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anomaly : 682-694

elliptic curve $\det(\partial_{\bar{z}} - \omega)$ 647-652

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$$\mathrm{Tr}\left(e^{-tD^*D}D^{-1}\delta\alpha\right) = \int d^2y \int d^2x \; tr\left(\langle y|e^{-tD^*D}|x\rangle\langle x|D^{-1}|y\rangle\alpha(y)\right)$$

(Actually the $\int d^2y, \int d^2x$ mean just integrating over y, x respectively; from an invariant viewpoint $\alpha(y) \in \mathrm{End}(E_y)\otimes T_y^{0,1}$ and $\langle x|D^{-1}|y\rangle \in E_x \otimes E_y^* \otimes T_y^{1,0}$, so that $\langle x|D^{-1}|y\rangle\alpha(y)$ already is a 1,1 form, so it incorporates $dy\,d\bar{y}$).

Our goal is take the limit as $t\downarrow 0$: [redacted]

$$\int [redacted] \langle y|e^{-tD^*D}|x\rangle d^2x \langle x|D^{-1}|y\rangle \longrightarrow PF\langle y|D^{-1}|y\rangle$$

where PF stands for partie fini. Yesterday we saw that it was possible to [redacted] replace the heat kernel by its WKB approx:

$$\langle y|e^{-tD^*D}|x\rangle d^2x \cong \frac{1}{2\pi t} e^{-\frac{r(y,x)^2}{2t}} \xi(y,x)d^2x \cdot F_0(y,x)$$

The first factor is the WKB approx. to the scalar heat kernel, the second factor is the radial transport isomorphism $E_y \xleftarrow{\sim} E_x$ relative to the connection on E. I think the first factor is just the image under the exponential map of the standard Gaussian measure on the tangent space at the point y.

Let's work locally in $\mathbb{C}$ around $y=0$. We saw yesterday that [redacted] when E is trivialized by an [redacted] orthonormal frame so that $D = (\partial_{\bar{z}}+\alpha)d\bar{z}$, that

$$F_0(z,0) = 1 - z\alpha_0^* + \bar{z}\alpha_0 + \cdots$$

hence

$$F_0(0,z) = 1 + z\alpha_0^* - \bar{z}\alpha_0 + \cdots$$

On the other hand $\langle z|D^{-1}|0\rangle = \dfrac{F_1(z,0)}{\pi z}dz_0$ where

$$(\partial_{\bar{z}}+\alpha)F_1(z,0) = 0 \quad \text{so} \quad F_1(z,0) = 1 + z\beta - \bar{z}\alpha_0 + \cdots$$

for a suitable matrix β.

Thus
$$F_0(0,z)F_1(\bar z,0) = \left(1 - z\alpha_0^* + \bar z\alpha_0 + \cdot\right)\left(1 + z\beta - \bar z\alpha_0 + \cdots\right)$$
$$= 1 + z(-\alpha_0^* + \beta) + O(z^2)$$

We want to divide this by πz and then average with respect to the WKB gaussian measure and let $t \downarrow 0$.

The first guess as to what we get is $-\alpha_0^* + \beta$, however this is not independent of the choice of the coordinate z. For if we put $z = w(1 + cw + \cdots)$ then

$$\frac{F_0(0,z)F_1(\bar z,0)}{z} = \frac{1 + w(-\alpha_0^* + \beta) + O(w^2)}{w(1+cw)}$$

$$= \frac{1}{w}\Big(\underbrace{\frac{1}{1+cw+}}_{1-cw+\cdots} + w(-\alpha_0^* + \beta) + O(w^2)\Big)$$

which adds the constant c to $-\alpha_0^* + \beta$.

The conjecture is now that we want to choose the coordinate z at O to be as nice as possible with respect to the Riemannian structure. So the metric is defined by

$$|\partial_z|^2 = g(z) = g_0 + g_1 z + g_{\bar 1}\bar z + \cdots$$

and if we change coordinates to w, then

$$\partial_z = \frac{\partial}{\partial z} = \frac{\partial}{\partial w}\frac{dw}{dz} = \left(\frac{dz}{dw}\right)^{-1}\partial_w$$

$$\therefore \quad |\partial_w|^2 = |\partial_z|^2\left|\frac{dz}{dw}\right|^2 = g\left|\frac{dz}{dw}\right|^2$$

so now it is clear that we can choose the coordinate z so that $g_1 \; g_{\bar 1} \; g_2 \; g_{\bar 2}$ are 0. The only this we can't change is $\partial\bar\partial \log g$ the curvature form.

~~Summarizing [illegible] can now [illegible]~~.

I want now to work out carefully the fmlas
in an arbitrary coord. system.

$$|\partial_z|^2 = g(z) = g_0 + g_1 z + g_{\bar 1}\bar z + \cdots$$

$$u(z) = |z|^2 (u_0 + u_1 z + u_{\bar 1}\bar z + \cdots)$$

$$\partial_z u = \bar z (u_0 + (2u_1 \quad) z + (u_{\bar 1} \quad)\bar z + \cdots)$$

$$|\partial_z u|^2 = |z|^2 (u_0^2 + u_0(3u_1) z + u_0(3\bar u_1)\bar z + \cdots)$$

$$g(z)\, u = (g_0 + g_1 z + g_{\bar 1}\bar z + \cdots)|z|^2 (u_0 + u_1 z + u_{\bar 1}\bar z + \cdots$$

$$= |z|^2 (g_0 u_0 + (g_1 u_0 + g_0 u_1) z + (g_{\bar 1} u_0 + g_0 u_{\bar 1})\bar z + \cdots)$$

$$\therefore\ g_0 u_0 = u_0^2 \Longrightarrow u_0 = g_0$$

$$g_1 u_0 + g_0 u_1 = 3u_0 u_1 \Rightarrow g_1 = 2u_1 \Rightarrow u_1 = \tfrac{1}{2} g_1$$

$$\therefore\ \boxed{u = |z|^2 (g_0 + \tfrac{1}{2} g_1 z + \tfrac{1}{2} g_{\bar 1}\bar z + \cdots)}$$

$$g^{-1}\partial_z u = \frac{1}{g_0 + g_1 z + g_{\bar 1}\bar z + \cdots}\,\bar z (g_0 + g_1 z + \tfrac{1}{2} g_{\bar 1}\bar z + \cdots)$$

$$\boxed{\begin{aligned} g^{-1}\partial_z u &= \bar z (1 - \tfrac{1}{2} g_0^{-1} g_{\bar 1}\bar z + \cdots) \\ g^{-1}\partial_{\bar z} u &= z (1 - \tfrac{1}{2} g_0^{-1} g_1 z + \cdots) \end{aligned}}$$

Next we need ξ which satisfies

$$g^{-1}((\partial_{\bar z} u)\partial_z + (\partial_z u)\partial_{\bar z}) \log \xi + (-1 + g^{-1}\partial^2_{z\bar z} u) = 0.$$

$$\partial_z u = \bar z \overset{g_0}{u_0} + \overset{g_1}{2u_1} z\bar z + \overset{\frac{1}{2}g_{\bar 1}}{u_{\bar 1}} \bar z^2 + \cdots$$

$$\partial^2_{z\bar z} u = g_0 + g_1 z + g_{\bar 1}\bar z + \cdots$$

$$\therefore\ \boxed{g^{-1}\partial^2_{z\bar z} u = 1 + O(z^2)}$$

hence $\boxed{\xi(z) = 1 + O(z^2)}$

[redacted] In order to define self-adjoint for a differential operator one needs to have a volume, hence we should think of the heat kernel as a symmetric matrix function:

$$\langle x | e^{-tD^*D} | y \rangle = \frac{1}{2\pi t} e^{-\frac{r(x,y)^2}{2t}} \xi(x,y) F_t(x,y)$$

where r, ξ [redacted] are symmetric functions and $F_t(x,y) \in \mathrm{Hom}(E_y, E_x)$ is also symmetric:

$$F_t(x,y)^* = F_t(y,x)$$

From the theory of the heat kernel we know that $F_0(x,y)$ is the radial transport isomorphism from E_y to E_x; its inverse is $F_0(x,y)^{-1} = F_0(y,x) = F_0(x,y)^*$, since it is unitary. Now let's go back to our integral

$$\int \langle 0 | e^{-tD^*D} | z \rangle \, g(z) \, i\,dz\,d\bar z \, \langle z | D^{-1} | 0 \rangle$$

$$= \int \frac{1}{2\pi t} e^{-\frac{u(z)}{t}} \xi(0,z)\, g(z)\, F_0(0,z) \frac{F_1(z,0)}{\pi z} \, i\,dz\,d\bar z$$

$$= \int \frac{1}{\text{[redacted]}\, t} e^{-\frac{|z|^2}{t}(g_0 + \frac{1}{2} g_1 z + \frac{1}{2} g_{\bar 1} \bar z + \cdots)} (1 + O(z^2))(g_0 + g_1 z + g_{\bar 1}\bar z + \cdots) \times \frac{(1 + (-\alpha^* + \beta) z + \cdots)}{\pi z} \frac{dx\,dy}{\pi}$$

$$= \int e^{-|z|^2 (g_0 + \frac{g_1}{2} \sqrt{t} z + \frac{1}{2} g_{\bar 1} \sqrt{t} \bar z + \cdots)} (1 + O(t z^2))(g_0 + g_1 \sqrt{t} z + g_{\bar 1} \sqrt{t} \bar z) \frac{(1 + (-\alpha^* + \beta)\sqrt{t} z)}{\pi \sqrt{t} z} \frac{dx\,dy}{\pi}$$

$$= \int e^{-|z|^2 g_0} \left[1 - \sqrt{t}\, \text{[redacted]} |z|^2 \tfrac{1}{2}(g_1 z + g_{\bar 1} \bar z) \right] \left[g_0 + g_1 \sqrt{t} z + g_{\bar 1} \sqrt{t} \bar z \right] \left[1 + (-\alpha^* + \beta)\sqrt{t} z \right] \frac{''}{\pi \sqrt{t} z}$$

$$= \int e^{-|z|^2 g_0} g_0 \frac{dx\,dy}{\pi\pi} \left[\text{[redacted]} - |z|^2 \tfrac{1}{2} g_1 + g_1 g_0^{-1} + (-\alpha^* + \beta) \right]$$

Have used that $\frac{\bar{z}}{z}$ will gives a vanishing integral.

$$\int e^{-t|z|^2}\frac{dx\,dy}{\pi} = \frac{1}{t} \Rightarrow \int e^{-t|z|^2}|z|^2\frac{dx\,dy}{\pi} = \frac{1}{t^2} \qquad \frac{1}{g_0^2}$$

Thus

$$\lim_{t\downarrow 0}\int \langle 0|e^{-tD^*D}|z\rangle g(z)\,i\,d\bar{z}dz\,\langle z|D^{-1}|0\rangle$$
$$= \frac{1}{\pi}\left[\frac{1}{2}g_0^{-1}g_1 + (-\alpha_0^* + \beta)\right]dz_0$$

So let's check that this is invariant under the choice of the coordinate z. Let's try to understand each of the terms invariantly as an element of $E_y \otimes E_y^* \otimes T_y^{1,0}$. Let φ denote a coordinate function at the point y, so that $\varphi: U \hookrightarrow \mathbb{C}$ where U is a nbd. of y. [redacted]

Let's take an element $e \in E_y$. [redacted] We know the Green's fn. has the form

$$\langle x|D^{-1}|y\rangle = \frac{F_1(x,y)}{\pi(\varphi(x)-\varphi(y))}d\varphi_y$$

hence $F_1(x,y)e$ will be a holomorphic section of E near y. Then we can apply the covariant derivative $\nabla: E \longrightarrow E\otimes T^*$, and since we have a holom. section we know that

$$\nabla(F_1(x,y)e) \in E\otimes T^{1,0}$$

and so evaluating at y gives an element of $E_y \otimes T_y^{1,0}$. This is obviously the [redacted] part $\frac{1}{\pi}(-\alpha_0^*+\beta)dz_0$ above. How does it depend on the choice of φ?

Do it with z:

$$\langle z|D^{-1}|y\rangle = \frac{F_1(z,y)}{\pi(z-y)}dz_y = G(z,y)\,dz_y$$

so

$$F_1(z,y)e = \pi(z-y)\,G(z,y) = (1+\beta_y(z-y)+\cdots)e$$
$$\nabla(F_1(z,y)e) = (\partial_z - \alpha^*)(1+\beta_y(z-y)+\cdots)e_{dz_y} = (-\alpha_y^*+\beta_y)e\,dz_y \quad ?$$

At this point I understand the regularized trace, when regularized by means of the heat kernel. Hence it should now be possible to compute the curvature. Recall the formula

$$\delta \log |s|^2 = \delta(-\zeta'(0)) = \mathrm{Tr}_{reg}(D^{-1}\delta\alpha) + c.c.$$

$$= \int tr\left(FP\langle y|D^{-1}|y\rangle\, \delta\alpha(y)\right) + c.c.$$

hence $\quad \partial \log |s|^2 = \int tr\left(FP\langle y|D^{-1}|y\rangle\, d\alpha(y)\right).$ The curvature form is $\bar\partial\partial \log|s|^2$, which means we look at the derivatives of the coefficients with respect to $\bar\alpha$. Analogy: if $\omega = \sum a_i dz^i$, then $\bar\partial\omega = \sum \partial_{\bar z_j} a_i \, d\bar z^j dz^i$. Clearly the β terms is holomorphic in α as it comes from the Green's fn., also the $\frac{1}{2}g^{-1}\partial g$ -terms. So we get

$$\bar\partial\partial \log |s|^2 = -\frac{1}{\pi}\int tr(d\alpha^* d\alpha) = \frac{1}{\pi}\int tr(d\alpha\, d\alpha^*)$$

<u>May 23, 1982</u>

Review yesterday. We have this $\bar\partial$-operator $D: E \to E\otimes T^{0,1}$ which is assumed to be invertible. By choosing metrics on M and E we can form the Laplacean D^*D and use the associated heat operator to obtain a finite part for the Schwarz kernel $\langle x|D^{-1}|y\rangle$ along the diagonal. Yesterday I found a formula for this $FP\langle y|D^{-1}|y\rangle$.

To describe this formula choose a local coordinate in a nbd. of y. Then we can think of the nbd. as being an open subset of $\mathbb{C}$ with the usual coordinate z. Let's trivialize the bundle E by means of an orthonormal frame. Then $D = (\partial_{\bar z}+\alpha)d\bar z$ where $\alpha(z)$ is a matrix of C^∞-functions. Also the metric on M is described by $g(z) = |\partial_z|^2$. Also

$$\langle z|D^{-1}|y\rangle = \frac{F_1(z,y)}{\pi(z-y)}\, dz_y$$

where $F_1(z,y)$ is a smooth matrix fn. of z such that $F_1(y,y) = 1$, and $(\partial_{\bar z}+\alpha)F_1(z,y) = 0$. Then

$$FP\langle y|D^{-1}|y\rangle = \frac{1}{\pi}\left[\frac{1}{2}\partial_z \log g\big|_y - \alpha_y^* + \partial_z F_1(z,y)\big|_y\right](dz)_y$$

Another description: The connection on E is

$$\nabla = (\partial_z - \alpha^*)dz + (\partial_{\bar z}+\alpha)d\bar z$$

hence the trivialization of E to first order at y is

$$F_b(z,y) = 1 + \alpha_y^*(z-y) - \alpha_y\overline{(z-y)} + \cdots$$

On the other hand

$$F_1(z,y) = 1 + \beta_y(z-y) - \alpha_y\overline{(z-y)} + \cdots$$

so that

$$\frac{F_1(z,y)}{z-y} - \frac{F_b(z,y)}{z-y} = -\alpha_y^* + \beta_y + O(z-y).$$

Again: To describe $FP\langle y|D^{-1}|y\rangle \in E_y \otimes E_y^* \otimes T_y^{1,0}$ I start with an element ω of $E_y \otimes (T_y^{1,0})^*$. Then I get a unique meromorphic section s of E with simple pole at y having residue ω. Our problem is to define the finite part of s at y. So we choose a local section of E flat to first order at y, call it s_b, and a local coordinate φ_1 centered at y flat to first order relative to the Riemannian metric such that

$$s - \frac{s_b}{\varphi}$$

has a finite limit at y. Then

$$\lim_{x\to y} \left(s - \frac{s_b}{\varphi}\right)(x) = FP\langle y|D^{-1}|y\rangle\, \omega$$

It might be very useful to relax the assumption that φ be holomorphic on the Riemann surface, in particular to be able to take it to be the coordinate on the tangent space. Let's work locally with coord. z and take $\varphi(z) = z + \cdots$.

$$\frac{F_1(z)}{z} - \frac{F_b(z)}{z} \longrightarrow FP \qquad \text{as } z \to 0$$

$$\frac{F_1(z)}{z} - \frac{F_b(z)}{\varphi} \longrightarrow ? \qquad \text{as } z \to 0$$

So we have to look at

$$F_b(z)\left(\frac{1}{z} - \frac{1}{\varphi(z)}\right) \qquad F_b(z) \longrightarrow 1$$

and see when it approaches 0. Suppose $\varphi(z) = z + az^2 + bz\bar{z} + c\bar{z}^2 + \cdots$

$$\frac{1}{z} - \frac{1}{\varphi(z)} = \frac{az^2 + bz\bar{z} + c\bar{z}^2 + \cdots}{z(z + az^2 + bz\bar{z} + c\bar{z}^2 + \cdots)} \sim \frac{az^2 + bz\bar{z} + c\bar{z}^2 + \cdots}{z^2}$$

so it's clear that φ must have no quadratic terms. Hence φ doesn't have to be holom.

In general it should be the case that if w is a linear coordinate on T_y such that z, w agree to first order under the exponential map, and if z is adapted to the metric at y, so that $g(z)$ has no linear terms and only a $z\bar{z}$ quadratic term, then w and z agree to 2nd order. This ultimately should be checked, but is clear in the constant curvature cases since $\frac{w}{z} = 1 + O(|z|^2)$.

Now that I understand the finite part, I should make some progress on the determinant functions.

$$\mathcal{O} \xrightarrow[\text{canon section}]{s} \mathcal{L}$$

$$|s|^2 = \text{anal. torsion}^{\tau} = e^{-\zeta'(0)}$$

Choose h a function on $\mathcal{A}$ such that

$$\bar{\partial}\partial \log \tau = \bar{\partial}\partial h$$

and then define metric $|1|^2 = e^h$ on $\mathcal{O}$. Since the curvatures agree we get an isom. of bundles with connection

$$\mathcal{L} \simeq \mathcal{O}$$

whose composition with s will give a holom. fn. f on $\mathcal{A}$ satisfying

$$|s|^2 = |f|^2 e^h ,$$

and then f is a holomorphic choice for $\det(D)$. So

$$\partial \log \tau = \int \operatorname{tr}\left(FP\langle y|D^{-1}|y\rangle\, d\alpha(y)\right)$$
$$= \partial h + \partial \log \det(D)$$

We take h to a quadratic function like $\frac{1}{\pi}\|\alpha\|^2$ and this involves choosing a basepoint: $D = D_0 + \alpha$. The effect of this choice of h should be to cancel the part of $FP\langle y|D^{-1}|y\rangle$ which is non-holomorphic in α. Notice that $\partial h = \frac{1}{\pi}\langle \alpha|\delta\alpha\rangle = \int \frac{1}{\pi_n}\operatorname{tr}(\alpha_y^* \, \delta\alpha_y)$, and that the effect of changing basepoint is to alter α_y^* by a constant form. So the goal will be to find a finite part of $\langle y|D^{-1}|y\rangle$ which is holomorphic in D but depends upon a choice similar to h. (Now I can work locally if I want since already there is something to do for when $\delta\alpha$ has small support.)

I want to get straight the different meanings of α. In particular you need a good notation. Let's begin by fixing a basepoint in $\mathcal{A}$, call it D_0, which is invertible. Then another point can be written $D = D_0 + \delta D$ where $\delta D_y \in \operatorname{End}(E_y) \otimes T_y^{0,1}$. On the other when I work locally and trivialize E by an orth. frame I have $D = (\partial_{\bar z} + \alpha)d\bar z$ where α is a matrix of functions. Thus one has $\alpha\, d\bar z = \alpha_0\, d\bar z + \delta D$, where $D_0 = (\partial_{\bar z} + \alpha_0)d\bar z$ locally. Now what I want is the function $h(D)$. Notice that from a practical viewpoint, δD is equivalence to α. So $h(D)$ will be a quadratic fn. of α:

$$h(D) = -\frac{1}{\pi}\|\delta D\|^2 + (\gamma|\delta D) + (\delta D|\gamma) + \text{const.}$$

$$= -\frac{1}{\pi}\int \operatorname{tr}(\alpha^*\alpha)\, i\,dz\,d\bar z + \left(\int \operatorname{tr}(\tilde\gamma^*\alpha)\, i\,dz\,d\bar z + \text{c.c.}\right)$$

Thus
$$\partial h = -\frac{1}{\pi}\int \operatorname{tr}(\alpha^*\delta\alpha)\, i\,dz\,d\bar z + \int \operatorname{tr}(\tilde\gamma^*\delta\alpha)\, i\,dz\,d\bar z$$

and this has to be removed from $\partial \log \tau$ to get $\delta \log\det D$. We have the local formula

$$\partial \log \tau = \int tr\left(\tfrac{1}{\pi}\left[\tfrac{1}{2}\partial_z \log g - \alpha^* + \beta\right] \delta\alpha\right) dz\, d\bar{z}.$$

Somehow there is an i missing.

The real problem is to define a finite part for $\langle y | D^{-1} | y\rangle$ which varies holomorphic in D. The heat kernel FP is constructed by taking a section of E with simple pole at y and removing a section of the form $\frac{s_b}{z-y}$ where s_b is flat for the hermitian connection assoc. to D. So a possibility is to choose for each D a connection varying holomorphically in D, the connection ∇ being required to extend D:

$$\begin{tikzcd} & E\otimes T^* \arrow[d] \\ J_1(E) \arrow[ur, "\nabla"] \arrow[r, "D"'] & E\otimes T^{0,1} \end{tikzcd}$$

commutes.

If the complex structure is given by $D = (\partial_{\bar z} + \alpha) d\bar z$, then a connection compatible with it is given by $\nabla = (\partial_z - \gamma) dz + (\partial_{\bar z} + \alpha) d\bar z$. Again we have

$$\langle z | D^{-1} | y \rangle = \frac{F_1(z,y)}{\pi(z-y)} dy \qquad F_1(z,y) = 1 + \beta(y)(z-y) - \alpha(y)\overline{(z-y)} + \dots$$

and the flat trivialization to first order for ∇ is

$$F_b(z,y) = 1 + \gamma(y)(z-y) - \alpha(y)\overline{(z-y)} + \dots$$

Hence
$$\frac{F_1 - F_b}{\pi(z-y)} \longrightarrow \beta(y) - \gamma(y) \qquad \text{as } z \to y.$$

Thus we do get a valid regularization process which will be holom. in α provided γ is.

So now it is clear that we fix a ∂-operator on E, and then associate to any $\bar\partial$ operator D, the unique ∇ compatible with the two.

Curious problem: According to the above we get a det(D) function defined up to a multiplicative constant once we choose a ∂-operator $E \to E \otimes T^{1,0}$, which is given by

$$\frac{\delta \log\det(D)}{\delta D(y)} = FP\langle y \,|\, D^{-1} \,|\, y\rangle$$

where FP depends on the ∂-op. Changing the ∂-operator by a $\gamma : E \to E \otimes T^{1,0}$ amounts to adding γ to the above finite part, and it changes det(D) by multiplying by exp of the linear fn. $D \mapsto (\gamma, D - D_0) + \text{const.}$). On the other hand if I propose to proceed by

$$\log\det D - \log\det D_0 = \log\det(1 - D_0^{-1}\phi)$$

$$= -\operatorname{tr}(D_0^{-1}\phi) - \frac{1}{2}\operatorname{tr}(D_0^{-1}\phi)^2 - \frac{1}{3}\operatorname{tr}(D_0^{-1}\phi)^3 - \cdots$$

I must fix the first two terms somehow, which makes it appear that log det D is only defined up to a quadratic fn. of $-\phi = D - D_0$. Hence there ought to be a way to fix up $\operatorname{tr}(D_0^{-1}\phi)^2$ up to a linear fn. of ϕ.

Notice that the same problem occurs with the Weierstrass functions: If I put

$$\tilde{\zeta}(z) = \sum_{\mu \in \alpha + \Lambda} \left(\frac{1}{z-\mu} + \frac{1}{\mu} + \frac{z}{\mu^2} \right) \qquad \alpha \notin \Lambda$$

Then $$\tilde{\zeta}(z+\omega) - \tilde{\zeta}(z) = \sum_{\mu \in \alpha+\Lambda} \left(\frac{1}{\mu+\omega} - \frac{1}{\mu} + \frac{\omega}{(\mu+\omega)^2} \right) + z \sum_{\mu \in \alpha+\Lambda} \frac{1}{(\mu+\omega)^2} - \frac{1}{\mu^2}$$

is a priori linear in z. However the last term may be identified with $z(\wp(\alpha+\omega) - \wp(\alpha))$, which is 0 as the $\wp$-function is doubly periodic.

May 24, 1982

Yesterday I finally reached some understanding of the regularization process required to define $\det(D)$. We use the formula

$$\delta \log \det(D) = Tr(D^{-1}\delta D) = \int tr(\langle y|D^{-1}|y\rangle \delta D(y))$$

where to be precise $\langle y|D^{-1}|y\rangle$ denotes the finite part of the Schwarz kernel $\langle x|D^{-1}|y\rangle$ as $x\to y$. The finite part is obtained by using a connection $\nabla : E\to E\otimes T^*$ having $\nabla'' = D$ and $\nabla' =$ a fixed ∂-operator [redacted] $E\to E\otimes T^{1,0}$, and also uses a metric on M. [redacted] If we choose a local coordinate φ near y, then we can write

$$\langle x|D^{-1}|y\rangle = \frac{F_1(x,y)}{\pi(\varphi(x)-\varphi(y))}\, d\varphi_y \qquad x \text{ near } y$$

where $F_1(x,y)\in \mathrm{Hom}(E_y,E_x)$ is holomorphic in x for the given complex ~~structure~~ and $F_1(y,y) = id$ on E_y. Let $F_b(x,y)$ be smooth with values in $\mathrm{Hom}(E_y,E_x)$, [redacted] $F_b(y,y) = id$, and $\nabla_x F_b(x,y)\big|_y = 0$, i.e. F_b is flat to first order at y. Then

$$\langle x|D^{-1}|y\rangle - \frac{F_b(x,y)}{\pi(\varphi(x)-\varphi(y))}\, d\varphi_y$$

has a limit as $x\to y$. This depends on the choice of φ by a multiple of the identity, [redacted] which one compensates for using the metric as follows.

Let's work in $\mathbb{C}$ with $y=0$ and $x = z\in\mathbb{C}$.

Let
$$\langle z|D^{-1}|0\rangle = \frac{F_1(z,0)}{\pi z}(dz)_0 \qquad \begin{aligned} F_1(z,0) &= 1+\beta z-\alpha\bar z+\cdots \\ F_b(z,0) &= 1+\gamma z-\alpha \bar z+\cdots\end{aligned}$$

$$FP\langle 0|D^{-1}|0\rangle = \frac{1}{\pi}(\beta-\gamma)(dz)_0$$

If $w = z + cz^2+\cdots$ is a new holom. coordinate, then this finite part changes by $\left(\frac{1}{z}-\frac{1}{w}\right)\frac{(dz)_0}{\pi} \longrightarrow c$ [redacted].

Let the metric be given by

$$\left|\frac{\partial}{\partial z}\right|^2 = g(z).$$

$$\frac{dw}{dz}\frac{\partial}{\partial w} = \frac{\partial}{\partial z} \qquad \left|\frac{dw}{dz}\right|^2 \underbrace{\left|\frac{\partial}{\partial w}\right|^2}_{\tilde{g}} = g(z)$$

$$\partial \log g = \partial \log \tilde{g} + \underbrace{\partial \log \left|\frac{dw}{dz}\right|^2}$$

$$d \log \frac{dw}{dz} = \frac{d^2w}{dz^2} \Big/ \frac{dw}{dz} = 2c.$$

Thus

$$\lim_{z\to 0}\left(\langle z | D^{-1} | 0 \rangle - \frac{F_b(z,0)}{\pi z}(dz_0)\right) + \frac{1}{2}(\partial \log g)_0$$

will be independent of the choice of coordinate.

Ultimately one should fix the underlying C^∞ surface and vector bundle, and allow both the holomorphic structure on the surface and on the bundle to vary. This brings up the whole question of the different holom. structures on a surface M, i.e. Teichmüller spaces. Notice that fixing a volume on M sets up a 1-1 correspondence with Riemannian metrics on M with this volume and holomorphic structures on M. Also the volume is what is needed for the adjoint of a differential operator. So we get a nice correspondence for (M,E) C^∞ equipped with (vol, herm. metric):

holom. str. on (M,E)	$\longleftrightarrow$	Riem. metric on M with correct vol. + hermitian connection

Ultimately I will want to define the line bundle $\mathcal{L}$ in this more general situation.

Problem: Let's go back to the expansion

$$D^{-1} = D_0^{-1} + D_0^{-1}\phi D_0^{-1} + D_0^{-1}(\phi D_0^{-1})^2 + \cdots \quad , \phi = D_0 - D.$$

I have a procedure for restricting the kernels of D^{-1} and D_0^{-1} to the diagonal. The part of the series from $D_0^{-1}(\phi D_0^{-1})^2$ on is a trace class operator, hence has a natural restriction to the diagonal. It follows therefore that there should be a natural way to restrict $D_0^{-1}\phi D_0^{-1}$ to the diagonal, and the problem is to find what this is.

This is obviously a first order calculation in ϕ.

$$\langle z | D_0^{-1} | 0 \rangle = \frac{F_1^0(z,0)}{\pi z} = \frac{1 + \beta z - \alpha \bar{z} + \cdots}{\pi z} \quad , \quad F_b^0(z) = 1 + \gamma z - \alpha \bar{z}$$

$$\langle z | D^{-1} | 0 \rangle = \frac{1 + \tilde{\beta} z - (\alpha - \phi)\bar{z} + \cdots\cdots}{\pi z} \qquad F_b(z) = 1 + \gamma z - (\alpha - \phi_0)\bar{z}$$

Subtract

$$\langle z | D^{-1} | 0 \rangle - \langle z | D_0^{-1} | 0 \rangle = \frac{\tilde{\beta} - \beta}{\pi} + \frac{\phi(0)}{\pi}\frac{\bar{z}}{z} + O(z).$$

Thus the nature of the singularity of $\langle z | D_0^{-1}\phi D_0^{-1} | 0 \rangle$ at $z = 0$ becomes clear. Namely it is cancelled exactly by $\dfrac{F_b(z) - F_b^0(z)}{\pi z} = \phi(0)\dfrac{\bar{z}}{\pi z} + O(z)$.

Removing this singularity seems to be an intrinsic process because

$$\frac{\bar{z} + \bar{c}\bar{z}^2 + \cdots}{z + c z^2 + \cdots} = \frac{\bar{z}}{z}\left(1 + \bar{c}\bar{z} - c z + O(z^2)\right) = \frac{\bar{z}}{z} + O(z)$$

So now we should try to make sense of the Fredholm formulas for

$$\det(D_0^{-1}D) = \det(1 - D_0^{-1}\phi) = 1 - tr(D_0^{-1}\phi) + tr\,\Lambda^2(D_0^{-1}\phi) - \cdots$$

This can be written

$$e^{-\sum_1^\infty \frac{1}{n} tr (D_0^{-1}\phi)^n}$$

but the Fredholm series ought to be convergent for all ϕ. However a proof has to be found, since the usual one based on Hadamard's inequality probably won't work. Maybe we can use the recursion formulas.

Put $\Delta = \det(1-K) = \sum_{n\geq 0} \Delta_n$ $\qquad \Delta_n = (-1)^n \operatorname{Tr} \Lambda^n K$

and let $C = \sum_{n \geq 0} C_n$ be the cofactor matrix of $I-K$ so that

$$\begin{cases} \dfrac{1}{1-K} = \dfrac{C}{\Delta} & \text{by Cramer's rule.} \\ \delta \Delta = -\operatorname{Tr}(\delta K\, C) \end{cases}$$

These lead to the recursion formulas

$$\boxed{\begin{aligned} C_n &= \Delta_n \cdot \mathrm{Id} + K C_{n-1} & \Delta_0 &= 1 \\ n \Delta_n &= -\operatorname{Tr}(K C_{n-1}) & C_0 &= I \end{aligned}}$$

which one can also establish by diagrams having vertices  weighted by the matrix element K_{ba}.

May 27, 1982

It's clear we have to understand the case when cohomology appears. Let's begin with the case where the index $= -g$ and in the generic situation $H^0 = 0$. Then we get a canonical family of sections from

$$\Lambda^g V_0 \longrightarrow \mathcal{L} \qquad V_0 = \Gamma(E \otimes T^{0,1})$$

and hence if we choose a $F^g \subset V_0$, and a volume on F^g, then we get a section s_F of $\mathcal{L}$ which is given by the map

$$\lambda^{\blacksquare} F \hookrightarrow \Lambda^g V_0 \longrightarrow \lambda^{\blacksquare} \operatorname{Cok} T = \mathcal{L}_T$$

over the open set where $H^0 = 0$ and zero outside. The section s_F is non-zero when $F \to \operatorname{Cok} T$ is an isom., or equivalently where the composed map

$$\Gamma(E) \xrightarrow{D} \Gamma(E \otimes T^{0,1}) \xrightarrow{\pi} V_0/F$$

is an isomorphism. Hence trivializing $\mathcal{L}$ should be equivalent to defining a $\det(\pi D)$ function.

So we want to use

$$\delta \log \det(\pi D) = \operatorname{tr}\left((\pi D)^{-1} \delta \pi D\right)$$

and so it is first necessary to understand $(\pi D)^{-1}$. D is not an isomorphism hence we can't find everything in $\operatorname{Im} D$. Hence if we want the kernel $\langle z | D^{-1} | z' \rangle$ we have to do something. This kernel is unique modulo a smoothing operator, which is where the π comes in maybe.

Perhaps the case index $= p > 0$ with $H^1 = 0$ is easier. Here to get a section of $\mathcal{L}$ one chooses a subspace W of V_1 of codim p and the section is non-vanishing over the set of D such that $\operatorname{Ker} D \cap W = 0$. In this case D^{-1} is defined to have its values in W. Thus

I use $i : W \hookrightarrow V_1$ and $([redacted]Di)^{-1}$ is a nice operator from V_0 to $W \subset V_1$ so there is no problem with its kernel. In fact we are really taking the kernel of the operator [redacted] $i(Di)^{-1}$.

Thus it is clear that the kernel $\langle z | i(Di)^{-1} | z' \rangle$ assigns to each element of $E_{z'} \otimes T^{0,1}_{z'}$ a section of E which is holomorphic except for a simple pole at z' with the given residue and which lies in W in some sense. For example if W is specified by conditions that a section vanishes [redacted] at specified points, then for z' not one of these points, it's clear what is meant by a section with pole at z' lying in W. [redacted] This kernel has the same local description

$$\langle z | \, i(Di)^{-1} | z' \rangle = \frac{F(z,z')}{\pi(z-z')} dz'$$

as before.

(I think that cutting down by vanishing [redacted] conditions might be the same as picking out a bundle of index 0 contained inside. However it's not so simple because if one looks at C^∞ fns. satisfying $f(0)=0$, then these are not sections of a C^∞ bundle, only the ones divisible by z are.)

Consider the example of line bundles of degree 1 over an elliptic curve and take W to be the subspace of fns. vanishing at $z=0$. Then you should be able to describe the kernel $\langle z | i(Di)^{-1} | z' \rangle$. [redacted] Lines bundles of degree 1 are described up to isomorphism by the point P of M [redacted] at which the unique non-zero section vanishes. Thus for $P \neq 0$ there will be no section of $\mathcal{O}(P)$ vanishing at O and for $z' \neq 0$ a unique (up to scalar multiple) section with simple pole at z' and vanishing at O. Fix $\mathcal{O}(P)$, $P \neq O$ and $W \subset \Gamma(\mathcal{O}(P))$. So I want a merom. fn. with simple poles at P, z' and a simple zero at O, whence another zero at $P + z'$. This must be

$$\frac{\sigma(z)\,\sigma(z-P-z')}{\sigma(z-z')\,\sigma(z-P)}$$

which is the kernel $\langle z | \, i(Di)^{-1} | z' \rangle$, at least, when $z' \neq O$. But the kernel is smooth for $z \neq z'$, hence we get the value $\langle z | \, i(Di)^{-1} | O \rangle = 1$.

In the case now where $H^0 = O$ the correct Green's function is $\langle z | \, (\pi D)^{-1} \pi \, | z' \rangle$ which will give sections of E such that when D is applied one gets a δ-function plus something in the subspace F. Hence the Green's function will not be an analytic section of E in the variable z.

<u>May 30, 1982</u>

One idea obtained during the Heidelberg-Konstanz trip is that the line bundle $\mathcal{L}$ can be defined by giving a divisor in $\mathcal{a}$. For example in the slope $(g-1)$ case the divisor is the subset of D which are not invertible. The problem of trivializing $\mathcal{L}$ is then one of finding an entire function on $\mathcal{a}$ whose zero-subvariety is this divisor.

Next let's go back ~~to~~ the case where the slope $>(g-1)$, so that generically D is onto. To get a divisor we fix a $W \subset V_1 = \Gamma(E)$ of codim = index and then the divisor contains those D such that Di is invertible where $i : W \hookrightarrow V_1$ is the inclusion. I recall that the simplest W's occur by specifying that the sections vanish at points in certain ways.

Example: $M =$ elliptic curve $\mathbb{C}/\Gamma$, and take $E = \mathcal{O}(P)$, and consider the family of operators $\frac{\partial}{\partial \bar{z}} - w$ with $w \in \mathbb{C}$. Here a section of E is to be thought of as a smooth periodic function f on $\mathbb{C} -$ the coset $p + \Gamma = P$ such that $(z-p)f(z)$ is smooth near p for all $p \in P$.

With the operator $\frac{\partial}{\partial \bar{z}} - w$, E becomes isomorphic as holomorphic line bundle to $\mathcal{O}(Q)$ where $Q = P +$ some (linear?) fn. of w. To find what Q is we must find the generator for the kernel of $\frac{\partial}{\partial \bar{z}} - w$ on $\Gamma(E)$ and see where it vanishes. This generator is of the form

$$f(z) = e^{\bar{z}w} h(z) \qquad h \text{ holom. with simple poles at } p' \in P.$$

and it is doubly-periodic. Try

$$f(z) = e^{\bar{z}w + \alpha z} \frac{\sigma(z-q)}{\sigma(z-p)}$$

and recall that for $\omega \in \Gamma$

$$\frac{\sigma(z+\omega)}{\sigma(z)} = e^{a_\omega z + b_\omega} \qquad a_\omega = l\omega + m\bar{\omega}$$

Thus

$$\log \frac{f(z+\omega)}{f(z)} = \varpi\bar{\omega} + \alpha\omega + a_\omega(z-q) - a_\omega(z-p)$$
$$= \varpi\bar{\omega} + \alpha\omega + (l\omega + m\bar{\omega})(p-q) = 0$$

$$\varpi + m(p-q) = 0 \qquad \text{or} \qquad mq = mp + \varpi$$

$$\text{or} \quad \boxed{q = p + \tfrac{1}{m}\varpi}$$

Let's suppose now that $\varpi = 0$, and let's choose the subspace $W \subset \Gamma(E)$ to consist of C^∞ ~~[illegible]~~-sections which ~~consist~~ locally are divisible by $z-\gamma$ around each $\gamma \in \Gamma$. I assume $P \neq 0$.

<u>May 31, 1982</u>

Consider the line bundle $\mathcal{O}(P)$ over $M = \mathbb{C}/\Gamma$ whose sections are periodic functions^f on $\mathbb{C}$, smooth off the ~~coset~~ P, such that $(z-p)f(z)$ is smooth near p for each $p \in P$. I consider the family of operators $\partial_{\bar z} - w$ on $\mathcal{O}(P)$. The ess. unique holom. section for this complex structure is of the form

$$f = e^{w\bar z + \alpha z} \frac{\sigma(z-q)}{\sigma(z-p)} \qquad P = p + \Gamma$$

where α, q are chosen so that f is periodic, i.e.

$$w\bar\omega + \alpha\omega + a_\omega(z-q) - a_\omega(z-p) = 0$$

$$w\bar\omega + \alpha\omega = \underbrace{a_\omega}(q-p)$$

$$(l\omega + \bar\omega)(q-p) \qquad \text{assume } m = 1.$$

$$\therefore \quad w = q - p \quad \text{or} \quad \boxed{q = p + w}$$

$$\alpha = lw$$

$$\therefore \quad f = e^{w(\bar z + lz)} \frac{\sigma(z-p-w)}{\sigma(z-p)}$$

Next I [redacted] fix the subspace $i: W \subset C^\infty(\mathcal{O}(P))$ to be sections vanishing at 0. Then for $p + w \notin \Gamma$ ~~[illegible]~~ the above function $f \notin W$, and $Di: W \to C^\infty(\mathcal{O}(P) \otimes T^{0,1})$ should be invertible. The Green's fn. $\langle z | i(Di)^{-1} | z' \rangle$ has poles at p, z' and a zero at 0, so is of the form

$$e^{w\bar z + \alpha z} \frac{\sigma(z)\sigma(z-q)}{\sigma(z-p)\sigma(z-z')} \quad \times \text{ constant ind of } z$$

where

$$w\bar\omega + \alpha\omega + \overset{l\omega + \bar\omega}{a_\omega}(0 - q + p + z') = 0$$

or

$$q = p + z' + w \qquad \alpha = lw.$$

The constant is determined so that this $\sim \frac{1}{\pi(z-z')}$ as $z \to z'$.

Thus
$$\langle z | i(Di)^{-1} | z' \rangle = e^{w(\overline{z-z'} + l(z-z'))} \frac{\sigma(z)}{\sigma(z')} \frac{\sigma(z-g)}{\sigma(z'-g)} \frac{\sigma(z'-p)}{\sigma(z-p)} \frac{1}{\pi\sigma(z-z')}$$

Next we must compute the finite part of this as $z \to z'$. Review the formulas

$$\langle z | \quad | 0 \rangle = \frac{1+\beta z - \alpha\bar{z} + \cdots}{\pi z} \qquad D = (\partial_{\bar{z}} + \alpha) d\bar{z}$$

$$F_b(z) = 1 + \gamma z - \alpha \bar{z} + \cdots \qquad \nabla = \underbrace{(\partial_z - \gamma) dz}_{\text{fixed } \partial \text{ operator}} + (\partial_{\bar{z}} + \alpha) d\bar{z}$$

and the finite part is $\beta - \gamma$.

So on $\mathcal{O}(P)$ I need to fix a ∂-operator, and $\frac{\partial}{\partial z}$ doesn't work because it increases the simple pole at P to a double pole.

I want $\frac{\partial}{\partial z} - \gamma$ to take a section like $\frac{1}{z-p}$ with simple pole at p into a function with simple pole at p, hence
$$-\gamma = \frac{1}{z-p} + \text{smooth near } p.$$
At the same time we want γ to be periodic. Hence the simplest choice seems to be
$$-\gamma = \zeta(z-p) - \bar{z} - lz$$
(Check signs. $\zeta(z) = \frac{d}{dz} \log \sigma$, hence $\zeta(z+\omega) - \zeta(z) = a_\omega = l\omega + m\bar{\omega}$)

Thus take the ∂-operator on $\mathcal{O}(P)$ to be

$$\boxed{\nabla_z = \frac{\partial}{\partial z} + \zeta(z-p) - \bar{z} - lz}$$

Here is the recipe for the finite part. Put
$$G(z) = \frac{1 + \beta z - \alpha \bar{z} + O(z^2)}{\pi z}$$
Then take $\frac{\partial}{\partial z} \log G = -\frac{1}{z} + \beta + O(z)$, take its f.p. as $z \to 0$ which is β and then add $-\gamma$. The nice thing about this is that it doesn't change if G is multiplied by a constant. So let's take

$$\frac{\partial}{\partial z} \log \left\{ e^{w(\bar{z} + lz)} \frac{\sigma(z)\sigma(z-p-z'-w)}{\sigma(z-p)\sigma(z-z')} \right\} = wl + \zeta(z) + \zeta(z-p-z'-w) - \zeta(z-p) - \zeta(z-z')$$

Now I take the finite part at $z \to z'$. Recall that

$$\zeta(z) = \frac{1}{z} + \sum{}'\left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2}\right) = \frac{1}{z} + O(z^3)$$

so its finite part as $z \to 0$ is 0. Thus for the f.p. of the log der, I get $\quad w\ell + \zeta(z') + \zeta(-p-w) - \zeta(z'-p)$

and so the final answer is

$$\boxed{F.P.\ \langle z' | i(Di)^{-1} | z' \rangle = \frac{1}{\pi}\left(w\ell + \zeta(z') + \zeta(-p-w) - \bar{z}' - \ell z'\right)}$$

which is periodic in z' as it should be.

Next we use this formula to compute the corresponding determinant function. Use

$$\frac{d}{dw} \log \det(\partial_{\bar z} - w) = -\operatorname{Tr}(\partial_{\bar z} - w)^{-1}$$

and interpret the latter by the above finite part. Then I need to compute

$$\int_M \left(\zeta(z') - \bar{z}' - \ell z'\right) dz'\, d\bar{z}'$$

but this is zero because the integrand is an odd periodic function. Thus $\quad \operatorname{Tr}(\partial_{\bar z} - w)^{-1} = w\ell - \zeta(p+w)$, so

$$\frac{d}{dw} \log \det(\partial_{\bar z} - w) = -w\ell + \zeta(w+p)$$

$$\boxed{\det(\partial_{\bar z} - w) = \text{const}\ e^{-\ell w^2/2}\, \sigma(w+p)}$$

Simpler example which should have been done first. Consider $\partial_{\bar z} - w$ on $\mathcal{O}$ with $\nabla_z = \partial_z$. Then

$$\langle z | D^{-1} | z' \rangle = e^{w(\bar z + \ell z)} \frac{\sigma(z - z' - w)}{\sigma(z - z')} \cdot \text{const dep. on } z'$$

so $\quad \pi\, F.P. \langle z' | D^{-1} | z' \rangle = w\ell + \zeta(-w)$, so this time

$$\boxed{\det(\partial_{\bar z} - w) = \text{const}\ e^{-\ell w^2/2}\, \sigma(w)}$$

Next go back to the $O(P)$ example but change W to be sections vanishing at a point $P' = p' + \Gamma$.

$$\frac{\partial}{\partial z} \log \left\{ e^{w(\bar z + lz)} \frac{\sigma(z-p')\sigma(z-p+p'-z'-w)}{\sigma(z-p)\sigma(z-z')} \right\}$$

$$= wl + \zeta(z-p') + \zeta(z-p+p'-z'-w) - \zeta(z-p) - \zeta(z-z')$$

$$\xrightarrow{F.P.} \quad wl + \zeta(z'-p') + \zeta(-p+p'-w) - \zeta(z'-p)$$

So

$$\pi \, F.P. \, G(z',z') = wl + \zeta(z'-p') - \bar z' - lz' + \zeta(-p+p'-w)$$

so
$$\mathrm{Tr}(G) = wl - \bar p' - lp' + \zeta(-p+p'-w)$$

$$\frac{d}{dw} \log\det D = -lw + (\bar p' + lp') + \zeta(w+p-p')$$

$$\det D = \text{const} \; e^{-lw^2/2 + (\bar p' + lp')w} \sigma(w+p-p')$$

The answer should be periodic in p', because p' enters in defining the space W only through the coset $p'+\Gamma$.

Things become simpler if I set $p=0$.

$$\frac{d}{dw}\log\det = -lw + \underbrace{\zeta(w-p') + (\bar p' + lp)}_{\text{periodic in } p'.}$$

$$\boxed{\det D = \text{Const}_{p'} \; e^{-lw^2/2 + (\bar p' + lp')w} \sigma(w-p')}$$

If we normalize the determinant to be 1 when $w=0$, then the constant becomes $\dfrac{1}{\sigma(-p')}$ and then it is clearly periodic in p'. Thus the formula is

$$\det D = e^{-lw^2/2 + (\bar p' + lp')w} \frac{\sigma(w-p')}{\sigma(-p')}$$

$\uparrow$

$D = \frac{\partial}{\partial \bar z} - w$ on $O(0)$
with W = sections vanishing at $p'+\Gamma$

Analogous calculation for $\partial_{\bar z} - w$ on $\mathcal{O}(2\cdot 0)$ with W defined by vanishing at $p+\Gamma, q+\Gamma$ yields

$$\boxed{\det = e^{-\ell w^2/2 + (\overline{p+q} + \ell(p+q))w} \frac{\sigma(w-p-q)}{\sigma(-p-q)}}$$

Next I want to consider the case where the line bundle is negative. In this case to define a Green's function we select a finite dimensional subspace F of $C^\infty(E \otimes T^{0,1})$ such that $\pi D : V_\phi \longrightarrow V_0/F$ is invertible. Then the Green's function $\langle z | (D\pi)^{-1} \pi | z' \rangle$ doesn't give the δ-function when D is applied but rather the δ function $+$ something in F. But the analogue of the above W are the case when F is spanned by δ-functions, so F not really a subspace of V_0. (This is maybe possible, because

$$\mathcal{L}^* \subset \mathrm{Hom}^{(-p)}(\Lambda V_0, \Lambda V_1) \longrightarrow\!\!\!\!\to (\Lambda^p V_0)^*$$

and hence elements of $(\Lambda^p V_0)^{**}$ give sections of $\mathcal{L}$. ?)

Hence the Green's function will be analytic except at the points belonging to F. Let's consider the example where $E = \mathcal{O}(-0)$ and $D = \left(\frac{\partial}{\partial \bar z} - w\right) d\bar z$ and $\nabla_z = \frac{\partial}{\partial z} - [\zeta(z) - \ell z - \bar z]$. Take F to be a δ-function at P. The Green's function will have a zero at 0, and poles at P, z', so is of the form

$$e^{w(\bar z + \ell z)} \frac{\sigma(z)\,\sigma(z-p-z'-w)}{\sigma(z-p)\,\sigma(z-z')} \cdot \mathrm{const}_{z'}$$

$$\frac{\partial}{\partial z} \log G = \ell w + \zeta(z) + \zeta(z-p-z'-w) - \zeta(z-p) - \zeta(z-z')$$

$$\pi\, F.P.\, G = \ell w + \zeta(-p-w) - \zeta(z'-p) + \ell z' + \bar z'$$

$$\therefore \mathrm{Tr}\big((D\pi)^{-1}\pi\big) = \ell w - \zeta(p+w) + \ell p + \bar p$$

$$\therefore \frac{d}{dw} \log\det = -\ell w + \zeta(w+p) - \bar p - \ell p$$

$$\det = e^{-\ell\frac{w^2}{2} - (\bar{p} + \ell p)w} \frac{\sigma(w+p)}{\sigma(p)}$$

which is clearly the same kind of formula.

is of the same type as $F(t; y, x)$. So I want to know about the behavior as $r \downarrow 0$ of

$$\int_0^\infty dt \, \frac{e^{-\frac{r^2}{2t}}}{2\pi t} f(t)$$

where f decays as $t \to +\infty$ and $\sim a_0 + a_1 t + a_2 t^2 + \dots$ as $t \downarrow 0$.

$$\int_0^\infty = \int_0^1 + \underbrace{\int_1^\infty}_{\text{has limit at } r \to 0.}$$

So we worry about $\displaystyle\int_0^1 \frac{dt}{2\pi t} e^{-r^2/2t} = \int_1^\infty \frac{dt}{2\pi t} e^{-\frac{r^2 t}{2}} = \int_{r^2}^\infty \frac{dt}{2\pi t} e^{-t/2}$

$$= \int_{r^2}^1 \frac{dt}{2\pi t}\left(1 + e^{-t/2} - 1\right) + \int_1^\infty = \left[\frac{1}{2\pi}\log t\right]_{r^2}^1 + \text{something with limit}$$

$$= -\frac{1}{\pi}\log r + \text{something with limit as } r \to 0.$$

This whole business isn't very clear. I dont understand the role of s. Suppose we put $s = 0$. I want to define

$$\int_0^\infty e^{-tA}\, dt = \frac{1}{A}$$

on the diagonal. So I want to use

$$\int_0^\infty \left\{ \langle x | e^{-tA} | x \rangle - \frac{a_0(x)}{t} \right\} dt$$

but this won't necessarily work because $\langle x | e^{-tA} | x \rangle$ should decay exponentially as $t \to \infty$ when $A \geq \varepsilon > 0$

June 1, 1982

Direct way ~~to~~ get the metric on the [redacted] line bundle $\mathcal{L}$ in general. Let's consider the eigenvalues of the Laplaceans D^*D and DD^*, where $D: E \to E\otimes T^{0,1} = F$. Can form subspaces $H^i_{\leq\lambda}$, $i = 0, 1$ where the Laplacean has eigenvalues $\leq \lambda$. Then we have

$$\mathcal{L}_D \cong \lambda(H^1_{\leq\lambda}) \otimes \lambda(H^0_{\leq\lambda})^*$$

a canonical isomorphism. On the right side we have a natural metric which we get from the natural [redacted] inner products on $H^i_{\leq\lambda}$. Let's check how these metrics change as we increase λ to λ'. Then we tensor with

$$(1) \qquad \lambda(H^1_{(\lambda,\lambda']}) \otimes \lambda(H^0_{(\lambda,\lambda']})^*$$

Suppose there is a single eigenvector ϕ with $D^*D\phi = \mu\phi$ and μ in the range $(\lambda,\lambda']$. Then $\|D\phi\|^2 = (\phi|D^*D\phi) = \mu\|\phi\|^2$. The line (1) is trivialized by the element $D\phi\otimes\phi^{-1}$ which has [redacted]

$$\|D\phi\otimes\phi^{-1}\|^2 = \mu$$

So now we define a metric on $\mathcal{L}_D$ by choosing an orth. basis ψ_j of $H^1_{\leq\lambda}$ and an orth. basis ϕ_i for $H^0_{\leq\lambda}$ and saying that

$$\Lambda\psi_j / \Lambda\phi_i \cdot e^{\frac{1}{2}\zeta'_{>\lambda}(0)}$$

has [redacted] unit length. [redacted] Here

$$\zeta_{>\lambda}(s) = \sum_{\substack{\mu \text{ eigenvalue} \\ \text{of } D^*D \;>\lambda}} \mu^{-s}$$

Check the sign as follows. Suppose D invertible and we take $\lambda = 0$. Then the formula says

$$\underbrace{\text{canon. section}}_{s} \cdot e^{\frac{1}{2}\zeta'_{>0}(0)}$$

has unit length, or that $\|\text{canon. sect}\|^2 = e^{-\zeta'_{>0}(0)}$ which is correct. If λ is increased to admit one eigenvalue μ, then we have

$$s\, D\phi/\phi \; e^{\frac{1}{2}\zeta'_{>0}(0)} = s \cdot \tfrac{1}{\sqrt{\mu}} D\phi/\phi \; e^{\underbrace{\frac{1}{2}\log\mu + \frac{1}{2}\zeta'_{>0}(0)}_{\frac{1}{2}\zeta'_{>\mu}(0)}}.$$

Therefore it is clear using the continuity of eigenvalues, eigenspaces, etc. that we get a smooth metric on $\mathcal{L}$ defined in this way. Next we want to compute its curvature.

To simplify suppose D invertible so that I can form $(D^*D)^{-s}$ without any problem. Let P^i_λ be the projection on $H^i_{<\lambda}$. We assume of course that λ is not an eigenvalue of D^*D. Then

$$\zeta_{>\lambda}(s) = \mathrm{Tr}\left((D^*D)^{-s}(I - P_\lambda)\right)$$

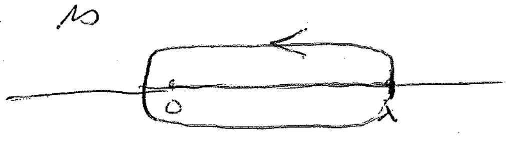
Now we vary the operator D^*D and look at the effect to first order:

$$\delta\zeta_{>\lambda}(s) = \mathrm{Tr}\left((-s)(D^*D)^{-s-1}\,\delta(D^*D)(I-P_\lambda)\right)$$
$$+ \mathrm{Tr}\left((D^*D)^{-s}(-\delta P_\lambda)\right)$$

Now P_λ hence δP_λ is an operator of finite rank, hence the last term is entire in s. But actually it should be identically zero: Put $H = D^*D$

$$P_\lambda = \frac{1}{2\pi i}\oint \frac{1}{s-H}\,ds$$

where the contour is



Thus $\delta P_\lambda = \frac{1}{2\pi i}\oint \frac{1}{s-H}\,\delta H\,\frac{1}{s-H}\,ds$

hence if $|a\rangle$, $|b\rangle$ are eigenvectors for H we have

$$\langle a|\delta P_\lambda|b\rangle = \frac{1}{2\pi i}\oint \frac{1}{s-\varepsilon_a}\langle a|\delta H|b\rangle \frac{1}{s-\varepsilon_b}\,ds$$

$$= \begin{cases} 0 & \text{if } \varepsilon_a, \varepsilon_b \text{ both } > \lambda, \text{ or both } < \lambda \\[2mm] \dfrac{\langle a|\delta H|b\rangle}{\varepsilon_b - \varepsilon_a} & \text{if } \varepsilon_b < \lambda > \varepsilon_a \\[2mm] \dfrac{\langle a|\delta H|b\rangle}{\varepsilon_a - \varepsilon_b} & \text{if } \varepsilon_a < \lambda > \varepsilon_b \end{cases}$$

In particular δP_λ has no diagonal matrix elements in the basis of eigenvectors for D^*D, hence

$$\mathrm{Tr}\left((D^*D)^{-s}\,\delta P_\lambda\right) = 0.$$

I am assuming first time around that D is inver-tible. Then

$$-\frac{\delta \zeta_{>\lambda}(s)}{s} = \mathrm{Tr}\left((D^*D)^{-s} D^{-1}\delta D\,(I-P_\lambda)\right)$$
$$+\,\mathrm{Tr}\left((D^*D)^{-s} D^{-1} D^{*-1}\delta D^* D\,(I-P_\lambda)\right)$$

$$= \mathrm{Tr}\left(D^*D^{-s} D^{-1}\delta D\right) - \mathrm{Tr}\left((D^*D)^{-s} D^{-1}\delta D\,P_\lambda\right) + \cdots$$

The first term has a good limit as $s\to 0$ which I have already computed. The second is OK because P_λ is of finite rank, hence we get

$$-\mathrm{Tr}\left(D^{-1}\delta D\,P_\lambda\right)$$

Actually it seems as if there is no particular reason not to work with $H = D^*D$. What is

$$\mathrm{Tr}\left(H^{-1}\delta H P_\lambda\right) = \sum_{\varepsilon_a<\lambda} \frac{1}{\varepsilon_a}\langle a|\delta H|a\rangle\ ?$$

One knows that $\delta\varepsilon_a = \langle a|\delta H|a\rangle$, hence

$$\mathrm{Tr}(H^{-1}\delta H P_\lambda) = \sum \frac{1}{\varepsilon_a}\delta\varepsilon_a = \delta \log\Big(\prod_{\varepsilon_a<\lambda}\varepsilon_a\Big).$$

. One point is the following: For bundles of index 0 you have now given a metric on $\mathcal{L}$ globally, which is smooth, and you know the curvature form is $\frac{1}{\pi}$(Kähler form) over the dense open set where D is invertible, hence ~~globally~~ by continuity you have now established that the curvature globally is $\frac{1}{\pi}$(Kähler form). So now everything works at least for index 0.

June 2, 1982:

I have been looking holomorphic structures D on a C^∞ vector bundle of index 0 such that D is invertible. Are the resulting holomorphic bundles stable or semi-stable? So let E have slope $g-1$ with zero cohomology and consider an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0.$$

$$0 \to H^0(E') \to \underset{0}{\underset{\parallel}{H^0(E)}} \to H^0(E'') \to H^1(E') \to \underset{0}{\underset{\parallel}{H^1(E)}} \to H^1(E'') \to 0$$

Suppose the slope of $E' > (g-1)$. Then

$$\underset{0}{\underset{\parallel}{h^0(E')}} - \underbrace{h^1(E')}_{\geq 0} = \deg E' - rg E'(g-1) > 0$$

which is a contradiction. Thus we have

Prop: If E has [redacted] zero cohomology, then E is semi-stable of [redacted] slope $g-1$.

Try the converse. Given an E of slope $g-1$ with $h^0 \neq 0$, we want to produce a subbundle of slope $> g-1$. However all we can do with a section is produce a sub-line bundle of positive (≥ 0) degree. So I dont seem able to prove the converse. In fact the converse is already false for line bundles of degree $g-1$.

Next problem is to trivialize $\mathcal{L}$ in general using the analytic-torsion metric. Assume index < 0 and work near a D where $H^0 = 0$. Then D^*D is invertible and I have no trouble defining

$$\zeta_{D^*D}(s) = \operatorname{Tr}\left((D^*D)^{-s}\right)$$

Now I want to consider its variation:

$$\delta \zeta(s) = -s \operatorname{Tr}\left(\left(D^*D\right)^{-s-1} \delta(D^*D)\right)$$

I want to evaluate $\lim_{s\to 0} \operatorname{Tr}\left(\left(D^*D\right)^{-s}\delta(D^*D)\right)$; the fact this is finite will show that $\delta\zeta(0) = 0$. So what comes next is

$$\delta(D^*D) = (\delta D^*)D + D^*(\delta D)$$

$$\operatorname{Tr}\left(\left(D^*D\right)^{-s-1} D^*\delta D\right) = \operatorname{Tr}\left(\left(D^*D\right)^{-s}\left(D^*D\right)^{-1} D^*\,\delta D\right)$$

$$\operatorname{Tr}\left(\left(D^*D\right)^{-s-1}\,\delta D^* D\right) = \operatorname{Tr}\left(\delta D^* D \left(D^*D\right)^{-1}\left(D^*D\right)^{-s}\right).$$

So provided $(\delta D)^* = \delta(D^*)$, which is OK if I don't vary the complex structure, the two trace terms are complex conjugates. Hence my problem is to understand the heat kernel regularization of the diagonal part of $(D^*D)^{-1}D^*$.

Clearly $(D^*D)^{-1}D^*$ is projection onto $\operatorname{Im} D$ followed by taking the inverse of D. Hence the Green's fn. is smooth off the diagonal and locally of the form

$$\langle z \,|\, (D^*D)^{-1}D^* \,|\, z' \rangle = \frac{F(z,z')\,dz'}{\pi(z-z')}$$

where

$$\left(\frac{\partial}{\partial \bar z} + \alpha\right)\frac{F(z,z')}{\pi(z-z')} = \delta(z-z') - \underbrace{\sum_a \langle z|a\rangle\langle a|z'\rangle}_{\text{smooth finite rank kernel.}}$$

$$\therefore \left(\frac{\partial}{\partial \bar z} + \alpha\right) F(z,z') = -\pi(z-z')\sum_a \langle z|a\rangle\langle a|z'\rangle$$

Recall $F(z,z') = 1 + \beta_{z'}(z-z') + \gamma_z \overline{(z-z')} + O(z-z')^2$

I guess it's clear that $\gamma = -\alpha$ as before. Put $z'=0$ and check

$$\left(\partial_{\bar z}+\alpha\right)F(z) = -\pi z\sum \langle z|a\rangle\langle a|0\rangle$$

$$F(z) = 1+az+b\bar z+c\frac{z^2}{2}+dz\bar z+e\frac{\bar z^2}{2}+\cdots$$

$$\left(\partial_{\bar z}+\alpha\right)F(z) = (b+\alpha)+(\alpha a+d)z+(\alpha b+e)\bar z+\cdots = -\pi z f$$

so the only relation you get is $b=-\alpha$. Thus

$$F(z) = 1+\beta z-\alpha\bar z+\cdots\cdots$$

$$F_b(z) = 1+\alpha^* z-\alpha\bar z+\cdots\cdots$$

and so the finite part is it seems

$$\lim_{z\to 0}\frac{F-F_b}{\pi z} = \frac{1}{\pi}\left(\beta-\alpha^*\right).$$

As before this must be combined with something coming from the metric to make things independent of the choice of z.

The next project is to figure out how β varies as the holomorphic structure α is changed. It seems unlikely that β is holom. in α since the Green's function using the orthogonal projection onto $\operatorname{Im} D$. The thing to do is to fix a finite-dimensional subspace complementary to $\operatorname{Im} D$, and use this to define a Green's function for D which is obviously analytic in D. Denote it $\frac{F_a(z,z')}{\pi(z-z')}$.

Digression: The question is whether you expect the curvature of $\mathcal{L}$ to always be $\frac{1}{\pi}\cdot$ Kähler form even when the index is $\neq 0$. Let's check whether this is the case for line bundles over $\mathbb{C}/\Gamma$. I recall considering $\mathcal{O}(2\cdot 0)$ with operator $\partial_{\bar z}-\omega$. This means I am getting a family of degree 2 line bundles, but tensoring $\mathcal{O}(2\cdot 0)$ by the variable degree 0 line bundle given by $\mathcal{O}$ with $\partial_{\bar z}-\omega$.

Now to define a Green's fn. what I do is to choose a subspace of $C^\infty(E)$ of codim 2, say those sections vanishing at p, q $(+\Gamma)$. This gives a Green's function when the degree 2 line bundle has no non-zero sections vanishing at p and q. If it does have such a section then the line bundle is $\sim \mathcal{O}(\tilde{p}+\tilde{q})$, i.e. $w = p+q \pmod{\Gamma}$. Moreover such a section is unique up to scalar because the other sections give the other positive divisors $p'+q'$ of degree 2 with $w = p'+q' \pmod{\Gamma}$. (Precisely: the flat line bundle given by $\mathcal{O}$ with $\partial_{\bar{z}} - w$ belongs to the divisor $(w)-(0)$, so $\mathcal{O}(2\mathcal{O})$ with $\partial_{\bar{z}} - w$ belongs to $(w)-(0)+2(0) = (w)+(0)$. On the other hand we can calculate tensor products of line bundles of degree 0 by adding in the elliptic curve. Thus $\mathcal{O}((w)+(0)) \cong \mathcal{O}((p)+(q)) \iff \mathcal{O}((w)-(0)) \cong \mathcal{O}((p)-(0)) \otimes \mathcal{O}((q)-(0)) \iff w = p+q \bmod \Gamma$.) The intersection of H^0 with ~~the~~ subspace given by p, q is 1-dim'l, hence the determinant should ~~have~~ simple zeroes at $w \in p+q+\Gamma$.

June 3, 1982.

Resolution of a contradiction. Consider line bundles of degree 1 over $M = \mathbb{C}/\Gamma$. I can a family of these parameterized by $w \in \mathbb{C}$ by picking one, say $\mathcal{O}((0))$, with $\partial_{\bar z}$, and then changing the holom. structure to $\partial_{\bar z} - w$. Associating to w the isom. class of the line bundle, which is determined by the unique zero in M of the unique (up to scalar) non-zero section, gives us a map $\mathbb{C} \longrightarrow M$, which I know is $w \mapsto w + \Gamma$. Now the cohomology-determinant-line bundle over M is trivial if I **think** of the universal bundle over M as being $m \mapsto \mathcal{O}((m))$, since $H^0 = \mathbb{C}$ canonically. On the other hand the cohomology-determinant-line bundle $\mathcal{L}$ over $\mathbb{C}$ is supposed to have the Kähler form for its curvature, which would not be the case if it were trivializable invariantly under Γ.

What's wrong is to think of the universal line bundle over the Jacobian in too canonical a way. It is only unique up <u>to line bundles on the base</u> at best.

Next, automorphy factors for σ.

$$\sigma(z) = z \prod{}'\left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}}$$

We know
$$\frac{\sigma(z+\lambda)}{\sigma(z)} = e^{a_\lambda z + b_\lambda}$$

where $\quad a_\lambda = \ell\lambda + m\bar\lambda \qquad m = 1$ if $\mathrm{vol}\,\Gamma = \pi$

and
$$e^{b_{\lambda+\lambda'} - b_\lambda - b_{\lambda'}} = e^{a_\lambda \lambda'} = e^{\ell\lambda\lambda'} \underbrace{e^{\bar\lambda\lambda'}}_{\substack{\text{symmetric as} \\ \bar\lambda\lambda' - \lambda\bar\lambda' \in 2\pi i\mathbb{Z}}}$$

I claim that
$$(\circledast)\qquad e^{-|z|^2}\left|e^{-\ell\frac{z^2}{2}}\sigma(z)\right|^2$$
is periodic. First calculate its autom. factor. For $e^{-\ell z^2/2}\sigma(z)$ we get exp of
$$-\ell\left(z\lambda+\frac{\lambda^2}{2}\right)+(\ell\lambda+\bar\lambda)z+b_\lambda=\bar\lambda z+\left(b_\lambda-\ell\frac{\lambda^2}{2}\right)$$
For $e^{-|z|^2}$ we get exp of $-z\bar\lambda-\bar z\lambda-|\lambda|^2$. So for $\circledast$ we get
$$e^{-|\lambda|^2}\left|e^{b_\lambda-\ell\frac{\lambda^2}{2}}\right|^2=e^{c_\lambda}.$$
Then
$$e^{c_{\lambda+\lambda'}-c_\lambda-c_{\lambda'}}=e^{-\lambda\bar\lambda'-\bar\lambda\lambda'}\,e^{[\ell\lambda\lambda'+\bar\lambda\lambda'-\ell\lambda\lambda'+c.c.]}=1,$$
hence $\lambda\mapsto c_\lambda$ is a homomorphism $\Gamma\to\mathbb{R}$. On the other hand $\circledast$ is invariant under $z\mapsto -z$. Hence
$$e^{c_\lambda}=\frac{\circledast(z+\lambda)}{\circledast(z)}=\frac{\circledast(-z-\lambda)}{\circledast(-z)}=e^{c_{-\lambda}}$$
so $c_\lambda=c_{-\lambda}$. Thus $c_\lambda\equiv 0$.

Now on $\mathcal{O}((0))$ we can define a metric by
$$|1|^2=e^{-|z|^2}\left|e^{-\ell\frac{z^2}{2}}\sigma(z)\right|^2$$
and the curvature is obviously
$$\bar\partial\partial\log|1|^2=-d\bar z\,dz=+dz\,d\bar z$$
which gives for the degree $=\displaystyle\int_M\frac{i}{2\pi}dz\,d\bar z=\int_M\frac{dx\,dy}{\pi}=\frac{\mathrm{vol}(\Gamma)}{\pi}=1$, as it should.

For the holom. structure $\partial_{\bar z}-w$ we have the global holom. section
$$s=e^{w(\bar z+\ell z)}\frac{\sigma(z-w)}{\sigma(z)}$$
and $\nabla s=s\theta$, $\theta=\partial\log|s|^2$ for the canonical connection associated to this ∂. structure.
$$s\theta=\nabla s=\nabla(s1)=s\nabla 1+ds \qquad \therefore\ \nabla 1=\partial\log|s|^2-d\log s$$

[redacted] $\underset{\text{as a section}}{|s|^2} = e^{-|z|^2} \left| e^{-lz^2/2}\cancel{\sigma(z)} \right|^2 \left| e^{w(\bar z + lz)} \frac{\sigma(z-w)}{\cancel{\sigma(z)}} \right|^2$

$$\partial \log |s|^2 = -\bar z\, dz - lz\, dz + \bar w\, dz + \cancel{wl\, dz} + \zeta(z-w)\, dz$$

as a fn $d \log s = w\, d\bar z + w\cancel{l}\, dz + [\zeta(z-w) - \zeta(z)]\, dz$

$$\therefore \quad \nabla 1 = (\zeta(z) - \bar z - lz)\, dz + (\bar w\, dz - w\, d\bar z).$$

Thus $\nabla_z = \frac{\partial}{\partial z} + \bar w + (\zeta(z) - \bar z - lz)$, $\nabla_{\bar z} = \frac{\partial}{\partial \bar z} - w$

What I need now is the Green's function for the operator $\partial/\partial\bar z - w$ whose image is [redacted] in the subspace $\perp$ to the H^0. This seems to be hard to calculate. Recall that if we [redacted] pick a complement to the H^0 by requiring sections to vanish at p we get the Green's function

$$\frac{1}{\pi}\, e^{w(\bar z + lz)} \frac{\sigma(z-p)\,\sigma(z+p-z'-w)}{\sigma(z)\,\sigma(z-z')} \Big/ e^{w(\bar z' + lz')} \frac{\sigma(z'-p)\,\sigma(p-w)}{\sigma(z')}$$

The simplest choice is to take $p = 0$, whence we get

$$\frac{1}{\pi}\, e^{w(\overline{z-z'} + l(z-z'))} \frac{\sigma(z-z'-w)}{\sigma(z-z')\,\sigma(-w)}$$

To this can be added a multiple depending on z', w of the unique holom. section

$$\circledast \qquad f(z', w)\, e^{w(\bar z + lz)} \frac{\sigma(z-w)}{\sigma(z)}$$

So now I want to take the finite part of the Green's function, which one [redacted] can do by using the connection ∇. The f.p. should be

$$lw + \zeta(z'-p) + \zeta(p-w) - \cancel{\zeta(z')} + \cancel{\zeta(z')} - \bar z' - lz'$$

plus from the arbitraire $\circledast$ the term $f(z', w)\, e^{w(\bar z' + lz')} \frac{\sigma(z'-w)}{\sigma(z')}$?

A different approach: Ultimately I hope to construct a [redacted] Hilbert space containing these determinant functions so I will need to know something about inner products of determinant functions which hopefully will be some other kind of determinants. Let's work out the ideas in the simple case of line bundles of degree 0, over M given as usual by $\partial_{\bar z} - w$, $w \in \mathbb{C}$. Over $\mathbb{C}$ we get this line bundle $\mathcal{L}$ with canonical section, which I can identify with the divisor Γ. Thus $\mathcal{L} = \mathcal{O}(\Gamma)$ and the canonical section is 1. [redacted] The metric on $\mathcal{L}$ is given by
$$|1|^2 = e^{-\rho'(0)} = \text{const.}\ e^{-|w|^2} \left| e^{-\ell\frac{w^2}{2}} \sigma(w) \right|^2$$

Now what should be the inner product on sections of $\mathcal{L}$? The good trivialization of $\mathcal{L}$ is

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "s", "\substack{\text{canon}\\ \text{section}}"'] \arrow[rr, bend left, "\det"] & \mathcal{L} & \mathcal{O} \arrow[l, "\sim", "\frac{s}{\det}"']
\end{tikzcd}$$

$$\left| \frac{s}{\det} \right|^2 = e^{-|w|^2}$$

Thus sections of $\mathcal{L}$ can be described as $s \cdot f / e^{-\ell\frac{w^2}{2}}\sigma(w)$ with $f(w)$ analytic in w, [redacted] and the norm of this section is $|f|^2 e^{-|w|^2}$.

Now we can construct other ~~[illegible]~~ determinant functions by fixing f. codim. subspaces of V_1 and f. dim. subspace of V_0. Thus ask for sections vanishing at $p+\Gamma$ with poles at $q+\Gamma$. The Green's fn. is then
$$e^{w(\bar z + \ell z)} \frac{\sigma(z-p)\,\sigma(z+p-q-z'-w)}{\sigma(z-q)\,\sigma(z-z')} \Big/ \text{const.}$$

and using the standard $\nabla_z = \partial_z$ we compute FP as usual

$$\ell\omega + \underset{-(\bar{z}'+\ell z')}{\delta(z'-p)} + \delta(p-q-\omega) - \underset{+(\bar{z}'+\ell z')}{\delta(z'-q)}.$$

Integrate over z' to get

$$\ell\omega - \left[\overline{p-q} + \ell(p-q)\right] + \delta(p-q-\omega)$$

so finally

$$\det = e^{-\ell\frac{\omega^2}{2} + \omega\left[\overline{p-q} + \ell(p-q)\right]} \frac{\delta(\omega-p+q)}{\sigma(-p+q)}$$

The setup: $M = \mathbb{C}/\Gamma$, $\mathrm{vol}(M) = \pi$ relative to $dxdy$. Consider the family of line bundles $(\mathcal{O}, \partial_{\bar z} - w)$ param. by $w \in \mathbb{C}$, and the associated coh. det. $\mathcal{L}$ over $\mathbb{C}$. In this case I can identify $\mathcal{L}$ with the line bdle belonging to the divisor Γ, using the canonical section. In general the gauge gp. Γ acts on $\mathcal{L}$ over $\mathbb{C}$. In the present case s is an invariant section.

There are other holomorphic sections of $\mathcal{L}$ which one obtains by specifying subspaces of V_1, V_0. In the case of line bundles where these subspaces are given by vanishing conditions, these subspaces will be gauge inv. and so the corresponding sections of $\mathcal{L}$ will change under Γ by characters. The obvious question is what sort of holomorphic sections of $\mathcal{L}$ are obtained.

Recall the formula

$$\begin{tikzcd} V_1 \arrow[rr, "T"] \arrow[dr, "\pi"'] & & V_0 \\ & F \arrow[ur, hook, "i"'] & \end{tikzcd}$$

$$\det(T)\,\det(\pi T^{-1} i) = \det(\bar T : \mathrm{Ker}\,\pi \to \mathrm{Cok}\, i)$$

So if $T = \partial_{\bar z} - w$, take i, π to be inclusion + projection onto the subspace spanned by $e^{\mu\bar z - \bar\mu z}$. Then

$$\det(\pi T^{-1} i) = \frac{1}{\mu - w}$$

so the corresponding section of $\mathcal{L}$ is $\frac{s}{\mu - w}$. It's clear that if I take F ~~to~~ be spanned by several exponential fns. we get the sections

$$\frac{s}{\prod_j (\mu_j - w)}$$

with $\{\mu_j\}$ a finite subset of Γ. On the other hand let's take F to be spanned by a C^∞ fn.

$$\sum a_\mu e^{\mu\bar{z}-\bar{\mu}z} \qquad a_\mu \text{ rapidly decrease}$$

and π to be a distribution $\frac{1}{\pi}\sum b_\mu$ [redacted] $e^{\mu\bar{z}-\bar{\mu}z}$, b_μ tempered.
Then we get

$$(*) \qquad \det(\pi T^{-1} i) = \sum \frac{b_\mu a_\mu}{\mu-\omega}$$

where $b_\mu a_\mu$ rapidly decreases.

I really want to take a δ function for $i(F)$ at q and the δ function at p for π. Thus

$$b_\mu a_\mu = \frac{1}{\pi} e^{\mu(\overline{p-q})-\bar{\mu}(p-q)}$$

up to some signs. Then $(*)$ perhaps should be interpreted as a Green's function for the operator $\partial_{\bar{z}}-\omega$ [redacted] evaluated [redacted] $p-q$. Formally it should be something like

[redacted]

$$e^{\omega(\overline{p-q}+\ell(p-q))} \frac{\sigma(p-q-\omega)}{\sigma(p-q)\sigma(\omega)}$$

which is consistent with yesterday's calculation. So [redacted] summarize.

We have this line bundle $\mathcal{L}$ with canonical section s which we think of as $\det(T)$, $T=\partial_{\bar{z}}-\omega$. Then via π, i I get other holom. sections

$$\det(\bar{T}) = \det(\pi T^{-1} i)\det(T).$$

Now the ultimate description is to trivialize [redacted] $\mathcal{L}$

$$\mathcal{L} \simeq \mathcal{O} \qquad s \longmapsto e^{-\ell\omega^2/2}\sigma(\omega)$$

whence the metric on $\mathcal{L}$ will corresp. to $e^{-|\omega|^2}|f|^2$ on $\mathcal{O}$. Then

$$\det(\bar{T}) = \det(\pi T^{-1} i)\, s \longmapsto e^{-\ell\frac{\omega^2}{2}+\omega(\overline{p-q})+\ell(p-q))} \frac{\sigma(p-q-\omega)}{\sigma(p-q)}$$

Question: Are the sections square-integrable?
Simplest case: Is

$$\int e^{-|w|^2} \left| e^{-\ell w^2/2} \sigma(w) \right|^2 < \infty$$

No because the integrand is periodic.

Therefore to get a Hilbert space out of these sections I have to integrate out modulo the gauge group.

Question: $|s|^2$ is precisely defined by analytic torsion once the volume on the Riemann surface and the inner product on the bundle is given. Hence $\int |s|^2$ is a canonical constant. What is it?

Falting explains that the constant curvature metric on a Riemann surface is not arithmetically interesting since it doesn't distinguish between special points like Weierstrass points. Better is to note that $H^0(M, \Omega^1)$ carries a canonical inner product $\|\alpha\|^2 = \int \alpha \wedge * \alpha$ and that $H^0(M, \Omega^1)$ maps onto Ω^1 at each point, so induces a metric on the bundle Ω^1. To see Ω^1 is spanned by global sections use

$$H^0(\Omega^1) \to \Omega^1 \otimes k(p) \to H^1(\Omega^1(-p)) \to H^1(\Omega) \to 0$$

and use that

$$H^1(\Omega(-2p)) \twoheadrightarrow H^1(\Omega(-p)) \twoheadrightarrow H^1(\Omega) \twoheadrightarrow 0$$

is dual to

$$H^0(\mathcal{O}(2p)) \hookleftarrow H^0(\mathcal{O}(p)) \hookleftarrow H^0(\mathcal{O}) \hookleftarrow 0$$

and $H^0(\mathcal{O}(p)) = \mathbb{C}$ unless $g = 0$ which is ruled out.

(Same arg. as Ω^1 gives an embedding in projective space $\Longleftrightarrow$ not hyperelliptic)

Next project: vary the holomorphic structure on M.

[redacted] For elliptic curves we have [redacted] translation invariant structures described by a single complex parameter. This time I want to fix a real torus $M = \mathbb{R}^2/\Gamma$ with a volume element on it. Then look at the different complex structures on $\mathbb{R}^2$ with the correct orientation. One gets $SL_2(\mathbb{R})/U_1$ [redacted] $=$ upper half plane.

Let's proceed more generally. I fix a C^∞ compact oriented surface M, and then I want to consider all holomorphic structures on M. Because we are dealing with a surface there are no integrability conditions, hence a holom. structure is the same as an almost cx. structure i.e. reducing the tangent bundle from $GL_2(\mathbb{R})_+$ to [redacted] $\mathbb{C}^*$. But

$$GL_2(\mathbb{R})_+/\mathbb{C}^* \longleftarrow SL_2(\mathbb{R})/SO(2) = UHP.$$

Thus we can form a fibre bundle [redacted] over M with fibre the UHP whose sections [redacted] are the different holomorphic structures on M. [redacted] From the standard holomorphic structure on the UHP one gets a holomorphic structure on the space of sections $\mathcal{S}$ of this fibre bundle. If I fix a volume on M, then a holom. structure can be identified with a Riemannian metric with this volume. Then we have the two gauge[redacted] groups of all orientation-preserving diffeos. and the subgroup of volume-preserving ones.

Next, how am I going to deal with [redacted] connections versus holom. structures on a vector bundle?

I want to consider a fixed torus, say $\mathbb{R}^2/\mathbb{Z}^2$, the trivial bundle over it with the connection $\nabla = d + A$, $A = A_x dx + A_y dy$ and then I want to vary the complex structure on $\mathbb{R}^2$ as well as A. It seems to be simpler to allow the lattice to vary. So let's do it that way.

So I let $\Gamma \subset \mathbb{C}$ be generated by ω_1, ω_2 where $\omega_2/\omega_1 = \tau \in UHP$ and where $\omega_1 \in \mathbb{R}_{>0}$ if I want. As ω_1, ω_2 vary over $\mathbb{C}$ in this way the real tori I get are canonically isomorphic. I assume the covolume of $\Gamma = \dfrac{\omega_2\bar\omega_1 - \bar\omega_2\omega_1}{2i} = \dfrac{2\pi i}{2i}$. For example $\omega_1 = 1$, $\omega_2 = \tau$, then volume $= \dfrac{\tau - \bar\tau}{2i} = \operatorname{Im}\tau = \pi$. Then it follows that $\Gamma^* = \{\mu \in \mathbb{C} \mid \mu\bar\gamma - \bar\mu\gamma \in 2\pi i\mathbb{Z}\}$ is equal to Γ. Next consider the operator $\dfrac{\partial}{\partial\bar z} - \omega : \mathcal{O} \to \mathcal{O}$. It has the eigenvalues $\lambda - \omega$, $\lambda \in \Gamma$ and so we get

$$\zeta(s) = \sum_\lambda \frac{1}{|\omega - \lambda|^{2s}}.$$

But now the lattice Γ varies in addition to ω.

$$-\delta\zeta(s) = s\sum{}' \frac{1}{|\omega-\lambda|^{2s+2}}\left(\delta|\omega-\lambda|^2\right)$$

$$-\frac{\delta\zeta(s)}{s} = \sum \frac{1}{|\omega-\lambda|^{2s+2}}\left((\omega-\lambda)(\overline{\delta\omega-\delta\lambda}) + \overline{(\omega-\lambda)}(\delta\omega-\delta\lambda)\right)$$

Think of $\lambda = n_1\omega_1 + n_2\omega_2$. Then $\delta\lambda = n_1\delta\omega_1 + n_2\delta\omega_2$.

$$-\delta\zeta'(0) = \lim_{s\to 0}\sum{}' \frac{1}{|\omega-\lambda|^{2s}}\,\frac{1}{(\omega-\lambda)}\left(\delta\omega - \delta\lambda\right) + \text{c.c.}$$

So for example when $\omega_1 = 1$, $\omega_2 = \tau$ and $\delta\tau \in \mathbb{R}$ then we want to evaluate

$$\lim_{s\to 0}\sum \frac{1}{|\omega-\lambda|^{2s}}\,\frac{1}{(\omega-\lambda)}\,n_2$$

thus to make sense by heat **equation** methods of

$$\sum_{m,n\in\mathbb{Z}} \frac{m}{(\omega-m-n\tau)}, \quad \sum_{m,n\in\mathbb{Z}} \frac{n}{(\omega-m-n\tau)}.$$

This represents worse convergence than the $\sum \frac{1}{(\omega-A)}$ which I have learned to handle via the **Weierstrass** ζ functions.

June 5, 1982

Dirac equation: Physical derivation.

Start with $E = \sqrt{p^2+m^2}$ and you want to make the substitutions $p, E \longmapsto \frac{1}{i}\nabla, \frac{-1}{i}\partial_t$. Put

$$E = \alpha_i p_i + \beta m. \qquad \beta = \alpha_0$$

Then the α_i, β satisfy $\alpha_i^2 = 1$, $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$. Also $\alpha_i^* = \alpha_i$ for the operator to be self-adjoint. The Schroedinger equation becomes

$$-i\,\partial_t \psi = \left(\alpha_i \tfrac{1}{i}\partial_{x_i} + \beta m\right)\psi$$

Rewrite in a more invariant way

$$0 = \left(\alpha_0 \tfrac{1}{i}\partial_t + \alpha_0\alpha_i \tfrac{1}{i}\partial_{x_i} + m\right)\psi$$

$$(\alpha_0\alpha_i)^2 = \alpha_0\alpha_i\alpha_0\alpha_i = -\alpha_0^2\alpha_i^2 = -1$$

$$(\alpha_0\alpha_i)^* = \alpha_i\alpha_0 = -\alpha_0\alpha_i$$

Change to imaginary time by the rule $e^{-itH} = e^{-x_0 H}$ i.e. $x_0 = it$. Then get

$$0 = \Big(\underbrace{\alpha_0}_{\gamma_0} \frac{\partial}{\partial x_0} + \underbrace{\frac{\alpha_0\alpha_i}{i}}_{\gamma_i} \frac{\partial}{\partial x_i} + m\Big)\psi$$

where γ_μ are hermitian and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

Therefore the Euclidean Dirac equation is

$$(\gamma_\mu \partial_\mu + m)\psi = 0.$$

Now let us concentrate on the operator

$$\sum \alpha_i \tfrac{1}{i}\partial_{x_i}$$

and try ~~to~~ make sense of it over a manifold. The symbol is $\sigma_p = \sum \alpha_i p_i$ for $p \in T^*$ which is a self-adjoint operator on the v.b. E. Hence we want

$$T^* \xrightarrow{\sigma} \mathrm{End}(E)$$

such that $\sigma_p^2 = |p|^2$, $\sigma_p^* = \sigma_p$. So E must be a representation of the Clifford algebra of T^*. In even dimensions we have $T^* \otimes \mathbb{C} = W \oplus W^*$ and the Clifford module is $E = \Lambda W$. This has an odd-even grading such that $\sigma_p : E^\pm \to E^\mp$. In fact we can identify T^* with W and then $\sigma_p = e(|p\rangle) + i(\langle p|)$.

Hence if the manifold has an almost-complex structure then we get at least the symbol of a Dirac operator by simply taking the operator $\sigma_p = e(p) + i(p)$ on $\Lambda T^{0,1}$. Lift this symbol to an operator and average it with its adjoint to get a s.a. operator.

Thus over a Riemann surface, any Clifford module bundle is $E \oplus E \otimes T^{0,1}$, where E is a line bundle. I lift the symbol to a self-adjoint operator as follows. First lift the $\bar\partial$ symbol to a $\bar\partial$-operator

$$E \xrightarrow{D} E \otimes T^{0,1}$$

and then the Dirac operator is $D + D^*$.

I am missing why E should somehow be a square root of the canonical bundle. If $E \otimes E = \Omega$ as holomorphic line bundles, then $E = E^* \otimes \Omega$. I recall identifying the transpose of D:

$$E^* \otimes T^{1,1} \xleftarrow{{}^tD} E^* \otimes T^{1,0}$$

with the $\bar\partial$ operator for the dual bundle $E^* \otimes \Omega$, hence

$$E \xleftarrow{D^*} E \otimes T^{0,1}$$

is isom. to the $\bar\partial$ operator for $E^* \otimes \Omega$. So if E is a square root of Ω, then D can be interpreted as an operator [redacted] D: [redacted] $E \longrightarrow E^*$

which is self-adjoint.

What are the explicit spinor representations. Let's do over the reals, the point being to find a real operator $\sum \alpha_i \partial_i$ which is self-adjoint and has square $-\Delta$. (Hence $\alpha_i^* = -\alpha_i$ and $\{\alpha_i, \alpha_j\} = -2\delta_{ij}$). The other possibility is for it to be skew-adjoint with square Δ (hence $\alpha_i = \alpha_i^*$ and $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$.) The former Clifford modules appear in the periodicity thm.

[redacted] Review: One forms the Clifford alg. generated over $\mathbb{R}$ by $e_1, \ldots, e_n$ subject to $\{e_i, e_j\} = -2\delta_{ij}$. Then one get for

$n =$	0	1	2	3	4
	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$M_2(\mathbb{H})$

For $n=3$ note that $e_1 e_2 e_3$ is central with square 1.
For $n=4$ the commutator of $\mathbb{H} = \mathbb{R}(e_1, e_2)$ is spanned by $1, \overset{x}{e_1 e_2 e_3}, \overset{y}{e_1 e_2 e_4}, \overset{-xy}{e_3 e_4}$, where $x^2 = y^2 = 1$, $(xy)^2 = -1$. Thus x, y generate a dihedral group of order 8, so centralizer of $\mathbb{H}$ is $M_2(\mathbb{R})$ with $x = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ $y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $xy = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus full alg is $\mathbb{H} \otimes M_2(\mathbb{R}) = M_2(\mathbb{H})$. The Clifford ~~alg~~ module is $\mathbb{H} \oplus \mathbb{H}$ and is the restriction of the complex irred. repn.

Important point: Over a Riemann surface given a C^∞ vector bundle with hermitian product and holomorphic structure, $D: E \to E \otimes T^{0,1}$ I get a Dirac operator given by $D + D^*$ on $E \oplus E \otimes T^{0,1}$.

June 6, 1982

What the Dirac operator looks like: One is over a Riemannian manifold and the symbol of the operator is a map $T^* \otimes S \to S$ or $T^* \longrightarrow \mathrm{End}(S)$, $p \mapsto p\cdot\sigma$ such that $p\cdot\sigma$ is self-adjoint for p real and $(p\cdot\sigma)^2 = |p|^2$. Let's concentrate on the constant coeff. cases, in which case $p\cdot\sigma = \sum p_\mu \gamma_\mu$ where the γ_μ are self-adjoint operators on S satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

For example in 3 dims. we can take the γ_μ to be the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ acting on $\mathbb{C}^2$. In 3 dimensions the Clifford alg. has the central elt. $\gamma_1\gamma_2\gamma_3$ and so there are 2 Clifford modules each of dim. 2, which are distinguished by $\gamma_1\gamma_2\gamma_3 = \pm i$

In even dims. the simplest description of the Clifford module is to put a complex structure on T^* compatible with the inner product, i.e. choose an isom. of T^* with the underlying Euclidean v.s. of a f.d. Hilbert space V. Then $S = \Lambda V$ and $\sigma_v = e(v) + i(v^*)$. In this case $S = S^+ \oplus S^-$, where $S^+ = \Lambda^{\text{even}} V$, $S^- = \Lambda^{\text{odd}} V$, and σ_v is of odd degree. Choose an orthonormal basis $v_1, \ldots, v_n$ for V, then $v_1, iv_1, v_2, iv_2, \ldots$ is an orth. base for T^* and so we get anti-commuting operators $\gamma_1 = \sigma_{v_1}$, $\gamma_2 = \sigma_{iv_1}$, ... of square 1.

Let's do the calculation carefully for $\mathbb{R}^2 = \mathbb{C}$.

$$\gamma_1 = a_1^* + a_1 \qquad \gamma_2 = i a_1^* - i a_1$$

$$\gamma_1\gamma_2 = -i a_1^* a_1 + i a_1 a_1^* = \begin{cases} i & \text{on } \Lambda^0 V \\ -i & \text{on } \Lambda^1 V \end{cases}$$

$$\gamma_1 \partial_x + \gamma_2 \partial_y = a_1^*(\partial_x + i\partial_y) + a_1(\partial_x - i\partial_y) = 2\left(a_1^* \frac{\partial}{\partial \bar z} + a_1 \frac{\partial}{\partial z}\right)$$

Next the operators $\gamma_1\gamma_2, \gamma_3\gamma_4, \dots$ all commute and their common eigenvectors are the natural basis $v_{i_1}\wedge\dots\wedge v_{i_k}$ for ΛV. Especially interesting is the operator

$$\gamma_1\gamma_2\gamma_3\gamma_4\dots\gamma_{2n}$$

which anti-commutes with all the γ_i and has square $(-1)^n$. Its eigenspaces are S^+, S^-.

For example if $n=2$, then $\gamma_1\gamma_2\gamma_3\gamma_4$ has square 1, has eigenvalue -1 on $\Lambda^0+\Lambda^2$, 1 on Λ^1.

So now I have a picture of the constant coeff. Dirac operator $\sum_i \gamma_\mu \frac{1}{i}\partial_\mu$. I'd like next to understand the possible 0-th order terms that can be added to this operator, especially the ones obtained from gauge transf.

Gauge transformations are mult. by e^{iX}, whence

$$\sum \gamma_\mu \tfrac{1}{i}\partial_\mu \longmapsto \sum \gamma_\mu\left(\tfrac{1}{i}\partial_\mu + \partial_\mu X\right)$$

so the 0th order term is $\sum \gamma_\mu a_\mu$ where a_μ is a real vector. The possible 0th order operators B which of odd degree relative to the grading and self-adjoint are functions with values in $\mathrm{Hom}(S^+,S^-)\cong \mathbb{C}^4\cong\mathbb{R}^8$ in 4 dim. So we see gauge transf. don't give very much in this case.

But for 2-dims. $\mathrm{Hom}(S^+,S^-)=\mathbb{C}\cong\mathbb{R}$ so that operators of the form $e^{-iX}\left(\sum_i \gamma_\mu \frac{1}{i}\partial_\mu\right)e^{iX}$ will at least locally give all operators with the Dirac symbol.

Therefore I now know what Dirac operators are on a Riemann surface: They are operators of the form $D+D^*$ on $E\oplus E\otimes T^{0,1}$, where $D: E\to E\otimes T^{0,1}$ is a holomorphic structure on E, and E is given a metric.

So the space of Dirac operators $= \mathcal{A}$.

Now I want to understand what the Ward identities are. These are some kind of consequence of gauge or chiral invariance. So review the standard Fermion integration formulas

$$\int e^{-\bar{\psi}_i a_{ij} \psi_j} [d\psi][d\bar{\psi}] = \det(a_{ij})$$

$$\frac{\int e^{-\bar{\psi}_i a_{ij}\psi_j}(-\bar{\psi}_i \psi_j)}{\int e^{-\bar{\psi}_i a_{ij}\psi_j}} = \frac{\partial}{\partial a_{ij}} \log\det = (a_{ij})^{-1}_{ji}$$

$$\therefore \quad \boxed{\frac{\int e^{-\bar{\psi} A \psi}\ \psi_i \bar{\psi}_j}{\int e^{-\bar{\psi} A \psi}} = (A^{-1})_{ij}}$$

The action is going to be something like

$$S = -i \int d^4x\, \bar{\psi} (\underbrace{\gamma_\mu D_\mu}_{\not{D}} - m)\psi$$

where $\not{D}$ has the form

$$\not{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \qquad\qquad D : S^+ \longrightarrow S^-$$

Let γ_5 be the operator which is $+1$ on S^+ and -1 on S^-. Then $\gamma_5 \not{D} = -\not{D}\gamma_5$, so $\not{D}$ is conjugate to $-\not{D}$ and the eigenvalues occur in pairs. In fact γ_5 set up an isomorphism of the eigenspace for λ with the eigenspace for $-\lambda$. I am used to this calculation in terms of Hodge theory.

Let's go over this. I choose an orthonormal basis of eigenfns. φ_i for D^*D in S^+, and ψ_i for DD^* in S^- so arranged that φ_i $i \leq 0$, ψ_i $i \leq 0$ are the harmonics. Also can arrange $D\varphi_i = \lambda_i \psi_i$ $i \geq 1$, and $D^*\psi_i = \lambda_i \varphi_i$ for $i \geq 1$. Then

$$\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \begin{pmatrix} \varphi_i \\ \pm\psi_i \end{pmatrix} = \begin{pmatrix} \pm D^*\psi_i \\ D\varphi_i \end{pmatrix} = \lambda_i \begin{pmatrix} \pm\varphi_i \\ \psi_i \end{pmatrix} = \pm\lambda_i \begin{pmatrix} \varphi_i \\ \pm\psi_i \end{pmatrix}$$

For the zero eigenvalue for $\not{D}$ we get $\operatorname{Ker} D \oplus \operatorname{Ker} D^*$ and this splitting is given by the eigenvalues of γ_5.

In the above I am supposing $\not{D}$ self-adjoint which means that $\not{D} = \gamma_\mu (\tfrac{1}{i}\partial_\mu)$ probably, and the action should be

$$S = \int d^4x \, \bar{\psi} (\not{D} - m) \psi$$

and then we have the fermion integration formula

$$\int (d\psi)(d\bar{\psi}) \, e^{-S} = \det(\not{D} - m).$$

In order to derive the chiral Ward identity, we consider a change of variable in the fermion integral, namely

$$\psi \longmapsto e^{-i\alpha\gamma_5}\psi$$
$$\bar{\psi} \longmapsto \bar{\psi} e^{-i\alpha\gamma_5}$$

Presumably multiplying ψ, $\bar{\psi}$ by unitaries shouldn't change $(d\psi)(d\bar{\psi})$, and so we should get

$$\int (d\psi)(d\bar{\psi}) \, e^{-S} = \int (d\psi)(d\bar{\psi}) \, e^{-\int d^4x \, \bar{\psi} \left(e^{-i\alpha\gamma_5} (\not{D} - m) e^{-i\alpha\gamma_5} \right) \psi}$$

or

$$\det(\not{D} - m) = \det\left(e^{-i\alpha\gamma_5} (\not{D} - m) e^{-i\alpha\gamma_5} \right)$$
$$= \det\left(\not{D} - m e^{-2i\alpha\gamma_5} \right)$$

Treat α as an infinitesimal so that

$$\det\left(\not D - m(1-2i\alpha\gamma_5)\right) = \det\left((\not D - m)\left[1 + m\frac{2i\gamma_5}{\not D - m}\alpha\right]\right)$$

and since this is $\det(\not D - m)$ we get

$$2im\alpha \, \mathrm{Tr}\left(\frac{\gamma_5}{\not D - m}\right) = 0$$

Now for the non-zero eigenvalues of $\not D$, γ_5 is an off-diagonal operator, so the above trace is calculated over the 0 eigenvalues:

$$2im\alpha \, \frac{n_+ - n_-}{-m} = 0$$

This is false because $n_+ - n_-$ is the index and it [redacted] can be non-zero. What this means is that the assumption that $(d\psi)(d\bar\psi)$ is unchanged is wrong. I'll now go over the derivation of the chiral Ward identity to understand what the anomaly term is.

$$S = \int \bar\psi (\not D - m)\psi$$

$$\delta\psi = -i\gamma_5\psi\,\delta\alpha \qquad \delta\bar\psi = -i\bar\psi\gamma_5\,\delta\alpha$$

$$\cancel{\delta S = -i\delta\alpha \int \bar\psi(\not D - m)\gamma_5} \qquad \text{too hard.}$$

June 7, 1982.

$$\not{D} = \left(\begin{array}{c|c} 0 & D^* \\ \hline D & 0 \end{array}\right) \quad \text{on } S = S^+ \oplus S^- \quad \text{where } D : S^+ \to S^-$$

$$\gamma_5 = \left(\begin{array}{c|c} 1 & \\ \hline & -1 \end{array}\right)$$

Let's begin with $\int [d\psi\, d\bar{\psi}]\, e^{-\int \bar{\psi} \not{D} \psi\, d^4x} = \det(\not{D})$

and make the substitution $\psi \mapsto \theta\psi \quad \bar{\psi} \mapsto \bar{\psi}\theta$

where $\theta = 1 + \varepsilon\gamma_5$ and ε is an endomorphism of S commuting with the symbol of $\not{D}$. (Think of $D : E \to E \otimes T^{0,1}$; then ε is an endomorphism of E). Working to first order in ε

$$\int \bar{\psi}(1+\varepsilon\gamma_5)\not{D}(1+\varepsilon\gamma_5)\psi - \int \bar{\psi} \not{D} \psi$$

$$= \int \bar{\psi}\, [\not{D}, \varepsilon]\gamma_5 \psi$$

and $\quad [\not{D}, \varepsilon] = \sum [\gamma_\mu \partial_\mu, \varepsilon] = \sum \gamma_\mu (\partial_\mu \varepsilon)$. Hence

$$\det\left(\not{D} + [\not{D}, \varepsilon]\gamma_5\right) = \int [d\psi\, d\bar{\psi}]\, e^{-\int \bar{\psi}\not{D}\psi\, dx} \int \left(-\bar{\psi}[\not{D},\varepsilon]\gamma_5\psi\right) dy$$
$$- \det(\not{D})$$

The Ward identity without anomaly correction says this change of variable doesn't change the integral, hence the above expression is zero. This expression divided by $\det \not{D}$ is

$$\int dy\ \mathrm{tr}\Big(\underbrace{\langle \psi(y)\bar{\psi}(y)\rangle}_{\text{diagonal part of } \langle \psi(x)\bar{\psi}(y)\rangle = \langle x | \not{D}^{-1} | y\rangle} \gamma_\mu \partial_\mu \varepsilon\, \gamma_5\Big)$$

The way this is written is [redacted] to integrate by parts thinking of ε as a fn. You get

$$\partial^\mu \langle j^5_\mu(y) \rangle = 0$$

where $\quad j^5_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$. This is wrong because of the anomaly.

Now I can go back and try to make direct sense of this in the Riemann surface situation. So the first thing I should be able to do is to separate the D and the D^*. Since

$$\theta \not{D} \theta = \begin{pmatrix} e^{\varepsilon} & \\ & e^{-\varepsilon} \end{pmatrix} \begin{pmatrix} & D^* \\ D & \end{pmatrix} \begin{pmatrix} e^{\varepsilon} & \\ & e^{-\varepsilon} \end{pmatrix} = \begin{pmatrix} & e^{\varepsilon} D^* e^{-\varepsilon} \\ e^{-\varepsilon} D e^{+\varepsilon} & \end{pmatrix}$$

it's clear that what I am doing is to apply a gauge transformation to D, and a different one to D^* [redacted] (unless ε is [redacted] hermitian)

Now I can pose the following question. Define the current using the finite part of the Green's function. Hence

$$\delta \log \det D = \operatorname{Tr}(D^{-1} \delta D)$$
$$= \int \operatorname{tr}\left(FP\langle y | D^{-1} | y \rangle \, \delta D(y) \right)$$

Does this expression vanish when δD is the result of an infinitesimal gauge transformation? Do locally over $\mathbb{C}/\Gamma$ with $D = \partial_{\bar z} + \alpha$. Then

$$FP\langle y | D^{-1} | y \rangle = \tfrac{1}{\pi}(\beta - \alpha^*)(y)$$

For a gauge transformation $\delta D = [D, \varepsilon] = (\partial_{\bar z} \varepsilon + [\alpha, \varepsilon])\, d\bar z$ so we are asking about

$$\partial_{\bar z}(FP) + [\alpha, FP]\, d\bar z \in \Gamma(\operatorname{End}(E) \otimes T^{1,1})$$

and indeed this [redacted] could be the curvature of D.

Let's be more precise. Put $J(y) = FP\langle y | D^{-1} | y \rangle$, so that $J \in \Gamma(\operatorname{End}(E) \otimes T^{1,0})$. Then $\delta D = [D, \varepsilon] = \partial_{\bar z} \varepsilon + [\alpha, \varepsilon]\, d\bar z$ $\in \Gamma(\operatorname{End} E \otimes T^{0,1})$.

$$\int \operatorname{tr}(J [D, \varepsilon]) = \int \operatorname{tr}\left(J(\partial_{\bar z} \varepsilon + [\alpha, \varepsilon])\right) d\bar z$$
$$= \int \operatorname{tr}\left((-\partial_{\bar z} J + [J, \alpha]) \varepsilon\right) d\bar z = -\int \operatorname{tr}([D, J]\, \varepsilon)$$

Thus we are getting the image of J under the canonical map

$$\Gamma\, \mathrm{End}(E)\otimes T^{1,0} \xrightarrow{[D,\ ]} \Gamma\, \mathrm{End}(E)\otimes T^{0,1}$$

which is the $\bar\partial$-operator for $\mathrm{End}(E)\otimes T^{1,0}$. Now I recall that to construct the FP you choose a ∂-operator on E; different choices for ∂-operator are described by elts of $\Gamma(\mathrm{End}\,E \otimes T^{1,0})$ which get added onto J. Hence we can arrange $[D,J]$ to be non-zero, in fact locally anything we want.

Hence the anomaly formula must be for the FP calculated using the heat kernel regularization. ~~NO~~ NO, the $\bar\partial$-operator and the ∂ operator combine to give a connection on the bundle and the anomaly is the curvature of this connection essentially.

June 8, 1982:

Adler anomaly: Let $D: E \longrightarrow E \otimes T^{0,1}$ be invertible and [redacted] let $J(z) = FP\langle z|D^{-1}|z\rangle$ where FP is defined by the heat kernel regularization. Then $J \in \Gamma(\mathrm{End}\,E \otimes T^{1,0})$ and I want to compute $[D,J] \in \Gamma(\mathrm{End}\,E \otimes T^{1,1})$.

Recall [redacted] how J is computed locally. One has

$$\langle z|D^{-1}|z'\rangle = \frac{F(z,z')}{\pi(z-z')}dz' \qquad\qquad D = (\partial_{\bar z} + \alpha)\,d\bar z$$

and from $DD^{-1} = \mathrm{Id}$ we get

$$(1) \qquad (\partial_{\bar z} + \alpha(z))F(z,z') = 0.$$

[redacted] $D^{-1}D = I$ means that for φ [redacted] $\in \Gamma(E$[redacted]$)$ with small support

$$\int \frac{F(z,z')}{\pi(z-z')}dz'\left(\frac{\partial}{\partial\bar z'} + \alpha(z')\right)\varphi(z')\,d\bar z' \quad = \quad \varphi(z)$$

$$= \int \frac{F(z,z')}{\pi(z-z')}\left(-\frac{\overleftarrow{\partial}}{\partial\bar z'} + \alpha(z')\right)\varphi(z')\,dz'd\bar z'$$

and hence we get

$$(2) \qquad F(z,z')\left(-\overleftarrow{\partial}_{\bar z'} + \alpha(z')\right) = 0.$$

[redacted] The equations (1) and (2) say that $F(z,z')dz'$ as a section of $pr_1^*E \otimes pr_2^*(E^* \otimes T^{1,0})$ over $M \times M$ is holomorphic in both variables. To see this suppose we trivialize E locally by a holomorphic frame $g(z)$; recall that I have already trivialized E, so that holom. sections are vector fns. killed by $\partial_{\bar z} + \alpha$, hence $g(z)$ is a matrix fn. such that $(\partial_{\bar z} + \alpha)g(z) = 0$, i.e.

$$\alpha = -(\partial_{\bar z}g)g^{-1} = g\,\partial_{\bar z}g^{-1}$$

In the new frame $F(z,z')$ is replaced by

$$g^{-1}(z)\,F(z,z')\,g(z')$$

Then

$$\frac{\partial}{\partial \bar z}\left[g^{-1}(z)F(z,z')g(z')\right] = g^{-1}(\partial_{\bar z} + \text{[redacted]}\underbrace{g\,\partial_{\bar z} g^{-1}}_{\alpha})F\,g(z') = 0$$

$$\frac{\partial}{\partial \bar z'}\left[g^{-1}(z)F(z,z')g(z')\right] = g^{-1}(z)\left[\frac{\partial}{\partial \bar z'}F\cdot g(z') + F\cdot\partial_{\bar z'}g(z')\right]$$

$$= g^{-1}(z)\left[\partial_{\bar z'}F + F\cdot\underbrace{\partial_{\bar z'}g\;g(z')^{-1}}_{-\alpha(z')}\right]g(z') = 0$$

[redacted]

Recall the formula for the FP. Expand

$$F(z,z') = 1 + \beta(z')(z-z') + \gamma(z')\overline{(z-z')} + O((z-z')^2)$$

From (1) we know $\gamma = -\alpha$. We remove from this

$$F_b(z,z') = 1 + \alpha(z')^*(z-z') - \alpha(z')\overline{(z-z')}$$

which is the flat to first order trivialization. Then

$$\pi J = \Big(\beta - \alpha^* + \underbrace{\text{scalar part depending on the metric on } \mathcal{M}}_{\text{Call this } s.}\Big)dz$$

Then

$$\pi[D,J] = \underbrace{[\partial_{\bar z}+\alpha,\ \beta-\alpha^*+s]}_{\partial_{\bar z}\beta \overset{+[\alpha,\beta]}{\wedge} - \partial_{\bar z}\alpha^* - [\alpha,\alpha^*] + \partial_{\bar z}s}\,d\bar z\,dz$$

I need a formula for $\partial_{\bar z}\beta$, [redacted] which should come from the fact that F is [redacted] holomorphic in z'. In effect suppose $\alpha = 0$. Then $F(z,z')$ is holomorphic in ~~$F(z,z')$~~ z, z' and $F-1$ vanishes along the divisor ~~$(z-z')$~~ $z-z'=0$, hence $\frac{F-1}{z-z'}$ is a holomorphic function of z, z'. Hence it's restriction to $z-z'=0$, which is β, is a holom. fn. of z: $\partial_{\bar z}\beta = 0$.

The generalization of this should be [redacted] that β as an endom.

of E is holomorphic:

$$(*) \qquad \partial_{\bar z}\beta + [\alpha, \beta] = 0.$$

Let's check this with $g(z)$ as above. Then $g(z)^{-1}F(z,z')g(z')$ is holomorphic in both z, z', hence we get a holom. fn. of z' by diff. using $\partial/\partial z$ and setting $z = z'$.

$$\frac{\partial}{\partial z} g(z)^{-1}F(z,z')g(z')\Big|_{z=z'} = -g^{-1}\partial_z g\, \cancel{g^{-1}g} + g^{-1}\beta g$$

$$= g^{-1}\beta g - g^{-1}\partial_z g .$$

So
$$0 = \partial_{\bar z}\left(g^{-1}\beta g - g^{-1}\partial_z g\right) = \underbrace{\partial_{\bar z} g^{-1}}_{g^{-1}\alpha}\beta g + g^{-1}\partial_{\bar z}\beta g + g^{-1}\beta\underbrace{\partial_{\bar z} g}_{-\alpha g} - \partial_{\bar z}(g^{-1}\partial_z g)$$

$$\partial_{\bar z}\beta + [\alpha, \beta] - g\,\partial_{\bar z}(g^{-1}\partial_z g)g^{-1} = 0$$

So it seems $(*)$ above is wrong. But

$$\partial_{\bar z} g + \alpha g = 0 \implies \partial_{\bar z}\partial_z g + (\partial_z \alpha) g + \alpha\,\partial_z g = 0$$

$$-g\,\partial_{\bar z}(g^{-1}\partial_z g)g^{-1} = -\underbrace{g\,\partial_{\bar z} g^{-1}}_{\alpha}\cdot \partial_z g\, g^{-1} - \underbrace{\partial_{\bar z}\partial_z g}_{(\partial_z\alpha)g + \alpha\partial_z g}\cdot g^{-1}$$

= − [redacted] $\cancel{\alpha\,\partial_z g\, g^{-1}} + \partial_z\alpha + \cancel{\alpha\,\partial_z g\, g^{-1}}$

So the correct formula is

$$(3) \qquad \boxed{\partial_{\bar z}\beta + [\alpha, \beta] + \partial_z \alpha = 0}$$

And so

$\pi[D, J] = \big(-\partial_z\alpha - \partial_{\bar z}\alpha^* - [\alpha, \alpha^*]$ [redacted] $+ \partial_{\bar z} s\big)\, d\bar z\, dz$

$$[D, J] = \left(\partial_z\alpha + \partial_{\bar z}\alpha^* + [\alpha, \alpha^*] - \partial_{\bar z} s\right)\frac{dz\,d\bar z}{\pi}$$

Notice that we haven't used the fact that α^* is the adjoint of α, so consequently this formula will hold in general once a connection is given extending the given holomorphic structure.

Let's review the scalar term, which compensates for changes in the local coord. Thus if instead of z we use $w = h(z)$ to compute the FP we get a change of

$$-\frac{dw'}{w-w'} + \frac{dz'}{z-z'} = \frac{dz'}{z-z'}\left\{\frac{-h'(z')}{h'(z')+\frac{1}{2}h''(z')(z-z')+\cdots} + 1\right\}$$

$$= \frac{dz'}{z-z'}\,\frac{\frac{1}{2}h''(z')(z-z')}{h'(z')+\frac{1}{2}h''(z')(z-z')} \longrightarrow \frac{1}{2}\frac{h''(z')\,dz'}{h'(z')}$$

On the other hand if $|\partial_z|^2 = g(z)$ $\quad |\partial_w|^2 = \tilde{g}(w)$, then

$$\tilde{g}(w) = \left|\frac{\partial}{\partial z}\Big/\frac{dw}{dz}\right|^2 = g(z)/|h'(z)|^2$$

so $\quad \partial \log \tilde{g} = \partial \log h - \frac{h''}{h'}\,dz$. Thus if we remove $\frac{1}{\pi}\left(\frac{dz'}{z-z'} - \frac{1}{2}\partial \log g(z')\right)$ from the Green's function we get something invariant under coord. changes so

$$\boxed{S\,dz = \frac{1}{2}\partial \log g(z) \qquad g(z) = |\partial/\partial z|^2}$$

Let's check consistency of the formulas.

$$\nabla = (\partial_z - \alpha^*)\,dz + (\partial_{\bar z} + \alpha)\,d\bar z$$

$$\nabla^2 = [\partial_z - \alpha^*, \partial_{\bar z} + \alpha]\,dz\,d\bar z$$

$$= \left(\partial_z \alpha + \partial_{\bar z}\alpha^* + [\alpha, \alpha^*]\right)dz\,d\bar z$$

We know

$$\int \frac{i}{2\pi}\,\mathrm{tr}(\nabla^2) = \deg E.$$

On the other hand

$$-\partial_{\bar z} S\,(dz\,d\bar z) = \frac{1}{2}\bar\partial\partial \log|\partial/\partial z|^2$$

is the $\frac{1}{2}$ curvature form for the tangent bundle, so when multiplied by $\frac{i}{2\pi}$ and integrated you get

$$\int \frac{i}{2\pi}\left(-\partial_{\bar z} S \; dz d\bar z\right) = \frac{1}{2}(2-2g) = 1-g.$$

Thus

$$\int \frac{i}{2\pi} \mathrm{tr}[D, \pi J] = \deg E + \mathrm{rank} E (1-g)$$

which looks good. In the case I have been looking at D is invertible, so the above is 0, but clearly we hope to define J in the general case so that the anomaly formula holds.

There is an interesting connection with the index thm. Recall that given D one chooses a parametrix P and then $\mathrm{Index} = \mathrm{Tr}\,[D,P]$ in a certain sense. Precisely $DP = I - K'$, $PD = I - K^0$ and $\mathrm{Index} = \mathrm{tr}\, K^0 - \mathrm{tr}\, K' = \mathrm{tr}(DP-I) - \mathrm{tr}(PD-I)$. Now the regularization process assigns to P, which is just a Green's function, the finite part of P along the diagonal, which is the current J. Then the anomaly formula gives

$$\mathrm{Index} = \int \frac{i}{2\pi} \mathrm{tr}\,[D, \pi J]$$

Further consistency results. I have been proposing

$$\delta \log\det(D) = \mathrm{Tr}(J_D\, \delta D)$$

as a definition of $\det(D)$, but this is not possible unless integrability conditions are satisfied. I should write it

$$d \log\det(D) = \mathrm{Tr}(J_D\, dD),$$

and then you must know the diff'l form on the right is closed, i.e.

$$d\,\mathrm{Tr}(J_D dD) = \mathrm{Tr}(dJ_D\, dD) = 0.$$

If I use the local expression

$$\pi J_D = s - \alpha^* + \beta$$

then

$$d\,\pi J_D = \quad -d\alpha^* + d\beta.$$

Presumably if I use the same ∂-operator as α varies, (then $d\alpha^* = 0$) and I should get a $\det D$, hence it is probably generally true that $\operatorname{Tr}(d\beta\, d\alpha) = 0$, however if α^* = conj of α, then it won't work.

What about gauge invariance? The formula

$$\delta \log \det(D) = \operatorname{Tr}(J_D\, \delta D)$$

computes the change for any δD in particular ones of the form $\delta D = [D, \varepsilon]$ comming from infinitesimal gauge transformations. Then

$$\operatorname{Tr}(J_D [D,\varepsilon]) = -\operatorname{Tr}([D, J_D]\, \varepsilon)$$

needn't be zero because of the anomaly formula. However it should be true that the analytic torsion is invariant under gauge transformations, and we should check this.

$$\delta\left(-\zeta'_{D^*D}(0)\right) = \operatorname{Tr}(J_D\, \delta D) + \text{c.c.}$$

so we want $\operatorname{Tr}(J_D\, \delta D)$ to be purely imaginary when $\delta D = [D, \varepsilon]$ and $\varepsilon = -\varepsilon^*$. Calculation above shows that $[D, J] = K \in \Gamma(\operatorname{End} E \otimes T^{1,1})$ satisfying $K^* = -K$.

$$\overline{\int \operatorname{tr}(K\varepsilon)} = \int \operatorname{tr}(\varepsilon^* K^*) = \int \operatorname{tr}(\varepsilon K) = \int \operatorname{tr}(K\varepsilon)$$

so something is funny.

The problem is caused by the fact that the formula

$$\partial_{\bar z} \frac{1}{\pi(z - z')} = \delta(z - z')$$

is defined by integration wrt $dx\,dy$, and you have been using it with respect to $dz\,d\bar z$ (see p. 685 middle).

So I seek a function $G(z,z')$ such that $G(z,z')dz'$ is a Schwarz kernel for $\bar\partial$. Thus I want

$$\bar\partial \int G(z,z')\, dz'\, \alpha(z')d\bar z' = \alpha(z)dz$$

or

$$\int \partial_{\bar z} G(z,z')\, \alpha(z')\, \underbrace{dz'd\bar z'}_{-2i\,dx'dy'} = \alpha(z)$$

so we want

$$\partial_{\bar z} G(z,z') = \frac{1}{-2i}\,\delta(z-z').$$

Since I know that

$$\partial_{\bar z}\frac{1}{\pi(z-z')} = \delta(z-z')$$

this means that $G(z,z') = \dfrac{i}{2\pi}\dfrac{1}{z-z'}$. Therefore the correct formula is

$$\boxed{\langle z|\,\bar\partial^{-1}|z'\rangle = \frac{i}{2\pi}\,\frac{dz'}{z-z'}}$$

and lots of formulas have to be modified. But at least the gauge invariance of $-\zeta'_{D^*D}(0)$ for infinitesimal gauge transformations ($\varepsilon = -\varepsilon^*$) checks out.

And now calculation gives

$$J(z) = \frac{i}{2\pi}\left[\tfrac{1}{2}\partial_z \log|\partial/\partial z|^2 + \beta - \alpha^*\right] dz$$

$$[D, J] = \frac{i}{2\pi}\left[\tfrac{1}{2}\,\text{curvature of } T + \text{curv. of } E\right]$$

$$\therefore \int \operatorname{tr}[D,J] = \text{index} \qquad \text{as expected.}$$

June 9, 1982

Yesterday I saw that for an invertible D, if J_D is the current constructed from D^{-1} using a connection extending D, then $[D, J_D] = \frac{i}{2\pi}$ (curvature $\nabla + \frac{1}{2}$ curv. T). ~~So $[D, J_D]$ doesn't depend on D.~~ Now we can construct a current J by regularizing any parametrix G for D, so a natural question is whether one gets the same answer for $[D, J]$.

Another parametrix is of the form $G + K$ where $K(z, z')dz'$ is a smooth kernel. Since the current is a sort of finite part along the diagonal, it should follow that the change in J is the restriction $K(z,z)dz$ of K to the diagonal. Hence the change in $[D, J]$ is an arbitrary element in the image of

$$\Gamma(\mathrm{End}(E) \otimes T^{1,0}) \xrightarrow{[D,\ ]} \Gamma(\mathrm{End}(E) \otimes T^{0,1}).$$

This suggests that there might be a special class of parametrics for which the anomaly formula holds even when the index isn't zero. The obvious first candidate is to require $(z - z')\,G(z, z')dz'$ to be holomorphic in a neighborhood of Δ. The argument yesterday should show that in this case $[D, J]$ is given by the curvature etc. as in the anomaly formula. However one can't require this because then for

$$DG = I - K^1$$
$$GD = I - K^0$$

K^0, K^1 will have 0 restriction to Δ, and hence index $= 0$.

If I compute locally I can assume $\alpha = 0$, in which case

$$\frac{2\pi}{i}(z-z')G(z,z') = 1 + \beta_{z'}(z-z') + O((z-z')^2).$$

The anomaly formula in this case amounts to

$$\partial_{\bar z}\beta(z) = 0.$$

Now
$$G(z,z') = \frac{i}{2\pi}\left(\frac{1}{z-z'} + \beta_{z'} + \frac{O((z-z')^2)}{z-z'} + \cdots\right)$$

$$\partial_{\bar z} G(z,z') = \frac{i}{2}\delta(z-z') + ?$$

Better:

$$G(z,z') = \frac{i}{2\pi}\left(\frac{1}{z-z'} + \beta_{z'} + a_{z'}\frac{z-z'}{2} + b_{z'}\overline{z-z'} + c_{z'}\frac{\overline{z-z'}^2}{2(z-z')} + \cdots\right)$$

$$\partial_{\bar z}G(z,z') = \frac{i}{2\pi}\left(\pi\delta(z-z') + b_{z'} + c_{z'}\frac{\overline{z-z'}}{z-z'} + \cdots\right)$$

A really good parametrix has $c_{z'} = 0$.

$$\partial_{\bar z'}G(z,z') = \frac{i}{2\pi}\Big(-\pi\delta(z-z') + \frac{\partial\beta_{z'}}{\partial\bar z'} + \frac{\partial a_{z'}}{\partial\bar z'}\frac{z-z'}{2}$$ [redacted]

$$+ \frac{\partial b_{z'}}{\partial\bar z'}\overline{z-z'} - b_{z'} + \frac{\partial c_{z'}}{\partial\bar z'}\frac{\overline{z-z'}^2}{2(z-z')} + c_{z'}\frac{\overline{z-z'}}{z-z'} + \cdots$$

Hence if $DG = 1 - K^1$ we conclude

$$K^1(z',z') = -b_{z'} \qquad \times \frac{i}{2\pi}$$

and if $GD = 1 - K^0$ we conclude

$$K^0(z',z') = \frac{\partial\beta_{z'}}{\partial\bar z'} - b_{z'}. \qquad \times \frac{i}{2\pi}$$

[redacted] Therefore the validity of the anomaly formula as above formu[redacted]lated can only hold when the index is zero.

Actually you have been looking at things wrong. In general one has

$$\int \operatorname{tr}[D,J] = 0$$

because J is the difference of two parametrices for D. Let's go over this carefully. We just calculated for an arbitrary parametrix G the index and got (using coords where $\alpha = 0$)

$$\mathrm{Tr}\, K^0 - \mathrm{Tr}\, K^1 = \int \frac{i}{2\pi} \operatorname{tr}(\partial_{\bar z}\beta)\, dz\, d\bar z \quad = \text{Index}$$

On the other hand defining J from this parametrix using the connection $\nabla = (\partial_z - \gamma)dz + \partial_{\bar z} d\bar z$ gives

$$J = \frac{i}{2\pi}\left[(\beta - \gamma)dz + \frac{1}{2}\partial \log |\partial_z|^2\right]$$

hence

$$[D, J] = \frac{i}{2\pi}\Big[\, \partial_{\bar z}\beta\, d\bar z\, dz \underbrace{- \partial_{\bar z}\gamma\, d\bar z\, dz}_{\text{[redacted] curvature of } \nabla} + \frac{1}{2}\bar\partial\partial \log|\partial_z|^2 \Big].$$

Now integrating the 2nd + 3rd term gives $\deg E + \text{rank}\, E \times (1-g)$, and the first gives $-$Index, so $\int \operatorname{tr}[D, J] = 0$ as claimed.

June 10, 1982

Consider the case where D has index 1, and in fact is generic so that $\operatorname{Ker} D$ is 1-diml. and D is onto. In this case the line bundle $\mathcal{L}^*$ is the subline bundle $\operatorname{Ker} D$. We begin by computing the curvature of this line bundle with respect to the obvious metric one gets from the L^2-metric on $V = \Gamma(E)$. This is essentially just calculation the curvature for the subline bundle $\mathcal{O}(-1)$ on projective space $\mathbb{P}V$.

Recall if one has orthonormal coordinates $z_n = \langle e_n |$ on V, then the connection form η is proportional to $\sum \bar{z}_n dz_n$ since parallel translation moves a vector v $\perp$ to $\mathbb{C}v$. Specifically, let's work around the line $\mathbb{C}e_0$ and use the coordinates z_n $n>0$ to describe lines: the line is $\mathbb{C}(e_0 + z_1 e_1 + \cdots)$. Then we have the holom. section $s = e_0 + z_1 e_1 + \cdots\cdots$ of $\mathcal{O}(-1)$ so the curvature is

$$\bar\partial \partial \log |s|^2 = \bar\partial\partial \log\Big(1 + \sum_{n>0} |z_n|^2\Big) = \bar\partial \, \frac{\sum \bar{z}_n dz_n}{1 + |z|^2}$$
$$= \sum_{n>0} d\bar{z}_n dz_n \quad \text{at the point } z_n = 0 \ n>0.$$

We want to pull this form back to our space of $\bar\partial$-operators and see what it is at the operator of interest D. Thus we suppose e_1 is an orthonormal gen. for $\operatorname{Ker} D$ and we will choose the remaining e_n to be eigenfunctions for D^*D. Then we will get an orth. basis for $\Gamma(E \otimes T^{0,1})$, call it e_n' $n>0$, with for $n>0$

$$\begin{cases} De_n = \lambda_n e_n' \\ D^* e_n' = \lambda_n e_n \end{cases} \qquad \lambda_n > 0.$$

Next consider a change in D to $D+\delta D$ and compute the change $\delta\psi$ in the generator $\psi = e_0$ for $\operatorname{Ker} D$.

$$(D+\delta D)(\psi+\delta\psi) = 0 \qquad \langle \psi | \delta\psi\rangle = 0$$

$$\delta D\psi + D\delta\psi = 0 \quad \Longrightarrow \quad \delta\psi = -G\,\delta D\psi$$

where G is the half-inverse for D such that $\operatorname{Im} G = (\operatorname{Ker} D)^\perp$. So $\operatorname{Ker}(D+\delta D)$ is spanned by

$$\psi + \delta\psi = e_0 - \sum_{n>0} e_n \frac{1}{\lambda_n}\langle e_n' | \delta D | e_0\rangle$$

so

$$\delta z_n = \langle e_n | \delta\psi\rangle = -\frac{1}{\lambda_n}\langle e_n' | \delta D | e_0\rangle$$

and hence the curvature of $\mathcal{L}^*$ at D is

$$\sum d\bar z_n \, dz_n = \sum_{n>0}' \frac{\overline{\langle e_n' | dD | e_0\rangle}\,\langle e_n' | dD | e_0\rangle}{\lambda_n^2}.$$

This is to be interpreted in the following way. The quantity $\langle e_n' | \delta D | e_0 \rangle$ is a $\mathbb{C}$-linear function of δD, which we think of as a tangent vector to $\mathcal{A}$ at D. Hence $\delta D \longmapsto \langle e_n' | \delta D | e_0\rangle$ is a $1,0$ form on $\mathcal{A}$ at D which we denote $\langle e_n' | dD | e_0 \rangle$.

Another thing we could say is that the $1,1$ curv. form we obtain is the one corresponding to the hermitian inner product whose norm^2 is

$$\delta D \longmapsto \sum' \frac{1}{\lambda_n^2} |\langle e_n' | \delta D | e_0\rangle|^2 = \| G\,\delta D | e_0\rangle \|^2$$

What would the generalization be for $\operatorname{Ker} D$ p-diml but D still onto? Then instead of e_0 we have e_j with $j \leq 0$ and $\operatorname{Ker}(D+\delta D)$ is spanned by

$$e_j - \sum_{n>0}' e_n \frac{1}{\lambda_n} \langle e_n' | \delta D | e_j \rangle$$

and the curvature form corresponds to

$$\sum_{n>0\geq j} \frac{1}{\lambda_n^2} |\langle e_n' | \delta D | e_j\rangle|^2 = \sum_j \| G\,\delta D e_j \|^2 = \operatorname{tr}(\delta D^* G^* G\,\delta D P)$$

P = projection on $\operatorname{Ker} D$.

So next let's go [redacted] on to the ζ function.

$$\zeta(s) = \mathrm{Tr}\,(DD^*)^{-s}$$

$$\delta\zeta(s) = -s\,\mathrm{Tr}\left((DD^*)^{-s-1}\,\delta(DD^*)\right)$$

$$= -s\,\mathrm{Tr}\left((DD^*)^{-s}(DD^*)^{-1}(\delta D\, D^* + D\,\delta D^*)\right)$$

$$\lim_{s\to 0} -\frac{\delta\zeta(s)}{s} = \mathrm{Tr}\left(\delta D\, D^*(DD^*)^{-1}(DD^*)^{-s}\right) + \mathrm{Tr}\left((DD^*)^{-s}(DD^*)^{-1}D\,\delta D^*\right)\Big|_{s\to 0}$$

So what you are going to have to prove is that the kernel of $D^*(DD^*)^{-1}(DD^*)^{-s}$ when restricted to the diagonal has a nice limit as $s\to 0$, which coincides with the finite part of the Green's function given by $D^*(DD^*)^{-1}$. This should be clear from the heat kernel method, I believe I proved the result from D^*D when D is injective.

(Perhaps a better approach works as follows. Choose a point λ which is not an eigenvalue for D^*D and let P project on the generalized eigenspace below λ, i.e. $P = E_\lambda$ in the sense of the spectral resolution.)

I am trying to calculate $\bar\partial\partial \log\tau$ where $\tau = e^{-\zeta'(0)}$ and I have found that

$$\delta \log\tau = \text{[redacted]} \int tr\,(\delta D\cdot J) + \text{c.c.}$$

where $J = \mathrm{FP}$ for $G = D^*(DD^*)^{-1}$. This shows

$$\partial \log\tau = \int tr\,(J\cdot dD)$$

and for $\bar\partial\partial\log\tau$ we need to understand the anti-holom. variation [redacted] of J in D. I know that J is given by a local formula

$$J = \frac{i}{2\pi}\left((\beta-\alpha^*)\,dz + \partial_z \tfrac{1}{2}\log|\partial_z|^2\right) \qquad D = \partial_{\bar z}+\alpha$$

and the real problem is how β varies wrt α.

The idea to use is that two Green's functions differ by a smooth kernel and that the associated currents have β term differing by the restriction of the smooth kernel to the diagonal.

So now we make a variation δD and calculate δG. The first idea is to use the Green's function $\quad G - G\delta D G + G\delta D G \delta D G - \cdots$ which varies holomorphically in δD.

$$(*) \qquad (G - G\delta D G)\, e_n' = \frac{1}{\lambda_n} e_n - \sum_{m>0} \frac{e_m}{\lambda_m} \langle e_m' | \delta D | e_n \rangle \qquad n>0$$

This has image contained in $\operatorname{Im} G = (\operatorname{Ker} D)^\perp$, so we have to modifying it becomes $\perp$ to $\operatorname{Ker}(D+\delta D)$ which is spanned by

$$(**) \qquad e_0 - \sum_{m>0} \frac{e_m}{\lambda_m} \langle e_m' | \delta D | e_0 \rangle$$

So add $\rho_n e_0$ to $(*)$, so that the inner product of $(*) + \rho_n e_0$ and $(**)$ is zero, i.e.

$$\rho_n - \frac{1}{\lambda_n^2} \overline{\langle e_n' | \delta D | e_0 \rangle} = 0$$

Thus we have

$$(G + \delta G)\, e_n' = \frac{1}{\lambda_n} e_n - \sum_{m>0} \frac{e_m}{\lambda_m} \langle e_m' | \delta D | e_n \rangle + e_0 \frac{1}{\lambda_n^2} \overline{\langle e_n' | \delta D | e_0 \rangle}$$

Hence the part of δG which is anti-holom. in δD is the operator

$$+ |e_0\rangle \sum \frac{1}{\lambda_n^2} \overline{\langle e_n' | \delta D | e_0 \rangle} \langle e_n' |$$

So $\qquad \bar\partial \partial \log \tau = \bar\partial \int \operatorname{tr}(J\, dD) = \int \operatorname{tr}(\bar\partial D \cdot dD)$

$$\text{the } \bar\partial\beta \text{ part} = + \int \operatorname{tr}\Big(\langle x | e_0 \rangle \sum_{n>0} \frac{1}{\lambda_n^2} \overline{\langle e_n' | dD | e_0 \rangle} \langle e_n' | dD | x \rangle \Big)$$

$$= +\sum_{n>0} \frac{1}{\lambda_n^2} \overline{\langle e_n' | dD | e_0 \rangle} \langle e_n' | dD | e_0 \rangle$$

So this cancels the curvature of $\mathcal{L}$ and so we end up with

$$\text{curvature of usual metric on } \mathcal{L} \;+\; \bar\partial \partial \log \tau \;=\; \int \frac{i}{2\pi} \operatorname{tr}\Big(\overbrace{-\delta\alpha^* dz}^{\delta D^*} \; \overbrace{\delta\alpha \, d\bar z}^{\delta D} \Big)$$