November 1, 1981

General stuff: I have the feeling that many objects in topology can be quantized in the sense that you can put a measure on them and get out numbers. There is the example of a classical gas which is a measure on the infinite symmetric product $SP(X)$.

Problem: Can you make a gas theory out of Segal chains on X?

Recall how Feynman relates a quantum situation to a classical situation using paths. The coordinate x describes a basis for the ~~Hilbert space~~, so that in computing time-evolution one wants to sum over ways of picking an intermediate state x at a given t. Thus one gets paths $x(t)$ over which one must sum.

Now we can view classical gas configurations as a basis in a quantum gas, and hence one [redacted] might be able to compute some examples of [redacted] measures on <u>paths</u> in a classical gas.

Recall [redacted] an old idea that placing a system in contact with a reservoir is analogous to taking a G-space X and multiplying by P_G.

Consider a particle on a line governed by a potential $U(x)$. [redacted] Let's [redacted] apply a constant force J to it and find its position. The new potential is $U(x) - Jx$ and its position $\bar{x}$ is

$$\bar{x} = \frac{\int x e^{-\beta(U-Jx)} dx}{\int e^{-\beta(U-Jx)} dx} = \frac{1}{\beta} \frac{\partial}{\partial J} \log Z(J)$$

where $Z(J) = \int e^{-\beta U + (\beta J)x} dx$.

The pseudo-potential $V(x)$ is the potential which when combined with the applied force J gives the correct position: $\frac{d}{dx}(V - Jx) = 0$ at $x = \bar{x}$.

One knows that V is the Legendre transform of $\frac{1}{\beta}\log Z(J)$. Thus

$$V(\bar{x}) = J\bar{x} - \frac{1}{\beta}\log Z(J)$$

where J is regarded as a fn. of $\bar{x}$ via $\bar{x} = \frac{\partial}{\partial J}\left(\frac{1}{\beta}\log Z\right)$.

Clearly the pseudo-potential is what one measures in the lab, and so the simplest possible mathematical description of nature should use models for which the potential and pseudo-potential are the same. Look at simple harmonic oscillator:

Classically: $H_0 = \frac{p^2}{2m} + \frac{\alpha}{2}q^2$.

$$Z(J) = \int e^{-\beta H_0 + \beta J x}\, dp\, dx \Big/ \int e^{-\beta H_0}\, dp\, dx$$

$$= \int e^{-\frac{\beta\alpha}{2}x^2 + \beta J x}\, dx \Big/ \int e^{-\frac{\beta\alpha}{2}x^2}\, dx$$

$$= e^{\frac{1}{2}\frac{(\beta J)^2}{\beta\alpha}} = e^{\frac{1}{2}\frac{\beta}{\alpha}J^2} \qquad \frac{\log Z}{\beta} = \frac{1}{2\alpha}J^2$$

So

$$\bar{x} = \frac{\partial}{\partial J} = \frac{J}{\alpha}$$

$$V = J\cdot\frac{J}{\alpha} - \frac{1}{2\alpha}J^2 = \frac{J^2}{2\alpha} = \frac{(\alpha\bar{x})^2}{2\alpha} = \frac{\alpha}{2}\bar{x}^2$$

which is the same as U.

Quantally: $H = \frac{p^2}{2m} + \frac{\alpha}{2}q^2 - Jq$

$$= \frac{p^2}{2m} + \frac{\alpha}{2}q^2 - \frac{J}{\sqrt{\alpha}}\sqrt{\alpha}\,q + \frac{J^2}{2\alpha} - \frac{J^2}{2\alpha}$$

$$= \underbrace{\frac{p^2}{2m} + \frac{\alpha}{2}\left(q - \frac{J}{\sqrt{\alpha}}\right)^2}_{\text{unitarily equivalent to } H_0} - \frac{J^2}{2\alpha}$$

So therefore you see that the eigenvalues of H are shifted from those of H_0, so

$$Z(J) = \frac{tr\left(e^{-\beta H}\right)}{tr\left(e^{-\beta H_0}\right)} = e^{\frac{\beta J^2}{2\alpha}}$$

which is the same as in the classical case.

Notice that in both examples you can't see the temperature in $\bar{x}(J)$, however, it appears in the fluctuations. Thus one has classically

$$\overline{x^2} - \bar{x}^2 = \left(\frac{1}{\beta}\frac{\partial}{\partial J}\right)^2 \log Z = \frac{1}{\alpha\beta} = \frac{kT}{\alpha}$$

Quantum mechanically the fluctuation (at least at $J=0$) of x is: Take $H_0 = \frac{p^2}{2} + \frac{1}{2}\omega^2 q^2$

$$\langle q^2 \rangle = \left\langle \left(\frac{a+a^*}{\sqrt{2\omega}}\right)^2 \right\rangle = \frac{1}{2\omega}\langle \underbrace{aa^* + a^*a}_{2a^*a+1} \rangle$$

$$= \frac{1}{2\omega}(2\bar{n}+1) \qquad \bar{n} = \frac{1}{e^{\beta\omega}-1}$$

$$\therefore \quad \langle q^2 \rangle = \frac{1}{\omega}\left(\frac{e^{\beta\omega}+1}{e^{\beta\omega}-1}\right) \sim \frac{1}{\beta\omega^2} = \frac{kT}{\omega^2} \quad \text{as } T \to \infty.$$

(The general formula for $\frac{p^2}{2m} + \frac{1}{2}a q^2$ is probably

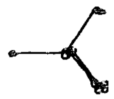
$$\langle q^2 \rangle = \frac{\hbar\omega}{a}\left(\frac{e^{\beta\hbar\omega}+1}{e^{\beta\hbar\omega}-1}\right) \sim \frac{1}{a\beta} \quad \text{as above.}$$)

Let's look formally at when the potential and pseudo-potential coincide.

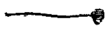
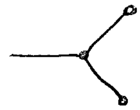
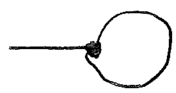
$$e^{\beta F(J)} = Z(J) = \int e^{\beta\left(Jx - \frac{a}{2}x^2 - \frac{b}{3!}x^3 - \cdots\right)} dx \Big/ \int e^{\beta\left(-\frac{a}{2}x^2 - \frac{b}{3!}x^3 - \cdots\right)} dx$$

Then $\beta F(J)$ is a sum of terms indexed by connected diagrams with vertices and edge

One can take $\beta = 1$ because the effective power of β is the number of loops. Thus

$F(J) =$  $+$  $+$  $+ \cdots\cdots$

$$= \frac{J^2}{2a} + \frac{(-b)}{a^3}\frac{J^3}{6} + \frac{(-b)J}{\beta a^2 2} + \cdots\cdots$$

$\bar{x} = F'(J) =$  $+$  $+$  $+ \cdots$

$$= \frac{1}{a}J + \frac{1}{a}(-b)\left(\frac{1}{a}J\right)^2\frac{1}{2} + \frac{1}{a}(-b)\frac{1}{a}\cdot\frac{1}{2}\frac{1}{\beta}$$

$$= \frac{1}{a}J + \frac{1}{a}\Gamma_1 + \frac{1}{a}\Gamma_2\,\bar{x} + \frac{1}{a}\Gamma_3\frac{\bar{x}^2}{2!} + \cdots$$

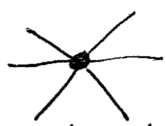
where Γ_n is the sum over irreducible graphs with n external lines and no J terms, with the proviso that Γ_1 doesn't contain , but rather starts with  $= \frac{-b}{a\cdot 2}$. (Put β in later). Thus

$$\frac{dV}{d\bar{x}} = J = (-\Gamma_1) + (a-\Gamma_2)\bar{x} + (-\Gamma_3)\frac{\bar{x}^2}{2} + \cdots\cdots$$

$$V(\bar{x}) = (-\Gamma_1)\bar{x} + (a-\Gamma_2)\frac{\bar{x}^2}{2} + (-\Gamma_3)\frac{\bar{x}^3}{3!} + \cdots\cdots$$

When is this the same as $U(x) = a\frac{x^2}{2} + b\frac{x^3}{3} + \cdots$.

The only irreducible graph without loops having n-external lines is



so one wants irred graphs with loops not to contribute. The only way to do this for different β is it seems for b, c, .. etc to be 0.

November 3, 1981.

General Feynman approach illustrated in one degree of freedom: $H_0 = \frac{p^2}{2} + U(x)$, and we add to this a source term $J(t)x$ [redacted] with J of compact support. Then we can form the imaginary-time action

$$iS_J(x) = -\int \left[\frac{1}{2}\dot{x}^2 + U(x) + J(t)x\right] d\tau$$

and form

$$Z(J) = \frac{\int \mathcal{D}x\, e^{iS_J(x)}}{\int \mathcal{D}x\, e^{iS_0(x)}}$$

where the path integrals are done as follows. One considers paths $x(\tau)$ starting with $x(\tau_0) = x_0$, ending with $x(\tau_1) = x_1$. Then you let $\tau_0 \to -\infty$, $\tau_1 \to +\infty$ and the limit should be independent of x_0, x_1. In effect we know that

$$\int_{\substack{x(\tau_0)=x_0 \\ x(\tau_1)=x_1}} \mathcal{D}x\, e^{iS_J(x)} = \langle x_1 | U_J(\tau_1, \tau_0) | x_0 \rangle$$

$$= \int dx_a\, dx_b\, \langle x_1 | e^{-(\tau_1-\tau_a)H_0} | x_a \rangle$$

$$\times \langle x_a | U_J(\tau_a, \tau_b) | x_b \rangle \langle x_b | e^{-(\tau_b-\tau_0)H_0} | x_0 \rangle$$

and $\langle x_1 | e^{-(\tau_1-\tau_a)H_0} | x_a \rangle = \sum_\alpha$ [redacted] $e^{-(\tau_1-\tau_a)E_\alpha} \langle x_1 | \alpha \rangle \langle \alpha | x_a \rangle$

So that as $\tau_1 \to +\infty$ [redacted] the ground state term dominates [redacted]. So as long as $\langle x_0 | 0 \rangle, \langle x_1 | 0 \rangle \neq 0$, the actual [redacted] values for x_0, x_1 won't matter.

Now that one has $Z(J)$ one obtains the Green's functions from it. But I forgot to mention that we have the following formula

$$Z(J) = \langle 0 | S_J | 0 \rangle$$

where here S_J is the scattering operator

$$S_J = e^{\tau_1 H_0}\, U_J(\tau_1,\tau_0)\, e^{-\tau_0 H_0}$$

where $\operatorname{Supp} J \subset [\tau_0, \tau_1]$.

Now the Green's functions are things like

$$\langle 0 | x(t) | 0 \rangle \qquad\qquad \langle 0 | T[x(\tau)x(\tau')] | 0 \rangle$$

$$\overset{\shortparallel}{\langle 0 | x | 0 \rangle} \qquad\qquad \overset{\shortparallel}{\langle 0 | x\, e^{-(\tau-\tau')H_0} x | 0 \rangle}\, \cancel{e^{(\tau-\tau')E_0}}$$

say $E_0 = 0$.

so that one can go back to real time and get

$$\langle 0 | x\, e^{-i(t-t')H_0} x | 0 \rangle$$

first for $t > t'$ and then for $t < t'$ by taking the conjugate. So therefore just from the 1 and 2 point Green's function one recovers the subspace of $L^2(\mathbb{R})$ spanned by $|0\rangle$, $x|0\rangle$ and their translates under time.

So far I have been assuming there is a unique discrete ground state which isn't the case for a periodic potential. Also one should review the double-well and instantons.

So next look at the EM field. Working in imaginary time [redacted] a field configuration is a 1-form $A = \sum A_\mu dx^\mu$ and the imaginary-time action is

$$iS_0 = -\tfrac{1}{2}\|dA\|^2$$

Sources in this situation are currents J so that the perturbed action is

$$iS_J(A) = -\tfrac{1}{2}\|dA\|^2 + (J, A)$$

where

$$(J, A) = \int \sum_\mu J^\mu A_\mu \qquad \text{is integrated}$$

over space-time. Usually J is a cycle in the sense that $\nabla\cdot J = \sum \partial_\mu J^\mu = 0$. This means that $(J, df) = 0$, so that $iS_J(A)$ is invariant under gauge transformations.

Now because the [redacted] action is quadratic in A we have Gaussian path integrals, so formally

$$Z(J) = \frac{\int e^{-\frac{1}{2}\|dA\|^2 + (J,A)}}{\int e^{-\frac{1}{2}\|dA\|^2}} = e^{\frac{1}{2}(J,GJ)}$$

where GJ is a solution of the equation $d^*dA = J$. Let's go over this abstractly.

[redacted] We have a vector space V consisting of the A and the subspace W consisting of the df. We have the form $\frac{1}{2}\|dA\|^2 = \frac{1}{2}(A, d^*dA)$ which is non-degenerate on V/W. The current J belongs to $(V/W)^\vee = W^\perp$. The form $\frac{1}{2}(J,GJ)$ should be the quadratic form on $(V/W)^\vee$ one gets by the isom $V/W \longrightarrow (V/W)^\vee$ associated to d^*dA. So G should be an operator $(V/W)^\vee = W^\perp \longrightarrow V/W$ which you obtain by solving the equation $d^*dA = J$. As far as the generating function $Z(J)$ is concerned, which A you choose doesn't matter.

Do the analysis by using Fourier transform:

$$A = \sum_k A_k e^{ikx} \qquad \overline{A_k} = A_{-k}$$

where A_k is a 4-vector. Then

$$dA = i \sum_k e(k) A_k e^{ikx} \qquad d^*J = -i\sum i(k) J_k e^{ikx} = 0$$

$$d^*dA = [redacted] \sum_k i(k)e(k) A_k e^{ikx}$$

Thus $d^*dA = J$ becomes $\boxed{i(k)e(k)A_k = J_k}$ which can be solved iff $i(k)J_k = 0$. Since

$$i(k)e(k) = |k|^2 - e(k)i(k)$$

the simplest solution is in the Lorentz gauge: $i(k)A_k = 0$ or

$$A_k = \frac{1}{|k|^2} J_k$$

Thus we can take [redacted] $G = \frac{1}{\Delta}$.

And now we have the problem of getting back to the real-time [redacted] situation, and the Hilbert space.

November 5, 1981

Connes pseudo-torus: Let $\mathcal{A}$ be the algebra generated by two invertible elements u, v with the relation
$$vu = \zeta uv$$
where ζ is a scalar. Clearly $\mathcal{A}$ has a basis consisting of the monomials $u^m v^n$ with the mult. rule
$$(u^m v^n)(u^{m'} v^{n'}) = \zeta^{nm'} u^{m+m'} v^{n+n'}.$$
Hence
$$[u^m v^n, u^{m'} v^{n'}] = (\zeta^{nm'} - \zeta^{n'm})\, u^{m+m'} v^{n+n'}$$
If $m+m'=0$, $n+n'=0$, then $\zeta^{nm'} = \zeta^{-nm} = \zeta^{n'm}$ and so one sees that the monomial $u^0v^0 \notin [\mathcal{A},\mathcal{A}]$, and consequently there is a trace on $\mathcal{A}$ given by

$$(*) \qquad \begin{aligned} \mathcal{A} &\longrightarrow \mathbb{C} \\ \sum a_{mn} u^m v^n &\longmapsto a_{00} \end{aligned}$$

Let's now compute the universal trace $\mathcal{A} \to \mathcal{A}/[\mathcal{A},\mathcal{A}]$. Put $m+m'=a$ and $n+n'=b$; then $u^a v^b$ will be in $[\mathcal{A},\mathcal{A}]$ when an m, n can be found such that
$$\zeta^{n(a-m)} \neq \zeta^{(b-n)m} \qquad \text{or} \qquad \zeta^{na} \neq \zeta^{mb}$$
Because $na - mb \in \mathbb{Z}$, if ζ is not a root of unity, then $\zeta^{na-mb} = 1 \Rightarrow na - mb = 0$ and if this holds for all $m, n \in \mathbb{Z}$ we must have $a = b = 0$. If ζ is a primitive l-th root of 1, then $\zeta^{na-mb} = 1 \Leftrightarrow na - mb \equiv 0 \pmod{l}$, and this holds for all $m, n \in \mathbb{Z} \Leftrightarrow a, b \equiv 0 \ (l)$.

Hence we see that for ζ not a root of unity that $(*)$ is the universal trace for the algebra $\mathcal{A}$.

Recall the "crossed product" construction. One has a field F with a primitive l-th root of unity ζ so that one has $H^1(F, \mu_l) = F^\cdot/(F^\cdot)^l$ by Hilbert's Thm. 90 (in general) and hence by cup product a pairing

$$H^1(F,\mu_\ell)\otimes H^1(F,\mu_\ell)\to H^2(F,\mu_\ell^{\otimes 2})\overset{\text{uses }\zeta}{\cong} H^2(F,\mu_\ell)$$

$= $ elements of order $|\ell$ in $Br(F)=H^2(F,\mathbb{G}_m)$.

According to Milnor's book one takes $f,g\in F^\circ$ and forms the algebra with generators u,v over F and relations

$$\begin{cases} u^\ell=f\,, & v^\ell=g \\ vu=\zeta\, uv \end{cases}$$

Next try to generalize as follows: We want to replace $\{u^n\}$ by an abelian group, also $\{v^n\}$, and the commutation relation by a pairing ~~into~~ the multiplicative group. For example

$$e^{iap}\,e^{ibq}=e^{iab}\,e^{ibq}\,e^{iap}.$$

In this case when we form the algebra $\mathcal{A}$ we get operators

$$\int A(b,a)\,e^{ibq}\,e^{iap}$$

where the coefficients $A(b,a)$ should go to zero sufficiently fast.

Take a pseudo-differential operator in the Hormander style notation:

$$\begin{aligned}
P(x,D)f &= \int\frac{d\xi}{2\pi}\,P(x,\xi)\,e^{-i\xi x}\hat f(\xi)\\
&= \int\frac{d\xi}{2\pi}\,P(x,\xi)\,e^{-i\xi x}\int dy\; e^{-i\xi y}f(y)\\
&= \int\frac{d\xi\,dy}{2\pi}\,P(x,\xi)\,e^{i\xi(x-y)}f(y)\\
&= \int\frac{d\xi\,dy}{2\pi}\,P(x,\xi)\,e^{-i\xi y}f(x+y)
\end{aligned}$$

Now $\quad \left(e^{ibq}e^{iap}f\right)(x)=e^{ibx}\left(e^{iap}f\right)(x)=e^{ibx}f(x+a)$

so to write $P(x,D)$ in the form of a linear combination of the operators $e^{ibq}e^{iap}$ one writes

$$P(x,\xi) = \int \frac{db}{2\pi} e^{ibx} P_1(b,\xi) \qquad P_1(b,\xi) = \int dx\, e^{-ibx} P(x,\xi)$$

$$\int \frac{d\xi}{2\pi} P(x,\xi) e^{-i\xi y} = \int \frac{db}{2\pi} e^{ibx} \underbrace{\int \frac{d\xi}{2\pi} P_1(b,\xi)^{-i\xi y}}_{Q(b,y)}$$

Thus

$$\begin{aligned} P(x,D)f &= \int \frac{d\xi\, dy}{2\pi} P(x,\xi) e^{-i\xi y} f(x+y) \\ &= \int \frac{db\, dy}{2\pi} \cdot Q(b,y) e^{ibx} f(x+y) \end{aligned}$$

where

$$\begin{aligned} Q(b,y) &= \int \frac{dx\, d\xi}{2\pi} e^{-ibx} P(x,\xi) e^{-i\xi y} \\ P(x,\xi) &= \int \frac{db\, dy}{2\pi} e^{ibx} Q(b,y) e^{i\xi y} \end{aligned}$$

Now I should be in a position to understand the trace. [redacted] Our algebra of operators has as basis the simple operators $e^{ibq}e^{iap}$ which multiply as follows:

$$e^{ibq}e^{iap}\,e^{ib'q}e^{ia'p} = e^{iab'}\,e^{i(b+b')q}\,e^{i(a+a')p}$$

so the subspace $[a,a]$ [redacted] is generated by

$$[e^{ibq}e^{iap}, e^{ib'q}e^{ia'p}] = (e^{iab'} - e^{ia'b})\,e^{i(b+b')q}e^{i(a+a')p}$$

Set again $\beta = b+b'$ $\quad \alpha = a+a'$.

$$\begin{aligned} e^{iab'} - e^{ia'b} &= e^{ia(\beta-b)} - e^{i(\alpha-a)b} \\ &= e^{-iab}[e^{ia\beta} - e^{i\alpha b}] \end{aligned}$$

For fix β,α this [redacted] vanishes identically for all b,a [redacted] only when $\beta,\alpha = 0$. So the trace of $P(x,D)$ should be $Q(0,0)$:

$$\operatorname{tr}\{P(x,D)\} = Q(0,0) = \int \frac{dx\,d\xi}{2\pi}\, P(x,\xi)$$

which is consistent with the idea that the kernel of $P(x,D)$ is

$$K(x,y) = \int \frac{d\xi}{2\pi}\, P(x,\xi)\, e^{i\xi(x-y)} .$$

Consider next the example of the circle; precisely $\{u^n\}$ becomes the operators $\{z^n\}$, and $\{V\}$ becomes the translation operators e^{iap} where $p = \frac{1}{i}\frac{d}{d\theta} = z\frac{d}{dz}$. Thus we are dealing with the basis

$$e^{ibq}\, e^{iap} \qquad a \in \mathbb{R}/2\pi\mathbb{Z} \qquad b \in \mathbb{Z}$$

[redacted] and we have the same trace. We [redacted] can also write elements of our algebra of operators in the form

$$p(x,D) \quad \text{where} \quad D = \frac{1}{i}\frac{d}{dx}$$

$$\begin{aligned} p(x,D)f &= \frac{1}{2\pi}\sum_{\xi}{}' \, p(x,\xi)\, e^{i\xi x}\hat{f}(\xi) \\ &= \frac{1}{2\pi}\sum_{\xi}{}' \, p(x,\xi)\, e^{i\xi x} \int_0^{2\pi} dy\, e^{-i\xi y} f(y) \\ &= \sum_{\xi\in\mathbb{Z}} \int \frac{dy}{2\pi}\, p(x,\xi)\, e^{i\xi(x-y)} f(y) \end{aligned}$$

and the trace is

$$\operatorname{tr}(p(x,D)) = \sum_{\xi\in\mathbb{Z}} \int \frac{dx}{2\pi}\, p(x,\xi)$$

The next thing I want[redacted] to do is to give examples of Type II algebras which are inductive limits of finite matrix rings. The basic idea is to use the map

$$(*) \qquad [redacted] \ \operatorname{End}(V) \longrightarrow \operatorname{End}(W\otimes V)$$

which on K_0 level is $\mathbb{Z} \xrightarrow{\dim W} \mathbb{Z}$. In effect one has an equivalence

$$\mathcal{P}_k \simeq \mathcal{P}_{\operatorname{End}(V)}, \qquad \operatorname{Hom}_{\operatorname{End}(V)}(V,M) \longleftarrow M$$

$$W \longmapsto V\otimes W \quad ,$$

and relative to this equivalence both K_0 groups are $\mathbb{Z}$. But the base extension map $W \longmapsto \mathrm{End}(V) \otimes_k W$ sends k to $\mathrm{End}(V) = V \otimes V^*$ which is $\dim(V^*)$ copies of V.

So we see that if we take an inductive system of homomorphisms of the form $\circledast$ we obtain an algebra with $K_0 \subset \mathbb{Q}$. Now I believe that the C^*-algebra inductive limit maybe classified by this subgroup, but the weakly closed algebra has $K_0 = \mathbb{R}$. (work of Glimm)

Now try to use cross-products to get a handle on the finite-dimensional matrix rings, and see if one can construct a good ~~[illegible]~~ inductive system.

Suppose we start with the algebra of rank n^2 generated by u, v subject to $vu = \zeta uv$, $u^n = v^n = 1$ where ζ is a primitive nth root of unity. Then we take a divisor m of n, say $n = md$ and we try to find inside a similar algebra of rank m^2. The simplest thing to try is the algebra generated by u^d, v^d. The relation is

$$v^d u^d = \zeta^{d^2} u^d v^d$$

and for this subalg to be a matrix ring we must have that ζ^{d^2} is a primitive m-th root of unity. But ζ^d is a primitive m-th root already, so $(\zeta^d)^d = \zeta^{d^2}$ is a primitive m-th root if and only if $(d, m) = 1$. Thus we can't get an interesting p-adic tower in this way.

November 6, 1981

Last night I tried to find a locally compact abelian group A such that the algebra of operators on $L^2(A)$ generated by multiplication by characters and translations has both a trace and lots of idempotents, so that one gets a Type II situation. This didn't work probably because this algebra has an irreducible repn, namely $L^2(A)$, so its weak closure is a Type I algebra.

November 8, 1981

The connections between ζ functions and cohomology and K-theory:

First review the Weil calculation of the ζ fn. for a smooth complete variety over $\mathbb{F}_q$

$$\zeta(X,s) \overset{\text{defn}}{=} \prod_{x \in X_{cl}} \frac{1}{1-(Nx)^{-s}} \qquad \begin{aligned} Nx &= \operatorname{card} k(x) \\ &= q^{\deg x} \end{aligned}$$

$$\underset{\zeta(X,q^{-s})}{\overset{\|}{}}$$

$$\log Z(X,z) = \sum_x -\log\left(1-z^{\deg x}\right)$$

$$= \sum_x \sum_k \frac{z^{k \deg x}}{k \deg x} \cdot \deg x = \sum_m \frac{z^m}{m} \underbrace{\sum_{\deg(x)\mid m} \deg x}_{\text{Fix } F^m \text{ on } \bar{X}(\bar{\mathbb{F}}_q)}$$

Use Lefschetz + graph Frob always transverse to Δ

$$= \sum_m \frac{z^m}{m} \operatorname{tr}\{F^m \text{ on } H^\cdot(\bar{X})\}$$

So

$$Z(X,z) = 1/\det\left(1 - z(F \text{ on } H^\cdot(\bar{X}))\right)$$

Thus for $X = \mathbb{F}_q$ we get

$$\boxed{\zeta(\mathbb{F}_q, s) = \frac{1}{1-q^{-s}}}$$

Take a complete n.s. curve C of genus g over $\mathbb{F}_q$. Then

$F = 1$ on H^0

$F = q$ on H^2

F has eigenvalues $\alpha_1, \dots, \alpha_{2g}$ on H^1 with $|\alpha_i| = q^{1/2}$

so that

$$\zeta(C,s) = \frac{\prod_1^{2g}(1-z\alpha_i)}{(1-z)(1-qz)}$$

Recall $z = q^{-s}$ and the product defining ζ converges nicely for $\operatorname{Re}(s) > 1$. So the pole first encountered occurs at $s=1$ and is due to the H^2. Removing points from the curve kills the H^2 but not the $H^2_!$ which has to be in the formula for ζ for an incomplete variety.

Let's look at conjectural values of the K-groups. We have

$K_{2i-1}(\mathbb{F}_q)$ cyclic order q^i-1 weight i $i \geq 1$

$K_0(\mathbb{F}_q) = \mathbb{Z}$ weight 0.

$$\therefore \quad |\zeta(-i)| = 1/(q^i-1) = \frac{1}{|K_{2i-1}|} \qquad i \geq 1.$$

$$|\zeta(0)| = \infty = |K_0|$$

Look at C and remove off two copies of $K_*(\mathbb{F}_q)$. Let's work with $\bar{C}$. We have a map

$$K(\bar{C}) \longrightarrow K(\mathbb{F}_q) \times K(\mathbb{F}_q)$$

given by restricting to a point and by integrating over $\bar{C}$. The map is onto because one can map the other way by lifting to $\bar{C}$ or pushing in via a point

Original idea I had was to look at the K-groups $K_i(\mathbb{C})$ and break them up into pieces according to the action of the Adams operations. The part of weight i should be related to $|\zeta(-i)|$. So let's conjecturally compute $K_i(C)$. First do $K_i(\bar{C})$ and then use descent, i.e. take the Galois invariants. Also

$$K_*(\bar{C}) = K_*(k) \overset{L}{\otimes} K_0(C) \qquad k = \bar{\mathbb{F}}_q$$

where $K_0(\bar{C}) = \mathbb{Z} \oplus \mathrm{Pic}(\bar{C})$ and $0 \to \mathrm{Pic}^0(\bar{C}) \to \mathrm{Pic}(\bar{C}) \xrightarrow{\text{degree}} \mathbb{Z} \to 0$

Since $K_{odd}(k)$ are $\cong \mathbb{Q}/\mathbb{Z}[\frac{1}{p}]$ and $\mathrm{Pic}(\bar{C}) \cong (\mathbb{Q}/\mathbb{Z}[\frac{1}{p}])^{2g}$ + some p torsion

we get
$$K_{2i}(\bar{C}) = \mathrm{Tor}_1(\mathrm{Pic}^0(\bar{C}), K_{2i-1}(\bar{k})) \quad \text{has weight } 1+i \quad i \geq 1$$

$$K_{2i-1}(\bar{C}) = K_{2i-1}(\bar{k}) \otimes K_0(C) \qquad i \geq 1$$

$$K_{2i-1}(\bar C) = \underbrace{K_{2i-1}(\bar k)}_{\text{weight } i} \oplus \underbrace{K_{2i-1}(\bar k)\otimes \underbrace{\frac{\mathrm{Pic}\,\bar C}{\mathrm{Pic}^\circ \bar C}}_{\mathbb{Z} \text{ with } \psi^k = k}}_{\text{weight } i+1}$$

Now try to take Galois invariants, and one must be careful not to confuse ~~Galois~~ Galois and Adams operations. So for example on the 2nd factor above, one can obtain it as a map

$$K_{2i-1}(\bar k) \xrightarrow{i_*} K_{2i-1}(\bar C)$$

induced by a rational point over $\mathbb{F}_q$, so i_* will commute with Galois although it doesn't commute with the Adams operations. Thus taking invariants under Galois gives at least two copies of $K_{2i-1}(\mathbb{F}_q)$ in $K_{2i-1}(C)$ one having weight i the other of weight $i+1$. When one takes invariants one really is thinking in terms of

$$\longrightarrow K_\cdot(C) \longrightarrow K_\cdot(\bar C) \xrightarrow{1-\mathrm{Frob}} K_\cdot(\bar C) \longrightarrow \cdots$$

~~and hence the $K_{2i}(\bar C)$ which are of rank 2g over $\prod_{\ell\neq p}\mathbb{Z}_\ell$ + p-torsion part will contribute in degree K_{2i-1}. So~~

$$\cancel{K_{2i-1}(C) = \underbrace{K_{2i-1}(\mathbb{F}_q)}_{\text{weight } i} + \underbrace{K_{2i-1}(\mathbb{F}_q)}_{i+1} + \underbrace{K_{2i}(\bar C)_{1-\mathrm{Frob}}}_{i+1}}$$

So if we put all this together we get

$$|\zeta(-i)| = \frac{|K_{2i-2}(C)|}{\underbrace{|K_{2i-1}(\mathbb{F}_q)|}_{q^i-1}\ \underbrace{|K_{2i-3}(\mathbb{F}_q)|}_{q^{i-1}-1}}$$

~~Which somewhat laughable. The denominator factor in $\zeta(i)$ is $(1-q^i)$~~

The denominator factor in $\zeta(-i)$ is

$$(1-q^{-(-i)})(1-q\,q^{-(-i)}) = (1-q^i)(1-q^{1+i})$$

so something is slightly off.

Let's check carefully in the case of $\mathbb{P}^1_{\mathbb{F}_q}$, where everything can be computed. I want to compute ψ^p on the image of

$$i_* : K_{2i-1}(\mathbb{F}_q) \longrightarrow K_{2i-1}(\mathbb{P}^1_{\mathbb{F}_q})$$

where i is the embedding of ∞ say. We have

$$i_*(\alpha) = i_*1 \cdot \pi^*\alpha \qquad \pi : \mathbb{P}^1_{\mathbb{F}_q} \to \mathbb{F}_q.$$

So

$$\Psi^p(i_*(\alpha)) = \Psi^p(i_*1) \cdot \pi^*\Psi^p(\alpha)$$

$$= p \cdot p^i \; i_*1 \cdot \pi^*\alpha$$

so the image of i_* has weight $i+1$. Thus the weight i part of $K_*(\mathbb{P}^1_{\mathbb{F}_q}) = \pi^*(K_*(\mathbb{F}_q)) \oplus i_*(K_*(\mathbb{F}_q))$ is

$$\underbrace{\pi^* K_{2i-1}(\mathbb{F}_q)}_{q^i-1} \oplus \underbrace{i_* K_{2i-3}(\mathbb{F}_q)}_{q^{i-1}-1}$$

order

which doesn't give the right denominator in the ζ functions.

What seems to be wrong is simply that the weight i part contributes to $\zeta(-i+1)$. Thus

$$|\zeta(-i)| = \frac{|K_{2i}(C)|}{\underset{q^{i+1}-1}{|K_{2i+1}(\mathbb{F}_q)|}\;\underset{q^i-1}{|K_{2i-1}(\mathbb{F}_q)|}}$$

is made up of weight $i+1$

Check this by observing that if we pass to an affine curve by removing a rational point, then the factor q^i-1 disappears. And with the ζ the factor $1-z$ belonging to $H^0_!$ disappears.

So it seems that from the point of view of ζ something interesting is going on with the K-groups. Somehow Thomason's theory says otherwise. There is a straightforward connection between K-theory and coh.

According to Thomason

K is connected to $H^\bullet$

and by Weil etc.

ζ is connected to $H_!^\bullet$

and Poincaré duality gives a connection between

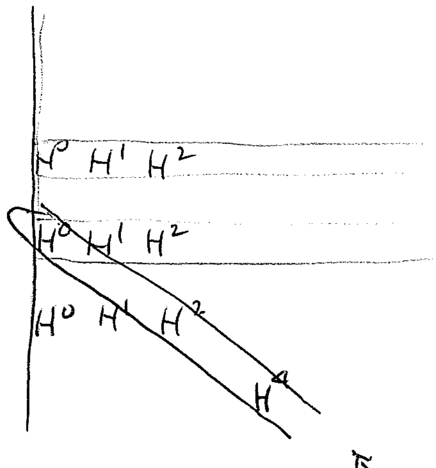
$H^\bullet$ and $H_!^\bullet$

So what one sees maybe is a dimension shift. The point is that by Poincaré the eigenvalues of Frobenius on $H^\bullet$ and $H_!^\bullet$ are related.

Let's try to write down a clear set of conjectural formulas. First we have the spectral sequence

$$E^2 = H^\bullet(\bar{X})(*) \Longrightarrow K^T(\bar{X}) \qquad T = \text{top or Thomason}$$

and we have to take coinvariants under Frobenius to get $K(X)$.



↖ where K_0 ends up.

So in the spectral sequence K_0 is related to the $\bigoplus_p H^{2p}(p)$ and in fact $W_p(K_0) \doteq H^{2p}(p)$. Thus if we take the horizontal line $H^*(i)$ we are getting the weight i part of the K-theory, and so

$$\left| \frac{1}{\det(1 - q^{-i} F)} \right| = \left| \text{weight } i \text{ part of } K(X) \right|$$

But here we are talking about F on $H^\bullet(\bar{X})$ and for ζ we want F on $H_!^\bullet(\bar{X})$. For X complete they are the same so I am still confused ?

Do again:

$$K_j(X) \sim \bigoplus_i \underbrace{H^{2i-j}(\bar{X})(i)}_{\text{weight } i} \Big/ \text{Frob}$$

$$\therefore \left| K(X)_{\text{weight } i} \right| = \det\left(1 - F \text{ on } H^{\bullet}(\bar{X})(i)\right)^{\pm 1}$$
$$= \det\left(1 - q^{-i}F \text{ on } H^{\bullet}(\bar{X})\right)^{\pm 1}$$

because $K_0(X) \sim \bigoplus H^{2i}(i)$ should have invariants which means $1 - q^{-i}F$ shoulds have 0's on $H^{2i}(\bar{X})$. Thus for $X = \mathbb{P}^d$, we would get

$$\left| K(X)_{\text{weight } i} \right| = \frac{1}{(1-q^{-i}) \cdots (1-q^{-i+d})}$$

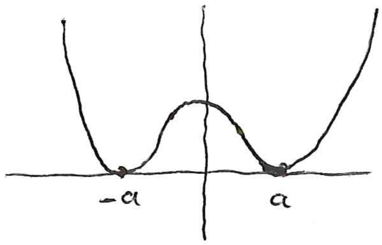
which is $\zeta(i)$, which in [redacted] turn is $\zeta(d-i)$ by Poincaré duality. [redacted] So maybe this is a correct principle that K groups compute ζ's at positive integers primarily.

November 9, 1981

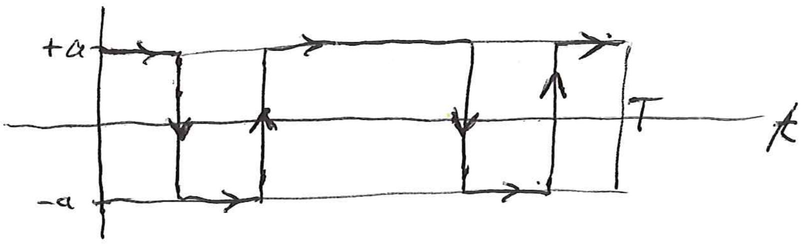
Consider the ring of integers A in a number field F. Yesterday it appeared that there is an analogy between the K-theory of A, or maybe it would be better to say the K-theory of F, and between Langlands style gadgets. In the K-theory one has Adams operations and in Langlands theory the Hecke operators. It should be possible to find a uniform viewpoint

November 17, 1981

Back to instantons. First consider the double-well



and let's adopt the viewpoint of the Feynman lectures, where the electron in a linear lattice is treated. The idea is that we approximate the above by a 2-state system with base states $|a\rangle$, $|-a\rangle$, which we can think of as simple harmonic oscillator ground states. Then tunnelling allows one state to pass to the other with a certain amplitude γ. Now we want to compute $\langle \pm a|e^{-TH}|\pm a\rangle$ which we can do as a sum over histories:



so you see the instantons. There is no difficulty doing this sum in many ways.

$$H = \begin{pmatrix} \omega_0 & -\gamma \\ -\gamma & \omega_0 \end{pmatrix}$$

eigenvalues	eigenvector
$\omega_0 - \gamma$	$\|a\rangle + \|-a\rangle$
$\omega_0 + \gamma$	$\|a\rangle - \|-a\rangle$

$$\langle a|e^{-TH}|a\rangle = \sum_{n \text{ even}} e^{-\omega_0 T}(+\gamma)^n \int_{0<t_1<\cdots<t_n<T} dt_1 \cdots dt_n \qquad \frac{T^n}{n!}$$

$$= e^{-\omega_0 T}\cosh(+\gamma T)$$

(γ should be >0, so the symmetric eigenstate $|a\rangle + |-a\rangle$ has the lower energy.)

Similarly one does the periodic potential, and this is done in the Feynman lectures. Thus one has base states $|n\rangle$ for $n \in \mathbb{Z}$ and the Hamiltonian is

$$H = \omega_0 - \gamma(T + T^{-1}) \qquad T = \text{shift}$$

so one works instead of the states $|n\rangle$ with the wave states $\langle n|\theta\rangle = e^{in\theta}$. Then

$$H|\theta\rangle = \left(\omega_0 - \gamma(e^{i\theta} + e^{-i\theta})\right)|\theta\rangle$$
$$= (\omega_0 - 2\gamma\cos\theta)|\theta\rangle$$

so the degenerate level ω_0 is split by tunneling into a band.

Finally one comes to gauge theories. Here one has to make some transformations. The role of the variable x now becomes the field $A(x)$ over space. If we were to proceed exactly as before, then an instanton would be a field $A(t,x)$ joining at the ends $A_0(x)$, $A_1(x)$ which are like $\pm a$.
However in the former case the action is

$$iS = -\int [\tfrac{1}{2}\dot{x}^2 + U(x)]\, dt$$

so that the ends of an instanton (= finite action configuration) are constant paths with values at points x where $U(x) = 0$. For gauge fields the action is

$$iS(A) = -\int \tfrac{1}{2}\|dA\|^2$$

so that finite action means dA goes to zero sufficiently fast, both in t and x directions. So it's sort of clear that the fields at the ends $t = 0$ or T say have to be zero (up to a gauge transformation). So we no longer worry about different a's, now we have a single a, but lots of different ways of tunneling – one for each instanton.

<u>November 18, 1981</u>

Let us consider the case of a gauge field in 2-dimensions with the idea of understanding instantons and the different vacuua. [redacted] The simplest case is with the scalar field. We have to distinguish between connection forms belonging to a line bundle and [redacted] 1-forms.

Fix a complex line bundle L over a manifold X. [redacted]. A connection on L determines a curvature which is a closed 2-form. Two connections differ by a [redacted] 1-form. [redacted] Do this precisely: select sections $\{s_i\}$ trivializing L over $\{U_i\}$ whence you get a 1-cocycle $f_{ij} = s_i s_j^{-1}$ with values in $\mathcal{O}^*$. A connection is given by a family of 1-forms A_i such that

$$D(s_i) = [redacted]\ A_i s_i$$

Hence $A_i f_{ij} s_j =$ [redacted] $A_i s_i$ [redacted] $= D(f_{ij} s_j) = (df_{ij}) s_j + f_{ij} A_j s_j$ so that

$$\frac{df_{ij}}{f_{ij}} = A_i - A_j .$$

Now taking a different connection $D' s_i = A_i' s_i$, one sees that $A_i' - A_i$ is a global 1-form. Invariantly, we know the connections form a torsor under $\mathrm{Hom}(L, L\otimes T^*) \cong \Gamma(T^*)$. Finally the curvature is the global closed 2-form dA_i. Invariantly it is a section of $\mathrm{Hom}_{\mathcal{O}}(L, L\otimes \Lambda^2 T^*) \cong \Gamma(\Lambda^2 T^*)$, which is closed. Also we know that the coh. class of this 2-form is equivalent to the Čech cocycle

$$\pm \frac{1}{2\pi i}\left[\log f_{ij} - \log f_{ik} + \log f_{jk}\right]$$

So we should think of the line bundle L as given, but that we can choose the connection arbitrarily. Thus on the curvature level one sees a closed 2-form varying by an arbitrary 1-form differential. We also have the gauge group $\Gamma(\mathcal{O}^*)$ which is acting on the connection by adding $\frac{df}{f}$, $f \in \Gamma(\mathcal{O}^*)$.

Now you see how different this picture is from the way you used to think about the EM field, namely, as just a closed 2-form.

Recall what was done in the case of the trivial bundle. In this case $D = d + A$, so a connection is a 1-form A and the curvature is dA. Now what we did was to look at the action

$$iS(A) = -\frac{1}{2}\int |dA|^2$$

on the space of A of compact support, and then the functional integral

$$Z(J) = \int e^{iS(A) + \int JA} \Big/ \int e^{iS(A)}$$

is a Gaussian which can be evaluated.

Other approach is to look not at A with compact support, but to require that dA has compact support. Then A defines a flat connection far out, which means there is a winding numbers. If the winding number is integral there is a natural way to trivialize the line bundle near ∞. In effect $e^{\int_{p}^{p'} A}$ is independent of the path joining P' to P, provided we stay outside the support of dA, and gives a canonical isomorphism of the fibres over P' and over P.

Now we are trying to calculate a functional integral and we know we have to work modulo gauge group orbits. So where are we? I can fix the winding number and try to understand the action integral over all A with dA of comp. support having this winding number. If A, A' are two such, that $d(A'-A)$ is compactly supported 2 form with integral 0, so $d(A'-A) = d\omega$ where ω is a compact supp. 1-form. Then $A'-A-\omega = df = \frac{d(e^f)}{e^f}$ where f is a function, so that up to a gauge transform A' is the same as $A+\omega$. So this seems to imply that we fix A_0 and then we calculate the action fnl. integral over all $A_0+\omega$. This present problems because if I were to try to minimize $\|d(A_0+\omega)\|^2$, then I get the condition

$$d^*(dA_0 + d\omega) = 0$$

so because $dA_0 + d\omega$ has compact support, or at least decays, we get

$$dA_0 + d\omega = 0$$

which is impossible as the winding no. $\neq 0$.

So what else can I do? Put in a current J and you get

$$d^*(d(A_0+\omega)) = J. \qquad ?$$

November 22, 1981

Recall vertex operators $X(\gamma, z)$ for the Kac-Fraenkel fundamental repn. are rigged up to commute for different z, so that one can form

$$X(\gamma, f) = \int f(z)\, X(\gamma, z)$$

and get a commuting family of operators belonging to the elements $f(z) X_\gamma$ $\quad X_\gamma =$ root vector. Then to the group element $\exp(f X_\gamma) = 1 + f X_\gamma$ belongs the operator

$$e^{X(\gamma, f)} = e^{\int f(z) X(\gamma, z)}$$

Idea: This formula appears as a S-operator provided we think of z as a time variable. What would be nice is for there to exist a Hilbert space with a time-evolution and a basic set of operators $X(\gamma, 0)$ at time zero, such that to an element of the Lie algebra belongs a time-zero operator. Then given a path $X(t)$ depending on time in the Lie algebra of compact support, one associates the S-operator

$$S = T\{e^{\int X(t)\,dt}\}$$

à la Dyson. This operator should then be the operator corresponding to the path $T\{e^{\int X(t)\,dt}\}$ in the group.

What is one doing? We are trying to represent the group of paths $g(t)$ in G which have compact support. Look at representations of the Lie algebra of paths $X(t)$ in $\mathfrak{g}$. There should be idealized elements belonging to $\delta_a(t) X$ with $X \in \mathfrak{g}$

So we are looking for representations of time and of the time-zero operators $\delta(t)X$.

So let's carefully look at the case of the group S^1. Suppose given $f : S^1 \longrightarrow S^1$, $z \longmapsto f(z)$ where $z = e^{it}$ and t is to be thought of as time. I am supposed to build up the unitary operator belonging to f as a time-ordered product:

$$S = T\{e^{\int g(t)dt}\}$$

corresponding to a similar way that f is built up. Now we know

$$f(t) = e^{\int_{-\infty}^{t} \frac{df}{f}}$$

hence $\quad g(t) = \dfrac{f'(t)}{f(t)} = \dfrac{d}{dt}\log f.$

For example suppose $f = 1$ for $t < 0$, and then it jimps around the circle to $f = 1$ for $t > 0$, i.e. $\log f(t) = 2\pi i\, H(t)$. Then

$$g(t) = 2\pi i\, \delta(t)$$

and so we want to think of a general f as built up out of such δ-functions.

Recall what one does for f of degree 0. In this case

$$\log f = i \sum c_n z^n \qquad \overline{c_n} = c_{-n}$$

and one associates to f the operator

$$e^{-\frac{1}{2}\sum |c_n|^2 n}\; e^{a^*((\log f)^-)}\; e^{a^*((\log f)^+)}$$

$$= e^{-\frac{1}{2}\sum |c_n|^2 n}\; e^{\sum_1^\infty i\bar{c}_n a_n^* \sqrt{n}}\; e^{\sum_1^\infty i c_n a_n \sqrt{n}} \qquad \text{up to a scalar of modulus 1}$$

There also a diagonal factor which maybe we worry about latter.

So what I want to do is to associate to the value of $g(t) = \frac{f'(t)}{f(t)}$ at t an operator which will be a linear combination of a, a^* so that the S-matrix is this previous operator. Recall the formula that

$$T\left\{ e^{\int (J a^* + \tilde{J} a)^{(4)}_{n} dt} \right\} = \text{const}\; e^{\gamma a^*} e^{\tilde{\gamma} a}$$

where $\quad \gamma = \int J e^{i\omega t} dt \qquad \tilde{\gamma} = \int \tilde{J} e^{-i\omega t} dt$

Since

$$\int \frac{d\theta}{2\pi} \sum_{m=1}^{\infty} i c_m z^m \sum_{n=1}^{\infty} z^{-n} a_n \sqrt{n} = \sum_{1}^{\infty} i c_n a_n \sqrt{n}$$

it appears that

$$\tilde{J} a = \underbrace{\left(\sum_{m=1}^{\infty} i c_m z^m \right)}_{\tilde{J}(t)} \left(\frac{1}{2\pi} \sum_{n=1}^{\infty} a_n \sqrt{n} \right)$$

so that the basic operator belonging to $\delta(t)$ is

$$2\pi \delta(t) \longleftrightarrow \sum_{1}^{\infty} a_n \sqrt{n} + \sum_{1}^{\infty} a_n^* \sqrt{n}$$

Similarly

$$2\pi \delta(t-\varepsilon) \longleftrightarrow \sum_{1}^{\infty} e^{-in\varepsilon} a_n \sqrt{n} + \sum_{1}^{\infty} e^{in\varepsilon} a_n^* \sqrt{n}$$

The commutator of these two is

$$\sum_{1}^{\infty} e^{in\varepsilon} n - \sum_{1}^{\infty} e^{-in\varepsilon} n$$

which is essentially the derivative of the δ-function.

$$2\pi \delta(t) = \sum e^{int}$$

?

It appears that we take

$$T\left\{ e^{\int_0^{2\pi} (\log f)(t) \left[\frac{1}{2\pi}\sum_{n=1}^{\infty} a_n^* \sqrt{n} + a_n \sqrt{n}\right](t)\, dt} \right\}$$

to get the operator belonging to f. Thus to an element of the Lie algebra, i.e. a map $S^1 \xrightarrow{g} i\mathbb{R}$ we take the operator belonging to $f = 1 + \varepsilon g$ where $\varepsilon^2 = 0$ and it seems that I get the operator

$$\int_0^{2\pi} g(t) \left[\sum_1^{\infty} a_n^* \sqrt{n} + a_n \sqrt{n} \right](t)\, \frac{dt}{2\pi}.$$

$$= \sum_1^{\infty} \left(b_{-n} a_n^* \sqrt{n} + b_n a_n \sqrt{n} \right) \qquad g = \sum b_n z^n$$

Check that

$$\left[\sum_1^{\infty} b_{-n} a_n^* \sqrt{n} + b_n a_n \sqrt{n},\ \sum_1^{\infty} b'_{-n} a_n^* \sqrt{n} + b'_n a_n \sqrt{n} \right]$$

$$= \sum_1^{\infty} b_n b'_{-n} n - b_{-n} b'_n n = \sum_{-\infty}^{\infty} b_n b'_{-n} n .$$

Nov. 23, 1981:

What does it mean to take a quadratic fn. of a and a^* and take the normal-ordered exponential e.g. what is $:e^{ta^*a}:$?

$$:e^{ta^*a}: = :\sum \frac{t^n (a^*a)^n}{n!}:$$

$$= \sum_n t^n \frac{(a^*)^n a^n}{n!}$$

Thus $\left(:e^{ta^*a}:\right) e^{\lambda z} = \sum t^n \frac{z^n \lambda^n}{n!} e^{\lambda z} = e^{(t+1)\lambda z}$.

or $\boxed{\left(:e^{ta^*a}:\right) e_\lambda = e_{(t+1)\lambda}}$ so it is not unitary even if t is purely imaginary.

Compare this with the formula

$$\boxed{e^{ta^*a}\, e_\lambda = e_{e^t\lambda}}$$

Let's construct the free fermion field over the real line. This means we want to find operators $\psi(x)$, $x \in \mathbb{R}$ satisfying the commutation relations

$$\tfrac{1}{2}\{\psi(x), \psi(x')\} = \delta(x-x').$$

More precisely to each $f(x)$ on $\mathbb{R}$ with compact support we want an operator $\psi(f) = \int f(x)\psi(x)\,dx$ such that

$$\begin{aligned}
\psi(f)^2 &= \int dx\,dx'\, f(x) f(x')\, \psi(x)\psi(x') \\
&= \tfrac{1}{2}\int dx\,dx'\, f(x) f(x')\, \{\psi(x), \psi(x')\} \\
&= [redacted] \int dx\; f(x)^2
\end{aligned}$$

Thus we are after "the" Clifford module belonging to the real Hilbert space of square-integrable fns. on the line.

In general if W is a real even-diml (Euclidean) orthogonal vector space, then choose a subspace V of $W \otimes \mathbb{C}$ with $V \oplus \bar{V} = W \otimes \mathbb{C}$. Then the Clifford module is ΛW with V acting as multiplications and $\bar{V}$ as interior multe:

$$\big(i(\langle v|) + e(|v\rangle)\big)^2 = \langle v|v\rangle.$$

November 24, 1981

Try to construct the irreducible reprs. of a loop group as holomorphic sections of a line bundle over the orbit. The first problem is to construct the line bundle, then produce the holomorphic section which is to be the eigenvector [redacted] for the parabolic. First consider what happens in the case of ordinary compact groups.

Consider $K = U_n$. The [redacted] orbits are the orbits of K on hermitian matrices. Hence if the eigenvalues are $\lambda_1 < \cdots < \lambda_r$ the orbit can be identified with orthogonal decompositions

$$(*) \qquad V = W_1 \oplus \cdots \oplus W_r$$

where $\dim W_j =$ mult. of λ_j. ← call this d_j. The line bundles over the orbit one considers are of the form

$$\{W_j\} \longmapsto [redacted] \; L_1^{\otimes k_1} \otimes \cdots \otimes L_r^{\otimes k_r} \qquad \text{where } L_j = \Lambda^{d_j} W_j$$

and $k_1, \ldots, k_r \in \mathbb{Z}$. Whence cometh the holomorphic structure? One has to use the ordering of the eigenvalues, whence the decomp. $(*)$ can be identified with a flag

$$0 \subset F_1 \subset \cdots \subset F_r = V$$

$$F_i = \bigoplus_{j \leq i} W_j$$

and so [redacted] $L_i = \Lambda^{d_i}(F_i/F_{i-1})$. [redacted] Next when do we get holomorphic sections of this line bundle?

Let's consider the simple case where $r = 2$. The vector [redacted] bundle with fiber $F_2/F_1 = V/F_1$ has sections given by elements of V, so we get sections of $\Lambda^{d_2}(V/F_2) = L_2$. On the other hand we have a [redacted] canonical isom.

$$L_1 \otimes L_2 \simeq \Lambda^{d_1} F_1 \otimes \Lambda^{d_2}(V/F_2) \simeq \Lambda^n V$$

which gives us sections of $L_1 \otimes L_2$. Thus in general when $k_1 \leq k_2$ we can write

$$L_1^{\otimes k_1} \otimes L_2^{\otimes k_2} = (L_1 \otimes L_2)^{\otimes k_1} \otimes L_2^{\otimes k_2}$$

and so we get holomorphic sections of this line bundle. This process obviously generalizes. Notice also that when a complementary flag is given

$$0 \subset A \subset V \qquad \dim A = d_2$$

then we have a canonical section of $L_2 = \Lambda^{d_2}(V/F_2)$.

In general suppose we consider all flags

$$0 < F_1 < \cdots < F_r = V \qquad \dim(F_i/F_{i-1}) = d_i$$

and put $L_i = \Lambda^{d_i}(F_i/F_{i-1})$. Then

$$L_{i+1} \otimes \cdots \otimes L_r = \Lambda^{d_{i+1}}(F_{i+1}/F_i) \otimes \cdots \otimes \Lambda^{d_r}(F_r/F_{r-1})$$

$$\cong \Lambda^{d_{i+1}+\cdots+d_r}(V/F_i)$$

has holomorphic sections and we get a canonical section belonging to a subspace of dim. $d_{i+1}+\cdots+d_r$. This means that if I give a flag of complementary dims.

$$\underbrace{0 \subset A_{r-1}}_{d_r} \underbrace{\subset A_{r-2}}_{d_{r-1}} \subset \cdots \underbrace{\subset A_1}_{d_2} \underbrace{\subset V}_{d_1}$$

I get a canonical section (up to scalars) of all the line bundles $L_{i+1} \otimes \cdots \otimes L_r$.

For example, if I take the projective space of all lines $0 \subset L \subset V$, then a hyperplane in V determines a section (up to a scalar) of the holomorphic line bundle $\mathcal{O}(1)$.

Clearly it must be true that

$$\Gamma_{hol}(\mathcal{L}_{i+1} \otimes \cdots \otimes \mathcal{L}_r) = \Lambda^{d_{i+1}+\cdots+d_r}(V)$$

These are the so-called fundamental repns.

of U_n or GL_n, because they correspond to the case where the sequence $k_1 \le \cdots \le k_n$ has a single jump.

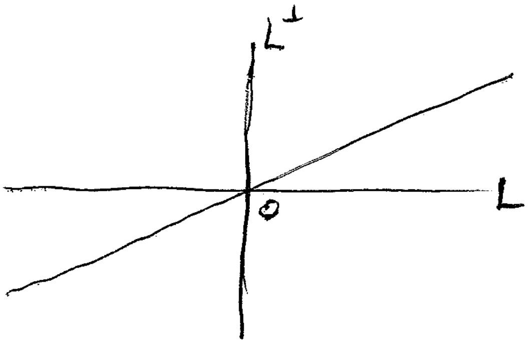
Next we want to look at the function $\langle 0|g|0\rangle$ in the case of one of these irreducible representations. Here $|0\rangle$ denotes the highest weight vector, i.e. an eigenvector for the Borel B. Now we have $B = HN$, $H = TA$ and N acts trivially on $|0\rangle$, so that $\langle 0|g|0\rangle$ is invariant under right multiplication by N and left multiplication by $\bar{N}$ = the unipotent part of the opposite Borel $\bar{B}$. The Bruhat decomposition tells us that

$$G = \coprod \bar{N} w B \qquad w \in \text{Weyl gp.}$$

but we also know that $\langle 0|w|0\rangle = 0$ for $w \neq 1$.* Therefore we see that $\langle 0|g|0\rangle = 0$ unless g is in the fat cell $\bar{N}B = \bar{N}HN$ for the Bruhat decomposition in which case its value is obtained by taking the character on the component in H.

*Actually this is true only when the character we induce from is strictly inside the Weyl chamber.

So the idea now is to look at the loop group case, so that now we have $\mathcal{K}$ = maps $S^1 \to U_n$. I am going to concentrate on the orbit $\mathcal{K}/K$ which I can identify with outgoing subspaces L, and K is the stabilizer of the standard one L_0. This gives me a nice way to think of elements of $\mathcal{K}/K = \mathcal{O}$. Now I can easily see the tangent space to a point L in $\mathcal{O}$; it is the set of maps $L \to L^\perp$ which commute with the unitary operator (?), or equivalently the set of lattices which are complementary to $L^\perp$. This equivalence comes from the graph construction, which identifies maps $L \to L^\perp$ with subspaces in $L \oplus L^\perp$ complem. to $L^\perp$



Moreover, I know that [redacted] an L (complementary to $L_0^\perp$) when combined with zL_0^- defines a vector bundle $\mathcal{E}$ over $\mathbb{P}^1$ with [redacted] $\mathcal{E} \cong \mathcal{O}^n$ because we have $h^0(\mathcal{E}\otimes\mathcal{O}(-1)) = 0$ and $h^1(\mathcal{E}\otimes\mathcal{O}) = n$ $h^1(\mathcal{E}\otimes\mathcal{O}(1)) = 2n$ etc. Thus I know the S matrix carrying L_0 to L factorizes, [redacted] so that $S \in G_- G_+$, so that L is in the fat cell for the Bruhat decomposition.

So now I want to see the symplectic structure on this tangent space. First do over the Grassmannian. The tangent space is $\mathrm{Hom}(F, V/F) = \mathrm{Hom}(F, F^\perp)$. Suppose $F, F^\perp$ are eigenspaces of a hermitian operator A with only two eigenvalues. Tangent vectors are of form

$$\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \qquad \text{where } T : F \to F^\perp$$

[redacted] If $A = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, then

$$\left[\begin{pmatrix} 0 & \frac{T}{\lambda_2-\lambda_1} \\ \frac{T^*}{\lambda_1-\lambda_2} & 0 \end{pmatrix}, \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \right] = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

so that the tangent vector $\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$ comes from the Lie alg. element $\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} 0 & -T \\ T^* & 0 \end{pmatrix}$. The skew form takes $\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$ into $\mathrm{tr}\left(A\left[\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} 0 & -T \\ T^* & 0 \end{pmatrix}, \frac{1}{\lambda_1-\lambda_2}\begin{pmatrix} 0 & -S \\ S^* & 0 \end{pmatrix}\right]\right)$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}\, tr\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\cdot\left[\begin{pmatrix}-TS^*+ST^* & \\ & -T^*S+S^*T\end{pmatrix}\right]$$

$$= \frac{1}{(\lambda_1-\lambda_2)^2}(\lambda_1-\lambda_2)\ tr(ST^*-TS^*) = \frac{1}{\lambda_1-\lambda_2}\, tr(ST^*-TS^*)$$

$$= \frac{1}{\lambda_1-\lambda_2}\, 2i\, \mathrm{Im}\{tr(T^*S)\}$$

(i occurs because A should be iA)

In other words one has this hermitian product $T,S \longmapsto tr(T^*S)$ on the tangent space and the skew-form is the imaginary part. ∎

November 26, 1981

$K = U_n$, $\mathcal{K} = \mathrm{Maps}(S^1, U_n)$ and we want to think of the orbit $\mathcal{K}/K$ as the outgoing subspaces L in $L^2(S^1)^n$ with the identity coset corresponding to $L_0 = (H_+^2)^n$. This makes the holomorphic structure on the orbit visible. Now the main problem is to see the line bundle over this orbit.

For example, take $n=1$ and restrict $\mathcal{K}$ down to the maps of degree 0. Then there is an obvious line bundle which assigns to each L of index 0 relative to L_0 the 1-dimensional subspace $zL_0^\perp \cap L$. ~~However this bundle is trivial because one has~~

$$\cancel{zL_0^\perp/L_0^\perp \xleftarrow{\sim} zL_0^\perp \cap L \xrightarrow{\sim} L/zL}$$

~~and if one uses the last isom. to regard it as a holomorphic~~

Because $L_0^\perp \cap L = 0$, one has an isom

$$zL_0^\perp \cap L \xrightarrow{\sim} L/zL$$

which makes evident the holomorphic structure of the bundle. However the line bundle is trivial, because we also have

$$zL_0^\perp \cap L \xrightarrow{\sim} zL_0^\perp/L_0^\perp.$$

Now our orbit is an affine space. In effect given $f: S^1 \to S^1$ of degree zero, it has a logarithm $\log f$ which we can adjust by a constant so that it has no constant term, in which case

$$\log f = i \sum_n{}' c_n z^n \qquad \overline{c_n} = c_{-n} \qquad c_0 = 0$$

and so you have the independent complex parameters $c_1, c_2, \ldots$. So if we take holomorphic sections we do seem to get the required representation.

However something is slightly fishy because it is clear that the group K acts on this line bundle, and I want a representation of the central extension. [redacted] Perhaps one has a multiplier somewhere.

Analogy. Take the group $\mathbb{C}$ and the trivial bundle, whose holom. sections are holom. functions on $\mathbb{C}$. Then one has the translation action of the group on functions but it must be supplemented by multiplications in order to give the [redacted] good unitary representation.

[redacted] ~~Nice point I have overlooked. If I fix lattices L of a given index with respect to L_0, then the corresponding vector bundles over $\mathbb{P}^1$ have the same degree and so I get a vector bundle by taking sections provided the degree is ≥ 0.~~ NO ~~In effect a vector bundle of type $\mathcal{O} \oplus \mathcal{O}$ can degenerate into $\mathcal{O}(k) \oplus \mathcal{O}(-k)$, but this is bad for H^0 once $k \geq 2$.~~ NO

Here's a way to produce a line bundle over the space of lattices, namely, we know that $R\Gamma$ over $\mathbb{P}^1$ gives a perfect complex and that a perfect complex has a well-defined determinant line bundle. So what do we find?

It seems that we get a holomorphic line bundle over the space of lattices. Given L the complex giving the cohomology is

[redacted]

$$\begin{matrix} L \\ \oplus \\ zL_0^{\perp} \end{matrix} \longrightarrow V$$

or

$$L \longrightarrow V/zL_0^{\perp}$$

and we want to take its highest exterior power. Now the nice thing about this is that there is no immediately obvious way to make the group $\mathcal{K}$ act. We can make the subgroup G_- of G fixing $zL_0^\perp$ work. On the other hand two lattices L, L' related by a group element g give rise to isomorphic vector bundles $\underline{\text{NO}}$ and so ?

So here's the idea: Consider lattices L of degree 0. On any limited family of such L we know there exists an N such that twisting N times kills the first homology, hence $L \cap z^{N+1} L_0^\perp$ is of constant rank over this family. What I am doing is to form the sequence

$$0 \longrightarrow \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}(N)) \longrightarrow \Gamma(\mathcal{E} \otimes \mathcal{O}/m^{N+1}) \longrightarrow H^1(\mathcal{E}) \to 0.$$

Concretely

$$0 \longrightarrow L \cap z L_0^\perp \longrightarrow L \cap z^N L_0^\perp \longrightarrow z^N L_0^\perp / z L_0^\perp \longrightarrow \quad \longrightarrow 0$$

$$z^N L_0^\perp \Big/ \underbrace{L \cap z^N L_0^\perp + z L_0^\perp}_{(L + zL_0^\perp) \cap z^N L_0^\perp} = \frac{z^N L_0^\perp + L}{L + z L_0^\perp} = V / L + zL_0^\perp$$

and so the line bundle we want has fibres

$$\lambda(L \cap z L_0^\perp) \otimes \lambda(V / L + z L_0^\perp)^* \underset{\text{canon isom.}}{\cong} \lambda(L \cap z^N L_0^\perp) \otimes \lambda\left(z^N L_0^\perp / z L_0^\perp\right)^*$$

the right-side being well-defined as a line bundle over the limited family. ~~On the other hand one also has the exact sequence~~

November 27, 1981

General remark: Fix a C^∞ hermitian vector bundle E over a Riemann surface and consider the space $\mathcal{a}$ of connections, or equivalently, holomorphic structures on E. It seems that $\mathcal{a}$ should have a holomorphic structure, because given a map of a complex manifold U into $\mathcal{a}$ the map is holomorphic iff the family of bundles param. by U is holomorphic. In particular this should make sense for tangent vectors to $\mathcal{a}$.

Let's go back to the problem of constructing sections of the canonical line bundle over the space of lattices L. Here's how to construct the line bundle. Fix the degree of our lattices. For any n the subset of lattices such that $V = z^n H_- \cap L$ is open and as n increases it exhausts all the lattices. This condition means that the corresponding bundle $\mathcal{E}_L$ formed from H_- and L satisfies $H^1(\mathcal{E}_L(n)) = 0$, and so on this space of L we get a vector bundle with fibre $\Gamma(\mathcal{E}_L(n)) = z^n H_- \cap L$. In virtue of the sequence

$$0 \longrightarrow \Gamma(\mathcal{E}) \longrightarrow \Gamma(\mathcal{E}_L(n)) \longrightarrow \Gamma(\mathcal{E}_L(n)/\mathcal{E}_L) \longrightarrow H^1(\mathcal{E}_L) \to 0$$

$$0 \longrightarrow H_- \cap L \longrightarrow z^n H_- \cap L \longrightarrow z^n H_-/H_- \longrightarrow V/L+H_- \to 0$$

one gets a canonical isomorphism

$$\lambda(H_- \cap L)\,\lambda(V/H_- + L)^* \cong \lambda(z^n H_- \cap L)\,\lambda(z^n H_-/H_-)^*$$

The right-side is a line bundle over the open set and this isomorphism shows that it is an extension of $\lambda(H_- \cap L)$ on the open set where $H_- + L = V$ i.e. $H^1(\mathcal{E}_L) = 0$. So now it's clear we get a holomorphic line bundle by taking n larger and larger to include all possible lattices.

Next I want to show that the dual of this line bundle has global holomorphic sections. The idea is that if L ranges over lattices of fixed degree such that $H^1(\mathcal{E}_L)=0$, then

$$H^0(\mathcal{E}_L) = H_- \cap L \quad \text{is a subspace of } H_-$$

of fixed rank p and so one has a surjection

$$\Lambda^p(H_-)^\vee \twoheadrightarrow \Lambda^p(H_-\cap L)^\vee$$

and hence sections coming from elements of $\Lambda^p(H_-)^\vee$. Now I want to show these sections can be extended as I go to the larger set of L such that $H^1(\mathcal{E}_L(1))=0$. Not quite correct, because the degree should be fixed. It's OK, because we still work with $H^1(\mathcal{E}_L)$.

So over the larger set the line bundle has the fibre $\Lambda^{p+r}(zH_-\cap L)\otimes\Lambda^r(zH_-/H_-)^\vee$ because of the preceding page. We can get sections of the dual of this by choosing $\omega\in\Lambda^r(zH_-/H_-)$ and then elements of $\Lambda^{p+r}(zH_-)^\vee$. But we have

$$0 \to H_- \to zH_- \to zH_-/H_- \to 0$$

so that there is an epim.

$$\Lambda^{p+r}(zH_-) \twoheadrightarrow \Lambda^p H_- \otimes \Lambda^r(zH_-/H_-)$$

Hence we find this diagram

valid when $H^1(\mathcal{E}_L)=0$

$$\begin{array}{ccc}
\Lambda^{p+r}(zH_-\cap L)\otimes\Lambda^r(zH_-/H_-)^\vee & \simeq & \Lambda^p(H_-\cap L) \\
\cap & & \cap \\
\Lambda^{p+r}(zH_-)\otimes\Lambda^r(zH_-/H_-)^\vee & \twoheadrightarrow & \Lambda^p(H_-)
\end{array}$$

Thus the sections obtained from $\Lambda^p(H_-)^\vee$ on the small set are extended by sections over the larger open set.

November 28, 1981

Algebraic version of fermion Fock space: Let V be an infinite-dimensional vector space and let F_0 be a fixed subspace. I know that any finite-dimensional subspace of V determines a line in ΛV. I want to construct an analogue of ΛV which will have a line associated to any subspace L of V which is commensurable with F_0 in the sense that $F_0 \cap L$ is of finite codim in both F_0 and L.

For any $F' \subset F$ of finite codim r we have an embedding

1) $$\Lambda^r(F/F') \otimes \Lambda^p(V/F) \hookrightarrow \Lambda^{p+r}(V/F')$$

hence if we had a way of picking a generator for $\Lambda^r(F/F')$ we would get an embedding

$$\Lambda(V/F) \longrightarrow \Lambda(V/F')$$

raising degree by $r = \dim(F/F')$. We want to use these maps to construct an ~~inverse~~ inductive system. The way to avoid the choices is to assume $F' \subset F \subset F_0$ and to use the canonical isom

2) $$\Lambda^r(F/F') \otimes \Lambda^s(F_0/F) \xrightarrow{\sim} \Lambda^{r+s}(F_0/F') \qquad s = [F_0:F].$$

Combined with 1) we get a canonical embedding

$$\Lambda^{[F_0:F]}(F_0/F)^\vee \otimes \Lambda^p(V/F) \hookrightarrow \Lambda^{[F_0:F']}(F_0/F')^\vee \otimes \Lambda^{p+[F:F']}(V/F')$$

raising degree by $[F:F']$. Thus we can form the inductive limit.

Now to what extent does this limit depend on F_0. Clearly the limit is taken over all F commensurable with F_0, so the role of F_0 is to trivialize the factors $\Lambda^{[F:F']}(F/F')$. If we have a subspace F_1 of finite codim in F_0, then the inductive system for F_0 and F_1 at F is

$$\Lambda^{[F_0:F]}(F/F_0)^\vee \otimes \Lambda(V/F) \quad , \quad \Lambda^{[F_1:F]}(F/F_1)^\vee \otimes \Lambda(V/F)$$

and so tensoring the former with $\Lambda^{[F_0:F_1]}(F_0/F_1)^\vee$ gives the latter up to a canonical isomorphism.

<u>Notation:</u>

$$\Lambda(V;F_0) = \varinjlim_{F\subset F_0} \lambda(F_0/F)^\vee \otimes \Lambda(V/F)$$

$$\Lambda(V;F_1) = \varinjlim_{F\subset F_1} \lambda(F_1/F)^\vee \otimes \Lambda(V/F)$$

so that we have

$$\lambda(F_0/F_1) \otimes \Lambda(V;F_0) \simeq \Lambda(V;F_1)$$

A nice way to think is that $\Lambda(V;F_0)$ has a distinguished <u>element</u> belonging to the subspace F_0 in V. Thus if F_0 is finite-dimensional we get

$$\Lambda(V;F_0) = \lambda(F_0)^\vee \otimes \Lambda(V)$$

so that there is a canonical <u>element</u> (not just line) belonging to F_0.

So now let's consider an automorphism θ of V such that $\theta(F_0)$ is commensurable with F_0. Then θ induces an isomorphism of $\Lambda(V;F_0)$ with $\Lambda(V;\theta F_0)$ so that if we have a way of picking an element of $\lambda(F_0|\theta F_0)$, then we get an automorphism of $\Lambda(V;F_0)$. I guess it's clear that we get a projective representation of the group of autos θ of V preserving the commensurability class of F_0. I want next to see cases where there is a non-trivial central extension.

Suppose that we choose a complement [redacted] W to F_0 in V:

$$V = F_0 \oplus C.$$

Then for $F \subset F_0$ of finite-codim we have

$$\Lambda(V/F) = \Lambda(F_0/F) \otimes \Lambda(C)$$

Now [redacted] $\lambda(F_0/F)^\vee \otimes \Lambda^p(F_0/F) = \Lambda^d(F_0/F)^\vee \otimes \Lambda^p(F_0/F)$ is canonically isomorphic to $\Lambda^{d-p}(F_0/F)^\vee$ $\quad d = [F_0 : F]$

and hence we have a canonical isomorphism

$$\lambda(F/F_0)^\vee \otimes \Lambda(V/F) = \Lambda(F_0/F)^\vee \otimes \Lambda(C)$$

which on passing to the limit gives

$$\Lambda(V; F_0) = \Lambda(F_0)^\vee_{cont} \otimes \Lambda(C).$$

Now take $V = L^2(S^1)$ and let S^1 act via translations. Let $F_0 = H_+$ [redacted] spanned by $1, z, z^2, \cdots$ and $C = (H_+)^\perp =$ spanned by $z^{-1}, z^{-2}, \ldots$. Thus $\Lambda(V; F_0) = \Lambda(F_0^\vee)$ [redacted] $\otimes \Lambda(C)$ is bigraded according to the degree in $\Lambda(V; F_0)$, which is the degree in $\Lambda(C)$ minus the degree in $\Lambda(F_0^\vee)$, and [redacted] according to the character under the S^1 action. So we get the following Poincare series

for the z^n in C

$$\prod_{n \geq 0} (1 - q^{n} t^{-1}) \prod_{n \geq 1} (1 - q^n t)$$

and according to the Jacobi identity this is

$$\frac{\sum q^{\frac{n(n+1)}{2}} (-t)^n}{\prod_{n \geq 1} (1 - q^n)}$$

Thus if we look at the degree zero part [redacted] $\Lambda^0(V; F_0)$ we get the character $\frac{1}{\prod_{n \geq 1}(1-q^n)}$ which is that of the symmetric algebra of C.

November 29, 1981

If V is a vector space and F_0 is a subspace we have defined $\Lambda(V;F_0)$ which is to be thought of as the exterior algebra of V tensored with the dual of the highest exterior power of F_0, so that there is a canonical element $1 \in \Lambda^0(V;F_0)$ corresponding to the subspace F_0. If F_1 is a subspace commensurable with F_0, then there is a canonical isomorphism

$$\lambda(F_0;F_1) \otimes \Lambda(V;F_0) \xrightarrow{\sim} \Lambda(F;F_1)$$

where $\lambda(F_0;F_1)$ is defined as follows:

$$\begin{aligned} \lambda(F_0;F_1) &= \lambda(F_0/F_1) && \text{if } F_1 \subset F_0 \\ &= \lambda(F_0/F)\,\lambda(F_1/F)^{\vee} && \text{if } F \subset F_0 \cap F_1 \end{aligned}$$

Degrees: $1 \in \Lambda(V;F_0)$ is a line of degree $[F_1:F_0]$ isomorphic to $\lambda(F_1;F_0)$. Hence the above map raises degrees by $[F_0:F_1]$, i.e.

$$\lambda(F_0;F_1) \otimes \Lambda^p(V;F_0) \xrightarrow{\sim} \Lambda^{p+[F_0:F_1]}(V;F_1)$$

Now let g be an auto. of V preserving the commensurability class of F_0. Then g induces an isomorphism

$$\Lambda(V;F_0) \xrightarrow{\sim} \Lambda(V;gF_0) = \lambda(F_0;gF_0) \otimes \Lambda(V;F_0)$$

If g_1 is another auto. we have

$$\begin{tikzcd}
\Lambda(V;F_0) \arrow[r, "\sim", "g"'] \arrow[dr, "\sim", "g_1 g"'] & \cancel{\lambda(F_0;gF_0)\otimes}\,\Lambda(V;gF_0) = \lambda(F_0;gF_0) \otimes \underbrace{\Lambda(V;F_0)}_{\lambda(F_0;g_1F_0)\otimes\Lambda(V;F_0)} \arrow[d, "g_1", "\wr"'] \\
 & \Lambda(V;g_1gF_0) = \lambda(F_0,g_1gF_0) \otimes \Lambda(V;F_0)
\end{tikzcd}$$

and so we should see a canonical isomorphism

$$\lambda(F_0; gF_0)\,\lambda(F_0; g_1F_0) \simeq \lambda(F_0; g_1 g F_0)$$

By obvious transitivity this will follow if one has

$$\lambda(F_0; g_1F_0) = \lambda(F_1; g_1F_1)$$

for any F_0, F_1. [redacted]

It occurs to me that we have to be careful about the signs when dealing with odd vector spaces. Recall the problem: [redacted] One has [redacted] canon. isomorphisms

$$\begin{array}{ccc} \lambda(V \oplus W) & \xleftarrow{\sim} & \lambda(V) \otimes \lambda(W) \\ \wr\! | & & \wr\! | \\ \lambda(W \oplus V) & \xleftarrow{\sim} & \lambda(W) \otimes \lambda(V) \end{array}$$

but the square commutes, only up to the sign $(-1)^{(\dim V)(\dim W)}$.

Perhaps the way to get around this problem is to think of $\lambda(V)$ as a being a complex of total dimension 1, i.e. consisting of $\Lambda^n(V)$ in [redacted] degree n if $n = \dim(V)$. Then we one has $W \subset V$, $\dim W = p$ one has canonical isomorphisms

$$\lambda(V/W) \otimes \lambda(W) \xrightarrow{\sim} \lambda(V) \xleftarrow{\sim} \lambda(W) \otimes \lambda(V/W)$$

[redacted] but the composite [redacted] isomorphism involves the sign. Better check this later.

[redacted] In any case we want to establish a canonical isomorphism

$$\lambda(F_0; gF_0) \simeq \lambda(F_1; gF_1)$$

for any two F_0, F_1 which are commensurable. Clear:

$$\lambda(F_0; gF_0)\,\lambda(gF_0; gF_1) = \lambda(F_0; gF_1) = \lambda(F_0; F_1)\,\lambda(F_1; gF_1)$$

$$[redacted] = \lambda(F_1; gF_1)\,\lambda(F_0; F_1)$$

and now use the obvious isomorphism

$$\lambda(F_0;F_1) \simeq \lambda(gF_0, gF_1)$$

induced by g. Notice that one has used the two ways of writing $\lambda(F_0;gF_1)$ in terms of $\lambda(F_0;F_1)$ and $\lambda(F_1;gF_1)$.

Let G be the group of autos. of V which preserve the commensurability class of F_0. Then [redacted] we can make a central extension $\tilde{G}$ of G act on $\Lambda(V;F_0)$. An element of $\tilde{G}$ is an element g of G ~~together~~ with a non-zero element of $\lambda(F_0;gF_0)$.

Now we must compute some examples. The simplest case is to take $V = \mathbb{C}[z,z^{-1}]$ and $F_0 = \mathbb{C}[z]$. Somehow I [redacted] am confused by the generality of the fact that we [redacted] use all subspaces of F_0 of finite codimension. Actually I [redacted] only want to look at lattices, i.e. $\mathbb{C}[z]$-submodules. But recall

$$\Lambda(V;F_0) = \Lambda(F_0)^\vee \otimes \Lambda C$$

where $F_0^\vee = \varinjlim (F_0/F)^\vee$ are the continuous linear functionals for the sort of [redacted] F one is using. Now

$$\varprojlim (F_0/F) = \text{completion of } \mathbb{C}[z] \text{ for all submodules}$$
$$= \prod_{m} \varprojlim_{n} \mathbb{C}[z]/m^n$$

Thus one finds that

$$(F_0)^\vee = \bigoplus_{m} \text{[redacted]} \underbrace{\varprojlim_{n} \left(\mathbb{C}[z]/m^n\right)^\vee}_{\substack{\text{looks [redacted] like rational functions} \\ \text{with poles at } m \text{ mod } \mathbb{C}[z].}}$$

So therefore it looks like I should be using $V = \mathbb{C}(z)$

so that V/F_0 and $F_0^\vee$ have the same **size**.
The point maybe is that if I consider autos of $\mathbb{C}[z,z^{-1}]$ which commute with z i.e. $\mathbb{C}[z,z^{-1}]^* = \mathbb{C}^\times \times \mathbb{Z}$, then under such an auto F_0 goes into $z^n F_0$, so only these F **need** be used in order to get a representation stable under the group.

So the natural thing to do in the case of $V = \mathbb{C}[z,z^{-1}]$ and $F_0 = \mathbb{C}[z]$ is to just use $F = z^n \mathbb{C}[z]$ inside of F_0 and to complete. Thus $\hat{V} = \mathbb{C}[[z]][z^{-1}]$ and $\hat{F}_0 = \mathbb{C}[[z]]$ and the group $(\mathbb{C}[[z]][z^{-1}])^*$ will act projectively on the representation. Similarly in the situation $V = \mathbb{C}(z)$, $F_0 = \mathbb{C}[z]$ the natural thing seems to be to complete with respect to all submodules in which case we get $\hat{V} =$ adeles $\hat{F}_0 = \widehat{\mathbb{C}[z]} = \prod_m$ m-adics, and the **idele** group will act projectively on the representation.

Next we want to compute the commutator pairing in interesting cases. Let f, g be two automorphisms of V preserving F_0 up to commensurability. Let's pick autos of $\Lambda(V;F_0)$ covering f and g. We have

$$\lambda(fF_0;F_0) \otimes \Lambda(V;fF_0) = \Lambda(V;F_0)$$

and

$$f_*: \Lambda(V;F_0) \xrightarrow{\sim} \Lambda(V;fF_0)$$

so a covering auto, will be of the form αf_* where α denotes the isomorphism

$$\Lambda(V;fF_0) \xrightarrow{\alpha \otimes \mathrm{id}} \lambda(fF_0;F_0) \otimes \Lambda(V;fF_0) = \Lambda(V;F_0)$$

belonging to $\alpha \in \lambda(fF_0;F_0)$. Similarly βg_* covers g where $\beta \in \lambda(gF_0;F_0)$. Another way of saying this is that there is in $\Lambda(V;F_0)$ a line belonging to fF_0 which is canonically isom. to $\lambda(fF_0;F_0)$. We then get a unique ~~auto~~ auto. of $\Lambda(V;F_0)$ covering f such that 1 goes to $\alpha \in \lambda(fF_0;F_0)$.

So now let's compose: $(\beta g_*)\circ(\alpha f_*)$

$$\begin{tikzcd}
\Lambda(V;F_0) \arrow[d, "\wr\, f_*"] & & \\
\Lambda(V;fF_0) \arrow[r, "\alpha"] \arrow[d, "\wr\, g_*"] & \Lambda(V;F_0) \arrow[d, "\wr\, g_*"] & \\
\Lambda(V;gfF_0) \arrow[r, "g_*(\alpha)"] & \Lambda(V;gF_0) \arrow[r, "\beta"] & \Lambda(V,F_0)
\end{tikzcd}$$

so we get

$$(\beta g_*)(\alpha f_*) = \beta\, g_*(\alpha)\,(g\circ f)_*$$

where one uses the map

$$\overset{\beta}{\lambda(gF_0;F_0)} \overset{\otimes}{\otimes} \overset{\alpha}{\lambda(fF_0;F_0)} \longmapsto \overset{\alpha \quad \otimes\, g_*(\beta)}{\lambda(gF_0;F_0)\otimes\lambda(gfF_0;gF_0)}$$
$$= \lambda(gfF_0,F_0)$$

Thus when say

$$\begin{array}{ccc} F_0 & \overset{\alpha}{\subset} & fF_0 \\ \beta\cap & & \cap \\ gF_0 & \subset & gfF_0 \end{array}$$

we take the volume element $\alpha \in \lambda(fF_0/F_0)$ transport it by the automorphism g to $g_*(\alpha) \in \lambda(gfF_0/gF_0)$ and combine with the volume elt. β.

Now when $gfF_0 = fgF_0$ we will get two different volume elements whose difference is what I want to compute. Take $f = z^{-n}$ and $g = c \in \mathbb{C}^\times$. Then $gF_0 = F_0$ so we can take $\beta = 1$, and $fF_0 = z^{-n}H_+$ so we can take $\alpha = z^{-1}\wedge\cdots\wedge z^{-n}$. Then $g_*(\alpha) = c^n\alpha$, so the commutator is $\langle z^{-n}, c\rangle = c^n$. (up to sign)

Conventions: One should think as follows

$$\Lambda(V; F) = \lambda(F)^{\vee} \otimes \Lambda(V)$$

$$\lambda(L; F) = \lambda(F)^{\vee} \otimes \lambda(L).$$

Precisely one defines

$$\lambda(L; F) = \lambda(F/F')^{\vee} \otimes \lambda(L/F') \qquad F' \subset F \cap L.$$

Now when one uses

$$\lambda(gF_0; F_0) \otimes \lambda(gfF_0; gF_0) = \lambda(gfF_0; F_0)$$

we think of a basis running from F_0 to gF_0 being combined with a basis from gF_0 to gfF_0 to get a basis of gfF_0/F_0.

Better set of conventions:

$$\Lambda(V; F) = \Lambda(V) \otimes \lambda(F)^{\vee}$$

Think of this as having a distinguished generator represented by the infinite exterior product of a basis of F.

<u>November 30, 1981.</u>

An [redacted] interesting problem is whether one can construct the tame central extension of [redacted] GL_n in the local field or number-theoretic case. This needs clarification to see if it makes sense. The idea is that perhaps we can make sense of $\lambda(L:L')$ for lattices as some object of a Picard category, and so be able to [redacted] mimic the construction I gave yesterday.

This seems to be the case at least in the local field case. Be careful. The problem is to define for two lattices $L_0 \supset L_1$ something that behaves like the highest exterior power of L_0/L_1 as far as exact sequences and <u>automorphisms</u> are concerned.

Let's consider a Riemann surface, for example $\mathbb{P}^1$. The idea is that [redacted] the sort of [redacted] $\Lambda(V, F_0)$ I am looking at where F is a $\mathbb{C}[z]$-lattice in $\mathbb{C}(z)^n$ is the tensor product of the separate exterior algebras for each of the localizations. The idea maybe is that

$$\Lambda(V; F_0) = \varinjlim_{\substack{F \subset F_0 \\ F' \supset F}} \Lambda(F'/F) \otimes \lambda(F_0/F)^\vee$$

and now $F'/F = \coprod_{w} (F'/F)_w$ and also for F_0/F. So it is going to work.

Now from this point of view it should be possible to add in the point at ∞. In fact, and this is a really nice point, one can [redacted] generalize F_0 to be a vector bundle E_0 with generic fibre $\mathbb{C}(z)^n$, and then take the limit over $E \subset E_0$ and $E' \supset E$. Notice that $\Gamma(E'/E)$ really makes sense whereas

$\Gamma(E)$ might be zero. Thus we will get a nice projective representation of the group Gl_n(adeles).

The obvious conjecture to try is to show that the group Gl_n(fn. field) actually acts on $\Lambda(V; E_0)$, that is, that there is a coherent way to trivialize $\lambda(E_0; E_1)$ [redacted] when one is given an isomorphism of the bundle E_0 with E_1.

We [redacted] would like to do the number-theoretical analogue, in which case we have to concoct projective representations of the groups $Gl_n(\mathbb{R})$, $Gl_n(\mathbb{Z}_p)$. K-theory tells me that $K_2(\mathbb{R}) \to KO^{-2}(pt) = \mathbb{Z}_2$, i.e. that one gets a double covering of $Gl_n(\mathbb{R})$ because O_n has the Pin_n double-covering so we should check out the Pin_n representation. Actually I would really like to find a projective representation of $Gl_n(\mathbb{R})$ in which there were [redacted] lines canonically attached to the different quadratic forms on $\mathbb{R}^n$.

December 1, 1981

Let H_0 be a self-adjoint operator on a Hilbert space V and let [redacted] V_- be the negative energy eigenspace, so that $\Lambda(V;V_-)$ is the Dirac construction with $1 =$ ground state. I consider a perturbation $H = H_0 + U$ on the one-particle space V and extend H to the Fock space $\Lambda(V;V_-)$. Notice that $\Lambda(V;V_-)$ is a Clifford module belonging to the underlying real Hilbert space of V, and that we are restricting attention to the unitary group of V, or its Lie algebra the self-adjoint operators on $V \times \sqrt{-1}$, instead of the full orthogonal group of V.

Now let's assume that wave operators for H_0, H on V exist, [redacted] and that they preserve the commensurability class of V_-. Actually the wave operators intertwine H_0, H and hence must carry V_- to the negative energy eigenspace for [redacted] H. In particular the scattering matrix has to preserve all of the eigenspaces $\leq\lambda$ of H_0 ~~(which means (at least in the case of simple spectrum) that~~ which doesn't say much because these subspaces are not commensurable with V_-. But it is easy to describe the scattering matrix because

$$\Lambda(V;V_-) = \Lambda V_+ \otimes \Lambda(V_-)^*$$

and S leaves $V_\pm$ invariant. So the only real question might be whether there is a way to assign a number of abs. value 1 to what S does to the ground state.

What I really should be understanding at the present time is the geometry of two subspaces in Hilbert space. If L and F are two subspaces of the same dimension p in the Hilbert space V, then what sort of invariants exist for these subspaces. Each determines a line in $\Lambda^p V$ and the angle between these lines is an invariant. Before this there is the [redacted] index $\dim(F) - \dim(L)$.

It seems good to look at the orthogonal projection $V \xrightarrow{pr}$ [redacted] L. If $F \cap L^\perp$ [redacted] $\neq 0$, then $\Lambda^p F \perp \Lambda^p L$. Because to compute the angle θ [redacted] one uses

$$\cos\theta = |\langle f_1 \wedge \cdots \wedge f_p | l_1 \wedge \cdots \wedge l_p \rangle|$$
$$= |\det \langle f_i | l_j \rangle|$$

where $\{f_i\}$ resp. $\{l_j\}$ are orthonormal bases for F, L resp. This gives 0 if one of the f_i is $\perp L$.

So we suppose $pr:$ [redacted] $F \xrightarrow{\sim} L$. Up to unitary transformations in both F and L one can allways put a linear transformation in diagonal form with [redacted] decreasing positive eigenvalues. So one can [redacted] find orthonormal bases of F, L such that $pr(f_j) = \lambda_j l_j$ whence

$$\langle f_1 \wedge \cdots \wedge f_p | l_1 \wedge \cdots \wedge l_j \rangle = \prod \lambda_j$$

More generally if $f_1, \ldots, f_p$ is any ortho. basis of F and $l_1, \ldots, l_p$ any orthonormal basis of L, then

$$\langle f_1 \wedge \cdots \wedge f_p | l_1 \wedge \cdots \wedge l_j \rangle = \det \langle f_i, l_j \rangle$$
$$= \det \langle pr f_i, l_j \rangle$$
$$= \langle pr(f_1 \wedge \cdots \wedge f_p) | l_1 \wedge \cdots \wedge l_p \rangle$$

and so therefore we see that [redacted] $\cos(\theta)$ is always the abs. val. of determinant of the orthogonal projection map from either F on L, or L on F.

Let's return to the physics. [redacted] One tends to be interested in the following. One has calculated out

H_0 and its eigenfunctions so one knows how to describe the Fock space entirely in terms of the eigenfunctions of H_0. Also once one knows S on V one extends it to Fock space. The interesting quantities left seem to be the ground energy shift the renormalization constant, and maybe the structure of the H eigenfunctions. All these have to do with the wave operators.

Draw a picture of Fock space:

$$\begin{array}{ccccc}
\Lambda^2 \bar{V}_- & \bar{V}_- & |0\rangle & V_+ & \Lambda^2 V_+ \\
 & \Lambda^2 \bar{V}_- \otimes V_+ & \bar{V}_- \otimes V_+ & \bar{V}_- \otimes \Lambda^2 V_+ & \\
 & & \Lambda^2 \bar{V}_- \otimes \Lambda^2 V_+ & &
\end{array}$$

In the middle we have a situation analogous to the model where we add a discrete state to a continuous family, as in the Lee model.

Green's functions?

December 2, 1981

Take $V = L^2(S^1)$, $L_0 = H_+$ and form the Hilbert space version of Fock space $\Lambda(V; L_0)$. Take a "scattering" operator $S: S^1 \to S^1$, which is a rational function so that $S(L_0)$ is commensurable with L_0 and hence the is a line $\underset{n}{\tilde{S}|0\rangle}$ in $\Lambda(V; L_0)$ attached to $S(L_0) = L$. Now I want to assume the degree of S is zero and to compute $|\langle 0|\tilde{S}|0\rangle|$ which is the cosine of the angle between the lines gen. by $|0\rangle$, $\tilde{S}|0\rangle$. In finite dimensions one picks orthonormal bases for L_0 and $S(L_0)$, call them $\{e_i\}$ and $\{f_j\}$, and one has

$$|\langle 0|\tilde{S}|0\rangle| = |\det \langle e_i|f_j\rangle|$$

In particular if S is unitary, one could take $f_j = Se_j$ and then

$$|\langle 0|\tilde{S}|0\rangle| = |\det \langle e_i|S|e_j\rangle|$$

So if we take the basis z^n $n \geq 0$ for L_0, then

$$\langle z^i|S|z^j\rangle = \int z^{j-i} S \frac{d\theta}{2\pi} \qquad i,j \geq 0$$

is a Toeplitz matrix.

However this perhaps is not a good way to think because Toeplitz matrices don't have determinants - one has to extract the N-th root of the N×N determinant to get a limit.

Nevertheless when S is a rational function of z invertible on the unit circle, of degree 0, one should be able to compute $|\langle 0|\tilde{S}|0\rangle|$ explicitly.

December 3, 1981

One should be able to use Grassman integrals to represent operators on ΛV:

$$\mathrm{Hom}(\Lambda V, \Lambda W) = \Lambda V \otimes (\Lambda W)^\vee$$
$$= \Lambda V \otimes \Lambda W \otimes \lambda(W)^\vee [-\dim W]$$
$$= \Lambda(V \oplus W) \otimes \lambda(W)^\vee [-\dim W]$$

Therefore if I choose a volume in W, operators from ΛW to ΛV are in one-one correspondence with elements of $\Lambda(V \oplus W)$, i.e. functions in non-commuting variables.

For example, a subspace of $V \oplus W$ when interpreted as a correspondence "maps" subspaces of W into subspaces of V. Actually the map is rational, i.e. it will take subspaces of W which are not transversal to the correspondence into the zero line in ΛV.

Question: We know that $\mathrm{Hom}(\Lambda V, \Lambda V)$ is the Clifford algebra of the hyperbolic quadratic space $V \oplus V^*$. What sort of elements of $\Lambda(V \oplus V)$ correspond to elements of the Clifford group?

I am particularly interested in the automorphisms of ΛV produced by an auto. of V. There seems to be a very simple formula that goes as follows. Suppose given a map $A : W \to V$ and suppose we want to extend A functorially to $\Lambda(A) : \Lambda(W) \to \Lambda(V)$. Pick basis $\{w_j\}$ and $\{v_i\}$ so that

$$A = \sum_{i,j} |v_i\rangle\, a_{ij} \,\langle w_j^*| \qquad a_{ij} = \langle w_j^* | A | v_j \rangle$$

where $\{w_j^*\}$ is the dual basis of $W^\vee$. Now form an exterior algebra with generators ψ_i and ψ_j^*. Better is to consider the exterior algebra $\Lambda(V \oplus W^*)$ of $V \oplus W^*$ and to consider $A \in V \otimes W^* \subset \Lambda^2(V \oplus W^*)$. Then we can form $\quad e^A = \sum_p \frac{1}{p!} A^p \quad$ where

$$A^p \in \Lambda^p V \otimes \Lambda^p(W^*) \subset \Lambda^{2p}(V \oplus W^*).$$

Now the claim is that

$$\Lambda^p A \in \mathrm{Hom}(\Lambda^p W, \Lambda^p V) = \Lambda^p V \otimes \Lambda^p(W^*)$$

is the element $\frac{1}{p!} A^p$ computed in $\Lambda(V \oplus W^*)$.

What this seems to mean is that

$$\boxed{\Lambda(A) = \,:e^A:}$$

The two colons denote normal ordering which is the canon. isomorphism

$$\begin{aligned}\Lambda(V \oplus W^*) &\Rightarrow \Lambda(V) \otimes \Lambda(W^*) \\ &\Rightarrow \Lambda(V) \otimes (\Lambda W)^* \\ &\Rightarrow \mathrm{Hom}(\Lambda W, \Lambda(V))\end{aligned}$$

The proof of this formula is analogous to

$$\int e^{+\psi^* A \psi} \, \mathcal{D}\psi^* \mathcal{D}\psi = \det(A)$$

The next point is that we are going to get operators on ΛV in the same way belonging to elements of

$$\Lambda^2(V \oplus V^*) = \Lambda^2 V \oplus V \otimes V^* \oplus \Lambda^2 V^*$$

and these up to scalars should be the elements of the Clifford group. By analogy with the symmetric algebra case this is clear: ~~[illegible]~~ When I write a Clifford operator in the form $:e^A:$, then A is the "scattering" matrix representation for the induced operator on $V \oplus V^* \subset \mathrm{End}(\Lambda V)$.

The way to write down operators in the Clifford group is as follows: We start with a vector space $\tilde{V}$ having a non-degenerate quadratic form. [redacted] We suppose given an orthogonal transformation θ of $\tilde{V}$ and we want to lift it to [redacted] a Clifford group element $\tilde{\theta}$. We choose an isomorphism $\tilde{V} = V \oplus V^*$. Then θ will determine a [redacted] Cayley transform element A in $\Lambda^2(\tilde{V})$ provided θ is not too far from the identity and then $\tilde{\theta} = :e^A:$ where again the normal product is [redacted] relative to the decomposition $\tilde{V} = V \oplus V^*$, i.e.

$$\Lambda(V \oplus V^*) = \Lambda(V) \otimes \Lambda(V^*) \hookrightarrow \Lambda(V) \otimes \Lambda(V)^*$$

$$\hookrightarrow \mathrm{Hom}(\Lambda(V), \Lambda(V)).$$

The idea is that

$$\begin{array}{ccccccc} \Lambda^2(\tilde{V}) & = & \Lambda^2(V) & \oplus & V \otimes V^* & \oplus & \Lambda^2(V^*) \\ A & = & A_1 & + & A_2 & + & A_3 \end{array}$$

[redacted]

and that

$$e^A = \underbrace{e^{A_1}}_{\substack{\text{pure} \\ \text{creation}}} e^{A_2} \underbrace{e^{A_3}}_{\substack{\text{pure} \\ \text{destruction}}}$$

so that

$$:e^A: = (:e^{A_1}:)(:e^{A_2}:)(:e^{A_3}:)$$

So [redacted] now the program is to compute the components A_1, A_2, A_3 which belong to the orthogonal transformation θ. The idea is that $\theta(V)$ is [redacted] a [redacted] maximal isotropic subspace of $V \oplus V^*$ which remains complementary to V^*, since I assume that θ is close to 1. Thus $\theta(V)$ is the graph of a map $V \to V^*$ which must be skew-symmetric; this gives the $A_3 \in \Lambda^2(V^*)$. Similarly $A_1 \in \Lambda^2(V)$ should come from the isotropic subspace $\theta(V^*)$. Finally $A_2 \in V \otimes V^*$ should maybe be the endom.

$V \to V$ which one gets by $V \xrightarrow{\theta} \theta(V) \hookrightarrow V \oplus V^* \xrightarrow{pr_1} V$?

$$\tilde{V} = V \oplus V^* \longrightarrow \mathrm{End}(\Lambda V)$$
$$\begin{pmatrix} v \\ \lambda \end{pmatrix} \xmapsto{\ \rho\ } e(v) + i(\lambda)$$

hence $\quad (e(v)+i(\lambda))^2 = e(v)\,i(\lambda) + i(\lambda)\,e(v) = \langle \lambda, v \rangle$

we get a canon. algebra map

$$C(\tilde{V}) \longrightarrow \mathrm{End}(\Lambda(V))$$

which is an isomorphism in the finite-diml. case.

Let $\theta : \tilde{V} \to \tilde{V}$ be an orthogonal transf. We want to find $\tilde{\theta} \in C(\tilde{V}) = \mathrm{End}(\Lambda(V))$ such that

$$\tilde{\theta}\, \rho\begin{pmatrix} v \\ \lambda \end{pmatrix} \tilde{\theta}^{-1} = \rho\left(\theta\begin{pmatrix} v \\ \lambda \end{pmatrix}\right)$$

(which implies that the action map $\tilde{V} \times \Lambda(V) \to \Lambda(V)$ is equivariant for $\theta, \tilde{\theta}$.)

Let's compute the θ belonging to $\tilde{\theta} = e^{i(\lambda_1) i(\lambda_2)}$

First $\quad [i(\lambda_1) i(\lambda_2), e(v)] = i(\lambda_1)\langle \lambda_2 | v\rangle - i(\lambda_2)\langle \lambda_1 | v \rangle$

and $i(\lambda_1) i(\lambda_2)$ is of square zero, hence

$$\tilde{\theta}\, e(v)\, \tilde{\theta}^{-1} = e(v) + i(\lambda_1)\langle \lambda_2 | v\rangle - i(\lambda_2)\langle \lambda_1 | v\rangle$$

Thus

$$\theta\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} v + \lambda_1 \langle \lambda_2 | v \rangle - \lambda_2 \langle \lambda_1 | v \rangle \\ \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \lambda_1 \wedge \lambda_2 & 1 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix}$$

Next compute the θ belonging to $\tilde{\theta} = e^{e(v_1) e(v_2)}$

$$[e(v_1) e(v_2), i(\lambda)] = e(v_1)\langle \lambda | v_2 \rangle - e(v_2)\langle \lambda | v_1 \rangle$$

so $\quad \tilde{\theta}\, i(\lambda)\, \tilde{\theta}^{-1} = i(\lambda) + e(v_1)\langle \lambda | v_2 \rangle - e(v_2) \langle \lambda | v_1 \rangle$

and thus we find

$$\Theta\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} v \\ \lambda + v_1 \langle \lambda | v_2 \rangle - v_2 \langle \lambda | v_1 \rangle \end{pmatrix}$$

$$= \begin{pmatrix} 1 & v_1 \wedge v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ \lambda \end{pmatrix}$$

Finally we compute the Θ belonging to $:e^{e(v)i(\lambda)}:$

$$[\, e(v)i(\lambda),\ \begin{matrix} e(v') \\ i(\lambda') \end{matrix} \,] = \begin{matrix} + e(v) \langle \lambda | v' \rangle \\ - i(\lambda) \langle \lambda' | v \rangle \end{matrix}$$

~~which gives~~

$$\cancel{\Theta\begin{pmatrix} v' \\ \lambda' \end{pmatrix} = \begin{pmatrix} v \otimes \lambda & 0 \\ 0 & -\lambda \otimes v \end{pmatrix} \begin{pmatrix} v' \\ \lambda' \end{pmatrix}}$$

Now $\quad :e^{e(v)i(\lambda)}: \; = 1 + e(v)\, i(\lambda)$

but one must be careful because $e(v)i(\lambda)$ is not necessarily of square 0. In fact its inverse is

$$\frac{1}{1 + e(v)i(\lambda)} = 1 - e(v)i(\lambda) + e(v)\langle \lambda | v \rangle i(\lambda) - e(v) \langle \lambda | v \rangle^2 i(\lambda) + \ldots$$

$$= 1 - e(v) \frac{1}{1 + \langle \lambda | v \rangle} i(\lambda)$$

In any case the point is to observe that if $\begin{pmatrix} v \\ \lambda \end{pmatrix} \mapsto \begin{pmatrix} Av \\ B\lambda \end{pmatrix}$ is orthogonal, then $\langle \lambda | v \rangle = \langle B\lambda | A v \rangle$ for all λ, v or that $B = (A^t)^{-1}$.

~~[illegible]~~