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October 28, 1982

Problem: The key to determinants is the formula
$$\delta \log \det D = \operatorname{Tr}_{reg}(D^{-1}\delta D)$$
a regularization process for the trace defines a 1-form on the space of operators, and [redacted] when this form is closed a determinant function is locally defined up to a constant factor. [redacted].

So clearly an interesting question is to understand the possible regularization processes and when they give closed 1-forms. Let's look at this for a $\bar\partial$-operator where the variations in the operator occur in a small coordinate patch. I would like eventually to handle variations which are first order operators.

We begin with a $\bar\partial$-operator in [redacted] $|z| \leq 1$ and suppose boundary conditions are given which specify a Green's function for the $\bar\partial$-operator. [redacted] Let D_0 be the $\bar\partial$-operator, G_0 the Green's function. Then for $D = D_0 + B$ we have the Green's fn.
$$G = G_0 - G_0 B G_0 + G_0 B G_0 B G_0 - \cdots\cdots .$$
The one-form of interest is $\operatorname{Tr}_{reg}(G\delta B)$.

It seems that the first thing to understand is the nature of the singularities of $G\delta B$. [redacted] From a [redacted] local viewpoint a $\bar\partial$-operator D is of the form
$$D = \varphi\, \partial_{\bar z}\, \varphi^{-1} \qquad \varphi \text{ smooth invertible}$$
hence
$$G(z,z') = \frac{\varphi(z)\varphi(z')^{-1}}{z-z'} + \text{smooth kernel}$$
$$\equiv \frac{1}{z-z'}\left\{ \varphi\big|_{z'} \text{[redacted]} + \partial_z\varphi\big|_{z'}(z-z') + \partial_{\bar z}\varphi\big|_{z'}(\overline{z-z'}) + \cdots \right\}\varphi(z')^{-1}$$
$$\equiv \frac{1}{z-z'}\sum_{n=0}^{\infty}\frac{(\overline{z-z'})^n}{n!}\left(\partial_{\bar z}^n\varphi\cdot\varphi^{-1}\right)\Big|_{z'} \qquad \text{mod smooth kernels}.$$
Here I have dropped the π and will do integrals wrt $\frac{d^2z}{\pi}$.

Next consider δB of 0-th order. Then $\delta B = \delta\alpha(z)$ and so

any parametrix has the form $\frac{1}{z-z'}$ + smooth.
Thus
$$G_b = \frac{F_b(z,z')}{z-z'} + \text{smooth}$$
and this will be a parametrix $\Leftrightarrow$ $F_b - I$ is divisible by $z-z'$.

Let's look at the case where $r^2(z,z') = |z-z'|^2$, $z'=0$ so that the geodesic from 0 to z is the straight line. Then $F_b(z,0)$ satisfies
$$\nabla_{r\frac{d}{dr}} F_b = 0 \qquad F_b = I \quad \text{at} \quad z=0.$$
If
$$\nabla = (\partial_z + \beta)\,dz + \partial_{\bar z}\,d\bar z, \text{ this becomes}$$
$$\left[z(\partial_z+\beta) + \bar z \partial_{\bar z}\right] F_b = 0$$
or
$$\left[z\partial_z + \bar z \partial_{\bar z}\right] F_b = -z\beta F_b.$$
Now you solve this by a power series
$$F_b = I + F_1 + F_2 + \cdots$$
where F_n is homogeneous of degree n in $z, \bar z$. Then
$$nF_n = \left(z\partial_z + \bar z \partial_{\bar z}\right) F_n = -z\left(\beta F_b\right)_{n-1}$$
which shows that F_n is divisible by z.

General case: Put $u = r^2(z,0)$. Then we have
$$|\partial_z u|^2 = \rho u \qquad u = |z|^2 e^w$$
where w is smooth. Also
$$\nabla_{r\frac{d}{dr}} = \frac{2}{\rho}\left[\partial_{\bar z} u \cdot (\partial_z + \beta) + \partial_z u \cdot \partial_{\bar z}\right]$$
$$\partial_z u = \bar z\left(1 + z\partial_z w\right)e^w$$
so $\nabla_{r\frac{d}{dr}} F_b^{\emptyset} = 0$ becomes
$$\left[z\left(1 + \bar z \partial_{\bar z} w\right)(\partial_z + \beta) + \bar z\left(1 + z\partial_z w\right)\partial_{\bar z}\right] F_b = 0$$

or $$(z\partial_z + \bar{z}\partial_{\bar{z}})F_b = -z\underbrace{\left[\beta + \partial_{\bar{z}}w\,\bar{z}\partial_z + \partial_{\bar{z}}w\beta\,\bar{z} + \partial_z w\,\bar{z}\partial_{\bar{z}}\right]}_{\text{does not lower degree.}}F_b$$

Hence as before you get
$$nF_n = (z\partial_z + \bar{z}\partial_{\bar{z}})F_n = z \cdot \text{expression in } F_j \quad j<n$$
which shows that F_n is divisible by z, and hence the flat Green's fn. is a parametrix.

It would be nice to understand the algebra behind parametrices and determinants. Suppose we have an operator D and a quasi-inverse P, better an inverse modulo compacts:
$$PD = I - K^0 \qquad DP = I - K^1$$
Here K^0, K^1 are compact but not of trace class, so you can't compute the index as $\mathrm{tr}(K^0) - \mathrm{tr}(K^1)$. However one has an algebraic process for refining P to a P_n such that
$$P_n D = I - (K^0)^{n+1} \qquad DP_n = I - (K^1)^{n+1}$$
and then the index can be computed as
$$\mathrm{tr}\,(K^0)^{n+1} - \mathrm{tr}\,(K^1)^{n+1}.$$
The algebra is explained by thinking of P as a chain homotopy between Id and $K = K^0, K^1$ for the complex
$$E \xrightarrow{D} F.$$
Then P_n is a homotopy between I and K^{n+1}.
$$\begin{aligned} I - K^2 &= I - K + K - K^2 \\ &= PD + DP + K(PD + DP) \\ &= (P + KP)D + D(P + KP) \end{aligned}$$
so $\quad P_1 = P + KP.$

(One of the reasons this is interesting comes from the fact that with a suitable anti-symmetrization $\mathrm{tr}(K^0)^{n+1} - \mathrm{tr}(K^1)^{n+1}$ will depend only on the symbol of D and only on first derivatives.)

If $D = D_0 - B$ and $P = D_0^{-1}$, then $K^0 = I - D_0^{-1}(D_0 - B) = D_0^{-1}B$

and so $P_1 = \underbrace{D_0^{-1}}_{P} + \underbrace{D_0^{-1}B}_{K^0}\underbrace{D_0^{-1}}_{P}$. Hence the refinement process is just giving us the geometric series: $D^{-1} = D_0^{-1} + D_0^{-1}BD_0^{-1} + \cdots\cdots$

Now you want to understand determinants. It would seem that the only general statement possible is that $\mathrm{tr}(D_0^{-1}B)^n$ is defined for large n and hence one can define locally

$$\log \det_n (D_0 - B) = -\sum_{n > n_0} \frac{1}{n}\, \mathrm{tr}(D_0^{-1}B)^n.$$

Then

$$\delta \log\det_n (D_0 - B) = -\sum_{n>n_0} \mathrm{tr}\left((D_0^{-1}B)^{n-1}D_0^{-1}\delta B\right)$$

$$= -\,\mathrm{tr}\left(\left(D^{-1} - D_0^{-1} - D_0^{-1}BD_0^{-1} - \cdots - (D_0^{-1}B)^{n-1}D_0^{-1}\right)\delta B\right).$$

Hence we see this kind of determinant corresponds to regularizing by subtracting off from D^{-1} the approximation

$$P_n = D_0^{-1} + D_0^{-1}BD_0^{-1} + \cdots\cdots + (D_0^{-1}B)^{n-1}D_0^{-1}.$$

What happens in the Riemann surface case? Here the regularization is done by subtracting a flat Green's fn. This is somehow much more subtle. Technically the same effect is achieved by removing D_0^{-1} and then using a principal value interpretation for the diagonal values of $D_0^{-1}BD_0^{-1}$.

October 29, 1982

I should perhaps understand the analysis behind the heat kernel à la Seeley. One has

$$e^{-tA} = \oint \frac{1}{\lambda - A} e^{-t\lambda} \frac{d\lambda}{2\pi i}$$

over a contour circling the positive real axis. Here A is a positive definite operator. An interesting point is that $\operatorname{Tr}(e^{-tA})$, $t > 0$ is defined although $\operatorname{Tr}\left(\frac{1}{\lambda - A}\right)$ is not. We even $\langle x | e^{-tA} | x \rangle$ defined with an asymptotic expansion

$$(*) \qquad \langle x | e^{-tA} | x \rangle \sim \sum{}' a_n(x) t^n \qquad t \downarrow 0$$

although we have no clear idea of $\langle x | \frac{1}{\lambda - A} | x \rangle$.

One might try to define

$$\langle x | \frac{1}{\lambda - A} | x \rangle = \int_0^\infty \langle x | e^{-tA} | x \rangle e^{\lambda t} \, dt$$

however there are problems because of the terms with positive powers of t in the asymptotic expansion $(*)$.

However it is clear from just these formulas that the singularities of $\langle x | \frac{1}{\lambda - A} | x' \rangle$ as $x' \to x$ should be intimately connected with the expressions in the asymptotic exp. $(*)$.

October 31, 1982

$$e^{-tA} = \oint \frac{1}{\lambda - A} e^{-t\lambda} \frac{d\lambda}{2\pi i}$$

What's interesting is that the diagonal [redacted] values $\langle x | e^{-tA} | x \rangle$ are defined, but those for $\langle x | \frac{1}{\lambda - A} | x \rangle$ are not. Ideas: 1) Up to now I have thought in terms of A being a 2nd order positive elliptic operator, whence e^{-tA} is a heat kernel (e.g. $A = D^*D$). However Hormander considers $A = (D^*D)^{1/2}$ a [redacted] first order positive elliptic ψDO, and this leads to interesting results concerning spectrum and characteristics.

2) Seeley's analysis of the heat kernel assumes the symbol is positive, hence one could take $A = D_0^*D$.

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determinant formalism: It seems that the following heuristic formula

$$\log \det(A) = -\int_0^\infty \operatorname{Tr}(e^{-tA}) \frac{dt}{t} + \text{const}$$

is good for the intuition, because it "explains" more precise formulas. E.g.

$$\delta \log \det(A) = -\int_0^\infty \operatorname{Tr}\left(e^{-tA}(-t\,\delta A)\right)\frac{dt}{t}$$

$$= \int_0^\infty \operatorname{Tr}(e^{-tA}\delta A)\, dt = \operatorname{Tr}(A^{-1}\delta A)$$

or

$$\begin{array}{ccccc} \dfrac{\zeta(s) - \zeta(0)}{s} & = & \dfrac{1}{s\Gamma(s)} \displaystyle\int_0^\infty \operatorname{Tr}(e^{-tA})\, t^s \frac{dt}{t} & - & \dfrac{\zeta(0)}{s} \\ \downarrow & & \downarrow & & \downarrow \\ \zeta'(0) & = & \displaystyle\int_0^\infty \operatorname{Tr}(e^{-tA}) \frac{dt}{t} & + & \text{const} \end{array}$$

This seems to show that one should not think of $-\zeta'(0)$ as $\log\det(A)$ unless it is clear that $\zeta(0)$ is constant for the variations of A considered.

What I want to do next is to understand both the Seeley and Hormander representations. In both cases you have a kernel $K(xt, x't')$ [redacted] which you represent as an oscillatory integral. Oscillatory integral means an integral of terms

$$\underbrace{e^{iS}}_{\substack{\text{rapidly varying}\\ \text{phase as } \xi\to\infty}} \quad \underbrace{A}_{\substack{\text{asymptotic exp. in homog. fns.}\\ \text{of } \xi \text{ as } \xi\to\infty.}}$$

Let's consider $A = -\partial_x^2 + q(x)$ on $\mathbb{R}$ and try to [redacted] derive the representations for the kernels of $e^{-itA^{1/2}}$ and e^{-itA}. [redacted]

Now $A^{1/2} = \sqrt{D^2+q}$, $D = \frac{1}{i}\partial_x$, is a ψDO with

$$\sqrt{D^2+q} = |D|\left(1 + \tfrac{1}{2} q D^{-2} + \cdots\right)$$

so $K(xt) = \langle x | e^{-itA^{1/2}} | 0 \rangle$ is a solution of the pseudo-diff'l equation

$$\left(i\partial_t + A^{1/2}\right) K(xt) = \delta(xt).$$

How does one solve this? Suppose q is a constant and let's use F.T.

$$\langle x | e^{-it\sqrt{D^2+q}} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x - it\sqrt{\xi^2+q}}$$

Now use $\sqrt{\xi^2+q} = |\xi|\left(1 + q|\xi|^{-2}\right)^{1/2} = |\xi|\left(1 + \tfrac{1}{2}q|\xi|^{-2} + \cdots\right)$

Thus in this case at least we have

$$\langle x | e^{-itA^{1/2}} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i(\xi x - |\xi| t)}\, \underbrace{e^{it[|\xi| - \sqrt{\xi^2 + g}]}}$$

This has an asymptotic expansion in homogeneous fns. of ξ as $|\xi| \to \infty$. So in the general case I want to do something similar. This means I want to construct asymptotic solutions of $(i\partial_t + A^{1/2})\psi = 0$ of the form

$$e^{i(\xi x - \omega(\xi) t)} \left(1 + t[\,\cdots\cdots\,]\right)$$

where $\omega(\xi)$ is the symbol of $A^{1/2}$. Then integrating these should give a parametrix for $i\partial_t + A^{1/2}$.

Now this method is very close to the way one constructs the heat kernel, where one conjugates by an exponential factor so as to be able to recursively grind out an asymptotic expansion in t.

November 1, 1982

Heat kernel e^{-tA} where A = Laplacean + lower order terms. For example $A = -\partial_x^2 + q$. If A has constant coefficients, then

$$\langle x | e^{-tA} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} e^{-t(\xi^2+q)}$$

The idea is to construct a parametrix for the operator $\partial_t + A$ as such an integral. This should be an oscillatory integral made up out of asymptotic solutions.

The logical idea is that one writes down an oscillatory integral

$$\sum_i e^{S_i} F_i \,.$$

which will have singularities because of the growth of the F_i. Applying the operator

$$\sum_i P(e^{S_i} F_i) = \sum e^{S_i} G_i$$

will give a smooth function because the G_i have rapid decrease.

So let's see how this idea might work for

$$\partial_t + A = \partial_t - \partial_x^2 + q \,.$$

I need a supply of asymptotic solutions and I proceed as in WKB. This seems geared to the Schroedinger operator

$$\frac{1}{i}\partial_t - \partial_x^2 + q$$

and the resulting asymptotic solutions have phase

$e^{iS(xt)}$ where S satisfies the Hamilton-Jacobi equation $+\partial_t S + (\partial_x S)^2 = 0$.

Up to now I have always taken the solution $S = \dfrac{x^2}{4t}$ and then derived the rest of the asymptotic expansion. This time I want to use other solutions such as

$$S(x,t) = \xi x - t|\xi|^2$$

There are some confusing aspects here. I want to construct solutions of

$$\left[\frac{1}{i}\partial_t - \partial_x^2 + q\right]\psi = 0$$

which have the form

$$\psi = e^{iS(xt)}\,u$$

where S is rapidly-varying and u is slowly-varying. The function u satisfies

$$\left[+\left(\frac{1}{i}\partial_t + \partial_t S\right) + \left(\frac{1}{i}\partial_x + \partial_x S\right)^2 + q\right] u = 0$$

$$\left[(\partial_x S)^2 + \partial_t S + 2\partial_x S\cdot\frac{1}{i}\partial_x + \frac{1}{i}\partial_x^2 S + \frac{1}{i}\partial_t - \partial_x^2 + q\right] u = 0$$

Because S is rapidly-varying one wants

$$\partial_t S + (\partial_x S)^2 = 0$$

which is the HJ equation, and in the particular case of $S(xt) = \xi x - |\xi|^2 t$ one has $\partial_x^2 S = 0$. So u now satisfies

$$\left[2\xi\frac{1}{i}\partial_x + \left(\frac{1}{i}\partial_t - \partial_x^2 + q\right)\right] u = 0$$

Now we are thinking of u as slowly-varying, and ξ

as very large, so that our first approximation for u is

$$\partial_x u = 0$$

i.e. u is a function of t. [redacted]
[redacted] But this is for one space dimension. In general one has

$$\xi \cdot \partial_x u = 0$$

This isn't quite accurate because the actual first approximation equation for u on the basis of the physics should be

$$\left(2\xi \cdot \partial_x + \partial_t\right) u = 0$$

The point is that the solution S of the H-J equation corresponds to a family of trajectories. [redacted] These trajectories in the case of $S = \xi x - |\xi|^2 t$ are solutions of $\dot{x} = \frac{\partial H}{\partial p} = 2p \quad \dot{p} = -\frac{\partial H}{\partial x} = 0$, since $H(p) = p^2$. Hence we get the family of trajectories

$$\dot{x} = 2\xi$$

which are the integral curves for the vector field $2\xi \partial_x + \partial_t$.

In some sense S is more rapidly varying in time than space so the ∂_t is of the same order as ξ. So now it is clear that I can solve the transport equations

$$\frac{1}{i}\left(2\xi \cdot \partial_x + \partial_t\right) u_n(x,t) = -\left(-\partial_x^2 + q(x)\right) u_{n-1}(x,t)$$

to obtain some kind of asymptotic series

$$u(x,t) \sim \sum_{n=0}^{\infty} u_n(x,t)$$

Of course the u_n are not unique.

Let's compare the above ideas with the Feynman path integral approach. This says that

$$\langle x | e^{-it(-\partial_x^2+g)} | 0 \rangle$$

is an integral over paths starting at 0 and ending at x. We have

$$(*) \quad \langle x | e^{-itA} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} \langle \xi | e^{-itA} | 0 \rangle$$

and presumably $\langle \xi | e^{-itA} | 0 \rangle$ is the integral over paths ~~starting~~ at 0 and ending with momentum ξ. We are thinking of ξ as huge, hence g is negligible in the first approximation. Thus the path is a free motion with $H(p) = p^2$, so $x = 2\xi t'$ for $0 < t' < t$ is the trajectory and the action over it is

$$\int L\,dt \qquad L = p\dot{q} - H = p2p - p^2 = |p|^2$$

or $S = \int L\,dt = |\xi|^2 t$. This would seem to give $\langle \xi | e^{-itA} | 0 \rangle = e^{iS} = e^{i|\xi|^2 t}$ which has the wrong sign.

Another problem with $(*)$ is that $\langle \xi | e^{-itA} | 0 \rangle$ is a function of t but not x, so I can't see how to get it from the transport equations.

November 2, 1982

Possible idea: [redacted] One can obtain the ζ fn. of A from either $\mathrm{Tr}(e^{-tA})$ or $\mathrm{Tr}(e^{-tA^{1/2}})$, hence it might be possible to use the known relations between $\mathrm{Tr}(e^{-itA^{1/2}})$ and geodesics in the theory of determinants.

Today I want to understand Fourier integral operators. I think the basic idea is the following. Take a Schroedinger operator e^{-itH} where H is a pseudo-differential operator. Then fix t [redacted] and try to understand the operator e^{-iHt}. Look at the Schwarz kernel $\langle x|e^{-itH}|x'\rangle$ and look at its singularities. Now we know what the singularities look like from the theory of hyperbolic equations. If one starts with a δ-function at $t=0$, then there is a light cone of singularities. Singularities propagate along bicharacteristics.

[redacted] I want to assume P is of order 1 so that $\frac{1}{i}\partial_t + P$ is hyperbolic. Then the symbol of P is a real function $H(x,\xi)$ homogeneous of degree 1 in ξ. Then you get a Hamiltonian flow on (x,ξ) space in particular a symplectic diffeomorphism $x'\xi' \longmapsto x\xi$ by letting the flow work for the time t. This symplectic diffeomorphism controls the singularities of e^{-itP}. There is a Huyghens' principle operating on the level of singularities.

Example: Suppose $H(x,\xi) = H(\xi)$ is homogeneous of degree 1. The Hamilton flow is

$$\dot{x} = \partial_\xi H \quad \text{and} \quad \dot{\xi} = -\partial_x H = 0.$$

For example, take $H(\xi) = |\xi|$ whence the symplectic diffeomorphism is $\quad \dot{x} = \frac{\xi}{|\xi|}$

$$\begin{cases} x = x' + \frac{\xi}{|\xi|} t \\ \xi = \xi' \end{cases}$$

Notice that one has $|x-x'| = t$ and hence the graph of this symplectic diffeomorphism does not project nicely onto x, x' space, and hence it can't be described [redacted] in terms of an action fn. $S(x,x')$. [redacted]

This behavior will be typical of any of the flows belonging to $H(x,\xi)$ which is homogeneous of degree 1 in ξ. The equations

$$\dot{x} = \partial_\xi H$$
$$\dot{\xi} = -\partial_x H$$

will be preserved under homotheties in ξ, so the graph of the map $x'\xi' \longmapsto x\xi$ will contain directions projecting trivially in x, x' space. The graph should [redacted] project onto a hypersurface in x, x' space with 1-dimensional fibres which are [redacted] orbits under homothety.

[redacted] Pseudo-differential operators correspond to the identity $x'\xi' \longmapsto (x\xi) = (x'\xi')$ which has image the diagonal in (x,x') space and n-dimensional fibres. So things are slightly more degenerate than in the case of e^{-itH}.

Let's take the case of e^{-itH} where $H = H(\xi)$ has constant coefficients. Then

$$\langle x | e^{-itH} | x' \rangle = \int \frac{d^n \xi}{(2\pi)^n} \, e^{i\xi(x-x') - itH(\xi)}$$

so we actually are getting the Fourier transform of the kernel. Where are its singularities? Let's use the method of stationary phase, i.e. look when the exponent is stationary in ξ:

$$0 = \nabla_\xi \left[\xi\cdot(x-x') - H(\xi)t \right] = (x-x') - \nabla_\xi H(\xi)\,t .$$

This happens when x,ξ is the image of $x'\xi$ under the symplectic transformation.

What is it that I want to understand? Somehow I have to see the singularities exactly in this constant coefficient case before I can generalize. This is a simple question, namely, the singularities of a sum of exponentials.

It might be easier to first look at what happens over (xt) space and then restrict to $t =$ constant, since restriction is a delicate process for distributions. So what we have is a hypersurface $\xi \longmapsto (\xi, -H(\xi))$ in the dual of (x,t)-space, in fact a cone because it's the graph of the homogeneous fn. $-H(\xi)$.

Review: I start with a positive ψDO operator P of order 1, say $P = A^{1/2}$ where $A = -\Delta + g$, and want to understand the unitary operator e^{-itP} with t fixed. The symbol of P, call it H, gives a Hamiltonian flow on (x,ξ) space, hence taking the flow at time t we get a symplectic diffeomorphism of (x,ξ) space which is the classical counterpart of e^{-itP}. Now the hope would be that via the singularities the classical + quantum pictures can be related. For example I can look at the singular support of the Schwarz kernel of e^{-itP} which is a closed subset of (x,x') space. This should be the image of the graph of symplectic diffeomorphism.

In some generic way, the graph of the diffeom. sits over a hypersurface in (x, x') space. Where the hypersurface is smooth one gets a line in the cotangent space which should be in the graph. This is a classical ~~statement~~ which I should be able to prove.

One wants to understand Lagrangian submanifolds of the cotangent bundle. [redacted] Those projecting non-singularly on the base I believe are locally sections of the form df. Then perhaps you can classify ones which project non-degenerately onto a smooth hypersurface.

Consider the example $H(\xi) = |\xi|$. Then the diffeom. is

$$x = x' + \frac{\xi}{|\xi|} t$$

$$\xi = \xi'$$

and the hypersurface is $|x - x'| = t$. ~~An obvious covector vanishing on the hypersurface is~~

$$\cancel{(x - x')\cdot(dx - dx') = 0}$$

~~writing this~~ $\cancel{\xi\, dx + \xi' \cdot dx' = 0}$

The tangent space to this hypersurface at (x, x') is

$$\{(\delta x, \delta x') \mid (x - x')\cdot(\delta x - \delta x') = 0\}.$$

and a linear function describing this hyperplane is $\xi\, dx - \xi' dx'$ where $\xi = \xi' = x - x'$.

All I am doing so far is classical geometry which depends only on the symbol of the operator P. It's clear that I will get, at least for t small, a nice hypersurface in (x, x') space and if I use the known results about propagation of singularities for hyperbolic equations, then the singularities of the kernel of e^{-itP} [redacted] are contained in this hypersurface.

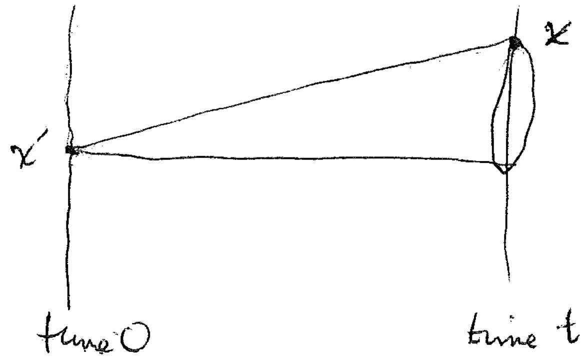
November 3, 1982

My goal is to understand what is known, mainly thru Hormander's work, about $\mathrm{Tr}(e^{-itA^{1/2}})$ where A is a positive 2nd order elliptic operator. The idea is that $e^{-itA^{1/2}}$ solves the Schroedinger equation $\left(\frac{1}{i}\partial_t + A^{1/2}\right)\psi = 0$ with the Hamiltonian operator $A^{1/2}$. This is a quantum mechanical situation whose corresponding classical situation is the Hamiltonian flow

$$\dot{x} = \partial_\xi H \qquad \dot{\xi} = -\partial_x H$$

associated to the symbol $H(x,\xi)$ of $A^{1/2}$. Hence on one hand we have a unitary operator and on the other a symplectic diffeomorphism. Because there is no Planck's constant here, the QM-classical correspondence here is given by singularities, namely the singularities of the kernel $\langle x | e^{-itA^{1/2}} | x' \rangle$ should be described in classical terms.

The precise relationship is what I want to work out. The singularities of $K(x,t;x') = \langle x | e^{-itA^{1/2}} | x' \rangle$ lie (in general) on a hypersurface in x, t, x' space. A point lies on this hypersurface iff there is a trajectory of the Hamilton flow starting at x' and ending at x' after the time t. Light cone picture



Fix t and x' and look at $K(x,t;x')$. Then it is singular on some hypersurface in x-space.

Hence I want to understand what distributions with singularities along a hypersurface might look like. I want the singularities to be in the transversal direction, to involve things like $|f|^\alpha$, where $f=0$ is the hypersurface, but not a δ function supported at a point of the hypersurface.

Look first at dim 1, i.e. $\mathbb{R}$, and suppose the hypersurface is $x=0$. Then the singularities I am interested in are the sort of things encountered with ψDO's:

$$K(x) = \int_{-\infty}^{\infty} e^{i\xi x} f(\xi) \frac{d\xi}{2\pi}$$

where f has an asymptotic expansion in homogeneous functions of ξ as $|\xi| \longrightarrow \infty$. (Note: homogeneous functions on $\mathbb{R}^*$ are very simple, [redacted] of two types for each degree supported on either side of 0. [redacted] So the [redacted] possible singularities at $x=0$ are like

$$\int_0^\infty e^{i\xi x} \xi^n d\xi = \frac{\Gamma(n+1)}{(-ix)^{n+1}} \quad .)$$

Anyway I might hope to show that the singularities of $K(x,t;x')$ for fixed t, x' could be described by an integral

$$K(x) = \int_{-\infty}^{\infty} e^{i\xi\varphi(x)} F(y,\xi)\, d\xi + \text{Smooth}$$

where $\varphi = 0$ is the hypersurface and y [redacted] $=(y_i)$ are coordinates along the hypersurface. F should have a nice expansion as $|\xi| \longrightarrow \infty$.

So let's try this out in a constant coefficient case say $A^{1/2} = P(D)$, e.g. $\sqrt{D^2+g}$, $g \in \mathbb{R}_{\geq 0}$. Then

$$\langle x | e^{-itP(D)} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i[\xi\cdot(x-x') - tP(\xi)]}$$

Let's take $x'=0$, $P(\xi) = |\xi|$. Then we have to understand the singularities of

$$K(x,t) = \int \frac{d^n\xi}{(2\pi)^n} \; e^{i(\xi\cdot x - |\xi| t)} .$$

This time let's allow t to vary also. The exponent is stationary when

$$0 = \partial_\xi \left(\xi\cdot x - |\xi| t \right) = x - \frac{\xi}{|\xi|} t$$

and so no such ξ will exist unless

$$|x| = t .$$

Hence $\varphi = t - |x|$ describes what should be the singularity locus of $K(x,t)$.

Let's try to understand the nature of the singularities of $K(x,t)$. Let's do the radial integral first

$$K(x,t) = c \int_{\omega\in S^{n-1}} d^{n-1}\omega \; \underbrace{\int_0^\infty r^{n-1} dr \; e^{ir(\omega\cdot x - t)}}_{\dfrac{\Gamma(n)}{[i(t-\omega\cdot x)]^n}}$$

I see from this that if for all $\omega \in S^{n-1}$ one has $t - \omega\cdot x \neq 0$, this is equivalent to $t > |x|$, then $K(x,t)$ is smooth. But this doesn't show that K is smooth when $|x| > t$, (which one would think would be easier by causality).

Let's instead do the radial integral last

$$K(x,t) = \int_0^\infty r^{n-1} dr \; e^{-irt} \int_{\omega\in S^{n-1}} c\, d^{n-1}\omega \; e^{ir(\omega\cdot x)}$$

Apply stationary phase to the ω-integral. There are two critical points $\omega = \pm \frac{x}{|x|}$ hence one has

$$\int_{\omega \in S^{n-1}} c \cdot d^{n-1}\omega \, e^{-ir(\omega \cdot x)} \sim \frac{e^{ir|x|}}{r^{(n-1)/2}} \left[a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots \right]$$
$$+ \frac{e^{-ir|x|}}{r^{(n-1)/2}} \left[\tilde{a}_0 + \frac{\tilde{a}_1}{r} + \frac{\tilde{a}_2}{r^2} + \cdots \right]$$

So it's clear that we get

$$K(xt) = \int_0^\infty e^{-ir(t+|x|)} F_1(r)\,dr + \int_0^\infty e^{-ir(t-|x|)} F_2(r)\,dr .$$

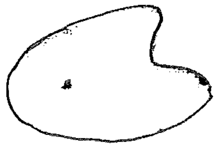
(Use a partition of unity over S^{n-1} to get sum over critical points), where F_1, F_2 have asymptotic expansions in powers of r as $r \to \infty$. For $t > 0$, the first integral is smooth.

This example convinces me of the essential correctness of the idea that $\langle x | e^{-itA^{1/2}} | x' \rangle$ near points where the hypersurface is smooth will be a 1-diml oscillatory integral modulo smooth functions. However, for the applications to $\mathrm{Tr}(e^{-itA^{1/2}})$ I want the behavior for $t \downarrow 0$, and I think this involves one <u>with a conical singularity</u>.

Let's consider a more complicated case with $A^{1/2} = P(D)$, $P(\xi)$ homogeneous of degree 1 and > 0. Then

$$\langle x | e^{-itP(D)} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i(\xi x - P(\xi)t)}$$

Now $P : \mathbb{R}^n - 0 \longrightarrow \mathbb{R}_{>0}$ has no critical points by homogeneity, hence $\{P(\xi) = 1\}$ is a smooth star-like surface surrounding the origin:



and we can write

$$\langle x | e^{-itP(D)} | 0 \rangle = \int_0^\infty r^{n-1} dr \; e^{-itr} \int_{P(\omega)=1} e^{ir(\omega \cdot x)} d\mu$$

where $d\mu$ is a suitable "solid angle" type measure on the surface $P(\omega) = 1$. (i.e. $\frac{d^n\xi}{(2\pi)^n} = r^{n-1} dr \cdot d\mu$).

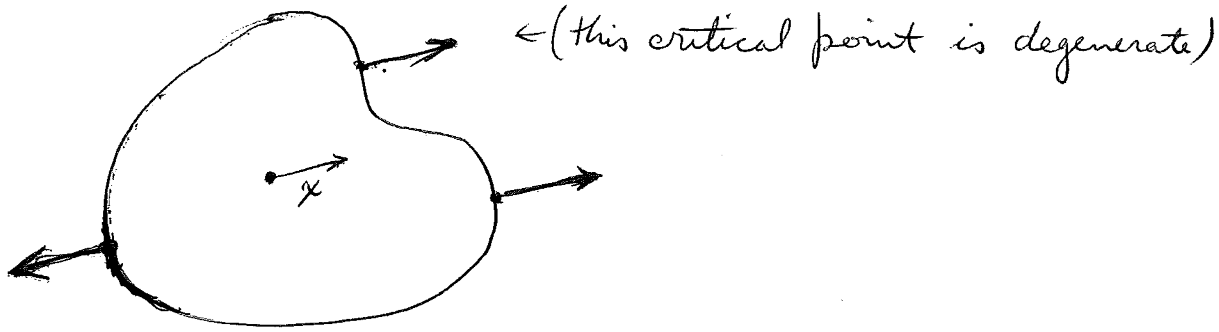
Now apply ~~stationary~~ phase to the last integral so as to get an asymptotic expansion as $r \to +\infty$. First we look for critical points. Now use Lagrange multipliers to understand critical pts. for $\xi . x$ subject to $P(\xi) = 1$. Consider

$$\xi \cdot x \; - \; \lambda (P(\xi) - 1)$$

and then we get the conditions

$$x = \lambda \, \partial_\xi P(\xi) \quad , \qquad P(\xi) = 1.$$

In other words ω is a critical point if grad P points in the same direction as x.



Actually $\omega \longmapsto \omega . x$ is "the height above the table" type of function, so we can easily visualize the critical points.

Consider the case where $P(\xi) = 1$ is strictly convex. In this case there are 2 critical points for each x, a maximum and a minimum. Call $m_+(x)$ the maximum value of $\omega . x$ on $P(\omega) = 1$ and $m_-(x)$

the minimum value. Then I have the asymptotic expansion

$$\int_{P(\omega)=1} e^{ir(\omega\cdot x)}\,d\mu \approx \frac{e^{-ir\,m_+(x)}}{r^{(n-1)/2}}\left[a_0(x)+\frac{a_1(x)}{r}+\cdots\right]$$

$+$ similar thing with $e^{ir m_-(x)}$.

So $\langle x|e^{-itP(D)}|0\rangle = \int_0^\infty e^{-i(t-m_+(x))r} * \text{something with asympt.}\ dr$

+ smooth

assuming $t>0$.

Actually I should make this more precise, namely I can write the kernel as 1-diml oscillatory integral

$$\langle x|e^{-itP(D)}|0\rangle = \int_0^\infty dr\ e^{-ir(t-m_+(x))}\ F(r,x)$$

+ smooth

where $F(r,x) \sim r^{(n-1)/2}\left\{a_0(x)+\frac{a_1(x)}{r}+\cdots\right\}$ as $r\to\infty$.

This is exactly the form I wanted, namely the singular hypersurface is $t-m_+(x)=0$, and the coefficients of the asymptotic expansion are expressed in the coordinates x which are complementary to $\varphi = t-m_+(x)$.

Question: I know that

$$e^{-itP} = \oint \frac{1}{\lambda-P}\, e^{-it\lambda}\,\frac{d\lambda}{2\pi i}$$

$$= \int_0^\infty e^{-it\lambda}\,\frac{dE_\lambda}{d\lambda}\,d\lambda$$

where $\frac{dE_\lambda}{d\lambda}$ is the jump in $\frac{1}{\lambda-P}$ across the real axis

Do I get any useful information about the resolvent in this situation?

$$\langle x | e^{-itP} | 0 \rangle = \int_0^\infty e^{-it\lambda} \langle x | \frac{dE}{d\lambda} | 0 \rangle \, d\lambda$$

$$= \int_0^\infty e^{-itr} \left[\int_{P(\omega)=1} e^{ir(\omega \cdot x)} \, d\mu \right] r^{n-1} dr$$

Hence for $x \neq 0$ I would get an asymptotic expansion for $\langle x | \frac{dE}{d\lambda} | 0 \rangle$ as $\lambda \longrightarrow \infty$.

The above formulas in the simple case $P(D)$ homogeneous of degree 1 are exact for $x = 0$. So we get

$$\langle 0 | e^{-itP} | 0 \rangle = \int_0^\infty e^{-itr} \left[\mathrm{vol}\{P = 1\} \right] r^{n-1} dr$$

$$= \frac{\Gamma(n) \left[\mathrm{vol} \right]}{(it)^n}$$

Also

$$\langle 0 | \frac{1}{\lambda - P} | 0 \rangle = \int_0^\infty \frac{1}{\lambda - r} \left[\mathrm{vol} \right] r^{n-1} dr$$

is undefined.

My idea is that as I pass to the general case I will perhaps be able to get an asymptotic expansion for $\langle x | e^{-itP} | x \rangle$ as $t \downarrow 0$.

How do I handle the general case? I should consider things like $P = \sqrt{D^2 + g}$, or $\sqrt{D^* D}$ where D is a $\bar{\partial}$-operator on a torus. The idea is that the symbol is constant coefficients, and so you will have to understand the changing lower order terms.

November 4, 1982

What might be the analogue of the action in the Hörmander theory? In QM one has

$$\langle x | e^{-\frac{i}{\hbar}tH} | x' \rangle \sim e^{\frac{i}{\hbar}S(xt, x'0)}$$

so the action appears using an asymptotic expansion w.r.t. $\hbar$. The action should be a classical gadget.

It might be useful to forget about x' and to think of solutions of the Schroedinger equation as having the form

$$\psi(xt) \sim e^{\frac{i}{\hbar}S(xt)}$$

where S is a solution of the HJ equation.

In the Hörmander situation solutions of the hyperbolic DE have singularities, not an asymptotic expansion. So what might the action be? A key idea I think is that one represents the solution of the hyperbolic DE as an oscillatory integral where the integration takes place over a cone, i.e. has this homogeneity. Thus

$$\psi(xt) = \int e^{i\varphi(xt, \xi)} \{ - - - \}$$

where $\varphi(xt, \xi)$ is homogeneous of degree 1 in ξ. Thus when one applies stationary phase one will get a ray of critical points

$$\nabla_\xi \varphi(xt, \xi) = 0 \qquad \text{homogeneous of deg } 0$$

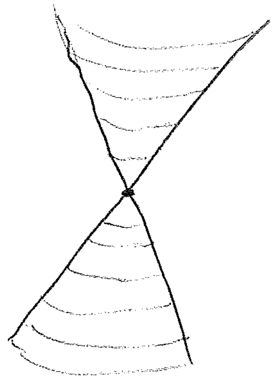
$$\Longrightarrow 0 = \xi \cdot \nabla_\xi \varphi(xt, \xi) = \varphi(xt, \xi) \qquad \text{by Euler.}$$

I have the feeling that the action is going to be zero on the singular surface and that if we have a smooth singular hypersurface, then it is given by $S = 0$ where S is the action. My idea now is that action is an equation for the singular hypersurface in some sense.

I should be thinking of S as a solution of the HJ equation. But then $S +$ constant will also be a solution.

This tells me that S is more [redacted] than the singularities of a particular solution. [redacted] Usually an S is [redacted] associated to a family of trajectories for the Hamilton flow which fill up x-space at any fixed time, assuming we are not at a bad point.

Picture of the singularities of a solution of the wave equation



Picture of a family of solutions of Hamilton's equations would be the union of these cones [redacted] at different times.

At time $t=0$ you could start with [redacted] $S = \xi' x$[redacted], i.e. all the particles have momentum ξ'. Then you solve the HJ equation, which amounts determining all trajectories with initial momentum ξ'. What is then $S(xt)$ assuming $H(x\xi) = |\xi|$? Clearly

$$S(xt) = \xi' \cdot x - |\xi'| t.$$

[redacted]

I want next to work out the Hormander theory for $e^{-itA^{1/2}}$ where A is a simple positive second order operator such as $-\partial_x^2 + q(x)$. The first step is to construct asymptotic solutions of

$$\left(\frac{1}{i}\partial_t + A^{1/2}\right) u = 0$$

of the form

$$u = e^{irS(xt)}\left[a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots\right].$$

Because A doesn't involve t, it follows that any solution of the Schroedinger equation is also a soln. of the wave eqn.

$$\left(-\frac{1}{i}\partial_t + A^{1/2}\right)\left(\frac{1}{i}\partial_t + A^{1/2}\right) u = \left(\partial_t^2 + A\right) u = 0,$$

and hence I think I can avoid having to explicitly work with the symbol of $A^{1/2}$. So if we put

$$u = e^{irS(xt)}\psi$$

then the wave equation for u becomes

$$\left[-\left(\frac{1}{i}\partial_t + r\partial_t S\right)^2 + \left(\frac{1}{i}\partial_x + r\partial_x S\right)^2 + q(x)\right]\psi = 0$$

$$-r^2(\partial_t S)^2 - 2r\,\partial_t S\cdot D_t \qquad r^2(\partial_x S)^2 + 2r\,\partial_x S\cdot D_x$$

$$-D_t^2 \quad - r\frac{1}{i}\partial_t^2 S \qquad + r\frac{1}{i}\partial_x^2 S \;+ D_x^2$$

The HJ equation is then

$$-(\partial_t S)^2 + (\partial_x S)^2 = 0$$

and I am interested in the branch

$$+\partial_t S + |\partial_x S| = 0$$

which is the HJ equation for $\frac{1}{i}\partial_t + A^{1/2}$. The simplest solution is

$$S(xt) = \xi x - |\xi| t$$

and I think in Hormander's method we solve this asymptotically for each ξ. Take $\xi > 0$ and call it k; then we are after an asymptotic solution of the wave equation

$$e^{ik(x-t)}\left[a_0 + \frac{a_1}{k} + \cdots\right].$$

The a_n which in general are functions of x, t are subject to the following equations (in general)

$$\left[-2\,\partial_t S \cdot D_t + 2\partial_x S \cdot D_x + \frac{1}{-i}\partial_t^2 S + \frac{1}{i}\partial_x^2 S\right] a_n$$

$$= -\left[-D_t^2 + D_x^2 + g(x)\right] a_{n-1}$$

In the present case where $S = x - t$ we get simply

$$2\left(D_t + D_x\right) a_n = -\left[-D_t^2 + D_x^2 + g(x)\right] a_{n-1}$$

which determines a_n up to a function of $x-t$.

At this point I have some understanding of the possible asymptotic solutions of the wave equation, and I have to figure out how these are combined to get $\langle x | e^{-itA^{1/2}} | x' \rangle$.

Example: Suppose that g is a constant (i.e. m^2 where m is a "mass.") Then the exact solution is

$$\langle x | e^{-itA^{1/2}} | x' \rangle = \int \frac{d\xi}{2\pi} e^{i[\xi(x-x') - \sqrt{\xi^2+g}\,t]}$$

$$= \int \frac{d\xi}{2\pi} e^{i[\xi(x-x') - |\xi| t]} \quad e^{i(|\xi| - \sqrt{\xi^2+g})t}$$

Because $\sqrt{\xi^2+g^2} - |\xi| = |\xi|\left[\left(1 + \frac{g}{|\xi|^2}\right)^{1/2} - 1\right] = \frac{g}{|\xi|} + \frac{?}{|\xi|^3} + \cdots$

the last factor has an asymptotic expansion

$$e^{i(|\xi| - \sqrt{\xi^2+g})t} = \left(1 - \frac{gt}{|\xi|} + \cdots\right)$$

in which the coefficients vanish at $t = 0$, except for the first

If we were to solve

$$2(D_t + D_x)\, a_n = \left[-D_t^2 + D_x^2 + g\right] a_{n-1}$$

requiring $a_n = 0$ for $t=0$ $n \geq 1$, and $a_0 = 1$ then clearly a_n is a polynomial in t. Thus this [redacted] should be the appropriate boundary condition in general.

Let's now summarize what ought to be true for $\langle x | e^{-itA^{1/2}} | x' \rangle$ where $A = -\partial_x^2 + g$. For each $\xi \neq 0$ I can construct an asymptotic solution of the wave equation $(\partial_t^2 - \partial_x^2 + g)u = 0$ of the form

$$e^{ir(\xi x - |\xi| t)} \left(1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots\right)$$

where the a_n are determined recursively by solving

$$2\left(|\xi| D_t + \xi D_x\right) a_n = -\left(-D_t^2 + D_x^2 + g(x)\right) a_{n-1}$$

subject to the boundary condition $a_n = 0$ when $t=0$. (Note that a_n is homogeneous of degree $-n$ in ξ). [redacted] Next select a $F(xt, \xi)$ which is smooth, vanishes near $\xi = 0$, and which has the above asymptotic expansion (with $r=1$ but $|\xi| \to \infty$). The claim is then that

$$\langle x | e^{-itA^{1/2}} | x' \rangle \equiv \int \frac{d\xi}{2\pi}\, e^{i(\xi(x-x') - |\xi| t)} F(xt, \xi)$$

modulo smooth functions. [redacted]

One thing that is clear is that if the wave operator $\partial_t^2 - \partial_x^2 + g(x)$ is applied to the integral, then one gets a smooth function of xt. I believe the same is true for the operator $\frac{1}{i}\partial_t + A^{1/2}$. Next if I [redacted] set $t=0$ I get $\delta(x-x')$ mod smooth fns.

What is mainly of interest to me is the diagonal behavior

$$\langle x | e^{-itA^{1/2}} | x \rangle \equiv \int \frac{d\xi}{2\pi} e^{-i|\xi|t} \underbrace{F(xt,\xi)}_{\sim\left[1 + a_1(xt,\xi) + a_2(xt,\xi) + \cdots\right]}$$

In one dimension what actually happens?

$$\int_0^\infty \frac{d\xi}{2\pi} e^{-i\xi t} = +\frac{1}{2\pi i}\frac{1}{t}$$

$$\int_0^\infty \frac{d\xi}{2\pi} e^{-i\xi t} \frac{1}{\xi} = \frac{-1}{2\pi} \log(t) + \infty\text{-renormalization constant}$$

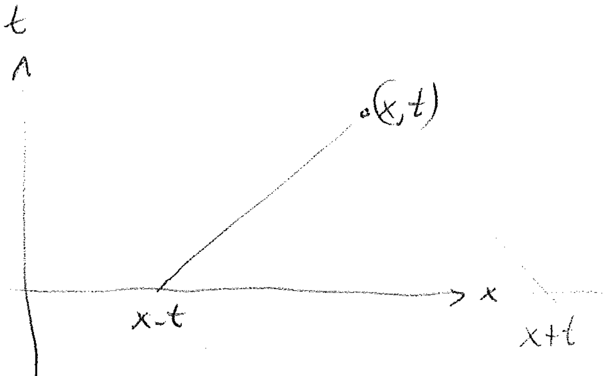
Let's determine $a_1(xt,\xi)$. It satisfies

$$2\left(|\xi| D_t + \xi D_x\right) a_1 = -g(x)$$

First take $\xi > 0$.

$$(\partial_t + \partial_x) a_1 = \frac{-i}{2\xi} g(x) \qquad a_1 = 0 \text{ when } t = 0$$

$$a_1(xt) = -\int_0^t \frac{i}{2\xi} g(x-t+t', t')\, dt'$$



Next take $\xi < 0$

$$(+\partial_t - \partial_x) a_1 = \frac{-i}{2|\xi|} g(x)$$

$$a_1(x,t) = -\int_0^t \frac{i}{2|\xi|} g(x+t-t', t')\, dt'$$

General formula $\quad a_1(x,t) = -\int_0^t \frac{i}{2|\xi|} g\left(x - \frac{\xi}{|\xi|}(t-t'), t'\right) dt'$

But a_1, a_2, etc. vanish at $t=0$, hence the worst singularity will be like $t\log(t)$ as t goes to zero. In fact we have

$$a_1(xt,\xi) \sim \frac{-i}{2|\xi|}\, g(x)\, t \qquad \text{as } t\to 0$$

so that

$$\int \frac{d\xi}{2\pi}\, e^{-i|\xi|t}\, a_1(xt,\xi) \sim \frac{+1}{2\pi}\log t \; i \; g(x)\, t$$
$$= \frac{1}{2\pi i}(t\log t)\, g(x).$$

Actually for later reference, it might be worthwhile to understand more precisely the function

$$g(t) = \int \frac{d\xi}{2\pi}\, e^{-i\left(\sqrt{\xi^2+g}\right)t}$$

November 5, 1982

$$\zeta_A(s) = \operatorname{Tr}(A^{-s}) = \operatorname{Tr}\left((A^{1/2})^{2s}\right)$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty \operatorname{Tr}(e^{-tA})\, t^s \frac{dt}{t} = \frac{1}{\Gamma(2s)}\int_0^\infty \operatorname{Tr}(e^{-tA^{1/2}})\, t^{2s}\frac{dt}{t}$$

I should be able to use what I know about the asymptotic expansion of $\operatorname{Tr}(e^{-tA})$ to say something similar about $\operatorname{Tr}(e^{-tA^{1/2}})$. Let's assume that $\exists$ asymptotic exp.

$$(*) \qquad \operatorname{Tr}(e^{-tA^{1/2}}) \sim \frac{a_{-n}}{t^n} + \cdots + a_0 + a_1 t + \cdots$$

Then
$$\int_0^\infty \operatorname{Tr}(e^{-tA^{1/2}})\, t^{2s}\frac{dt}{t} \sim \sum a_k \int_0^1 t^{k+2s}\frac{dt}{t}$$

$$\sim \sum_{k\geq -n} \frac{a_k}{2s+k}$$

has simple poles at $\frac{n}{2}, \frac{n-1}{2}, \ldots$ provided the coefficients are $\neq 0$. $\frac{1}{\Gamma(2s)}$ is 0 for $s = 0, -\frac12, -1,$ so consequently assuming $(*)$ we see that $\zeta(s)$ will be regular at these points $0, -\frac12, -1, \ldots$ and so the singularities will be at most simple poles at $s = \frac{n}{2}, \frac{n-1}{2}, \ldots, \frac12$.

On the other hand I know $\exists$ an asymptotic expansion

$$\operatorname{Tr}(e^{-tA}) = \frac{b_{-n/2}}{t^{n/2}} + \frac{b_{-n/2+1}}{t^{n/2-1}} + \cdots + b_{-\frac{n}{2}+k}\, t^{-\frac{n}{2}+k}$$

hence
$$\int_0^\infty \operatorname{Tr}(e^{-tA})\, t^s \frac{dt}{t} \sim \frac{b_{-n/2}}{s - n/2} + \frac{b_{-\frac{n}{2}+1}}{s - (\frac{n}{2}-1)} + \cdots$$

has at most simple poles at the points $\frac{n}{2}, \frac{n}{2}-1, \ldots$. $\frac{1}{\Gamma(s)}$ vanishes at $s = 0, -1, -2, \ldots$, hence for n odd $\zeta(s)$ might have poles at $-\frac12, -\frac32, \ldots\ldots$ Hence it might be the case that the asymptotic expansion for $\mathrm{Tr}(e^{-tA^{1/2}})$ exists only through the a_0 term. Probably we encounter a $t \log t$ term, and this I should see already when $A = D^2 + q$, q constant.

Next I want to understand $\mathrm{Tr}(e^{-itP(D)})$ where $P(D)$ is a constant coefficient operator over a torus $\mathbb{R}^n / L\mathbb{Z}^n$, e.g. $P(\xi) = \sqrt{\xi^2 + q}$, $q > 0$. I can get at the kernel $\langle x | e^{-itP} | x' \rangle$ in two ways, either by eigenfunctions

$$\langle x | e^{-itP} | x' \rangle = \frac{1}{L^n} \sum_{\xi \in \frac{2\pi}{L}\mathbb{Z}^n} e^{i\xi(x-x')} e^{-itP(\xi)}$$

or by taking the kernel in $\mathbb{R}^n$ and making it periodic

$$\sum_{\mu \in L\mathbb{Z}^n} \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi(x-x'+\mu)} e^{-itP(\xi)} .$$

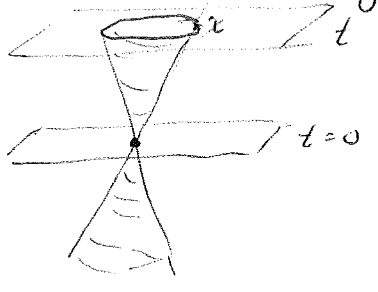
The equality of these is the Poisson summation fmla.

We have

$$\mathrm{Tr}(e^{-itP}) = \sum_{\xi \in \frac{2\pi}{L}\mathbb{Z}^n} e^{-itP(\xi)}$$

$$= L^n \sum_{\mu \in L\mathbb{Z}^n} \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi\mu} e^{-itP(\xi)}$$

Let's put $\quad Q(x t) = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} e^{-itP(\xi)}$. This is the fundamental solution in $\mathbb{R}^n$ of $\left(\frac{1}{i}\partial_t + P\right) u = 0$, and has its singularities on the light cone:



We have

$$\langle x | e^{-itP} | x \rangle = \sum_{\mu \in \frac{2\pi}{L}\mathbb{Z}^n} Q(\text{[redacted]}\mu, t)$$

and the point of Hormander's analysis is that the essential term is $Q(0,t)$, provided we work in $|t| < \varepsilon$.

Each [redacted] term $Q(\mu, t)$ for $\mu \neq 0$ is a smooth around $t = 0$. We have

$$(\partial_t)^n Q(\mu, t) = \int \frac{d^n\xi}{(2\pi)^n} e^{-i\xi\mu} (-i)^n P(\xi)^n e^{-itP(\xi)}$$

So when $t=0$ we get the F.T. of $P(\xi)^n$. If $P(\xi) = \sqrt{\xi^2 + g}$, then for n even, [redacted] $P(\xi)^n$ is a polynomial in ξ, hence its F.T. is supported at 0. This means that for $x \neq 0$, $Q(x t)$ is an odd smooth function of t if it were analytic; in any case we have

$$Q(x,t) \sim c_1 t + c_3 t^3 + \cdots \quad \text{as [redacted]} \quad t \to 0.$$

Modulo then a result about $Q(xt) \to 0$ fast enough as $|x| \to \infty$ which will guarantee uniform convergence [redacted] of the μ-sum, we see that the asymptotic expansion for $\langle x | e^{-itP} | x \rangle$ is [redacted] essentially the same as for [redacted]

$$Q(0t) = \int \frac{d^n\xi}{(2\pi)^n} e^{-itP(\xi)}.$$

So let's understand this for $P(\xi) = \sqrt{|\xi|^2 + g}$

$$\sqrt{|\xi|^2 + g} = |\xi| \left(1 + \frac{g}{|\xi|^2}\right)^{1/2}$$

$$= |\xi| + \frac{1}{2} g \frac{1}{|\xi|} - \frac{1}{8} g^2 \frac{1}{|\xi|^3} + \cdots$$

$$Q(0t) = \int \frac{d^n\xi}{(2\pi)^n} e^{-it|\xi|} e^{-it\left(\sqrt{|\xi|^2+g} - |\xi|\right)}$$

$$= \int \frac{d^n\xi}{(2\pi)^n} e^{-it|\xi|} \left[1 - it\frac{g}{2}\frac{1}{|\xi|} + \frac{1}{2!}\left(it\frac{g}{2}\frac{1}{|\xi|}\right)^2 + it\frac{g^2}{8}\frac{1}{|\xi|^3} + \cdots \right]$$

So we need

$$\int \frac{d^n\xi}{(2\pi)^n} e^{-it|\xi|} \frac{1}{|\xi|^k} = c\int_0^\infty r^{n-1}dr\, e^{-itr}\frac{1}{r^k}$$

$$= c\,\frac{\Gamma(n-k)}{(it)^{n-k}}$$

This will hold for $k = 0, 1, \ldots, n-1$. There won't be any possibility of $\log t$ terms until $k = n$, and in this case they come multiplied by t. from the numerator.

Let's return to the general situation. I have this 2nd order pos. elliptic operator A and I am going to use Hormander's way to get at the kernel $\langle x|e^{-itP}|x'\rangle$ where $P = A^{1/2}$. For each $\xi \neq 0$ one lets $S(xt,\xi)$ be the solution of the HJ equation such that $S(xt,\xi) = \xi\cdot x$ when $t = 0$. Then we find a formal solution

$$e^{iS(xt,\xi)}\left(1 + a_1(xt,\xi) + a_2(xt,\xi) + \cdots\right)$$

of the hyperbolic equation $(\partial_t^2 + A)u = 0$ where $a_n(xt,\xi)$ is homogeneous of degree $-n$ in ξ, and $a_n(xt,\xi) = 0$ when $t = 0$. Then

$$\langle x|e^{-itP}|x'\rangle \equiv \int \frac{d^n\xi}{(2\pi)^n} e^{iS(xt,\xi) - i\xi x'} \left(1 + a_1(xt,\xi) + \cdots\right)$$

smooth version of

To fix the ideas let A have symbol $|\xi|^2$. Then

$$S(xt,\xi) = \xi\cdot x - |\xi|t$$

Digression: Consider a Dirac-type self-adjoint operator on $\mathbb{R}$

$$\begin{pmatrix} \frac{1}{i}\partial_x & p(x) \\ \bar{p}(x) & -\frac{1}{i}\partial_x \end{pmatrix}$$

where the spectrum is discrete. For p not real, I believe the spectrum is not symmetric under $\lambda \mapsto -\lambda$ necessarily. Can you calculate the η-invariant?

Let's try to understand the formal solutions a bit better. Let me fix an origin x' around which to work. The idea is to describe the solution $\langle x | e^{-itA^{1/2}} | x' \rangle$ whose singularities will lie on the light-cone issuing from the origin x'. This means that we are solving $\left(\frac{1}{i}\partial_t + A^{1/2}\right)u = 0$ with $u(xt) = \delta(x-x')$ when $t=0$. Now the normal way to do this is to resolve $\delta(x-x')$ in terms of eigenfunctions of $A^{1/2}$ and then propagate these the obvious way in time. This works for constant coeffs.

$$\delta(x-x') = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi(x-x')}$$

$$e^{-itP(D)} e^{i\xi x} = e^{-itP(\xi)+i\xi x}$$

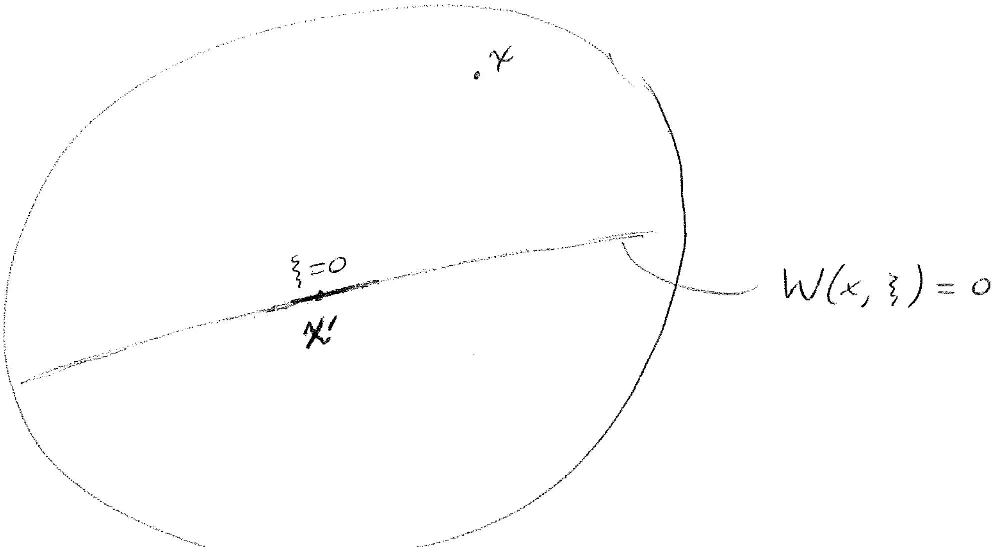
and finally $$\langle x | e^{-itP(D)} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i(\xi(x-x') - tP(\xi))}$$

So now I want to do something similar in general for variable coeffs. Work with large momenta + energy so that the coeffs. are varying very slowly. See if you can work in the Hamilton W-function.

Recall that when $H(x,\xi)$ is independent of t, then one looks for solutions of the HJ equation of the form $S(xt) = W(x) - Et$

where E is a constant. What does this mean in the Riemannian case where $H(x,\xi) = |\xi|$ at x.

I am after $S(xt,\xi) = W(x,\xi) - E(\xi)t$ which is homogeneous of degree 1 in ξ and such that $\xi = (\partial_x W)\big|_{x'}$. Here $E(\xi) = |\xi|$. Thus $W(x,\xi)$ is a fn. whose gradient has size $|\xi|$ and also coincides with ξ at x'. Picture:



I guess it is clear that you get a $W(x,\xi)$ by choosing nicely a surface thru x' with normal vector ξ, and then W will be the distance of x from that surface suitably scaled by $|\xi|$.

The problem with using this W is that we will not have

$$\int \frac{d^n\xi}{(2\pi)^n} e^{iW(x,\xi)} = \delta(x-x')$$

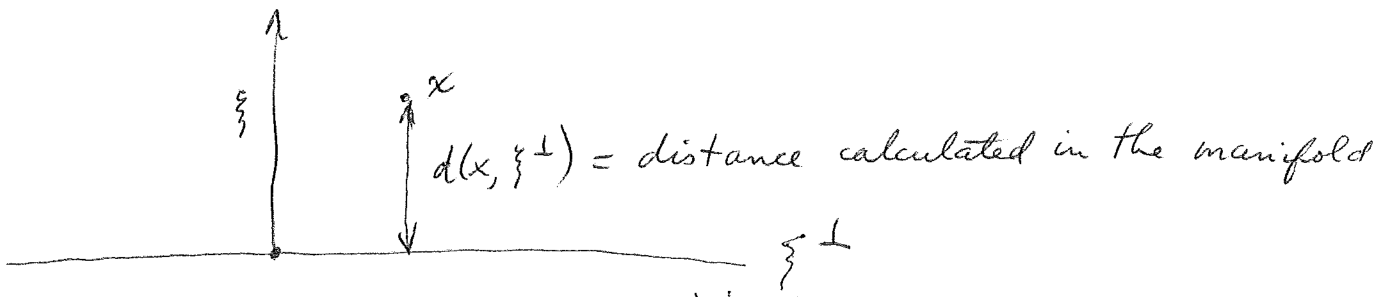
even modulo smooths? ~~Maybe this is OKAY. I should pick a $\rho(x) \equiv 1$ near x' of compact support and consider~~

$$\cancel{\int \frac{d^n\xi}{(2\pi)^n} e^{iW(x,\xi)} \rho(x)}$$

At least this isn't clear to me.

Now why would I like an $S(xt,\xi)$ of the form $W(x\xi) - |\xi|t$? Because then when I compute the trace I put $x = x'$ and I get a very simple time dependence. What might be the simplest type of $S(xt, \xi)$ from a geometric viewpoint? This would have to be a family of trajectories all having the initial momentum ξ. The only way I can see to do this is to use the exponential map to extend ξ to covectors in a nbd. of x'. You can also parallel translate along radial lines.

Formula for W in the Riemannian case: Think of ξ as being a tangent vector in the tangent space at x'. Let $\xi^\perp$ be the perpendicular hyperplane and also use the same notation for its image under the exponential map



Then $\quad W(x, \xi) = d(x, \xi^\perp)\,|\xi|$.

Evidently Hormander actually uses $S(xt, \xi)$ of the form $W(x, \xi) - E(\xi)\,t$. Therefore he must have a way to understand

$$\left[\int \frac{d^n\xi}{(2\pi)^n}\, e^{iW(x,\xi)}\right] - \delta(x - x').$$

At this point I get less enthusiastic about being able to apply the wave equation methods to understanding $\mathrm{Tr}(e^{-tA})$ where $A = D^*D$, where the symbol of A is given by the Riemannian metric. I don't see a nice choice for the $W(x, x', \xi)$ function.

November 6, 1982

Essential problem for me is to understand the asymptotic expansion of

$$\circledast \qquad \operatorname{Tr}(e^{-tA}B)$$

as $t \downarrow 0$. The reason is because I need to know about regularizing the trace of the operator B. Here B is a ψDO, hence it has singularities along the diagonal. One can't pull B back to the diagonal and integrate, because this is not a transversal situation. Hence one smooths out the identity to e^{-tA} and then tries to extract something as $t \downarrow 0$.

The extraction proceeds via taking the constant term in the asymptotic expansion. This is reasonable because

$$\operatorname{Tr}(A^{-s}B) = \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-tA}B)\, t^s \frac{dt}{t}$$

and so provided $\circledast$ has an asymptotic expansion one gets

$$\operatorname{Tr}(A^{-s}B)\Big|_{s=0} = \text{constant term.}$$

Let's go over the Hörmander theory for future understanding, even though it doesn't seem relevant to the problem of determinants. One has a first order positive elliptic ψDO P and one wants to understand its eigenvalue distribution: $N(\lambda) =$ no. of eigenvalues $\leq \lambda$. In the case of $P(D)$ on a torus, say where $P(\xi)$ is homog. of degree 1 one has

$$N(\lambda) = \text{no. of lattice points } \xi \text{ with } P(\xi) \leq \lambda.$$

Let the torus be $\mathbb{R}^n / L\mathbb{Z}^n$, then $\xi \in \frac{2\pi}{L}\mathbb{Z}^n$ and so

$$N(\lambda) \cdot \left(\frac{2\pi}{L}\right)^n \sim \operatorname{vol}\{P(\xi) \leq \lambda\} = \lambda^n \cdot \operatorname{vol}\{P(\xi) \leq 1\}$$

$$\therefore \quad N(\lambda) \sim \frac{\lambda^n}{(2\pi)^n}\, L^n \cdot \operatorname{vol}(P(\xi) \leq 1) = \frac{\lambda^n}{(2\pi)^n} \operatorname{vol}\{(x,\xi) \mid P(\xi) \leq 1\}$$

Hormander's general thm. is that the error term is $O(\lambda^{n-1})$. The method of proof uses the quantity

$$\mathrm{Tr}(e^{-itP}) = \sum_{\lambda \text{ eigenvalue}} e^{-it\lambda} = \int e^{-it\lambda} \frac{dN}{d\lambda} d\lambda$$

which for $|t| < \varepsilon$ can be shown to be of the form

$$L^n \int e^{-itP(\xi)} \frac{d^n\xi}{(2\pi)^n} + \text{smooth } r(t)$$

(I am writing out the formula in the constant coeff. case)

WAIT. It seems that Hörmander's asymptotic solutions are of the form

$$e^{i(W(x,\xi) - tE(\xi))} \left[1 + a_1(x,\xi) + a_2(x,\xi) + \cdots \right]$$

where the second term is independent of t. If I consider this in the case $P(x,D) = \sqrt{D^2+q}$ on $\mathbb{R}$ then I am looking at an asymptotic solution of

$$(D^2+q)u = \xi^2 u$$

of the form $u = e^{i\xi x}\left(1 + \frac{a_1(x)}{\xi} + \frac{a_2(x)}{\xi^2} + \cdots\right)$, and I know such an asymptotic solution is unique up to multiplying by a series in ξ^{-1}! So now the problem is to reconstruct $\delta(x)$ out of these modulo smooth fns. Thus I want to see that it is possible to find a formal series $f(\xi)$, such that

$$\int \frac{d\xi}{2\pi} e^{i\xi x}\left(1 + \frac{a_1(x)}{\xi} + \frac{a_2(x)}{\xi^2} + \cdots\right) \equiv \delta(x) \quad \text{mod smooths.}$$

Now in fact the problem will be that $a_1(x) = cx$ so

$$\int \frac{d\xi}{2\pi} e^{i\xi x} \frac{x}{\xi} \sim x \log x$$

has to be cancelled by the constant term in a_2

November 7, 1982:

There is an aspect of the Hörmander theory that I completely missed by looking at constant coeff. operators. Consider for example $P(D) = \sqrt{D^2+q}$, q constant. Then

$$\langle x | e^{-itP(D)} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{-i[\xi(x-x') - tP(\xi)]}$$

is an exact formula which can be written in the form of an oscillatory integral

$$\int \frac{d^n\xi}{(2\pi)^n} e^{i[\xi(x-x') - |\xi| t]} \underbrace{e^{-it[\overbrace{\sqrt{\xi^2+q} - |\xi|}^{\frac{1}{2}\frac{q}{|\xi|} - \frac{1}{8}\frac{q^2}{|\xi|^3} + \cdots}]}}_{1 - \frac{itq}{2}\frac{1}{|\xi|} - \frac{t^2q^2}{8}\frac{1}{|\xi|^2} + \cdots}.$$

However [redacted] Hörmander's representation is in the form

$$\int \frac{d^n\xi}{2\pi^n} e^{i(\xi(x-x') - |\xi| t)} \left(1 + a_1(x,x',\xi) + a_2(x,x',\xi) + \cdots \right)$$

which means he is using asymptotic eigenfunctions for $P(D)$. [redacted]

In general given $P(x,D)$ of order 1 one can look at the eigenvalue equation

$$P(x,D)u = Eu$$

and construct asymptotic solutions as $E \to \infty$. Look for

$$u = e^{irW(x)}\left[1 + \frac{a_1(x)}{r} + \cdots\right]$$

satisfying

$$P(x,D)u = rEu. \quad \text{[redacted]}$$

[redacted] So if $u = e^{irW}v$, then

$$\underbrace{e^{-irW} P(x,D) e^{irW}}_{(r\,p(x,dW) + \cdots\cdots)} v = rEv$$

and so we get the HJ equation

$$p(x, dW) = \text{[redacted]}E$$

I have already seen that I can do this in the case of the ordinary DE

$$(D^2 + q)\, u = k^2 u$$

whence we get asymptotic solutions

$$u = e^{-ikx}\left(1 + \frac{a_1(x)}{k} + \cdots\right).$$

Here's how this works for $P(D) = \sqrt{D^2+q}$. Write $\xi = r\omega$ with $r > 0$, $|\omega| = 1$, and then change variables $\xi \longmapsto \eta = r'\omega$, $r' = \sqrt{r^2+q}$. (In general you want the change of variables

$$\xi \longmapsto \eta = P(\xi)\frac{\xi}{|\xi|}\ .)$$

The point is that this is diffeomorphism for $|\xi|$ large. In the specific example

$$\int d^n\xi \; e^{i[\xi\Delta x - t\sqrt{\xi^2+q}]} = \int_0^\infty r^{n-1}dr \int_{|\omega|=1} e^{i[r\omega\cdot\Delta x - t\sqrt{r^2+q}]}\, d\mu(\omega)$$

$$= \int_{\sqrt{q}}^\infty \left(\sqrt{r'^2-q}\right)^{n-1} \frac{r'dr'}{\sqrt{r'^2-q}} \int_{|\omega|=1} e^{i[\sqrt{r'^2-q}\;\omega\cdot\Delta x - tr']}\, d\mu(\omega)$$

$$= \int_{\sqrt{q}}^\infty \frac{\left(\sqrt{r'^2-q}\right)^{n-2}}{r'^{\,n-2}}\, r'^{\,n-1}dr' \int_{|\omega|=1} e^{-i[\sqrt{r'^2-q}\;\omega\cdot\Delta x - tr']}\, d\mu(\omega)$$

$$= \int_{\sqrt{q}}^\infty \left(\frac{\sqrt{|\eta|^2-q}}{|\eta|}\right)^{n-2} d^n\eta \;\; e^{i\left[\left(\frac{\sqrt{|\eta|^2-q}}{|\eta|}\right)\eta\cdot\Delta x - t|\eta|\right]}.$$

Now you see that it is possible to write the exponent $i[\eta\cdot\Delta x - t|\eta|]$, and the rest an asymptotic expansion in η whose coefficients are fns. of $x-x'$.

Next I would like to see if these asymptotic eigenfunctions can be related to the resolvent. I want to use the formula

$$e^{-itP} = \oint \frac{1}{\lambda - P} e^{-i\lambda t} \frac{d\lambda}{2\pi i},$$

and to compute $\langle x | \frac{1}{\lambda - P} | x' \rangle$ by ΨDO methods, treating λ as having a certain homogeneity. In the case $P(D) = \sqrt{D^2 + q}$, the exact formula is

$$\langle x | \frac{1}{\lambda - P} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi \cdot \Delta x} \frac{1}{\lambda - P(\xi)}$$

i.e. the full symbol of $\frac{1}{\lambda - P}$ is $\frac{1}{\lambda - \sqrt{|\xi|^2 + q}}$. If you think of the λ as coming from ∂_t for the elliptic operator $\partial_t + P$, then the obvious expansion for the symbol $\frac{1}{\lambda - \sqrt{|\xi|^2 + q}}$ involves powers of $\lambda - |\xi|$ in the denominator. But

$$\oint \frac{1}{(\lambda - |\xi|)^m} e^{-i\lambda t} \frac{d\lambda}{2\pi i} = \oint e^{-i(\lambda + |\xi|)t} \frac{d\lambda}{\lambda^m \, 2\pi i}$$

$$= e^{-i|\xi|t} \frac{(-it)^{m-1}}{(m-1)!}$$

introducing these factors of t that the Hormander method gets rid of.

I have since looked at various papers including the Guillemin-Duistermatt one. It seems that Hormander originally used

$$\langle x | e^{-itP} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i(W(x,x';\xi) - p(x';\xi)t)} a(x,t,x';\xi)$$

so that t does appear in the amplitude. None of the papers seemed to say that a could be taken independent of t, so perhaps there exists an obstruction.

Summary of the wave equation approach:

Let A be a positive 2nd order elliptic operator e.g. a Laplacean D^*D or $-\partial_x^2 + q$. Then

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \underbrace{\operatorname{Tr}(e^{-tA})}_{\text{asymptotic exp. } t^{-n/2}[a_0 + a_1 t + \cdots]} t^s \frac{dt}{t}$$

$$\sim \frac{1}{\Gamma(s)} \left[\frac{a_0}{s-\frac{n}{2}} + \frac{a_1}{s-\frac{n}{2}+1} + \cdots\cdots \right]$$

so $\zeta(s)$ has at most simple poles at $\frac{n}{2}, \frac{n}{2}-1, \ldots$ which for n even get cancelled by those of the Γ-fn. This is because the heat kernel has an asymptotic exp. in powers of t.

On the other hand

$$\zeta_A(s) = \operatorname{Tr}\left((A^{\frac{1}{2}})^{-2s}\right) = \frac{1}{\Gamma(2s)} \int_0^\infty \operatorname{Tr}(e^{-tA^{1/2}})\, t^{2s} \frac{dt}{t}$$

so for n odd when ζ has a pole at say $-\frac{1}{2}, -\frac{3}{2}$, then there must be a $t \log t$, $t^2 \log t$, etc. term in the asymptotic expansion for $\operatorname{Tr}(e^{-tA^{1/2}})$.

How to understand the kernel $\langle x | e^{-itP} | x' \rangle$, where $P = A^{1/2}$. One constructs approximate, or asymptotic eigenfunctions for the operator $P(x, D)$ of the form

$$e^{i(W(x,x',\xi) - p(x',\xi)t)} \left[1 + a_1(x,x';\xi) + \cdots\cdots \right].$$

Here $p(x,\xi)$ is the symbol of P and W is Hamilton's W-function, i.e. a solution of

$$p(x, \nabla W) = p(x';\xi)$$

such that $W = 0$ at x' and $\nabla W = \xi$ at x'. In other words I take the momentum ξ at x' and construct a family of trajectories near x' of the same energy, and use this to define the idea of momentum ξ near x'. [redacted] The only point I don't understand well is whether I can get enough approximate eigenfunctions to build up the δ fn. $\delta(x-x')$ at [redacted] least mod smooth functions.

However Hörmander proves that there is a (homogeneous) change of variables $\xi = \xi(x,\eta)$ such that (take $x'=0$)

$$W(x, \xi[redacted](x,\eta)) = \eta\cdot x$$

Then

[redacted]

$$\delta(x) = \int \frac{d^n\eta}{(2\pi)^n} e^{i\eta x} = \int \frac{d^n\eta}{(2\pi)^n} e^{-iW(x,\xi(x,\eta))}$$

$$= \int \frac{d^n\xi}{(2\pi)^n} \frac{1}{\det\left(\frac{\partial\xi(x,\eta)}{\partial\eta}\right)} e^{-iW(x,\xi)}$$

so that if one takes $a(x,\xi) = \dfrac{1}{\det\left(\frac{\partial\xi}{\partial\eta}(x,\eta)\right)}$, then one has[redacted]

$$\delta(x) = \int \frac{d^n\xi}{(2\pi)^n} e^{-iW(x,\xi)} a(x,\xi)$$

From this viewpoint the amplitude is fixed at $t=0$ and so to integrate the transport equations one [redacted] needs an amplitude $a(xt,\xi)$. However I still would like to understand these approximate eigenfns. better in connection with $-\partial_x^2 + q(x)$, and in particular whether the above difficulty is the same as encountered with KdV versus MKdV.

Continue with the summary. From Hörmander's repn.

$$\langle x|e^{-itP}|x'\rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\{W(x,x';\xi) - p(x'\xi)t\}} a(xt,x';\xi[redacted]) + \text{smooth}$$

one gets a formula for the diagonal element

$$\langle x|e^{-itP}|x\rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{-it\,p(x,\xi)} \underbrace{a(x t x \xi)}_{\left[1 + a_1(xt\xi) + a_2(xt\xi) + \cdots\right]} + r(t)$$

where $r(t)$ is smooth. Now from this one can read off an asymptotic expansion in t, e.g., look at

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{-it\,p(\xi)}\, t^k a_{-l}(\xi)$$

where $a_{-l}(\xi)$ is homogeneous of degree $-l$ far out.

Do this radially:

$$\int_0^\infty r^{n-1}\,dr\; e^{-itr}\cdot t^k \underbrace{\int_{p(\omega)=1} a_{-l}(r\omega)\, d\mu(\omega)}_{\sim\, c\cdot r^{-l}}$$

So this will be the same essentially as

$$\sim\; c t^k \int_0^\infty r^{n-1}\,dr\; e^{-itr}\, r^{-l} = c t^k \frac{\Gamma(n-l)}{(it)^{n-l}}$$

except when l becomes $n, n+1, \ldots$ when one gets $\log t$ terms.

The main virtue of the above expression for $\langle x|e^{-itP}|x\rangle$ is that it valid in a interval $|t| < \varepsilon$ and hence tells more about the spectral fn. The point is that it is smooth in the interval except at $t = 0$. The asymptotic expansion in t is equivalent to knowing its ~~structure~~ as a distribution, more precisely the expression as a F.T. is given.

Finally because of the remainder term

$$\langle x | e^{-itP} | x \rangle = \int \cdot \quad - \quad - \qquad + r(t)$$

one knows nothing about the constant term in the asymptotic expansion, which makes the method uninteresting for the index thm.

<u>Remark:</u> If n is even, then from the regularity of

$$\zeta(s) = \frac{1}{\Gamma(2s)} \int_0^\infty \operatorname{Tr}\left(e^{-tA^{1/2}}\right) t^{2s} \frac{dt}{t} \qquad \text{for } Re(s) \le 0$$

one concludes that $\operatorname{Tr}\left(e^{-tA^{1/2}}\right)$ has no $t^k \log t$ terms in its asymptotic expansion. Notice that it looks like one should be able to get values for $\zeta(s)$ at $0, \frac{-1}{2}, -1, \ldots$ from the terms in the asymptotic expansion, provided one had complete control over $r(t)$.

November 8, 1982

Today I want to understand why and when $\mathrm{Tr}(e^{-tA}B)$ has an asymptotic expansion as $t\downarrow 0$. Here A is a Laplacean and B is a ψDO.

(Question: Is this a distribution in t with support in $t\geq 0$ in a natural way?)

Let's suppose $A=\frac{1}{2}\left(\frac{1}{i}\partial\right)^2$ on $\mathbb{R}^n$ whence

$$\langle x|e^{-tA}|x'\rangle = \frac{e^{-\frac{|x-x'|^2}{2t}}}{(2\pi t)^{n/2}}\,\theta(t)$$

and suppose

$$\langle x|B|x'\rangle = \int\frac{d^n\xi}{(2\pi)^n}\,e^{i\xi(x-x')}\,b(x,\xi)$$

Then

$$\langle 0|Be^{-tA}|0\rangle = \int d^nx'\left(\int\frac{d^n\xi}{(2\pi)^n}\,e^{-i\xi x'}b(0,\xi)\right)\frac{e^{-\frac{|x-x'|^2}{2t}}}{(2\pi t)^{n/2}}\,\theta(t)$$

$$=\int\frac{d^n\xi}{(2\pi)^n}\,b(0,\xi)\,e^{-\frac{t}{2}|\xi|^2}\,\theta(t)$$

Hence the essential problem is to understand when something like

$$(*)\qquad \int\frac{d^n\xi}{(2\pi)^n}\,e^{-\frac{t}{2}|\xi|^2}\,b(\xi)$$

has an asymptotic expansion in t as $t\downarrow 0$, where $b(\xi)$ is assumed to have an asymptotic expansion in ξ as $|\xi|\longrightarrow\infty$.

Important Observation: Normally when one works with ψDO's, one works with them modulo smooth operators, which means that one identifies symbols $b(x,\xi)$ which have the same asymptotic expansion as $|\xi|\longrightarrow\infty$. So if I work in the usual framework, the integral $(*)$ is determined modulo the

same integral with $b(\xi) = O(|\xi|^N)$ for all N.
What is it that one gets, what sort of function of t?

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{-\frac{t}{2}|\xi|^2}\, b(\xi) = \int_0^\infty r^{n-1}dr\, e^{-\frac{t}{2}r^2} \int_{|\omega|=1} b(r\omega)\, d\mu(\omega)$$

Now $r \longmapsto \int_{|\omega|=1} b(r\omega)\, d\mu(\omega) = \beta(r)$ is a smooth fn. of r which in the present case decays as $r \to +\infty$. It follows that

$$f(t) = \int_0^\infty dr\, e^{-\frac{t}{2}r^2}\, r^{n-1}\beta(r)$$

is a smooth function of t for $t \geqslant 0$, and it has its Taylor series at $t=0$ for the asymptotic expansion.
(Notice that the fact that $b(\xi)$ is smooth in ξ is not used.)

The important observation is that the normal ambiguity in $b(\xi)$ leads to

$$(*) \qquad \int \frac{d^n\xi}{(2\pi)^n}\, e^{-\frac{t}{2}|\xi|^2}\, b(\xi)$$

changing by a smooth function on $t \geqslant 0$. This would affect the value of a regularized trace (i.e. the coefficient of t^0), but it does not affect the regularization process. Hence maybe these symbol methods can be used to get at things like the curvature.

In general we can write $(*)$ as a radial integral. (Change $\frac{t}{2}$ to t):

$$\int_0^\infty dr\, e^{-tr^2}\, r^{n-1}\beta(r)$$

where β is smooth. (If $b(\xi)$ is smooth at 0, then $\beta(r)$ will have a power series expansion in r^2 at $r=0$.) Also $\beta(r)$ has an asymptotic expansion as $r \to \infty$.

Suppose $\beta(r)$ contains r^k in its asymptotic expansion. Then

$$\int_0^\infty \frac{dr}{r}\, e^{-tr^2} r^n r^k = \int_0^\infty \frac{dr}{2r}\, e^{-tr}\, r^{\frac{n+k}{2}} = \frac{1}{2}\, \frac{\Gamma\left(\frac{n+k}{2}\right)}{t^{(n+k)/2}}$$

which is fine for $k < -n$, but gives trouble for $k = -n, -n-2, \ldots\ldots$. These are exactly the places where the above formula would give powers $t^0, t^1, \ldots$

Earlier work on this question: 159-163, 196-200. In this work I found the following examples where $\mathrm{Tr}(e^{-t\Delta}K)$ had a $\log t$ term in the asymptotic expansion. Take $\Delta = A^2$ where A is an odd-dim'l self-adjoint operator such as $\frac{1}{i}\partial_x + \sigma$. Then in computing $\delta\eta_A(s)$ we encountered

$$\mathrm{Tr}\left(e^{-t\Delta}\, \frac{1}{|A|}\, \delta A\right)$$

which gives a lot t term. Also from

$$\eta_A(s) = \mathrm{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}\left(e^{-tA^2}\frac{A}{|A|}\right) t^{s/2}\frac{dt}{t}$$

there would be no problem in showing $\eta_A(0)$ finite if one were to know for simple reasons that $\mathrm{Tr}\left(e^{-tA^2}\frac{A}{|A|}\right)$ had only a t^0 term in its expansion.

Review: If $A = D^2 = -\nabla^2$, and $B = B(x,D)$ has full symbol $b(x,\xi)$, then I saw that

$$\langle 0 | B e^{-tA} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n}\, b(0,\xi)\, e^{-t|\xi|^2}$$

and I should be able to construct the asymptotic expansion in t as $t \downarrow 0$. To do this write the integral in the form

$$\int_0^\infty r^{n-1}dr\, e^{-tr^2} \underbrace{\int_{|\omega|=1} b(r\omega)\, d\mu(\omega)}_{\beta(r)}$$

Put $b(0,\xi)=b(\xi)$

Now from the asymptotic expansion of $b(\xi)$ we get

$$b(\xi) \sim \sum_m^{-\infty} b_k(\xi) \qquad b_k \text{ homog of deg } k$$

hence

$$\beta(r) = \int_{|\omega|=1} b(r\omega)\, d\mu(\omega) \sim \sum_m^{-\infty} r^k \underbrace{\int_{|\omega|=1} b_k(\omega)\, d\mu(\omega)}_{\beta_k}$$

Thus

$$\langle 0 | Be^{-tA} | 0 \rangle = \int_0^\infty r^{n-1} dr\, e^{-tr^2} \beta(r)$$

$$\text{where} \quad \beta(r) \sim \sum_{k=m}^{-\infty} \beta_k r^k$$

and from this you must deduce the asymptotic expansion.

Formally
$$\int_0^\infty r^{n-1} dr\, e^{-tr^2} r^k = \int_0^\infty \frac{dr^2}{2r^2} e^{-tr^2} r^{2\frac{n+k}{2}}$$

$$= \frac{1}{2}\Gamma\left(\frac{n+k}{2}\right) \frac{1}{(t)^{\frac{n+k}{2}}}$$

so the homogeneity degrees giving trouble should be $k=-n, -n-2, \ldots\ldots$, and these correspond to the powers $t^0, t^1, t^2, \ldots$ which can't be determined from the asymptotic expansion of $\beta(r)$. So the conjecture is that I get an asymptotic expansion with terms $t^k \log t$, $k=0,1,2,\ldots$ whose coefficients are simply expressed in terms of the β_k.

Which kinds of ψDO's B are going to give these $\log t$ terms? In particular I am interested in understanding this regularization method for Greens function's ~~of~~ $\bar{\partial}$-operators.

<u>Example:</u> Consider $\mathrm{Tr}(e^{-tA}A^{-1})$ where $A = -\Delta$ in the plane. Up to a constant we get

$$\langle 0|e^{-tA}A^{-1}|0\rangle = \int_0^\infty \frac{e^{-\frac{r^2}{4t}}}{4\pi t}\cancel{\frac{1}{2\pi}}\log r\; r\,dr\cdot\cancel{2\pi} = \frac{1}{4\pi t}\int_0^\infty e^{-\frac{r^2}{4t}}\log r^2\, dr^2\,\frac{1}{4}$$

$$= \frac{1}{4\pi t 4}\int_0^\infty e^{-\frac{u}{4t}}\log u\, du = \frac{1}{4\pi t 4}\int_0^\infty e^{-u}\log(4tu)\,4t\,du$$

$$= \frac{1}{4\pi}\log t\int_0^\infty e^{-u}\,du + \frac{1}{4\pi}\int_0^\infty e^{-u}\log(4u)\,du$$

$$= \frac{1}{4\pi}\log t + \text{constant}$$

Idea: The sort of $b(\xi)$ that give a $\log t$ term in

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{-t|\xi|^2} b(\xi)$$

are those having a term homogeneous of degree $-n$. Such b also cause log singularities in the Fourier transform

$$\langle x|B(D)|0\rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, b(\xi).$$

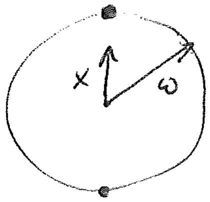
Calculate

$$F(x) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, a(\xi) \qquad a(\xi) \text{ of degree } -n.$$

$$= \int_0^\infty \frac{dr}{r}\underbrace{\int_{|\omega|=1} e^{ir(\omega\cdot x)}\, a(\omega)\, d\mu(\omega)}_{f(rx)}$$

Formally $F(\lambda x) = F(x)$. Notice that $(\omega \mapsto (\omega\cdot x))$ as a fn.

on $\{|\omega|=1\}$ has two non-degenerate critical points, so by stationary phase



$$f(rx) \sim \frac{e^{ir|x|}}{r^{(n-1)/2}}(\dots) + \frac{e^{-ir|x|}}{r^{(n-1)/2}}(\dots)$$

so if $n>1$, then the integral for $F(x)$ converges at the $r \to \infty$ end. I think this holds even for $n=1$ because of the oscillatory factors. [redacted] [redacted]

At the $r=0$ end the integral will diverge unless [redacted] we have

(+) $$f(0) = \int_{|\omega|=1} a(\omega)\, d\mu(\omega) = 0$$

In this case $F(x)$ should be a well-defined homogeneous function of degree 0.

Check tomorrow that (+) is independent of the norm $|\xi|$.

November 9, 1982

Homogeneous distributions. Let $\mathcal{E}_d$ be the space of tempered distributions on $\mathbb{R}^n$ which are smooth except at 0, and which are homogeneous of degree d. For example a homogeneous function of degree d smooth except at 0 will be integrable near the origin if $d > -n$, and so will define a homogeneous distribution. Let $\mathcal{F}_d$ be the space of homogeneous functions of degree d smooth except at 0. Then we have a map

$$(1) \qquad \mathcal{E}_d \longrightarrow \mathcal{F}_d$$

which is an isomorphism for $d > -n$. A distribution in the kernel is one having support at 0 and is of the form $P(\partial)\delta$ where P is a homogeneous polynomial. The δ distribution has degree $-n$. Hence the kernel of the above map is non-zero ~~degree~~ when $d = -n, -n-1, -n-2, \ldots$ and its dimension is given by a space of polynomials.

The Fourier transform gives an isomorphism

$$(2) \qquad \mathcal{E}_d \xrightarrow{\sim} \mathcal{E}_{-n-d}$$

hence a homogeneous distribution of degree d can be thought of as a homogeneous function if $d > -n$ or the F.T. of a homog. fn. of degree $-n-d$ if $d < 0$.

Let's now look at the F.T. by integrating radially.

$$F(x) = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} f_d(\xi) = \int_0^\infty r^{n+d} \frac{dr}{r} \int_{|\omega|=1} e^{ir(\omega \cdot x)} f_d(\omega)\, d\mu(\omega)$$

~~The last integral is a kind of transform on smooth functions on S^{n-1} (?)~~ The last integral depends only on the function f_d restricted to S^{n-1}, but $F(x)$ for $|x|=1$ also depends on d. However it is obviously analytic in d, provided there are no convergence problems.

Conjecture: The map [redacted] $\mathcal{E}_d \to \mathcal{F}_d$ from smooth homog. distributions to smooth homog. fns. is an isomorphism for $d \neq -n$, $-n-1$, etc. For $d = -n-k$ a function $f \in \mathcal{F}_d$ is in the image iff for any homogeneous polynomial P of degree k one has

$$(+) \qquad \int_{|\omega|=1} P(\omega) f(\omega)\, d\mu(\omega) = 0.$$

Hence $\mathcal{E}_d \to \mathcal{F}_d$ is of index 0 for $d = -n, -n-1, \dots$

(Invariance of (+): $P(\xi) f(\xi)\, d^n\xi$ is an n-form on $\mathbb{R}^n - \{0\}$ homogeneous of degree 0. $\mathbb{R}^n - \{0\}$ is a $\mathbb{R}_{>0}$ principal bundle over S^{n-1}. Analogous ~~situation~~ is a form [redacted] $f(x)\,dx\,dy$ on $\mathbb{R}^2$ which is invariant under translation in the y direction. [redacted] I claim [redacted] $f(x)\,dx$ is invariantly defined on the base $\mathbb{R}$. In effect under $(x,y) \mapsto (x, y+u(x))$, the form goes to $f(x)\,dx(dy + u'dx) = f(x)\,dx\,dy$, so the $f(x)dx$ part doesn't change. The good way to describe this is to notice that one has the vector field ∂_y along the fibres which is invariant for the action, hence $i(\partial_y) f(x)\,dx\,dy$ will [redacted] come from the base.)

Review: a ψDO kernel $\langle x | B | 0 \rangle$ is a distribution smooth away from 0, and is given by

$$\langle x | B | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi x} b(\xi)$$

where b has an asymptotic expansion as $\xi \to \infty$

$$b(\xi) \sim \sum_{r}^{-\infty} b_k(\xi). \qquad r = \text{order of } b$$

On the other hand I can ask if $\langle x | B | 0 \rangle$ has an asymptotic expansion as $x \to 0$ in homogeneous fns:

$$\langle x | B | 0 \rangle \sim \sum_{-n-n}^{\infty} \beta_m(x).$$

One might hope that the Fourier transform of the homogeneous function $b_k'(\xi)$ would be $\beta_m(x)$ for $m = -n-k$. I have seen however that for $k = -n, -n-1, -n-2, \dots$ and hence $m = 0, 1, \dots$ there are difficulties with defining the Fourier transform for $b_k(\xi)$, unless $b_k(\xi)$ is orthogonal to the homogeneous polynomials of degree m (see (+) on the preceding page). In the case that these orthogonality conditions are satisfied, the actual $\beta_m(x)$ is unspecified up to a homogeneous polynomial of degree m.

<u>Example:</u> If I take a $\bar\partial$-operator $\partial_{\bar z} + \alpha$, then I can get its parametrix in the form

$$G(z, z') = \frac{\varphi(z)\,\varphi(z')^{-1}}{z - z'} + \text{smooth kernel}$$

and hence it is asymptotically a sum of homogeneous functions of the form $\frac{\overline{(z-z')}^n}{z-z'}$ for $n \geq 0$. On the other hand, the Laplacean in 2-dims. has Green's fn. $\sim \log r$ which is not homogeneous.

What I would like to do next is to assume that $\langle x | B | 0 \rangle$ has an asymptotic expansion as $x \to 0$ and show then that $\langle 0 | e^{+t\Delta} B | 0 \rangle$ has an asymptotic expansion in powers of t. So I should look at

$$\int \frac{e^{-\frac{|x|^2}{4t}}}{(\sqrt{4\pi t})^n}\, \beta_m(x)\, d^n x$$

and begin with the case $m > -n$ so there is no problem at $x = 0$. Clearly we can change variable $x \mapsto \sqrt{t}\, x$

and get

$$\underbrace{t^{-n/2}\, t^{m/2}\, t^{+n/2}}_{t^{(m)/2}} \int \frac{e^{-\frac{|x|^2}{4}}}{(4\pi)^{n/2}} \beta_m(x)\, d^n x$$

which is a power of t increasing with m.

For $m < 0$ we can use the F.T. picture which is necessary to make sense of β_m as a distribution for $m \leq -n$ at least. Then

$$\int \frac{e^{-\frac{|x|^2}{4t}}}{(\sqrt{4\pi t})^n} \beta_m(x)\, d^n x = \int \frac{d^n\xi}{(2\pi)^n} e^{-t|\xi|^2} b_{-n-m}(\xi)$$

$$= t^{-n/2}\, t^{\frac{n+m}{2}} \int \frac{d^n\xi}{(2\pi)^n} e^{-|\xi|^2} b_{-n-m}(\xi)$$

is also $t^{m/2} \times$ const.

Therefore since heat kernels are Gaussians times smooth functions of $x t$, one concludes that $\mathrm{Tr}(e^{-tA}B)$ will have an asymptotic expansion in powers of t when B is a ψDO of the good type.

Possible application: Take A to be a self-adjoint elliptic operator, say invertible. Then

$$\eta_A(s) = \mathrm{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \mathrm{Tr}\left(\frac{A}{|A|}(A^2)^{-s/2}\right)$$

$$= \frac{1}{\Gamma(s/2)} \int_0^\infty \mathrm{Tr}\left(\frac{A}{|A|} e^{-tA^2}\right) t^{s/2} \frac{dt}{t}.$$

Hence if $\frac{A}{|A|}$ were of the good type, it would follow $\eta_A(0)$ is defined. So therefore it should be possible to prove this directly by a symbol calculation.

But simpler perhaps is

$$\eta_A(s) = \mathrm{Tr}\left(A|A^2|^{-\frac{s+1}{2}}\right) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty \mathrm{Tr}\left(Ae^{-tA^2}\right) t^{\frac{s+1}{2}} dt$$

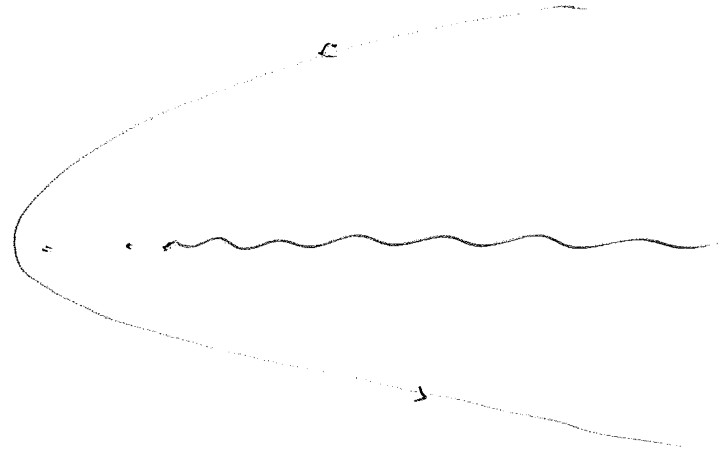
whence the residue of $\eta_A(s)$ at $s=0$ is the coefficient of $t^{-1/2}$ in $\mathrm{Tr}(Ae^{-tA^2})$

November 10, 1982

Let's try to understand the ~~cases of~~ heat kernel e^{-tA} when $A = -\partial_x^2 + q(x)$ on $\mathbb{R}$. Seeley gets at this via the contour integral formula

$$\langle x | e^{-tA} | x' \rangle = \oint \langle x | \frac{1}{\lambda - A} | x' \rangle e^{-\lambda t} \frac{d\lambda}{2\pi i}$$

and a calculation of the resolvent by ΨDO techniques. The contour goes around the spectrum of A which is mainly on the positive real axis, hence is nicely parametrized by $\lambda = k^2$ where k goes from $\infty + ia$ to $-\infty + ia$ and a is sufficiently positive.



So making this change a variable gives

$$\langle x | e^{-tA} | x' \rangle = \int_{-\infty+ia}^{\infty+ia} \langle x | \frac{1}{k^2 - A} | x' \rangle e^{-tk^2} \frac{i}{\pi} k\,dk$$

November 10, 1982 (cont.)

Program: I need to understand the heat kernel $\langle x | e^{-tA} | x' \rangle$ well enough to calculate

$$\langle 0 | B e^{-tA} | 0 \rangle = \int dx \underbrace{\langle 0 | B | x \rangle}_{\int \frac{d^n\xi}{(2\pi)^n} e^{-i\xi x} b(0,\xi)} \langle x | e^{-tA} | 0 \rangle$$

(1)
$$= \int \frac{d^n \xi}{(2\pi)^n} b(0,\xi) \int dx\, e^{-i\xi x} \langle x | e^{-tA} | 0 \rangle$$

So what I want is to work with coordinates so that this last integral has the form $e^{-t|\xi|^2} f(t\xi)$, where f has a smooth asymptotic expansion as $|\xi| \to \infty$.

Now I believe the usual asymptotic expansion for e^{-tA}, $A = D^2 + \mathcal{E}$ takes the form

$$\langle x | e^{-tA} | 0 \rangle = \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \underbrace{F(tx)}_{\sim a_0(x) + a_1(x)t + \cdots \quad \text{as } t \to 0}$$

It maybe is possible in computing the Fourier transform of this to work with the formal power series of $F(tx)$ at $xt = 00$. So therefore I should first understand

$$\int dx\, e^{-i\xi x} \frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} x^\alpha$$

for a given monomial x^α. But this I can [redacted] calculate via

$$\int dx\, e^{-i\xi x} \frac{e^{-\frac{|x|^2}{4t}}}{(\sqrt{4\pi t})^{n/2}} = e^{-t|\xi|^2}$$

and differentiating wrt ξ. These are just Gaussian moments. [redacted]

So we conclude that corresponding to a term

$t^k x^\alpha$ in F we expect a term

$$t^k e^{-t|\xi|^2} \times (\text{polynomial in } t, \xi).$$

Hence what I would like to prove is that there is some sort of asymptotic formula

$$(2) \quad \int d_x^n \, e^{-i\xi x} \langle x | e^{-tA} | 0 \rangle = e^{-t|\xi|^2} \sum t^k \xi^\alpha a_{k\alpha}.$$

I want then to be able to substitute this in (1) above to get

$$\langle 0 | B e^{-tA} | 0 \rangle = \sum a_{k\alpha} t^k \int \frac{d^n\xi}{(2\pi)^n} \, b(0,\xi) \, \xi^\alpha e^{-t|\xi|^2}$$

However (2) is perhaps unreasonable because upon inversion one gets

$$\langle x | e^{-tA} | 0 \rangle = \int \frac{d^n\xi}{(2\pi)^n} \, e^{i\xi x - t|\xi|^2} \underbrace{\sum a_{k\alpha} t^k \xi^\alpha}$$

and the amplitude at the end, being independent of x, is not the sort of this one would get by integrating the transport equations. (Recall that if $u = e^{i\xi x - t|\xi|^2} v$ then $(\partial_t + A) u = 0$ becomes

$$\left[\partial_t - |\xi|^2 + (D_x + \xi)^2 + q(x) \right] v = 0$$

or

$$\left[\partial_t + 2\xi \cdot D_x + D_x^2 + q(x) \right] v = 0.$$

Hence this can't be solved by a function $v(t,\xi)$, although it can formally be solved with a $v(x,\xi)$.

Let's compute the resolvent $\frac{1}{\lambda - A}$ where $A = D^2 + q$ on the line. Proceed formally. Put

$$P(x, D) = \sum_{n \geq 0} a_n(x, D) \frac{1}{(\lambda - D^2)^{n+1}}$$

$$= \frac{1}{\lambda - D^2} + a_1 \frac{1}{(\lambda - D^2)^2} + \cdots$$

and ask that

$$(\lambda - D^2 - q) P(x, D) = 1.$$

Then $(\lambda - D^2$ [redacted] $) a_n \frac{1}{(\lambda - D^2)^{n+1}} = [\lambda$ [redacted] $- D^2, a_n] \frac{1}{(\lambda - D^2)^{n+1}} + a_n \frac{1}{(\lambda - D^2)^n}$

leading to the recursion relations

$$[\lambda - D^2, a_n] + a_{n+1} = \text{[redacted]}\; q a_n$$

or more simply

$$\boxed{a_{n+1} = [D^2, a_n \text{[redacted]}] + q a_n}$$

Notice that if we use

$$\frac{(-s)(-s-1)\cdots(-s-k+1)}{k!} A^{-s-k} = \frac{1}{2\pi i} \oint \frac{1}{(\lambda - A)^{k+1}} \lambda^{-s} \, d\lambda$$

then we get

$$(D^2 + q)^{-s} = (D^2)^{-s} + (-s) a_1 (D^2)^{-s-1} + \frac{(-s)(-s-1)}{2!} a_2 (D^2)^{-s-2} + \cdots$$

Thus we relate the expansion of the resolvent to the ψDO expansion for A^{-s}. Next use

$$\frac{(-t)^k}{k!} e^{-tA} = \frac{1}{2\pi i} \oint \frac{1}{(\lambda - A)^{k+1}} e^{-t\lambda} \, d\lambda$$

and one gets the expansion for the heat kernel

$$\boxed{\cancel{e^{-t(D^2+q)} = e^{-tD^2} + (-t) a_1}}$$

$$\circledast \qquad e^{-t(D^2+g)} = \left(1 + (-t)a_1 + \frac{(-t)^2}{2!}a_2 + \cdots\right) e^{-tD^2}$$

Let's apply this expansion for the resolvent:

$$\langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{i\xi x}\left\{ \frac{1}{\lambda - \xi^2} + a_1(x,\xi)\frac{1}{(\lambda-\xi^2)^2} + \cdots \right\}$$

to compute the heat kernel:

$$\langle x | e^{-tA} | \xi \rangle = \oint \langle x | \frac{1}{\lambda - A} | \xi \rangle e^{-t\lambda} \frac{d\lambda}{2\pi i}$$

$$= e^{i\xi x}\left\{ e^{-t\xi^2} + a_1(-t)e^{-t\xi^2} + \cdots \right\}$$

$$= e^{i\xi x - t\xi^2}\left\{ 1 + a_1(x,\xi)(-t) + a_2(x,\xi)\frac{(-t)^2}{2!} + \cdots \right\}$$

which is exact what one obtains using $\circledast$ above. Thus Seeley's method of construction for the heat kernel is given by integrating the transport equations as a power series in t.

November 11, 1982:

Generalities about ψDO's: Composition formula:

$$(*) \qquad P(x,\xi)\circ Q(x,\xi) \doteq \sum_\alpha \frac{1}{\alpha!}\,\partial_\xi^\alpha P(x,\xi)\cdot D_x^\alpha Q(x,\xi)$$

on the total symbol level. Recall that the total symbol of a ψDO $P = P(x,D)$ is the asymptotic expansion of $P(x,\xi)$ ~~$P(x,D)$~~ as $\xi \to \infty$ where

$$e^{ix\xi}P(x,\xi) = \langle x|P|\xi\rangle.$$

The total symbol is therefore a formal series of homog. fns.

$$P(x,\xi) \doteq \sum p_i(x,\xi)$$

such that there are only finitely many p_i of degree $\geq -N$ for any N. In the above formula, because differentiation ∂_ξ lowers the degree in ξ by 1, the right side is such a formal series assuming P, Q are.

Next start with an elliptic differential operator

$$P(x,D) = p_m + p_{m-1} + \cdots\cdots$$

where each $p_i(x,\xi)$ is a homogeneous polynomial in ξ of degree i. To construct on the symbol level a

$$Q = q_{-m} + q_{-m-1} + \cdots\cdots$$

such that $P\circ Q = 1$. I want to understand exactly what the q_j look like.

$$P\circ Q = \sum_{j,k,\alpha} \frac{1}{\alpha!}\,\underbrace{\partial_\xi^\alpha p_j\; D_x^\alpha q_k}_{\text{homog. in } \xi \text{ of degree } j+k-|\alpha|}$$

$$(P\circ Q)_0 = p_m q_{-m} = 1$$

$$(P\circ Q)_{-1} = p_m q_{-m-1} + p_{m-1} q_{-m} + \partial_\xi p_m\, D_x q_{-m} = 0$$

$$(P \circ Q)_{-2} = p_m q_{-m-2} + p_{m-1} q_{-m-1} + p_{m-2} q_{-m}$$
$$\partial_\xi p_m D_x q_{-m-1} + \partial_\xi p_{m-1} D_x q_{-m}$$
$$\frac{1}{2!} \partial_\xi^2 p_m D_x^2 q_{-m}$$

Since $p_m(x,\xi) \neq 0$ for $\xi \neq 0$, $\frac{1}{p_m}$ is a nice homogeneous function of degree $-m$, so one can solve:

$$q_{-m} = \frac{1}{p_m}$$

$$q_{-m-1} = -\frac{1}{p_m}\left[p_{m-1} q_{-m} + \partial_\xi p_m D_x q_{-m} \right]$$

From the recursion relations one sees that the terms q_{-m-j} are obtained by the following processes: 1) differentiating wrt x, 2) multiplying by the $\partial_\xi^i p_j(x,\xi)$ which are polynomials in ξ, 3) dividing by p_m. Since

$$D_x\left(\frac{1}{p_m^r} u\right) = -r \frac{1}{p_m^{r+1}}(D_x p_m \cdot u) + \frac{1}{p_m^r}(D_x u)$$

~~it~~ is clear that q_{-m-j} is of the form $\frac{u}{p_m^r}$ where u is a polynomial in ξ. Another thing that follows that if p_m is of the form $\lambda \xi_0^m - a_m$, where the variables are $x_0, \dots, x_n$ and everything else (a_m, p_j) is independent of x_0, ξ_0, then the $q_{-m-j} = \frac{u}{p_m^r}$ are independent of x_0 and u is independent of ξ_0.

Let's go back to the heat kernel e^{-tA} where A is a 2nd order operator like the Laplacean, and hence its symbol is a positive-definite form

$$a(x,\xi) = g^{\mu\nu}(x)\, \xi_\mu \xi_\nu .$$

The above calculation gives a[redacted] formal expansion

$$\langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{i\xi x}\left[\frac{1}{\lambda - a} + \frac{u_1(x,\xi)}{(\lambda - a)^2} + \cdots\cdots \right]$$

where $u_n(x,\xi)$ is a polynomial in ξ. By taking a contour integral in λ we then get a formal expansion

$$\langle x | e^{-tA} | \xi \rangle = e^{ix\xi - ta(x,\xi)}\left[1 + (-t)u_1(x,\xi) + \frac{(-t)^2}{2}u_2(x,\xi) + \cdots \right]$$

which in turn gives

$$\langle x | e^{-tA} | x' \rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i(x-x')\xi - ta(x,\xi)} \left[1 + \cdots\cdots\cdots \right].$$

[redacted] Because the $u_n(x,\xi)$ are polynomials in ξ, it follows that the above [redacted] integral involves moments of a Gaussian measure.

For simplicity suppose $a(x,\xi) = a(x)|\xi|^2$, i.e. the metric is conformally equivalent to the standard one. Then

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{i\Delta x\,\xi - ta(x)|\xi|^2} = \frac{e^{-\frac{|\Delta x|^2}{4a(x)t}}}{(4\pi a(x) t)^{n/2}}.$$

I can get the moments of this measure by differentiating wrt Δx, which brings down $\frac{1}{t}$ factors.

Here is a nice observation in Seeley's paper. Suppose the top symbol $p_m(x,\xi)$ is a scalar, or more generally commutes with the other terms in the symbol and their derivatives. Then look at $q_{-m-j}(\lambda,x,\xi)$ which is homogeneous of degree $-m-j$ in $(\xi,\lambda^{1/m})$. Also it can be written as a sum of terms

$$\frac{u_{jk}(x,\xi)}{(\lambda - p_m)^k}$$

where [redacted] $k \le 2j+1$ and u_{jk} ~~has degree~~ is a polynomial

in ξ of degree l, where $l - km = -m - j$.
Now when we do the contour integral

$$\frac{\xi^l}{(\lambda - p_m)^k} \longmapsto e^{-t p_m(x,\xi)} \frac{(-t)^{k-1}}{(k-1)!} \xi^l$$

Now integrate over ξ to get $\langle x | e^{-tA} | x \rangle$
and you get

$$\int \frac{d^n \xi}{(2\pi)^n} e^{-t p_m(x,\xi)} \frac{(-t)^{k-1}}{(k-1)!} \xi^l$$

which gives t^e where the exponent is

$$e = -\frac{n+l}{m} + k - 1 =$$

$$= -\frac{n + (km - m - j)}{m} + k - 1 = -\left(\frac{n-j}{m}\right)$$

Thus the different homogeneous parts $q_{-m-j}(\lambda, x, \xi)$ of the symbol for $(\lambda - A)^{-1}$ contribute the various coeffs. in the asymptotic expansion in powers of t.

Review: Viewpoint: I remain fundamentally suspicious of the Seeley method because for the Laplacean on a curved Riemannian manifold, I expect a Gaussian factor

$$\frac{e^{-\frac{r(x,x')^2}{4t}}}{\left(4\pi\, d(x,x')\, t\right)^{n/2}}$$

instead of the type of factor $e^{-\frac{q(x-x')}{t}}$ where q is a quadratic function. In any case it seems to me that the Seeley is limited to an asymptotic expansion for the diagonal part of the heat kernel. However these objections might not be of any practical importance.

Virtues of his approach. It seems be equivalent to knowing the full symbol of A^{-s} as an analytic fn. of s. Thus from

$$\langle x | \frac{1}{\lambda - A} | \xi \rangle = e^{i\xi x}\left\{ \frac{1}{\lambda - a} + \frac{u_1(x,\xi)}{(\lambda-a)^2} + \cdots\cdots \right\}$$

one gets by contour integration

$$\langle x | A^{-s} | \xi \rangle = e^{i\xi x}\, a(x,\xi)^{-s}\left\{ 1 + (-s)a^{-1}u_1 + \frac{(-s)(-s-1)}{2}a^{-2}u_2 + \cdots \right\}$$

as well as

$$\langle x | e^{-tA} | \xi \rangle = e^{i\xi x - t a(x,\xi)}\left\{ 1 + (-t)u_1 + \frac{(-t)^2}{2!}u_2 + \cdots \right\}$$

November 12, 1982

$$\langle x| \frac{1}{\lambda - A} |\xi\rangle \doteq e^{i\xi x}\left\{ \frac{1}{\lambda - a} + \frac{u_1(x,\xi)}{(\lambda-a)^2} + \cdots \right\}$$

Plug this into $\quad A^{-s} = \frac{1}{2\pi i}\oint \frac{1}{\lambda - A}\lambda^{-s}\,d\lambda$

to get

$$\langle x|A^{-s}|\xi\rangle \doteq e^{i\xi x}\left\{ a^{-s} + (-s)a^{-s-1}u_1 + \frac{(-s)(-s-1)}{2!}a^{-s-2}u_2 + \cdots \right\}.$$

Next you try to use this to get

$$\langle 0|A^{-s}|0\rangle = \int \frac{d^n\xi}{2\pi}\langle 0|A^{-s}|\xi\rangle$$

or more generally

$$\langle x|A^{-s}|x'\rangle \doteq \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi(x-x')}\left\{ a(x,\xi)^{-s} + \cdots \right\}$$

I want the diagonal terms: $x' = x$. Since $u_k(x,\xi)$ is a polynomial in ξ, its clear that formally $\langle x|A^{-s}|x\rangle$ is a sum of terms of the form

$$\int \frac{d^n\xi}{(2\pi)^n}\, a(x,\xi)^{-s-k}\, u_{jk}(x,\xi).$$

(Recall that

$$\langle x| \frac{1}{\lambda - A} |\xi\rangle = g_{-2}(x,\lambda,\xi) + g_{-2-1}(x,\lambda,\xi) + \cdots$$

where

$$g_{-2-j}(x,\lambda,\xi) = \sum_{k=0}^{2j} \frac{u_{jk}(x,\xi)}{(\lambda - a(x,\xi))^{k+1}}$$

and $\quad \deg(u_{jk}) - 2(k+1) = -2-j \quad$ or

$$\deg(u_{jk}) = 2k - j. \quad \text{Then}$$

we have $\quad u_k(x,\xi) = \sum_{j\geq k/2}^{2k} u_{jk}(x,\xi).\quad$ How can this be a polynomial? Clear, because each u_{jk} is one.)

So look at

$$(*)\qquad \underbrace{\int \frac{d^n\xi}{(2\pi)^n}\; a(x,\xi)^{-s-k}\; u_{jk}(x,\xi)}_{\text{has degree } n+2(-s-k)+2k-j \;=\; n-2s-j.}$$

Now one can't integrate a homogeneous function, hence there is something wrong with integrating the formal symbol for A^{-s} wrt ξ. What actual happens is that we have

$$\langle x \,|\, A^{-s} \,|\, \xi \rangle = e^{i\xi x}\, \tilde{A}(x,s,\xi)$$

where $\tilde{A}$ has an asymptotic expansion

$$\tilde{A}(x,s,\xi) \sim a(x,\xi)^{-s} + (-s)\, a(x,\xi)^{-s-1} u_1(x,\xi) + \cdots$$

as $\xi \to \infty$. So it will be true that

$$\langle x \,|\, A^{-s} \,|\, x \rangle = \int \frac{d^n\xi}{(2\pi)^n}\; \tilde{A}(x,s,\xi).$$

To understand this, I should understand $(*)$ with a cutoff around $\xi = 0$, and also the case when $\tilde{A}$ is rapidly decreasing in ξ. In the last case one gets something analytic in s. (?)

If one uses a cutoff in $(*)$ of the form $\rho(a(x,\xi))$ where $\rho = 0$ near 0 and $\rho = 1$ far out, then we can do the integral radially

$$\int_0^\infty r^{n-1}dr \int_{|\omega|=1} \rho(r^2 a(x,\omega))\,(r^2 a(x,\omega))^{-s-k}\, r^{2k-j}\, u_{jk}(x,\omega)\, d\mu(\omega)$$

$$= \int_0^\infty \frac{dr}{r}\, r^{n-2s-j} \underbrace{\int_{|\omega|=1} \rho(r^2 a(x,\omega))\; a(x,\omega)^{-s-k}\, u_{jk}(x,\omega)\, d\mu(\omega)}_{\text{entire function of } s}$$

Let's compare the Hormander construction of $e^{-itA^{1/2}}$ with the Seeley construction of e^{-tA}. Thus I first need to get at

$$\langle x|e^{-tA}|0\rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x - t a(x,\xi)}\{1+\cdots\cdots\}$$

by a WKB method. Seems hard because you can't get $i\xi x - tE(\xi)$ to be a solution of the H.J. equation, you need $\underline{iW(x,\xi) - E(\xi)t}$.

Question for tomorrow:

$$\langle x|e^{-tA}|x'\rangle = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi(x-x') - t a(x,\xi)}\{1+(-t)u_1(x,\xi)+\cdots\}$$

Doing the ξ integral in this Gaussian converts the polys. in ξ in $u_n(x,\xi)$ into

$$\partial_{x'}^{\alpha}\ \frac{e^{-\frac{(x-x')^2}{t a(x)}}}{\cdots}$$

which brings down powers of $\frac{1}{t}$. When does one still get a power series in t?

November 14, 1982

Given an elliptic operator on a compact manifold M one can tensor it with a vector bundle over M and take the index. This gives us a map

$$K^0(M) \longrightarrow \mathbb{Z}$$

which one can interpret as capping with a class in $K_0(M)$ determined by the elliptic operator.

More generally if I have a family of elliptic operators on the fibres of X/Y, then I get a map

$$(*) \qquad K^q(X) \longrightarrow K^q(Y).$$

I feel that these ideas are susceptible to various refinements of a non-topological character involving connections and differential forms. What are some examples?

Let's begin with

$$K^{-1}(X) = K^0(S^1X) = [X, U]$$

and try to describe the map $(*)$ when $q=-1$. Let's suppose then we have an $f: X \longrightarrow U_1$. Then over $S^1 \times X$ we can form a twisted version of the symbol of the operator $D: E \longrightarrow F$ over X we begin with. Thus if $S^1 = [0,1]/0=1$, we use f to form the Mobius bundle versions of E, F over $S^1 \times X$. Now try to compute the index of this family. The index is a virtual bundle over $S^1 \times Y$. What I might try to do is to make a slightly more rigid version of this virtual bundle, which first suffers from the fact that only the symbol is really defined over $S^1 \times X$. If I really want a bona fide operator on $S^1 \times X$, I must join together D and fDf^{-1}. So what might this mean? Take a linear path $D + tB$, understand how the index of $D+tB$ isn't changing, and combine this with the isomorphism of the index of D with that of fDf^{-1} given by f. Here the index means the virtual bundle $(\mathrm{Ker})-(\mathrm{Cok})$ in some sense, and in particular one has the determinant line associated to it, which is precisely defined.

So it's clear now that I am getting exactly the stuff I looked at before.

<u>November 16, 1982</u>

Let us go over the heat kernel formulas with a view toward obtaining an expression for the anomaly. The idea I have is that I should be able to write down an expression for the current J_D defined by

$$\operatorname{Tr}\left(\left(D^*D\right)^{-s}D^{-1}\delta D\right)\Big|_{s=0} = \int \operatorname{tr}(J_D\,\delta D)$$

and then $[D, J_D]$ should be the measure which gives the index.

I will assume that D is a kind of Dirac operator, that is, its symbol is the same as

$$\sum \gamma^k \xi_k$$

where $\quad \left(\sum \gamma^k \xi_k\right)^*\left(\sum \gamma^k \xi_\ell\right) = |\xi|^2$, hence

$$\sum_{k,\ell} (\gamma^k)^*\gamma^\ell = \delta_{k\ell}.$$

For example $\quad D = \frac{1}{i}\partial_x + \partial_y \quad$ in 2 dims. has symbol

$$\xi_1 + (i\,\xi_2)$$

and $$\overline{(\xi_1 + i\xi_2)}(\xi_1 + i\xi_2) = \xi_1^2 + \xi_2^2 .$$

Let's now go over the calculation of the heat kernel for a 2nd order positive elliptic operator A, whose top symbol is $a_2(x,\xi) = a$

$$\langle x|A|\xi\rangle = e^{i\xi x}\left\{a_2(x,\xi) + a_1(x,\xi) + a_0(x,\xi) + \cdots\right\}$$

We calculate the symbol of the resolvent formally

$$\langle x|\frac{1}{\lambda - A}|\xi\rangle = e^{i\xi x}\left\{ g_{-2}(x,\lambda,\xi) + g_{-1}(x,\lambda,\xi) + \cdots\right.$$

where $q_{-2-j}(x, \lambda, \xi)$ is homogeneous of degree $-2-j$ in $\lambda^{1/2}, \xi$. Recursion formulas:

$$a_2(x,\xi)\, q_{-2}(x,\xi) = 1$$

$$a_2 q_{-3} + a_1 q_{-2} + \partial_\xi a_2 \cdot D_x q_{-2} = 0$$

In general

$$\left(\lambda - \sum a_i\right) \circ \sum q_j = \sum_{i,j,\alpha} \frac{1}{\alpha!} \underbrace{\partial_\xi^\alpha(\lambda - a_i)\, D_x^\alpha q_j}_{\text{homog. of degree } i+j-|\alpha|.} = 1$$

So what is the structure of q_{-2-j}? You differentiate wrt x the earlier q_{-2-j}, multiply by ξ-derivatives of the a_j and divide by $\lambda - a(x,\xi)$. Hence

$$q_{-2-j} = \sum_k \frac{u_{jk}(x,\xi)}{(\lambda - a)^{k+1}} \qquad \deg u_{jk} = -2-j+2(k+1) = 2k-j$$

Do the contour integral

$$\langle x | e^{-tA} | \xi \rangle = e^{i\xi x} \sum_{j,k} u_{jk}(x,\xi)\, e^{-t a(x,\xi)} \frac{(-t)^k}{k!}$$

and then

$$\langle x | e^{-tA} | x \rangle = \int \frac{d^n\xi}{(2\pi)^n} \sum u_{jk}(x,\xi)\, e^{-t a(x,\xi)} \frac{(-t)^k}{k!}$$

$$= \sum \underbrace{t^{-\frac{n}{2} - \frac{1}{2}(2k-j)} (+t)^k}_{t^{-\frac{n}{2}+\frac{j}{2}}} \frac{(-1)^k}{k!} \int u_{jk}(x,\xi)\, e^{-a(x,\xi)}$$

$$= \sum_{j,k} t^{\frac{-n+j}{2}} \int \frac{d^n\xi}{(2\pi)^n} \sum_k \frac{(-1)^k}{k!} u_{jk}(x,\xi)\, e^{-a(x,\xi)}$$

A question I had is whether in order to calculate this series one needs to know the symbol of $\frac{1}{\lambda - A}$ or whether one can do with the symbol of $\frac{1}{A}$. So can $\langle x | e^{-tA} | x \rangle$ [redacted] or at least the constant term be expressed in terms of

$$\langle x | \frac{1}{A} | \xi \rangle = e^{i\xi x}\left\{ \sum \frac{(-1)^k u_{jk}(x,\xi)}{a^{k+1}} \right\}$$

Here the u_{jk} are not [redacted] well defined, because we could multiply $\frac{u_{jk}}{a^{k+1}}$ in the numerator and denominator by a. As a test case try

[redacted]

$$\frac{(-1)^k a^{k-\frac{j}{2}}}{(\lambda - a)^{k+1}} = \frac{u_{jk}}{(\lambda - a)^{k+1}}$$

and note this is independent of k when $\lambda = 0$. Then

$$\int \frac{d^n\xi}{(2\pi)^n} \frac{(-1)^k}{k!} u_{jk} \, e^{-a} = \int \frac{d^n\xi}{(2\pi)^n} \frac{1}{k!} a^{k-j/2} e^{-a}$$

$$= \int_0^\infty r^{n-1} dr \, \frac{1}{k!} r^{2k-j} e^{-r^2} \int_{a(\omega)=1} d\mu(\omega)$$

$$= \frac{1}{k!} \int_0^\infty \frac{dr}{2r} r^{\frac{n+2k-j}{2}} e^{-r} c_1 = c \frac{1}{k!} \Gamma\left(\frac{n+2k-j}{2}\right)$$

This obviously depends on k unless $j =$ [redacted] $n-2$.

Now that I understand something about the heat kernel maybe I can understand the anomaly formula in general. Also I think it's true that the whole heat kernel can now be excised from the index thm. proof. Not quite - you still can't make sense of the diagonal values [redacted] $\langle x | \frac{1}{\lambda - A} | x \rangle$.

Here's a little check with Seeley's work. If A were a ψDO, then in the recursion relation you get

$$(a_2 q_{-2-j} + a_0 q_{-1-j} + a_0 q_{-j} + a_{-1} q_{1-j} + \cdots + a_{-j} q_{-2}) + \cdots$$

and since $q_{-2} = \frac{1}{\lambda - a}$, [redacted] q_{2-j} can contain a $\frac{1}{(\lambda-a)^2}$ term i.e.

$$q_{-2-j} = \sum_{k\geq 1} \frac{u_{jk}}{(\lambda-a)^{k+1}}$$

instead of the $2k-j \geq 0$. So the integral

$$\int d^n\xi \; u_{jk}(x,\xi)\, e^{-a(\xi)}$$

which gives trouble when $-n = 2k-j$ or $\frac{-n+j}{2} = k$, leads to a $t^k \log t$ term in the asymptotic expansion for $k \geq 1$.

November 17, 1982

Proof of the anomaly formula: First suppose D is an invertible elliptic operator. I suppose we can regularize D^{-1} along the diagonal, i.e. $\exists$ J such that

$$\int J \cdot \delta D = \left[\mathrm{Tr}\left(e^{-tD^*D} D^{-1} \delta D\right)\right]_{\text{const. term}} .$$

Then take $\delta D = [f, D]$. Then

$$\int J \cdot [f, D] = \int [D, J] f$$

$$\mathrm{Tr}\left(e^{-tD^*D} D^{-1} [f, D]\right) = \mathrm{Tr}\Big(\underbrace{De^{-tD^*D}}_{e^{-tDD^*}D} D^{-1} f - e^{-tD^*D} f$$

$$= \mathrm{Tr}\left(\left(e^{-tDD^*} - e^{-tD^*D}\right) f\right)$$

On the other hand we know from the proof of the index thm. that

$$\left[\langle x | e^{-tD^*D} | x \rangle - \langle x | e^{-tDD^*} | x \rangle \right]_{\text{const. term in } t} = + \omega_D(x)$$

is a measure one integrates to get the index of D. Hence combining one gets

$$[D, J_D] = - \omega_D$$

which is <u>the anomaly formula</u>.

Now there has to be a simple relation between the curvature of the determinant line bundle and the anomaly formula given via the moment map.

Recall in general that if a Lie alg. $\mathfrak{g}$ acts on (M, L) preserving a connection on L, then the action on L can be given by first order operators on sections of L. Then $X \in \mathfrak{g}$

becomes the operator

$$D_X = \nabla_{\bar{X}} + \varphi_X$$

where ∇ is the connection and $\bar{X}$ is the vector field induced by X on M. The fact that X preserves the connection is

$$[D_X, \nabla_Y] = \nabla_{[\bar{X}, Y]} \qquad Y \in \Gamma(T_M)$$

or simply

$$\boxed{Y\varphi_X = \Omega(\bar{X}, Y)}$$

where Ω is the curvature. This last fmla says that

$$d\varphi_X = i(\bar{X})\Omega$$

so that φ_X is a moment map for the action.

Now take the determinant line bundle over operators $D+B$ and the action of gauge transformations. The anomaly fmla gives

$$\varphi_X(D) = +\,\mathrm{Tr}\left(\left[(D^*D)^{-s} - (DD^*)^{s}\right]X\right)\Big|_{s=0}$$
$$= +\int \omega_D \cdot X.$$

Where does this come from? If s is the canonical section, then

$$(\nabla_Y s) = \theta(Y)s \qquad \theta(Y)(D) = \mathrm{Tr}^{reg}(D^{-1}Y)$$

since s is preserved under the gauge action

$$0 = D_X s = (\nabla_{\bar{X}} + \varphi_X)s = (\theta(\bar{X}) + \varphi_X)\, s$$

and hence

$$\varphi_X(D) = -\mathrm{Tr}^{(reg)}(D^{-1}[X, D])$$
$$= \int \omega_D \cdot X \qquad \text{by previous page}$$

So therefore the moment map is given by the measure ω_D. Since I know φ_X, then I can use the formula

$$Y\varphi_X = \Omega(\bar{X}, Y)$$

perhaps to determine Ω. One gets

$$\Omega(\bar{X},Y)(D) = \varphi_X(D+Y) - \varphi_X(D) \qquad \text{mod } Y^2$$

$$= \int (\omega_{D+Y} - \omega_D)\cdot X \qquad \text{mod } Y^2$$

but this would have to be written in terms of $\bar{X} = [X, D]$. Not useful as it stands, because we lack information on ω_D.

Next project: Find ω_D for a Dirac equation in $\mathbb{R}^n$ (n even) with arbitrary gauge potential.

Witten: Consider a Riemannian manifold M. Then one can use a path integral over paths in M to compute e^{-tH} where H is the energy $(= \frac{1}{2}\Delta)$. Hence a path integral over all maps $S^1 \to M$ will give $\mathrm{Tr}(e^{-tH})$, however one can't hope this to be an integral over the fixpts like in the Duistermaat fmla.

The reason is that the Duistermaat fmla involves

$$\int e^{-tH}\cdot\omega^n$$

where ω^n is the symplectic volume, whereas the path integral uses the Riemannian volume. The two are related by a Pfaffian.

So on the space of loops $S^1 \to M$ one considers a closed 2-form $\Omega(X,Y) = \int (X\cdot\frac{d}{dx}Y)\,dx$

where $\frac{d}{dx}$ denotes covariant differentiation. Then the vector field X: $q(x) \longmapsto \dot{q}(x)$ gives rise to the energy function $H(q) = \int \frac{1}{2}|\dot{q}|^2 dx$ in the sense that

$$dH = i(X)\Omega.$$

(Let Y be a tangent vector at q, then

$$\big(i(Y)dH\big)(q) = H(q+Y) - H(q) \qquad \text{mod } Y^2$$

$$= \int \dot{q}\cdot\dot{Y} = \Omega(X_q, Y) \quad .)$$

(Notice that in this example, because we start with the vector field X, the form Ω doesn't have to be non-degenerate.)

The generalization of the Duistermaat formula thus becomes

$$\int e^{-tH(q)} \cdot \mathrm{Pfaff}_q \cdot (\text{Riem. volume})$$

and the Pfaffian is to be realized by a fermion integral. Thus ghost fermions are to be put into the Lagrangian before getting the good quantum theory.

Next the restriction of H to the fixpts. is 0 and hence the result must be independent of t. Somehow what Witten gets is

$$\mathrm{Tr}\left(e^{-tD^2}\gamma_5\right) = \text{Index of Dirac operator.}$$

In some general way, he is able to use super-symmetric ideas to view the standard elliptic complexes ($d, \bar{\partial}$ maybe, signature + Dirac) with the difference of traces in quantum mech. terms.

<u>November 18, 1982</u>

Review the Duistermaat thm: We have a circle action on M with infinitesimal generator X, a closed 2-form ω on M invariant under the action, and the moment fn. $f : M \to \mathbb{R}$ defined by

$$df = i(X)\omega$$

(Assume the closed form on the right is exact). On S^{2N-1} we have the connection form η with $i(X)\eta = 1$ and $\theta(X)\eta = 0$, and so $d\eta$ is the generator for $H^2(S^1\backslash S^{2N-1})$. Then extend ω to the equivariant form $\tilde{\omega} = \omega + d(f\eta) = \omega + df\,\eta + f\,d\eta$; $\tilde{\omega}$ reps a class in $H^2_{S^1}(M)$. Cohomological fixpt formula:

$$\begin{array}{ccc} M^{S^1} & \xrightarrow{i} & M \\ & & \downarrow{\scriptstyle \pi} \\ & & pt \end{array} \qquad \pi_*(\alpha) = \sum_p \frac{i_p^*\alpha}{e(\nu_p)} \quad \text{in } H^*_{S^1}[u^{-1}].$$

Apply this to $\alpha = \tilde{\omega}^{n+k} = (\omega + df\,\eta + f\,d\eta)^{n+k}$. Since $\eta^2 = 0$ and M has even dimension

$$\pi_*(\alpha) = \left[\int_M \binom{n+k}{k} \omega^n f^k\right](d\eta)^k$$

so we get

$$\left(\int_M \frac{f^k}{k!}\frac{\omega^n}{n!}\right)\cancel{(d\eta)^k} = \sum_p \frac{f(p)^{n+k}}{(n+k)!}\frac{(d\eta)^{\cancel{n+k}}}{e(\nu_p)}$$

or using a generating fn.

$$\int_M e^{tf}\frac{\omega^n}{n!} = \sum_p e^{-tf(p)}\frac{(d\eta)^n}{e(\nu_p)^n}$$

Next we want to look at an infinite diml.

version of this. Let's take the examples we get from loop groups. Here one takes a compact connected Lie group G and looks at connections on the principal bundle $S^1 \times G \longrightarrow G$ with a given monodromy. This gives one a symplectic manifold, and we can ask about fixpts. For this example we should want to take a torus action. The trouble with this example is that the fixpoints will correspond to an affine Weyl gp. orbit, so maybe I won't see the relation with the index thm, but rather with the Lefschetz fixpt. formula.

So let's try to bring in the Borel-Weil thm. Over this symplectic manifold M will be a line bundle with a holomorphic polarization, [redacted]. and the representation is the space of holomorphic sections. Now [redacted] you want to apply the ABL-formula to determine the character of this representation. Now I know that the fixpts P are all conjugate under the Weyl group, and the contribution of a fixpt should have the Weyl character formula denominator. Hence I should see [redacted] what $\frac{(d\eta)^n}{e(\nu_p)}$ amounts to in this case.

Setup: T maximal torus inside of G, then $S^1 \times T$ acts on the connections with a given generic monodromy. A connection $\frac{d}{dx}$ [redacted] A will be a fixpt when [redacted] A is constant and centralized by T, hence A must be in $Lie(T)$, and $\exp(2\pi A) =$ monodromy μ. So if $\mu = (\mu_j)$ is diagonal, then the possible A are $\left(\frac{1}{2\pi}\log \mu_j + \right.$ [redacted] $\left. i\mathbb{Z}\right)$. Now I want to determine the [redacted] tangent space to such a fixpoint. I am working with an orbit under the group [redacted] $\mathcal{G} = Map(S^1, G)$ with stabilizer T. So I know that the tangent space is $Lie(\mathcal{G})/Lie\,T$, and so the different characters on [redacted] $S^1 \times T$ are the [redacted] affine roots essentially.

November 19, 1982

Last night I thought about the fact that field theory doesn't have a covariant Hamiltonian formulation. I know that single particle motion can be given a covariant Hamiltonian formalism as follows. Inside of the cotangent bundle of space-time is a hypersurface Y

(1) $$H(q,p,t) = \omega$$

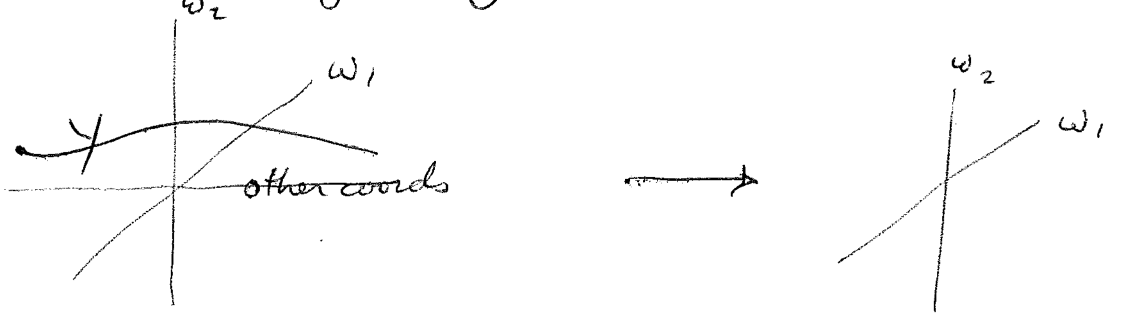
where the canonical 1-form on the cotangent bundle is
$$\eta = p\,dq - \omega\,dt.$$
If space-time has dimension d, then the hypersurface has dimension $2d-1$. Inside the hypersurface is a 1-dimensional distribution which is the kernel for the symplectic 2-form $d\eta$ restricted to Y. Integral curves of this distribution are the solutions of Hamilton's equations. Finally Lagrangian submanifolds contained in Y are [redacted] given by solutions of the Hamilton-Jacobi equation.

Let's next consider 2 independent particles moving in space-time. Think of the hypersurface (1) as giving the allowed possibilities [redacted] of local energy-momentum for the particle at a given space-time point. So with 2 independent particles the allowed possibilities are $Y_1 \times Y_2 \subset T^* \times T^*$. [redacted] In this situation the Hamilton distribution is 2 dim'l and solutions of the Hamilton-Jacobi equations are [redacted] generically described by Lagrangian subm. of $T^* \times T^*$ contained in $Y_1 \times Y_2$.

An interesting point is that upon quantizing this situation one gets a wave function $\psi(x_1^t, x_2^t)$ satisfying two Schroedinger type equations. In other words the Schroedinger equation becomes an over-determined system.

An interesting question is what an interaction between 2-particles looks like from this viewpoint. We can define an interacting system of 2 particles to be a codimension 2 submanifold Y of $T^* \times T^*$ which is integrable. Integrable means that the space of functions vanishing on Y is closed under Poisson bracket, and then the vector fields belonging to these functions give rise to an integrable 2-diml distribution in Y.

I need some simple examples of interacting particles and then I could decide perhaps whether it is possible to reduce things to the standard form of a single Hamiltonian $H(x_1, p_1, x_2, p_2, t)$. Thus I should think of a $Y \subset T^* \times T^*$ as being the zero set of a pair of Hamiltonian functions $\tilde{H}_1, \tilde{H}_2$ on $T^* \times T^*$ with $\{\tilde{H}_1, \tilde{H}_2\} = f_1 \tilde{H}_1 + f_2 \tilde{H}_2$. These functions $\tilde{H}_i$ depend on x_i, p_i, t_i, ω_i. We assume that the distribution generated by the vector fields $X_{\tilde{H}_i}$ $i = 1, 2$ project non-singularly on t_1, t_2 space. This means that given a trajectory of our system, [redacted] which is a 2-diml manifold, we get two curves (world lines) in space-time representing the trajectories of the two particles. Because the vector fields $X_{\tilde{H}_i}$ have indep. projections in the t_1, t_2 [redacted] space, it follows that $\partial_{\omega_i} \tilde{H}_j$ is non-singular, thus the normal bundle to Y projects non-singularly on the ω_1, ω_2 plane



Consequently we can describe Y by two Hamiltonian functions:

$$\omega_i = H_i(x_j, t_j, p_j).$$

Next I should write down the integrability condition. [redacted] Maybe a better thing to do first is to look at the Hamilton-Jacobi equations.

$$(*) \qquad \frac{\partial S}{\partial t_i} + H_i\left(x_1, x_2, \frac{\partial S}{\partial x_1}, \frac{\partial S}{\partial x_2}, t_1, t_2\right) = 0$$

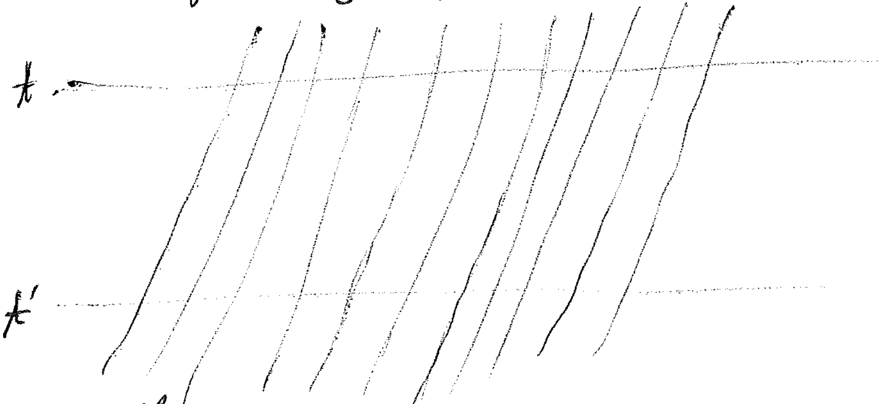
which are two first order equations for $S(x_1, x_2, t_1, t_2)$. (Here S is something like a wave function. Ultimately it should be the phase in a wave function.)

The simplest way to get two [redacted] equations like $(*)$ is to use vector fields. Thus I am going to assume H_i is linear in the p_j. Thus I get two vector fields

$$X_i = \partial_{t_i} + \sum a_{ij}(x_1 t_1 x_2 t_2)\, \partial_{x_j}$$

which generate an integrable distribution, i.e. foliation in $M \times M$. [redacted]

I should first understand a foliation in space-time M. This is the thing you get from a 1-parameter family of diffeomorphisms of space



Like other Hamiltonians which are homogeneous of degree 1, this doesn't correspond to a typical mechanical system, because given [redacted] $(x t) \in M$, you can't prescribe the momentum arbitrarily. Anyway this is the sort of trajectories we get when we look at the 1 space dim'l Dirac equation.

Now the next point is that the leaves of the foliation in $M \times M$ must project into curves in the two factors in order that a leaf be considered a trajectory. So a leaf must be a product of two curves in M which are time-like curves. I find that the leaves have the following description. Each leaf contains a unique point $(x_0, x'_0, 0, 0)$ such that $t = t' = 0$. The leaf projects into two curves in M. One curve is $x = f(t)$ and the other is $x' = g(t')$, and so the leaf consists of all $(f(t), g(t'), t, t')$. Hence if I understand the trajectory $(f(t), g(t), t, t)$, then I recover the whole leaf.

<u>Question</u>: Does $\exists$ any relation with the eigenvalues of $\gamma^{2n+1} = \gamma^1 \cdots \gamma^{2n}$ on the Clifford module and the Gauss sign?

November 20, 1982

Local index formula ~~over~~ $\mathbb{R}^{2n}$ for the Dirac operator with coefficients in a vector bundle. Start with the space of spinors $S = S^+ \oplus S^-$ and the Dirac matrices γ^μ which are hermitian, of square 1, and which anti-commute. Put $Q_5 = \pm 1$ on $S^\pm$ so that the γ^μ anti-commute also with Q_5 and hence have the form

$$\gamma^\mu = \left(\begin{array}{c|c} 0 & (\sigma^\mu)^* \\ \hline \sigma^\mu & 0 \end{array}\right)$$

where $\sigma^\mu : S^+ \longrightarrow S^-$. Next take a vector bundle E over $\mathbb{R}^{2n}$ with a hermitian metric and connection ∇. Then the Dirac operator with coeff. in E is

$$\not{D} = \frac{1}{i} \gamma^\mu \nabla_\mu \quad \text{on} \quad S \otimes E$$

and since it anti-commutes with Q_5 we have

$$\not{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \qquad D : S^+ \otimes E \longrightarrow S^- \otimes E .$$

I propose now to compute the local index expression

$$\langle x | e^{-t\not{D}^2} Q_5 | x \rangle \Big|_{t=0} = \Big[\langle x | e^{-tD^*D} | x \rangle - \langle x | e^{-tDD^*} | x \rangle \Big]_{t=0}$$

which should involve the character of E calculated via differential forms from the connection ∇. This should be done exactly by the Seeley method using the fact that the top symbol of $\not{D}^2$ is $|\xi|^2$.

We must first understand traces of products of the γ-matrices. Think of $\mathbb{R}^{2n}$ as being $\mathbb{C}^n$. Better is to take the even diml real Euclidean space $V = \mathbb{R}^{2n}$, and to describe the spinor representation of $C(V)$ we choose a "polarization" i.e. complex structure on V. Then the spinor module is ΛV and

$$C(V) = \mathrm{End}(\Lambda V)$$

$$\gamma : v \longmapsto e_v + i_{\langle v |}$$

Now choose an orthonormal basis for V and let the corresponding γ operators be $\gamma^1, \ldots, \gamma^{2n}$. Then "$\gamma^5$" is

$$\gamma^{2n+1} = \gamma^1 \cdots \gamma^{2n}$$

Changing the orientation changes Q to $-Q$ because the γ^i anti-commute. Also

$$(\gamma^{2n+1})^2 = (-1)^{2n(2n-1)/2} = (-1)^n.$$

One can check that eigenspaces of $S = \Lambda V$ under γ^{2n+1} are $S^+ = \Lambda^{\text{even}}$ and $S^- = \Lambda^{\text{odd}}$.

I want to know about traces of the form

$$\mathrm{tr}\left(\gamma^{i_1} \cdots \gamma^{i_\ell}\, \gamma^{2n+1}\right)$$

and most of these should be zero. One can reduce to the case where $i_1, \ldots, i_\ell = 1, \ldots, 2k$. An odd ℓ gives an operator $\gamma^{i_1} \cdots \gamma^{i_\ell}$ carrying $S^\pm$ to $S^\mp$ and hence 0 trace. Also because of the fact that the problem depends on the orthogonal ~~structure~~ of V the actual values of $i_1, \ldots, i_\ell$ are irrelevant.

So use the decomposition $V = V_k \oplus V_{n-k}$ where $\gamma^1 \ldots \gamma^{2k}$ come from a real orth. basis of V_k. Then on

$$\Lambda V = \bigoplus_{p,q} \Lambda^p V_k \otimes \Lambda^q V_{n-k}$$

the operator $\gamma^1 \cdots \gamma^{2k}$ is essentially $(-1)^p \otimes \mathrm{id}$ while γ^{2n+1} is essentially $(-1)^{p+q}$ (essentially means factor of $\pm i$). Thus the product has the trace

$$\mathrm{tr}(\gamma^1 \cdots \gamma^{2k} \gamma^{2n+1}) = \sum (-1)^q \dim \Lambda^p V_k \otimes \Lambda^q V_{n-k}$$

$$= \left(\sum_p \binom{k}{p}\right)\left(\sum_q (-1)^q \binom{n-k}{q}\right) = 2^p \cdot 0^q$$

which gives 0 unless $k = n$. Thus

<u>Lemma</u>: $\mathrm{tr}\left(\gamma^{i_1} \cdots \gamma^{i_l} \gamma^{2n+1}\right)$ for $1 \le i_j \le 2n$
$= 0$

when $l < 2n$.

Here is an alternative way to understand the lemma. The Clifford algebra $C(V)$ of the real Euclidean space $V_{\mathbb{R}}$ has a ~~canonical~~ increasing filtration such that $gr\, C(V_{\mathbb{R}}) = \Lambda(V_{\mathbb{R}} \otimes \mathbb{C})$ so what we are proving is that $\alpha \mapsto \mathrm{tr}(\alpha Q)$, $Q = \gamma^{2n+1}$, kills $F_{2n-1} C(V_{\mathbb{R}})$. So instead of using the basis $\gamma^1, \ldots, \gamma^{2n}$ for the space of operators $\gamma(V_{\mathbb{R}} \otimes \mathbb{C}) \subset C(V_{\mathbb{R}})$ we can use the creation and annihilation operators. So I write the operators in normal product form

$$\psi^*_{i_1} \cdots \psi^*_{i_a} \psi_{j_1} \cdots \psi_{j_b}$$

and want to compute the Q-trace, which after all is the alternating sum of traces on ΛV. Clearly I need $a = b$ to have something of degree 0. Then I should think of $\psi_{j_1} \cdots \psi_{j_a}$ as ~~giving~~ an a-diml $W' \subset V^*$ while $\psi^*_{i_1} \cdots \psi^*_{i_a}$ gives an a-dim $W \subset V$. Generically I can suppose $W' \simeq W^*$. In this case I get a splitting $V = W \oplus (W')^\perp$ and the computation of the trace proceeds as before.

Perhaps what's important for future work is that Clifford algebra traces correspond to intersections. I've seen something like this in the Japanese work, and it also is part of the Lefschetz fixpoint business. Let's review the Sato etc. stuff.

We are interested in endomorphisms of ΛV, that is, elements of

$$C(V_{\mathbb{R}}) = \mathrm{End}(\Lambda V).$$

An interesting way to obtain such endomorphisms is by means of a correspondence:

$$W \subset V \times V.$$

It would be better to write this

$$\begin{tikzcd} & W \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ V & & V \end{tikzcd}$$

and to assume p_1 is "oriented" in the sense that we trivialize $\lambda(p_1) = \lambda(W)^* \otimes \lambda(V)$. Then pull-back via p_2 leads to a map

$$p_2^* : \Lambda V \longrightarrow \Lambda W$$

which can be followed by the usual $(p_1)_* = \Lambda(p_1)$ [redacted] More generally I can [redacted] use the formula

$$(p_1)_* \, K \, p_2^*$$

to associate an endomorphism to any $K \in \Lambda W$. But then one might as well make $W = V \times V$. In this way I am thinking of an isomorphism

$$\Lambda(V \times V) \xrightarrow{\sim} \mathrm{End}(\Lambda V),$$

[redacted] whereby subspaces W correspond to endos.

It should be true then that the alternating sum of traces of a degree 0 endom. of ΛV corresponding to intersection of the subspace with the diagonal. This relates a non-geometric trace operation to a geometric intersection.

Let's apply the lemma to prove a local index formula for the Dirac operator in flat space. We have

$$\not{D} = \frac{1}{i}\sum \gamma^\mu \nabla_\mu$$

$$\not{D}^2 = -\sum_{\mu,\nu} \gamma^\mu \nabla_\mu \gamma^\nu \nabla_\nu = -\left[\nabla^2 + \frac{1}{2}\sum_{\mu\neq\nu} \gamma^\mu\gamma^\nu[\nabla_\mu,\nabla_\nu]\right]$$

$$= -\nabla^2 - \frac{1}{2}\sigma F \qquad \sigma^{\mu\nu} = \begin{cases} \gamma^\mu\gamma^\nu & \mu\neq\nu \\ 0 & \mu=\nu \end{cases}$$

Now look at the resolvent

$$\frac{1}{\lambda - \not{D}^2} = \frac{1}{\lambda + \nabla^2 + \frac{1}{2}\sigma F}$$

$$= \frac{1}{\lambda+\nabla^2} + \frac{1}{\lambda+\nabla^2}\left(-\frac{1}{2}\sigma F\right)\frac{1}{\lambda+\nabla^2} + \cdots$$

Let tr_S denote the spinorial trace. All these operators work on $S\otimes E$ and we have

$$End(S\otimes E) = End(S)\otimes End(E) \xrightarrow{tr_S\otimes 1} End(E)$$

Thus

$$tr_S\left(\frac{1}{\lambda-\not{D}^2}Q\right) = \frac{1}{\lambda+\nabla^2}tr_S(Q) + \frac{1}{\lambda+\nabla^2}\left(-\frac{1}{2}\right)tr_S(\sigma Q)F\frac{1}{\lambda+\nabla^2}$$
$$+ \cdots\cdots$$

and the first term to give a possible non-zero contribution is the nth. This is a ψDO of order $-2(n+1)$ in $\mathbb{R}^{2n}$, so it has a continuous kernel, and the diagonal values make sense. So I can at least conclude that

$$\langle x|\, tr_S\left(\frac{1}{\lambda-\not{D}^2}Q\right)|x\rangle = \langle x|\underbrace{\frac{1}{\lambda-D^*D}}_{tr(S^+)}|x\rangle - \langle x|\underbrace{\frac{1}{\lambda-DD^*}}_{tr(S^-)}|x\rangle$$

is a well-defined endomorphism of E.

To understand the next point, let us put $x=0$ and assume that $\nabla^2 = \partial^2$, F constant.

Then I have to understand

$$\left(-\frac{1}{2}\right)^n tr_s\left((\sigma F)^n Q\right) \quad \langle 0 | \frac{1}{(\lambda+\nabla^2)^{n+1}} | 0 \rangle .$$

But

$$\langle 0 | \frac{1}{(\lambda+\nabla^2)^{n+1}} | 0 \rangle = \int \frac{d^{2n}\xi}{(2\pi)^{2n}} \frac{1}{(\lambda+|\xi|^2)^{n+1}}$$

$\leftarrow$ wrong sign

$$= \frac{vol(S^{2n-1})}{(2\pi)^{2n}} \int_0^\infty r^{2n} \frac{dr}{r} \frac{1}{(\lambda+r^2)^{n+1}}$$

$$= \frac{vol(S^{2n-1})}{(2\pi)^{2n}} \frac{1}{2} \int_0^\infty \frac{u^n}{(\lambda+u)^{n+1}} \frac{du}{u}$$

November 22, 1982:

Atiyah talks on Witten's mod 2 anomaly.

Background: If A is a skew-symmetric real matrix, then its eigenvalues are of the form $\pm i\lambda$ with $\lambda \in \mathbb{R}$. If D is a real skew-symmetric elliptic operator (more generally Fredholm operator), then $\dim(\mathrm{Ker}\, D) \bmod 2$ is a homotopy invariant. This is because non-zero eigenvalues must disappear in pairs under specialization.

So next take a family of real skew-symmetric elliptic operators. Then I should be able to define over the parameter space a real line bundle whose fibre at D_A is the highest exterior power of $\mathrm{Ker}(D_A)$. (Moreover, analytic torsion, better the ζ fn. of $-D_A^2$ should give a metric on this line bundle.) I forgot to mention that this line bundle has a canonical section which vanishes where D_A has $\mathrm{Ker} \neq 0$.

Now we take the Dirac operators as examples. The Dirac operator written as a real operator looks like

$$\not{D} = \sum \left(\tfrac{1}{i}\gamma^\mu\right)\partial_\mu \qquad ?\quad ?$$

where $\frac{1}{i}\gamma^\mu$ are symmetric anticommuting matrices of square -1, so that $\not{D}^2 = -\partial_\mu^2$. Now modify this by a gauge potential: $\partial_\mu \longmapsto \partial_\mu + A_\mu$ where A_μ is skew-symmetric, and then one can ask questions about gauge invariance. ? ?

General example: Given $D: E \to F$ on complex vector spaces I can consider the self-adjoint op. $\not{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ on $E \oplus F$ with the $Q_5 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ operator. Because Q_5 anti-commutes with $\not{D}$ it establishes isomorphisms between the $+\lambda$ and $-\lambda$ eigenspaces for $\lambda \neq 0$. Also one knows that $\mathrm{Ker}\,\not{D} \cong \mathrm{Ker}\, D \oplus \mathrm{Ker}\, D^*$ so that $\dim(\mathrm{Ker}\,\not{D}) = \mathrm{Index}\, D \bmod 2$.

Idea: We have the gauge group $\mathcal{G}$ acting on the space $\mathcal{A}$ of connections and we want to construct gauge invariant determinants for the operator D_A associated to a connection A. A determinant function is a trivialization of the determinant line bundle $\mathcal{L}$ over $\mathcal{A}$. A gauge invariant determinant is a trivialization of the descended bundle over $\mathcal{A}/\mathcal{G}$. The point is that $\mathcal{L}$ is defined over $\mathcal{A}/\mathcal{G}$ (modulo assuming $\mathcal{G}$ acts freely). So one can see that gauge invariant determinants don't exist by showing $\mathcal{L}$ over $\mathcal{A}/\mathcal{G}$ is non-trivial. Because of

$$H^0(\mathcal{G}, \mathbb{Z}/2) = H^1(\mathcal{A}/\mathcal{G}, \mathbb{Z}/2)$$

one might construct an example using a disconnected gauge group.

Actual talk:

Anomaly = quantity conserved classically but not when quantized.

① Index theory: M compact $D: C^\infty(M,E) \to C^\infty(M,F)$

$\text{index}(D)$ = top formula

family D_t $t \in$ Compact space T. Then

$$\text{index}(D) \in K(T)$$

Real operators

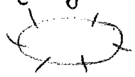
$$\text{ind } D \in KO(T)$$

Examples:

ⓐ $T = S^1$ $\text{ind} \in KO(S^1) = (\mathbb{Z}/2) \oplus \mathbb{Z}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underset{w_1(\text{Ind } D)}{\cup}$

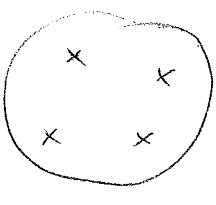
Assume D_t gen. invertible



w_1 = no. mod 2 of $t \ni D_t$ not invertible

ⓑ $T = S^2$ $\text{ind} \in K(S^2) = \mathbb{Z} \oplus \mathbb{Z}$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \underset{c_1(\text{Ind } D)}{\cup}$



c_1 = alg. no. of $t \ni D_t$ not invertible

~~more sophisticated~~

$\text{ind } D \begin{array}{l}\in KO(T) \\ \in K(T)\end{array} \Rightarrow \det(\text{ind } D)$ real / cx. line bundle

∞-determinants: Δ Laplace type op. elliptic s.a. >0

$\det \Delta = \prod \lambda$ Regularize

$\zeta(s) = \mathrm{Tr}(\Delta^{-s}) = \sum \lambda^{-s}$ $\mathrm{Re}\, s > \dfrac{\dim M}{\text{order } \Delta}$

$\zeta(0) =$ "formal. dim Hilbert space"

$\zeta'(0) = -\log \det \Delta$

Ex: $-\dfrac{d^2}{dx^2}$ on S^1

$\zeta(s) = 2 \sum \dfrac{1}{n^{2s}}$

$-\dfrac{d^2}{dt^2} - \alpha^2$ $\quad n^2 - \alpha^2$

$$\frac{\prod(n^2 - \alpha^2)}{\prod n^2} = \prod\left(1 - \frac{\alpha^2}{n^2}\right) = \frac{\sin \pi\alpha}{\pi\alpha}$$

$\det D = (\det D^* D)^{1/2} e^{i\theta}$ can this be done

Assume $w_1(\text{index}) \neq 0$. Claim $\exists$ no possible $\det(D_t) \in \mathbb{R}$ varying smoothly with t.

Also complex cases if $c_1(\text{index}) \neq 0$ over S^2 can't construct smooth $\det(D_t) \in \mathbb{C}$.

$M = S^4$ Basic operator $D: S^+ \longrightarrow S^-$

$G \to P$, $P \downarrow M = S^4$ $\quad G \longrightarrow \mathrm{Aut}(V)$ $\quad G = SU_2, SU_3$

$V = P \times^G V$

need connection A on P $\quad D_A : S^+ \otimes V \longrightarrow S^- \otimes V$

$\mathcal{G}$ = gauge transformations = id at ∞.

$P = S^4 \times G \Longrightarrow \mathcal{G} = \Omega^4 G$

$\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$ free action

index $((D_A)) \in K(\mathcal{A}/\mathcal{G})$

$$B\mathcal{G} = \Omega^3 G$$

What does index thm. say? get a map $\Omega^3 G \to \mathbb{Z}\times BU$, $\longrightarrow \mathbb{Z}\times BO$

(a) $G = SU(2)$ $V = \mathbb{C}^2$ $P = M \times G$

$$\Omega^3(SU(2)) = \Omega^3(Sp(1)) \longrightarrow \Omega^3(Sp) \overset{\text{Bott}}{\sim} \mathbb{Z}\times BO$$

Claim the index of this family of Dirac operators is this Bott map. We know

$$\pi_1(\Omega^3 SU(2)) = \pi_4(SU(2)) = \pi_4(S^3) = \mathbb{Z}_2 \xrightarrow{\sim} \pi_1(BO)$$

In this ~~ex~~ example the Dirac op. over S^4 is quaternionic, but also $SU(2)$ on $\mathbb{C}^2$ is quaternionic so the Dirac on $S^+\otimes V \longrightarrow S^-\otimes V$ is real.

(b) $G = SU(3)$ $V = \mathbb{C}^3$

$$\pi_2(\Omega^3 SU(3)) \longrightarrow \pi_2(BU)$$
$$\| $$
$$\pi_5(SU(3))$$

Condition for anomoly to disappear

$\sum \lambda_i^3 = 0$ eigenvalue

November 23, 1982

Consider a compact Kähler manifold M, a C^∞ vector bundle E with metric over M, and let $\mathcal{A}$ be the space of holomorphic structures on E. These can be identified with certain connections of E preserving the metric, namely, the ones whose curvature $F = \nabla^2$ [redacted] is of type $(1,1)$, so that ∇ induces a $\bar\partial$ operator.

Over $\mathcal{A} \times M$ we obtain then a holomorphic bundle $\tilde{E}$ which is $pr_2^*(E)$ equipped with the appropriate varying holomorphic structure. Let's compute $c_1(pr_{1!}\tilde{E})$ using GRR. The conjecture I have is that this determinant line bundle equipped with the analytic torsion metric has curvature form given by the GRR formula on the differential form level.

I want to compute $ch(\tilde{E})$ on the differential form level. It's enough for calculating $c_1(pr_! E)$ to work with a 1-cx. diml. family. So I will suppose given a 1-parameter family ~~depending~~ of holomorphic structures on E depending holom. in t.

~~Locally these can be described by $\bar\partial$ op. of the form~~

$$\bar\partial_t : E \longrightarrow E \otimes T_M^{0,1}.$$

Locally using coords z_μ on M and a local trivialization of E with metric we have

$$\bar\partial_t = \left[\partial_{\bar z_\mu} + \alpha_\mu(t,z)\right] d\bar z^\mu$$

The $\bar\partial$ on $\tilde{E}$ is going to be

$$\bar\partial = \partial_{\bar t}\, d\bar t + \left[\partial_{\bar z_\mu} + \alpha_\mu\right] d\bar z^\mu$$

because we want holom. sections of $\tilde{E}$ to be a module over holom. fns. in t. The connection on $\tilde{E}$ is then

$$\nabla_{\tilde{E}} = \partial_t dt + \partial_{\bar t} d\bar t + \left[\partial_{z_\mu} - \alpha_\mu^*\right] dz^\mu + \left[\partial_{\bar z_\mu} + \alpha_\mu\right] d\bar z^\mu$$

which can be written

$$\nabla_{\tilde{E}} = d_t + \nabla_E \qquad \text{on } \tilde{E} = pr_2^* E$$

Hence the curvature is

$$\nabla_{\tilde{E}}^2 = \nabla_E^2 + \left(d_t \nabla_E + \nabla_E d_t \right)$$

or

$$F_{\tilde{E}} = F_E + d_t A.$$

In local coords.

$$\nabla_{\tilde{E}}^2 = \left(\partial_{z_\mu} \alpha_\nu + \partial_{\bar{z}_\nu} \alpha_\mu^* - [\alpha_\mu^*, \alpha_\nu] \right) dz^\mu d\bar{z}^\nu$$

$$+ \quad \partial_t \alpha_\mu \, dt \, d\bar{z}^\mu \; - \; \partial_{\bar{t}} \alpha_\nu^* \, d\bar{t} \, dz^\nu$$

where $\partial_{\bar{t}} \alpha_\mu = 0$ as we assume α is holom. in t.
Then

$$ch(\tilde{F}) = tr(e^{\tilde{F}}) = ch(F)\, ch(d_t A)$$

except maybe we should be careful because these are matrices.

Let's suppose first that we are working with line bundles. Then the curvature form take values in the commutative ring of even forms, and so the above works. Thus we get in general

$$e^{d_t A} = 1 + d_t A + \tfrac{1}{2}(d_t A)^2 + \cdots$$

$$= 1 + d_t A + \tfrac{1}{2}\left(\partial_t \alpha_\mu \, dt \, d\bar{z}^\mu - \partial_{\bar{t}} \alpha_\nu^* \, d\bar{t} \, dz^\nu \right)^2$$

$$= 1 + d_t A + \tfrac{1}{2}\Big[\partial_t \alpha_\mu \circ \partial_{\bar{t}} \alpha_\nu^* \quad dt\, d\bar{t}\, dz^\nu d\bar{z}^\mu$$
$$\partial_{\bar{t}} \alpha_\nu^* \; \partial_t \alpha_\mu \Big]$$

so for line bundles

$$e^{d_t A} = 1 + d_t A + \partial_t \alpha_\mu \, \partial_{\bar{t}} \alpha_\nu^* \quad dt \, d\bar{t} \quad dz^\nu d\bar{z}^\mu$$

GRR gives

$$ch(pr_{1!}\tilde{E}) = (pr_1)_*\left(ch(\tilde{E})\, Todd(M)\right)$$

where $(pr_1)_*$ is now to be interpreted as integration over the fibre M. $Todd(M)$ is a collection of forms of type (p,p) over M in $dz^\mu, d\bar{z}^\mu$. For line bundles we have

$$e^{\tilde{F}} = \underbrace{e^{F}}_{\text{forms of type } (p,p) \text{ in } dz^\mu,\, d\bar{z}^\mu} e^{d_t A}$$

so it's clear that we get a contribution only from the $(d_t A)^2$ term, because an odd product of $dz^\mu, d\bar{z}^\mu$ integrates to 0. Thus

$$c_1(pr_{1!}\tilde{E}) = \left\{ \int_M \underbrace{\partial_t \alpha_\mu \partial_{\bar{t}} \alpha^*_\nu \, dz^\nu d\bar{z}^\mu}_{\frac{1}{2}(d_t A)^2} \cdot \left[e^F \, Todd\, M\right]_{2n-2} \right\} dt\, d\bar{t}$$