is obviously wrong for scalars.

Thus the 2-form I seek has the property that when integrated against an element X of $H^0(\text{End}\, E)$, it gives the trace of X on the cohomology of E. (It seems almost as if in general [redacted] given an elliptic operator $D: E \to F$ and an infinitesimal automorphism X of E and F commuting with the symbol, then one should be able to define the trace of X relative to D.) So when $X = \text{id}$ the integral of the 2-form is to give the index. This suggests that we get the same form as for [redacted] index 0, namely

$$\frac{i}{2\pi}\left[\text{curv. of } D + \tfrac{1}{2}\text{curv. of } T_M\right].$$

Except it is now necessary to reformulate the [redacted] critical point problem, because the complex gauge group acts non-trivially in the fibrewise direction. So instead of considering all $X \in \Gamma(\text{End}\, E)$ we will [redacted] only consider those which [redacted] act at ξ the same as the lift of $[D, X]$. Thus I want the



So I should do some examples of stability in Mumford's sense. One starts with a representation of a reductive alg. gp G in a vector space V. [redacted] Mumford starts with [redacted] G acting on a projective variety X

and an equivariant line bundle $\mathcal{O}(1)$ which gives a projective embedding. This should be the same with $V = \Gamma(X, \mathcal{O}(1))$. In any case the important thing is to understand on what subset of $\mathbb{P}(V)$ you can expect to construct an orbit space. Supposedly the good subset is given by the points of $V - \{0\}$ whose orbits are closed in V. Then if one takes the maximal compact K of G and chooses a K-invariant metric on V, each closed G-orbit will contain points closest to 0, and these divided out by K give the orbit space.

Example: Take GL_n acting by conjugation on $\mathfrak{gl}_n$. The orbits are the different Jordan forms and one knows that only the semi-simple matrices have closed orbits. In each semi-simple class is a unique U_n conjugacy class of normal matrices (eigenspaces are $\perp$) which presumably is the part closest to 0 in the orbit

Ex: Let $\mathbb{G}_m$ act on $\mathbb{C}^2$ by $t(z_n) = (tz_1, t^{-1}z_2)$. Then there are 3 orbits on $\mathbb{P}^1$.

June 20, 1982

Let's check some ideas of Atiyah. The first is that the space of connections, a gauge group $\mathcal{G}^c$ are the same for E and for $L \otimes E$, only the canonical line bundles differ. To get an isomorphism $\mathcal{a}_E \xrightarrow{\sim} \mathcal{a}_{L\otimes E}$ we of course need a holomorphic structure on L. The gauge groups $\mathcal{G}^c_E$, $\mathcal{G}^c_{L\otimes E}$ are obviously the same, so one should check the isom. is compatible with the actions. The good way is to see there are canonical maps

$$\begin{cases} \mathcal{a}_L \times \mathcal{a}_E \longrightarrow \mathcal{a}_{L\otimes E} \\ \mathcal{G}^c_L \times \mathcal{G}^c_E \longrightarrow \mathcal{G}^c_{L\otimes E} \end{cases}$$

and all we are doing is to fix a point of $\mathcal{a}_L$ and $id \in \mathcal{G}^c_L$.

Locally suppose L, E trivialized and $D_L = \bar\partial + \omega$, $D_E = \bar\partial + A$. Then the map is $\bar\partial + A \longmapsto \bar\partial + \omega + A$, and if g is a gauge transf of E, then

$$g(\bar\partial + A)g^{-1} = \bar\partial + g\bar\partial g^{-1} + gAg^{-1} \longmapsto \bar\partial + \omega + g\bar\partial g^{-1} + gAg^{-1}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad g(\bar\partial + \omega + A)g^{-1}$$

because ω operating on L commutes with g operating on E.

Next let us consider the different line bundles $\mathcal{L}_E$, $\mathcal{L}_{L\otimes E}$. The group $\mathcal{G}^c_E$ doesn't act faithfully on $\mathcal{a}_E$, and the elements $g \in \mathcal{G}^c_E$ which do act trivially are the scalars $\mathbb{C}^*$. Hence I can ask about how $\mathbb{C}^*$ acts on $\mathcal{L}_E$ and the answer is via the character $z \longmapsto z^{-\text{Index}}$. Now

$$\text{Index}(E) = \deg E + (\text{rank})(1-g)$$
$$\text{Index}(E\otimes L) = \deg E + (\text{rank})(1-g+\deg L)$$

Things to work out when you're better:

Dilog. Real part = hyperbolic simplex volume. Imag. part given as follows: Take $P_3(\mathbb{R})$ and the 4 points coming from coord. lines. Then given a line in $P_3(\mathbb{R})$ can join it ~~by~~ planes to these points and take cross-ratio. This gives a map of most of $\mathrm{Grass}_2(\mathbb{R}^4)$ to $\mathbb{R}$ which is also orbits under diagonals in $SL_4(\mathbb{R})$. Then image of the class p_1 is the differential of the dilog.

Atiyah: If a compact torus acts on a compact Kähler manifold M having a fixpt (always true if M alg.), then the restriction of a harmonic form on M to both the torus orbits, and the "orthogonal" real torus orbits is identically 0.

MacPherson + Gelfand use this ($\Rightarrow p_1$ vanishes on $(\mathbb{R}^*)^3$ orbits) to [redacted] identify the dilog from its fnl. eqn.

Have seen how dilog enters into the construction of the central extension of ΩU_n. This extension acts on the line bundle over ΩU_n, which via $H^2(\Omega U_n) = H^4(BSU_n)$ is essentially the first Pontryagin class.

Idea: Given G reductive acting on a proj. variety X one gets a moment map $X \longrightarrow k^*$ and one can define the YM Morse fn. by $\| \|^2$ in k^*. This gives a perfect Morse fn. in rational cohomology. One can also look in the line bundle L over X at points whose orbits are closed. These are the semi-stable ones = minimum for the Morse fn. So you single out the fat [redacted] semi-stable set [redacted] with L.

June 22, 1982: (42 years old)

To understand what Coleman means by denom. free Green's fns. and the associated anomaly identities.

What is the functional integral [redacted] $\int [d\psi][d\bar{\psi}]\, e^{\bar{\psi} D \psi}$? In finite-dimensions $\psi = \{\psi_i\}$ $\bar{\psi} = \{\bar{\psi}_j\}$ and $D = (D_{ji})$ is a matrix. One works in the exterior alg. with generators $\psi_i, \bar{\psi}_j$, and then $[d\psi], [d\bar{\psi}]$ denote [redacted] generators for the highest exterior powers, whereas the integral means one [redacted] takes the appropriate highest component in some sense. If D has index $\neq 0$, then the above integral is 0. In fact the only moments, (or Green's functions), which are non-zero are for the monomials $\psi_{i_1} \cdots \psi_{i_p} \bar{\psi}_{j_1} \cdots \bar{\psi}_{j_q}$ where $p - q =$ index (up to sign).

June 24, 1982

To understand the height function for rational points on an elliptic curve (or abelian variety) defined over a number field. First have to understand the situation over finite fields.

So let E be an elliptic curve (curve of genus 1 equipped with a rational point) defined over $k = \mathbb{F}_q$, (better with $k = \Gamma(\mathcal{O})$). Then $E(\bar{k})$ is a divisible abelian group with an action of Frobenius; for $\ell \neq p = \text{char}(k)$, $E_{(\ell)} \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^2$ and $E_{(p)} = \mathbb{Q}_p/\mathbb{Z}_p$ or 0, the former in the ordinary case. There is an obvious kind of measure one can attach to a point of $E(\bar{k})$, namely, the degree over k of its field of rationality = size of its orbit under Frobenius. I want to see if this degree function has any kind of quadratic behavior.

Recall the Lefschetz formula

$$\text{card } E(k_n) = tr\,(\text{Frob})^n \text{ on } H^*(\bar{E})$$
$$= 1 - (\omega_1^n + \omega_2^n) + q^n$$

where ω_1, ω_2 are conjugate complex numbers of abs. value $\sqrt{q}$. (In fact we have $(X-\omega_1)(X-\omega_2) = X^2 - cX + q$ where $c = \omega_1 + \omega_2$ satisfies

$$h = 1 - c + q \qquad h = \text{card } E(k).$$

The fact that the ω_i are conjugate complex is equivalent to the discriminant being <0:

$$c^2 - 4q = (h-1-q)^2 - 4q \leq 0$$

or

$$-2\sqrt{q} \leq h-1-q \leq 2\sqrt{q}$$

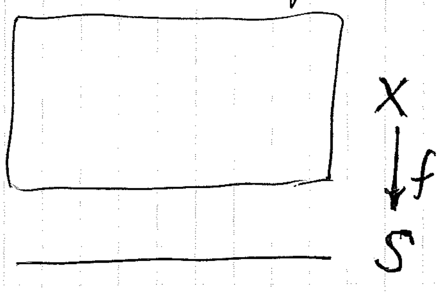
or

$$h = 1-c+q \quad \text{where} \quad |c| \leq 2\sqrt{q}\,.)$$

It doesn't seem to be possible to make a quadratic function out of this degree function in any simple way.

Let's go on to the Arakeloff arithmetic surface situation. We want to think in terms of a surface X over a base curve S. [redacted] From a birational viewpoint a surface is a finitely generated field extension F of $\mathbb{C}$ of transcendence degree 2, so one can pick $\mathbb{C} \subset K \subset F$ with K of transc. deg 1. Then K gives a curve S and F gives a curve C over K, and somehow one fits the curve thus given over the generic point of S into a surface. One maybe should assume that K is maximal algebraic [redacted] inside of F so that it really is the ground field of C.

So our picture is



except the map $f : X \longrightarrow S$ has some singular fibres, e.g. if I were to take an elliptic curve C over K, then its j-invariant becomes singular at certain points of S.

Given a vector bundle E over X one can ask about RR. One takes $Rf_*(E)$ and then [redacted] $R\Gamma_S$ to get $R\Gamma_X$. In the arithmetic surface case S will be a ring of integers completed by the real places. $Rf_*(E) = f_*(E)$ if $E \gg 0$ will be a module over the ring of integers, and the structure at ∞ will provide a volume on $f_*(E) \otimes \mathbb{R}$, maybe even a hermitian form.

So what I have to be careful about perhaps is [redacted] replacing the complex $Rf_*(E)$ by its highest exterior power. $Rf_*(E)$ is a perfect complex over S, say [redacted] quasi-isomorphic to $N^0 \to N^1$, where these are vector bundles over S. Then clearly

$$\dim R\Gamma_S(Rf_* E) = \dim R\Gamma_S(N^0) - \dim R\Gamma_S(N^1)$$

because of the evident triangles, so

$$= \deg(N^0) + rg(N^0)(1-g) \qquad g = \text{genus of } S$$
$$- \deg(N^1) - rg(N^1)(1-g).$$

However $\deg(N^0) - \deg(N^1) = \deg(\lambda N^0 \otimes \lambda N^{1*})$, hence the RR formula reads

$$\dim(R\Gamma_X(E)) = \deg(\lambda Rf_*(E)) + rg([redacted] Rf_*(E))\,(1-g).$$

[redacted] The point is that the K-element of $Rf_*(E)$ is in

$$K_0(S) \xrightarrow{(rg,\,\deg)} \mathbb{Z} \times \mathrm{Pic}(S).$$

Therefore it is clear that the left-side of RR is known once I [redacted] know about the canonical line bundle $\mathcal{L}$ over S.

<u>July 25, 1982</u>

I consider a surface X which is the total space of a fibration $X \xrightarrow{f} S$ where S and the fibres are curves. I am interested in the RR theorem for X. This [redacted] comes out of the RR theorem for S and for the map f by Grothendieck. For the map f one has

$$\begin{tikzcd} K(X) \arrow[r, dashed, "ch"] \arrow[d, "f_!"] & A(X) \arrow[d, "f_*"'] = \mathbb{Z} \oplus \operatorname{Pic} X \oplus A_2(X) \\ K(S) \arrow[r, dashed, "ch"] & A(S) = \mathbb{Z} \oplus \operatorname{Pic} S \end{tikzcd}$$

Question: Is there a local RR thm. for f, more generally for a [redacted] family of curves? The above is a global theorem. In some sense NO because Groth. would have found **any** obvious answer. However we can [redacted] look for a refined answer as follows.

Start with a v.b. E over X. Then $f_!(E) = Rf_*(E)$ is a perfect complex over S. So far have attached to this perfect complex its rank and its highest exterior power. These determine the K-element when S is a curve, but one is missing the [redacted] higher-codimensional information. I have seen [redacted] how to make sense of the highest exterior power as a local object over S and Spencer Bloch tells me somewhat that the remaining info. has a Zariski-local interpretation: $H^p(S, \mathcal{K}_{-p})$.

Something else one can do is to try [redacted] to capture the fibrewise K-nature of a v.b. E as follows. One forms $\operatorname{Pic}(X/S)$ which is a fiber bundle over S with fibres $\simeq \mathbb{Z} \times \text{Jacobian}(X_s)$. There evidently is a way to make the fibres $s \mapsto \operatorname{Pic}(X_s)$

into a algebraic family over S. For example if all the fibres have $g = 1$, then $\mathrm{Pic}^0(X/S)$ is the associated translation group of X/S, $\mathrm{Pic}^1(X/S) = X$[redacted], [redacted] and the rest can be constructed by symmetric products.

The idea might be that to E belongs a section of this bundle, which might have some sort of curvature information which is non-trivial. This is a very standard <u>approach - Griffith's Jacobians</u>.

Return to the case where S is a curve and try to understand the RR thm. We start with E over X. Then $f_!(E) = [Rf_*(E)]$ is determined by its rank and the determinant [redacted] line bundle of cohomology over S. The rank is given by the RR thm. on the fibres of $X \to S$. Thus we need the degree of E on the fibres, which is the same as of $c_1(E) = \lambda(E)$ in $\mathrm{Pic}(X)$. How can I compute this? To simplify suppose E is a line bundle. [redacted] To compute the degree you choose a divisor D with $E = \mathcal{O}(D)$ and intersect D with the fibres and count points.

Intersection of curves on a surface X + R.R. Let D be a curve on X i.e. positive divisor of [redacted] dimension 1. Because the local rings of X are UFD's one knows D is locally defined by a single equation, hence it gives rise to a line bundle $\mathcal{O}(D)$. Another point is that if D_1, D_2 have no irreducible component in common, then tensoring the sequences

$$0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0$$

together leads to an exact sequence, resolution of $\mathcal{O}_{D_1} \otimes \mathcal{O}_{D_2}$:

$$0 \longrightarrow \mathcal{O}(-D_1-D_2) \longrightarrow \mathcal{O}(-D_1)\oplus\mathcal{O}(-D_2) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{D_1}\otimes\mathcal{O}_{D_2} \longrightarrow 0$$

(This means $\mathrm{Tor}_1^{\mathcal{O}}(\mathcal{O}_{D_1},\mathcal{O}_{D_2}) = 0$, and is equivalent to

$$0 \longrightarrow \mathcal{O}(-D_1)\otimes\mathcal{O}_{D_2} \longrightarrow \mathcal{O}_{D_2} \longrightarrow \mathcal{O}_{D_1}\otimes\mathcal{O}_{D_2} \longrightarrow 0$$

being exact. Locally one has

$$A/f_2A \xrightarrow{f_1} A/f_2A$$

and $f_1 a = f_2 a'$ & f_1, f_2 having no common irred. factors $\Rightarrow$ $f_2 | a$ in a UFD.) Also $\mathcal{O}_{D_1}\otimes\mathcal{O}_{D_2} = \mathcal{O}_{D_1\cap D_2}$ by defn. of $D_1\cap D_2$ as a subscheme.

So now one defines $D_1\cdot D_2 = \dim H^0(\mathcal{O}_{D_1\cap D_2})$, and because $D_1\cap D_2$ is zero diml, this is the same as the $\chi(\mathcal{O}_{D_1\cap D_2})$ and the length of $\mathcal{O}_{D_1\cap D_2}$. We then have the formulas

$$\begin{aligned} D_1\cdot D_2 &= \chi(\mathcal{O}) - \chi(\mathcal{O}(-D_1)) - \chi(\mathcal{O}(-D_2)) + \chi(\mathcal{O}(-D_1-D_2)) \\ &= \chi(\mathcal{O}_{D_2}) - \chi(\mathcal{O}(-D_1)\otimes\mathcal{O}_{D_2}) \end{aligned}$$

But for a curve (even singular provided the correct defns. are made) one has

$$\chi(L) = \deg L + \chi(\mathcal{O}).$$

$L \mapsto \deg L$ additive

hence we get

$$\boxed{D_1\cdot D_2 = \deg(\mathcal{O}(D_1)\otimes\mathcal{O}_{D_2})}.$$

From this follows the bilinearity of $D_1\cdot D_2$ and the fact that it extends to divisor classes. (In the arithmetic case one uses this formula as starting point to define $D_1\cdot D_2$, and one must work to get the symmetry.)

RR: $$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

$$\begin{aligned} \chi(\mathcal{O}(-D)) &= \chi(\mathcal{O}) - \chi(\mathcal{O}_D) \\ &= \chi(\mathcal{O}) - (1-g) \end{aligned}$$

g = genus of $\mathcal{O}_D$

We need to compute K_D. But

$$0 \longrightarrow T_D \longrightarrow T_X|D \longrightarrow N_{D\subset X} \longrightarrow 0$$

and because D = zeroes of a section of $\mathcal{O}(D)$ we have

$$N_{D\subset X} = \mathcal{O}(D)|D = \mathcal{O}(D)\otimes\mathcal{O}_D .$$

Dualizing

$$0 \longrightarrow \mathcal{O}(-D)\otimes\mathcal{O}_D \longrightarrow \Omega^1_X\otimes\mathcal{O}_D \longrightarrow \Omega^1_D \longrightarrow 0$$

hence

$$\deg(\mathcal{O}(-D)\otimes\mathcal{O}_D) + 2g-2 = \deg(K_X\otimes\mathcal{O}_D) \qquad K_X=\Omega^2_X$$

$$\therefore\ g-1 = K_X\cdot D + \underbrace{\deg(\mathcal{O}(D)\otimes\mathcal{O}_D)}_{D\cdot D}$$

$\therefore$ get

$$\boxed{\chi(\mathcal{O}(-D)) = \chi(\mathcal{O}) + \tfrac{1}{2}(D\cdot D + K_X\cdot D)}$$

Check for $\mathbb{P}^2$. $K_X = \mathcal{O}(-3)$, $\chi(\mathcal{O}) = 1$. $\chi(\mathcal{O}(-1)) = 0$.
where $\mathcal{O}(-1) = \mathcal{O}(-H)$, H = hyperplane section. $H\cdot H = 1$.
Then

$$\chi(\mathcal{O}(-1)) = 1 + \tfrac{1}{2}(H^2 + (-3H)H)$$
$$= 1 + \tfrac{1}{2}(1-3) = 0$$

which works. In general we have

$$\frac{(n+1)(n+2)}{2} = \chi(\mathcal{O}(nH)) = 1 + \tfrac{1}{2}(n^2 + (-3)(-n))$$

which checks.

Used in the above proof

$$\boxed{D\cdot D = \deg(N_{D\subset X}) \quad \text{for a non-singular curve}}$$

(In effect $D\cdot D = \deg(\mathcal{O}(D)|D)$ and because D is the zero set of the canonical section of $\mathcal{O}(D)$, $\mathcal{O}(D)|D = N_{D\subset X}$.)

Next project is the Hodge index theorem. But first one should connect things up with the Hirzebruch version of RR. Actually the above is not even a proof of RR since it assumes D is a non-singular curve.

Let's go over some of the general nonsense connected with RR:

$$\begin{tikzcd} K(X) \arrow[r, "ch"] \arrow[d, "f_!"'] & A(X) \arrow[d, "f_*"] \\ K(Y) \arrow[r, "ch"] & A(Y) \end{tikzcd}$$

not. comm.

In general given a coh. theory like A we can twist the Gysin homomorphism by an arbitrary multiplicative characteristic class Θ:

$$f_!(\alpha) = f_*\left(\Theta(\nu_f)\,\alpha\right).$$

To see what choice of Θ goes with RR one considers the case of the embedding of X as zero section of a line bundle L: $\quad i : X \longrightarrow L, \quad \nu_i = L.$ One has

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_L \longrightarrow \mathcal{O}_X \to 0$$

hence $\quad i^* i_!(1) = 1 - L^{-1}.$ Thus if

$$ch(i_! 1) = i_*(\Theta(\nu_i))$$

we have

$$\underbrace{ch(i^* i_!(1))}_{ch(1-L^{-1})\,=\,1-e^{-c_1(L)}} = i^* i_*(\Theta(\nu_i)) = \underbrace{(i^* i_* 1)}_{c_1(L)} \cdot \Theta(L)$$

so that $\quad \Theta(L) = \dfrac{1-e^{-c_1(L)}}{c_1(L)}.\quad$ Thus RR is

$$\chi(\alpha) = \int_X ch(\alpha)\,\mathrm{Todd}(X)$$

where $\quad \mathrm{Todd}(L) = \dfrac{c_1(L)}{1-e^{-c_1(L)}}\quad$ and Todd is multiplicative

Now

$$\mathrm{Todd}(x) = \frac{x}{1-e^{-x}} = \frac{1}{-(-1+\frac{x}{2}-\frac{x^2}{6})} = \frac{1}{1-\frac{x}{2}+\frac{x^2}{6}+\cdot}$$

$$= 1 + \left(\frac{x}{2}-\frac{x^2}{6}\right) + \left(\frac{x}{2}\right)^2 + \cdots = 1 + \frac{x}{2} + \frac{x^2}{12} + \cdots$$

So for a surface X we put $c_1(X) = \xi_1 + \xi_2$, $c_2(X) = \xi_1\xi_2$ and

$$\mathrm{Todd}(X) = \left(1 + \frac{\xi_1}{2} + \frac{\xi_1^2}{12}\right)\left(1 + \frac{\xi_2}{2} + \frac{\xi_2^2}{12}\right)$$

$$= 1 + \frac{1}{2}c_1(X) + \frac{\xi_1^2 + \xi_2^2 + 2\xi_1\xi_2}{12} + \xi_1\xi_2\left(\frac{1}{4} - \frac{1}{6}\right)$$

$$\boxed{\mathrm{Todd}(X) = \text{[redacted]}\ 1 + \frac{1}{2}c_1(X) + \frac{1}{12}\left(c_1(X)^2 + c_2(X)\right)}$$

Thus RR says

$$\chi(L) = \int \left(1 + c_1(L) + \tfrac{1}{2}c_1(L)^2\right)\left(1 + \tfrac{1}{2}c_1(X) + \tfrac{1}{12}(c_1(X)^2 + c_2(X))\right)$$

$$\boxed{\chi(L) = \text{[redacted]}\ \frac{(c_1^2 + c_2)(X)}{12} + \frac{c_1(L)\cdot c_1(X)}{2} + \frac{c_1(L)^2}{2}}$$

This agrees with

$$\chi(L) = \chi(\mathcal{O}) + \frac{1}{2}(L\cdot L - L\cdot K) \qquad \text{since } c(K) = -c_1(X).$$

Check for $\mathbb{P}_2$: $\chi(\mathcal{O}) \overset{?}{=} \frac{1}{12}(c_1^2 + c_2)$. $\quad T = \mathrm{Hom}(\mathcal{O}(-1), V^3/\mathcal{O}(-1)) = 3\cdot\mathcal{O}(1)/\mathcal{O}$

$$c(T) = (1+H)^3 = 1 + 3H + 3H^2$$

Thus $\frac{1}{12}(c_1^2 + c_2) = \int \frac{1}{12}\left((3H)^2 + 3H^2\right) = \frac{1}{12}(9+3) = 1.$

Next project is the Hodge index thm. which says that the signature of the intersection pairing on $\mathrm{Pic}(X)$ is $(+, - - - - -)$, i.e. Lorentzian. I've always wanted to understand this from the point of view of Mumford's result on the intersection form of the irreducible components in the exceptional locus obtained by resolving the [redacted] singularities of [redacted] a singular point on a surface. Let the different exceptional curves be $C_1, \dots, C_n$. ~~I think what happens is that if one takes the maximal ideal of the singular point and lifts up one obtains the line bundle of the exceptional locus.~~

Better is to let $X \longrightarrow X'$ be the resolution and $P \in X'$ the singular point. Then $\mathcal{O}_X \otimes_{\mathcal{O}_{X'}} k(P) = \mathcal{O}_D$, where D is the singular locus. $\blacksquare$

Now one has $D = \sum n_i C_i$ with $n_i > 0$. Also one has $C_i \cdot D = 0$ [redacted] for some reason (having to do with $x \cdot f^*(y) = f_*(x) \cdot y$ and the fact that $f_* D = 0$. ??)

So one reaches the following algebraic situation: A symmetric matrix $C_i \cdot C_j \geq 0$ positive off the diagonal, such that $\exists$ a positive linear combination $D = \sum n_i C_i$, $n_i > 0$ with $C_i \cdot D = 0$. One then wants to conclude that this matrix is negative semi-definite.

Look at the negative of this intersection matrix A. It has off-diagonal entries ≤ 0 and diagonal entries ≥ 0, and we want to show it is ≥ 0. So use induction to get all symmetric-about-the-diagonal submatrices are ≥ 0. Then take a vector $v = \sum a_i e_i$ and split it into

$$v^+ = \sum_{a_i > 0} a_i e_i \qquad \text{and} \qquad v^- = \sum_{a_i < 0} a_i e_i$$

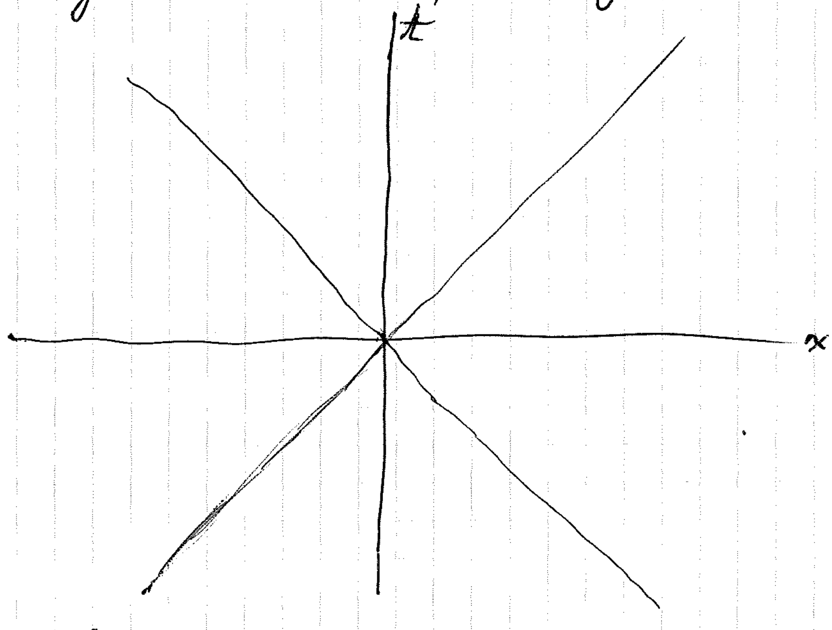
Then
$$\langle v^+ + v^- | A | v^+ + v^- \rangle = \underbrace{\langle v^+ | A | v^+ \rangle}_{\geq 0 \text{ by ind.}} + \underbrace{\langle v^- | A | v^- \rangle}_{\geq 0} + \underbrace{2 \langle v^- | A | v^+ \rangle}_{\geq 0}$$

because
$$\langle v^- | A | v^+ \rangle = \sum_{\substack{a_i > 0 \\ a_j < 0}} a_j \underbrace{A_{ji}}_{\leq 0} a_i \geq 0.$$
This will work provided both v^+ and v^- are $\neq 0$. $\blacksquare$ Now if we use this vector $D = \sum n_i e_i$ with all $n_i > 0$, [redacted] which is in the null-space of A, we can remove a multiple from v and get one entry $= 0$. This unfortunately destroys the induction.

The correct ~~inductive assumption~~ assertion is as follows. One supposes that A is a real symmetric matrix with off-diagonal entries ≤ 0 and such that there exists a vector v with strictly positive entries such that $v^*Av \geq 0$. Then $A \geq 0$. For the proof we can suppose $v = \sum e_i$ whence the rows sums of A are ≥ 0. Induction shows all symmetric proper submatrices are ≥ 0. On the other hand we can lower the diagonal entries of A until the row sums are zero, and then we are in the previous situation.

June 26, 1982

Hodge Index Thm. for surfaces. One works with line bundles over X, i.e. the group $\mathrm{Pic}(X)$, on which we have the intersection form with values in $\mathbb{Z}$. In $\mathrm{Pic}(X)$ one has line bundles which are effective, i.e. $h^0(L) \neq 0$ hence $L \simeq \mathcal{O}(D)$ for a divisor $D \geq 0$. These form a kind of cone in $\mathrm{Pic}(X)$. One also has the cone of ample L, which is described say by Nakai's criterion that L ample $\iff$ $L \cdot D > 0$ for all $D > 0$. The index theorem says $NS(X) = \mathrm{Pic}(X)/$Kernel of intersection form is Lorentzian



$$Q(tx) = t^2 - x^2$$

I think that what one tries to prove is that the interior of the positive light cone coincides with the ample L. This interior is described by $L \cdot L > 0$, $L \cdot H > 0$.

We want to use RR.

$$\chi(F \otimes L^n) = h^0(F \otimes L^n) - h^1(F \otimes L^n) + h^2(F \otimes L^n)$$
$$\leq h^0(F \otimes L^n) + h^2(F \otimes L^n)$$

Now assume L effective i.e. $h^0(L) \neq 0$, so that we have an embedding

$$0 \longrightarrow \mathcal{O} \longrightarrow \overset{\mathcal{O}(D)}{\overset{\parallel}{L}} \longrightarrow \underbrace{\mathcal{O}(D) \otimes \mathcal{O}_D}_{\text{dim Support} = 1} \longrightarrow 0$$

Then we get embeddings

$$0 \to F\otimes L^{n-1} \to F\otimes L^{n} \to F\otimes L^{n-1}\otimes \mathcal{O}_D \to 0$$

which shows that $H^2(F\otimes L^{n-1}) \twoheadrightarrow H^2(F\otimes L^n)$ and hence the $h^2(F\otimes L^n)$ is always bounded as $n \to +\infty$ for L effective

RR says

$$\chi(F\otimes L^n) = \chi(\mathcal{O}) + \tfrac{1}{2}\left\{(F+nL)^2 - K\cdot(F+nL)\right\} \sim \tfrac{1}{2}L^2 n^2$$

hence if $L^2 = L\cdot L > 0$ and L is effective we get that $h^0(F\otimes L^n) > 0$ for $n \gg 0$. This shows there is an embedding $F^* \hookrightarrow L^n$ and so taking F^* to be the hyperplane bundle one sees that L is ample. $\therefore$

$$L\cdot L > 0 + L \text{ effective} \implies L \text{ ample}$$

Now let's assume that $L\cdot L > 0$ and that $L\cdot H > 0$. The condition $L\cdot H > 0$ means that L restricted to the hyperplane H is of positive degree and hence ample. So if we take

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_H \to 0$$

$$0 \to F\otimes\mathcal{O}(-1)\otimes L^n \to F\otimes L^n \to F\otimes L^n\otimes\mathcal{O}_H \to 0$$

$$\underset{\substack{\shortparallel \\ 0}}{H^1(F\otimes L^n\otimes\mathcal{O}_H)} \ \ n \text{ large} \to \underset{?}{H^2(F\otimes\mathcal{O}(-1)\otimes L^n)} \to H^2(F\otimes L^n) \to 0$$

Better: Suppose $F\cdot H > 2g-2-d$, g = genus of H, $d = H\cdot H$

Then

$$0 \to \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}_H(1) \to 0$$

$$0 \to F \to F(1) \to (F|H)\otimes\mathcal{O}_H(1) \to 0$$

$$H^1\big((F|H)\otimes\mathcal{O}_H(1)\big) \to H^2(F) \to H^2(F(1)) \to 0$$

now $\deg\big(F|H\otimes\mathcal{O}_H(1)\big) = \deg(F|H) + \deg(\mathcal{O}_H(1))$

$$= F\cdot H + H\cdot H > 2g-2-d+d = 2g-2$$

so the $H^1 = 0$ and we conclude that $H^2(F) \xrightarrow{\sim} H^2(F(1))$.
So repeat for $F(1)$ etc. and you conclude

<u>Lemma</u>: If $F \cdot H > 2g-2-d$, $g = \text{genus } H$, $d = H \cdot H$ then $H^2(F) = 0$.

Then if we know that $L \cdot H > 0$ we conclude that for any F, $H^2(F \otimes L^n) = 0$ for n large. Thus if in addition $L^2 > 0$ the above argument shows L is ample. Thus we get

$$L \cdot H > 0, \; L^2 > 0 \iff L \text{ is ample.}$$

Proof of the Hodge index theorem: Consider the "2-plane" spanned by H and another line bundle F. We have to show that the intersection form is not positive definite on this 2-plane. If the form were positive definite then $L^2 > 0$ is automatic for $L \neq 0$, and then the ample cone would be a half-space $L \cdot H > 0$. But this half space clearly depends on H and the ample cone doesn't. Q.E.D.

So now on to the arithmetic surface situation

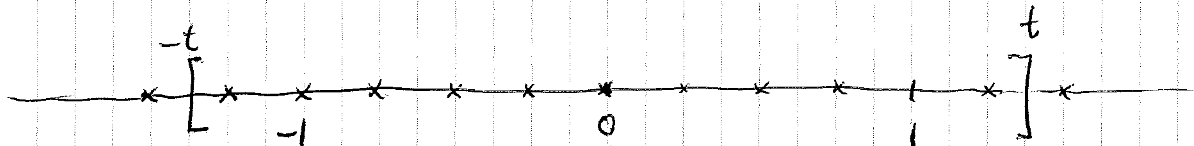
$$\begin{array}{c} X \\ \downarrow f \\ S \end{array}$$

X is an algebraic curve over a number field extended to a non-singular surface (2 diml scheme) over the ring of integers in some way. Over each infinite place of S we get a Riemann surface. Simplest example is an elliptic curve over $\mathbb{Q}$ defined by a Weierstrass equation.

I have to go over the RR thm. for S which is now an arithmetic curve. For simplicity I assume S is either $\mathbb{Q}$ or a quadratic imaginary, so that there is only one infinite place. (One maybe should think of a [redacted] quadratic extension of $\mathbb{Q}$ as an arithmetic analogue of a hyperelliptic curve?)

Now a vector bundle over $S = \mathbb{Z} \cup \{\infty\}$ is a proj. fg $\mathbb{Z}$-module M together with a positive-definite quadratic form over $M \otimes \mathbb{R}$. In generale it is a proj. fg module M over the ring A of integers together with a positive-definite quad. form over $M \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ invariant under the maximal compact subgroup of $(A \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$. Thus when $A \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{C}$ we get a positive-definite form invariant under S^1, which is of course equivalent to a hermitian form.

We need some pictures to get our intuition straight. Let's stick to line bundles, in fact, to divisors. On a curve we look at $\mathcal{O}(D)$ say where $D > 0$ and expect sections. In the arithmetic ~~situation~~ M gives a lattice in $M \otimes \mathbb{R}$ and $D > 0$ corresponds to the lattice M being $A \subset M < A \otimes_{\mathbb{Z}} \mathbb{Q} = F$. Typical model is $\mathbb{Z} \cdot \frac{1}{n} \subset \mathbb{R}$.



In addition we give the positive definite form on $\mathbb{R}$ which we think of as giving a box indicated as [ ]. The condition [redacted] that the divisor is very positive means the box is big. The box is $Q \leq 1$ and so $Q = \frac{x^2}{t^2}$. Then the analogue of h^0 (or really q^{h^0}) is the Θ-fn

$$\sum_{m \in M} e^{-\pi Q(m)} \sim \frac{1}{\text{covol of } M \text{ wrt } Q} \underbrace{\int e^{-\pi |x|^2} d^n x}_{1}$$

For $M = \mathbb{Z}$ with $Q(x) = \frac{x^2}{t^2}$ on $\mathbb{R}$ we get

$$\theta\left(\frac{1}{t^2}\right) = \sum_{n \in \mathbb{Z}} e^{-\pi \frac{n^2}{t^2}} \overset{\text{fnl}}{\underset{\text{eqn.}}{=\!=}} t\,\theta(t^2).$$

So with this convention the analogue of the degree of the divisor is

$$\frac{1}{\text{covol of } M \text{ wrt } Q} \;\text{"="}\; \text{number of elements of } M \text{ in } Q \le 1.$$

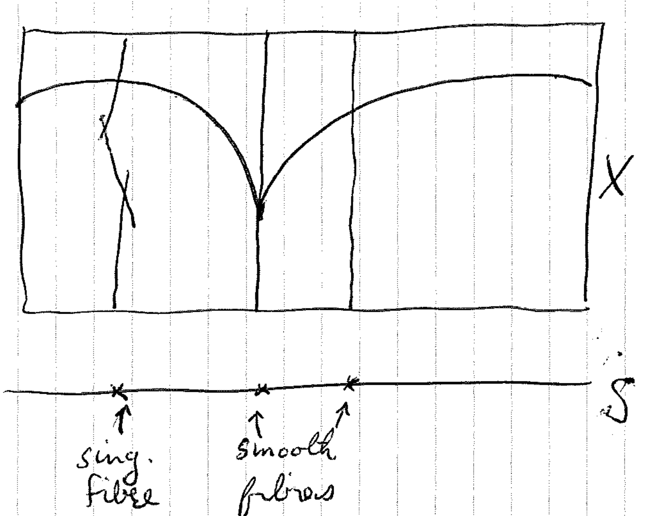
The analogue of h^1 is essentially the θ-fn. of the dual lattice modified by the canonical bundle, and RR is the transformation formula for the θfunction.

Suppose that the above notion of vector bundle over $\mathrm{Sp}(A) \cup \{\text{inf. places}\}$ is a good notion. Is the Grothendieck group of these vector bundles what one would want: $\mathbb{Z} \times \{\text{Divisor classes}\}$?

~~Let's go on to Riemann Roch. We take a vector bundle over X~~

Let's describe curves on X. Recall X starts out as a curve defined over a number field and then one extends it to a non-singular 2-diml scheme over the ring of integers. Thus X will be a smooth family of curves everywhere except [redacted] for a finite number of singu[redacted]lar fibres. An irreducible curve on X will either lie over [redacted] a closed point of S, or else the generic point of S. In the latter case the curve is a point of the original curve X [redacted] defined over a finite extension of the number field.

We have the picture



The map f_* on divisors and what it induces

$$f_* : \underset{\mathrm{Pic}(X)}{\overset{\|}{A^1(X)}} \longrightarrow \underset{\mathbb{Z}}{\overset{\|}{A^0(S)}}$$

is clear. Curves in closed fibres go to zero, and a curve D projecting onto S goes to its degree.

It's clear that the normal bundle to a smooth fibre is trivial, hence the self-intersection of a smooth fibre is 0. The same is true for a singular fibre provided the multiplicities are incorporated.

Also the image of $f^* : A^1(S) \longrightarrow A^1(X)$ is isotropic for the intersection pairing. $f^*(x) \cdot f^*(y) = x \cdot \underbrace{f_* f^*(y)}_{0}$. Actually the result is geometrically clear.

Next want to understand the intersection number for curves on X and RR. Normally in the geometric case where X and S are complete for any coherent sheaf F on X we have number $h^i(F)$ and $\chi(F)$ defined, and we need some kind of analogue. Now in the arithmetic situation we have $R^i f_*(F)$ $i = 0, 1$ which are f.g. A-modules, but we need metrics at the ∞-places.

If I take

$$\begin{tikzcd} & X \arrow[d] \\ S' \arrow[ur, "i"] \arrow[r, "\text{finite}"'] & S \end{tikzcd}$$

then what I [redacted] should get from a vector bundle E on X is a vector bundle $i^*(E)$ over S'. This means I want some sort of metric on the fibres over the infinite places. The only thing that bothers me is the

Galois invariance.

To explain this suppose we start with a curve defined over $\mathbb{Q}$, e.g. an elliptic curve in P_2 described by a Weierstrass equation [redacted] with rational coefficients. Then I get a nice Riemann surface [redacted] over $\mathbb{C}$. Now take as line bundle over this curve defined over $\mathbb{Q}$, e.g. take a divisor rational over $\mathbb{Q}$. I want to put a metric on this line bundle which will be invariant under Galois. This means that look at a Galois orbit on the algebraic points of the curve and the fibres of the line bundle over this orbit, the Galois gp of $\bar{\mathbb{Q}}/\mathbb{Q}$ acts on this, and [redacted] I want the metric to be invariant.

For example complex conjugation should preserve the metric. If you think of the real points as forming a 1-diml., it's clear that most metrics on the line bundle do not have this Galois invariance property.

Maybe the Galois invariance property is not essential.

~~So~~ Recapitulate. We have a curve defined over $\mathbb{Q}$, a line bundle F over it also defined over $\mathbb{Q}$, and we put [redacted] metrics on the curve + line bundle over $\mathbb{C}$. Average to make it invariant under conjugation, hence defined over $\mathbb{R}$. Now [redacted] to define $f_!(F)$.

Interesting point. Even when [redacted] F is sufficiently positive that $R^1f_*(F) = 0$ and $f_*(F)$ is locally-free, I don't have a way to put a metric on $f_*(F)$. Only a volume by means of the analytic torsion. So I won't get an f_* on the vector bundle level, but only on the K-theory level.

I should understand Grothendieck RR for the map $f: X \longrightarrow S$ from a surface to a curve.

For a curve we have

$$(*) \qquad K(S) \xrightarrow{\sim} \mathbb{Z} \times \mathrm{Pic}(S)$$

$$x \longmapsto (rg(x), c_1(x)) \qquad c_1(E) = \text{highest ext. power.}$$

So for example $L \longmapsto (1, c_1(L))$ and these generate. Thus

$$K(S) \xrightarrow{ch} A(S) = \mathbb{Z} \oplus \mathrm{Pic}(S)$$

is the identity wrt $(*)$, since $ch(L) = e^{c_1(L)} = 1 + c_1(L)$.

Now GRR says

$$ch(f_! \alpha) = f_*(ch(\alpha) \cdot Todd(X/S))$$

and we know

$$ch(f_! E) = \begin{pmatrix} \text{index of } E & \text{coh. det.} \\ \text{along fibres} \;, & \text{line bdle,} \\ \text{i.e. rel. } S & \lambda(R^0 f_*) \otimes \lambda(R^1 f_*)^* \end{pmatrix}$$

But also

$$Todd(T) = 1 + \tfrac{1}{2} c_1(T) + \tfrac{1}{12}(c_1(T)^2 + c_2(T)) + \cdots.$$

So we need to know

$$\begin{tikzcd} A(X) \arrow[d, "f_*"] & = & \mathbb{Z} \oplus & \mathrm{Pic}(X) \arrow[dll] & \oplus \; A^2(X) \arrow[dll] \\ A(S) & = & \mathbb{Z} \oplus \mathrm{Pic}(S) \end{tikzcd}$$

The component $f_*: \mathrm{Pic}(X) \to \mathbb{Z}$ is easy: Take the degree of the line bundle over any fibre of f. (Any two fibres are algebraically equivalent in particular homotopic). So the first component of f_* gives the RR for the index of E relative to S:

$$\mathrm{ind} = \int_{\text{fibre}} \left(c_1(E) + rg(E)\tfrac{1}{2} c_1(X/S)\right) = \deg E + rg(E)(1-g).$$

The map $f_*: A^2(X) \longrightarrow \mathrm{Pic}(S)$ is also easy: Given a zero cycle on X, just push it down to S.

so GRR says

$$c_1(f_! E) = rg(E)\cdot f_*\left[\tfrac{1}{12}\left(c_1(X/S)^2 + c_2(X/S)\right)\right]$$

$f_*(ch_2(E))$ in general

$$+ \tfrac{1}{2} f_*\left[c_1(E)\cdot c_1(X/S)\right] + \tfrac{1}{2} f_*\left[c_1(E)^2\right].$$

[redacted] You should get a differential form version of this formula in some way. The curvature of the coh. determinant line bundle you calculate in terms of the variation of the connection integrated over the Riemann surface. This explains the $\frac{1}{2} f_*[c_1(E)^2]$ term, and the others will come from varying the metric on the Riemann surface.

Now you [redacted] begin to see the general context of the calculations you did over Riemann surfaces, namely, you will be able to calculate $c_1(f_! E)$ as a [redacted] curvature [redacted] for X/S of arbitrary dimension. [redacted]

June 27, 1982

Take an elliptic curve X over $\mathbb{Q}$, a [redacted] divisor rational over $\mathbb{Q}$, L the corresponding line bundle. The idea is to analyze this situation locally. Over $\mathbb{C}$ we put a [redacted] volume on the Riemann surface $M = X_{\mathbb{C}}$ and a metric on the line bundle $L_{\mathbb{C}}$. Then by analytic torsion I get a volume in the cohomology-determinant line of $L_{\mathbb{C}}$.

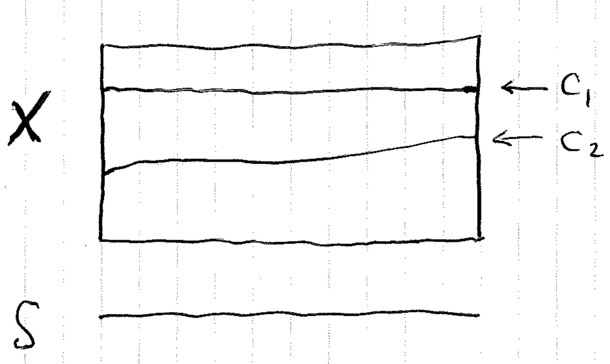
In more detail, we have $f: X \longrightarrow \mathbb{Q}$ and so we can take $c_1(f_! L) = \lambda(H^0(X,L)) \otimes \lambda(H^1(X,L))^*$, which is a 1-diml $\mathbb{Q}$-vector space. The analytic torsion gives a volume in $c_1(f_! L) \otimes_{\mathbb{Q}} \mathbb{C}$.

If $\deg(L) = 0$, then $c_1(f_! L)$ [redacted] has a canonical generator, assuming $h^0 = h^1 = 0$, so this volume is just a number which is given by the Singer-Ray formula for the torsion.

In general in the [redacted] 0-cohomology case one should think of $c_1(f_! L)$ as being a "divisor" over $\mathbb{Q}$. Here 0-cohomology means at the generic pt. of $\mathbb{Q}$, so that the divisor is located at the primes where cohomology appears. This still isn't very clear.

So let us consider the geometric situation where X is a family of curves parameterized by S, and E is a [redacted] vector bundle over X of index 0 rel. S, generically without cohomology. Then $\mathcal{L} = c_1(f_! L)^*$ has a canonical section s which we know is not identically zero, hence $\mathcal{L}, s$ defines a divisor on S. This is the Θ-divisor.

The problem will be to do some calculations in the elliptic curve case. How to proceed? I first have to understand what is going on at a point of S, [redacted] where we are are getting a pos. real [redacted] number from the analytic torsion. Over S I have a family of elliptic curves and the sort of line bundle E I want to look at will be of the form $\mathcal{O}($[redacted]$C_1 - C_2)$, where C_1, C_2 are sections of X over S. Then modulo choice of metrics + vol. we get a function on S.



This still isn't very clear.

X is a family of curves over S, E is a vector bundle over X. Then we have this line bundle $\mathcal{L}^* = c_1(f_! E)$. After choosing a metric on E and a volume on the fibres of X/S, we get a metric on $\mathcal{L}$ with known curvature. [redacted] This determines $\mathcal{L}$ up to a flat line bundle over S. In the index 0 case, you have a canonical section σ of $\mathcal{L}$ which vanishes where cohomology appears in the fibres. [redacted] The torsion gives me $|\sigma|^2$, so as we approach a point of S where cohomology occurs I can read off the order of σ. It seems that the order of vanishing is just $\dim H^0$? No, only in good cases: Clearly can pull back by a ramified map $S' \to S$ and change the multiplicity.

Riemann-Roch says

$$\chi(X, E) = \chi(S, f_! E) = \underbrace{\deg(f_! E)}_{-\text{number of zeroes of } \sigma} + \underbrace{rg(f_! E)}_{0}(1 - g_S)$$

which obviously agrees with the idea that $R^0 f_*(E) = 0$, $R^1 f_*(E)$ = sheaf supported at zeroes of σ.

Geometric situation: One has a family of curves $X \xrightarrow{f} S$ and a vector bundle E over X. One chooses volumes on the fibres of f and a metric on E. Then one gets a metric on $c_1(f_! E)$. (Perhaps better is to say that [redacted] the metric on $c_1(f_! E)$ at a point of S depends only on the volume + metric on (X, E) chosen on the fibre over that point. One then chooses vol. + metric smoothly on the fibres so as to get a metric on $c_1(f_! E)$.)

Arithmetic situation: One has a family of curves $X \to S$, where S is a ring of integers with maybe a few places removed, and also a vector bundle E over X. Here one doesn't seem to have much choice - the [redacted] integral structure at the finite places is determined. [redacted] One makes the same choices [redacted] of vol. + metric on (X, E) at the infinite places. Then $c_1(f_! E)$ has integral structure and a metric at the infinite places.

Next project is to understand the intersection theory for curves on the arithmetic [redacted] surface. A [redacted] vector bundle E on the arithmetic surface is a line bundle in the usual sense plus metric at the infinite places. I define the Euler characteristic by

$$\chi(E) = \chi(f_! E) = \deg(f_! E) + rg(f_! E)(1 - g_S)$$

Here to define $\deg(f_!(E))$ we need to have chosen a volume on the curve at the infinite places. $c_1(f_! E)$ is the determinant of the cohomology $\lambda(H^0(E)) \otimes \lambda(H^1(E))^*$, which is a [redacted] line over the number field with an integral lattice and volume at ∞ ~~places~~. Its degree is the logarithm of the covolume, up to sign.

A first thing we need is to know that $\chi(E)$, $\deg(f_! E)$ are additive for exact sequences. This follows from the corresponding statement for the analytic torsion.

<u>Assertion</u>: Let M be a Riemann surface, $0 \to E' \to E \to E'' \to 0$ an exact sequence of v.b. over M. Then we have a ~~can~~. canon isom.

$$\mathcal{L}_{E'} \otimes \mathcal{L}_{E''} \simeq \mathcal{L}_E$$

Suppose we choose a metric on E and equip E', E'' with the induced metrics. Choose also a volume on X, so that analytic torsion gives volumes on $\mathcal{L}_E, \mathcal{L}_{E'}, \mathcal{L}_{E''}$. Then the above canonical isom. is compatible with these volumes.

Pf: Because of the metrics I have an isom. $E' \oplus E'' \simeq E$ of C^∞ v.b. with metric. Hence the holom. structure D_E is of the form $D_{E'} \oplus D_{E''} \oplus B$ where $B \in \Gamma(\mathrm{Hom}(E'', E' \otimes T^{0,1}))$.
Now consider the [redacted] family of holom structures $D_{E'} \oplus D_{E''} \oplus tB$, $t \in \mathbb{C}$, and looks at the line bundles $\mathcal{L}_{\tilde{E}'}$ etc. over [redacted] the parameter space. Now we know the curvature form for $\mathcal{L}_{\tilde{E}}$ is [redacted] given by [redacted]

$$f_*(ch_2(\tilde{E}))$$

which is the <u>same for the line bundle $\mathcal{L}_{\tilde{E}'} \otimes \mathcal{L}_{\tilde{E}''}$</u>. Hence it follows that the two metrics differ by a constant under the canonical isom.

need to check

$$\mathcal{L}_{\tilde{E}'} \otimes \mathcal{L}_{\tilde{E}''} \simeq \mathcal{L}_{\tilde{E}}.$$

But they agree at the point $t = 0$, etc.

(note: April 3, 1983. This argument shows, at best, $\exists$ an canon. isom $\mathcal{L}_{E'} \otimes \mathcal{L}_{E''} \simeq \mathcal{L}_E$ compatible with holom. structure + metric, not the known canon. isom is compatible with metric.)

Next thing to understand is the analogue of the exact sequence

$$(1) \qquad 0 \longrightarrow L(-D) \longrightarrow L \rightarrow L \otimes \mathcal{O}_D \longrightarrow 0$$

and its consequence in the geometric case

$$\chi(L) - \chi(L(-D)) = \chi(L \otimes \mathcal{O}_D)$$
$$= \deg(L \otimes \mathcal{O}_D) + \chi(\mathcal{O}_D)$$
$$\chi(\mathcal{O}) - \chi(\mathcal{O}(-D)) = \chi(\mathcal{O}_D)$$

$$\therefore \ \deg(L \otimes \mathcal{O}_D) = \chi(L) - \chi(L(-D)) - \chi(\mathcal{O}) + \chi(\mathcal{O}(-D))$$

or changing L to L^{-1} gives

$$\deg(L \otimes \mathcal{O}_D) = \chi(\mathcal{O}) - \chi(L^{-1}) - \chi(\mathcal{O}(-D)) + \chi(L^{-1} \otimes \mathcal{O}(-D)).$$

This means that if we define

$$L_1 \cdot L_2 = \chi(\mathcal{O}) - \chi(L_1^{-1}) - \chi(L_2^{-1}) + \chi(L_1^{-1} \otimes L_2^{-1})$$

then we have $L_1 \cdot \mathcal{O}(D) = \deg(L_1 \otimes \mathcal{O}_D)$ and therefore this restriction definition is symmetric.

The exact sequence (1) has a clear meaning at finite places. Suppose D has no components contained in a fibre over S. Then $\mathcal{O}_D$ is finite flat over S. In practice an irreducible D is just a algebraic point of the algebraic curve over the number field. From the exact sequence of v.s. over the no. field

$$0 \rightarrow H^0(L(-D)) \rightarrow H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_D) \rightarrow H^1(L(-D)) \rightarrow H^1(L) \rightarrow 0$$

one gets an isom.

$$c_1(f_! L(-D)) \otimes \lambda(H^0(L \otimes \mathcal{O}_D)) \xrightarrow{\sim} c_1(f_! L).$$

One needs to understand the volumes at ∞.

We can formulate the problem already for a Riemann surface. Namely given a v.b. E over M with metrics on both and a point P of M. Given also a metric on $\mathcal{O}(-P)$. Then the analytic torsion gives volumes on $\mathcal{L}_E, \mathcal{L}_{E(-P)}$ and hence there has to be an induced volume on the fibre $E \otimes k(P)$. What is it?

Go back to the geometric situation: X is a family of curves over S, E is a v.b. over X, P is a section of X over S. Then we will have an isom. of line bundles over S:

$$\mathcal{L}^*_{E\otimes\mathcal{O}(-P)} \otimes \lambda(E|P) = \mathcal{L}^*_E$$

Now I want to understand the metrics. Have a metric on E the fibres of X/S and $\mathcal{O}(-P)$, so you want to understand the induced volume on $\lambda(E|P)$.

What is the curvature of each of these bundles?

<u>June 28, 1982:</u>

Review ζ functions and K-groups for varieties over $\mathbb{F}_q$.

In general given X a scheme of finite type over $\mathbb{Z}$ one defines
$$\zeta_X(s) = \prod_{\substack{x \text{ closed} \\ \text{point}}} \frac{1}{1-(Nx)^{-s}} \qquad Nx = \text{card } k(x).$$

[redacted] Now suppose X is a smooth projective variety defined over $\mathbb{F}_q$, whence $\bar{X}$ is a smooth proj. variety over $\bar{\mathbb{F}}_q$ with Frobenius endom. $Fr: \bar{X} \longrightarrow \bar{X}$. $\qquad z = q^{-s}$

$$\log \zeta(s) = \sum_{x\text{ cl.}} \sum_{n=1}^{\infty} \frac{1}{n} z^{(\deg x)n} \qquad m = (\deg x)\, n$$

$$= [redacted] \sum_{m \geq 1} z^m \sum_{\substack{x\text{ cl.} \\ \deg x \mid m}} \frac{\deg x}{m} = \sum_{m\geq 1} \frac{z^m}{m} \sum_{\substack{x\text{ cl.} \\ \deg x \mid m}} \deg x$$

[redacted]

Now $\displaystyle\sum_{\substack{x\text{ cl.} \\ \deg x \mid m}} \deg x$ = no. of fixpts of $(Fr)^m$ on $\bar{X}$

$$= \sum_q (-1)^q \operatorname{Tr} (Fr)^m \text{ on } H^q(\bar{X})$$

by the Lefschetz Fixpt Formula, where one uses that $(Fr)^m$ always has non-degenerate fixpts. $\therefore$

$$\log \zeta(s) = \sum_q (-1)^q \sum_{m\geq 1} \frac{1}{m} \operatorname{Tr}\left[(z \cdot Fr)^m \text{ on } H^q(\bar{X}) \right]$$

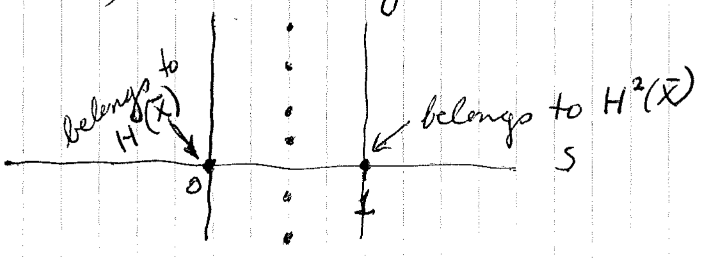
or

$$\log \zeta(s) = \sum_q (-1)^q (-1) \log\Big(\det\big(1 - z Fr \text{ on } H^q(\bar{X})\big)\Big)$$

Example: ① X a curve. Then

$$\zeta(s) = \frac{\det\{1 - zFr \text{ on } H^1(\bar{X})\}}{(1-z)(1-qz)} = \frac{\prod_1^{2g}(1 - z\alpha_i)}{(1-z)(1-qz)}$$

where by the RH, $|\alpha_i| = \sqrt{q}$. Since $z = q^{-s}$ we get:



$$q^{-s}\alpha = 1$$
$$s = \log_q \alpha$$
$$\operatorname{Re} s = \log_q |\alpha| = \tfrac{1}{2}$$

② X a surface. Then

$$\zeta(s) = \frac{(\qquad)(\qquad)}{(1-z)\ \det\{1-z\,\mathrm{Fr} \text{ on } H^2\}\quad (1-q^2 z)}$$

According to the Tate conjecture which he proved for divisors, the subspace of $H^2(\bar{X})$ on which Fr has value q is the subgroup $NS(X)$ of algebraic cycles on X. Hence the order of the pole of ζ at $s=1$ is the rank of $NS(X)$.

The strong Lefchety theorem tells one that one has an isom.

$$H^1(\bar{X}) \xrightarrow[\sim]{\cup L} H^3(\bar{X})$$

where L is the class of the hyperplane section. Hence Fr on H^3 corresp. to qF on H^1, so the numerator of ζ is

$$\det\{1-zF \text{ on } H^1\}\cdot \det\{1-zqF \text{ on } H^1\}.$$

I want to put $s=1$ or $z=q^{-1}$; but this doesn't help. Duality says that

$$H^1(\bar{X})\otimes H^3(\bar{X}) \longrightarrow H^4(\bar{X}) \qquad \mathrm{Fr}=q^2 \text{ here}$$

is a perfect pairing, so

$$\begin{aligned}
\det\{1-zF \text{ on } H^3\} &= \det\{1-zF \text{ on } (H^1)^*\otimes H^4\}\\
&= \det\{1-zq^2F \text{ on } (H^1)^*\}\\
&= \det\{1-zq^2(F)^{-1} \text{ on } H^1\}\\
&= \det\{1-zqF \text{ on } H^1\}
\end{aligned}$$

where one uses the RH on H^1 to identify the eigenvalues of F^{-1} with those of $q^{-1}F$. Anyway it doesn't seem to be possible to cancel out the effects of the H^1 and H^3.

However H^1 can be understood via the Picard variety of the [redacted] surface. Hence there is probably a way to compute the value at $s=1$ of the part of ζ not involving the H^2. But I don't see how to extract a number out of the part of H^2 where $\mathrm{Fr}\neq q$.

Next I want to understand how the K-theory is supposed to behave. Let's begin with finite fields. The key idea is that Thomason has identified the periodic l-adic K-theory with the etale cohomology essentially, so what we must do is understand the deviation of the actual theory from the periodic one.

For $\mathbb{F}_q$ the K-groups are

$$\cdots \quad 0 \quad \mathbb{Z}/(q-1)^2 \quad 0 \quad \mathbb{Z}/(q-1) \quad \mathbb{Z}$$

and for $\overline{\mathbb{F}}_q$ the groups are

$$\cdots \quad 0 \quad \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}] \quad 0 \quad \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}] \quad \mathbb{Z}.$$

Next consider a curve X over $\mathbb{F}_q$ and $\bar{X}$ over $k = \overline{\mathbb{F}}_q$. Then one has the exact sequence of localization

$$\to \mathrm{Div} \otimes K_1(k) \longrightarrow K_1(\bar{X}) \longrightarrow K_1(F) \longrightarrow \mathrm{Div} \otimes K_0(k) \longrightarrow K_0(\bar{X}) \to K_0(F) \to 0.$$

Now one wants to compare this with the sequence

$$0 \longrightarrow F^\times/k^\times \longrightarrow \mathrm{Div} \longrightarrow \mathrm{Pic}(\bar{X}) \longrightarrow 0$$

and

$$0 \to \mathrm{Tor}_1(\mathrm{Pic}\,\bar{X}, K_n k) \to F^\times/k^\times \otimes K_n k \to \mathrm{Div} \otimes K_n k \to \mathrm{Pic}(\bar{X}) \otimes K_n(k) \longrightarrow 0$$

We can split $K(\bar{X})$ into three pieces using the maps.

$$\begin{array}{ccc} \mathrm{Sp}(k) & \xrightarrow{i} & \bar{X} \\ & & \downarrow \pi \\ & & \mathrm{Sp}(k) \end{array}$$

These give us two copies of $K(k)$ in $K(\bar{X})$:

$$K(k) \underset{i_*}{\overset{\pi_*}{\rightleftarrows}} K(\bar{X}) \qquad K(k) \underset{i^*}{\overset{\pi^*}{\rightleftarrows}} K(\bar{X})$$

One has

$$K(k)^2 \xrightarrow{(\pi^*, i_*)} K(\bar{X}) \xrightarrow{\binom{i^*}{\pi_*}} K(k)^2$$

$$\begin{pmatrix} i^* \\ \pi_* \end{pmatrix} \begin{pmatrix} \pi^* & i_* \end{pmatrix} = \begin{pmatrix} id & i^* i_* \\ \pi_* \pi^* & id \end{pmatrix}$$

$i^* i_* = c(\nu_i) = 0$

$\pi_* \pi^* =$ mult by $\pi_* 1 = 1-g$

hence this 2×2 matrix is invertible and it splits off from $K(\bar X)$ two copies of $K(k)$. This corresponds to

$$K_0(\bar X) = \mathbb{Z} \oplus \mathrm{Pic}(\bar X) = \mathbb{Z} \oplus \mathrm{Pic}^\circ(\bar X) \oplus \mathbb{Z}.$$

Next we have because k is algebraic closed that $K_n(k) \hookrightarrow K_n(F)$. (Write F as an inductive limit of f.g. k-algebras and use the Nullstellensatz.) This gives a uniform way to cancel off the π^*. So we define

$$\tilde K(F) = K(F)/K(k) \qquad \tilde K(\bar X) = \tilde K(X)/K(k).$$

and then I can further divide out by the image of i_* to get groups $\tilde{\tilde K}(\bar X)$. Then we will have an exact sequence

$$\begin{tikzcd}
\mathrm{Div}^\circ \otimes K_n k \arrow[r] & \tilde{\tilde K}_n \bar X \arrow[r] & \tilde K_n F \arrow[r] & \mathrm{Div}^\circ \otimes K_{n-1} k \arrow[r] & \tilde{\tilde K}_{n-1} \bar X \arrow[r] & \tilde K_{n-1}(F) \\
0 \arrow[r] & \mathrm{Tor}_1(\mathrm{Pic}^\circ, K_{n-1}k) \arrow[r] & F^\times/k^\times \otimes K_{n-1}k \arrow[r] \arrow[u] & \mathrm{Div}^\circ \otimes K_{n-1} k \arrow[r] \arrow[u, equal] & \mathrm{Pic}^\circ \otimes K_{n-1} k \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$= 0$ if n even

$= 0$ for $n \geq 2$ because Pic° is divisible and $K_{n-1}k$ is torsion.

The basic conjecture is

$$\boxed{F^\times/k^\times \otimes K_{n-1} k \xrightarrow{\sim} \tilde K_n F} \qquad n \geq 1.$$

for [redacted] a curves over $k = \bar{\mathbb{F}}_q$. If so then

$$\begin{cases} \tilde{\tilde K}_n(\bar X) = 0 & \text{for } n \text{ odd} \\ \tilde{\tilde K}_n(\bar X) = \mathrm{Tor}_1(\mathrm{Pic}^\circ(\bar X), K_{n-1}k) & n \text{ even} \geq 2 \\ \tilde{\tilde K}_0(\bar X) = \mathrm{Pic}^\circ(\bar X) \end{cases}$$

Now take the limit as $F \longrightarrow \overline{k(T)}$ and you get that

$$\tilde K_n(\overline{k(T)}) = \begin{cases} \overline{k(T)}^\times / k^\times & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Fix a C^∞ vector bundle E over our Riemann surface M, and let $\mathcal{A}$ be the family of all holom. structures on E. Let P be a point of M. Then by associating to a holom. structure on E the holom. [redacted] bundle $E \otimes \mathcal{O}(P) = E(P)$, I get a new family of holom. bundles over M parametrized by $\mathcal{A}$. Moreover we have a canonical isom.

$$(*) \qquad \lambda(f_!(E)) \otimes \lambda(E \otimes k(P)) \;\tilde{\longrightarrow}\; \lambda(f_!(E(P)))$$

better to work with $\mathcal{O}(-P)$.

so that the bundles $\mathcal{L}_E$, $\mathcal{L}_{E(P)}$ are isomorphic. Note that $\lambda(E \otimes k(P)) = \lambda(\text{fibre of } E \text{ at } P)$ is constant over $\mathcal{A}$.

Next I want to choose metrics on $M, E, \mathcal{O}(P)$ whence I get metrics on $\mathcal{L}_E$, $\mathcal{L}_{E(P)}$ whose curvature forms are the same. This is because the map

$$\mathcal{A}_E \longrightarrow \mathcal{A}_{E\otimes\mathcal{O}(P)}$$

is [redacted]

$$D_E \longmapsto D_E \otimes id_{\mathcal{O}(P)} + id_E \otimes D_{\mathcal{O}(P)}$$

i.e.

$$D_E^0 + B \longmapsto B + D^0_{E\otimes\mathcal{O}(P)}.$$

This is just an affine space isomorphism, hence it preserves the Kähler structure.

Since the obvious metric on $\lambda(E \otimes k(P))$ doesn't change over $\mathcal{A}$, it follows that $(*)$ preserves the metric up to a constant scalar multiple, i.e. a positive real number. This number will depend upon the rank + degree of E and the metric on $\mathcal{O}(P)$. If we take a direct sum structure on E, things multiply, so it is enough to worry about E being a line bundle of a given degree.

Husemoller tells me the following proof (due to Groth.) of the Hodge index thm. using RR + Serre duality. One has $h^0(nD) + h^0(K-nD) \geq \chi(nD) = \frac{1}{2}n^2D^2 + O(n)$. If $D^2>0$, then $h^0(nD) + h^0(K-nD) \uparrow \infty$ as $n \to \infty$. Once say $h^0(K-nD)>0$, then using $\mathcal{O}(nD) \otimes \mathcal{O}(K-nD) \simeq \mathcal{O}(K)$ one gets an embedding $\mathcal{O}(nD) \hookrightarrow K$ hence $h^0(\mathcal{O}(nD)) \leq h^0(K)$. Thus for large n either $h^0(nD) \uparrow \infty$ and $h^0(K-nD) = 0$ or the opposite. Once $h^0(nD) > 0$ one has $\mathcal{O} \hookrightarrow \mathcal{O}(nD)$, hence $H \cdot nD > 0$ and $H \cdot D > 0$. Similarly the case $h^0(K-nD) \uparrow \infty$ implies $H \cdot D < 0$. Thus for $D^2 > 0$ one has either $H \cdot D > 0$ or < 0, but not $= 0$. Thus if $H \cdot D = 0$ we must have $D^2 \leq 0$ which is the Hodge signature thm.

A critical point in both proofs is that one has an $h^0(F)$ which if > 0 says that F comes from a divisor, i.e. $\exists\ \mathcal{O} \hookrightarrow F$.

Assume $D^2 > 0$, $H \cdot D = 0$

(Point: $h^0(nD) + h^0(K-nD) \geq \frac{1}{2}n^2D^2 + O(n) \longrightarrow +\infty$ as $|n| \to \infty$.
If $h^0(nD) > 0$ for some $n \neq 0$, then $H \cdot nD > 0 \Rightarrow H \cdot D \neq 0$, cont.
Thus $h^0(nD) = 0$ all $n \neq 0$, and $h^0(K-nD) \longrightarrow \infty$ as $|n| \to \infty$.
But then for large $|n|$ one has $h^0(K+nD) > 0$, hence an embedding $\mathcal{O}(K-nD) \subset \mathcal{O}(2K)$, so $h^0(K-nD)$ is bdd.
a contradiction.)

July 29, 1982

Go back to the geometric ~~situation~~ of a surface X over a curve S and let D be a divisor on X finite over S.

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, "g"'] & X^E \arrow[d, "f"] \\ & S \end{tikzcd}$$

Then we have

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

$$0 \to E(-D) \to E \to E|D \to 0$$

$$\chi(E) - \chi(E(-D)) = \chi(E|D) = \deg(E|D) + (\operatorname{rg} E)\,\chi(\mathcal{O}_D)$$

I was thinking of using the arithmetic analogue to get a definition of $\deg(E|D)$. Also we have

$$\chi(E) - \chi(E(-D)) = \chi(f_! E) - \chi(f_!(E(-D)))$$

$$= \chi(f_!(E|D))$$

$$= \deg(f_!(E|D)) + \operatorname{rg}(f_!(E|D))\,\chi(\mathcal{O}_S)$$

Now g has a certain degree $=$ [redacted] $\operatorname{rg}(f_*\mathcal{O}_D)$. ~~Denote~~ this $[D:S]$. Then

$$\operatorname{rg}(f_!(E|D)) = \operatorname{rg}(E)\,[D:S].$$

Putting the above together

$$\chi(E|D) = \deg(E|D) + (\operatorname{rg}E)\,\chi(\mathcal{O}_D)$$

$$\|$$

$$\chi(g_!(E|D)) = \deg(g_!(E|D)) + (\operatorname{rg}E)\,[D:S]\,\chi(\mathcal{O}_S)$$

one gets a relation between the degrees of a [redacted] v.b. E' over D and $g_* E'$ over S, which could be established directly.

Return to the arithmetic situation. Here one has a Riemann surface with some sort of integral structure. D is an effective divisor on the Riemann surface which is rational over the number field. I have defined $f_!(E(-D))$ using the metric on E and one on $\mathcal{O}(-D)$. [redacted] I also have

$$\lambda(f_! E) = \lambda(f_! E(-D)) \otimes \lambda(f_!(E|D))$$

up to a scalar independent of E. But I need to have this formula working exactly which means that the metric on $\mathcal{O}(-D)$ has to influence the metric on $E|D$ in some definite way.

Let's concentrate on this phenomenon. Consider all metrics on $\mathcal{O}(-P)$. To each one we get a positive real number by comparing torsions under the canonical isom.

$$\lambda(f_! \, \mathcal{O}(-P)) \otimes k(P) \simeq \lambda(f_! \mathcal{O}).$$

[redacted] This suggests a simpler question. Namely, fix a holomorphic vector bundle and vary the metric. How does the torsion change? This should be the same problem as asking how the torsion changes under complex gauge transformations. The connection goes as follows. Suppose we [redacted] write a new metric on E in the form $\langle u|g|u\rangle$ where $g \in \Gamma(\mathrm{Hom}(E,E))$ is positive s.a. in each fibre. Then we compute the adjoint of

$$D : E \longrightarrow E \otimes T^{0,1}$$

relative to the new metric:

$$\langle Du|g|v\rangle = \langle u|D^* g v\rangle = \langle u|g|g^{-1}D^* g v\rangle.$$

Hence the new adjoint is $g^{-1}D^*g$, and so the new Laplacean is

$$g^{-1}D^*gD = g^{-1/2}(g^{-1/2}D^*g^{1/2})(g^{1/2}Dg^{-1/2})g^{1/2}.$$

Hence the torsion for the new Laplacean is just the torsion for the transformed [redacted] holom. structure $g^{1/2}Dg^{-1/2}$. (As usual I am thinking of the index 0 case, where I can work over the dense set where D is invertible.)

[redacted] Next let $g = 1 + X$ where $X = X^*$ is infinitesimal, then the infinitesimal change in the torsion is

$$[redacted] \; \delta \log |s|^2 = \mathrm{Tr}^{(\mathrm{reg})}\left(D^{-1}\left[\tfrac{1}{2}X, D\right]\right) + \text{c.c.}$$

$$= \tfrac{1}{2}\int \mathrm{tr}\left(J[X,D]\right) + \text{c.c.}$$

$$= \frac{1}{2}\int \operatorname{tr}\left([D,J]X\right) + \text{c.c.}$$

Actually I checked once that the quantity is real. Now I know $[D,J] = \frac{i}{2\pi}(\text{curv of } E + \frac{1}{2}\text{curv } T)$.

It seems that by varying X one can make any desired change in $\log|s|^2$. So the only reasonable thing it seems is to work at a stationary point. Let's see what this means.

We start with a holom. bundle E without cohomology and then look at all metrics on E, and pick a metric at which the torsion is stationary, which means that $\text{curv}(E) + \frac{1}{2}\text{curv } T = 0$. I assume such a ~~stationary~~ metric exists. The set of such metrics is acted on by the automorphism group of the holom. bundle. But if we use a fixed square root of the canonical bundle K to get rid of the $\frac{1}{2}\text{curv } T$ term, we find that we have a fixed holom. bundle of degree 0 and we are looking for all metrics on it which are flat.

For a line bundle of degree 0, there is a unique connection such that the holonomy group is contained in S^1, because

$$H^1(M,S^1) \xrightarrow{\sim} H^1(M,\mathcal{O}^*)^0 .$$

The possible metrics compatible with this connection differ by elts of [redacted] $\mathbb{R}_{>0} = \mathbb{C}^*/S^1$. [redacted]

I want to go back to elliptic curves and work out formulas for line bundles of degree $\neq 0$. So pick a holomorphic line bundle of degree 1, say $\mathcal{O}(P)$. I work over $M = \mathbb{C}/\Gamma$ where the volume of M is π with respect to $dx\,dy = \frac{i}{2}dz\,d\bar{z}$. Thus I find the first Chern class of $\mathcal{O}(P)$ is cohomologous to $\frac{i}{2\pi} dz d\bar{z}$.

The is a unique metric on $\mathcal{O}(P)$, up to a multiplicative pos. real constant, having the curvature form $\frac{i}{2\pi} dz d\bar{z}$. I want to find it. Can suppose $P = 0$, then translate in general. $\mathcal{O}(0)$ has a canonical section 1 vanishing at 0, hence we want to define $\rho = |1|^2$ on M such that

$$\bar{\partial}\partial \log \rho = \quad dz\, d\bar{z}.$$

I've solved this problem in a different form, by means of the analytic torsion metric on the $\mathcal{L}$ associated to the family of holom. structures $(\partial_{\bar{z}} - w)\, d\bar{z}$ on the trivial bundle over M. $\mathcal{L}$ is this case is the line bdle. $\mathcal{O}(\Gamma)$ belonging to Γ in the w-plane. $\rho(w)$ is the analytic torsion, so

$$\partial \log \rho(w) = \quad = \mathrm{Tr}^{(\mathrm{reg})}\left(\frac{1}{\partial_{\bar{z}} - w}(-dw)\right)$$

$$= \left(\sum_{\mu}^{\mathrm{reg.}} \frac{1}{w - \mu}\right) dw = \left(\zeta(w) - \ell w - \bar{w}\right) dw$$

and

$$\bar{\partial}\partial \log \rho(w) = -\,d\bar{w}\, dw = dw\, d\bar{w}.$$

Hence

$$\rho(w) = \text{const}\ \sigma(w)\, e^{-\ell \frac{w^2}{2} - \bar{w}w} \cdot \text{anti holom.}$$

except ρ is to be real, so that

$$\boxed{\rho(w) = \text{const}\ e^{-|w|^2} \left| e^{-\ell \frac{w^2}{2}} \sigma(w) \right|^2}$$

So now it is clear how to put good metrics on $\mathcal{O}(D)$ for any divisor D on M. These metrics are defined up to a positive real multiple.

Let's recall what I discovered yesterday, namely that having chosen a metric on $\mathcal{O}(-P)$ one can then compare volumes relative to the canonical isomorphism

$$\circledast \quad \lambda(f_!(E(-P))) \otimes \lambda(E|P) \simeq \lambda(f_! E)$$

and one gets a pos. real number which depends only on the rg and deg of E. It is compatible with exact sequences so one will get a homomorphism $\mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{R}_{>0}$ which depends on the metric chosen on $\mathcal{O}(-P)$.

[redacted] An ordinary rescaling of the metric on $\mathcal{O}(-P)$ acts on $\lambda(f_!(\mathcal{O}(-P)))$ by $c \mapsto c^{h^0-h^1} = c^{-1+(1-g)}$ essentially so if $g \neq 0$, one can adjust the metric so that the homomorphism $\mathbb{Z}\times\mathbb{Z} \to \mathbb{R}_{>0}$ vanishes on $(1,0)$ = class of $\mathcal{O}$. Assume this [redacted] done. [redacted] Precisely, we choose for each P the unique metric on $\mathcal{O}(-P)$ with constant curvature such that the canonical isomorphism

$$\lambda(f_!\, \mathcal{O}(-P)) \otimes \lambda(\mathcal{O}|P) \simeq \lambda(f_! \mathcal{O})$$

preserves volumes.

Now compare two points P, Q.

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-P-Q) & \longrightarrow & \mathcal{O}(-P) & \longrightarrow & \mathcal{O}(-P)|Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\wr & & \\
0 & \to & \mathcal{O}(-Q) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}|Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}(-Q)|P & \xrightarrow{\sim} & \mathcal{O}|P & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By construction the "volume in" $\mathcal{O}$ is the same as the "volume in" $\mathcal{O}(-P)$ or $\mathcal{O}(-Q)$. The "volume in" $\mathcal{O}(-P-Q)$ will define from the "volume in" $\mathcal{O}(-P)$ by 2 factors, the former coming from the norm at Q of the canonical section, the latter due to the [redacted] possible volume change for [redacted] the isom. $\circledast$ with [redacted] bundles of degree -1.

So it is now clear that I need to know about

⊛ as P varies. Gets more & more complicated.

June 30, 1982: (Faltings talks today).

Recall that the exact sequence

$$0 \longrightarrow \mathcal{O}(-P) \longrightarrow \mathcal{O} \longrightarrow k(P) \longrightarrow 0$$

leads to a canonical isomorphism

$$\lambda\left[f_! E(-P)\right] \otimes \lambda(E|P) \simeq \lambda(f_! E)$$

for any vector bundle E. Now I choose a metric on the R.S. M and a metric on $\mathcal{O}(-P)$. Then I get volumes in all three of the above lines ~~and hence~~ assuming E has a metric a number

$$\Phi(\mathcal{O}(-P), E) = \cancel{\log \text{ of}}\ \text{ratio of the volumes.}$$

I believe that Φ does not depend on the (holomorphic) ~~complex~~ structure or metric of E. Certainly the curvature argument shows that for a given metric on E it is independent of the holom. structure, so it depends only on the rank + degree of the bdle E. If you take two metrics on the same holom. bdle, then choose a metric-preserving isomorphism, and then you have two holom. structures on the same bundle with metric, etc.

Since Φ depends only on the rank + deg of E I can write

$$\Phi(\mathcal{O}(-P), E) = \Phi(\mathcal{O}(-P), \operatorname{rg} E, \deg E).$$

Next consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}(-P-Q) \arrow[r] \arrow[d] & \mathcal{O}(-Q) \arrow[r] \arrow[d] & \mathcal{O}(-Q)|P \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathcal{O}(-P) \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & \mathcal{O}|P \arrow[r] & 0 \\
 & \mathcal{O}(-P)|Q \arrow[r, phantom, "\simeq"] \arrow[d] & \mathcal{O}|Q \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Now

$$\lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-P-Q)) = \lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-Q)) \cdot \lambda(f_!\mathcal{O}(-Q))/\lambda(f_!\mathcal{O}(-P-Q))$$

$$= \overbrace{\quad}^{\Phi(\mathcal{O}(-Q);\,1,0)} \cdot \underbrace{\lambda(\mathcal{O}|Q)}_{1} \cdot \overbrace{\quad}^{\Phi(\mathcal{O}(-P);\,1,-1)} \lambda(\mathcal{O}(-Q)|P)$$

What am I doing? I am working in the category of lines over $\mathbb{C}$ with a volume, and 1 denotes $\mathbb{C}$ with $|z|$. You are missing something. It is the canonical isom.

(1) $$\lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-P-Q)) = \lambda(\mathcal{O}|P \oplus \mathcal{O}|Q)$$

There is this canonical isom. and the isom.

(2) $$\lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-P-Q)) = \lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-Q)) \otimes \lambda(f_!\mathcal{O}(-Q))/\lambda(f_!\mathcal{O}(-P-Q))$$
$$= \lambda(\mathcal{O}|Q) \cdot \lambda(\mathcal{O}(-Q)|P)$$

as well as the similar isom.

(3) $$\lambda(f_!\mathcal{O})/\lambda(f_!\mathcal{O}(-P-Q)) = \lambda(\mathcal{O}|P) \cdot \lambda(\mathcal{O}(-P)|Q).$$

If I compare (1) and (2) and use that $\lambda(\mathcal{O}|P)$, $\lambda(\mathcal{O}|Q)$ have canonical generators I get a canonical generator for $\lambda(\mathcal{O}(-Q)|P)$. Put another way, just using (2) and (3) gives a canonical isomorphism of

$$\lambda(\mathcal{O}(-Q)|P) = \lambda(\mathcal{O}(-P)|Q)$$

This is God-given, so the only real possibility is the obvious one, namely, you use the canonical section 1 of $\mathcal{O}(-Q)$ which is non-zero at P for $P \neq Q$.

It is more or less clear that by this procedure I will end up with the formula

$$\Phi(\mathcal{O}(-P-Q),1,0) = \Phi(\mathcal{O}(-Q),1,0)\,\Phi(\mathcal{O}(-P),1,-1)\left(\begin{matrix}\text{norm of } 1 \text{ in } \mathcal{O}(-Q)\\ \text{at } P\end{matrix}\right)$$

$$= \Phi(\mathcal{O}(-P),1,0)\,\Phi(\mathcal{O}(-Q),1,-1)\left(\text{norm of } \underset{\text{at } Q}{1} \text{ in } \mathcal{O}(-P)\right)$$

Problems: 1) Find a formula for this Φ function. I could hope to define any metric on $\mathcal{O}(-P)$ to the YM metric and calculate how Φ changes. Then I can suppose that $\mathcal{O}(-P)$ is given the unique YM metric such $\Phi(\mathcal{O}(-P),1,0) = 1$.

I think it true in the case of elliptic curves that

$$\begin{matrix}\text{norm of } 1 \text{ in } \mathcal{O}(-P) \\ \text{at } Q\end{matrix} = \begin{matrix}\text{norm of } 1 \text{ in } \mathcal{O}(-Q) \\ \text{at } P\end{matrix}$$

If so, it follows that

$$\Phi(\mathcal{O}(-P),1,-1) = \Phi(\mathcal{O}(-Q),1,-1)$$

which is obvious by translation anyway.

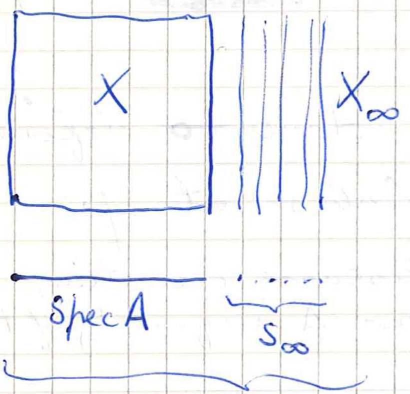
2) One should understand the p-adic situation, or better, the arithmetic situation, in order to get some feeling for the range of possibilities

June 30, 1982 Gerd Faltings

(Arakeloff work on Shafarevich – Parsin conjecture)

K ~~number~~ field, $A \subseteq K$ integers

X/K curve X/A semi-stable model



Need a complete surface to define intersection of curves.

$$v \in S \text{ place of } K \Rightarrow K_v$$

$X(K_v)$. For v finite, you translate arithmetic mod v to statements for $X(K_v)$. Then guess analogue for $K_v = \mathbb{C}$.

$S = \{\text{all places}\}$

First to understand <u>dim 1</u>. (above is dim 2)

Sheaf on $S = S_f \cup S_\infty$ $=$ A-module M + hermitian metric on $M \otimes_A K_v$ for $v \in S_\infty$.

$$\prod_{v \in S_\infty} M \otimes_A K_v = M \otimes_{\mathbb{Z}} \mathbb{R}$$

Define
$$\chi(M) = -\log\left(\frac{\operatorname{vol}(M \otimes \mathbb{R}/M)}{\# M_{tors}}\right)$$

Then have RR
$$\chi(M) = \deg M + c \cdot \operatorname{rank}(M)$$

where
$$\deg(M) = \deg(\Lambda^{\max} M) = \sum \text{local degrees}$$

<u>dim 2</u>: a line bundle on $X \cup X_\infty$ = line bundle $\mathcal{L}$ on X + hermitian metrics on $\mathcal{L} \otimes \overline{K}_v$ on $X(\overline{K}_v)$, $v \in S_\infty$.

(This is consistent with what happens for v finite)

Put restriction on the metrics: The curvature form is a multiple of $\sum \omega_k \wedge \overline{\omega}_k$ where ω_k is an orth. basis of $\Gamma(X(\overline{K}_v), \Omega^1)$

where for $v \in S_f$ · $\langle D_1, D_2 \rangle_v = \log (\#_v)$ usual mult.

$\left(= -\log d_v(D_1, D_2) \right)$

So for $v \in S_\infty$ $\langle D_1, D_2 \rangle_v = -\log d_v(D_1, D_2)$

In general $\langle D_1, D_2 \rangle = \deg(\mathcal{O}(D_1)|D_2)$.

Above all due to Avakeloff. Now Faltings

Want to prove ⓐ RR ⓑ Hodge signature thm.

(Get more: a description of the − divisors as the obvious ones.)

ⓐ RR $\quad \chi(\mathcal{O}(D)) = \dfrac{\langle D, (D-K) \rangle}{2} + \text{constant}$

$$\begin{gathered} \chi(\mathcal{O}(D)) \\ \| \\ \chi(H^0(\mathcal{O}(D)) - \chi(H^1(\mathcal{O}(D))) \end{gathered}$$

Have to put a volume on $\lambda(H^0(\mathcal{O}(D))) \otimes \lambda(H^1(\mathcal{O}(D)))^*$ which will make RR automatic. For Y a R.S. need volume forms on $R\Gamma(Y, \mathcal{O}(D))$ such that

with hermitian metric.

ⓐ it is invariant under isomorphisms

ⓑ $D \to D + P$ $\quad 0 \to H^0(\mathcal{O}(D)) \to H^0(\mathcal{O}(D+P)) \to \underset{\text{herm. metric}}{\mathcal{O}(D+P)(P)}$

$\to H^1(\mathcal{O}(D)) \to H^1(\mathcal{O}(D+P)) \to 0$

is volume exact.

ⓑ shows uniqueness

ⓐ For line bundles of degree $g-1$ the scaling of the metric shouldn't matter. So its enough to prove ⓐ for $\deg(D) = g-1$.

$$\begin{array}{ccc} X \times \mathrm{Div}_{g-1} & \longrightarrow & X \times \mathrm{Jac} \\ \downarrow p_1 & & \downarrow p_1 \\ \mathrm{Div}_{g-1}(X) & \longrightarrow & \mathrm{Jac}_{g-1}(X) \supset \Theta \text{ divisor} \end{array}$$

not aff.; take finite family

Have metric on $\lambda(Rp_{1*}(\mathcal{L})) = \mathcal{O}(-\Theta)$. Faltings calculates the curvature of ↑ and shows it comes from Jac_{g-1}.

More Faltings:

$$\int \|f\|^2 < 1 \Longrightarrow \int \log |f| \leq 0$$

Suppose $\mathcal{L}$ is of degree $d+g-1$. The idea will be to correlate the volume of the L^2- unit ball in $H^0(M, \mathcal{L})$ with what? Szegö theorem.

General viewpoint: Fix a line bundle L and then look at L^n as $n \to \infty$. Assume degree $L > 0$. Then from the torsion one gets a good volume on $H^0(L^n)$, once degree $(L^n) > 2g-2$. On the other hand one has the L^2 inner product on this space of sections, hence one can hope to compare the two volumes.

Faltings' method goes somewhat as follows: The estimate will depend only on the degree of L because the metrics have been fixed in advance and the line bundles of a given degree form a compact family. So let the degree of L be $d+g-1$ and pick d points $x_1, \ldots, x_d$ in general position so that $H^0(L(-x_1 - \cdots - x_d)) = 0$. Then we have

$$H^0(L) \xrightarrow{\sim} \bigoplus_{i=1}^{d} L|x_i$$

and so we can compare volumes. We have the L^2 inner product on $H^0(L)$, the torsion volume, and the obvious inner product on $\oplus L|x_i$. First compare the L^2 and other inner products. This leads to a factor

$$\det f_i(x_j)$$

where $f_1, \ldots, f_d$ is an orthonormal basis for $H^0(L)$. Secondly compare the torsion volume on $H^0(L)$ with the volume on $\oplus L|x_i$. By using $L = \otimes \mathcal{O}(x_i) \otimes L(-\Sigma x_i)$, somehow the torsion volume on $H^0(L)$, the obvious volume on $\oplus (L|x_i)$ are related to a combination of the numbers

$$\prod_{i\neq j} d(x_i, x_j) \quad , \quad \begin{array}{l}\text{analytic torsion}\\ \text{of the line bundle } L(-\Sigma x_i)\\ \text{of degree } g-1.\end{array}$$

Somehow the last factor is bounded in the right direction. Thus one has a comparison of

$$\left|\det\{f_i(x_j)\}\right|^2 , \quad \prod_{i\neq j} d(x_i,x_j) , \quad C$$

where C = torsion volume of the L^2-unit ball in $H^0(L)$. Now integrate over M^d and use

$$\int_{M^d} \left|\det\{f_i(x_j)\}\right|^2 = d!$$

and this relates C to $\int \left[\prod_{i\neq j} d(x_i,x_j)\right]^{\pm 1}$. Now he uses some kind of inequality between

$$\log \int f \qquad \int \log f$$ NO, just a sup inequality

and the fact that $\log d(x_i, x_j)$ is the Green's function for $\bar\partial\partial$.

Faltings has another proof of the Hodge index thm. Namely he chooses a map $X \xrightarrow{f} S$ to a curve so that at the generic point things are reduced. Then for L on X of degree $= g-1$, g = genus of the fibre one has a Θ divisor. If L is off the Θ divisor at the generic point, then $f_* L = 0$ and $R^1 f_*(L)$ is torsion, so that $H^0(L) = H^0(S, f_* L) = 0$ and $H^2(L) = H^1(S, R^1 f_*(L)) = 0$, so $\chi(L) = -h^1(L) \leq 0$. So now given a divisor D of degree 0 (degree = degree rest. to fibres), choose a generic E of degree $g-1$ (so that $E+nD$ doesn't meet the Θ divisor over the generic point for arb. large n). Then

$$0 \geq \chi(E+nD) \sim \tfrac{1}{2} n^2 \langle D^2\rangle + O(n)$$

so $\langle D^2 \rangle \leq 0$.

July 1, 1982

$$\begin{array}{c} X \;\; E \\ \downarrow f \\ S \end{array}$$

I want to look at a holom. family of curves X/S and a vector bundle E over S. Now I choose metrics and get an analytic torsion volume on $f_! E$. I then have the conjecture that the **curvature** of $c_1(f_! E)$ is given by RR-formula $f_*(ch(E)\,Todd(X/S))$ on the differential form level. To prove this conjecture I can trivialize the family on the C^∞-level, and assume that $(X,E) = S \times (X_0, E_0)$. ~~with the metrics constant.~~ ~~This won't be true because the volume of the fibres may change~~ Then I have a **varying** metric on X_0 and a varying holom. structure on E_0 and metric on E_0. I think I can make the metric on E_0 **constant**. It would be nice if one could assume the volume on X_0 changed only by a **constant** scaling **as** one varies over S. This is a question about how the **diffeomorphism** group acts on the volumes.

I think it is fairly natural to assume that the volume of the fibres of X/S is constant over S. In fact the natural Kähler metrics come from a closed 2-form on X giving the volume form on each fibre.

July 3, 1982

What is the relation between the family of vector bundles constructed by clutching on a given holomorphic vector bundle and the family of holom. structures on a fixed C^∞ vector bundle? Fix the Riemann surface M and a small disk U about a point ∞ on M. Fix a holomorphic bundle E over M, and consider for each autom f of E over $U-\{\infty\}$ the clutched bundle E_f.

This is a bit abstract and it is probably better to consider line bundles and $E=\mathcal{O}$. Then from the fact that any v.b. E becomes holom. trivial over $M-\{\infty\}$, we know any line bundle is an $\mathcal{O}_f$:

$$\Gamma(M-\infty,\mathcal{O}^*)\times\Gamma(U,\mathcal{O}^*)\longrightarrow\Gamma(U-\infty,\mathcal{O}^*)\longrightarrow H^1(M,\mathcal{O}^*)\longrightarrow 0$$

$$f\longmapsto\mathcal{O}_f$$

Elliptic curves can be computed explicitly. Here ∞ is the origin O, and I can think of z as a local coordinate at O. Then $f=z^n$ goes to $\mathcal{O}(nO)$ and $f=e^{w/z}$ as $w\in\mathbb{C}$ will give line bundles of degree 0. There has to be a correspondence between the bundle $\mathcal{O}_f$ with $f=e^{w/z}$ and the bundles obtained from $\mathcal{O}$ with $\bar\partial-w\,d\bar z$. It would seem to be useful to get the formulas straight.

We have two descriptions of line bundles of degree 0 over $M=\mathbb{C}/\Gamma$. One is using $\bar\partial$-operators on the trivial bundle 1 of the form $D=(\partial_{\bar z}-w)\,d\bar z$. The other is to use the clutching fns. $e^{w/z}$ on the trivial holom. line bundle $\mathcal{O}$. For each w we have to identify the isomorphism class of the corresponding holomorphic line bundle. [redacted] We can do this by constructing a [redacted] meromorphic section with simple pole at $\bar{0} \in \mathbb{C}/\Gamma$. This section has a zero at some point Q, whence the line bundle is isomorphic to $\mathcal{O}(Q-\bar{0})$.

Review how this works for $D = (\partial_{\bar z} - \omega)\, d\bar z$. Recall some formulas for $\sigma(z)$:

$$\frac{\sigma(z+\gamma)}{\sigma(z)} = e^{a_\gamma z + b_\gamma} \qquad \gamma \in \Gamma$$

where $a_\gamma = \frac{d}{dz} \log \frac{\sigma(z+\gamma)}{\sigma(z)} = \zeta(z+\gamma) - \zeta(z)$

has the form

$$a_\gamma = l\gamma + m\bar\gamma .$$

$$l = \sum{}' \frac{1}{\gamma^2 |\gamma|^{2s}} \Big|_{s=0}$$

$$m = \left(s \sum{}' \frac{1}{|\gamma|^{2+2s}} \right)\Big|_{s=0} = \frac{\pi}{\mathrm{vol}(\mathbb{C}/\Gamma)}$$

Also $$e^{b_{\lambda+\gamma} - b_\lambda - b_\gamma} = e^{a_\gamma(\lambda)}$$

(There seems to be some virtue in replacing the function $\sigma(z)$ by $e^{-l\frac{z^2}{2}}\sigma(z)$, because

$$\frac{e^{-l\frac{(z+\gamma)^2}{2}}\sigma(z+\gamma)}{e^{-l\frac{z^2}{2}}\sigma(z)} = e^{-\cancel{lz\gamma} - l\frac{\gamma^2}{2} + (\cancel{l\gamma} + m\bar\gamma)z + b_\gamma}$$

hence $a_\gamma = l\gamma + m\bar\gamma$ gets replaced by $\tilde a_\gamma = m\bar\gamma$ and b_γ by $\tilde b_\gamma = b_\gamma - l\frac{\gamma^2}{2}$. Unfortunately however l is not analytic in Γ.)

So let me [redacted] find next a holom. section of 1 for $D = (\partial_{\bar z} - \omega)\, d\bar z$ with simple pole at $z = 0$.

$$s = e^{\omega \bar z + cz} \frac{\sigma(z-q)}{\sigma(z)}$$

For periodicity we need

$$\omega\bar\gamma + c\gamma + a_\gamma(z-q) - a_\gamma z = 0$$

$$\omega\bar\gamma + c\gamma - (l\gamma + m\bar\gamma) q = 0$$

$\omega = qm \qquad q = \frac{\omega}{m}$

$c = \frac{l}{m}\omega$

So the section we want is

$$s(z) = e^{\frac{w}{m}(lz + m\bar{z})} \frac{\sigma(z - \frac{w}{m})}{\sigma(z)}$$

and therefore we get the holom. line bdle. $\mathcal{O}(Q - \bar{0})$ where $Q \equiv \frac{w}{m} \bmod \Gamma$.

The next thing is to do the case of the clutching function $e^{w/z}$. This means we are looking for a merom. periodic fn. of z with only singularities at $\bar{0} = 0 + \Gamma$ and with the local form $\frac{e^{w/z}}{z}(1 + a_1 z + a_2 z^2 + \cdots)$ as $z \to 0$.

It's clear we try $e^{w\zeta(z)}$ which changes under $z \mapsto z+\gamma$ by $e^{w a_\gamma}$ since $\zeta(z+\gamma) - \zeta(z) = a_\gamma$. So the holom. section we want is

$$s = e^{w\zeta(z)} \frac{\sigma(z - w)}{\sigma(z)}$$

Possible generalizations. One might take all $\bar{\partial}$-operators $D = (\partial_{\bar{z}} - w(z))\, d\bar{z}$. This will involve only minor changes because one has complex gauge transformations

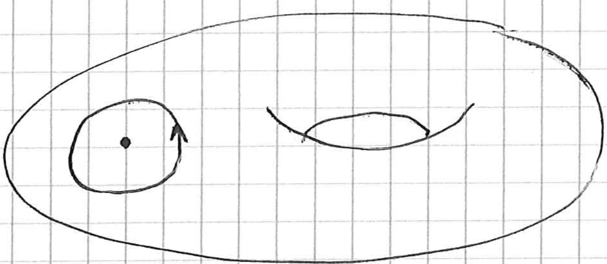
$$f D f^{-1} = (\partial_{\bar{z}} - \partial_{\bar{z}} \log f - w)\, d\bar{z},$$

so one can easily kill all non-constant Fourier terms in w.

We might also consider clutching functions around $z=0$ of the form $e^{w/z + w_2/z^2 + w_3/z^3 + \cdots}$. However because we have periodic functions like $\wp(z) = \frac{1}{z^2} + \cdots$ it is clear that w_2, w_3, etc. do not move the isom. class of the line bundle.

We have a line of line bundles over M and hence we get the line bundle $\mathcal{L} = c_1(L)^*$ over the parameter space. I want to identify $\mathcal{L}$ somehow. I know it has a canonical section, which vanishes when $w \in \Gamma$ so this pins $\mathcal{L}$ down. Still however aren't clear what you are after.

I should try to put a metric on $\mathcal{L}$ in the clutching case and calculate the curvature. So try the



approach with the unitary representation of the loop group. We start with the trivial bdle 1 over M, then have a little circle around the point $\bar{0}$ in M. We somehow choose a measure on this circle, then can [redacted] form $L^2(S^1)$, and get closed subspaces

$W =$ closure of functions analytic [redacted] outside S^1

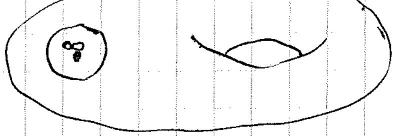
$H = H^2(S^1)$.

a [redacted] fn. f on S^1 then gives rise to a line bundle $\mathcal{O}_f$ whose sections are $W \cap fH$. Now form the wedge Fock space of $L^2(S^1)$ with decomposable vectors belonging to subspaces which are "L^2-commensurable in a certain sense" with H. [redacted] Assume for the moment that $W^\perp$ is L^2-commensurable with H, hence we get a unit vector u_W.

[redacted] The problem ultimately is to identify $\mathcal{L}_f^*$ with the line [redacted] corresponding to fH. Because W is complementary to fH for most f (of degree 0), the canonical section of $\mathcal{L}_f$ should be [redacted] given by $\langle u_W|$ on $\mathcal{L}_f^*$.

July 4, 1982

Fix a Riemann surface M and a little$_{\wedge}^{\text{open}}$ disk D around a point ∞ on M. Then for any $f: \partial D \to \mathbb{C}^*$ we can clutch to get a line bundle $\mathcal{O}_f$ of M, and so we get a family of line bundles of M parameterized by $\{f\}$. I'd like to put a metric on the coh. det line bundle and calculate its curvature which would then$_{\wedge}^{\text{be}}$ a 2 form on the space of f.



Method. Put $V =$ sections of 1 over ∂D, and then define subspace $V^- =$ sections $\overset{\text{of } 1}{}$ holom. outside of D, $V^+ =$ sections of 1 holom. inside of D. Then the Čech$_{\wedge}^{\text{cochain}}$ complex relative to the covering $M = \bar{D} \cup (M - D)$ gives

$$0 \to H^0(\mathcal{O}_f) \longrightarrow V^- \oplus fV^+ \longrightarrow V \longrightarrow H^1(\mathcal{O}_f) \longrightarrow 0.$$

Hence the cohomology determinant line depends only on the subspace fV^+ and not f; hence on image f in $\Gamma(\partial D, \mathcal{O}^*)/\Gamma(\bar{D}, \mathcal{O}^*)$.

(Actually the line bundle $\mathcal{O}_f$, up to isomorphism, also depends only on the image of f in

$$\Gamma(M-D, \mathcal{O}^*) \Big\backslash \Gamma(\partial D, \mathcal{O}^*)\Big/\Gamma(\bar{D}, \mathcal{O}^*) \qquad \cong \quad H^1(M, \mathcal{O}^*)$$

which is clear from the Čech complex for computing $\uparrow$.)

Now we forget about f and simply work with the "outgoing" subspace $W = fV^+$. These subspace form a sort of Grassmannian. I want now to pass to a Hilbert space situation, so I choose a volume on ∂D, so that V acquires an L^2 inner product.

For the moment we want to forget about M because we are interested in the cohomology determinant line bdle.

which now sits over the Grassmannian of outgoing subspaces in V. M provides us with the subspace V^-.

Here's the situation now. We have V = functions on S^1 holomorphic in a nbd. of $S^1 \subset \mathbb{C}$. We have a fixed space V^- and the family of outgoing subspaces $W = fV^+$, V^+ = holom. functions in a nbd of $|z| \leq 1$, $f : S^1 \to \mathbb{C}^*$ holom. near S^1. From V^- we construct a coh. det. line bundle over the Grassmannian of these outgoing subspaces.

[I seem to recall that Graeme prefers to work with the f on the other side. In other words, with $f^{-1}V^-$ varying (in practice $f^{-1} = e^{x_1 z + x_2 z^2 + \cdots\cdots}$) and V^+ fixed.]

The next thing is to introduce Fock space and the line bundle over the Grassmannian. Here we start with $L^2(S^1)$ and form the fermion Fock space $\mathcal{F}$ with vacuum vector $|0\rangle$ belonging to the subspace $H^+ = \overline{V^+}$. Then inside the Fock space $\mathcal{F}$ are "decomposable" lines belonging to closed subspaces W of $L^2(S^1)$ which are "ℓ^2-commensurable" with H^+. Hence over this Hilbert-Schmidt Grassmannian is a line bundle obtained by associating to W the corresponding line in $\mathcal{F}$. Because $\mathcal{F}$ is a Hilbert space this line bundle has an inner product, and because the Grassmannian has some kind of complex structure, there should be a curvature form for this line bundle. Call this line bundle $\mathcal{O}(-1)$, because it is going to give the Plücker embedding in the finite dimensional case.

Problems: 1) Curvature for $\mathcal{O}(-1)$.

2) Why given a V^-, we can identify $\mathcal{O}(-1)|_W$ with the

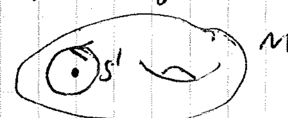
coh. det. line belonging to the map V^-[redacted]$\oplus W \to V$.

Let's understand 2) in the case of index 0 using the divisor approach. First suppose V is finite-dimensional, and we consider the Grassmannian of all W of the same dimension p, as a [redacted] subspace H^+. In this case $\mathcal{O}(-1)|_W = \lambda(W)$, and so $\mathcal{O}(-1)|_W = \Lambda^p W \subset \Lambda^p V$ and we get a nice family of holomorphic sections of $\mathcal{O}(1)$ by taking p-diml. quotients [redacted] V/V^- of V. The section belonging to the quotient V/V^- vanishes when

$$W \longrightarrow V/V^- \quad \text{is singular,}$$

i.e. where the canonical section for the coh. det. line belonging to $V^- \oplus W \longrightarrow V$ vanishes.

The above argument is perfectly general and solves 2) modulo specifying conditions on V^-. [redacted] It's clear also that each V^- is giving a section of $\mathcal{O}(1)$, hence when I am in a situation where $\mathcal{O}(1)$ has been trivialized, each V^- will be giving rise to a holomorphic function of W.

<u>Review</u>: I start with  and associate to $f \in \Gamma(S^1, \mathcal{O}^*)$ a clutched line bundle $\mathcal{O}_f$ over M. Taking coh. det. I then get a line bundle $\mathcal{L}$ over $\Gamma(S^1, \mathcal{O}^*)$. [redacted] The point is that $\mathcal{L}$ is independent of M. [redacted] It is a canonical line bundle over the group $\Gamma(S^1, \mathcal{O}^*)$ which comes with a hermitian metric.

[redacted] To get to index 0 you should fix a line bundle L_0 on M of degree $g-1$, [redacted] equipped with a trivialization over S^1 + interior. Then the family $(L_0)_f = L_0 \otimes \mathcal{O}_f$ will give a Θ-section of [redacted] $\mathcal{O}(1)$.

So the Riemann surface disappears, in fact, one can generalize[redacted] so that (M, L_0) have singularities, in particular so you are dealing with generalized Jacobians. [redacted]

Let's concentrate our original program which is to understand the cohomology determinant line bundle, and not to get involved with representations of loop groups, or KdV.

Summary: The clutching fn. construction over a curve [redacted] $S^1 \hookrightarrow M$ [redacted] allows one to start from a holomorphic vector bundle E_0 over M and construct a family $(E_0)_f$, parameterized by $f \in \mathrm{Aut}(E_0|S^1)$, of hol. v.b. over M. Hence one gets a coh. det. line bundle $\mathcal{L}$ over this parameter space. What I want to do in general is to describe this [redacted] line bundle $\mathcal{L}$.

When $(E_0)_f$ has slope $g-1$, the line bundle $\mathcal{L}$ has a canonical section and so one gets a Θ-divisor in the parameter space.

If I can put a metric on $\mathcal{L}$, then the curvature determines $\mathcal{L}$ (ignoring 1-conn. problems). [redacted] Assuming the parameter space has no H^1 or H^2, then $\mathcal{L}$ can be trivialized by lifting its curvature form to a 1-form, and putting a suitable metric on the trivial bundle. Then the Θ-divisor becomes a Θ-fn.

When $S^1 \hookrightarrow M$ is a small circle, then [redacted] I can associate to f a subspace W in $\Gamma(S^1, E_0)$ and thereby map $\{f\}$ to a Grassmannian. The line bundle $\mathcal{L}$ over $\{f\}$ comes from a canonical line bundle $\mathcal{O}(1)$ over the Grassmannian, and [redacted] I get a metric by choosing a volume on S^1 and a metric on E_0 over S^1. (Maybe one wants to trivialize E_0 holomorphically over S^1 + interior)

In the small circle case the line bundle $\mathcal{L}$ comes from a canonical line bundle on the Grassmannian, so we know its curvature, and can probably construct a trivialization of some sort. But the actual sections of $\mathcal{L}$, the τ or θ-functions depend on the part of (M, E_0) outside of S^1.

It should be important to see if there are natural trivializations of $\mathcal{L}$. First look at topological restrictions. $\{f\} = \text{Maps}(S^1, GL_n) \sim \text{Maps}(S^1, U_n)$. Now the constants don't have any effect on the line bundle, so we can assume $\{f\} = \Omega U_n$ in which case there is a 1-1 correspondence between $f \in \Omega U_n$ and outgoing subspaces. Restrict to degree 0, whence $\{f\} = (\Omega U_n)_{(0)}$ which has the homotopy type of ΩSU_n. But $H^2(\Omega SU_n) = \mathbb{Z}$ for $n \geq 2$, and presumably $c_1 \mathcal{O}(1)$ generates this $\mathbb{Z}$. Hence for $n \geq 2$ we will not be able to trivialize $\mathcal{L}$ over $(\Omega U_n)_{(0)}$ nor the bigger $\text{Maps}(S^1, GL_n)_{(0)}$. What we have to do is to lift back to the central extension. The central extension of ΩU_n acts on the line bundle $\mathcal{O}(1)$, so the map $\widetilde{\Omega U_n} \longrightarrow \Omega U_n \xrightarrow{\sim}$ Grass lifts thru $\mathcal{O}(1)$, hence $\mathcal{O}(1)$ becomes trivial over $\widetilde{\Omega U_n}$. In fact you get a unique equivariant trivialization. Thus given an f you get an identification of $\mathcal{O}(1)_W$ with $\mathcal{O}(1)_{fW}$ by lifting f to $\tilde{f}$ which moves $\mathcal{O}(-1)_W$ to $\mathcal{O}(-1)_{fW}$ inside the Fock space.

Idea: (Recall the Mumford idea:) Let V be holomorphic sections in a punctured disk $0<|z|<\varepsilon$ for some ε, V^{+} = ones holomorphic in a disk $|z|<\varepsilon$, and V^{-} = sections holomorphic on $M-\{0\}$. (Actually Mumford does this formally.) Then f the clutching fn. is to be analytic in a punctured disk. This way we get a nice family of vector bundles over the curve, and there are obvious difficulties doing things in a unitary fashion. So the idea is to try to vary the size of the circle and see how the metric on $\mathcal{L}$ and its curvature change.

July 5, 1982

Dirac matrices in 2 dimensions. We want herm. matrices γ^1, γ^2 satisfying $(\gamma^1)^2 = (\gamma^2)^2 = 1$, $\gamma^1\gamma^2 + \gamma^2\gamma^1 = 0$, and then we look at the operator $\not\partial = \gamma^1 \partial_{x_1} + \gamma^2 \partial_{x_2}$. Let's make the choice

$$\gamma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} & -i \\ +i & \end{pmatrix}$$

so that

$$\not\partial = \begin{pmatrix} & \partial_{x_1} - i\partial_{x_2} \\ \partial_{x_1} + i\partial_{x_2} & \end{pmatrix} = 2\begin{pmatrix} 0 & \partial_z \\ \partial_{\bar z} & 0 \end{pmatrix} \quad \text{if} \quad z = (x_1 + ix_2)$$

I want to get rid of the 2 so put instead

$$\partial_z = \tfrac{1}{2}(\partial_x - i\partial_y) = \partial_{x_1} - i\partial_{x_2} \qquad x = \tfrac{x_1}{2}, \quad y = x_2/2$$

Thus we have

$$\not\partial = \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar z} & 0 \end{pmatrix} \quad \text{on} \quad \mathbb{C}^2 \otimes \mathbb{C}^n$$

This is a skew-adjoint operator. Gauge transformations.

$$e^{i\chi} \not\partial e^{-i\chi} = \begin{pmatrix} 0 & \partial_z - i\partial_z \chi \\ \partial_{\bar z} - i\partial_{\bar z}\chi & 0 \end{pmatrix} \qquad g \not\partial g^{-1} = \begin{pmatrix} 0 & \partial_z - g\partial_z g^{-1} \\ \partial_{\bar z} - g\partial_{\bar z} g^{-1} & 0 \end{pmatrix}$$

So

$$\not D = \not\partial + \not A = \begin{pmatrix} 0 & \partial_z + \alpha^\dagger \\ \partial_{\bar z} - \alpha & 0 \end{pmatrix}$$

where α will be arbitrary smooth matrix function of x, y.

Formally

$$\det(\not D) = \left| \det(\partial_{\bar z} - \alpha) \right|^2$$

$$\not D = \begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & D^\dagger \end{pmatrix}$$

hence $\det(\not{D}) = \det\begin{pmatrix} D & 0 \\ 0 & D^{+} \end{pmatrix} = \det D \ \det D^{+}$

$$= \det(D^{+}D)$$

The operator $\partial_{\bar{z}} - \alpha = D$ can be generalized as follows. Over a Riemann surface M or a vector bundle E we can consider an operator

$$D : E \longrightarrow E \otimes T^{0,1}$$

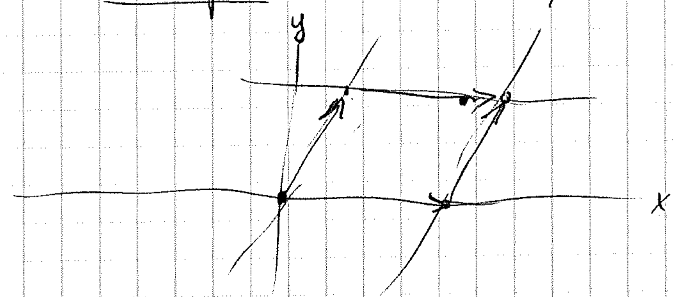
which locally looks

$$D\psi = (\partial_{\bar{z}} - \alpha)\psi \, d\bar{z} .$$

Assuming E is given an inner product then we can form its adjoint $D^{+} : E \otimes T^{0,1} \longrightarrow E$ and the (massless) Dirac operator

$$\not{D} = \begin{pmatrix} 0 & -D^{+} \\ D & 0 \end{pmatrix}$$

My problem is to define $\det(D)$ when M is compact. Example: $M = \mathbb{C}/\Gamma$ Γ period lattice



$D = \partial_{\bar{z}} - \alpha$ α Γ-periodic fn. of x, y.

Problem: Define $\det(D)$ so that
(i) analytic function of D
(ii) $\det(D) \neq 0 \iff D$ is invertible
(iii) invariance under gauge transf:

$$\det(gDg^{-1}) = \det(D)$$

Impossible: Look at $n = 1$. $\alpha \in \mathbb{C}$ constant
(i) (ii) $\underset{\shortparallel \atop f(\alpha)}{\det(\partial_{\bar{z}} - \alpha)}$ entire fn. of $\alpha \in \mathbb{C}$ with zeroes

exactly when $\alpha \in$ dual lattice Γ^*. $\qquad \mu \in \Gamma^*$

$$\det\left(e^{\mu\bar{z}-\bar{\mu}z}(\partial_{\bar{z}}-\alpha)e^{-\mu\bar{z}+\bar{\mu}z}\right) = \det(\partial_{\bar{z}}-\alpha-\mu) = f(\alpha+\mu)$$

$$\| $$

$$\det(\partial_{\bar{z}}-\alpha) = f(\alpha).$$

$$\det(\not{D}) = \det\begin{pmatrix} 0 & -D^\dagger \\ D & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & -I \\ I & 0\end{pmatrix}\begin{pmatrix} D & \\ & D^\dagger\end{pmatrix}$$

$$= \det D \det D^\dagger$$

$$= |\det D|^2 = \det(D^\dagger D)$$

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Corrigan, Goddard, Osborn + Templeton Zeta fn. reg. + multi-inst. det.

Idea: $\det(D^\dagger D)$ can be defined by ζ fn. regularization

$$\zeta_{D^\dagger D}(s) = \mathrm{Tr}\,(D^\dagger D)^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\left(e^{-tD^\dagger D}\right) t^s \frac{dt}{t}$$

known this is merom. in the whole s plane with simple pole at $s=1$. $\therefore$ Can define when $\not{D}$ has no 0 eigenvalue

$$\log \det(\not{D}) = -\frac{d}{ds}\zeta_{D^\dagger D}(s)\Big|_{s=0}$$

Reasonable [redacted] because in finite dims.

$$-\frac{d}{ds}\zeta(s)\Big|_{s=0} = \sum{}' \lambda^{-s}\log\lambda\Big|_{s=0} = \sum{}'\log\lambda = \log\det.$$

[redacted] Somehow you must get to the business of the line bundle over the space of gauge fields. ~~I could start with~~ The problem is to define $\det(\partial_{\bar{z}}-\alpha)$ as an entire analytic function of α. Can use

$$\log\frac{\det(D_0-\delta\alpha)}{\det(D_0)} = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\left(D_0^{-1}\delta\alpha\right)^n$$

provided one regularizes the first 2 terms. However this series converges only up to nearest zero and doesn't give an entire fn. So I wish to proceed differently to get a global determinant.

So

Thm: slope $g-1$. Fix $D_0 \in \mathcal{A}$. Then $\exists$ a [redacted] entire analytic function [redacted] $D \mapsto (\det D)$ on $\mathcal{A}$ such that

$$\det(\not{D}) = e^{-\zeta'_{D^\dagger D}(0)} = e^{-\|D-D_0\|^2} \left|\det D\right|^2$$

(both sides are 0 when D not inv.)

$\det D$ is unique up to a mult. constant.

Key ingredient of the proof is to calculate the curvature. After $\mathcal{L}$ and the canonical section are introduced and one defines the [redacted] metric on $\mathcal{L}$ using the ζ fn. determinant, one [redacted] must calculate the curvature

$$\bar\partial \partial \log |s|^2$$

You must go over the details carefully. So how does this work?

Assertion: $D = \partial_{\bar z} - \alpha$ $\qquad \partial_z + \alpha^*$

$$\delta\left(-\zeta'_{D^\dagger D}(0)\right) = \int_M d^2z\, d\bar z\; \mathrm{tr}\left(J(z)\,\delta\alpha(z)\right) + \text{c.c.}$$

where if $\langle z | D^{-1} | z' \rangle = \frac{i}{2\pi} \frac{1}{z-z'} \left(1 + \beta(z')(z-z') + \alpha(z')(\overline{z-z'}) + O(\overline{z-z'})^2\right)$

then

$$J(z) = \frac{i}{2\pi}\left[\beta(z) + \alpha(z)^*\right]$$

$$\partial\left\{-\zeta'_{D^\dagger D}(0)\right\} = \int_M \frac{i}{2\pi} dz\, d\bar z\; \mathrm{tr}\left(\left[\beta(z) + \alpha(z)^\dagger\right] \delta\alpha(z)\right)$$

Now β depends analytically on α provided we stay on the set where D is invertible

$$\bar\partial \partial \left\{- \qquad \right\} = \int \frac{i}{2\pi} dz\, d\bar z\; \mathrm{tr}\left(\delta\alpha(z)^\dagger\, \delta\alpha(z)\right)$$

Now we should work out the general case of your theorem for a family of curves. Let's first begin with where only the holom. str. on E varies.

$$\begin{array}{c} \mathcal{A}\times M \\ f\downarrow \\ \mathcal{A} \end{array}$$

$\mathcal{A}\times E$. On $\mathcal{A}\times E$ we have the $\bar\partial$ op.

$$\left(\partial_{\bar z}-\alpha\right)d\bar z + \partial_{\bar\alpha}\, d\bar\alpha$$

and hence we have the connection

$$\nabla = \left(\partial_z+\alpha^*\right)dz + \left(\partial_{\bar z}-\alpha\right)d\bar z + \partial_\alpha\, d\alpha + \partial_{\bar\alpha}\, d\bar\alpha$$

(Somehow $\partial_{\bar\alpha}\, d\bar\alpha$ is short for $\int_M \frac{\partial}{\partial\overline{\alpha(z)}}\, d\overline{\alpha(z)}$, i.e. you have one α variable for each point of M and component of α.) So

$$\nabla^2 = \left[\partial_z+\alpha^*, \partial_{\bar z}-\alpha\right]dz\,d\bar z + \left[\partial_{\bar z}-\alpha, \partial_\alpha\right]d\bar z\, d\alpha$$
$$+ \left[\partial_z+\alpha^*, \partial_{\bar\alpha}\right]dz\, d\bar\alpha$$

$$= -\left\{\partial_z\alpha+\partial_{\bar z}\alpha^*+[\alpha,\alpha^*]\right\}dz\,d\bar z + \left[\partial_\alpha,\alpha\right]d\bar z\,d\alpha - \left[\partial_{\bar\alpha},\alpha^*\right]dz\,d\bar\alpha$$

(How do matrices fit in? $\partial_\alpha\, d\alpha = \partial_{\alpha_{ij}}\, d\alpha_{ij}$

$$\left[\partial_{\alpha_{ij}},\alpha\right]d\bar z\, d\alpha_{ij} = E_{kl}\, d\bar z\, d\alpha_{kl}\,.)$$

Now the GRR formula is

$$c_1\, f_!(\tilde E) = \left[f_*\left(ch(\tilde E)\cdot \mathrm{Todd}\, f\right)\right]_{(1)}$$

$$\mathrm{Todd}\, f = 1 + \tfrac12 c_1(M)$$
$$ch\,\tilde E = g + tr\left(\tfrac{i}{2\pi}K\right) + \tfrac12 tr\left(\left(\tfrac{i}{2\pi}K\right)^2\right) + \cdots$$

Let's look at

$\frac12 \left(c_1(\tilde E)\, c_1(M)\right)$. $c_1(M)$ involves $dz\,d\bar z$ and f_* means we integrate over M. Then since K_E has no $d\alpha\, d\bar\alpha$ terms it's clear that $K_E\cdot c_1(M)=0$. So the only non-zero term is $f_*\left(\frac12\, tr\left(\frac{i}{2\pi}K_E\right)^2\right)$. Because of the $dz, d\bar z$ in $\nabla^2=K$ the terms to consider are just ?

$$\nabla^2 = -\left(\partial_z \alpha + \partial_{\bar z}\alpha^* + [\alpha, \alpha^*]\right) dz\, d\bar z + E_{k\ell}\, d\bar z\, d\alpha_{k\ell}(z) - E_{k\ell}\, dz\, d\overline{\alpha_{\ell k}(z)}$$

and z denotes the actual point of the Riemann surface M over which you are looking. Then

$$\frac{1}{2}\operatorname{tr}\left(\nabla^2\right)^2 = + d\bar z\, dz\, d\alpha_{k\ell}(z)\, d\overline{\alpha_{k\ell}(z)}$$

sign because dz is moved

$\therefore$ $c_1 f_!(\tilde E)$ is represented by the 2 form

$$\left(\frac{i}{2\pi}\right)^2 \int_M d\bar z\, dz\; d\alpha_{k\ell}(z)\, d\overline{\alpha_{k\ell}(z)} .$$

Let's try to understand this better. Suppose we worry about 1×1 matrices. I need the symplectic form on the tangent space to $\mathcal{A}$. The tangent vector is the 1-form $\alpha\, d\bar z$. According to my conjectures about GRR the curvature form of $\mathcal{L}$ will be given by

$$-\frac{i}{2\pi}\int_M d\bar z\, dz\;\; d\alpha(z)\, d\overline{\alpha(z)}$$

where the $-$ sign comes from the fact that $\mathcal{L}$ is dual to $c_1(f_! \tilde E)$.

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Idea: The curvature form of a line bundle is a form of type 1,1 and hence should be determined by its restriction to 1-parameter families. In more detail, given a hermitian form $h(x,y)$ on a complex vector space V, one knows by the polarization identity, that h can be recovered from the real-valued quadratic form $h(x,x)$. In practice one has a real-valued function $f(z_i)$ on $\mathbb{C}^n$ and I want to compute, or determine, $\bar\partial\partial f$ at 0. $\bar\partial\partial f = \sum \underbrace{\partial^2_{\bar z_i z_j} f|_0}_{a_{ij}} d\bar z_i dz_j$.

We consider $f(tv)$ where $v \in \mathbb{C}^n$ is fixed and t is a single complex variable. Then

$$f(tv) = f(0) + t(\partial_{z_i} f|_0 v_i) + \bar t(\partial_{\bar z_i} f|_0 \bar v_i) + \frac{t^2}{2}\partial^2_{z_i z_j} f|_0 v_i v_j$$
$$+ t\bar t\, \partial^2_{z_i \bar z_j} f|_0 v_i \bar v_j + \frac{\bar t^2}{2}\partial^2_{\bar z_i \bar z_j} f|_0 \overline{v_i v_j} + \cdots$$

better $\partial_{\bar t}\partial_t f(tv) = \partial_{\bar t}[\partial f_{z_j}(tv) v_j] = \partial^2_{\bar z_i z_j} f\, \bar v_i v_j$, which completely determines $\partial^2_{\bar z_i z_j} f$.

So now consider a 1-parameter family of $\bar\partial$ operators $D = \partial_{\bar z} - \alpha_t$ where $\alpha_t = \alpha_0 + t\alpha_1$ say. (all we need is that α_t should be holomorphic in t) Then over $\mathbb{C}\times M$ we have the connection

$$\nabla_E = (\partial_z + \alpha^*)dz + (\partial_{\bar z} - \alpha)d\bar z + \partial_t dt + \partial_{\bar t} d\bar t$$

$$\nabla_E^2 = -(\partial_z\alpha + \partial_{\bar z}\alpha^* + [\alpha,\alpha^*])dz d\bar z + \partial_t\alpha\, d\bar z dt - \partial_{\bar t}\alpha^* dz d\bar t$$

and according to the GRR conjecture

$$c_1 f_! E = f_* \operatorname{tr} \frac{1}{2}\left(\frac{i}{2\pi}\nabla_E^2\right)^2$$

Now $$\frac{1}{2}(\nabla_E^2)^2 = \partial_t\alpha\, \partial_{\bar t}\alpha^*\, d\bar z dz dt d\bar t$$

so

$$-c_1 f_! E = \frac{i}{2\pi}\underbrace{\left(\int_M \frac{i}{2\pi} \cancel{\operatorname{tr}}(\partial_t\alpha\, \partial_{\bar t}\alpha^*)\, dz d\bar z\right) dt d\bar t}_{\text{This should be the curvature form of } \mathcal{L}.}$$

<u>Curvature thm.</u> Define

$$\|B\|^2 = \int_M tr(\alpha^*\alpha)\,\frac{i}{2\pi}\,dz d\bar{z} \;=\; \frac{1}{2\pi}\int_M tr(B\wedge * B) \qquad \text{if } B = \alpha d\bar{z}$$

(Here $*$ = Hodge op. $\quad *dz = i\,d\bar{z}$ so that $dz\wedge *dz = i dz d\bar{z} > 0$.)
Then the curvature of $\mathcal{L}$ for the analytic torsion metric is the same as the curv. of 1 with $|1|^2_{D_0+B} = e^{-\|B\|^2}$.

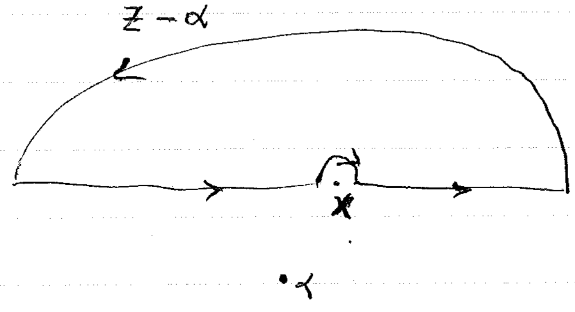
<u>Determinant thm:</u> Assume slope $= g-1$, and pick $D_0 \in \mathcal{A}$. Then there is a [redacted] holom. fn. $\det(D)$ on $\mathcal{A}$, unique up to a mult. const. such that

$$\begin{matrix}\text{anal. torsion}\\ \text{of } D^*D\end{matrix} = e^{-\|D-D_0\|^2}\,|\det D|^2$$

It follows from this that $\det D$ transforms under gauge transformations by a multiplier (exp (linear)).
<u>Question:</u> Is this also true for complex gauge transformations?

Hilbert transform. $\quad (Hf)(x) = P\int \frac{1}{\pi}\frac{1}{x-x'}\,f(x')\,dx'$
can be calculated via residues.

$\frac{1}{z-\alpha} \qquad \alpha \in LHP$



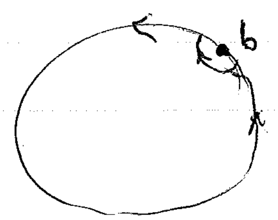
$$0 = P\int \frac{1}{z-x}\,\frac{1}{z-\alpha}\,dz \;-\; \pi i\,\frac{1}{x-\alpha}$$

In general if $f(z)$ is analytic in UHP/LHP and decays fast enough, then

$$P\int \frac{1}{z-x}\,f(z)\,dz = \begin{cases} -\pi i\,f(x) & UHP \\ +\pi i\,f(x) & LHP\end{cases}$$

Thus the Hilbert transform is $-i$ on H^+ and $+i$ on H^-; it's a unitary operator, hence defined from L^2 to L^2.

Over S^1:



$$\frac{1}{2\pi i}\int \frac{1}{z-b}\,f(z)\,dz = \tfrac{1}{2}f(b) \quad \text{for } f \text{ analytic in disk.}$$
$$= -\tfrac{1}{2}f(b) \quad \text{for } f \text{ anal. outside decaying at } \infty.$$

$$\text{Kernel} = -\frac{1}{2}\sum_{n\geq 0} z^n \overline{z'^n} + \frac{1}{2}\sum_{n<0} z^n \overline{z'^n}$$

$$= -\frac{1}{2}\frac{1}{1-z/z'} + \frac{1}{2}\frac{z'/z}{1-z'/z} = -\frac{1}{2}\frac{z'}{z'-z} + \frac{1}{2}\frac{z'}{z-z'}$$

$$= \frac{z'}{z-z'}$$

Thus

$$P\int \frac{1}{2\pi i}\frac{z'}{z-z'} f(z')\frac{dz'}{z'} = \begin{cases} \frac{1}{2} f(z) & f \text{ analy. outside} \\ -\frac{1}{2} f(z) & f \text{ anal. inside} \end{cases}$$

Now the idea to be investigated is to ask whether I can make sense of $\det(\partial_{\bar z} + \alpha)$ where α is to be a distribution supported on a curve, e.g. S^1. So how can this be understood.

So I want to look at the operator $\partial_{\bar z} - \alpha$ and try to understand what kind of holomorphic structure it defines. We know that

$$\partial_{\bar z}\left(\frac{1}{\pi}\frac{1}{z-z'}\right) = \delta(z-z')$$

So if we wanted to have $(\partial_{\bar z} - \alpha)f = 0$ with $\alpha = \delta(z-z')$, then $\partial_{\bar z}\log f = \delta(z-z')$, $\log f = \frac{1}{\pi}\frac{1}{z-z'} \Rightarrow f = e^{\frac{1}{\pi}\frac{1}{z-z'}}$

July 12, 1982

Compute determinants in the $\overset{\text{degree } 0}{\text{line bundle}}$ case over $M = \mathbb{C}/\Gamma$.

$$\delta \log\left(\frac{\det(e^f D_0 e^{-f})}{\det(D_0)}\right) = \mathrm{Tr}_{reg}\left(D^{-1}[\delta f, D]\right)$$
$$= \int tr(J[\delta f, D]) = \int tr([D,J]\delta f)$$

Now $[D,J] = -\left(\frac{i}{2\pi}\right)$ curvature of D. (In effect we have

$$[D,J] = \underset{\overset{\shortparallel}{0}}{[D,G]} - [D,G_b]$$

$$\mathrm{Index}(D) = \mathrm{Tr}\left(I - G_b D\right) - \mathrm{Tr}\left(I - DG_b\right) = \mathrm{Tr}\,[D, G_b]$$

This is zero but it has a local sense and shows that $[D, G_b]$ has to be $\frac{i}{2\pi}$. curvature D $+ \frac{i}{2\pi}\cdot\frac{1}{2}$ curv. T in general.)

$$\therefore\ [D,J] = -[D,G_b] = -\frac{i}{2\pi}(\text{curv} \ldots\ldots)$$

The next point is to write $D_0 = \partial_{\bar z} + \alpha_0$, $D = e^f D_0 e^{-f}$ $= \partial_{\bar z} + \underbrace{(\alpha_0 - \partial_{\bar z} f)}_{\alpha}$. I will take the determinants to be defined using the trivial ∂ operator ∂_z. Then the connection is

$$\nabla = \partial_z\, dz + (\partial_{\bar z} + \alpha) d\bar z \qquad\qquad \partial_z \alpha = \partial_z \alpha_0 - \partial^2_{z\bar z} f$$

curvature:
$$\nabla^2 = [\partial_z, \partial_{\bar z} + \alpha]\, dz\, d\bar z = (\partial_z \alpha)\, dz\, d\bar z$$

and so

$$\delta \log \frac{\det(e^f D_0 e^{-f})}{\det(D_0)} = -\int \left(\partial_z \alpha_0 - \partial^2_{z\bar z} f\right)\delta f \ \frac{i}{2\pi} dz\, d\bar z$$

$$= -\int +(\partial_z \alpha_0)\,\delta f + \partial_{\bar z} f\ \delta(\partial_z f) \quad \frac{i}{2\pi} dz\, d\bar z$$

the $\partial_z, \partial_{\bar z}$ appear symmetrically

$$\therefore \quad \boxed{\log\left(\frac{\det(e^f D_0 e^{-f})}{\det(D_0)}\right) = -\int\left[(\partial_z \alpha_0) f + \tfrac{1}{2}(\partial_{\bar z} f)(\partial_z f)\right]\frac{i}{2\pi} dz\, d\bar z}$$

Now can you rewrite this in terms of $\alpha = \alpha_0 - \partial_{\bar z} f$. You should be able to because you have defined the determinant function, so the expression on the right depends only on α.

Let's use Fourier series starting with α_0 and requiring α_0 to be the constant Fourier term of α.

$$\alpha = \sum \alpha_\mu e^{\mu\bar z - \bar\mu z} = \alpha_0 - \partial_{\bar z}\Big[\underbrace{-\sum{}' \frac{\alpha_\mu}{\mu} e^{\mu\bar z - \bar\mu z}}_{f}\Big]$$

$$-\int \tfrac{1}{2}\,\partial_{\bar z} f\,\partial_z f = +\int \tfrac{1}{2}\Big(\sum{}' \alpha_\mu e^{\mu\bar z-\bar\mu z}\Big)\Big(\sum{}' \alpha_\mu \frac{+\bar\mu}{\mu} e^{\mu\bar z - \bar\mu z}\Big)$$

$$= \frac{vol(M)}{2}\sum{}' \alpha_{-\mu}\frac{\bar\mu}{\mu}\alpha_\mu$$

$$\therefore \quad \boxed{\frac{\det(D+\alpha)}{\det(D+\alpha_0)} = e^{\frac{vol M}{2}\sum' \alpha_{-\mu}\frac{\bar\mu}{\mu}\alpha_\mu} \qquad \text{if } \alpha = \sum \alpha_\mu e^{\mu\bar z - \bar\mu z}}$$

The next project is to see if this has a meaning for distributions supported on a curve $S^1 \hookrightarrow M$. The idea is to see if the answer extends by continuity or whether a new regularization must be introduced.

So let's work with the lattice $\mathbb{Z} + \mathbb{Z}i$, whence

$$\overset{\mu}{(m+ni)}\overset{\bar z}{(x-iy)} - \overset{\bar\mu}{(m-ni)}\overset{z}{(x+iy)} = 2i(nx - my)$$

and an α supported on the curve $y=0$ looks like $\delta(y)f(x)$, hence

$$\alpha = \delta(y)f(x) = \sum e^{-2imy}\sum{}' b_n e^{2inx}$$

and hence $a_\mu = a_{m+ni}$ is independent of m. Then

$$\sum_{m,n}{}' b_{-n}\frac{m-ni}{m+ni} b_n \quad \text{is what we want to}$$

converge. However $\sum_m{}' \frac{m-ni}{m+ni}$ clearly diverges for any n because the summand approaches 1

Put another way we have for $\alpha = \delta(y)f(x)$

$$\partial_z\,\partial_{\bar z}^{-1}\alpha = \partial_z\int \frac{1}{\pi}\frac{1}{z-x'}f(x')\,dx' = \int -\frac{1}{\pi}\frac{1}{(z-x')^2}f(x')\,dx'$$

so $\displaystyle -\int \alpha\, \partial_z^2 \partial_{\bar z}^{-1} \alpha = \int f(x)\,dx \int \frac{1}{\pi(x-x')^2} f(x')\,dx'$.

Clearly for $f=1$ this has to be interpreted in some principal value way before it makes sense.

Now actually the kernel of $\partial_z \partial_{\bar z}^{-1}$ is $\dfrac{1}{-\pi(\bar z-\bar z')^2}$ and it has to be interpreted carefully. But there is no problem because integrating z' around circles about z give a definite number for monomials $(z-z')^k(\overline{z-z'})^l$, in fact zero unless $k-l=2$, and then $k\geq 2$, etc.

December 12, 1980

Recall that for a simple harmonic oscillator with a finite time perturbation

$$H = \frac{p^2}{2} + \frac{1}{2}(\omega^2 + \varepsilon(t))\frac{q^2}{2}$$

one gets an S-operator

$$\langle e_{\bar{\mu}} | S | e_\lambda \rangle = \langle 0 | S | 0 \rangle \, e^{\frac{1}{2}(R\lambda^2 + 2T\lambda\mu + \tilde{R}\mu^2)}$$

where one has the asymptotic behavior

$$Re^{i\omega t} + e^{-i\omega t} \longleftrightarrow Te^{-i\omega t}$$

$$Te^{i\omega t} \longleftrightarrow e^{i\omega t} + \tilde{R}e^{-i\omega t}$$

for the solutions of the classical equation of motion

$$\left(\frac{d^2}{dt^2} + \omega^2 + \varepsilon\right) q = 0$$

(see p. 247).

Next I want to apply this to the infinite dimensional oscillator represented by the wave eqn.

$$\left[\partial_t^2 + (-\partial_x^2)\right]\phi = 0$$

and the perturbation given by the wave equation with potential:

(*) $$\left[\partial_t + (-\partial_x^2 + V)\right]\phi = 0$$

The idea is that because of the continuous spectrum of $(-\partial_x)^2$ we can deal with V independent of t.

Let us first examine the asymptotic behavior of the solutions of the classical equation of motion (*). Let's work on $x \geq 0$ with bdry condition $\partial_x \phi = 0$ at $x = 0$. Then a solution of (*) is of the form

$$\phi(x,t) = \int \frac{d\omega}{2\pi}\, u(x,\omega)\, e^{-i\omega t} f(\omega)$$

where $\left(-\omega^2 - \partial_x + V\right) u_\omega(x) = 0$, $\partial_x u_\omega = 0$ at $x=0$

and

$$u_\omega(x) \sim e^{-i\omega x} + R(\omega) e^{i\omega x} \qquad x \to +\infty .$$

Notice that ϕ can be written

$$\phi(x,t) = \int_0^\infty \frac{d\omega}{2\pi} e^{-i\omega t}\, u(x,\omega) f(\omega) + \int_0^\infty \frac{d\omega}{2\pi} e^{i\omega t}\, u(x,-\omega) f(-\omega)$$

$$= e^{-iPt} \int_0^\infty \frac{d\omega}{2\pi} u_\omega f(\omega) + e^{iPt} \int_0^\infty \frac{d\omega}{2\pi} u_{-\omega} f(-\omega)$$

where $P = +\sqrt{-\partial_x^2 + V}$.

Because of the Riemann - Lebesgue lemma, we know that $\phi(x,t)$ has the following asymptotic behavior

$$\phi(x,t) \sim \int \frac{d\omega}{2\pi} \left(e^{-i\omega x} + e^{i\omega x}\right) e^{-i\omega t} f(\omega) \qquad t \to -\infty$$

$$\sim \int \frac{d\omega}{2\pi} \left(\quad '' \quad \right) e^{-i\omega t} R(\omega) f(\omega) \qquad t \to +\infty$$

where on the right are solutions of the "free" wave equation. Therefore if we use these asymptotics we have

$$e^{-iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_\omega f(\omega) \xleftrightarrow{\ \phi\ } e^{-iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_\omega R(\omega) f(\omega)$$

$$+ e^{iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_{-\omega} f(-\omega) \qquad\qquad + e^{iP_0 t} \int_0^\infty \frac{d\omega}{2\pi} u^0_{-\omega} R(-\omega) f(-\omega)$$

Hence we have

$$e^{-iP_0 t} \hat{f} \longleftrightarrow e^{-iP_0 t} \widehat{Rf}$$

$$e^{iP_0 t} \hat{g} \longleftrightarrow e^{-iP_0 t} \widehat{\overline{R}f}$$

where $\hat{f} = \int \frac{d\omega}{2\pi} u^0_\omega f(\omega)$

Thus there is no reflection, only transmission, and the T operator is multiplication by $R(\omega)$.

It follows that the S-operator on the quantum level is essentially trivial, in the sense that there is no creation and annihilation of particles, only a phase shift.

But it would be interesting to ask about $\langle 0|S|0\rangle$. We have

$$\langle e_{\bar{\mu}}|S|e_\lambda\rangle = \langle 0|S|0\rangle e^{\mu^t T\lambda}$$

and also

$$|\langle 0|S|0\rangle|^2 = |\det T| = 1.$$

Consider the fermion analogue of the perturbed oscillator. One is given H_0 on $\mathcal{H}$, then H_0 is extended to Fock space $\Lambda(\mathcal{H})$ as a derivation. Assuming H_0 does not have 0 as eigenvalue, the ground state of H_0 on $\Lambda(\mathcal{H})$ is the element [redacted] belonging to the subspace $\mathcal{H}^-$ of $\mathcal{H}$ on which $H_0 < 0$. Next suppose $H_0^{\text{on } \mathcal{H}}$ is perturbed:

$$H = H_0 + V(t)$$

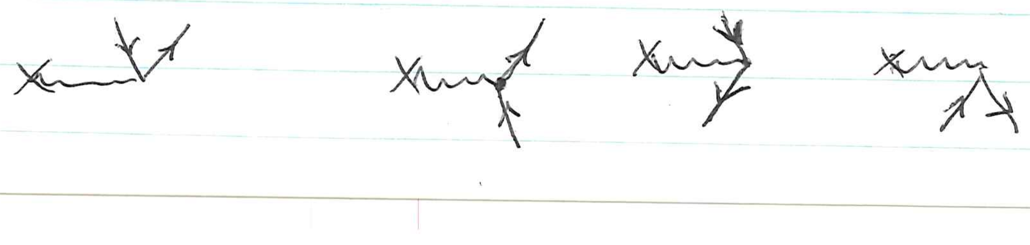
[redacted] where $V(t)$ has compact support. We want to compute the S-matrix of this perturbation on $\Lambda\mathcal{H}$.

Because H is extended as a derivation, the operators e^{-itH_0}, $U(t,t')$, $U_D(t,t')$ are the autos. of $\Lambda\mathcal{H}$ induced by the corresponding autos. of $\mathcal{H}$. Then the S-matrix on $\Lambda\mathcal{H}$ is the extension of the S-matrix on $\mathcal{H}$.

By analogy with boson case, we would like to write S in terms of creation + annihilation operators. The idea will be to work with particles and hole operators as follows. Let H_0 have the eigenvalues ω_α and $H_0 v_\alpha = \omega_\alpha v_\alpha$, and let a_α, a_α^* be the corresponding annihilation and creation operators. Let $b_\alpha = a_\alpha$ if $\omega_\alpha > 0$ and $c_\alpha = a_\alpha^*$ if $\omega_\alpha < 0$. Then $V(t)$ lies in the Lie algebra of operators spanned by $\begin{matrix} a_\alpha^* & a_\beta \\ b^* & b \\ c & c^* \end{matrix}$ and these fall into four types

[redacted]

$b^* c^*$ $\qquad$ $b^* b$ $\qquad$ $c b^*$ $\qquad$ $c b$



Consequently we expect the S-matrix to appear in the form

$$S = \langle 0|S|0\rangle \, e^{\alpha b^* c^*} e^{\beta b^* b} e^{\gamma c c^*} e^{\delta c b}$$

Furthermore we know that $S : \mathcal{H}^+ \oplus \mathcal{H}^- \longrightarrow \mathcal{H}^+ \oplus \mathcal{H}^-$ breaks up into 4 pieces, [redacted] hence we expect these pieces will be essentially $\alpha, \beta, \gamma, \delta$.

Here's how to determine α. Note that

$$S|0\rangle = \langle 0|S|0\rangle \, e^{\alpha b^* c^*} |0\rangle$$

Now if the eigenvectors of H_0 in increasing order of eigenvalue are $v_1, \ldots, v_p, \ldots, v_n$ with $\omega_1 < \ldots < \omega_p < 0 < \omega_{p+1} < \ldots < \omega_n$, then

$$|0\rangle = v_1 \wedge \ldots \wedge v_p$$

$$S|0\rangle = (Sv_1) \wedge (Sv_2) \wedge \ldots \wedge (Sv_p)$$

[redacted] Assume that we are in the nice case where [redacted] $S\mathcal{H}^- \cap \mathcal{H}^+ = 0$. Then $S\mathcal{H}^-$ is the graph of a linear map T from $\mathcal{H}^-$ to $\mathcal{H}^+$, and so we can find an orthonormal basis $e_1, \ldots, e_p$ for $\mathcal{H}^-$; $e_{p+1}, \ldots, e_n$ for $\mathcal{H}^+$ so that

$$T e_i = \lambda_i e_{p+i} \qquad \lambda_1 \geq \lambda_2 \geq \ldots .$$

Then

$$S(e_i) = \frac{e_i + \lambda_i e_{p+i}}{\sqrt{1+\lambda_i^2}}$$

and

$$S|0\rangle = \prod_{i=1}^{p} \frac{1}{\sqrt{1+\lambda_i^2}} \underbrace{(e_1 + \lambda_1 e_{p+1}) \wedge \ldots \wedge (e_p + \lambda_p e_{2p})}_{e^{\lambda_1 a_1^* a_{p+1} + \ldots + \lambda_p a_p^* a_{2p}} (e_1 \wedge \ldots \wedge e_p)}$$

for the e_i basis

But it is even simpler than this, maybe:

$$e^{\alpha b^* c^*}|0\rangle = e^{\sum \alpha_{ij} a_i^* a_j}(v_1 \wedge \cdots \wedge v_p) \qquad j \le p,\ i \ge p$$

[redacted]

$$= \left(e^{\sum \alpha_{i1} a_i^* a_1} e_1\right) \wedge \cdots \wedge \left(e^{\sum \alpha_{ip} a_i^* a_p} e_p\right)$$

$$= \left(e_1 + \sum \alpha_{i1} e_i\right) \wedge \cdots \wedge \left(e_p + \sum \alpha_{ip} e_i\right)$$

The point therefore seems to be that [redacted] any element of $\Lambda^p(\mathcal{H})$ belonging a subspace [redacted] complementary to $\mathcal{H}^+$ is uniquely represented in the form

$$e^{\alpha b^* c^*}|0\rangle.$$

In fact this is clear $\alpha b^* c^* = \sum_{\substack{j \le p \\ i > p}} \alpha_{ij} a_i^* a_j$ is a nilpotent (square zero) operator on $\mathcal{H}$ carrying $\mathcal{H}^-$ to $\mathcal{H}^+$.

It seems clear then that writing S in the form

$$S = \langle 0|S|0\rangle\, e^{\alpha b^* c^*}\, e^{(\log \beta) b^* b}\, e^{(\log \gamma) c c^*}\, e^{\delta c b}$$

$$e^{\alpha a_+^* a_-}\, e^{(\log \beta) a_+^* a_+}\, e^{(\log \gamma) a_-^* a_-}\, e^{\delta a_-^* a_+}$$

[redacted] corresponds to factoring $S : \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+$ in the form

$$S = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & \gamma \end{pmatrix}\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$$

December 15, 1980

Yesterday we looked at $H = H_0 + V$ on $\mathcal{H}$ extended to $\Lambda\mathcal{H}$. Then S on $\Lambda\mathcal{H}$ is the automorphism extending [redacted]

$$S = T\left\{e^{-i\int dt\, V_D(t)}\right\} \qquad \text{on } \mathcal{H}$$

The perturbation will create pairs when

$$S : \mathcal{H}^- \oplus \mathcal{H}^+ \longrightarrow \mathcal{H}^- \oplus \mathcal{H}^+$$

does not preserve this decomposition, because $S|0\rangle$ is the element of $\Lambda\mathcal{H}$ corresponding to the subspace $S(\mathcal{H}^-)$.

So now let us consider an infinite dimensional situation, say where H_0 is the Dirac operator and V is the perturbation represented by a weak external time-independent EM field. Then we know the scattering operator S exists on $\mathcal{H}$ and that it commutes with H_0. In particular S has to preserve the decomposition into positive and negative eigenspaces. Thus S [redacted] cannot create pairs. So what happens is that S separately moves the particles and holes around.

Consider two oscillators described by

$$H_0 = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2$$

and a coupling between them, say

$$H_{int} = \varepsilon q_1 q_2 \qquad q_i = \frac{a_i + a_i^*}{\sqrt{2\omega_i}}$$

In practice ω_1, ω_2 are large relative to $\omega_1 - \omega_2$, hence the terms involving $a_1^* a_2^*$, $a_1 a_2$ in H_{int} have rapidly changing phase in the Dirac picture. So if one were to compute

$$U_D(t, t') = 1 - i \int_{t'}^{t} dt_1 \; \varepsilon q_1(t_1) q_2(t_1)$$

to first order these terms would be small. Thus one make the "rotating wave" approximation and drops them.

So we consider $H_{int} = \varepsilon(a_1^* a_2 + a_2^* a_1)$. The classical equations of motion are

$$\dot{a}_1 = i[H, a_1] = i(-\omega_1 a_1 - \varepsilon a_2)$$
$$\dot{a}_2 = i(-\omega_2 a_2 - \varepsilon a_1)$$

or simply

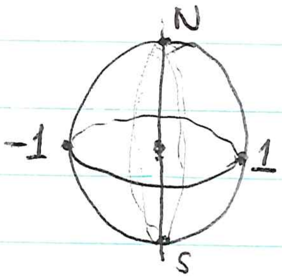
(★) $$i \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\cdot} = \begin{pmatrix} \omega_1 & \varepsilon \\ \varepsilon & \omega_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Notice that $H = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \varepsilon(a_1^* a_2 + a_2^* a_1)$ preserves "particle number". Hence the quantum situation is the symmetric algebra on the 1-particle space which is the same for both H_0 and H. The 1-particle space is described by the "classical" equations (★) which happens to be a Schroedinger equation for a 2-state system.

Think of the states of a 2-particle system as corresponding to points on the Riemann sphere, with $\begin{pmatrix}1\\0\end{pmatrix} \leftrightarrow$ north pole $\begin{pmatrix}0\\1\end{pmatrix} \leftrightarrow$ south pole. If $\varepsilon = 0$, these points stay fixed and

$$\frac{a_1}{a_2}(t) = \frac{e^{-i\omega_1 t}a_1}{e^{-i\omega_2 t}a_2} = e^{i(\omega_2-\omega_1)t}\left(\frac{a_1}{a_2}\right)$$

rotates with angular frequency $\omega_2 - \omega_1$.



Suppose $\omega_1 = \omega_2$. The eigenvalues of $\begin{pmatrix}\omega_1 & \varepsilon\\ \varepsilon & \omega_1\end{pmatrix}$ are $\omega_1 \pm \varepsilon$ and the eigenvectors are

$$\begin{pmatrix}1\\-1\end{pmatrix} \text{ for } \omega_1 - \varepsilon, \quad \begin{pmatrix}1\\1\end{pmatrix} \text{ for } \omega_1 + \varepsilon$$

Thus the perturbed system is described by [redacted] rotation [redacted] leaving 1, -1 fixed. [redacted] The state $\begin{pmatrix}1\\0\end{pmatrix}$ evolves as

$$\psi(t) = e^{-i\omega t}\begin{pmatrix}\cos \varepsilon t\\ -i\sin \varepsilon t\end{pmatrix}$$

and alternates between N and S. It's a mixture of the N, S states. It seems the the probability splitting

$$\cos^2 \varepsilon t + \sin^2 \varepsilon t = 1$$

is the [redacted] relative height, or level, between N and S.

On the other hand if $\varepsilon \ll |\omega_1 - \omega_2|$, then the eigenstates for the perturbed system are very close to those of the unperturbed system. Thus the perturbed system is described by rotation about an axis very close to NS. So one sees that the states N, S are nearly invariant in this case.

These two cases exhibit the transitions $N \longleftrightarrow S$ when $\omega_1 = \omega_2$, and the impossibility of such [redacted] transitions when $\omega_1 \neq \omega_2$ or more precisely $|\varepsilon| \ll |\omega_1 - \omega_2|$.

Review Nyquist's relation: Suppose we have a transmission line described by

$$\frac{\partial I}{\partial x} = -C_0 \frac{\partial V}{\partial t} \qquad \frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}.$$

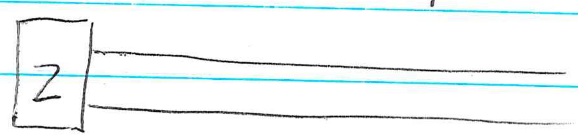
The travelling waves are $\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} \hat{V} \\ \hat{I} \end{pmatrix} e^{i(kx - \omega t)}$

with $\quad k\hat{I} = C_0 \omega \hat{V} \qquad k\hat{V} = L_0 \hat{I} \omega \quad$ or

$$\frac{\hat{V}}{\hat{I}} = \frac{k}{C_0\omega} = \frac{L_0 \omega}{k} \qquad \text{speed} = \frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}}$$

$$R_0 = \text{imped.} = \sqrt{\frac{L_0}{C_0}}$$

Suppose the transmission line is terminated by a circuit at $x=0$ with impedance $Z(\omega)$:



The normal mode are

$$V(x,t) = \mathrm{Re}\left(A\left(e^{-ikx} + S(\omega) e^{ikx}\right) e^{-i\omega t}\right)$$
$$I(x,t) = \frac{1}{R_0} \mathrm{Re}\left(A\left(-e^{-ikx} + S(\omega) e^{ikx}\right) e^{-i\omega t}\right) \qquad \frac{k}{\omega} = \sqrt{L_0 C_0}$$

where the reflection coefficient $S(\omega)$ satisfies

$$+Z(\omega) = R_0 \frac{S+1}{S-1}$$

The time-averaged energy density for this mode is

$$\left\langle \tfrac{1}{2} C_0 V^2 + \tfrac{1}{2} L_0 I^2 \right\rangle = \tfrac{1}{2} C_0 \left(\tfrac{1}{2} |A|^2 |e^{-ikx} + S e^{ikx}|^2 + \tfrac{1}{2} |A|^2 |-e^{-ikx} + S e^{ikx}|^2 \right)$$

$$= C_0 |A|^2$$

assuming the circuit is lossless, so that $|S| = 1$.

Fix a length ℓ of the line and terminate it say by a short circuit. Then the normal modes are given by frequencies ω satisfying

$$e^{-ik\ell} + S(\omega)e^{ik\ell} = 0 \qquad \text{or} \qquad S(\omega) + e^{-2ik\ell} = 0$$

If ℓ is large and we want the density of modes around a fixed frequency, the variation of $S(\omega)$ can be neglected, and so the modes are distributed for

$$k\ell = n\pi + \underset{\text{small change}}{\delta(\omega)} \qquad n \in \mathbb{Z}.$$

hence the number in the range dk or $d\omega$ is

$$dn = \frac{\ell}{\pi}\,dk = \frac{\ell}{\pi}\sqrt{L_0C_0}\,d\omega.$$

Lets now suppose the system is at the temperature T. Then the situation is described by the Maxwell-Boltzmann distribution in phase space. Since we have an oscillator, this means each amplitude A_ω is a Gaussian variable such that the average energy of the mode is kT: The energy is $C_0\ell\langle|A_\omega|^2\rangle$ + the energy inside the circuit. If ℓ is large then approximately we have

$$C_0\ell\langle|A_\omega|^2\rangle = kT$$

Now

$$V\big|_{x=0} = \sum_\omega \mathrm{Re}\big(A_\omega(1+S(\omega))\big)$$

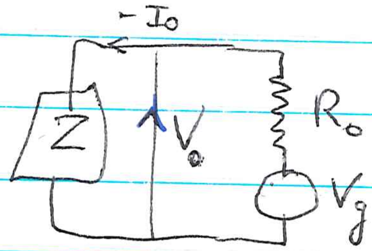
so

$$\langle V_0^2\rangle = \sum \frac{1}{2}\underbrace{\langle|A_\omega|^2\rangle}_{\sim\, kT/C_0\ell}\,|1+S(\omega)|^2\,\underbrace{dn}_{\frac{\ell}{\pi}\sqrt{L_0C_0}\,d\omega}$$

so

$$\langle V_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \cdot \frac{1}{2} k T R_0 \cdot |1 + S(\omega)|^2$$

On the other, hand we can think of the line as a resistor R_0 plus noise generator



whence

$$\hat{V}_0 = \frac{Z}{R_0 + Z} \hat{V}_g$$ [redacted]

[redacted]

$$\frac{Z}{R_0 + Z} = \frac{\frac{S+1}{S-1}}{1 + \frac{S+1}{S-1}} = \frac{S+1}{2S}$$

$$\left| \frac{Z}{R_0 + Z} \right|^2 = \left| \frac{S+1}{2} \right|^2 \qquad \text{since } |S| = 1.$$

$$\therefore \quad \langle V_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \underbrace{2kTR_0 \left| \frac{S+1}{2} \right|^2}_{|\hat{V}_0(\omega)|^2}$$

and so one concludes* that

$$\langle V_g^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \, 2kTR_0$$

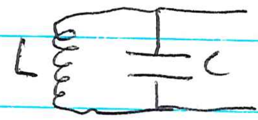
which is the Nyquist relation. [redacted]

[redacted]

The above is sloppy in many respects at the end. First of all one has used the Wiener-Khinchin theorem to get

$$\langle V_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} |\hat{V}_0(\omega)|^2$$

and secondly one argues that one can take various impedances Z. For example Wannier takes Z to be a tuned circuit



with a sharp resonance frequency. By equipartition one knows that

$$\frac{1}{2} C \langle V_0^2 \rangle = \frac{1}{2} kT,$$

so by computing $\left|\frac{Z}{R_0+Z}\right|^2$ one can check the Nyquist formula at the resonant frequency

Let's go over this in detail. For the tuned circuit

$$Z(\omega) = \frac{1}{Cs + \frac{1}{Ls}} \qquad s = -i\omega$$

so

$$\frac{S+1}{2S} \simeq \frac{Z}{R_0 + Z} = \frac{1}{R_0\left(Cs + \frac{1}{Ls}\right) + 1} = \frac{\frac{L}{R_0} s}{LCs^2 + \frac{Ls}{R_0} + 1}$$

Thus

$$\langle V_0^2 \rangle = 2kTR_0 \int_0^\infty \frac{d\omega}{\pi} \left| \frac{s}{R_0 C \left(s^2 + \frac{1}{R_0 C} s + \frac{1}{LC} \right)} \right|^2$$

$$= \frac{2kTR_0}{(R_0 C)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2}{(\omega^2 - \omega_0^2)^2 + g^2 \omega^2} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$g = \frac{1}{R_0 C}$$

The integral can be evaluated by residues and yields $\frac{1}{2\gamma}$. Thus

$$\langle V_0^2 \rangle = \frac{2kTR_0}{(R_0C)^2}\,\frac{R_0C}{2} = \frac{kT}{C}$$

so that $\quad \frac{1}{2}kT = \frac{1}{2}C\langle V_0^2\rangle \quad$ as required.

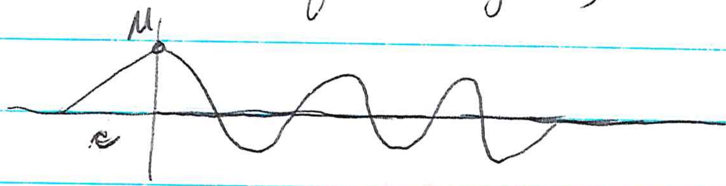
Let's go over why this has to hold. We are calculating $\langle V_0^2\rangle$ by the Maxwell-Boltzmann distribution. The phase space involved has points given by the pair of functions $\begin{pmatrix} V(x) \\ I(x) \end{pmatrix}$, [redacted] together with the current thru the inductance, call it I_L. Then on this vector space we have the energy

$$H: \quad \frac{1}{2}LI_L^2 + \frac{1}{2}CV_0^2 + \int\left\{\frac{1}{2}C_0V(x)^2 + \frac{1}{2}L_0I(x)^2\right\}dx$$

Therefore on this vector space goes the Gaussian measure $e^{-\beta H}$, and hence $\quad \langle \frac{1}{2}LI_L^2\rangle = \langle \frac{1}{2}CV_0^2\rangle = \frac{1}{2}kT$.

December 22, 1980

Consider a line of density λ, tension λ connected to an oscillator



$$\lambda \partial_t^2 u = \lambda \partial_x^2 u \qquad x > 0$$

$$M(\ddot{u}_0 + \omega_0^2 u_0) = \lambda (\partial_x u)_0$$

Suppose the string is tied down at $x = l$, where l is very large. The motion is of the form

$$u(x,t) = \sum_\omega \mathrm{Re}\left(A_\omega \left(e^{-i\omega x} + S(\omega) e^{i\omega x}\right) e^{-i\omega t}\right)$$

where ω runs over solutions > 0 of

$$e^{-i\omega l} + S(\omega) e^{i\omega l} = 0$$

and $S(\omega)$ is determined by

$$M(s^2 + \omega_0^2)(1+S) = \lambda s (1-S) \qquad s = -i\omega$$

$$\frac{1+S}{1-S} = \frac{\lambda s}{M(s^2+\omega_0^2)}$$

The energy density along the string is

$$\frac{1}{2}\lambda \dot{u}^2 + \frac{1}{2}\lambda (\partial_x u)^2$$

For the ω-th mode, the time-averaged energy density is

$$\lambda |A_\omega|^2 \omega^2$$

so proceeding as in the Nyquist problem

$$\langle u_0^2 \rangle = \int_0^\infty \frac{d\omega}{\pi} \frac{2kT}{\lambda \omega^2} \left| \frac{1+S_\omega}{2} \right|^2 .$$

Here $\quad \dfrac{1+S}{2} = $ [redacted] $\dfrac{Z}{Z+1} = \dfrac{\lambda s}{M(s^2+\omega_0^2)+\lambda s}$

and a residue [redacted] calculation again gives

$$\frac{1}{2} M\omega_0^2 \langle u_0^2 \rangle = \frac{1}{2} kT$$

as required by <u>equi-partition</u>.

It seems to be possible to understand emission classically as follows. Consider the string as an absorber, so that only outgoing waves are allowed. Then we have a damped harmonic oscillator. In effect

$$\partial_x u = -\dot{u} \qquad \text{for outgoing waves, so}$$

$$M(\ddot{u}_0 + \omega_0^2 u_0) = -\lambda \dot{u}_0$$

or

$$\ddot{u}_0 + \gamma \dot{u}_0 + \omega_0^2 u_0 = 0 \qquad \gamma = \frac{\lambda}{M}$$

The solutions are

where [redacted]

$$\boxed{\begin{aligned} u_0 &= e^{-\frac{\gamma}{2}t}\, Re\left(A e^{-i\omega_1 t}\right) \\ \omega_1 &= \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \end{aligned}}$$

In effect the roots of $\quad +\omega^2 + i\gamma\omega - \omega_0^2 = 0 \quad$ are

$$\omega = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}\,.$$

Thus we see that the energy, potential or kinetic, has the decay rate $e^{-\gamma t}$. (Assume $\gamma \ll \omega_0$).

December 25, 1980

The problem is still to find a simple example of emission and absorption. Let's consider a simple oscillator coupled to a continuous family of oscillators. The Hamiltonean is

$$\mathbb{H} = \underbrace{\omega_0 b^* b + \sum_\alpha \omega_\alpha a_\alpha^* a_\alpha}_{H_0} + \sum_\alpha b^*(\gamma_\alpha a_\alpha) + (a_\alpha^* \bar{\gamma}_\alpha) b.$$

Because this Hamiltonian is of degree 0, i.e. it commutes with the number operator $b^*b + \sum_\alpha a_\alpha^* a_\alpha$, it follows that it's effect on states of many particles is determined by what it does to 1-particle states. More precisely the Hilbert space is the symmetric algebra on the 1-particle space $\mathcal{H}$ which is spanned by the vectors $b^*|0\rangle$, $a_\alpha^*|0\rangle$. On this 1-particle space, time evolution is described by

$$i\frac{\partial}{\partial t}\psi = \mathbb{H}\psi \qquad \mathbb{H} = \left(\begin{array}{c|c} \omega_0 & \gamma^* \\ \hline \gamma & H_0 \end{array}\right)$$

where $H_0 = \sum_\alpha |\alpha\rangle \omega_\alpha \langle\alpha|$, $\gamma^* = \sum_\alpha \bar{\gamma}_\alpha \langle\alpha|$.

Set
$$\Phi = b^*|0\rangle.$$

The first thing is to understand how Φ decays. Thus we want to understand

$$\langle\Phi| e^{-i\mathbb{H}t} |\Phi\rangle$$

as $t \to +\infty$. One has

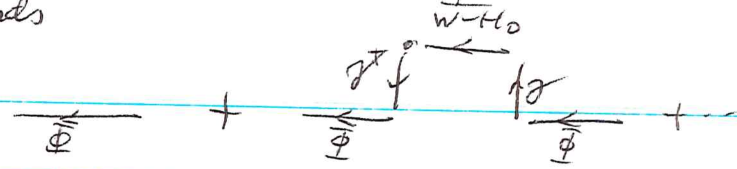
$$\int_0^\infty dt\, e^{+iWt} \langle\Phi| e^{-i\mathbb{H}t} |\Phi\rangle = \langle\Phi| \frac{-1}{i(W_{+i0^+} - \mathbb{H})} |\Phi\rangle.$$

hence

$$\langle\Phi| e^{-i\mathbb{H}t} |\Phi\rangle = \int_{-\infty}^{\infty} \frac{dW}{2\pi} \langle\Phi| \frac{-1}{i(W + i0^+ - \mathbb{H})} |\Phi\rangle e^{-iWt} \quad \text{for } t > 0.$$

Now we can compute $\frac{1}{W-H}$ by diagrams as in Weinberg's paper. One finds

$$\langle \Phi | \frac{1}{W-H} | \Phi \rangle =$$



$$= \frac{1}{W-\omega_0} + \frac{1}{W-\omega_0} \langle \gamma | \frac{1}{W-H_0} | \gamma \rangle \frac{1}{W-\omega_0} + \cdots$$

so

$$\boxed{\langle \Phi | \frac{1}{W-H} | \Phi \rangle = \frac{1}{W-\omega_0 - \langle \gamma | \frac{1}{W-H_0} | \gamma \rangle}}$$

<u>Remark</u> [redacted]: We can split the space on which H_0 [redacted] operates into the cyclic subspace spanned by γ and the orthogonal complement. We can forget about the latter in trying to understand H from H_0, and so can suppose γ is a cyclic vector for H_0.

[redacted] One has

$$\langle \gamma | \frac{1}{W-H_0} | \gamma \rangle = \int [redacted] \frac{d\mu(\omega)}{(W-\omega)}$$

for some measure on the line. Let's suppose

$$d\mu(\omega) = |\gamma_\omega|^2 d\omega$$

and put

$$g(W) = \int \frac{|\gamma_\omega|^2 d\omega}{W-\omega} = \langle \gamma | \frac{1}{W-H_0} | \gamma \rangle .$$

for W in the UHP. Define g by analytic continuation from the UHP. From the formula

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \int \frac{dW}{2\pi i} e^{-iWt} \left(\frac{(-1)}{W-\omega_0 - g(W)} \right) \qquad t>0$$

and the fact that e^{-iWt} decays in the LHP for $t>0$, we see that the behavior of $\langle \Phi | e^{-iHt} | \Phi \rangle$ is controlled by the singularities in the LHP nearest the real axis.

Suppose γ is small. Then $g(W)$ should be small so that near ω_0 is a root $\tilde{\omega}_0$ of $W-\omega_0-g(W)$, which can be found by iterating: $W = \omega_0 + g(W)$

$$W^{(0)} = \omega_0$$

$$W^{(1)} = \omega_0 + g(\omega_0)$$

$$W^{(2)} = \omega_0 + g(\omega_0 + g(\omega_0)) = \omega_0 + g(\omega_0) + g'(\omega_0)g(\omega_0) + \dots$$

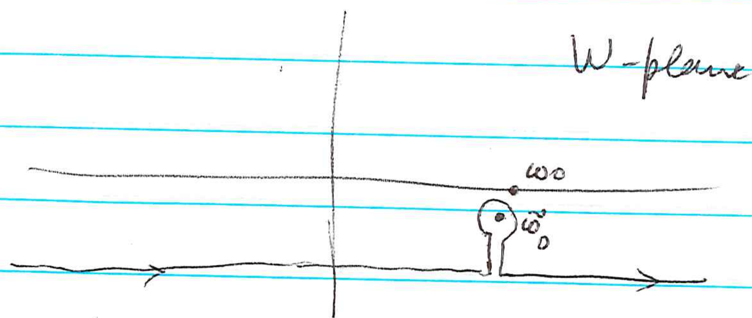
Then $W^{(n)} \longrightarrow \tilde{\omega}_0$. Now $g(W) \in LHP$ when $W \in UHP$, in fact we know that on the real W-axis

$$\mathrm{Im}(g(W)) = \int |\gamma_\omega|^2 d\omega \, \mathrm{Im}\underbrace{\left(\frac{1}{W+i0^+-\omega}\right)}_{\frac{-i\varepsilon}{(W-\omega)^2+\varepsilon^2}}$$

$$= \int |\gamma_\omega|^2 \left(-\pi\delta(W-\omega)\right) d\omega$$

$$= -\pi |\gamma_W|^2$$

Therefore $\tilde{\omega}_0$ is in the LHP. We can deform the contour



so as to get

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \frac{1}{1-g'(\tilde{\omega}_0)} e^{-i\tilde{\omega}_0 t} + \text{faster decaying exponentials}$$

Thus it should be true that

$$\langle \Phi | e^{-iHt} | \Phi \rangle \simeq (1+g'(\omega_0))\, e^{-i(\omega_0+g(\omega_0))t}\,(1+O(|\gamma|^4))$$

$$= e^{-i\omega_0 t}\left(1+g'(\omega_0) - i g(\omega_0) t + O(|\gamma|^4)\right)$$

Let's check this by perturbation theory.

$$\langle \Phi | e^{-iHt} | \Phi \rangle = e^{-i\omega_0 t} + (-i)\int 0$$

$$+ (-i)^2 \underbrace{\int_0^t dt_1 \int_0^{t_1} dt_2\, e^{-i\omega_0(t-t_1)} \langle \gamma | e^{-iH_0(t_1-t_2)} | \gamma \rangle \times e^{-i\omega_0 (t_2)}}_{} + \dots$$

$$= e^{-i\omega_0 t} \int d\omega\, |\gamma_\omega|^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, e^{i\omega_0(t_1-t_2)}\, e^{-i\omega(t_1-t_2)}$$

Now

$$\int_0^t dt_1 \int_0^{t_1} dt_2\, e^{i\alpha(t_1-t_2)} = \int_0^t dt_1\, e^{i\alpha t_1}\, \frac{e^{-i\alpha t_1}-1}{-i\alpha} = \int_0^t dt\, \frac{1-e^{i\alpha t_1}}{-i\alpha}$$

$$= \frac{t}{-i\alpha} + \frac{e^{i\alpha t}-1}{(i\alpha)^2} = \frac{1+i\alpha t - e^{i\alpha t}}{\alpha^2}$$

$$\therefore \langle \Phi | e^{-iHt} | \Phi \rangle = e^{-i\omega_0 t}\left\{1 - \int d\omega\, |\gamma_\omega|^2\, \frac{1+i(\omega_0-\omega)t - e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2} + O(|\gamma|^4)\right\}$$

But if we let ω_0 be approached from the UHP

$$-\int d\omega\, |\gamma_\omega|^2\, \frac{1+i(\omega_0-\omega)t}{(\omega_0-\omega)^2} = -i g(\omega_0) t + g'(\omega_0)$$

and the term

$$-\int d\omega\, |\gamma_\omega|^2\, \frac{-e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)^2},$$

for ω_0 above the real axis, can be evaluated by pushing the

ω-contour down till it catches the singularities of $|\gamma_\omega|^2$; hence it contributes [redacted] faster decaying exponentials.

So our conclusion is that the exact behavior is

$$\langle \Phi | e^{-iHt} | \Phi \rangle = \frac{1}{1-g'(\tilde{\omega}_0)} e^{-i\tilde{\omega}_0 t} + \text{faster decay terms}$$

where $\tilde{\omega}_0$ satisfies:

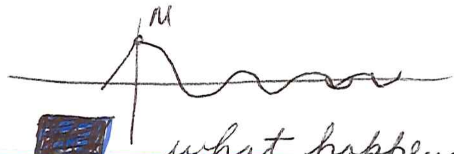
$$\tilde{\omega}_0 = \omega_0 + g(\tilde{\omega}_0)$$

Thus

$$\operatorname{Im}(\tilde{\omega}_0) = \operatorname{Im} g(\tilde{\omega}_0) = -\pi |\gamma_{\omega_0}|^2 + O(|\gamma|^4)$$

Next we could use some specific examples. Ultimately we want H_0 to have spectrum ≥ 0 so $g(W)$ [redacted] is to be an [redacted] analytic function with a [redacted] cut along the positive real axis. Unfortunately this puts a singularity on the real axis. So there is no way to construct an example of the above type where $\langle \Phi | e^{-iHt} | \Phi \rangle$ has exponential decay and such that H_0 has spectrum ≥ 0.

So let's look at some examples where the spectrum of H_0 is the whole line.

Let's return to 
and try to understand what happens on the quantum mechanics level. We are working with a harmonic oscillator hence we know that there is a boson particle structure on the quantum states.

Review the formalism for an oscillator with a finite number of degrees of freedom. Say

$$H = \frac{p^2}{2} + \frac{1}{2} q \cdot \omega^2 q$$

where ω^2 is a positive-definite matrix of which ω is the positive square root. The classical equation of motion is

$$\left(\frac{d^2}{dt^2} + \omega^2\right) q = 0$$

and it has solutions

$$q = e^{-i\omega t} A + e^{i\omega t} B \qquad B = \bar{A}.$$

or better
$$q = Re\,(e^{-i\omega t} A).$$

The energy of this solution is

$$E = \frac{1}{2}|\omega A|^2$$

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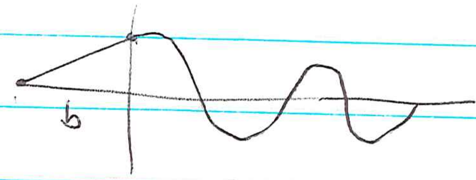
Lets look at some examples of light strings terminated differently. Define the impedance to be $Z = \frac{1+R}{1-R}$ where R is the reflection coefficient.

1) tied at $x=0$: Here $R = -1$

2) free at $x=0$: $R = 1$.

3) weightless stretch of length b tied at $x=-b$



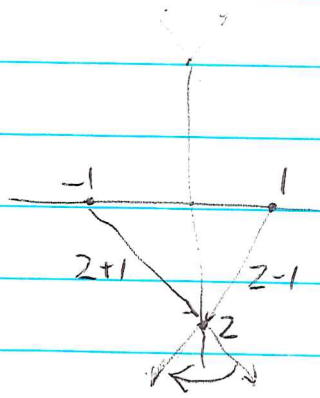
$$\lambda(\partial_x u)_0 = \frac{u_0}{b} \qquad \lambda s(1-R) = \frac{1}{b}(1+R)$$

$\therefore\ Z = \lambda b s$. As $s = -i\omega$ goes from 0 to $-i\infty$

Z goes from 0 to $-i\infty$, so

$$R = \frac{Z-1}{Z+1} \quad \text{goes from } -1 \text{ to } 1$$

$\arg(R)$ " " $-\pi$ to 0



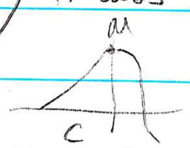
4) mass M at $x=0$:

$$M\ddot{u}_0 = \lambda(\partial_x u)_0 \Rightarrow Ms^2(1+R) = \lambda s(1-R)$$

$$\Rightarrow Z = \frac{1+R}{1-R} = \frac{\lambda s}{Ms^2} = \frac{\lambda}{Ms}$$

Z goes from $i\infty$ to 0 so $\arg(R)$ goes from 0 to π

5) mass M at $x=0$ + weightless stretch tied at $-c$.



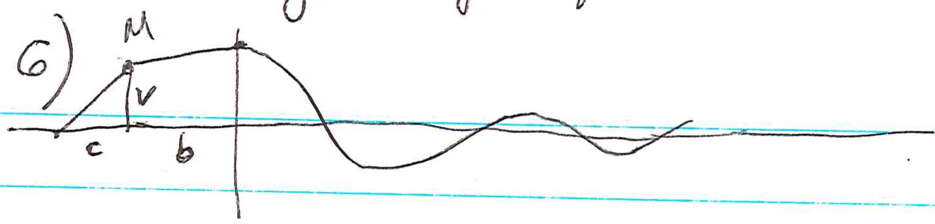
$$M\ddot{u}_0 + \frac{1}{c}u_0 = \lambda(\partial_x u)_0$$

$$\left(Ms^2 + \frac{1}{c}\right)(1+R) = \lambda s(1-R)$$

$$Z = \frac{\lambda s}{Ms^2 + \frac{1}{c}} \quad \text{goes from } 0 \text{ to } -i\infty, \text{ then } i\infty \text{ to } 0.$$

so $\arg R$ goes from $-\pi$ to π.

6)



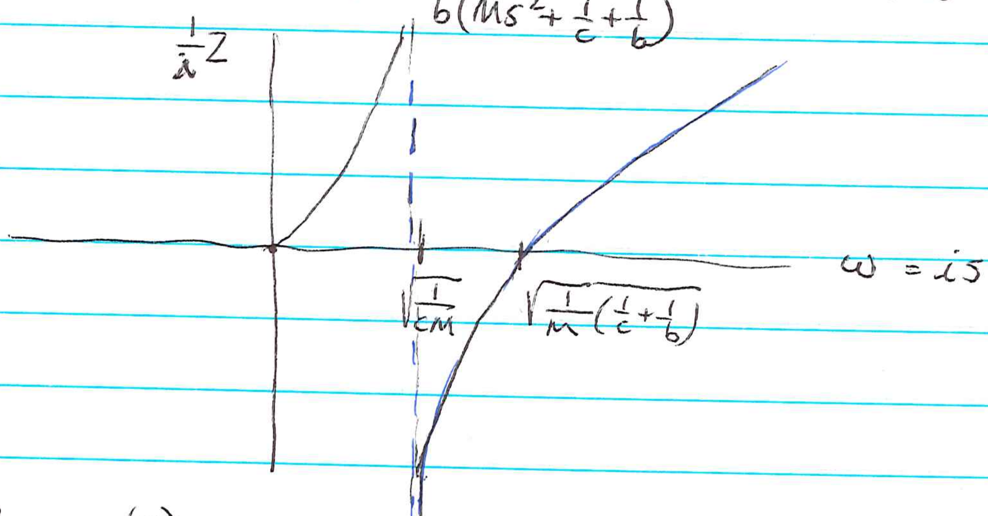
$$M\ddot{v} + \frac{1}{c}v = \frac{u_0 - v}{b} = \lambda\,(\partial_x u)_0 \qquad \text{say } v = Ae^{-i\omega t}$$

$$\left(Ms^2 + \frac{1}{c}\right)A = \frac{1+R-A}{b} = \lambda s\,(1-R)$$

$$\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)A = \frac{1}{b}(1+R)$$

$$(1+R)\left[1 - \frac{1}{b\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)}\right] = b\lambda s\,(1-R)$$

$$\therefore\; Z = \frac{b\lambda s}{1 - \frac{1}{b\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)}} = \frac{b\lambda s\left(Ms^2 + \frac{1}{c} + \frac{1}{b}\right)}{Ms^2 + \frac{1}{c}}$$



Thus $\arg(R)$ goes from $-\pi$ to 0 to π to 0.

A point that emerges from these examples is that $\arg(R)$ changes $n\pi$ in going from $\omega = 0$ to $\omega = \infty$ where n is the number of quadratic terms in the energy in the port. Thus we know that $\frac{3}{2}k_B T$ is the thermal energy one can store in the two strings and the kinetic energy of the mass M in example 6).

Let's now interpret these "strings" as (generalized) oscillators. First look at example 6): A configuration consists of $\phi = (v, u)$ where $v \in \mathbb{R}$, u is real fn in $C_0^\infty(\mathbb{R}_{\geq 0})$. The kinetic energy is $\frac{1}{2}\|\dot{\phi}\|^2$ where

$$\frac{1}{2}\|\phi\|^2 = \frac{1}{2}Mv^2 + \frac{\lambda}{2}\int_0^\infty u^2\,dx$$

and the potential energy is

$$\frac{1}{2}P(\phi) = \frac{1}{2c}v^2 + \frac{1}{2b}(u_0 - v)^2 + \frac{\lambda}{2}\int_0^\infty (\partial_x u)^2\,dx$$

We want to write

$$P(\phi) = \langle \phi | L\phi \rangle$$

where L is an operator and $\langle \phi|\phi\rangle = \|\phi\|^2$. So polarize P:

$$\frac{1}{2}P(\tilde{\phi}, \phi) = \frac{1}{2c}\tilde{v}v + \frac{1}{2b}(\tilde{u}_0 - \tilde{v})(u_0 - v) + \underbrace{\frac{\lambda}{2}\int_0^\infty (\partial_x \tilde{u})(\partial_x u)\,dx}_{-\frac{\lambda}{2}\tilde{u}_0(\partial_x u)_0 + \frac{\lambda}{2}\int_0^\infty \tilde{u}(-\partial_x^2 u)\,dx}$$

In order that this can be written $\langle \tilde{\phi} | L\phi\rangle$ we must have that $\phi = (v, u)$ makes the $\tilde{u}_0$ vanish, hence

$$\frac{1}{b}(u_0 - v) = \lambda(\partial_x u)_0$$

whence
$$\frac{1}{2}P(\tilde{\phi}, \phi) = \frac{1}{2}M\,\tilde{v}\left[\left(\frac{1}{c}+\frac{1}{b}\right)v - \frac{1}{b}u_0\right]\frac{1}{M} + \frac{\lambda}{2}\int_0^\infty \tilde{u}(-\partial_x^2 u)\,dx.$$

Thus
$$L(v, u) = \left(\frac{1}{M}\left[\left(\frac{1}{c}+\frac{1}{b}\right)v - \frac{u_0}{b}\right],\ -\partial_x^2 u\right)$$

The eigenfunctions of L with eigenvalue ω^2 are given by solutions of

$$\omega^2 v = \frac{1}{M}\left[\left(\frac{1}{c}+\frac{1}{b}\right)v - \frac{u_0}{b}\right]$$

$$\omega^2 u = -\partial_x^2 u$$

$$\frac{1}{b}(u_0 - v) = \lambda(\partial_x u)_0$$

and hence are of the form $\phi_\omega = (v_\omega, u_\omega)$

$$u_\omega = e^{-i\omega x} + R_\omega e^{i\omega x}$$
$$v = A_\omega$$

where

$$\left(M(-\omega^2) + \frac{1}{c}\right) A = \frac{1+R-A}{b} = \lambda(-i\omega)(1-R)$$

Example where $b = 0$:

$$\|u\|^2 = M u_0^2 + \lambda \int_0^\infty u^2\,dx$$

$$P(u) = \frac{1}{c} u_0^2 + \lambda \int_0^\infty (\partial_x u)^2\,dx$$

$$P(\tilde{u}, u) = \frac{1}{c} \tilde{u}_0 u_0 - \lambda \tilde{u}_0 (\partial_x u)_0 + \lambda \int_0^\infty \tilde{u}\,(-\partial_x^2 u)\,dx$$

$$\langle \tilde{u} | Lu \rangle = M \tilde{u}_0 (Lu)_0 + \lambda \int_0^\infty u\, Lu\,dx$$

leads to

$$Lu = -\partial_x^2 u$$

$$M(Lu)_0 = \frac{1}{c} u_0 - \lambda (\partial_x u)_0$$

Thus for u to be in the domain of L we must have

$$M(\partial_x^2 u)_0 + \frac{1}{c} u_0 = \lambda (\partial_x u)_0$$

The eigenfunctions are

$$u_\omega = e^{-i\omega x} + R_\omega e^{i\omega x}$$

where

$$\frac{1+R}{1-R} = \frac{\lambda s}{Ms^2 + \frac{1}{c}}$$

Now let's fix a length l and tie the string at $x = l$. We then get eigenvalues ω_n by solving

$$R_\omega + e^{-2i\omega l} = 0$$

and corresponding eigenfunctions $u_n = u_{\omega_n}$. Assuming these eigenfns. are complete we know that

$$\frac{u_n}{\|u_n\|}$$

is an orthonormal basis for functions on $[0,l]$ with the norm $\|f\|^2 = M|f_0|^2 + \lambda\int_0^l |f|^2 dx$. Hence we should have an expansion formula

$$f = \sum_n u_n \frac{1}{\|u_n\|^2}\langle u_n|f\rangle$$

Now let $l \to \infty$. For large l, the variation of R_ω can be neglected so that the number of eigenvalues dn in a range $d\omega$ is given by $dn = \frac{l}{\pi}d\omega$.

Also $$\|u_\omega\|^2 = M|u_\omega(0)|^2 + \lambda\int_0^l \underbrace{|u_\omega|^2}_{2\cos(\omega x-\delta)} dx \sim \lambda\cdot 4\,\frac{l}{2} = 2\lambda l$$

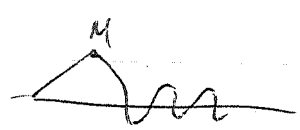
Thus the expansion formula becomes

$$f(x) = \int_0^\infty \frac{d\omega}{2\lambda\pi}\, u_\omega(x)\cdot\langle u_\omega|f\rangle$$

For example, take the free line: $u_\omega(x) = 2\cos(\omega x)$

$$f(x) = \int_0^\infty \frac{d\omega}{\lambda\pi}\cos\omega x\cdot\left(2\lambda\int_0^\infty \cos(\omega x')\,f(x')\,dx'\right)$$

which checks.

Another check is as follows. Take  again and let $f(x) = \begin{cases} 0 & x>0 \\ 1 & x=0\end{cases}$

Then $\langle u_\omega|f\rangle = M\overline{(1+R_\omega)}$ and so we should have the formula

$$f(x) = \int_0^\infty \frac{M\,d\omega}{2\lambda\pi} \underbrace{\left(e^{-i\omega x} + R_\omega e^{i\omega x}\right)\left(1+\overline{R_\omega}\right)}_{e^{-i\omega x} + e^{i\omega x} + R_\omega e^{i\omega x} + \overline{R_\omega} e^{-i\omega x}}$$

$$= \int_{-\infty}^{\infty} \frac{M\,d\omega}{\lambda\ 2\pi} \left(1+R_\omega\right) e^{i\omega x}$$

Now R_ω is analytic in the UHP, so this will vanish for $x>0$. If $x=0$ then use

$$\frac{1+R}{2} = \frac{\frac{1+R}{1-R}}{1+\frac{1+R}{1-R}} = \frac{\lambda s}{Ms^2+\lambda s + M\omega_0^2} \sim \frac{\lambda}{Ms}$$

$$f(0) = \int \frac{\cancel{M}}{\cancel{\lambda}2\pi} \frac{2\cancel{\lambda}\,d\omega}{\cancel{M}(-i\omega)} = \int_\pi^0 \frac{1}{\pi} \frac{\cancel{Re^{i\theta}}\,i\,d\theta}{-i\ \cancel{Re^{i\theta}}} = 1$$

which checks.