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July 15, 1987

Let's return to the problem of coupling superconnections to Dirac operators. I think it is necessary to work out analytically some simple examples.

The key problem seems to be to take the harmonic oscillator operator

$$\begin{pmatrix} 0 & \partial_x - x \\ \partial_x + x & 0 \end{pmatrix}$$

on the line and to compress it, or restrict it, in someway to a small interval $(-\delta, \delta)$.

Let's recall the link between the Dirac over the 2-torus and the family of Diracs on the circle

$$f(t) \longmapsto \sum e^{2\pi i n y} f(x+n) = F(x,y)$$

$$\partial_t \longleftrightarrow \nabla_x = \partial_x$$

$$2\pi i t \longleftrightarrow \nabla_y = \partial_y + 2\pi i x .$$

Thus for a given x, we have the Dirac operator $\partial_y + 2\pi i x$ acting on functions of $y \in \mathbb{R}/\mathbb{Z}$. For $x \in (-\frac{1}{2}, \frac{1}{2})$, this operator has an eigenvalue closest to zero

July 20, 1987

In Hörmander's book (vol 3) there is a discussion of the index theorem in which he carefully constructs a ψDO of order zero on $\mathbb{R}^n$ whose symbol is of compact support and represents the Bott class. This operator is obtained by deforming (maybe one should say "cutting-off") the Gaussian DR complex. The construction is not trivial.

Also he (as A.S. previously) has to work to handle the exterior product of two ψDO's, which isn't a ψDO.

I get the impression then that the problem of coupling a superconnection to a Dirac operator is non-trivial. I can still hope for a nice solution which would shed light on the index thm. for families etc.

Let's consider dimension 1, where we wish to couple a loop g to ∂_x. On the level of ψDO's we have two ways to proceed. The first is the Toeplitz operator construction. The second is the method coming from Kasparov type ideas. Here one forms a ψDO with symbol

$$\gamma^1(pF) + \gamma^2 q \qquad\qquad F = \mathrm{sgn}(\xi)$$

where p, q are functions applied to g which look as follows:

The simplest example is $p = \dfrac{g^{1/2}+g^{-1/2}}{2}$, $q = \dfrac{g^{1/2}-g^{-1/2}}{2i}$

There $p^2+q^2=1$ and

$$\frac{q}{p} = \frac{1}{i}\,\frac{g-1}{g+1}.$$

Now we have

$$\circledast \qquad \gamma^1 pF + \gamma^2 q = \begin{pmatrix} 0 & pF - iq \\ pF + iq & 0 \end{pmatrix}$$

and we should look at the kernel of $pF+iq$ if we are concerned with the index.

If

$$(pF+iq)\psi = 0$$

then

$$F\psi + \left(\frac{g-1}{g+1}\right)\psi = 0$$

actually put in $g^{1/2}p = \frac{g+1}{2}$ $g^{1/2}q = \frac{g-1}{2}$

$$gF\psi + F\psi + g\psi - \psi = 0$$

$$g(F+1)\psi + (F-1)\psi = 0$$

or

$$\boxed{(gP_+ - P_-)\psi = 0}$$

In other words setting $\psi_\pm = P_\pm\psi$ we have

$$g\psi_+ = +\psi_-$$

which is the Hilbert factorization of g.

Next instead of the operator let us consider the self-adjoint operator

$$\gamma^1 p^{1/2}Fp^{1/2} + \gamma^2 q = \begin{pmatrix} 0 & p^{1/2}Fp^{1/2} - iq \\ p^{1/2}Fp^{1/2} + iq & 0 \end{pmatrix}$$

If

$$\left(p^{1/2}Fp^{1/2} + iq\right)\phi = 0, \text{ then multiplying by}$$

$p^{1/2}$ gives

$$(pF + iq)(p^{1/2}\phi) = 0$$

whence $p^{1/2}\phi = \psi$.

Here's an interesting point. It seems that if I take g to be a loop which is -1 on some open interval, then necessarily ψ vanishes on this interval, which is clear from

$$(gP_+ - P_-)\psi = 0$$
$$(P_+ + P_-)\psi = \psi.$$

So there is no problem dividing ψ by $p^{1/2}$.

Summary: I seem to understand a bit more ~~about~~ the operator $\gamma^1(p^{1/2}Fp^{1/2}) + \gamma^2 q$ this time around than before. In fact I seem to be able to identify this operator with the Toeplitz operator more or less.

Notice that

$$pF + iq = \tfrac{1}{2} g^{-1/2}\left((g+1)F + (g-1)\right)$$

$$= g^{-1/2}\left(g\,\frac{F+1}{2} + \frac{F-1}{2}\right)$$

$$= g^{-1/2}\left(g P_+ - P_-\right)$$

Also $\langle x|gP_+ - P_-|y\rangle = g(x)\langle x|P_+|y\rangle - \langle x|P_-|y\rangle$

$$= -\delta(x,y)$$

when $g(x) = -1$. Thus I seem to learn that the operator $gP_+ - P_-$ is ~~essentially~~ the kind of operator constructed by Hörmander.

July 21, 1987

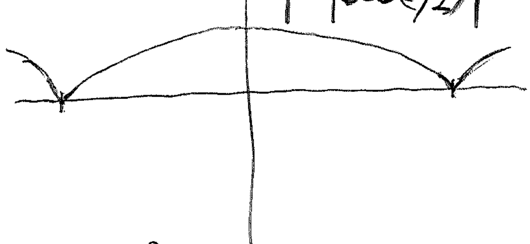
I propose to study the equation

$$(\ast) \qquad (p^{1/2} F p^{1/2} + iq)\psi = f$$

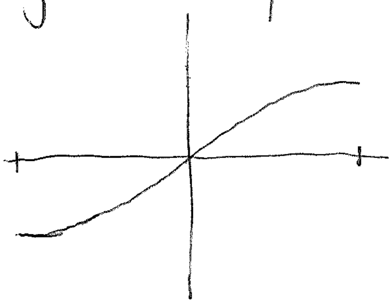
where $p = g^{-1/2}\left(\frac{g+1}{2}\right)$, $q = g^{-1/2}\left(\frac{g-1}{2}\right)$. Recall that $g^{1/2}$ is the square root of g with eigenvalues satisfying $\mathrm{Re}(\zeta) \geq 0$, hence there are problems with its definition when g has the eigenvalue -1. In the simple case where g is a blip approximant we have seen that q should be viewed as a map from the trivial line bundle to the Mobius line bundle on the circle. The same problem doesn't arise for p because p vanishes when $g = -1$, so there is no discontinuity.

A natural question is whether one can make sense of $g^{1/2}$ as a map from the trivial vector bundle and some other vector bundle. It's not clear this is worth exploring at this time, since even if we succeeded in making q smooth, we still have the singularity in p:

$p = |\cos(\theta/2)|$



$q = \sin(\theta/2)$



I think we should concentrate on solving the equation $(\ast)$, more precisely checking that it's a Fredholm operator, paying attention to the fact that ψ is restricted somehow where $g=-1$. Setting $\phi = p^{1/2}\psi$ we have

$$(pF + iq)\phi = p^{1/2} f$$

$$\left(\left(\frac{g+1}{2}\right)F + \left(\frac{g-1}{2}\right)\right)\phi = p^{1/2} g^{1/2} f$$

$$\|$$

$$(g P_+ - P_-)\phi = p^{1/2} g^{1/2} f$$

$$(P_+ + P_-)\phi = \phi$$

$$\Rightarrow \quad \boxed{\phi - (g+1)P_+\phi = -p^{1/2}g^{1/2} f}$$

This is a nice singular integral equation for ϕ which should have good solution properties. Note that the solution is divisible by $p^{1/2}$, so we ought to be able to get a similar equation for ψ.

$$\left(p^{1/2} F p^{1/2} + iq\right)\psi = f$$

$$\left[g^{1/2} p^{1/2} F p^{1/2} + \frac{(g-1)}{2}\right]\psi = g^{1/2} f$$

$$\left[g^{1/2} p \qquad - \frac{(g-1)}{2}\right]\psi = \psi$$

$$g^{1/2} p^{1/2} (F+1) p^{1/2}\psi = \psi + g^{1/2} f$$

$$\boxed{\psi - 2g^{1/2} p^{1/2} P_+ p^{1/2}\psi = -g^{1/2} f}$$

This shows that the operator to be inverted is of the form $1-K$ where K is supported where $g \neq -1$.

Analogy: Let's recall our earlier idea of defining a subspace

$$\mathrm{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix} = \mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix} \oplus \begin{pmatrix} \text{null space} \\ 0 \end{pmatrix}$$

and the conclusion that there was no way to construct G "=" $r\,(p\partial_x + q)^{-1}$ without constructing $(p\partial_x + q)^{-1}$, i.e. showing $p\partial_x + q$ is hypo-elliptic.

In the setup of the previous page we solve for ψ, while ϕ is the function with the good support.

Furthermore suppose we wish to solve

$$(gP_+ - P_-)\, u = f$$

and to simplify suppose $gH_+ \oplus H_- = L^2$. We assume know the factorization $g = g_-^{-1} g_+$, which corresponds to knowing about the homogeneous equation (roughly). To find

$$g\, h_+ + h_- = f$$

write

$$g_-^{-1} g_+ h_+ + h_- = f$$

$$g_+ h_+ + g_- h_- = g_- f$$

$$\therefore \quad g_\pm h_\pm = P_\pm (g_- f)$$

$$\therefore \quad h_\pm = g_\pm^{-1} P_\pm (g_- f)$$

This shows that we can construct the Green's function for $gP_+ - P_-$ starting from the factorization of g. This is analogous to constructing the Green's function for $p\partial_x + q$ ~~starting~~ from the solution of the homogeneous equation.

Now the point is that we know a lot about $gP_+ - P_-$ (which is essentially equivalent to $pF + iq$). We want to use the analogy to treat the Dirac case $p\partial_x + q$.

Further idea: Let us recall that

$$A = \gamma^1 p^{1/2} F p^{1/2} + \gamma^2 q$$

is a ψDO of order zero which is self-adjoint and involutive modulo compacts. Let's assume its a contraction, so that we have the Atiyah-Singer map to unitaries congruent to -1 mod compacts:

$$g = \left(\sqrt{1-A^2} + iA\right)^2$$

Since A is odd relative to ε, g is inverted by ε and so we have a point in the Grassmannian. On the other hand ~~starting~~ from a Dirac operator X we assign to it the unitary congruent to -1 mod compacts given by the Cayley transform

$$g = \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2$$

Thus the analogy should be that

$$A \sim \frac{\frac{1}{i}X}{\sqrt{1-X^2}}$$

i.e.

$$p^{1/2} F p^{1/2} + iq \sim \frac{\frac{1}{i}(p\partial_x - q)}{\sqrt{1 - (p\partial_x - q)^2}}$$

This last equation doesn't look very useful.

The important point is the following. We have a symbol

$$* \quad \gamma^1 p\, \mathrm{sgn}(\xi) + \gamma^2 q = \begin{pmatrix} 0 & p\,\mathrm{sgn}(\xi) - iq \\ p\,\mathrm{sgn}(\xi) + iq & 0 \end{pmatrix}$$

which is an odd involution. This operator can be lifted to an odd self-adjoint contraction.

$$A = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}$$

Thus B is unitary modulo compacts. Now we have a way to associate to A a subspace, namely the subspace corresponding to

$$g = \left(\sqrt{1-A^2} + iA\right)^2.$$

It's also the graph of the operator

$$B\left(1 - B^*B\right)^{-1/2}$$

The natural <u>problem</u> is whether one can carry out this idea in a concrete way. Thus I would like to there to be a good way to link the ψDO symbol $*$ with a suitable subspace.

<u>July 22, 1987</u>

The main problem is this: We have the symbol $g^{1/2}(p\, \mathrm{sgn}(\xi) + iq)$ = symbol of $gP_+ - P_-$ which is unitary. It can be lifted to an essentially unitary contraction operator T. If g is defined by

$$A = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} \qquad g = \left(\sqrt{1-A^2} + iA\right)^2$$

then g corresponds to a subspace of $L^2 \oplus L^2$ and determines a point in a suitable Grassmannian. The problem is to carry out this process concretely, and hopefully to find a link with gH_+.

Note that $gP_+ - P_-$ is not a contraction in general, since if $gh_+ = h_-$, then $\|h_+\| = \|h_-\|$, say these are 1. Then $h_+ - h_-$ has norm $\sqrt{2}$ and

$$(gP_+ - P_-)(h_+ - h_-) = gh_+ + h_- = 2h_- \text{ has norm } 2.$$

There are three things to relate:

Toeplitz operator $\quad i^* g\, i \qquad i : H_+ \to H$

Hilbert-Privalov " $\quad gP_+ - P_-$

Subspace $\quad gH_+ \longleftrightarrow$ involution $\quad gFg^{-1}$

$$\mathrm{Ker}(i^* g i) = \{v \in H_+ \mid gv \in H_-\} \cong gH_+ \cap H_-$$

$$\mathrm{Ker}(gP_+ - P_-) = gH_+ \cap H_-$$

$gH_+ \cap H_-$ is also the "kernel" of gH_+ rel. to F.

Similarly for the cokernel = kernel of adjoint we have

$$\operatorname{Ker} \, \iota^* g^{-1} \iota \simeq g^{-1} H_+ \cap H_- = H_+ \cap g H_-$$

$$\operatorname{Ker} \left(P_+ g^{-1} - P_- \right) = H_+ \cap g H_-$$

Let's write g relative to the decomposition $H = H_+ \oplus H_-$ in the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\iota^* g \iota = a$

and

$$g P_+ - P_- = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 \\ c & -1 \end{pmatrix}$$

which can be interpreted as saying we have an endomorphism of the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H_- \arrow[r] \arrow[d, "-1"] & H \arrow[r] \arrow[d, "gP_+ - P_-"] & H_+ \arrow[r] \arrow[d, "\iota^* g \iota"] & 0 \\
0 \arrow[r] & H_- \arrow[r] & H \arrow[r] & H_+ \arrow[r] & 0
\end{tikzcd}$$

Thus it appears that the good contraction operator is $\iota^* g \iota$. However the operator $g P_+ - P_-$ has nice localization properties.

July 23, 1987

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ relative to $H = H_+ \oplus H_-$ and assume that g is "the" minimal dilation of $a = \iota^* g \iota$. First note that a has the dilation

$$\tilde{g} = \begin{pmatrix} a & \sqrt{1-aa^*} \\ \sqrt{1-a^*a} & -a^* \end{pmatrix}$$

Since

$$\tilde{g}^*\tilde{g} = \begin{pmatrix} a^* & \sqrt{1-a^*a} \\ \sqrt{1-aa^*} & -a \end{pmatrix}\begin{pmatrix} a & \sqrt{1-aa^*} \\ \sqrt{1-a^*a} & -a^* \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \cancelto{0}{a^*\sqrt{1-aa^*} - \sqrt{1-a^*a}\,a^*} \\ \cancelto{0}{\sqrt{1-aa^*}\,a - a\sqrt{1-a^*a}} & 1 \end{pmatrix}.$$

$\tilde{g}$ is a minimal dilation if a^*a, aa^* do not have the eigenvalue 1, which is part of the assumption. Thus we know there is a unique embedding $H_+ \oplus H_+ \hookrightarrow H_+ \oplus H_-$ such that g induces $\tilde{g}$. The rest of the assumption is that this embedding is an isomorphism. In other words we have an isomorphism of $H_+ \simeq H_-$ such that b, c correspond to $\sqrt{1-a^*a}$, $\sqrt{1-aa^*}$ respectively. So it seems that the assumptions amounts to non-degeneracy conditions on b, c.

Next let us look at the subspace of $H_+ \oplus H_+$ corresponding to the unitary inverted by ε

$$\left(\sqrt{1-A^2} + iA\right)^2 \quad \text{where} \quad A = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$$

Let's recall that for the graph of iT the unitary is

$$\left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 \quad \text{where } X = \begin{pmatrix} 0 & iT^* \\ iT & 0 \end{pmatrix}$$

Thus
$$\frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} 1 & iT^* \\ iT & 1 \end{pmatrix}\begin{pmatrix} (1+T^*T)^{-1/2} & 0 \\ 0 & (1+TT^*)^{-1/2} \end{pmatrix}$$

$$\sqrt{1-A^2} + iA = \begin{pmatrix} (1-a^*a)^{1/2} & ia^* \\ ia & (1-aa^*)^{1/2} \end{pmatrix}$$

so
$$\boxed{\begin{aligned} a &= T(1+T^*T)^{-1/2} \\ T &= a(1-a^*a)^{-1/2} \end{aligned}} \qquad (1-a^*a)^{1/2} = (1+T^*T)^{-1/2}$$

Conclusein: The subspace of $H_+ \oplus H_+$ assoc. to $\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ is the graph of $a(1-a^*a)^{-1/2}$.

Now use the isomorphism $H_+ \oplus H_+ = H_+ \oplus H_-$, or better $H_+ \simeq H_-$ which transform $\sqrt{1-a^*a}$ into c. It would appear that we are interested in the subspace which is the graph of ac^{-1}. Thus it appears that the process leads to the subspace

$$\mathrm{Im}\begin{pmatrix} a \\ c \end{pmatrix} = gH^+$$

In general given a contraction $a : H_0 \to H_1$ I get a subspace of $H_0 \oplus H_1$ corresponding to the unitary operator reversed by ε given by

$$\Big(\underbrace{\sqrt{1+Y^2}+Y}_{h}\Big)^2 \qquad Y = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} \sqrt{1-a^*a} & -a^* \\ a & \sqrt{1-aa^*} \end{pmatrix}$$

Note that $\varepsilon h \varepsilon = \sqrt{1+Y^2} - Y = h^{-1}$, so h^2 is inverted by ε. Also we have

$$\underbrace{h^2\varepsilon}_{\substack{\text{involution} \\ \text{corresp. to } h^2}} \cdot h = h\varepsilon h^{-1} h = h\varepsilon$$

which means that the columns of h are the eigenspaces of the involution. Thus the subspace corresponding to h^2 is

$$\operatorname{Im}\begin{pmatrix} \sqrt{1-a^*a} \\ a \end{pmatrix}$$

Summary: The subspace belonging to $\left(\sqrt{1+Y^2}+Y\right)^2$ where $Y = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$ a contraction is $\operatorname{Im}\begin{pmatrix} \sqrt{1-a^*a} \\ a \end{pmatrix}$, in analogy to the fact that the subspace corresponding to

$$\left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 = \frac{1+X}{1-X} \quad \text{where} \quad X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

is the graph $\operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$.

July 26, 1987

I still want to make use of the following discovery. The operator

$$g^{1/2}(pF + iq) = \left(\frac{g+1}{2}\right)F + \left(\frac{g-1}{2}\right) = gP_+ - P_-$$

is an elliptic ΨDO of order zero, hence Fredholm, whose kernel is "localized" in the following sense

$$gP_+ - P_- = (g+1)P_+ - 1$$

$$(gP_+ - P_-)f = 0 \iff f = (g+1)P_+ f$$

so any f in the kernel vanishes where $g = -1$.

One idea that doesn't work is the following. In analogy with

$$\operatorname{Im}\begin{pmatrix} \frac{g+1}{2} \\ \partial_x \frac{g+1}{2} + \frac{g-1}{2} \end{pmatrix}$$

we can consider the subspace

$$\operatorname{Im}\begin{pmatrix} \frac{g+1}{2} \\ F\frac{g+1}{2} + \frac{g-1}{2} \end{pmatrix} = \operatorname{Im}\begin{pmatrix} \frac{g+1}{2} \\ gP_+ - P_- \end{pmatrix}$$

However this subspace is not always in the restricted Grassmannian, since for $g \equiv 1$ it is $\operatorname{Im}\begin{pmatrix} 1 \\ F \end{pmatrix}$.

Something else worth mentioning is the cokernel of $gP_+ - P_-$ which is also

$$\operatorname{Ker}(P_+ g^{-1} - P_-)$$

$$= H_+ \cap gH_-$$

As it stands $P_+ g^{-1} - P_- = P_+(g^{-1} + 1) - 1$, so a function in the kernel of $P_+ g^{-1} - P_-$ satisfies

$$f = P_+ (g^{-1}+1) f$$

and so is not supported where $g \neq -1$. However we have

$$\operatorname{Ker}(P_+ g^{-1} - P_-) = H_+ \cap g H_- \cong \operatorname{Ker}(P_+ - g P_-)$$

and the latter admits a "localized" description.

$$\operatorname{Ker}(g P_+ - P_-) \simeq g H_+ \cap H_-$$
$$f \longmapsto f_- = g f_+$$

$$\operatorname{Ker}(P_+ g^{-1} - P_-) = H_+ \cap g H_- \simeq \operatorname{Ker}(P_+ - g P_-)$$
$$f_+ = g f_- \longleftarrow f$$

Thus for the "operator" we seek

$$\operatorname{Ker} = g H_+ \cap H$$
$$\operatorname{Cok} = H_+ \cap g H_-$$

We could look at the subspace $g H_+$ which we know belongs to the restricted Grassmannian and which gives rise to this kernel and cokernel. However we seek something which is "localized" on the set where $g \neq -1$. Ideally I would like to find an operator defined over this set "extended by zero" to the rest of the circle.

Another point is that the subspace gH_+ doesn't see the difference between $g=+1$ and -1. Thus we [redacted] will have to use more information about g, say perhaps the Toeplitz operator i^*gi on H_+.

Perhaps it would be useful to specify what we would like to find. To a [redacted] loop g we would like to have a subspace $\Gamma_g \subset L^2 \oplus L^2$. It should be in the restricted Grassmannian, i.e. be close to $0 \oplus L^2$ in a suitable sense. Its kernel, i.e. $\Gamma_g \cap (L^2 \oplus 0)$ should be isomorphic to $gH_+ \cap H_-$ and its cokernel, namely the orthogonal complement of $\overline{\Gamma_g + (L^2 \oplus 0)}$ should be isomorphic to $H_+ \cap gH_-$. [redacted] Also we would like these isomorphisms to be the ones described above [redacted] which are compatible with localization; e.g.

$$gH_+ \cap H_- \simeq \mathrm{Ker}\ \underbrace{gP_+ - \text{[redacted]}\,P_-}_{(g+1)P_+ - 1}$$

$$gf_+ = f_- \longleftarrow\!\!\mid \quad f$$

[redacted] Finally I would like Γ_g to be localized, supported in the closure of the set where $g \neq -1$. To be precise let this set [redacted] be Z, then

$$\Gamma_g = \left(\Gamma_g \cap (L^2_Z \oplus L^2_Z)\right) \oplus \left(0 \oplus (L^2_Z)^\perp\right)$$

July 31, 1987

I am looking again at the differential operator case, where I propose to define a subspace by

$$\mathrm{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}.$$

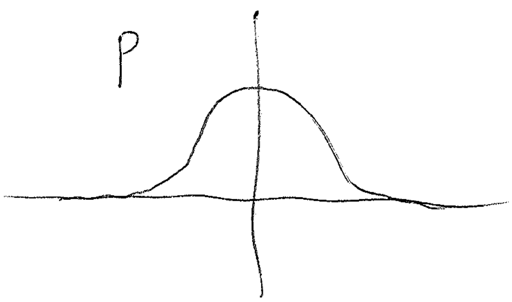
The hope is that under suitable conditions this is the same modulo finite dimensional subspaces as

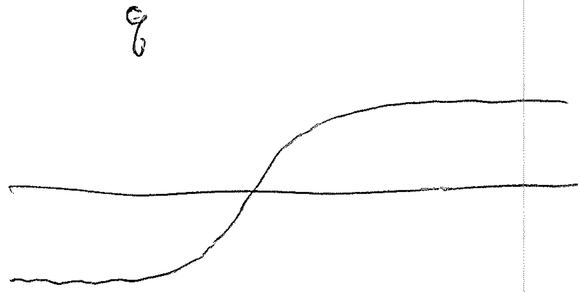
$$\mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$$

where G is a Green's function. Now the equality of this two spaces means one can solve

$$(p\partial_x + q)u = f$$

for any f. Thus I have to understand whether this can be the case in a typical index 1 ~~situation~~:





The problem occurs at the endpoints of $\mathrm{Supp}(p)$.

To simplify let's consider $q = -1$ and

$$p \begin{cases} = 0 & \text{for } x \leq 0 \\ > 0 & \text{for } x > 0 \end{cases}$$

and set

$$\varphi = e^{\int^x \frac{1}{p}}, \qquad \psi = \frac{1}{\varphi}$$

so that

$$p\varphi' - \varphi = 0 \quad , \qquad p\psi' + \psi = 0$$

A Green's function for $-p\partial_x + 1$ should be

$$K(x,y) = \begin{cases} 0 & y < x \\ \delta(x-y) & x < 0 \\ -\dfrac{\psi'(y)}{\psi(x)} & 0 < x \le y \end{cases}$$

Actually this is obviously a Green's function for $x > 0$, since

$$(Kf)(x) = \int_x^R K(x,y)\, f(y)\, dy$$

$$= \int_x^R \frac{-\psi'(y)}{\psi(x)} f(y)\, dy$$

obviously is the unique solution of $(-p\partial_x + 1)u = f$ such that $u(R) = 0$. Thus I want to see that Kf defined for $x > 0$ in this way and defined to be f for $x \le 0$, is a smooth function of x.

I claim it's continuous in x. Note that $-\frac{\psi'(y)}{\psi(x)}\, dy$ is a positive measure for $x \le y \le R$ and that

$$\int_x^R -\frac{\psi'(y)}{\psi(x)}\, dy = \frac{\psi(x) - \psi(R)}{\psi(x)}$$

$$= 1 - \frac{\psi(R)}{\psi(x)} \uparrow 1 \quad \text{as } x \downarrow 0.$$

It's clear therefore that

$$K(x,y) \downarrow \delta(y) \qquad \text{as } x \downarrow 0.$$

Let's recall that measures being continuous linear functionals on continuous functions are automatically distributions. Also the norm of a positive measure as a linear functional on continuous functions is the ~~total~~ measure (assuming compact support) since

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu \leq \|f\|_\infty \int d\mu$$

with equality for $f = 1$ on $\mathrm{Supp}(d\mu)$. Since

$$\int_x^R -\frac{\psi'(y)}{\psi(x)} \, dy = 1 - \varphi(x)\psi(R)$$

it follows that in the space of measures the distance between $K(x,y)\,dy$ for $x>0$ and $\delta(y)\,dy$ is $O(x^N)$ for all N. This indicates that $x \longmapsto K(x,y)\,dy$ might be a smooth path in the space of distributions.

Set

$$u(x) = \int_x^R -\frac{\psi'(y)}{\psi(x)} f(y) \, dy \qquad \text{for } x > 0$$

It is the unique solution of

$$(-p\partial_x + 1)\, u = f$$

which vanishes at $x = R$. Here f is smooth on $\mathbb{R}$ and R is fixed > 0.

Then integrating by parts

$$u(x) = \left[-\frac{\psi(y)}{\psi(x)} f(y)\right]_{y=x}^{y=R} + \int_x^R \frac{\psi(y)}{\psi(x)} f'(y)\, dy$$

$$* \qquad = f(x) - \frac{1}{\psi(x)} \psi(R) f(R) + \int_x^R \frac{\psi(y)}{\psi(x)} f'(y)\, dy$$

We would like to extend the definition of $u(x)$ by setting

$$u(x) = f(x) \qquad \text{for} \qquad x \leq 0$$

The question is whether this extension is smooth, in particular C^1. We have seen the extension is continuous.

We compute the derivative of the last term in $*$.

$$\partial_x \int_x^R \frac{\psi(y)}{\psi(x)} f'(y)\, dy = -\frac{\psi(x) f'(x)}{\psi(x)} + \int_x^R \underbrace{\partial_x\left(\frac{1}{\psi(x)}\right)}_{\partial_x \varphi = \frac{1}{p(x)\psi(x)}} \psi(y) f'(y)\, dy$$

$$\cancel{= -f'(x) + \int_x^R \frac{1}{p(x)}\left(-\frac{\psi'(y)}{\psi(x)}\right) p(y) f'(y)\, dy}$$

$$= -f'(x) + \int_x^R \frac{\psi(y)}{p(x)\psi(x)} f'(y)\, dy$$

We want to argue that $\overset{\chi_{[x,R]}(y)}{\cancel{\theta(y-x)}} \dfrac{\psi(y)}{p(x)\psi(x)} \longrightarrow \delta(y)$

as $x \downarrow 0$, hence we need to estimate

$$\int_x^R \psi(y)\, dy = \int_x^R -\psi'(y) p(y)\, dy$$

$$= \left[-p\psi\right]_x^R + \int_x^R p'(y)\psi(y)\, dy$$

error see p. 24

$$= p(x)\psi(x) - p(R)\psi(R) + \int_x^R p'(y)\psi(y)\, dy$$

<u>Lemma</u>: Assume $h(y)$ (h decreasing and) $\uparrow +\infty$ as $y \downarrow 0$.

Then
$$\frac{1}{h(x)} \int_x^R h(y)\,dy \to 0 \quad \text{as } x \downarrow 0.$$

<u>Proof</u>: Given $\varepsilon > 0$ write

$$\int_x^R h(y)dy = \int_x^\varepsilon h(y)dy + \int_\varepsilon^R h(y)dy$$
$$\leq h(x)(\varepsilon - x) + \text{const.}$$

$$\therefore \limsup_{x \downarrow 0} \frac{1}{h(x)} \int_x^R h(y)dy \leq \varepsilon \qquad \text{QED.}$$

So if we assume that $p(x)\psi(x)$ increases to ∞ as $x \downarrow 0$, then we conclude

$$\frac{\int_x^R \psi(y)\,dy}{p(x)\psi(x)} \longrightarrow 1 \quad \text{as } x \downarrow 0.$$

From this we conclude that as $x \downarrow 0$

$$\partial_x \int_x^R \frac{\psi(y)}{\psi(x)} f'(y)\,dy \longrightarrow -f'(0) + f'(0) = 0$$

and so $(\partial_x u)(x) \longrightarrow f'(0)$ as $x \downarrow 0$. Thus u is C^1.

Note $\partial_x(p(x)\psi(x)) = p'\psi + p\psi' = (p'-1)\psi$ and $p'(x) - 1 < 0$ for x near 0. $\therefore \partial_x(p\psi) < 0$ and $p\psi$ is decreasing near 0.

August 1, 1987

Here's how to understand yesterday's arguments. First integrating by parts:

$$-\psi'(y)p(y)$$

$$\int_x^R -\frac{\psi'(y)}{\psi(x)} f(y)\,dy = f(x) - \frac{\psi(R)}{\psi(x)} f(R) + \int_x^R \frac{\psi(y)}{\psi(x)} f'(y)\,dy$$

$$= f(x) - \frac{\psi(R)}{\psi(x)} f(R) + (pf')(x) - \frac{\psi(R)}{\psi(x)} (pf')(R)$$

$$+ \int_x^R -\frac{\psi'(y)}{\psi(x)} \big(p(pf')'\big)\,dy$$

This is just the geometric series

$$\frac{1}{1-p\partial} = 1 + \frac{1}{1-p\partial}\, p\partial = 1 + p\partial + \frac{1}{1-p\partial}(p\partial)^2$$

Also

$$v(x) = \int_x^R \frac{\psi(y)}{p(x)\psi(x)} f(y)\,dy$$

satisfies

$$\partial_x p v = \partial_x \int_x^R \frac{\psi(y)}{\psi(x)} f(y)\,dy \qquad \partial\varphi = \frac{1}{p}\varphi$$

$$= -f(x) + \int_x^R \frac{\psi(y)}{p(x)\psi(x)} f(y)\,dy$$

so

$$v = \frac{1}{1-\partial_x p} f$$

Now we showed that

$$f \in C^0 \implies \frac{1}{1-\partial_x p} f \in C^0$$

From

$$\partial_x \frac{1}{1-p\partial_x} = \frac{1}{1-\partial_x p}\partial_x$$

we conclude that $\frac{1}{1-p\partial_x} C^1 \subset C^1$. This is [redacted] essentially the argument above.

So at the moment all we know is that $\frac{1}{1-\partial_x p}$ yields continuous functions and $\frac{1}{1-p\partial_x}$ yields C^1 functions, when applied to smooth functions.

Most of [redacted] the above is incomplete because of the integration by parts error on p. 21. Thus we have not [redacted] proved that $\frac{1}{1-\partial_x p} f$ is continuous.

Let's review this. We have

$$\int_x^R \frac{\psi(y)}{p(x)\psi(x)} f(y)\,dy = \frac{1}{(p\psi)(x)} \int_x^R -\psi'(y)(pf)(y)\,dy$$

$$= \frac{1}{(p\psi)(x)} \left\{ \left[-\psi p f\right]_x^R + \int_x^R \psi (pf)'\,dy \right\}$$

$$= f(x) - \frac{(\psi p f)(R)}{(p\psi)(x)} + \int_x^R \frac{\psi(y)}{p(x)\psi(x)} (pf)'(y)\,dy$$

conforming to the formal identity

$$\frac{1}{1-\text{[redacted]}\partial_x p} = 1 + \frac{1}{1-\partial_x p}\partial_x p$$

Set $f \equiv 1$.

$$\int_x^R \frac{\psi(y)}{p(x)\psi(x)}\,dy = 1 - \frac{(p\psi)(R)}{(p\psi)(x)} + \int_x^R \frac{\psi(y)}{p(x)\psi(x)} p'(y)\,dy$$

We want to show this approaches 1 as $x \downarrow 0$. This is consistent with the fact that the latter [redacted] integral should have limit zero, as p' vanishes at 0.

Let's ~~write~~ the preceding as

$$\int_x^R \frac{\psi(y)}{p(x)\psi(x)}\,[1-p'(y)]\,dy = 1 - \frac{(p\psi)(R)}{(p\psi)(x)}$$

~~Choose $\varepsilon > 0$ so small that $|p'(x)| < a$~~

Given $\varepsilon > 0$ choose $\delta > 0$ so that $|p'(x)| < \varepsilon$ for $0 \le x < \delta$. Then

$$1-\varepsilon < 1-p'(y) < 1+\varepsilon$$

so

$$(1-\varepsilon)\, I_x \le \int_x^\delta \frac{\psi(y)}{p(x)\psi(x)}\,(1-p'(y))\,dy \le (1+\varepsilon)\, I_x$$

where $I_x = \int_x^\delta \frac{\psi(y)}{p(x)\psi(x)}\,dy$. Now because $p(x)\psi(x)$ is becoming $+\infty$ as $x \downarrow 0$, we have

$$\overline{\lim_{x\downarrow 0}} \int_x^R = \overline{\lim_{x\downarrow 0}} \int_x^\varepsilon$$

and similarly for $\underline{\lim}$. Thus

$$(1-\varepsilon)\,\overline{\lim}\, I_x \le 1 \le (1+\varepsilon)\,\overline{\lim}\, I_x$$

and similarly for $\underline{\lim}$, so that we do get

$$\lim_{x\downarrow 0} \int_x^R \frac{\psi(y)}{p(x)\psi(x)}\, f(y)\,dy = f(x)$$

for $f \equiv 1$ and then in general for any continuous f. This fixes the error on p. 21

August 2, 1987

We consider the ODE

$$(1) \qquad u - pu' = f$$

where p, f are smooth functions of x defined in an open interval I containing $x=0$. We assume

$$p(x) > 0 \quad \text{for} \quad x > 0$$
$$p(x) = 0 \quad \text{for} \quad x \leq 0.$$

We can solve the equation for $x > 0$ - we pick an initial value for u at some $a > 0$ in I, and then there is a unique solution $u(x)$ for $x > 0$ in I with this initial value. Formula

$$u(x) = c\varphi(x) + \varphi(x)\int_a^x \frac{-1}{\varphi(y)p(y)} f(y)\, dy$$

$$\varphi(x) = \exp\left(\int_a^x \frac{dy}{p(y)}\right).$$

Then we extend this solution to the whole of I by putting $u(x) = f(x)$ for $x \leq 0$. The <u>claim</u> is that the resulting $u(x)$ is smooth. It satisfies (1) for $x < 0$ and $x > 0$, and hence for all x in I.

We wish to prove this claim by induction; that is, to show $u \in C^r$ for $r \geq 0$ by induction on r. Let's check the induction step. Let $r > 0$.

Differentiating (1) yields

$$u' - p'u' - pu'' = f'$$

$$(1-p')u' - p(u')' = f'$$

$$u' - \frac{p}{1-p'}(u')' = \frac{1}{1-p'} f'$$

Because p is smooth and 0 for $x<0$, it follows that $p'=0$ at $x=0$, and so $1-p'>0$ near $x=0$. Thus if we shrink the right end of I, then we see u' satisfies an equation of the same type as ①. Moreover $u' = f'$ for $x<0$. By induction hypothesis u' is C^{r-1} hence u is C^r on the smaller interval. Hence u is C^r on I, proving the induction step.

Thus we need only prove the continuity of u at $x=0$. Notice that we haven't yet used the positivity of p yet.

First look at $f=0$, $u(x) = c\varphi(x)$.

$$\varphi(x) = \exp\left\{-\int_x^a \frac{1}{p(y)}\,dy\right\}$$

We have $0 < p(y) \leq Cy$ on $[0,a]$ so

$$-\frac{1}{p(y)} \leq \frac{-1}{Cy}$$

$$\varphi(x) \leq \exp\left\{-\int_x^a \frac{+1}{Cy}\,dy\right\} = \exp\left(-\frac{1}{C}\log\frac{a}{x}\right)$$

$$= \left(\frac{x}{a}\right)^{1/C} \qquad \text{with } C>0$$

This shows $\lim_{x \downarrow 0} \varphi(x) = 0$ proving continuity, hence smoothness by the inductive step.

Next we show for f continuous that

$$\lim_{x \downarrow 0} u(x) = f(0)$$

We have

$$u(x) = c\varphi(x) + \int_x^a \underbrace{\frac{\varphi(x)}{\varphi(y)p(y)}}_{>0} f(y)\,dy$$

Also we have

$$\int_x^a \frac{\varphi(x)}{\varphi(y)p(y)}\,dy = 1 - \frac{\varphi(x)}{\varphi(a)}$$

since both satisfy $(1 - p\partial_x)u = 1$ and vanish at $x = a$. Since $\varphi(x) \to 0$ as $x \downarrow 0$, it follows that the measures

$$d\mu_x = \chi_{[x,a]}(y)\,\frac{\varphi(x)}{\varphi(y)\,p(y)}\,dy$$

approaches the delta measure at $y = 0$. This implies $u(x) \longrightarrow f(0)$ as $x \downarrow 0$.

Another proof can be given using l'Hopital's rule in the form $\frac{\infty}{\infty}$:

$$u(x) = \frac{c + \int_x^a \frac{\psi(y)}{p(y)} f(y)\,dy}{\psi(x)}$$

$$\frac{\text{deriv of numer}}{\text{deriv of denom}} = \frac{-\frac{\psi(x)f(x)}{p(x)}}{\psi'(x)} = f(x)$$

except we have to check hypotheses of Hopital's Rule. Simpler appears to be to use the MVT for integrals directly

$$u(x) = \underbrace{\frac{c + \int_\varepsilon^a \cdots\cdots}{\psi(x)}}_{\downarrow \\ 0} + \left(\frac{\int_x^\varepsilon -\psi'(y) f(y)\, dy}{\int_x^\varepsilon -\psi'(y)\, dy}\right) \cdot \underbrace{\left(\frac{\psi(x) - \psi(\varepsilon)}{\psi(x)}\right)}_{\downarrow \\ 1}$$

Applying the MVT for integrals

$$\frac{\int_a^b f(y)\, d\rho(y)}{\int_a^b d\rho(y)} = f(\xi) \qquad \text{for some } \xi \in [a, b]$$

(Requires f continuous, $d\rho \geq 0$), yields the result.

Let's now discuss the case of opposite sign: [redacted] $u + p\partial_x u = f$.

This time the solution of the homogeneous equation $u + p\partial_x u$ on $x > 0$ is $\psi(x)$, which blows up. Thus there is no hope of extending solutions for $x > 0$ to $x < 0$. However we [redacted] can hope ~~to~~ extend the solution $u(x) = f(x)$ for $x < 0$ to the interval. Thus we look at the solution for $x > 0$ given by

$$u(x) = \psi(x) \int_0^x \overset{\varphi'(y)}{\frac{1}{p(y)\psi(y)}}\, f(y)\, dy$$

and try to show $u(x) \to f(0)$ as $x \downarrow 0$. We should write this in the form

$$u(x) = \frac{\int_0^x \varphi'(y) f(y)\,dy}{\varphi(x)} = \frac{\int_0^x f(y)\varphi'(y)\,dy}{\int_0^x \varphi'(y)\,dy}$$

Thus the desired result follows from the MVT for integrals, or Hopital's Rule in the case $\frac{0}{0}$.

Thus we know there is a unique continuous solution of $u + p\partial_x u = f$ for a given f. Now do the induction

$$u' + p'u' + pu'' = f'$$

$$u' + \frac{p}{1+p'}(u')' = \frac{f'}{1+p'}$$

Let v be the unique solution of

$$v + \frac{p}{1+p'} v' = \frac{f'}{1+p'}$$

continuous at $x = 0$, hence continuous. ∎

Set $\tilde{u}(x) = f(0) + \int_0^x v(y)\,dy$. This is C^1 and C^2 for $x > 0$. As $\tilde{u}' = v$ we know that

$$\tilde{u} + p\tilde{u}' = f + \text{const}$$

and the constant is zero by looking at $x<0$. Thus $u = \tilde{u}$ is C^1, etc.

At this point we have probably proved everything we need in order to handle the index 1 example

August 3, 1985

Let us now consider $(1-p\partial_x)u = f$ where $p(x)$ is real smooth of compact support and $p(x)=0 \implies p'(x)=0$. We would like to show the above DE has a unique smooth solution for any smooth f.

Consider first the case where p has a single hump, say $p(x)>0$ for $x\in(0,1)$ and $p(x)=0$ outside $(0,1)$. Then the unique solution is

$$u(x) = \begin{cases} \displaystyle\int_x^1 \frac{-\psi'(y)}{\psi(x)} f(y)\,dy & x\in(0,1) \\ f(x) & x\notin(0,1) \end{cases}$$

where $\psi(x) = \exp\left(-\int_{1/2}^x \frac{1}{p}\right)$. We saw yesterday that u is smooth.

Next suppose $p(x)<0$ for $x\in(0,1)$ and $p(x)=0$ for $x\notin(0,1)$. Let ψ be the same function, but note that it decays as $x\downarrow 0$ and blows up as $x\uparrow 1$. Then the solution is

$$u(x) = \begin{cases} \displaystyle\int_0^x \frac{\psi'(y)}{\psi(x)} f(y)\,dy & x\in(0,1) \\ f(x) & x\notin(0,1) \end{cases}$$

and this will be smooth also.

Let's recall that to prove smoothness we first prove continuity.

Here the idea is the following. The solution of $(1-p\partial_x)u = f$ is of the form

$$u(x) = \int f(y)\, d\mu_x(y)$$

where $d\mu_x$ is a probability measure depending on x which we explicitly know. Thus

$$d\mu_x(y) = \begin{cases} \delta(y-x)\,dy & \text{if } p(x)=0 \\ \theta(y-x)\,\frac{1}{p(y)}\,e^{-\int_x^y \frac{1}{p}}\cdot dy & \text{if } p(x)>0 \text{ and } p \text{ is } >0 \text{ on } [x,y] \\ & 0 \text{ if } p \text{ vanishes on } [x,y] \\ \theta(x-y)\,\frac{-1}{p(y)}\,e^{\int_y^x \frac{1}{p}} & \text{if } p(x)<0 \text{ and } p \text{ is } <0 \text{ on } [y,x] \\ & 0 \text{ otherwise} \end{cases}$$

We have a family of probability measures depending on x. I think it's then clear that $u(x)$ is continuous for f continuous. The point is that we have control over the support of $d\mu_x$, so that if $p(x_0)=0$ and x is close to x_0, then we know that the support of $d\mu_x$ is close to x_0.

Note that $d\mu_x$ has to be a probability measure because $u=1$ satisfies $(1-p\partial_x)u = 1$.

Now we know that the equation $(1-p\partial_x)u = f$ has a continuous solution u which is C^1 where $p\neq 0$ for any continuous f. Uniqueness of u is clear, even on an open interval provided we allow for some necessary bdry conditions when $p\neq 0$ at the ends.

Next I want to check the ~~iteration~~ so as to get u smooth when f is smooth. This is a local matter near a point x_0 where $p(x_0) = 0$, and here is where where have to ~~[illegible]~~ use the condition

$$p(x) = 0 \implies p'(x) = 0.$$

Let u be continuous, C^1 where $p \neq 0$ satisfying $u - pu' = f$ smooth. Solve

$$v - \frac{p}{1-p'} v' = \frac{f}{1-p'}$$

near the point x_0 where $p(x_0) = 0$. Note that $p_1 = \frac{p}{1-p'}$ satisfies the same condition since if $p_1(x) = 0$, then $p(x) = 0$, so $p'(x) = 0$ and

$$p_1'(x) = \frac{p'(x)}{1-p'(x)} - \frac{-p''(x)p(x)}{(1-p'(x))^2} = 0.$$

So if we use an induction we know that v is of class C^{r-1}. Now solve

$$u' = v$$
$$u(x_0) = f(x_0)$$

near x_0. Then u is C^r and C^∞ where $p \neq 0$. Also

$$(u - pu' - f)' = v - p'v - pv' - f'$$
$$= 0$$

on the open set where $p \neq 0$. ?

Before getting involved with fancy smooth functions p, we first ought to see if to rather simple loops g we can assign subspaces in a restricted Grassmannian. Let's recall that we want to define the subspace using a first order operator:

$$\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$$

where p, q, r are given by formulas in terms of the loop g.

In ~~[illegible]~~ order that this subspace lie in the restricted Grassmannian we want

$$p(x) = 0 \Longrightarrow r(x) = 0.$$

Because the orthogonal complement of this subspace should be on the same footing we want this orth. complement to have an analogous form

$$\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}^{\perp} = \operatorname{Im}\begin{pmatrix} 1 \\ (p\partial_x + q)r^{-1} \end{pmatrix}^{\perp} = \operatorname{Im}\begin{pmatrix} r^{-1}(\partial_x p - q) \\ 1 \end{pmatrix}$$

$$= \operatorname{Im}\begin{pmatrix} r^{-1}(\partial_x p - q)s \\ s \end{pmatrix}$$

The natural choice for s is r in which case because

$$r^{-1}(\partial_x p - q)r = \left(\partial_x + \frac{r'}{r}\right)p - q$$

~~[illegible]~~ we require $\boxed{p\frac{r'}{r} \text{ smooth}}$

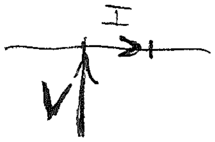
If we put in Planck's constant $\partial_x \to h\partial_x$, then q becomes more intrinsic, and we want

$$p(x) = 0 \Longrightarrow r(x) = 0, \quad q(x) \neq 0.$$

This for the orthogonal complement side forces us to take $s = r$.

August 5, 1987

Transmission lines intrinsically. Recall the equations:



$$-dV = (L\,dx)\,\partial_t I$$
$$-dI = (C\,dx)\,\partial_t V$$

and we can remove the sign by changing the direction of I.

Intrinsically we have a 1-manifold on which we are given two positive densities $L|dx|$, $C|dx|$. V is a function on the manifold and I is a section of the orientation line bundle, meaning that I becomes a function upon picking an orientation. The equations of motion are

$$dV = (L|dx|)\,\partial_t I$$
$$dI = (C|dx|)\,\partial_t V$$

and they make sense because the bundle of 1-forms Ω^1 is the tensor bundle of the bundle of densities and the orientation bundle.

The signal speed is the geometric mean of the inductance and capacitance densities

$$ds = \sqrt{LC}\,|dx|$$

and this gives us an intrinsic metric on the line.

Transmission line equations

$$\partial_t \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & C^{-1}\partial_x \\ L^{-1}\partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

The operator $\begin{pmatrix} 0 & C^{-1}\partial_x \\ L^{-1}\partial_x & 0 \end{pmatrix}$ is skew-adjoint relative to the energy norm

$$\int \left(\tfrac{1}{2} CV^2 + \tfrac{1}{2} L I^2\right) dx$$

Thus if we rewrite the equations

$$\partial_t \begin{pmatrix} C^{\frac{1}{2}} V \\ L^{1/2} I \end{pmatrix} = \begin{pmatrix} 0 & C^{-1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x C^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} C^{1/2} V \\ L^{\frac{1}{2}} I \end{pmatrix}$$

we obtain the skew-adjoint operator

$$\not{D} = \begin{pmatrix} 0 & C^{-1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x C^{-1/2} & 0 \end{pmatrix}$$

Note that $L^{-1/2}\partial_x C^{-1/2} = C^{1/2} \underbrace{\left(\frac{1}{\sqrt{LC}}\partial_x\right)}_{\partial_s} C^{-1/2}$.

The condition for limit point behavior at an endpoint is that not both basis solutions of the homogeneous equation $\not{D} u = 0$, namely $C^{1/2}$ and $L^{1/2}$ be square integrable. Thus

$$\text{not both } \int C\,dx, \int L\,dx \quad < \infty$$

Notice that this is compatible with Cauchy

$$\int ds = \int \sqrt{LC}\,dx \le \left(\int L\,dx\right)^{1/2} \left(\int C\,dx\right)^{1/2}$$

and finiteness of $\int ds$.

We want to go back to the case where we deal with the operator

$$\begin{pmatrix} 0 & r^{-1}(\partial_x p - q) \\ (p\partial_x + q) r^{-1} & 0 \end{pmatrix}$$

and where we must introduce h in order to get a sensible limit.

August 6, 1987

Here's a sensible way to put in Planck's constant. In general we want to consider an operator of the form

$$(*) \qquad h a \partial_x + hb + c$$

whose kernel is

$$u = e^{-\int \frac{hb+c}{ha}} = e^{-\int \frac{b}{a} - \frac{1}{h}\int \frac{c}{a}} .$$

The adjoint operator

$$h\partial_x a - hb - c = ha\partial_x + h(a'-b) - c$$

has kernel

$$v = e^{-\int \frac{a'-b}{a} + \frac{1}{h}\int \frac{c}{a}}$$

Thus we should write the operator $(*)$ in the form

$$h\, a_1 \partial_x a_2 + c$$

whence

$$u = a_2^{-1} e^{-\frac{1}{h}\int \frac{c}{a_1 a_2}} \qquad v = a_1^{-1} e^{\frac{1}{h}\int \frac{c}{a_1 a_2}}$$

Notice that

$$uv = \frac{1}{q_1 q_2} \qquad ds = \frac{1}{q_1 q_2}\,dx$$

gives the signal speed, or proper time, for the transmission lines.

Let's now return to the subspace

$$\mathrm{Im}\begin{pmatrix} r \\ hp_1\partial_x p_2 + q \end{pmatrix}.$$

We've split p into $p_1 p_2$ so as to allow for the dependence on Planck's constant.

We suppose that $r > 0$ on an open interval and 0 outside. On this interval we have

$$\textcircled{+} \qquad \mathrm{Im}\begin{pmatrix} 1 \\ (hp_1\partial_x p_2 + q)r^{-1} \end{pmatrix}$$

and we are dealing with a transmission line situation. The two homogeneous DE solutions are

$$\boxed{\begin{aligned} u &= \frac{r}{p_2}\, e^{-\frac{1}{h}\int \frac{q}{p_1 p_2}} \\ v &= \frac{1}{p_1}\, e^{\frac{1}{h}\int \frac{q}{p_1 p_2}} \end{aligned}}$$

The product $uv = \dfrac{r}{p_1 p_2} = \dfrac{r}{p}$ is related to the sound speed: $ds = \dfrac{r}{p}\,dx$.

The orthogonal complement of $\oplus$ should be

$$\mathrm{Im}\begin{pmatrix} r^{-1}(hp_2\partial_x p_1 - q) \\ 1 \end{pmatrix} = \mathrm{Im}\begin{pmatrix} r^{-1}(hp_2\partial_x p_1 - q)r \\ r \end{pmatrix}$$

so we want

$$r^{-1}(h p_2 \partial_x p_1 - q) r = h p_2 (\partial_x + \frac{r'}{r}) p_1 - q$$

to be smooth. Thus we obtain again the condition

$$p_2 \frac{r'}{r} p_1 = p \frac{r'}{r} \quad \text{smooth}$$

I want next to discuss things intrinsically as possible. For example, we can restrict to the open interval where $p, r > 0$, and according to our present thinking, the setup on the line is an extension by zero of the setup on this interval. So we should understand what happens on this interval intrinsically using the natural signal speed metric. Then we can worry about the embedding into the line.

So the first step will be to rewrite the operators in the intrinsic metric. Recall that the wave equation is of the form

$$\partial_t \psi = \begin{pmatrix} 0 & u^{-1}\partial_x v^{-1} \\ v^{-1}\partial_x u^{-1} & 0 \end{pmatrix} \psi$$

where

$$u = \frac{r}{p_2} e^{-\frac{1}{h}\int \frac{q}{p}} \qquad v = \frac{1}{p_1} e^{\frac{1}{h}\int \frac{q}{p}}$$

We want to rewrite this in terms of the independent variable s given by

$$\partial_s = \frac{1}{uv}\partial_x \qquad \text{or} \quad ds = uv\, dx$$

Note

$$\int \psi^2 dx = \int \psi^2 \frac{1}{uv} ds = \int \left((uv)^{-1/2}\psi\right)^2 ds$$

$$\partial_t (uv)^{-1/2}\psi = \underbrace{(uv)^{-1/2}\begin{pmatrix} 0 & u^{-1}uv\partial_s v^{-1} \\ v^{-1}uv\partial_s u^{-1} & 0 \end{pmatrix}(uv)^{1/2}}\,(uv)^{-1/2}\psi$$

$$= \begin{pmatrix} 0 & \left(\frac{v}{u}\right)^{1/2}\partial_s\left(\frac{u}{v}\right)^{1/2} \\ \left(\frac{u}{v}\right)^{1/2}\partial_s\left(\frac{v}{u}\right)^{1/2} & 0 \end{pmatrix}$$

Thus when we transform to intrinsic coordinates the two solutions become

$$\frac{u}{(uv)^{1/2}} = \underbrace{\frac{r}{p_2}\left(\frac{p_1p_2}{r}\right)^{1/2}}_{\alpha} e^{-\frac{1}{h}\int\frac{q}{p}dx}$$

$$\frac{v}{(uv)^{1/2}} = \underbrace{\frac{1}{p_1}\left(\frac{p_1p_2}{r}\right)^{1/2}}_{\alpha^{-1}} e^{\frac{1}{h}\int\frac{q}{p}dx}$$

Notice that $\displaystyle\int^x \frac{q}{p}dx = \int\frac{q}{r}ds$ and that $\frac{q}{r}$ is the function relevant for the local index calculation.

Although there may ultimately be some purpose in choosing p_1, p_2 subtle, at the moment it seems simplest and most symmetric to choose p_1, p_2 so that $\alpha = 1$. This means that in intrinsic coordinates the h dependence is simplest.

Thus we want to choose p_1, p_2 so that

$$\frac{r}{p_2}\left(\frac{p_1p_2}{r}\right)^{1/2} = \frac{r^{1/2}p_1^{1/2}}{p_2^{1/2}} = 1 \quad \text{or} \quad p_2 = rp_1$$

so $$p_1^2 = \frac{p_1 p_2}{r} = \frac{p}{r} \qquad \boxed{\begin{aligned} p_1 &= \left(\frac{p}{r}\right)^{1/2} \\ p_2 &= (pr)^{1/2} \end{aligned}}$$

$$p_2 = r p_1 = r\left(\frac{p}{r}\right)^{1/2}$$

Now we should check this leads to good differential operators

$$\begin{aligned} p_1 \partial_x p_2 &= \left(\frac{p}{r}\right)^{1/2} \partial_x (pr)^{1/2} \\ &= p\,(pr)^{-1/2} \partial_x (pr)^{1/2} \\ &= p\left(\partial_x + \frac{1}{2}\left(\frac{p'}{p} + \frac{r'}{r}\right)\right) \end{aligned}$$

$$\begin{aligned} r^{-1} p_2 \partial_x p_1 r &= r^{-1}(pr)^{1/2} \partial_x \left(\frac{p}{r}\right)^{1/2} r \\ &= p\,(pr)^{-1/2} \partial_x (pr)^{1/2} \\ &= p\left(\partial_x + \frac{1}{2}\left(\frac{p'}{p} + \frac{r'}{r}\right)\right) \end{aligned}$$

This is smooth assuming $p\frac{r'}{r}$ is smooth.

[redacted]

Let us try to construct some examples.

We start with a connected oriented 1-manifold I equipped with a metric ds and a "potential" function k. We then have the operator

$$\begin{pmatrix} 0 & h\partial_s - k \\ h\partial_s + k & 0 \end{pmatrix}$$

with [redacted] null-solutions $u = \varphi^{\frac{1}{h}}$, $v = u^{-1}$ where

$$\varphi = \exp\left(-\int^s k\, ds\right).$$

We assume that k goes to $\pm\infty$ at the ends of I so fast that one of u, v is not square-integrable. We can now choose $r > 0$ on I such that

$$r k = q = \pm 1 \quad \text{near the ends.}$$

Thus we start with $r = \frac{1}{|k|}$ near the ends and extend.

Next we suppose our 1-manifold I embedded as an interval ~~of~~ the line. Then we have the coordinate x on I and we can define p so that

$$ds = \frac{r}{p}\,dx$$

I think I now have obtained p, q, r from intrinsic data. The only choice involved is with r in the middle of the interval where it doesn't matter if we are interested in the subspace $\operatorname{Im}\left(\widetilde{hp\partial_x} + q\right)^{r}$.

What I have learned today is that there are essentially no choices once you specify the metric on the interval and the potential. The next stage is to pin down the conditions on $\frac{p}{r}$ and $\frac{q}{r}$.

Recall that the wave equation is

$$\partial_t \psi = \begin{pmatrix} 0 & u^{-1}\partial_x v^{-1} \\ v^{-1}\partial_x u^{-1} & 0 \end{pmatrix} \psi$$

where

$$v^{-1}\partial_x u^{-1} = \left(h p_1 \partial_x p_2 + q\right) r^{-1}$$

Now put in the values $p_1 = \left(\frac{\rho}{r}\right)^{1/2}$ $p_2 = (\rho r)^{1/2}$ and we have

$$v^{-1}\partial_x u^{-1} = h\left(\frac{\rho}{r}\right)^{1/2}\partial_x (\rho r)^{1/2} r^{-1} + \frac{q}{r}$$

$$= h\left(\frac{\rho}{r}\right)^{1/2}\partial_x\left(\frac{\rho}{r}\right)^{1/2} + \frac{q}{r}$$

and recall that a vector field $a\partial_x$ acts on half-densities by

$$\mathcal{L}_{a\partial_x}\left(f\,dx^{1/2}\right) = \underbrace{\left(af' + \tfrac{1}{2}a'f\right)}_{\left(a^{1/2}\partial_x a^{1/2}\right)f} dx^{1/2}.$$

This shows that our generator for the wave motion is the combination of the Dirac on half-densities (using the metric) with the potential.

It's also useful to note the formulas

$$\begin{cases} u = \left(\frac{r}{\rho}\right)^{1/2} e^{-\frac{1}{h}\int \frac{q}{\rho}dx} \\ v = \left(\frac{r}{\rho}\right)^{1/2} e^{\frac{1}{h}\int \frac{q}{\rho}dx} \end{cases} \qquad \frac{q}{r}ds$$

Note

$$\left(h\left(\frac{\rho}{r}\right)^{1/2}\partial_x\left(\frac{\rho}{r}\right)^{1/2} + \frac{q}{r}\right) r$$

$$= h\rho\underbrace{\frac{1}{(\rho r)^{1/2}}\partial_x (\rho r)^{1/2}}_{\partial_x + \frac{1}{2}\frac{\rho'}{\rho} + \frac{1}{2}\frac{r'}{r}} + q$$

is smooth provided we assume

$$p\frac{r'}{r} = \partial_s r$$

is smooth in x. Does this have any deeper meaning?

<u>August 7, 1987</u>

Yesterday we decided that the Dirac operator to be studied should be written in the form

$$* \qquad \begin{pmatrix} 0 & h\left(\frac{p}{r}\right)^{1/2}\partial_x\left(\frac{p}{r}\right)^{1/2} - \frac{q}{r} \\ h\left(\frac{p}{r}\right)^{1/2}\partial_x\left(\frac{p}{r}\right)^{1/2} + \frac{q}{r} & 0 \end{pmatrix}$$

over the interval where $p, r > 0$. One has

$$h\left(\frac{p}{r}\right)^{1/2}\partial_x\left(\frac{p}{r}\right)^{1/2} + \frac{q}{r} = v^{-1}\partial_x u^{-1}$$

where u, v are the null solutions

$$u = \left(\frac{r}{p}\right)^{1/2} e^{-\frac{1}{h}\int^x \frac{q}{p}dx}$$

$$v = \left(\frac{r}{p}\right)^{1/2} e^{+\frac{1}{h}\int^x \frac{q}{p}dx}$$

Let's now recall the conditions to put on p, q, r. First of all we want the C.T. of $*$ to correspond to the subspace on the whole of $L^2(\mathbb{R})^2$

$$\operatorname{Im}\begin{pmatrix} r \\ \underbrace{\left(h\left(\frac{p}{r}\right)^{1/2}\partial_x\left(\frac{p}{r}\right)^{1/2}+\frac{q}{r}\right)}_{L} r \end{pmatrix}$$

where the differential operator L is smooth. As

$$\left(\frac{p}{r}\right)^{1/2}\partial_x (pr)^{1/2} = p\,\partial_x + \tfrac{1}{2}\left(\frac{p'}{p}+\frac{r'}{r}\right)$$

this leads to the condition

$$p\frac{r'}{r} \text{ is smooth}$$

Secondly we want the subspace to be in the restricted Grassmannian. Since the subspace is also

$$\operatorname{Im}\begin{pmatrix} rL^{-1} \\ 1 \end{pmatrix}$$

we want the Green's function rL^{-1} to be Hilbert-Schmidt.

Strictly speaking, in order that L^{-1} be defined I want q to remain $\neq 0$, say $q<0$ to fix the ideas. Set

$$\varphi = e^{+\frac{1}{h}\int^x \frac{-q}{p}\,dx} \qquad\qquad \psi = \varphi^{-1}$$

so that φ decays at the left end $x=0$ and ψ decays at the right end $x=1$. We can write down L^{-1} as follows. The null solution of L is

$$\frac{1}{(pr)^{1/2}}\varphi = \frac{1}{(pr)^{1/2}}\frac{1}{\psi}$$

and the coefficient of ∂_x is hp in L, so

Thus

$$\boxed{\begin{aligned}(L^{-1}f)(x) &= \frac{1}{((pr)^{1/2}\psi)(x)}\int_x^1 \frac{-(pr)^{1/2}\psi(y)}{h\,p(y)}\, f(y)\,dy \\ (rL^{-1}f)(x) &= \left(\frac{r}{p}\right)^{1/2}(x)\int_x^1 \frac{-\psi(y)}{\psi(x)}\, f(y)\left(\frac{r}{p}\right)^{1/2}(y)\,dy\end{aligned}}$$

so the Schwartz kernel of rL^{-1} is

$$(rL^{-1})(x,y) = \begin{cases} \left(\frac{r}{p}\right)^{1/2}(x)\,\frac{-\psi(y)}{\psi(x)}\left(\frac{r}{p}\right)^{1/2}(y) & x<y \\ 0 & x>y \end{cases}$$

When is this Hilbert-Schmidt?

$$\iint (rL^{-1}(x,y))^2\,dx\,dy = \iint_{x<y} \frac{\psi(y)^2}{\psi(x)^2}\left(\frac{r}{p}(x)\,dx\right)\left(\frac{r}{p}(y)\,dy\right)$$

But

$$\frac{\psi(y)}{\psi(x)} = e^{\int_x^y \frac{+q}{p}} \leqslant 1, \text{ so that}$$

a sufficient condition for Hilbert-Schmidt is the total "signal length" is finite:

$$\int \frac{r}{p}\,dx < \infty$$

Remark: The above is obviously a transcription of a simpler proof in the intrinsic coordinates

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So far we have been making good progress with constructing analytically a point in the restricted Grassmannian, once p, q, r are given.

We still have to show how to construct p, q, r starting from a loop g. Now the obvious choice seemed to take $\frac{q}{p} = \frac{1}{i}\frac{g-1}{g+1}$; $\frac{q}{p}$ should be related to the loop. The Dirac operator was to be a version of

$$h\,\partial_x + \frac{1}{i}\frac{g-1}{g+1}$$

and its adjoint. This apparently doesn't work, and the idea was to introduce r as a cutoff or smoothing construction.

The simplest choice for r satisfying the two conditions

$$p\frac{r'}{r} \text{ smooth} \qquad\qquad \int \frac{r}{p}\,dx < \infty$$

is to take $r = p$. This doesn't work it seems. Another possibility is to take

$$r = p^n$$

However an interesting possibility is to set

$$p\frac{r'}{r} = 1 \quad \text{or} \quad r = e^{\int \frac{1}{p}dx}$$

Then we have $\frac{r}{p} = r'$, so the signal time coordinate is

$$s = \int_0^x \frac{r}{p}\,dx = \int_0^x r'\,dx = r$$

At the ends of $(0,1)$ we suppose $q = \pm 1$. Thus if $q = -1$ near $x = 0$, the potential is

$$\frac{q}{r} = \frac{-1}{s}$$

and the intrinsic Dirac is

$$\begin{pmatrix} 0 & h\partial_s + \frac{1}{s} \\ h\partial_s - \frac{1}{s} & 0 \end{pmatrix}$$

with null-solutions $s^{1/h}$ and $s^{-1/h}$.

So we conclude that the choice $r = e^{\int \frac{1}{p}}$ leads to a rather simple operator near the boundary in intrinsic coordinates.

August 9, 1987

At this point I feel that the program of defining a point in the restricted Grassmannian via a formula of the type $\mathrm{Im}\left(h\hat{p}\partial_x + q + O(h)\right)$ is reasonable. However proofs based on explicit formulas for the ~~[illegible]~~ Green's functions are not likely to generalize. For this reason I feel that it is necessary to find parametrix type proofs.

I think the basic problem is one of showing a certain ~~[illegible]~~ formally skew-adjoint operator is in fact skew-adjoint. Thus how does one show that a Dirac operator

$$\not{D} = \begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}$$

determines a decomposition of $(L^2)^{\oplus 2}$, or equivalently $\not{D}$ has a Cayley transform which is unitary? One ~~[illegible]~~ has to show that the operators $(1 \pm \not{D})^{-1}$ are well-defined on the Hilbert space and have common range.

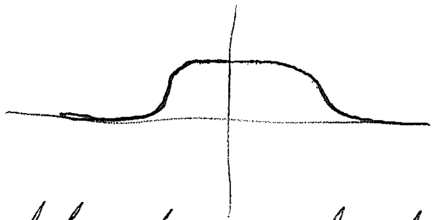
I have the idea that I might be able to use parametrix methods to construct

$$(\lambda \pm \not{D})^{-1}$$

for $\mathrm{Re}(\lambda) >> 0$. The approximations used in the parametrix construction become ~~[illegible]~~ insignificant as $\mathrm{Re}(\lambda)$ becomes large.

Let's discuss an example. Let's start with $\not{D} = \partial_x$ on the circles. Here we can explicitly construct the resolvent either directly or by the Fourier transform. However let us use the method of patching together local operators with the right singularities along the diagonal.

On $\mathbb{R}$ the operator $(\lambda+\partial_x)^{-1}$ has the Schwartz kernel $\theta(x-y)\,e^{-\lambda(x-y)}$. To put this on the circle let's choose $\rho(x) \in C_0^\infty(\mathbb{R})$ of the form



with small support and then put

$$P_\lambda(x,y) = \theta(x-y)\,e^{-\lambda(x-y)}\,\rho(x-y)$$

Assuming the support of ρ is disjoint from its translates relative to the deck transformation for $R \longrightarrow$ circle, this kernel descends to define an operator on the circle. Since everything is translation invariant, P_λ is multiplication by

$$\xi \mapsto e^{-i\xi x}\,P_\lambda\,e^{i\xi x} = \int \theta(x-y)\,e^{(\lambda+i\xi)(y-x)}\,\rho(x-y)\,dy$$

on the F.T. side. When considered over the circle we restrict to ξ such that $e^{i\xi}$ is a character on the circle.

So our parametrix is mult. by

$$\int_{-\infty}^{0} e^{(\lambda+i\xi)y}\,\rho(-y)\,dy = \int_0^\infty e^{-(\lambda+i\xi)y}\,\rho(y)\,dy$$

Composing with $\lambda + \partial_x$ we get the operator of multiplying the F.T. by

$$\int_0^\infty (\lambda + i\xi)\, e^{-(\lambda+i\xi)y} \rho(y)\, dy$$

$$= \underbrace{\left[- e^{-(\lambda+i\xi)y} \rho(y) \right]_0^\infty}_{1} + \int_0^\infty e^{-(\lambda+i\xi)y} \rho'(y)\, dy$$

Now

$$\left| \int_0^\infty e^{-(\lambda+i\xi)y} \rho'(y)\, dy \right| \leqslant \int_0^\infty e^{-\mathrm{Re}(\lambda) y} |\rho'(y)|\, dy$$

and this can be made small by requiring $\mathrm{Re}(\lambda) \gg 0$.

Unfortunately it's clear this won't work for $\mathrm{Re}(\lambda) \ll 0$. This means we have to find a way to distinguish between

$$\theta(x-y)\, e^{-\lambda(x-y)}$$

$$-\theta(y-x)\, e^{-\lambda(x-y)}$$

even locally near the diagonal.

August 13, 1987

Idea: Let's recall that we want to define a subspace of $L^2(\mathbb{R})^2$ by

$$\operatorname{Im}\begin{pmatrix} r \\ L \end{pmatrix} \qquad L = p\partial_x + q$$

and that the orthogonal complement is to be of the form

$$\operatorname{Im}\begin{pmatrix} L' \\ r \end{pmatrix}$$

where

$$L' = r^{-1}(\partial_x p - q)\, r = p\partial_x + p' + p\frac{r'}{r} - q,$$

which leads to the condition that $p\frac{r'}{r}$ be smooth. Moreover we would like the following decomposition on the C^∞ level

$$\circledast \qquad C^\infty(\mathbb{R})^2 = \begin{pmatrix} r \\ L \end{pmatrix} C^\infty(\mathbb{R}) \oplus \begin{pmatrix} L' \\ r \end{pmatrix} C^\infty(\mathbb{R}).$$

If this is true then

$$\begin{pmatrix} r & L' \\ L & r \end{pmatrix}$$

maps $C^\infty(\mathbb{R})^2$ surjectively onto itself.

Let's check it's injective. First we observe the decomposition $\circledast$ is orthogonal, since

$$-L^* r = (\partial_x p - q)\, r$$

$$r L' = r\left(p\partial_x + p' + p\frac{r'}{r} - q\right) = \overbrace{rp\partial_x + (rp' + r'p)}^{\partial_x(rp)} - rq$$

so

$$\boxed{-L^* r = r L'}$$

and hence

$$\int \left(\begin{pmatrix} r \\ L \end{pmatrix} f \right)^* \cdot \begin{pmatrix} L' \\ r \end{pmatrix} g = \int r\bar{f}\, L'g + \overline{Lf}\, rg$$

$$= \int \bar{f}\, (rL' + L^* r)\, g = 0$$

Thus if $\begin{pmatrix} f \\ g \end{pmatrix}$ is killed by $\begin{pmatrix} r & L' \\ L & r \end{pmatrix}$
we must have $rf = Lf = 0$ and $L'g = rg = 0$.
So f is supported $\overbrace{\text{where}}^{\text{in the set}}$ $r = 0$, and $p = 0$ and $q \neq 0$ on this set, so that $Lf = qf = 0$ implies $f = 0$. Similarly $g = 0$.

So we conclude that if what we hope to work does work, then the operator

$$(*) \qquad \begin{pmatrix} r & L' \\ L & r \end{pmatrix}$$

will be invertible on L^2 and C^∞. But this operator is a matrix version of irregular differential operators such as $1 - p\partial_x$, which we have perhaps succeeded in showing to be invertible on C^∞.

The idea is that perhaps we could show $(*)$ to be invertible by suitable variants of what we have already done. And this would hopefully work in case where L isn't invertible.

Idea for tomorrow: What is the unitary reversed by ε belonging to the subspace $\operatorname{Im}\begin{pmatrix} r \\ L \end{pmatrix}$? Formally this subspace is the graph of Lr^{-1}, and so the unitary is formally

$$G = \frac{1 + \begin{pmatrix} 0 & L'r^{-1} \\ Lr^{-1} & 0 \end{pmatrix}}{1 - \begin{pmatrix} 0 & L'r^{-1} \\ Lr^{-1} & 0 \end{pmatrix}} = \frac{1 + \begin{pmatrix} 0 & L' \\ L & 0 \end{pmatrix} r^{-1}}{1 - \begin{pmatrix} 0 & L' \\ L & 0 \end{pmatrix} r^{-1}}$$

$$= \left(\begin{pmatrix} r & L' \\ L & r \end{pmatrix} r^{-1} \right) \left(\begin{pmatrix} r & -L' \\ -L & r \end{pmatrix} r^{-1} \right)^{-1}$$

$$\boxed{G = \begin{pmatrix} r & L' \\ L & r \end{pmatrix} \cdot \begin{pmatrix} r & -L' \\ -L & r \end{pmatrix}^{-1}}$$

Also

$$\frac{G+1}{2} = \frac{1}{1 - \begin{pmatrix} 0 & L' \\ L & 0 \end{pmatrix} r^{-1}} = r \begin{pmatrix} r & -L' \\ -L & r \end{pmatrix}^{-1}$$

Now the latter should be a ΨDO of order -1, and might be easier to construct than $\begin{pmatrix} r & -L' \\ -L & r \end{pmatrix}^{-1}$.

<u>August 14, 1987</u>

Problem: Let D be an elliptic differential operator over a compact manifold, and let D^* be ~~the~~ (formal) adjoint differential operator relative to a choice of metrics. D, D^* then give rise to closed densely-defined operators on L^2 by the process of closing up the graph on C^∞. How do we show that these two closed densely defined operators on L^2 are adjoint in the Hilbert space sense? In particular if $D = D^*$, how do we see its closure on L^2 is self-adjoint?

A variant of this problem is to show that the minimal and maximal closed d.d. operators on L^2 associated to D coincide. The minimal op has the graph

$$\Gamma_{D_{min}} = \overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C^\infty}$$

The maximal operator has graph consisting of $\begin{pmatrix} u \\ f \end{pmatrix}$ in $(L^2)^2$ such that $Du = f$ <u>weakly</u>, that it $(u, D^*\varphi) = (f, \varphi)$ for all $\varphi \in C^\infty$. Equivalently

$$\Gamma_{D_{max}} = \left\{ \begin{pmatrix} -D^* \\ 1 \end{pmatrix} C^\infty \right\}^\perp = \left\{ \overline{\begin{pmatrix} -D^* \\ 1 \end{pmatrix} C^\infty} \right\}^\perp$$

which means that

$$D_{max} = \text{Hilbert space adjoint of } (D^*)_{min}$$

Thus we have equivalent conditions

1) $D_{min} = D_{max}$ (or $(D^*)_{min} = (D^*)_{max}$)

2) $(D_{min})^{*(HS)} = (D^*)_{min}$

3) $$\overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C^\infty} + \overline{\begin{pmatrix} -D^* \\ 1 \end{pmatrix} C^\infty} = (L^2)^2$$

Now I would like to see how to establish this condition when D is elliptic, using parametrix methods. Let's start with the case where we can produce a Green's function G for D, that is, an [redacted] operator G on C^∞ satisfying

$$DG = GD = 1.$$

Then we we have

$$\begin{pmatrix} 1 \\ D \end{pmatrix} C^\infty = \begin{pmatrix} G \\ 1 \end{pmatrix} C^\infty$$

If we know G is bounded in the L^2 norm, then denoting its extension to L^2 by $\bar{G}$ we have

$$\overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C^\infty} = \begin{pmatrix} \bar{G} \\ 1 \end{pmatrix} L^2$$

At the same time we should know that the adjoint operator G^* (defined by $G^*(x,y) = G(y,x)^*$ on the Schwartz kernel) is inverse to D^* and also bounded in the L^2 norm. Thus

$$\overline{\begin{pmatrix} -D^* \\ 1 \end{pmatrix} C^\infty} = \begin{pmatrix} +1 \\ -\overline{G^*} \end{pmatrix} L^2$$

But these subspaces

$$\begin{pmatrix} \bar{G} \\ 1 \end{pmatrix} L^2 \quad , \quad \begin{pmatrix} 1 \\ -\overline{G^*} \end{pmatrix} L^2$$

are $\perp$ and the only way this can happen is for $\overline{G^*} = \bar{G}^{*(HS)}$, and then the two subspaces are complementary.

Next we want to generalize. Because D is elliptic one construct a ΨDO P which is a parametrix for D in the sense that

$$DP = 1 - K \,, \qquad PD = 1 - L$$

where K, L are smooth kernel operators. Now the <u>hope would be to link</u> the subspaces $\begin{pmatrix} 1 \\ D \end{pmatrix} C^\infty$ and $\begin{pmatrix} -D^* \\ 1 \end{pmatrix} C^\infty$ to the complementary subspaces $\begin{pmatrix} \bar{P} \\ 1 \end{pmatrix} L^2$ and $\begin{pmatrix} 1 \\ -\bar{P}^* \end{pmatrix} L^2$.

<u>Example</u>: Consider $D = \partial_x$ on $(0,1)$ and take C^∞ to means $C_c^\infty(0,1)$. In this case we know $D_{min} \neq D_{max}$. Let's fix a $\rho \in C_c^\infty$ with $\int_0^1 \rho \, dx = 1$ and define P to be

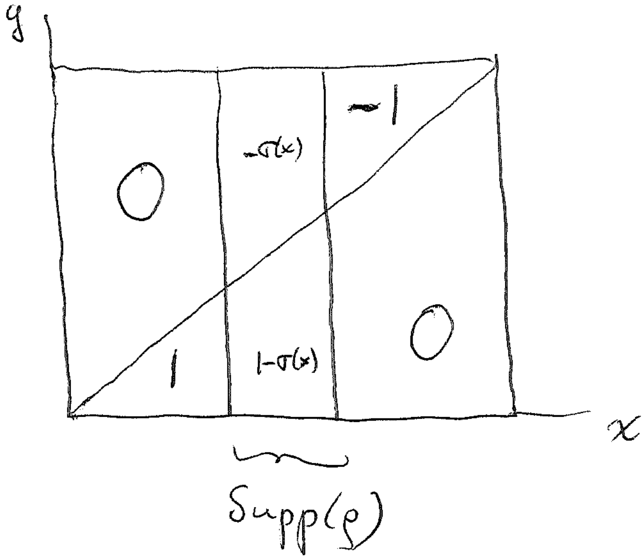
$$(Pf)(x) = \int_0^x \left(f(y) - \rho(y) \left(\int_0^1 f \right) \right) dy$$

for $f \in C_c^\infty$. Then $(Pf)(x) = 0$ for $x < \operatorname{supp} f$ and for $x > \operatorname{supp}(f)$

$$(Pf)(x) = \int_0^1 f(y)\,dy - \underbrace{\left(\int_0^1 \rho \, dy \right)}_{1} \left(\int_0^1 f \right) = 0.$$

Thus P is well-defined on C_c^∞. We have
$$\left(DPf\right) = f - \rho\left(\int_0^1 f\right) = (I-K)f$$
where K is of rank 1. Also
$$\left(PDf\right)(x) = \int_0^x \left(f'(y) - \rho(y)\left(\underbrace{\int_0^1 f'}_{0}\right)\right) dy = f(x)$$
so that $PD = 1$. All this, i.e. P, D and the identities $DP = 1-K$, $PD = 1$ hold on C_c^∞.

Next what is P^*.
$$P(x,y) = \chi_{(0,x]}(y) - \overbrace{\int_0^x \rho(x')dx'}^{\sigma(x)}$$

Picture


Clearly for each y, $P(x,y)$ has compact support in x. However $P(y,x)$ does not have this property which implies P^* probably doesn't (this is clear) preserve C_c^∞.

Summary: This example leaves hope that one could prove $D_{min} = D_{max}$ starting from a parametrix P for D

Another observation is that if we have
$$PD = 1 - L$$
with L smoothing, then any distribution u satisfying $Du \in C^\infty$ satisfies
$$u = P\underbrace{Du}_{\in C^\infty} + \underbrace{Lu}_{\in C^\infty} \in C^\infty.$$
where we use that $PC^\infty \subset C^\infty$

<u>August 15, 1987</u>

Let D be a differential operator and suppose there exists a distribution $P(x,y)$ on the product which is smooth off the diagonal such that
$$D_x P(x,y) = \delta(x,y) - K(x,y)$$
with K smooth.

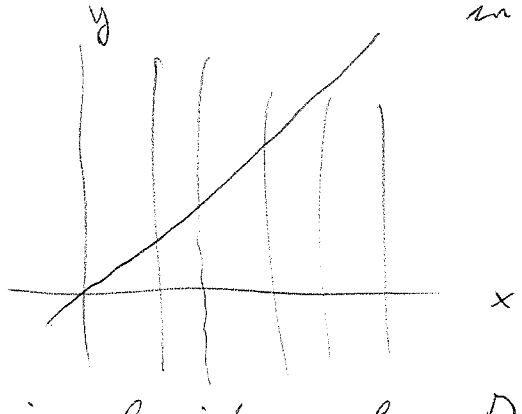
First we show that P may be chosen to have support in a nbd. of the diagonal Δ. Let ρ be a smooth function with support in this nbd and such that $\rho \equiv 1$ near Δ. Then φP is smooth off Δ and so $D_x(\varphi P)(x,y)$ is smooth off Δ. And near Δ, $\varphi P = P$ so that near Δ we have
$$D_x(\varphi P)(x,y) = D_x P(x,y) = \delta(x,y) - K(x,y).$$
Thus $D_x(\varphi P) - \delta$ is smooth on the product.

Next we show that any distribution u such that $D^t u$ is smooth is necessarily smooth. We use a P supported in a nbd. of Δ so that the integrals times u are defined:

$$\int u(x)\, D_x P(x,y)\, dx = \int (D^t u)(x) P(x,y)\, dx \quad ?$$

There are some problems with this which have to be sorted out before the above integrals make sense. The integral on the right is OK because $D^t u$ is a smooth function of x and $P(x,y)$ is a distribution with compact support in x for "fixed" y. More precisely $P(x,y)$ is a distribution in y with values in compactly supported distributions in x, so the integral is a dist in y.

But in the integral on the left $u(x)$ is a distribution, so we are trying to pair two distributions in x. Now the singularities of $u(x)$ are are in vertical direction



and the singularities of $D_x P$ are on the diagonal. So we expect because these are transversal that the product distribution $u(x)\,(D_x P)(x,y)$ is defined, and then because it is compactly supp. in x it can be integrated to give a distribution in y.

One begins to understand a bit about the necessity for wave front sets.

In any case we should have

$$\int u(x)\, D_x P(x,y)\, dx = \int u(x)\left[\delta(x,y) - K(x,y)\right] dx$$

$$= u(y) - \underbrace{\int u(x) K(x,y)\, dx}_{\text{smooth in } y}$$

Key point:

$$\int \underbrace{(D^t u)(x)}_{\text{smooth in } x} P(x,y)\, dx \quad \text{is smooth in } y$$

Using this it follows that u is smooth.

<u>Summary</u>: What's missing from the above is a precise definition of the kind of singularities P is allowed to have along the diagonal, together with a proof that the operators

$$u \longmapsto \int u(x) \begin{cases} P(x,y) \\ D_x P(x,y) \end{cases} dx$$

are defined on distributions and preserve smoothness.

Next let's return to our problem of showing $D_{min} = D_{max}$ for elliptic operators. If one replaces D by $\begin{pmatrix} 1 & -D^* \\ D & 1 \end{pmatrix}$ one is led to showing that when $\operatorname{Ker} D = \operatorname{Ker} D^* = 0$ on smooth functions, then D is invertible on smooth functions and D^{-1} is bounded in the L^2 norm. Let's look at this question.

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Schur's estimate (after Hormander's book)

Assume $\sup_x \int |K(x,y)| dy \le C_1$, $\sup_y \int |K(x,y)| dx \le C_2$,

Then $f(y) \longrightarrow \int K(x,y) f(y) dy$ has norm $\le \sqrt{C_1 C_2}$ on L^2.

Proof: $$|(Kf)(x)|^2 \le \left(\int |K(x,y)| \, |f(y)| dy \right)^2$$

$$= \left(\int |K(x,y)|^{1/2} |K(x,y)|^{1/2} |f(y)| dy \right)^2$$

$$\le \underbrace{\int |K(x,y)| dy}_{\le C_1} \int |K(x,y)| \, |f(y)|^2 dy$$

So $$\int |Kf(x)|^2 dx \le C_1 \int \underbrace{\left(\int |K(x,y)| dx \right)}_{\le C_2} |f(y)|^2 dy$$

whence the result.

Here's a harder way to obtain this ~~same~~ result. I work with matrices instead of integrals.

Let $p(x,y)$ be a measure on $X \times Y$ and let $\mu(x) = \sum_y p(x,y)$ be the induced measure on X. Then pull-back via $pr_1 : X \times Y \longrightarrow X$ induces an isometric embedding

$$L^2(X, \mu) \hookrightarrow L^2(X \times Y, p)$$

as $$\sum_{x,y} f(x)^2 p(x,y) = \sum_x f(x)^2 \mu(x)$$

The adjoint of this embedding is found as follows

$$\sum_{x,y} f(x)\, h(x,y)\, p(x,y) = \sum_x f(x) \left(\sum_y h(x,y) \frac{p(x,y)}{\mu(x)} \right) \mu(x)$$

(So far $X \times Y$ could have been any space over X). Thus if we combine the pull-back from Y with this adjoint, which is a contraction, we obtain a contraction

$$L^2(Y, \nu) \longrightarrow L^2(X, \mu)$$

$$g(y) \longmapsto \sum_y \frac{p(x,y)}{\mu(x)}\, g(y)$$

Now use the isometries

$$\begin{array}{cc} g\nu^{-1/2} & L^2(Y, \nu) \\ \uparrow & \uparrow S \\ g & L^2(Y) \end{array} \qquad\qquad \begin{array}{cc} L^2(X,\mu) & f \\ \downarrow S & \\ L^2(X) & f\mu^{1/2} \end{array}$$

and we find that

$$g(y) \longmapsto \sum_y \frac{p(x,y)}{\mu(x)^{1/2}\, \nu(y)^{1/2}}\, g(y)$$

is a contraction from $L^2(Y)$ to $L^2(X)$. Thus

Prop: If $p(x,y)$ is a matrix ≥ 0, then the matrix

$$\frac{p(x,y)}{\sqrt{\sum_{y'} p(x,y')}\ \sqrt{\sum_{x'} p(x',y)}}$$

is of norm ≤ 1 on L^2.

But of course we can use the "Schur" proof. In this case we don't have to assume $p \geq 0$. $\mu(x) = \sum_y |p(x,y)|$ similarly for ν

$$\left| \int \frac{p(x,y)}{\mu(x)^{1/2}\,\nu(y)^{1/2}} f(y)\,dy \right|^2 \leqslant \left\{ \int \frac{|p(x,y)|^{1/2}}{\mu(x)^{1/2}} \frac{|p(x,y)|^{1/2}}{\nu(y)^{1/2}} |f(y)|\,dy \right\}^2$$

$$\leqslant \underbrace{\int \frac{|p(x,y)|}{\mu(x)} dy}_{1} \cdot \int \frac{|p(x,y)|}{\nu(x)} |f(y)|^2\,dy$$

Now integrate over X and obtain.

<u>Prop:</u> If $\mu(x) = \int |p(x,y)|\,dy$, $\nu(y) = \int |p(x,y)|\,dx$ then

$$\int dx \left| \int \frac{p(x,y)}{\mu(x)^{1/2}\nu(y)^{1/2}} f(y)\,dy \right|^2 \leqslant \int |f(y)|^2\,dy$$

This is more sophisticated ~~[illegible]~~ than the Hilbert-Schmidt estimate

$$|(Kf)(x)|^2 = \left| \int K(x,y) f(y)\,dy \right|^2$$

$$\leqslant \int |K(x,y)|^2 dy \int |f(y)|^2 dy$$

$$\Longrightarrow \quad \int |Kf(x)|^2 dx \leqslant \int |K(x,y)|^2 dx\,dy \cdot \int |f(y)|^2 dy$$

Next I want to use the Frobenius theorem to get a best possible estimate. My idea here is that $\mu(x)$ and $\nu(y)$ as defined above are not optimal. Suppose to fix the ideas that $p(x,y) > 0$. I recall that the Frobenius eigenvector for p can be obtained by successively applying p ~~starting~~ from the constant function $\mathbb{1}$. Thus $(p\mathbb{1})(x) = \sum_y p(x,y)$ is just the first ~~step~~.

Let α, β be the right and left Frobenius eigenvectors

$$\sum_y p(x,y)\beta(y) = \lambda\beta(x)$$

$$\sum_x \alpha(x)p(x,y) = \lambda\alpha(y)$$

Here $\alpha(y), \beta(x) > 0$, and λ is the unique largest real eigenvalue for p. Then

$$\sum_x \frac{\alpha(x)}{\beta(x)}\left|\sum_y p(x,y)f(y)\right|^2$$

$$\leq \sum_x \frac{\alpha(x)}{\beta(x)}\left|\sum_y p(x,y)^{1/2}\beta(y)^{1/2}\frac{p(x,y)^{1/2}f(y)}{\beta(y)^{1/2}}\right|^2$$

$$\leq \sum_x \frac{\alpha(x)}{\cancel{\beta(x)}}\underbrace{\left(\sum_y p(x,y)\beta(y)\right)}_{\lambda\cancel{\beta(x)}}\left(\sum_y \frac{p(x,y)}{\beta(y)}|f(y)|^2\right)$$

$$= \lambda\sum_y\left(\sum_x \frac{\alpha(x)p(x,y)}{\beta(y)}\right)|f(y)|^2$$

$$= \lambda^2\sum \frac{\alpha(y)}{\beta(y)}|f(y)|^2$$

In other words $f \longmapsto pf$ has norm λ_1 on L^2 relative to the measure $\frac{\alpha}{\beta}$.

Notice that the previous inequality is sharp in the sense that it is an equality when $f = \beta$.

Thus if p is conjugated by the diagonal matrix $\left(\frac{\alpha}{\beta}\right)^{1/2}$ to obtain

$$\left(\frac{\alpha(x)}{\beta(x)}\right)^{1/2} p(x,y) \left(\frac{\alpha(y)}{\beta(y)}\right)^{-1/2}$$

then this matrix, although not symmetric, has its norm equal to the largest eigenvalue λ.

What this means is the best Schur type estimates for the L^2 norm of $f \longmapsto pf$ are to be obtained in terms of λ and

$$\frac{\operatorname{Max}\left(\frac{\alpha(x)}{\beta(x)}\right)}{\operatorname{Min}\left(\frac{\alpha(x)}{\beta(x)}\right)}$$

Specifically

$$\|p\| \leq \lambda \sqrt{\frac{\operatorname{Max}\left(\frac{\alpha(x)}{\beta(x)}\right)}{\operatorname{Min}\left(\frac{\alpha(x)}{\beta(x)}\right)}}$$

Let's record some earlier ideas. Consider $1 - p\partial_x$ where $\begin{cases} p = 0 & x < 0 \text{ or } x > 1 \\ p > 0 & 0 < x < 1 \end{cases}$

Then we found the inverse

of $1-p\partial_x$ to be

$$u(x) = \int_x^1 \frac{-\psi'(y)}{\psi(y)} f(y)\,dy \qquad \psi = \varphi^{-1} = e^{\int_x \frac{1}{p}dx}$$

$$= \int_0^1 K(x,y)\, f(y)\,dy$$

Then $K(x,y)\,dy$ is a probability measure depending on x, so we can view as a Markov chain. We can ask for a limiting measure $\mu(x)$ satisfying

$$\int \mu(x) K(x,y)\,dx = \mu(y)$$

It's easy to see the solution is (up to scalars)

$$\mu(x) = \frac{1}{p(x)}$$

and, as in the case of heat flow, this can not be normalized to be a prob. measure.

I hoped to get useful inequalities to see that K is bounded on L^2. What emerged was the general contraction result

$$\int (Kf)^2 \frac{1}{p}dx \leq \int f^2 \frac{1}{p}dx$$

i.e. $\frac{1}{\sqrt{p}} K \sqrt{p}$ is a contraction for L^2. This is the inverse of the operator $\frac{1}{\sqrt{p}}(1-p\partial_x)\sqrt{p}$

$= 1-\overbrace{\sqrt{p}\,\partial_x \sqrt{p}}^{X}$ which satisfies

$$\|(1-X)u\|^2 = \|u\|^2 + \|Xu\|^2 \geq \|u\|^2$$

as X is skew-adjoint. So it doesn't seem one

one gains anything from the Markov theory.

Notice that

$$\|(1-p\partial_x)u\|^2 = \|u\|^2 + \|p\partial_x u\|^2 - (u, p\partial_x u) - (p\partial_x u, u)$$

$$= \|u\|^2 + \|p\partial_x u\|^2 + \underbrace{(u, (\partial_x p - p\partial_x) u)}_{(u,\, p' u)}$$

And this last term will be ≥ 0 if $p' > 0$. Thus $K = \frac{1}{1-p\partial_x}$, should be a contraction for the kind of p which increases. ?