May 13, 1987

Weil's Acta paper seems concerned mainly with the following. One considers a quadratic character f on a locally compact abelian group G which is non-degenerate in the sense that the associated pairing of G and G^* gives an isomorphism $G \simeq G^*$. Then one can compare the Fourier transform of f with its transport under this isomorphism. There is a determinantal factor having to do with Haar measures and then a factor of absolute value 1 which is denoted $\gamma(f)$. The paper seems to be mainly devoted to studying $\gamma(f)$. When G is the adeles, or perhaps the adelic version of a vector space with quadratic form, then the equation $\gamma(f) = 1$ is a formulation of the law of quadratic reciprocity.

A key sentence in his paper is that the proof of quad. reciprocity based on the θ-function and Gauss sums is essentially the same as the one Weil gives. It seems one ought to have Cartier's version to understand this.

When it comes to discussing adeles Weil chooses a non-trivial character on A which vanishes on the number field F. This character then sets up a self-duality of the adeles, which is a symmetric pairing.

In the function field case, Weil defines a meromorphic differential to be linear functional on adeles which vanishes on F and which is continuous in the sense that it vanishes on some parallelotope. It turns out that the space of these linear functionals is 1-diml over F.

When one has defined residues one can identify meromorphic differentials in the usual sense with such linear functionals. Thus it's fairly clear that there is no canonical choice for $\chi : A/F \to \mathbb{T}$. In the number field case one can ask whether the vector space of such χ is one dimensional over F.

Let's consider the case $F = \mathbb{Q}$. Let χ be a character on $A = \left(\prod_p{}' \mathbb{Q}_p\right) \times \mathbb{R}$ which vanishes on $\mathbb{Q}$ embedded diagonally and a parallelotope $\prod p^{e_p}\mathbb{Z}_p = m \prod \mathbb{Z}_p$ where $m = \prod p^{e_p}$. Then $\chi \circ m^{-1}$ vanishes on $\mathbb{Q}$ and $\prod \mathbb{Z}_p$. Now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \prod \mathbb{Z}_p \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{Q} + \prod \mathbb{Z}_p \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{Q} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod' \mathbb{Q}_p \arrow[r] & A \arrow[r] & \mathbb{R} \arrow[r] & 0
\end{tikzcd}$$

By snake this gives

$$0 \to \oplus\, \mathbb{Q}_p/\mathbb{Z}_p \to A/\mathbb{Q} + \prod \mathbb{Z}_p \to \mathbb{R}/\mathbb{Q} \to 0$$

~~and the first map is an isomorphism necessarily. Thus~~

This shows that

~~Now one obviously has that the subgroups~~

When one has defined residues one can identify meromorphic differentials in the usual sense with such linear functionals. There's obviously no canonical choice for such a differential, hence probably also no canonical choice in the number field case for a character $\chi : A/F \longrightarrow \mathbb{T}$.

In the number field case we can ask whether the character group of A/F is 1-diml over F. ~~[illegible]~~

Let's consider the case $F = \mathbb{Q}$. Consider a character χ on $A = (\prod' \mathbb{Q}_p) \times \mathbb{R}$ vanishing on $\mathbb{Q}$ and a parallelotope $\prod p^{e_p}\mathbb{Z}_p = m \prod \mathbb{Z}_p$ where $m = \prod p^{e_p}$. Then $\chi \circ m^{-1}$ vanishes on $\mathbb{Q}$ and $\prod \mathbb{Z}_p$. Now $\mathbb{Q}$ here stands for $\Delta\mathbb{Q}$ where Δ is the diagonal embedding, and it's clear that inside A we have

$$\Delta\mathbb{Q} + \prod \mathbb{Z}_p = \Delta\mathbb{Q} + \Delta\mathbb{Z} + \prod \mathbb{Z}_p$$
$$= \Delta\mathbb{Q} + \left(\prod \mathbb{Z}_p\right) \times \mathbb{Z}$$

Yesterday we saw that

$$A / \Delta\mathbb{Q} + \prod(\mathbb{Z}_p) \times \mathbb{Z} = \bigoplus \mathbb{Q}_p/\mathbb{Z}_p * \mathbb{R}/\mathbb{Z} \Big/ \mathrm{Im}\,\Delta\mathbb{Q}$$

$$\cong \mathbb{R}/\mathbb{Z} \,/\, \mathrm{Im}\,\Delta\mathbb{Z} = \mathbb{R}/\mathbb{Z}$$

Thus $\chi \circ m^{-1} = \chi_0 \circ n$ so $\chi = \chi_0 \circ \overset{\in\mathbb{Q}}{(nm)}$. So it's OKAY.

Thus we have proved

<u>Prop</u>: <u>For the adeles over $\mathbb{Q}$ the character group of $A/\mathbb{Q}$ is a 1-dimensional $\mathbb{Q}$-vector space.</u> It has a fairly canonical generator χ_0 vanishing on $(\prod \mathbb{Z}_p) \times \mathbb{Z}$.

What's apparently missing in the number theory case is the map $f \mapsto df$ from merom. fns. to merom. diffl's. There are no derivations over $\mathbb{Q}$.

May 14, 1987

Real scalar field $\phi(x)$ on the circle with the Hamiltonian

$$\underbrace{\int \tfrac{1}{2}\dot{\phi}^2 dx}_{P.E.} + \underbrace{\int \tfrac{1}{2}\phi(-\partial_x^2+m^2)\phi\, dx}_{K.E.}$$

This is a harmonic oscillator. Recall how to quantize a simple oscillator

$$L = \tfrac{1}{2}\dot{q}^2 - \tfrac{1}{2}\omega^2 q^2 \qquad p = \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2$$

equation of motion $\quad \ddot{q} + \omega^2 q = 0$

One solves the classical problem by finding a <u>complex structure</u> on phase space such that time evolution has <u>positive</u> frequencies

$$\begin{cases} q = \mathrm{Re}\left(Ae^{-i\omega t}\right) \\ p = \dot{q} = \omega\,\mathrm{Im}\left(Ae^{-i\omega t}\right) \end{cases}$$

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2 = \frac{\omega^2}{2}|A|^2 \qquad = \omega a^* a$$

To quantize, replace the classical amplitude A by a so that H becomes $\omega a^* a$.

$$\frac{\omega^2}{2}|A|^2 \text{``}=\text{''} \omega a^* a \qquad A = \sqrt{\frac{2}{\omega}}\, a$$

$$q(t) = \frac{1}{\sqrt{2\omega}}\left(ae^{-i\omega t} + a^* e^{i\omega t}\right)$$

$$p(t) = -i\sqrt{\frac{\omega}{2}}\left(ae^{-i\omega t} - a^* e^{i\omega t}\right)$$

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip), \qquad a^* = \frac{1}{\sqrt{2\omega}}(\omega q - ip)$$

Now to handle the oscillator given by the potential energy $\int \frac{1}{2}\phi(-\partial_x^2+m^2)\phi\, dx$, we can diagonalize the potential energy i.e. use the basis

$\cos(kx)$, $\sin(kx)$ $k>0$ for real functions. But in the end we want a description of phase space as a complex vector space where time-evolution proceeds via positive frequencies. ~~[illegible]~~ It is simpler to use the complex basis e^{ikx} for all k. Write classical solutions

$$\phi(x,t) = \sum_k \mathrm{Re}\left\{A_k e^{i(kx-\omega_k t)}\right\} \qquad \omega_k = \sqrt{k^2+m^2}$$

so that now the classical solns are described by the complex parameters A_k which evolve by positive frequencies. Then the quantization is

$$\phi(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k}}\left(a_k e^{i(kx-\omega_k t)} + a_k^* e^{-i(kx-\omega_k t)}\right)$$

i.e.

$$\phi(x) = \sum_k \frac{1}{\sqrt{2\omega_k}}\left(a_k e^{ikx} + a_k^* e^{-ikx}\right)$$

$$\dot\phi(x) = \sum_k -i\sqrt{\frac{\omega_k}{2}}\left(a_k e^{ikx} - a_k^* e^{-ikx}\right)$$

and the commutation relations are

$$[\phi(x), \phi(y)] = [\dot\phi(x), \dot\phi(y)] = 0$$

$$[\phi(x), \dot\phi(y)] = i\,\delta(x-y)$$

Now let $m \to 0$ in which case our equation of motion is the wave equation

$$\partial_t^2\phi + (-\partial_x^2)\phi = 0$$

and its solutions split into left and right moving waves: $f(x-t) + g(x+t)$. (Throughout all of this discussion we ignore problems with $k=0$.) Thus we have $\omega_k = |k|$ and

$$\phi(x,t) = \sum_k \frac{1}{2}\left(A_k e^{ikx - i|k|t} + \bar{A}_k e^{-ikx + i|k|t}\right)$$

$$= \sum_{k>0} \frac{1}{2}\left(A_k e^{ik(x-t)} + \bar{A}_k e^{-ik(x-t)}\right)$$

$$+ \sum_{k<0} \frac{1}{2}\left(A_k e^{ik(x+t)} + \bar{A}_k e^{-ik(x+t)}\right)$$

When we pass to imaginary time $\tau = +it$ ~~$t=-i\tau$~~ (so that e^{-itH} becomes $e^{-\tau H}$), this becomes the sum $f(x+i\tau) + g(x-i\tau)$ where f, g are holomorphic. A similar formula holds on the operator level using $A_k \longrightarrow \sqrt{\frac{2}{|k|}}\, a_k$.

I want now to discuss some general issues concerning this real scalar field on the **circle**. More generally we are concerned with the maps from the circle S^1 to $\mathbb{T}$ and the **ones with** degree $= 0$ are of the form $e^{i\phi}$ where ϕ is a real fn on S^1 determined up to $2\pi\mathbb{Z}$.

There appear to be two ways to do physics on such fields. First of all we can view these fields as configurations of the theory, so that phase space consists of pairs $(\phi, \dot{\phi})$. It is then relevant to consider the topology of configuration space in as much as this topology affects the **quantization**. Configuration space is disconnected $(\pi_0 = \mathbb{Z})$ and each component has $\pi_1 = \mathbb{Z}$.

Secondly this loop group apparently has a symplectic structure and so it may be considered as phase space. Thus the loop group [redacted] has a sort of Heisenberg group as central extension and one may [redacted] form the associated irreducible representation. Dynamics come from rotation on the circle, so we have fields moving in one direction. This is a holomorphic or chiral theory.

Now I have to concentrate on what Vafa does. Let's try describing it. First of all [redacted] I should give a general discussion of bosonization.

There are difficulties from my viewpoint due to the fact that they work with a non-chiral theory and I am not used to this language. However on the circle this means I have the sort of thing described [redacted] above, namely a quantum [redacted] field theory in which the configurations are real-valued functions ϕ on the circle. More generally one works with circle-valued functions [redacted] $e^{i\phi}$.

[redacted]

Consider now a disk D on Σ with coordinate z and center p. One then consider the field theory of a real scalar field on the circle $S = \partial D$. This is an oscillator theory so that there are definite creation and annihilation operators. Roughly we have a set of creation + ann. operators $(a_n^+, a_n \quad n \geq 1)$ attach to holom. functions of z and another set $\bar{a}_n^+ \; \bar{a}_n$ attached to anti-holomorphic fns.

So the point I really want to understand is how to think of the field theory as associated to harmonic functions. Roughly given any function on a circle it gives an operator on the Fock space attached to that circle and it is harmonic functions that are consistent with "imaginary time evolution". But this isn't perhaps correct since an operator is attached to a pair $\phi, \dot{\phi}$.

May 15, 1987

Let us now return to the question of coupling Dirac on the circle to a loop in a unitary group. There are difficult problems concerned with finding the correct formulation. One reason this is hard is that you don't have enough experience with the analysis. It seems to be worth while to explore some simple examples, to write down carefully the estimates required.

For example to treat carefully $\not{D} = h\gamma\partial_x + \gamma^2 X$ where $X: S^1 \longrightarrow U(V)$ is smooth. It would be necessary to prove the existence of $(\lambda - \not{D})^{-1}$ for $Re(\lambda) \neq 0$. And to evaluate $\lim_{h\to 0} tr_s\left(\frac{1}{\lambda^2 - \not{D}^2}\right)$, better $\lim_{h\to 0} tr_s \langle x|\frac{1}{\lambda^2-\not{D}^2}|x\rangle$.

I think I know much of what goes into this and therefore regard the details as boring. But I must come back to this exercise if other things fail.

More interesting and potentially important is the following example. We consider an interval I of the line and the DR complex

$$C^\infty(I) \xrightarrow{\partial_x} C^\infty(I)$$

Following Hörmander and Witten we introduce weighted L^2 norms

$$\int |f|^2 e^{-2t\varphi}\, dx$$

and obtain then a closed densely defined op.

$$* \qquad L^2(I, e^{-2t\varphi}dx) \xrightarrow{\partial_x} L^2(I, e^{-2t\varphi}dx)$$

If we use the isom

$$\begin{array}{ccc} L^2(I, dx) & \xrightarrow{\sim} & L^2(I, e^{-2t\varphi}dx) \\ e^{-t\varphi}f & \longleftrightarrow & f \end{array}$$

then $*$ becomes

$$L^2(I, dx) \xrightarrow{e^{-t\varphi}\partial_x e^{t\varphi}} L^2(I, dx)$$

or

$$e^{-t\varphi}\partial_x e^{t\varphi} = \partial_x + t\varphi'$$

This operator together with its formal adjoint gives the formally skew-adjoint operator

$$** \qquad \begin{pmatrix} 0 & \partial_x - t\varphi' \\ \partial_x + t\varphi' & 0 \end{pmatrix}$$

on $L^2(I, dx)^{\oplus 2}$. Dividing by t and putting $h = \frac{1}{t}$ we get an operator in the form $h\gamma^1\partial_x + \gamma^2 X$. If φ grows sufficiently fast at the ends of I, then presumably $**$ is essentially skew-adjoint, and its Cayley transform is $\equiv -1 \mod \mathcal{L}^2$.

Thus the above example leads to a variant of $h\gamma^1\partial_x + \gamma^2 X$ on an open manifold where the lack of compactness must be compensated for by the growth of X. So the analytical problems of the existence of $\frac{1}{\lambda - \not{D}}$ and evaluating $\mathrm{tr}_s\left(\frac{1}{\lambda^2 - \not{D}^2}\right)$ become harder.

Let's recall the formula for the index (p690)

$$\mathrm{tr}_s \exp u(h^2\partial_x^2 + X^2 + h\gamma^1\gamma^2\partial_x X)$$

$$\xrightarrow[h\to 0]{} \int dx \int \frac{d\xi}{2\pi}\, \mathrm{tr}\left(e^{u(-\xi^2 + X^2)}\partial_x X\right) u2i$$

$$= \int dx\ \mathrm{tr}\left(e^{uX^2}\partial_x X\right)\sqrt{u}\,\frac{i}{\sqrt{\pi}}$$

Now let us recall that we must construct an operator $\not{D}$, such as $h\gamma^1\partial_x + \gamma^2 X$ depending on a loop g; to fix the ideas suppose $g: S^1 \to U(1)$. If I wish to use $h\gamma^1\partial_x + \gamma^2 X$, then X is a function of g. Now I have seen this doesn't work, because if I take g to be constant, then $\not{D}$ is constant coefficients and we can see that the graph of $\not{D}$ can't be smooth in g.

Let's check this. A constant coefficient operator becomes a multiplication operator in ξ. So $\not{D}$ is multiplication by

$$\begin{pmatrix} 0 & hi\xi - iX \\ hi\xi + iX & 0 \end{pmatrix}$$

Set $h = 1$ and remove the i factor. We are concerned with the graph of the operator of multiplying by the function $\xi + X$ where X is an imaginary scalar depending on g. It is important that as $|\xi| \to \infty$ this function go to infinity sufficiently fast that the graph is congruent to $\mathrm{Im}\begin{pmatrix}0\\1\end{pmatrix}$ mod $\mathcal{L}^p$. But if $X = ia^{-1}$

$$\frac{1}{\xi + X} = \frac{a}{a\xi + i}$$

This sequence for $\xi \in \mathbb{Z}$ is in ℓ^2 for all a but it's derivative at $a = 0$ is the const. sequence $\xi \longrightarrow \frac{1}{i}$.

Suppose now that a is a function of x. Then we consider $\frac{a(x)}{a(x)\xi + i}$ as a sequence $\underbrace{\ }_{\text{in } \ell^2}$ depending on x. Suppose that $a(x)$ is smooth and vanishes to infinite order at $x=0$. Is then $x \longmapsto \frac{a(x)}{a(x)\xi+i} \in \ell^2$ smooth at $x=0$?

$$\frac{d}{dx}\,\frac{a(x)}{a(x)\xi+i} = \frac{a'}{a\xi+i} + a(-1)\frac{a'\xi}{a\xi+i}$$

Better

$$\frac{a}{a\xi+i} = \frac{1}{\xi}\left(\frac{a\xi+i-i}{a\xi+i}\right) = \frac{1}{\xi} - \frac{i}{\xi(a\xi+i)}$$

$$\frac{d}{dx}\left(\frac{a}{a\xi+i}\right) = (+i)\frac{a'\xi}{\xi(a\xi+i)^2} = i\,\frac{a'}{(a\xi+i)^2}$$

Now
$$\int \frac{d\xi}{(a\xi+i)^4} = \int \frac{d(\xi/a)}{(a(\xi/a)+i)^4}$$

$$= \frac{1}{a}\int\frac{d\xi}{(\xi+i)^4}$$

Thus $\left\|\frac{1}{a\xi+i}\right\|_2 = O\left(\frac{1}{\sqrt{a}}\right)$, so

$$\left\|\frac{d}{dx}\left(\frac{a}{a\xi+i}\right)\right\|_2 = O\Big(\underbrace{\frac{a'}{a^{1/2}}}_{(a^{1/2})'}\Big) \qquad \text{OK.}$$

Next look at 2nd derivative

$$\frac{d^2}{dx^2}\left(\frac{a}{a\xi+i}\right) = i\,\frac{a''}{(a\xi+i)^2} + i(-2)\frac{(a')^2\xi}{(a\xi+i)^3}$$

$$\left\|\frac{1}{a\xi+i}\right\|_2 = O\left(\frac{1}{\sqrt{a}}\right), \quad \left\|\frac{\xi}{(a\xi+i)^3}\right\|_2 = O\left(\frac{1}{a^{3/2}}\right)$$

$$\left(\int \frac{d\xi\ \xi^2}{(a\xi+i)^6}\right)^{1/2} = \left(\int \frac{d(\xi/a)\ (\xi/a)^2}{(\xi+i)^6}\right)^{1/2} = \left(\frac{C}{a^3}\right)^{1/2} = O\left(\frac{1}{a^{3/2}}\right)$$

$$\left\| \frac{d^2}{dx^2}\left(\frac{a}{a\xi+i}\right)\right\|_2 \leq O\left(\frac{a''}{\sqrt{a}}\right) + O\left(\frac{(a')^2}{a^{3/2}}\right)$$

This seems OKAY, i.e. to go to zero as $x \to 0$.

Point: $(a^p)' = p\frac{a'}{a^{1-p}}$ if $0<p<1$.

So it might be true that when a vanishes to infinite order at $x=0$, we have that $x \longmapsto \frac{a(x)}{a(x)\xi+i}$ is smooth with values in ℓ^2.

However it is not the same as having a smooth function of x, ξ^{-1} near $0,0$.

$$\frac{a}{a\xi+i} = \frac{a}{a\xi\left(1+\frac{i}{a\xi}\right)} = \frac{1}{\xi}\left(1-\frac{i}{a\xi}+\left(\frac{i}{a\xi}\right)^2-\cdots\right)$$

Thus we don't have a Taylor series expansion up to the second order in $\frac{1}{\xi}$.

The preceding tells me roughly the following. It should be possible to construct resolvents etc. for $h\gamma'\partial_x + \gamma^2 X$ where X grows very fast as g approaches -1. However it is probably going to be harder than if I succeed in formulating a problem where ∂_x is scaled suitably as a function of g.

Now let us try to understand the general problem. I am trying to define certain subspaces of $L^2(S^1)^2$ which are $\mathcal{L}^2$-commensurable with the second factor. I want to use the graph of a differential operator like

$$\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$$

We don't want p, q, r to vanish simultaneously and we probably want $r = 0 \Rightarrow q \neq 0$. Also from consideration of the symbol we probably want r to be divisible by p^n for any n.

Let's look at an interval where $r > 0$. Thus we want the graph of $(p\partial_x + q)r^{-1}$. Note that as $r = p^n s$ for some s $r \neq 0 \Rightarrow p \neq 0$. Moreover

$$\frac{p}{r} = \frac{p}{p^n s} = \frac{1}{p^{n-1} s}$$

hence if r approach zero, then so does either p or s and so $\frac{p}{r} \longrightarrow \infty$. Thus we are dealing with a differential operator $a\partial_x + b$ on an interval where b, a become rapidly infinite at the ends and $a > 0$.

In fact we see there are very few possibilities. It should be possible to reduce the operator on the Hilbert space $L^2(I)$, I = interval, via changing coords to $y \ni dy = a^{-1}dx$ to the simple case of an operator $\partial_y + f$ where f grows at the endpoints. This model for different f's and different embeddings into the circle should describe all of the possibilities.

May 16, 1987

Fix a finite open interval on the line, say $(-1,1)$ and consider over it ~~the~~ differential operator on pairs of functions

$$\circledast \qquad \begin{pmatrix} 0 & h\, a^{1/2}\partial_x a^{1/2} - b \\ h a^{1/2}\partial_x a^{1/2} + b & 0 \end{pmatrix}.$$

Here $a = \frac{1}{r}$ where $r > 0$ on $(-1,1)$ and it extends by zero to a smooth function on the line. b is a smooth real function on $(-1,1)$ possibly depending on h. The problem is [redacted] to [redacted] show under suitable hypotheses that this differential operator is essentially self-adjoint on $L^2(-1,1)^2$ and that it has a well-defined index. Moreover we want to show the index can be evaluated by asymptotic methods as $h \downarrow 0$.

One thing that puzzled me yesterday was the operator

$$\begin{pmatrix} 0 & h a \partial_x \\ h\partial_x a & 0 \end{pmatrix} = \begin{pmatrix} 0 & h\, a^{1/2}\partial_x a^{1/2} - \frac{h}{2}a' \\ h\, a^{1/2}\partial_x a^{1/2} + \frac{h}{2}a' & 0 \end{pmatrix}$$

which has a well-defined index which is equal to 1. This is not susceptible to asymptotic [redacted] evaluation of the index. The ground state doesn't [redacted] change with h, whereas [redacted] it seems that when the asymptotics apply the ground state peaks at the zeroes of b à la Witten.

I conclude from this discussion that the following should be true.

1) As just mentioned the ground state peaks at the zeroes of b in the classical limit.

2) As far as writing $a^{1/2}\partial_x a^{1/2}$ instead of $a\partial_x$ is concerned in ⊛ this changes b by $\frac{\hbar a'}{2}$, which doesn't matter in the $\hbar \to 0$ limit.)

Remark: It appears that we are actually defining a Dirac operator on the line attached to a more general kind of potential than was previously considered. Thus are there KdV possibilities?

Let's work out the "asymptotic" evaluation of the index for ⊛.

$$\not{D} = \hbar\gamma^1 \underbrace{a^{1/2}\partial_x a^{1/2}}_{\text{call this } \nabla} + \gamma^2 X \qquad\qquad X = -ib$$

Then $$\text{Index} = Tr_s\left(e^{u\not{D}^2}\right) = Tr_s\, e^{u(\hbar^2\nabla^2 + X^2 + \hbar\gamma^1\gamma^2[\nabla,X])}$$

$$\sim \int dx \int \frac{d\xi}{2\pi}\, e^{-u\hbar^2 a^2\xi^2 + uX^2}\, u\,2i\hbar[\nabla,X]$$

(Here $\hbar^2\nabla^2 = \hbar^2 a^{1/2}\partial_x a\,\partial_x a^{1/2} \sim -\hbar^2 a^2\xi^2$. Thus

$$\text{Index} = \int dx\; e^{uX^2}\, \frac{1}{2\pi}\frac{\sqrt{\pi}}{\sqrt{u}\,\hbar a}\, u\,2i\hbar\,[\nabla,X]$$

But $$[\nabla, X] = [a^{1/2}\partial_x a^{1/2}, -ib] = -i[a^{-1/2}a\partial_x a^{1/2}, b]$$

$$= -i\,a^{-1/2}[a\partial_x, b]\,a^{1/2} = -i\,a\partial_x b = a\partial_x X$$

so

$$\text{Index} = \int dx\, e^{uX^2} \sqrt{u}\, \partial_x X \, \frac{i}{\sqrt{\pi}}$$

The marvellous thing is that a drops out at the end which means that if I can manage to keep $\lim_{h\to 0} X = \frac{g-1}{g+1}$, then I should end up with the superconnection forms.

It appears that there are some obvious problems with trying to an arbitrary a and X. For example if I take $X = -ix$ so that X remains bounded on the support of r, then the index will not be given by

$$\int_{\text{Supp}(r)} dx\, e^{uX^2} \sqrt{u}\, \partial_x X \, \frac{i}{\sqrt{\pi}} = \int_{-1} e^{-ux^2} \sqrt{u}\, \frac{1}{\sqrt{\pi}}\, dx$$

since this isn't 1. It is not clear what goes wrong with the formal calculation on the previous page.

~~This is related to the~~ Actually what goes wrong is that the index isn't defined when X is bounded on the support of r. Thus if we consider

$$(*) \quad \begin{pmatrix} 0 & h a^{1/2} \partial_x a^{1/2} - b \\ h a^{1/2} \partial_x a^{1/2} + b & 0 \end{pmatrix}$$

then we know this is equivalent via the change of variable $y = \int_0^x r\, dx$ to

$$\begin{pmatrix} 0 & h\partial_y - b \\ h\partial_y + b & 0 \end{pmatrix}$$

and this fails to be essentially skew-adjoint on $L^2(-1,1)^2$ when b is bounded, e.g. $b = x$.

The problem is the following. We consider a connected open oriented 1-diml Riemannian manifold of finite length together with a real function b on it. Then we can form a Dirac operator acting on pairs of half-densities:

$$h\gamma^1\nabla + \gamma^2(-ib).$$

If we let y be the natural coordinate on the manifold - length from the left end, this operator becomes

$$\begin{pmatrix} 0 & h\partial_y - b \\ h\partial_y + b & 0 \end{pmatrix} \quad \text{on} \quad L^2(0,l)^2$$

where l is the length. The problem is to determine when this operator is essentially skew-adjoint and such that its Cayley transform is congruent to -1 modulo $\mathcal{L}^2$. Also we want to know when its index can be found by asymptotic expansion in h as $h \downarrow 0$.

The answer to this problem, or at least sufficient conditions can probably be found by the methods, ODE methods, discussed in the Dym-McKean book. The conditions involve various integrals with the function $\phi = e^{-\int^x b}$.

However I think the real issue is one of compactification. Thus we should look at the symbol of the operator which goes from the manifold $(0,\ell) \times \mathbb{R}$ = cotangent bundle to $\mathbb{C}$. It is important that this symbol map represent a K class with compact support, which tells us that b must go to infinity at the ends $0, \ell$.

But I think I want to embed $(0,\ell)$ inside the line so that extending by ∞ will give a smooth map from $\mathbb{R} \times \{\mathbb{R} \cup \infty\}$ to S^2 which is ∞ outside $(0,\ell) \times \mathbb{R}$.

Let's now find conditions on

$$\begin{pmatrix} 0 & ha^{1/2}\partial_x a^{1/2} - b \\ ha^{1/2}\partial_x a^{1/2} + b & 0 \end{pmatrix}$$

that it be essentially skew-adjoint and its C.T. be $\equiv -1 \mod \mathcal{L}^2$. Here $a = r^{-1}$ where $r > 0$ on the open interval of interest and $r = 0$ outside this interval.

We make the change of variable

$$y = \int^x r$$

and then we know that the operator becomes

$$\begin{pmatrix} 0 & h\partial_y - b \\ h\partial_y + b & 0 \end{pmatrix}$$

over some interval. We are interested in the graph of the operator $h\partial_y + b$.

To simplify things I am going to suppose everything is symmetric about $x=0$. I suppose b is odd and that it goes to $+\infty$ at the right. Let φ be the solution of

$$\begin{cases} (h\partial_y + b)\varphi = 0 \\ \varphi(0) = 1 \end{cases}$$

Then

$$\varphi(y) = e^{-h^{-1}\int_0^y b(y')dy'}.$$

Let $y=1$ be the right endpoint. The critical point is ~~whether~~ $\varphi^{-1} \notin L^2(-1,1)$. This certainly will be true if $h^{-1}b(y)$ grows suff. fast as $y \to 1$.

Notice that $\varphi(y)$ decreases for $0<y<1$, so it goes to zero as y approaches 1. Now I can write a Green's function for $h\partial_y + b$ at least over $[0,1)$ with vanishing boundary condition at 0:

$$G(x,y) = \begin{cases} \frac{1}{h}\frac{\varphi(x)}{\varphi(y)} & x>y \\ 0 & x<y \end{cases}$$

Similarly I can work on the other side and put these together to get a Green's function for $h\partial_y + b$ with the condition $G(0,y)=0$ so as to eliminate multiplies of φ.

At this point one knows essentially skew-adjointness and the fact the index is 1.

Next we want to see that G is Hilbert-Schmidt. But

$$\int_0^1 dx \int_0^1 dy \; G(x,y)^2 = \frac{1}{h^2}\int_0^1 dx\, \varphi(x)^2 \int_0^x \frac{1}{\varphi(y)^2}\, dy$$

Since $\varphi(x)$ decreases for $x > 0$ we have

$$\frac{1}{\varphi(y)^2} \leqslant \frac{1}{\varphi(x)^2} \quad \text{for } 0 \leq y \leq x$$

$$\int_0^x \frac{1}{\varphi(y)^2}\, dy \leqslant x \frac{1}{\varphi(x)^2}$$

so

$$\int_0^1 dx \int_0^1 dy\, G(x,y)^2 \leqslant \frac{1}{h^2}\int_0^1 (dx)\, x = \frac{1}{2h^2}$$

Now that we understand what happens in the y coordinate we just have to make the translation

May 17, 1987

Let's work on the interval $0 < x < 1$, and consider

$$(1) \qquad \begin{pmatrix} 0 & hr^{-1/2}\partial_x r^{-1/2} - b \\ hr^{-1/2}\partial_x r^{-1/2} + b & 0 \end{pmatrix}$$

where $r(x) \in C^\infty(R)$ is positive on $(0,1)$ and 0 outside and where $b \in C^\infty(0,1)$. We suppose $b(x) \longrightarrow -\infty$ as $x \to 0$ and $b(x) \longrightarrow +\infty$ as $x \to 1$. We consider the solution of

$$\left(hr^{-1/2}\partial_x r^{-1/2} - b\right)\psi = 0 \qquad (h\partial_x - br)r^{-1/2}\psi$$

which is

$$\psi = r^{1/2} e^{\frac{1}{h}\int^x br}$$

We require that $\psi \notin L^2$. This implies that br has to go to infinity sufficiently fast to cancel the effect of the $r^{1/2}$ in front. For example we can take $r(x)$ to be $e^{-1/x}$ near $x=0$ and then we would want br to be like $\sim\frac{1}{x^2}$ with h suitable. Note that b is allowed to depend on h.

When we evaluate asymptotically the local index density depends on $\lim_{h\to 0} b$.

To keep things simple I will suppose b doesn't depend on h. In this case b must grow very fast at the ends i.e. $br \sim \frac{1}{x^3}$.

Now suppose we start with an $f: (0,1) \to R$ such that $f(0+) = -\infty$, $f(1-) = +\infty$. We would like to find h, b such that (1) is roughly an extension of

$$(2)\qquad \begin{pmatrix} 0 & h\partial_x - f \\ h\partial_x + f & 0 \end{pmatrix}$$

to the line. The idea is that the symbol of (1) which is a smooth map from T^* of $(0,1)$ to S^2 extends smoothly to the compactified T^* of the line. The symbol of (1) is (ignoring h)

$$z = \frac{i\xi}{r} + b$$

To check smoothness we look at the boundary of $(0,1)\times\mathbb{R} \subset \mathbb{R}\times(\mathbb{R}\cup\infty)$ where z is ∞.

$$\frac{1}{z} = \frac{1}{\frac{i\xi}{r}+b} = \frac{r}{i\xi + br} = \frac{r}{i\xi\left(1 - i\frac{br}{\xi}\right)}$$

$$\text{"="}\ \frac{r}{i\xi}\left\{1 + \left(i\frac{br}{\xi}\right)^2 + \left(i\frac{br}{\xi}\right)^3 + \cdots\right\}$$

This last series will probably show that $\frac{1}{z}$ is smooth near $\xi = \infty$ provided

$$r(br)^n \text{ is smooth } \quad n \geq 0.$$

Recall that br blows up at the ends. Next near a point where ξ is finite we have

$$\frac{1}{z} = \frac{r}{br\left(1+\frac{i\xi}{br}\right)} \text{"="} \frac{1}{b}\left(1 - \frac{i\xi}{br} + \left(\frac{i\xi}{br}\right)^2 + \cdots\right)$$

Thus we want to know that $r\frac{1}{(br)^n}$ is smooth.
$n \geq 1$

Let's review the problem. Given a loop $g: S^1 \longrightarrow \mathbb{T}$ we wish to associate an index problem which involves the operator

$$\star \qquad \begin{pmatrix} 0 & hr^{-1/2}\partial_x r^{-1/2} - qr^{-1} \\ hr^{-1/2}\partial_x r^{-1/2} + qr^{-1} & 0 \end{pmatrix}$$

where r, q are certain real-valued functions on $\mathbb{T}$ applied to g. The first stage of the problem is to find functions r, q which work. Previously we considered

$$r = 1 \qquad q = \frac{1}{i}\,\frac{g-1}{g+1}$$

but ran into trouble because the Cayley transform of the operator wasn't smooth in g.

(The reason for the $r^{-1/2}\partial_x r^{-1/2}$ is to preserve the skew-adjointness of the operator in more complicated cases, e.g. the ungraded case

$$\varepsilon\,\partial_x + \sum_{\mu=1}^{2} \gamma^\mu X_\mu$$

where apparently I can't do one thing on one side and something else on the other $(\partial_x r^{-1}, r^{-1}\partial_x)$.)

So the problem is to find functions q, r on $\mathbb{T}$ which work. Now we want to evaluate the index by $h \downarrow 0$ asymptotics. We have seen this ought to lead to a local index density based on the function $\frac{q}{r}$. Thus I would like the map from $\mathbb{T}$ to itself given by

$$\frac{1 + i\frac{q}{r}}{1 - i\frac{q}{r}} = \frac{r + iq}{r - iq}$$

to be of degree 1.

The original idea was that $r \geq 0$ and it vanishes to infinite order at $\xi = -1$. and is > 0 elsewhere. q is defined for $\xi \neq -1$ and goes monotonically from $-\infty$ to $+\infty$ as ξ runs counter-clockwises ~~starting~~ as -1. Actually until we have further knowledge q might go from $-R$ to R is also acceptable with a r that vanishes at -1.

Now requiring the symbol map to be a smooth map from the "spherical completion" of the cotangent bundle $S^1 \times (\mathbb{R} \cup \infty)$ to S^2 leads to the requirement~~s~~ (see 739):

$$r q^n \quad \text{smooth near } \xi = -1, \quad \forall n \in \mathbb{Z}.$$

So far we have listed conditions which come by looking at the symbol of the operator and the fact that this symbol is supposed to give the index. Next we have to look carefully at the analysis.

An important case of a loop is one which crosses -1 transversally. For example suppose we have a loop such that $\frac{1}{i}\frac{q-1}{q+1}$ has a simple pole at $x = 0$. Set $f = \underline{\qquad}$ so that

$f(x) = \frac{c}{x} + \text{smooth}$ near $x = 0$.

We now look at the operator $h r^{-1/2} \partial_x r^{-1/2} + q r^{-1}$ on an open interval $(0, \delta)$. Now the issue is whether we have limit point behavior as $x \downarrow 0$.

This means we want the solution of

$$\left(h r^{-1/2} \partial_x r^{-1/2} \pm q r^{-1}\right) \phi = 0$$

$$\left(h \partial_x \pm q\right)\left(r^{-1/2} \phi\right) = 0$$

$$\phi = r^{1/2} e^{\mp \int^x \frac{1}{h} q}$$

We want one of these solutions to fail to be in $L^2((0, \delta])$ because it grows too fast as $x \downarrow 0$.

To fix the ideas suppose $q > 0$ on $(0, \delta]$ and consider

$$\phi = r^{1/2} e^{-\frac{1}{h} \int_\delta^x q \, dx} = r^{1/2} e^{\frac{1}{h} \int_x^\delta q}$$

Recall that r is to vanish at zero to infinite order. Thus we see q must grow sufficiently fast to cancel the effect of $r^{1/2}$.

You also have to keep the growth of q as $x \downarrow 0$ small enough that $r q^n$ be smooth for all $n > 0$.

Something that works at least for large enough h is

$$r = e^{-f^2} \qquad q = f^3 \qquad f = \frac{1}{i}\left(\frac{J-1}{J+1}\right)$$

Thus if $f \sim \frac{c}{x}$ then

$$\frac{1}{h} \int_x^\delta \left(\frac{c}{x}\right)^3 dx = + \frac{c^3}{x^2 h 2}$$

$$\log r^{1/2} = -\frac{1}{2} f^2 = -\frac{1}{2} \frac{c^2}{x^2}$$

and so the integral wins for $h \le c$.

Other possibilities would be to take $g = f$ and $r = f^{-N}$. Thus

$$(x^N)^{1/2} e^{+\frac{1}{h}\int_x^\delta \frac{1}{x}} = \text{cnst}\ x^{N/2} e^{-\frac{1}{h}\ln x}$$
$$= O\left(x^{N/2-\frac{1}{h}}\right)$$

which fails to be in L^2 for large $\frac{1}{h}$. Here $r = f^{-N}$ won't give a C^∞ symbol but might give enough [redacted] smoothness to do the analysis.

[redacted]

Conclusion: It looks there is no obstruction to making $h\gamma' r^{-1/2}\partial_x r^{-1/2} + \gamma\frac{2i}{i\delta}q$ work with g, r chosen suitably. Unfortunately the whole process looks unwieldy

May 18, 1987

The problem is to make sense out of the operator $\not{D} = h\gamma^1 r^{-1/2}\partial_x r^{-1/2} + \gamma^2(-iqr^{-1})$ where are functions of the sort discussed yesterday. What I have to do is to construct a unitary operator U on $L^2(S^1)^2$ which is the C.T. of $\not{D}$. I have decided that the way to do this is to construct the resolvent $\frac{1}{\lambda-\not{D}}$ for large λ. The idea is to show the existence of $\frac{1}{\lambda\pm\not{D}}$ for one real value of $\lambda\neq 0$ and to define a unitary by $U = (\lambda+\not{D})\frac{1}{\lambda-\not{D}}$

In order for this to work, it seems we need to know the domain of $\not{D}$. You really need control over the image of $\frac{1}{\lambda-\not{D}}$. For example $r^2 f$ where f is smooth is in the domain of $\not{D}$, as

$$r^{-1/2}\partial_x\left(r^{-1/2}r^2 f\right) = r^{-1/2}\,\frac{3}{2}r^{1/2}r' f + r f'$$

$$q r^{-1} r^2 f = (qr)f$$

If one assumes r fits into a family r^t such as $r = e^{-1/x^2}$, then the domain of $\not{D}$ contains all $r^{1+\varepsilon} f$ with $\varepsilon > 0$.

May 19, 1987

Back to bosonization. The first task is to get a clean version of the theory of a real scalar field on a Riemann surface.

We have to start on the circle. Here the "theory" can be viewed as a kind of harmonic oscillator.

Thus we have to first understand the simple **harmonic** oscillator, especially its quantum mechanics and imaginary time.

Classically the simple harmonic oscillator is described by a configuration space $\{q \in \mathbb{R}\}$ with Lagrangian $\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2$, or a phase space $\{(q,p) \in \mathbb{R}^2\}$ with $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$. The equation of motion is either

$$\ddot{q} + \omega^2 q = 0 \qquad \text{or} \qquad \begin{cases} \dot{q} = p \\ \dot{p} = -\omega^2 q \end{cases}$$

Quantum mechanically it is described by herm. operators p, q with $[p,q] = \frac{1}{i}$ and the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 = \omega\left(a^* a + \frac{1}{2}\right)$$

$$a = \frac{\omega q + ip}{\sqrt{2\omega}}$$

I want to think in terms of the real symplectic space of real linear combinations of q, p; better in terms of the complex symplectic space $\mathbb{C}a + \mathbb{C}a^*$ and the real structure given by the adjoint map.

Quantum mechanically the oscillator can also be described by a path integral. This leads to the imaginary time equation of motion

$$\ddot{q} - \omega^2 q$$

in configuration space.

When we consider the real scalar field on the circle, the configurations are real fns $f(x)$ on the circle and points of phase space are pairs $(f(x), \dot{f}(x))$, i.e. Cauchy data for the equation of motion

$$\ddot{f} = \partial_x^2 f \qquad \text{(wave equation).}$$

(Digression: There is an annoying duality in this whole business already present for the simple oscillator. On one hand the easiest things to work with classically are points of phase space, but on the other it is the functions on phase space which corresponding to operators. Thus classically I like to think of (q,p) as a typical ~~points~~ point of phase space and not as linear functions on phase space. When it comes to the QM I want to actually parametrize the operators which are linear combinations of a, a^* by points in phase space, rather than the linear functions on phase space.)

Now when we quantize we have hermitian operators associated to pairs of real functions. Recall the formula

$$\phi(x) = \sum \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{ikx} + a_k^* e^{-ikx} \right)$$

$$\dot{\phi}(x) = \sum \sqrt{\frac{\omega_k}{2}} \left(-i a_k e^{ikx} + i a_k^* e^{-ikx} \right)$$

where $\omega_k = |k|$ and we ignore all zero mode problems. Then

$$[\phi(x), \phi(y)] = [\dot\phi(x), \dot\phi(y)] = 0$$

$$[\phi(x), \dot\phi(y)] = i\,\delta(x-y)$$

We can then define smeared operators

$$\phi(f) = \int f(x)\,\phi(x)\,dx$$

$$\dot\phi(g) = \int g(x)\,\dot\phi(x)\,dx$$

satisfying

$$[\phi(f), \phi(g)] = [\dot\phi(f), \dot\phi(g)] = 0$$

$$[\phi(f), \dot\phi(g)] = i\int f(x)g(x)\,dx$$

<u>Natural question</u>: When is $\phi(f_t) + \dot\phi(g_t)$ consistent with imaginary time translation? Does this have anything to do with harmonic functions: $\partial_t^2 f + \partial_x^2 f = 0$?

<u>Motivation</u>: If we have a circle on a Riemann surface then we can probably attach a Fock space to the circle as follows. We can consider pairs consisting of a function and a 1-form on the circle. These we can make into a symplectic vector space using the natural pairing of functions and 1-forms. So we have a real symplectic vector space attached to the circle, and hence, assuming the polarization business works, a

Fock space attached to the circle.

Next notice that because of the complex structure on the Riemann surface the 1-form gives a first order function in the normal direction to the circle. Thus the pair consisting of the function and 1-form is equivalent to a function on the first infinitesimal nbd. of the circle in the surface. Thus we have Cauchy data for a harmonic function.

Let's consider the simple oscillator. The problem is that imaginary time translation doesn't preserve the real space of hermitian operators $ca + \bar{c}a^*$.

Let us consider a family of operators

$$\phi(f_t) + \dot{\phi}(g_t)$$

which is consistent with imaginary time translation in the sense that

$$\phi(f_t) + \dot{\phi}(g_t) = \underset{\text{(wrong sign)}}{e^{tH}}\left(\phi(f_0)+\dot{\phi}(g_0)\right)e^{-tH}$$

Now by defn.

$$\phi(f) + \dot{\phi}(g) = \int f(x) \sum \frac{1}{\sqrt{2\omega_k}}\left(a_k e^{+ikx} + a_k^* e^{-ikx}\right)\frac{dx}{2\pi}$$

$$+ \int g(x) \sum \sqrt{\frac{\omega_k}{2}}\left(-ia_k e^{ikx} + i a_k^* e^{-ikx}\right)\frac{dx}{2\pi}$$

$$= \sum \frac{1}{\sqrt{2\omega_k}}\left(a_k \hat{f}_{-k} + a_k^* \hat{f}_k\right) + \sum \frac{1}{\sqrt{2\omega_k}}\omega_k\left(-ia_k \hat{g}_{-k} + i a_k^* \hat{g}_k\right)$$

$$= \sum \frac{1}{\sqrt{2\omega_k}}\left[a_k\left(\hat{f}_{-k} - i\omega_k \hat{g}_{-k}\right) + a_k^*\left(\hat{f}_k + i\omega_k \hat{g}_k\right)\right]$$

where $\quad \hat{f}_k = \int e^{-ikx} f(x) \frac{dx}{2\pi}$

For this to be consistent with imag. time evolution we must have

$$\hat{f}_{+k} - i\omega_k \hat{g}_{+k} = e^{-\omega_k t} u_k$$
$$\hat{f}_k + i\omega_k \hat{g}_k = e^{\omega_k t} v_k$$

for constants u_k, v_k. Thus

$$\hat{f}_k = \frac{1}{2}\left(e^{-\omega_k t} u_k + e^{\omega_k t} v_k\right)$$

and so

$$f(x,t) = \frac{1}{2}\sum e^{ikx}\left(e^{-\omega_k t} u_k + e^{\omega_k t} v_k\right)$$

$$= \frac{1}{2}\sum_k e^{ikx-kt} \begin{cases} u_k & k>0 \\ v_k & k<0 \end{cases}$$

$$+ \frac{1}{2}\sum_k e^{ikx+kt} \begin{cases} v_k & k>0 \\ u_k & k<0 \end{cases}$$

Thus we learn that $f(x,t)$ is an arbitrary harmonic function. What is g?

$$\hat{g}_k = \frac{i}{2\omega_k}\left(e^{-\omega_k t} u_k - e^{\omega_k t} v_k\right)$$

$$g(x,t) = \frac{i}{2}\sum \frac{1}{\omega_k} e^{ikx}\left(e^{-\omega_k t} u_k - e^{\omega_k t} v_k\right)$$

Thus $(\partial_t g)(x,t) = \frac{1}{i} f(x,t)$. So we learn that the family $\phi(f_t) + \dot{\phi}(g_t)$ is given by a single harmonic function

May 20, 1987

Consider a family of operators $c_1(t)q + c_2(t)p$ which is consistent with imag time evolution in the sense that

$$* \quad \left(c_1(t)q + c_2(t)p\right)e^{-tH} = e^{-tH}\left(c_1(0)q + c_2(0)p\right)$$

Then
$$\dot{c}_1 q + \dot{c}_2 p = -[H, c_1 q + c_2 p]$$
$$= -c_1\left[\tfrac{1}{2}p^2, q\right] - c_2\left[\tfrac{1}{2}\omega^2 q^2, p\right]$$
$$= c_1 ip - c_2 \omega^2 iq$$

so
$$\dot{c}_1 = -i\omega^2 c_2$$
$$\dot{c}_2 = i c_1$$

In other words c_2 is a solution of $\ddot{c}_2 = \omega^2 c_2$ and c_1 is $\frac{1}{i}\dot{c}_2$.

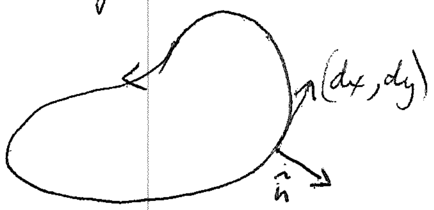
Similarly when we consider the real scalar field situation, a family
$$\phi(f_t) + \dot{\phi}(g_t)$$
is consistent with imag. time evolution when g is harmonic and $f = \frac{1}{i}\partial_t g$.

So we have a solution of the equations of motion

$$\partial_t\left(\phi(f_t) + \dot{\phi}(g_t)\right) = -[H, \phi(f_t) + \dot{\phi}(g_t)]$$

$$\iff \quad f_t = \frac{1}{i}\partial_t g_t \quad \text{and} \quad \left(\partial_t^2 + \partial_x^2\right) g_t = 0$$

Review of Green's formulas in the plane.



$$\hat{n}\,ds = \hat{\imath}\,dy - \hat{\jmath}\,dx$$

$$\oint P\,dx + Q\,dy = \iint (\partial_x Q - \partial_y P)\,dx\,dy$$

$$\oint y\,dx = -\text{area}.$$

$\hat{\imath}P + \hat{\jmath}Q$

$$\oint \vec{F}\cdot\hat{n}\,ds = \oint P\,dy - Q\,dx = \iint (\partial_x P + \partial_y Q)\,dx\,dy \text{ or}$$

$$\oint \vec{F}\cdot\hat{n}\,ds = \iint (\vec{\nabla}\cdot\vec{F})\,dA.$$

$$\oint \underbrace{\vec{\nabla}g\cdot\hat{n}}_{\frac{\partial g}{\partial n}}\,ds = \iint (\nabla^2 g)\,dA$$

$$\oint \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds = \oint (f\vec{\nabla}g - g\vec{\nabla}f)\cdot\hat{n}\,ds$$

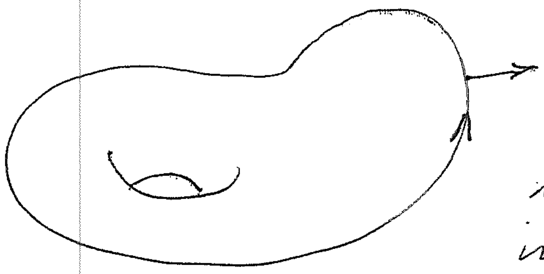
$$= \iint \vec{\nabla}\cdot(f\vec{\nabla}g - g\vec{\nabla}f)$$

$$= \iint (\cancel{\nabla f\cdot\nabla g - \nabla g\cdot\nabla f}) + (f\nabla^2 g - g\nabla^2 f)$$

$$\boxed{\oint \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)ds = \iint (f\,\Delta g - g\,\Delta f)}$$

$$\boxed{\oint \left(f\frac{\partial g}{\partial n} + g\frac{\partial f}{\partial n}\right)ds = \iint (2\nabla f\cdot\nabla g + f\,\Delta g + g\,\Delta f)}$$

I now want to consider a circle on a Riemann surface cutting it in two parts.



On the circle we consider pairs consisting of a function and a normal derivative, i.e. a function on the first infinitesimal nbd. Because of the complex structure the normal bundle to the curve is isom. to the tangent bundle, so the normal derivative which is a section of the dual normal (conormal) bundle is the same thing as a 1-form. This means we have a natural pairing between functions and conormal sections. Let's use the notation f for function and ω for a 1-form on the circle γ. Then we can look for representations of the Heisenberg algebra with generators $\pi(f)$ and $\phi(\omega)$ and relations

$$\pi(f)^\dagger = \pi(\bar{f}) \qquad \phi(\omega)^\dagger = \phi(\bar{\omega})$$

$$[\pi(f), \pi(f')] = [\phi(\omega), \phi(\omega')] = 0$$

$$[\pi(f), \phi(\omega)] = \frac{1}{i}\int_\gamma f\omega$$

Next we consider harmonic functions f on the inside of the circle γ and associate to such an f the operator

$$i\pi(f) + \phi(\partial_n f)$$

Then

$$[i\pi(f) + \phi(\partial_n f), i\pi(f') + \phi(\partial_n f')]$$

$$= \int (f\,\partial_n f' - f'\,\partial_n f) = \iint_{\text{inside}} (f\,\Delta f' - f'\,\Delta f) = 0.$$

so we get an isotropic subspace.

Also

$$[i\pi(f) + \phi(\partial_n f),\ -i\pi(\bar f) + \phi(\partial_n \bar f)]$$

$$= \int f\,\partial_n \bar f + \bar f\,\partial_n f \quad = \iint \nabla f \cdot \nabla \bar f \quad \geq 0$$

and it is > 0 unless f is constant.

Thus we see that the space of operators $i\pi(f) + \phi(\partial_n f)$ with f harmonic in the interior is a candidate for a space of annihilation operators.

Recall that for any function on the boundary there is a unique harmonic function on the inside having this function as boundary. What this means is that the subspace of operators $i\pi(f) + \phi(\partial_n f)$ f harmonic is the analogue of a space spanned by operators

$$i p_j + \sum_k \omega_{jk} q_k$$

with $\omega_{jk} = \omega_{kj}$ and $\omega_{jk} + \bar\omega_{kj} \geq 0$. It therefore should be ~~[illegible]~~ the annihilator of a unique state, except for the problem of the zero mode.

Problems: 1) Shale conditions

2) zero mode. If $f = 1$, then the subspace contains $i\pi(1)$ which has continuous spectrum.

Let's discuss the zero modes and what to do about it. This mode arises because $-\partial_x^2$ on the circle has a kernel. What it means is that our harmonic oscillator with potential $-\partial_x^2$ is degenerate, so that we can split off a degree of freedom q, p on the line with Hamiltonian $p^2/2$. This is like a free particle — the spectrum is continuous and there is no ground state (which is L^2)

One thing to do is to replace this free particle on the line by a free particle on the circle. So we now replace our real scalar field φ on the circle by a circle-valued field $e^{i\varphi}$.

Our first task is to make sense of the quantum mechanics of such a field. Basically the problems have to do with the quantum mechanics of a free particle on the circle. The configuration space is non-simply connected so one can twist any quantization by a flat line bundle. Also there is no position operator q only e^{iq} and this means momentum is quantized.

It seems that ~~one~~ wants to form a Heisenberg group out of the circle and its dual. Thus when I take circle-valued fields $e^{i\varphi}$ I combine this group with its dual, both of which will have different components. We presumably have a Fock space representation of this Heisenberg group. Now the problem comes when we try to find a state in the Fock space corresponding to harmonic φ in the interior. The different φ's in the interior fall into instanton components indexed by elements of $H^1(\Sigma, \mathbb{Z})$. Presumably the weightings are the same data required to specify a line bundle of degree $g-1$.

We've replaced our real scalar field with a circle-valued ~~field~~ u. The first problem will be to construct the quantum mechanics of such fields on the circle. The rough idea is that we take the group of these fields as configuration space and the dual group as momentum space. Both groups have more than one components. In addition to the oscillator modes described before we have two degrees of freedom stemming the simple loops $\{cz^n \mid c \in \mathbb{T},\ n \in \mathbb{Z}\}$.

Let us temporarily suppose we understand the Hilbert space to be attached to $\mathbb{T}$-valued fields on the circle. Now suppose that we take the circle to be the boundary of a small disk D on the Riemann surface Σ. We can now look at $\mathbb{T}$ valued fields on the complement of D. Our problem is to assign a line in the Hilbert space which corresponds to harmonic fields on $\Sigma - D$.

First we should understand what harmonic means. A map $u : \Sigma - D \to \mathbb{T}$ has a logarithm defined locally which is unique up to adding an element of $2\pi i\mathbb{Z}$, so it makes sense to ask that $\log(u)$ be harmonic.

Notice that when we remove D from Σ we don't change H^1.

$$\underset{\substack{\parallel \\ 0}}{H^1(D,\partial D;\mathbb{Z})} \longrightarrow H^1(\Sigma,\mathbb{Z}) \longrightarrow H^1(\Sigma - D,\mathbb{Z}) \longrightarrow H^1(D,\partial D;\mathbb{Z}) \xrightarrow{\cong} H^2(\Sigma,\mathbb{Z}) \longrightarrow \underset{\substack{\parallel \\ 0}}{H^2(\Sigma - D,\mathbb{Z})} \longrightarrow 0$$

Thus $H^1(\Sigma - D,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$

<u>May 21, 1987</u>

Calculate cohomology of harmonic and log harmonic functions. Let $\mathcal{H}$ be the sheaf of harmonic functions. Then we have

$$0 \longrightarrow \mathcal{H} \longrightarrow \underline{C}^\infty \xrightarrow{\Delta} \underline{C}^\infty \longrightarrow 0$$

so over a compact Riemann surface we have

$$H^0(\mathcal{H}) = \text{[redacted]}\ H^1(\mathcal{H}) = \mathbb{C}$$
$$H^2(\mathcal{H}) = 0.$$

Because Δ is self-adjoint it has the same kernel and cokernel.

Next consider the sheaf $\exp(\mathcal{H})$ of functions whose logarithm is harmonic.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{H} \longrightarrow \exp(\mathcal{H}) \longrightarrow 0$$

$$0 \to \mathbb{Z} \to \mathbb{C} \longrightarrow H^0(\exp(\mathcal{H})) \to$$
$$\to H^1(\text{[redacted]}\,\mathbb{Z}) \to \mathbb{C} \longrightarrow H^1(\exp(\mathcal{H})) \to$$
$$\to H^2(\mathbb{Z}) \longrightarrow 0$$

[redacted] Now if ω is a harmonic 1-form with integral periods, then

$$e^{2\pi i \int^x \omega}$$

is a log harmonic function. In effect the space of harmonic 1-forms splits into the sum of the holomorphic diffls and its conjugate space by Hodge theory, and $\int^x \omega$ is holom. (resp. antiholom.) when ω is.

Since any element of $H^1(\mathbb{Z})$ is represented by a harmonic 1-form with integral periods it is clear that $H^0(\exp \mathcal{H}) \longrightarrow H^1(\mathbb{Z})$ is onto. Thus

$$0 \longrightarrow \mathbb{C}^\times \longrightarrow H^0(\exp \mathcal{H}) \longrightarrow \underset{\substack{\parallel \\ \mathbb{Z}^{2g}}}{H^1(\mathbb{Z})} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(\exp \mathcal{H}) \longrightarrow \underset{\substack{\parallel \\ \mathbb{Z}}}{H^2(\mathbb{Z})} \longrightarrow 0$$

$$H^2(\exp \mathcal{H}) = 0$$

In what ways might this cohomology be relevant to [redacted] QFT on the Riemann surface?

<u>Idea:</u> It ought to be possible to make sense of QFT for sheaves having cohomology. It seems strange that there should be a theory associated to [redacted] log harmonic functions but not harmonic functions. Also there should be some sort of fermionic theory attached to a spin structure having nonzero cohomology. Certainly the latter statement is true and the theory which is just the fermion fcl. integral belonging to the Dirac operator is given by a point of the determinant line. Not clear for a spin structure but it's OKAY for the bigger system whose fields are (c, b), sections of ξ, $K \otimes \xi^{-1}$.

Let start with the "QFT" associated to harmonic functions. On a circle we can consider pairs (f, ω) with f a function and ω a 1-form. These [redacted] form a [redacted] symplectic vector space with real structure and it should be possible to associate a canonical representation of the Heisenberg algebra. In order to prove [redacted] this I would have to verify the Shale condition, that is, I have to specify a Hilbert-Schmidt class of polarizations.

Next we must link imaginary-time evolution with harmonic functions.

I think it's perhaps fastest if I push this through brutally and work on the circle with Fourier series. We start with functions and 1-forms on the circle. We associate to the function $f(x) = \sum' \hat{f}_k e^{ikx}$ the operator

$$\pi(f) = \int f(x)\, \overbrace{\pi(x)}^{\sum' \sqrt{\frac{\omega_k}{2}}\left(-i a_k e^{ikx} + i a_k^* e^{-ikx}\right)} \frac{dx}{2\pi}$$

$$= \sum \sqrt{\frac{\omega_k}{2}} \left(-i a_k \hat{f}_{-k} + i a_k^* \hat{f}_k\right)$$

and to a 1-form $g(x)\,dx$ the operator

$$\phi(g) = \cancel{\frac{1}{\sqrt{2\omega_k}}} \int g(x)\, \overbrace{\phi(x)}^{\sum \frac{1}{\sqrt{2\omega_k}}\left(a_k e^{+ikx} + a_k^* e^{-ikx}\right)} \frac{dx}{2\pi}$$

$$= \sum \frac{1}{\sqrt{2\omega_k}} \left(a_k \hat{g}_{-k} + a_k^* \hat{g}_k\right)$$

As written these are defined for f modulo constants and for g such that $\int g(x)\,dx = \hat{g}_0 = 0$.

$$[\pi(f), \phi(g)] = -i \sum_k \hat{f}_{-k} \hat{g}_k = \frac{1}{2\pi i} \int f(x) g(x)\,dx$$

Let us next solve the Dirichlet problem on the disk. Now x becomes θ.

$$f(\theta) = \sum \hat{f}_k e^{ik\theta}$$

extends inside the disk to the harmonic function

$$f(r,\theta) = \sum_{k \geq 0} \hat{f}_k \underbrace{r^k e^{ik\theta}}_{z^k} + \sum_{k<0} \hat{f}_k \underbrace{r^{-k} e^{ik\theta}}_{\bar{z}^k}$$

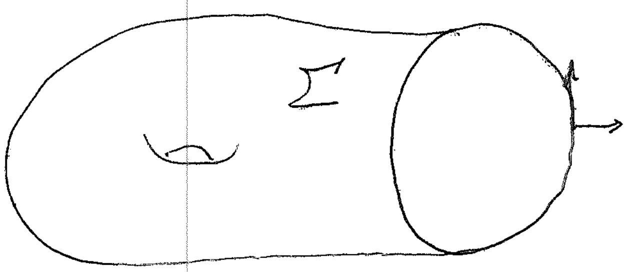
and the normal derivative is

$$\left.\frac{\partial f}{\partial r}\right|_{r=1} = \sum_{k\geq 0} k\hat{f}_k e^{ik\theta} + \sum_{k<0} (-k)\hat{f}_k e^{ik\theta}$$

$$= \sum_{k\geq 0} \omega_k \hat{f}_k e^{ik\theta} .$$

Thus the operator taking f on the circle to the normal derivative of its harmonic extension to the interior is just $|\partial_x| = (-\partial_x^2)^{1/2}$.

Now I have control on the Hilbert space and operators associated to the circle, and I wish to move on to the case where the circle is the boundary of a surface. I still have the operators $\pi(f)$ and $\phi(g\,dx)$ associated to functions on the circle and 1-forms respectively.



The problem is to produce a state in the Fock space corresponding to the harmonic functions on the surface. Specifically given a harmonic function f on Σ we associate the operator

$$i\pi(f) + \phi(\partial_n f)$$

Observe in the case where Σ is the disk that this is

$$\sum \sqrt{\tfrac{\omega_k}{2}}\left(a_k\hat{f}_{-k} - a_k^*\hat{f}_k\right) + \sum \frac{1}{\sqrt{2\omega_k}}\left(a_k\omega_k\hat{f}_{-k} + a_k^*\omega_{-k}\hat{f}_k\right)$$

$$= \sum \sqrt{2\omega_k}\cdot a_k\hat{f}_{-k}$$

that is, it is a linear combination of destruction ops.

Next look at the commutation relations

$$[i\pi(f) + \phi(\partial_n f),\ i\pi(f_1)+\phi(\partial_n f_1)]$$

$$= \oint (f\,\partial_n f_1 - f_1\,\partial_n f) = \iint_\Sigma (f \underbrace{\Delta f_1}_{0} - f_1 \underbrace{\Delta f}_{0}) = 0$$

$$[i\pi(f)+\phi(\partial_n f),\ -i\pi(\bar f)+\phi(\partial_n \bar f)]$$

$$= \oint (f\,\partial_n \bar f + \bar f\,\partial_n f) = \iint \nabla f\cdot\nabla\bar f \geq 0$$

with equality iff f is constant. Also I should notice that $\partial_n f$ is always a 1-form with integral 0, because

$$\int \partial_n f = \iint \Delta f = 0.$$

The theorem we want to establish is that there is a unique line in the Hilbert space which is annihilated by the operators

$$i\pi(f) + \phi(\partial_n f) \qquad f \text{ harmonic on } \Sigma$$

I think it is reasonable to expect in general that the operator from functions on $\partial\Sigma$ to 1 forms which takes a function to the normal derivative of its harmonic extension to Σ is a ΨDO of order 1 with the symbol $|\xi| = \operatorname{sign}(\xi)\cdot\xi$. Let's call this operator T. It's symmetric in the sense that

$$\langle f, Tf_1\rangle = \langle Tf, f_1\rangle$$

and positive: $\langle \bar f, Tf\rangle = \frac{1}{2}\iint |\nabla f|^2 \geq 0.$

From this it follows that its kernel is the constant functions. Because it is real symmetric and positive it should follow that the image of T is the space of 1-forms of integral zero. In any case one ought to know this from the Neumann problem theory.

Now we ought to be able to check the Shale condition. Thus when are the irreducible repns. with annihilation operators

$$a = \frac{1}{\sqrt{2}}\left(i(\lambda^t)^{-1}p + \lambda q\right) \qquad \lambda = \omega^{1/2}$$

and

$$a' = \frac{1}{\sqrt{2}}\left(i(\mu^t)^{-1}p + \mu q\right) \qquad \mu = \omega'^{1/2}$$

unitarily equivalent. The symplectic transformation

$$q \longmapsto \lambda^{-1}\mu q$$

$$p \longmapsto \lambda^t(\mu^t)^{-1}p = (\mu^{-1}\lambda)^t p$$

transforms a to a'. According to the Shale theorem this symplectic transf. is unitarily implementable iff $\lambda^{-1}\mu \equiv 1 \mod \mathcal{L}^2$. This certainly implies $\omega \equiv \omega' \mod \mathcal{L}^2$ and the rest should be functional calculus.

May 22, 1987

Let us consider on a finite interval I the operator

$$\not{D} = \begin{pmatrix} 0 & \partial_x - q \\ \partial_x + q & 0 \end{pmatrix}$$

where q is a real valued function becoming infinite at the ends (i.e. a proper function $q : I \to \mathbb{R}$). We would like to understand how to show $\not{D}$ is essentially skew adjoint. It should be enough to construct the resolvent operator $R_\lambda = \frac{1}{\lambda - \not{D}}$ for λ real and $|\lambda|$ large. We have

$$\frac{\lambda + \not{D}}{\lambda - \not{D}} = -1 + \frac{2\lambda}{\lambda - \not{D}}$$

and $g = -1 + m$ is unitary iff

$$gg^* = (-1+m)(-1+m^*) = 1 - m - m^* + mm^* = 1$$

g^*g etc.

Thus we want to construct the resolvent and prove it satisfies

$$\frac{1}{\lambda - \not{D}} + \frac{1}{\lambda + \not{D}} = 2\lambda \left(\frac{1}{\lambda - \not{D}}\right)\left(\frac{1}{\lambda + \not{D}}\right)$$

It's clear that we have to understand enough about the domain of $\not{D}$ in order to show $(\lambda - \not{D})\left(\frac{1}{\lambda - \not{D}}\right) = 1$

You propose to construct $\frac{1}{\lambda - \not{D}}$ essentially by writing a formula. One way to proceed would be to find the solutions of the homogeneous equation, then construct the Green's function. This amounts to a power-series-in-λ construction, and really I want an approach using large λ. This is what fits with the heat equation approach.

Important point: You must understand the domain of $\not{D}$ well enough to see that your candidate for $\frac{1}{\lambda - \not{D}}$ has its range in this domain.

Let's discuss generalities. $\not{D}$ starts out defined on $C_c^\infty(I)^2 \subset L^2(I)^2$ and it is skew-symmetric. Closing we obtain a closed densely defined skew-symmetric operator. This is the minimal closed operator and there is also the maximal closed operator. When the two are equal we have a skew-adjoint operator. When we have a skew-adjoint operator the resolvent $\frac{1}{\lambda - \not{D}}$ has for its image the domain of $\not{D}$.

I now want to take a simple example which I feel I have to be able to handle in order to make any further progress. We let r be a smooth function with compact support which is >0 on $(0,1)$ and zero outside $(0,1)$. We wish to consider

$$\not{D} = \begin{pmatrix} 0 & r^{-1}\partial_x \\ \partial_x r^{-1} & 0 \end{pmatrix}$$

acting on $C_c^\infty((0,1))$ and to show it is essentially skew-adjoint.

Domain in the C^∞ setting. A natural question is whether $(\lambda-\not{D})^{-1}$ carries C^∞ functions to C^∞ functions and what its image is. This is clear as ∂_x is elliptic, but then we really want to know the domain of $\not{D}$. The domain consists of smooth functions supported on the closed interval, but there are perhaps growth conditions depending on r.

Let's look at $\partial_x r^{-1}$. Let's find the graph of this operator on the C^∞ level. Thus I want to consider the subspace consisting of pairs

$$\begin{pmatrix} rf \\ \partial_x f \end{pmatrix}$$

where f is C^∞, say on the circle. I can close this up in $(C^\infty)^2$ thereby obtaining a closed subspace Γ. Similarly I can look for the orthogonal subspace. Formally this consists of pairs $\begin{pmatrix} r^{-1}\partial_x g \\ g \end{pmatrix}$ where g is a smooth function on the circle whose derivative is divisible by r.

Let's formulate the problem as follows. I am looking at a Dirac type operator on an open interval which turns out to be essentially skew-adjoint. It therefore has a Cayley transform which is a unitary operator on $L^2(I)^2$. Now I extend this unitary to $L^2(S^1)^2$ by -1. I want to know if the extended unitary preserves the smooth functions. If so it corresponds to a decomposition of $C^\infty(S^1)^2$ into orthogonal subspaces.

I now want to construct an example to see if the sort of behavior I am looking for really occurs. Thus within $L^2(I)^2$ I have to specify precisely what subspace to consider. It is going to be the graph of a first order differential operator $a\partial_x + b$ where a, b are smooth on I and $a > 0$.

The solution of

$$(a\partial_x + b)\varphi = 0 \qquad \text{is} \qquad \varphi = ce^{-\int^x \frac{b}{a}}$$

whereas the solution of the adjoint equation

$$(\partial_x a - b)\psi = 0 \qquad \text{or} \qquad \left(\partial_x - \frac{b}{a}\right)(a\psi) = 0$$

is

$$\psi = ca^{-1} e^{\int^x \frac{b}{a}}$$

What's important is whether these functions are L^2 at the boundary. NO.

What's important is that the closed graph of the differential operator be in the restricted Grassmannian. This ~~implies~~ that projection onto the second factor is Fredholm. Thus we want a parametrix for $a\partial_x + b$ which is Hilbert-Schmidt.

By standard regularity theorems the diffl operator will map its domain onto L^2 when $\psi \notin L^2$. Here I am assuming the graph of the minimal extension of the operator is in the restricted Grass, and I want to see what this says about b, a.

What interests me most is when $\varphi \in C_c^\infty(\mathbb{R})$ is > 0 on I and $= 0$ outside I. Then $\psi = ca^{-1}\varphi^{-1} \notin L^2$ for a large family of a's. We continue and construct the Green's fn. for $a\partial_x + b$, say the one which vanishes to the left.

$$G(x,y) = \begin{cases} \dfrac{\varphi(x)}{a(y)\varphi(y)} & x > y \\ 0 & x < y \end{cases}$$

This is Hilbert-Schmidt provided

$$\iint G(x,y)^2\,dx\,dy = \int dx\ \varphi(x)^2 \int^x \frac{1}{\varphi(y)^2 a(y)^2}\,dy < \infty$$

Example: Take $\varphi(x) \sim e^{-\frac{1}{2x}}$ as $x \downarrow 0$ and $a(y) = y$. Then

$$\int_\varepsilon^x e^{\frac{1}{y}}\ \frac{1}{y^2}\,dy = \left[\frac{+1}{\varphi(x)^2}\right]_x^\varepsilon$$

So the Green's fn. is Hilbert-Schmidt. However when $a = x$, then the natural time parameter $y = \int^x \frac{dx}{a}$ diverges.

So we learn it seems that we can maybe choose a to vanish instead of blowing up at the ends.

May 23, 1987 (Alice is 25)

Yesterday it occurred to me to look at the problem from the smooth viewpoint. The problem is to attach to a loop $g: S^1 \longrightarrow \mathbb{T}$ a unitary operator on $L^2(S^1)^2$ reversed by ε. I propose to do this by constructing the resolvent using ΨDO methods. This resolvent should preserve smooth functions, when ~~so~~ should the unitary. Thus the corresponding decomposition of $L^2(S^1)^2$ into the graph and its orth. complement should already take place in $C^\infty(S^1)^2$.

I now want to define a subspace of $C^\infty(S^1)^2$ by taking the image of an operator of the form

$$\begin{pmatrix} r(x) \\ h p(x)\partial_x + q(x) \end{pmatrix}$$

where p, q, r are functions applied to the loop g. For example

$$\begin{pmatrix} g+1 \\ h\partial_x (g+1) + \frac{1}{i}(g-1) \end{pmatrix}.$$

although this doesn't seem to work analytically.

We certainly want $\frac{q}{r} : S^1 \longrightarrow P^1(\mathbb{R})$ to be homotopy equivalent to g, when we identify $P^1(\mathbb{R}) = \mathbb{T}$ via $x \longmapsto \frac{1+ix}{1-ix}$.

Important Observation: This forces q, r to be complex-valued because the Mobius line bundle over the circle is non-trivial as a real line bundle. Thus

$$\begin{pmatrix} (g+1) \\ \frac{1}{i}(g-1) \end{pmatrix} = g^{1/2} \begin{pmatrix} g^{1/2}+g^{-1/2} \\ \frac{g^{1/2}-g^{-1/2}}{i} \end{pmatrix}$$

so you need the $g^{1/2}$ phase factor.

The goal at the moment is to produce an example of index 1 where everything works. I want to make sure there is no obstruction of a topological nature to the proposed construction.

We want our loop g to be a blip near $x=0$ on the circle. This means it is -1 except in an interval around zero and then goes counterclockwise around the circle as we move through 0.

Here's how to handle the problem with the Mobius line bundle. Take the upper bundle on the circle to be trivial and the lower line bundle to be Mobius. Then $g(x)=-1$ for $x<-\delta$ and $g(x)=+1$ for $x>\delta$, and $r(x)$ can be assumed to be >0 in $(-\delta,\delta)$ and zero outside.

Anyway we now attempt to construct an example of an index 1 situation. We define a subspace to be the image of

$$\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$$

where $r>0$ on $(0,1)$ and zero outside. We propose to identify this subspace with the graph of an operator going the other way which is given by a Green's function, at least up to the zero mode. The orthogonal subspace should then be given by the transposed Green's fn. essentially, and perhaps we can then identify it with the ~~[illegible]~~ graph of the adjoint operator. Note that over $(0,1)$

$$\mathrm{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}^{\perp} = \mathrm{Im}\begin{pmatrix} 1 \\ (p\partial_x+q)r^{-1} \end{pmatrix}^{\perp} = \mathrm{Im}\begin{pmatrix} r^{-1}(\partial_x p - q) \\ 1 \end{pmatrix}$$

$$= \mathrm{Im}\begin{pmatrix} r^{-1}(\partial_x p - q) r \\ r \end{pmatrix}$$

and
$$r^{-1}(\partial_x p - q) r = \left(\partial_x + \frac{r'}{r}\right) p - q$$
which means we want

(1) $\qquad p\frac{r'}{r}$ to be smooth.

Now let's turn to the construction of the Green's function. This will be an inverse, more or less for $(p\partial_x + q)r^{-1}$ and is constructed from the solution of the homogeneous equation

$$(p\partial_x + q)(r^{-1}u) = 0\ , \qquad r^{-1}u = \varphi = e^{-\int^x \frac{q}{p}}$$

$$u = r\varphi$$

Similarly we find the solution of the adjoint DE

$$r^{-1}(\partial_x p - q)v = 0$$

$$\left(\partial_x - \frac{q}{p}\right) pv = 0 \qquad pv = e^{\int^x \frac{q}{p}} = \varphi^{-1}$$

$$v = \frac{1}{p\varphi}$$

The Green's function for $(p\partial_x + q)r^{-1}$ vanishing at the left is

$$G(x,y) = \begin{cases} \dfrac{r(x)\varphi(x)}{p(y)\varphi(y)} & y < x \\ 0 & x < y \end{cases}$$

It helps to put

$$\alpha(x) = r(x)\varphi(x)$$

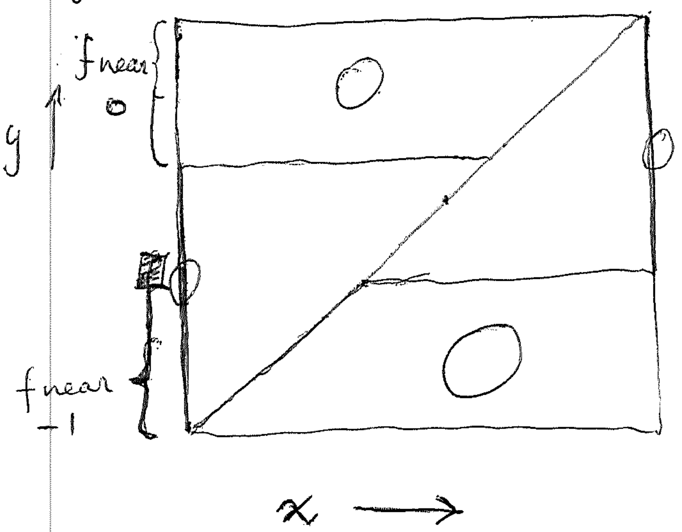
$$\beta(y) = (p\varphi)(y)$$

as these are intrinsically determined by the subspace unlike p, q, r. The above Green's function can then be altered by adding a multiple of $\alpha(x)$ depending on y.

$$(*) \qquad G(x,y) = \alpha(x)\beta(y)\left[\,\theta(x-y) + f(y)\right]$$

Now in the index 1 case we want $\alpha = r\varphi$ to decay at the ends of the interval and for $\beta = \frac{1}{p\varphi}$ to grow at the ends. From the L^2 viewpoint we want $\alpha \in L^2$, $\beta \notin L^2$. Let's choose $f(y) = \begin{cases} -1 & y \text{ near } 0 \\ 0 & y \text{ near } 1 \end{cases}$. Then for x fixed $G(x,y) = 0$ for y near 0 and 1.

Picture of G:



Near the corners $(1,1)$ and $(0,0)$ we need control over G. Near $(1,1)$ G has the jump from 0 to

$$\alpha(x)\beta(x) = \frac{r}{p}(x)$$

so we certainly want $\frac{r}{p}$ to go to zero at the ends. Recall that smoothness considerations for the symbol dictated this also. List this as ~~our~~ second condition on r, p:

(2) $\quad \dfrac{r}{p}$ smooth on $\mathbb{R}$ with support in $[0,1]$.

It seems clear but eventually has to be checked that with reasonable choices we get a ψDO from G.

Next we must check that when G is extended from $(0,1)$ to the circle or line, then the subspace $\operatorname{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ is equal to $\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$

Let's adopt a more intrinsic notation. p, q, r are not intrinsic to the subspace we are trying to describe, however α, β are. As r can be chosen in many ways, it is clear that $\operatorname{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ can not be described as $\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$. However we expect $\operatorname{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ to be the closure of $\operatorname{Im}\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$ in good cases.

Thus we want to start with functions $\alpha(x), \beta(x)$ where $\alpha \in C^\infty(\mathbb{R})$ is >0 on $(0,1)$ and 0 outside, and where $\beta \in C^\infty(0,1)$ is not in L^2. We then define the Green's function as on the preceding page. Things are still a bit delicate because of this $\beta \notin L^2$ condition; in fact I seem to need this condition at both ends of $(0,1)$.

To construct examples we must pick α, β. The simplest choice for β is

$$\beta(y) = \frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}$$

although it might be nicer to have

$$\beta(y) = \frac{1}{[y(1-y)]^{3/4}}$$

so that β is integrable but not L^2.

Now I perhaps should try to keep things very close to the Hilbert space situation.

Next we have to choose α somehow. Then we can define the Green's function $G(x,y)$. If this is L^2, then we know that we have a bounded operator in L^2, hence we get that the graphs for G and its adjoint are complementary. The C.T. is defined. In fact we have eigenfunctions which are smooth at least over the open interval. So the real question is what happens at the boundary — supposedly we are solving a 2nd order eigenvalue problem on the open interval with L^2 boundary conditions. The basic questions are how do the eigenfunctions look at the boundary, e.g. what sort of singularities.

Perhaps you can handle this by looking at the decaying eigenfunction at one end with eigenvalue λ.

May 24, 1987

Review. The problem is still to construct an examples of a first order smooth diff. op. $a\partial_x + b$ on $(0,1)$ with the following properties. Firstly, $\phi = \begin{pmatrix} 0 & \partial_x a - b \\ a\partial_x + b & 0 \end{pmatrix}$ is essentially skew-adjoint on $L^2((0,1))^2$ and its C.T. when extended by -1 from $L^2((0,1))^2$ to $L^2(\mathbb{R})^2$ preserves smooth fns. Secondly the index is to be 1.

I suppose $a > 0$ on $(0,1)$ of course. I can write $a\partial_x + b = \rho\,\partial_x\, r^{-1}$ and work with ρ, r. Then I can define a Green's function for $\rho\,\partial_x\, r^{-1}$

$$G(x,y) = \frac{r(x)}{\rho(y)}\Big(\theta(x-y) + f(y)\Big)$$

where $f \in C^\infty(0,1)$ is identically -1 near $y=0$ and identically zero near $y=1$. Thus for a fixed x, $G(x,y)$ is zero near $y=0$ and $y=1$. This defines G on $(0,1)^2$ and we extend by 0 to a function on $\mathbb{R}^2$.

Suppose G turns out to be a ΨDO of order -1. Then

$$\mathcal{G} = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$$

is a ΨDO operator of order -1, hence it is Hilbert-Schmidt. $\mathcal{G}$ is a bounded skew adjoint operator on $L^2(\mathbb{R})^2$ so its C.T. is well-defined. In fact $1 \pm \mathcal{G}$ are ΨDO's of order 0 which are invertible, so the C.T. $\frac{1+\mathcal{G}}{1-\mathcal{G}}$ is a ΨDO of order 0 congruent to $+1$ mod ΨD operators of order -1. What this means is that we have a decomposition of smooth functions $C^\infty(\mathbb{R})^2$

into the sum of the graphs $\mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ and $\mathrm{Im}\begin{pmatrix} 1 \\ G^t \end{pmatrix}$.

<u>Conclusion</u>: Once it is established that G is a ΨDO of order -1, then everything needed about smooth functions probably follows.

Next we have to look at the index 1 question. We want r which spans the kernel of $\rho\partial_x r^{-1}$ to be smooth on $\mathbb{R}$, zero outside $(0,1)$, and >0 inside. Then the line $\mathbb{C}\begin{pmatrix} r \\ 0 \end{pmatrix}$ is to be added to $\mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ in order to get the subspace Γ we want. It's then clear we get a decomposition into Γ and $\Gamma^\perp$ both on the L^2 and smooth levels.

What isn't clear from all of this is why $\mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix}$ over $(0,1)$ is contained in the closure of $\begin{pmatrix} 1 \\ D \end{pmatrix} C_c^\infty(0,1)$. Thus take $\begin{pmatrix} Gu \\ u \end{pmatrix}$ with u in L^2. We want to find $v_n \in C_c^\infty(0,1)$ such that $v_n \to Gu$, $Dv_n \to u$. We can take $u_n \in C_c^\infty(0,1)$ such that $u_n \to u$. Then $Gu_n \to Gu$, so $\begin{pmatrix} 1 \\ D \end{pmatrix} Gu_n \to \begin{pmatrix} G \\ 1 \end{pmatrix} u$. Thus we can suppose $u \in C_c^\infty(0,1)$.

Thus we have to see why we can approximate $\begin{pmatrix} Gu \\ u \end{pmatrix}$, u compact support by $\begin{pmatrix} Gu_n + c_n r \\ u_n \end{pmatrix}$ where both have compact support. Now if u has compact support, then Gu has to be a multiple of r near the ends.

Repeat: We want to see why $\mathrm{Im}\begin{pmatrix} G \\ 1 \end{pmatrix} + \mathbb{C}\begin{pmatrix} r \\ 0 \end{pmatrix} = \Gamma$ is the closure of $\mathrm{Im}\begin{pmatrix} 1 \\ D \end{pmatrix}$ on C_0^∞. The direction $\begin{pmatrix} 1 \\ D \end{pmatrix} C_0^\infty \subset \Gamma$ is obvious.

Look at whether $\begin{pmatrix} r \\ 0 \end{pmatrix} \subset \overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C_0^\infty}$. Actually we probably want to show that a smooth function starting at 0 and ending with r is in this closure, and also the reverse. So there ought to be estimates required at the two ends, just explicit construction. For example suppose f is  and consider $u_\varepsilon = f r^{1+\varepsilon}$ where $\varepsilon \downarrow 0$.

Then
$$Du_\varepsilon = p\,\partial_x r^{-1}(r^{1+\varepsilon} f)$$
$$= p\,\partial_x r^\varepsilon f = r^\varepsilon\left(p\,\partial_x + \varepsilon\, p\frac{r'}{r}\right) f$$

To the right where $f=1$, this is $r^\varepsilon \, \varepsilon\left(p\frac{r'}{r}\right)$, so if $p\frac{r'}{r}$ is bounded, Du_ε converges as $\varepsilon \downarrow 0$.

More details are needed but it seems likely that the condition that p kill the singularity of $\frac{r'}{r}$ should suffice to show fr and $(1-f)r$ can be approximated by fns. in C_0^∞ such that when D is applied one has convergence.

Now given $\begin{pmatrix} Gw \\ w \end{pmatrix}$ with $w \in L^2$, one can assume to show this belongs to $\overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C_0^\infty}$, that $w \in C_0^\infty$. Then Gw agrees with a multiple of r at either end. Thus we want to show that any pair $\begin{pmatrix} u \\ w \end{pmatrix}$ with $w \in C_0^\infty$ and $Du = w$ lies in the closure $\overline{\begin{pmatrix} 1 \\ D \end{pmatrix} C_0^\infty}$. Now within this closure are pairs $\begin{pmatrix} fr \\ D fr \end{pmatrix}$, $\begin{pmatrix} r \\ 0 \end{pmatrix}$ of the sort that we can alter $\begin{pmatrix} u \\ w \end{pmatrix}$ without the condition that $w \in C_0^\infty$ being changed, and assume that $u \in C_0^\infty$, whence we are done.

<u>Summary</u>: Although the above arguments

are not complete it is clear that I have to examine both endpoints in order to conclude that

$$\overline{\begin{pmatrix}1\\D\end{pmatrix}C_0^\infty} = \overline{\begin{pmatrix}G\\1\end{pmatrix}C_0^\infty} + \mathbb{C}\begin{pmatrix}r\\0\end{pmatrix}$$

Thus it is not unreasonable to expect the conditions that $\frac{1}{p} \notin L^2$ at either endpoint to be used in establishing this point.

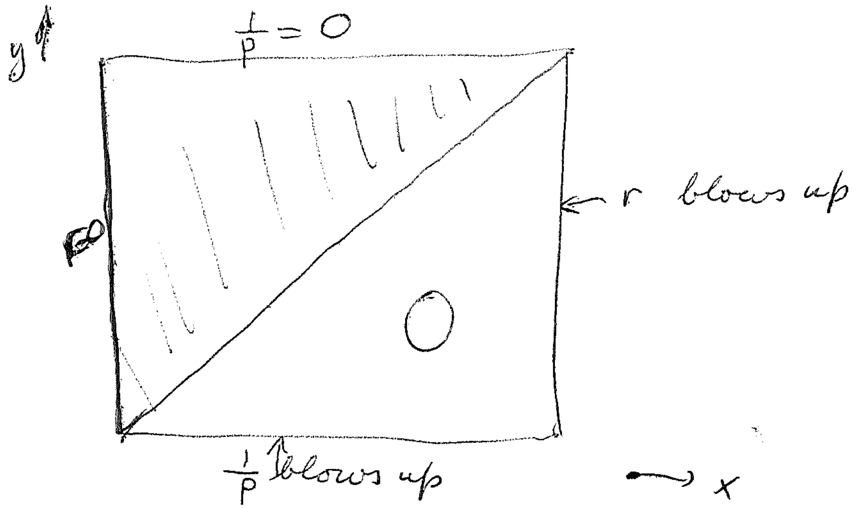
So the main thing to examine now is why G is a ψDO.

To simplify let's put ourselves in an invertible situation. Suppose $r(x)$ is smooth for $x<1$ vanishes for $x \leq 0$ positive on $(0,1)$ and has a pole as $x \uparrow 1$. Assume $\frac{1}{p(x)}$ has the opposite behavior, i.e. smooth for $x>0$, positive on $(0,1)$, zero for $x \geq 1$, a pole as $x \downarrow 0$. Then we take

$$G(x,y) = \begin{cases} -r(x)\dfrac{1}{p(y)} & x<y \\ 0 & x>y \end{cases}$$

$$= r(x)\frac{1}{p(y)}\left[\theta(x-y)-1\right]$$

Picture



Notice that what doesn't work (NO) is to try to obtain $G(x,y)$ as the product of a smooth function and the discontinuous function $\theta(x-y)-1$. At least this doesn't seem to work as $r(x)\frac{1}{p(y)}$ has problems along $y=0$. Precisely suppose that $F(x,y)$ is smooth and

$$F(x,y)\left[\theta(x,y)-1\right] = r(x)\frac{1}{p(y)}\left[\theta(x,y)-1\right].$$

This implies $F(x,y) = r(x)\frac{1}{p(y)}$ for $x \leq y$. Thus a natural question is whether the function defined on the closed half plane $x \leq y$ by

$$\begin{cases} r(x)\dfrac{1}{p(y)} & \begin{matrix} 0<y<1 \\ 0<x<1 \end{matrix} \\ \\ 0 & \text{Otherwise} \end{cases}$$

admits a smooth extension to the plane.

If we make a change of variable, then we want to know whether

$$* \qquad r(x)\left(\frac{1}{p}\right)(x+t)$$

defined for all x and $t \geq 0$ is smooth.

Slowly we are reaching by different routes the same problem. Here's another routes.

The Green's function $G(x,y)$ is an inverse for the differential operator $p\partial_x r^{-1}$. Note that the solution of $(p\partial_x r^{-1})u = f$ is

$$r(x)\int^{x}\frac{1}{p(y)}f(y)\,dy$$

However formally we produce the inverse as follows. Recall the formula

$$\frac{1}{A}B = B\frac{1}{A} + \left[\frac{1}{A},B\right]$$

$$= B\frac{1}{A} - \frac{1}{A}[A,B]\frac{1}{A}$$

$$= B\frac{1}{A} - [A,B]\frac{1}{A^2} + \frac{1}{A}B^{(2)}\frac{1}{A^3}$$

$$= \sum_{n\geq 0}(-1)^n B^{(n)}\frac{1}{A^{n+1}}$$

where $B^{(n)} = [A,\ldots,[A,B]\ldots]$ with n-A's. Thus

$$\frac{1}{\partial_x}\frac{1}{p} = \sum_{n\geq 0}(-1)^n\ \partial_x^n\!\left(\frac{1}{p}\right)\cdot\frac{1}{\partial_x^{n+1}}$$

Thus the full symbol of $(p\partial_x r^{-1})^{-1}$ as a ψDO is

$$\sum_{n\geq 0}(-1)^n\ r\cdot\partial_x^n\!\left(\frac{1}{p}\right)\ \frac{1}{(i\xi)^{n+1}}$$

Thus from two routes we reach the necessary condition that the functions

$$r\ \partial_x^n\!\left(\frac{1}{p}\right) \qquad n\geq 0$$

be smooth. This is at least for near the point $x=0$. However we might have ~~written~~ used a different change of variable and instead of $*$ p. 777 obtained

$r(y-t)\ \frac{1}{p}(y)$ should extend smoothly from $(y,t)\in\mathbb{R}\times\mathbb{R}_{>0}$ to $\mathbb{R}^2$

This leads to the condition

$$\partial_x^n(r) \cdot \frac{1}{\rho} \quad \text{smooth} \quad \forall n \geq 0.$$

Here is what we want to understand now. We have the formal expansions for the Green's function, better for the operator $r\,\partial_x^{-1} s$, where $s = \frac{1}{\rho}$. We want to really understand how these are related to the actual Green's function.

The formal expansions are

$$\boxed{\begin{aligned} \partial^{-1} s &= \sum_{n\geq 0} (-1)^n (\partial^n s)\, \partial^{-n-1} \\ r\,\partial^{-1} &= \sum_{n\geq 0} \partial^{-n-1} \circ (\partial^n r) \end{aligned}}$$

Proof of the second

$$\sum_n \partial^{-n-1} \circ (\partial^n r) \circ \partial = \sum_n \partial^{-n} \circ (\partial^n r) + \partial^{-n-1} \underbrace{[\partial^n r, \partial]}_{-\partial^{n+1} r} = r$$

Perhaps a mixture of the two is to be used.

The first expression leads to

(a) $$r\,\partial^{-1} s = \sum_{n\geq 0} (-1)^n (r \cdot \partial^n s)\, \partial^{-n-1}$$

What has this to do with an actual Green's function for $s^{-1}\partial r^{-1}$ e.g.

(b) $$(Gf)(x) = -r(x) \int_x^1 s(y) f(y)\, dy \quad ?$$

We can compare (a) (b) as follows using the Taylor expansion for s.

$$-r(x)\int_x^1 s(y)f(y)\,dy \sim -\sum_{n\geq 0} r(x)(\partial^n s)(x)\int_x^1 \frac{(y-x)^n}{n!}f(y)\,dy$$

Now $\displaystyle \partial_x \int_a^x \frac{(x-y)^n}{n!} f(y)\,dy = \int_a^x \frac{(x-y)^{n-1}}{(n-1)!} f(y)\,dy \qquad n>0$

so $\displaystyle \partial_x^{n+1} \int_a^x \frac{(x-y)^n}{n!} f(y)\,dy = \partial_x \int_a^x f(y)\,dy = f(x)$

Thus we see the above expansion for $(Gf)(x)$ is just $\sim\sum(-1)^n(r\partial^n s)\,\partial^{-n-1}$

using $\displaystyle\int_1^x \frac{(x-y)^n}{n!}f(y)\,dy$ for $\partial^{-n-1}f$.

To proceed further we need to recall Taylor's theorem with remainder. Let

$$g(x) = \int_a^x \frac{(x-y)^n}{n!} f^{(n+1)}(y)\,dy$$

From the above we know $g^{(n+1)} = f^{(n+1)}$ so that $g(x)-f(x)$ is a poly of degree n, and it's clear that at $x=a$ g vanishes to order n. Thus $f-g$ is the Taylor polynomial or

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \overbrace{\int_a^x \frac{(x-y)^n}{n!} f^{(n+1)}(y)\,dy}^{g}$$

This can be verified by integrating g by parts.

So now it is time to use this somehow

May 25, 1987

Fast summary of points learned

We know $\gamma^1 h\partial_x + \gamma^2 \frac{g-1}{g+1}$ doesn't work. We wish to replace it by an operator
$$\gamma^1 h\{\text{"}p\partial_x r^{-1}\text{"}\} + \gamma^2 \frac{q}{r}$$
where p, q, r are functions of g. ("$p\partial_x r^{-1}$" is perhaps a skew-adjoint version of the operator $p\partial_x r^{-1}$). The problem is to find p, q, r.

1) We learned that $\frac{q}{r}$ is to define a map of degree 1 from $\mathbb{T}$ to $P^1\mathbb{R}$, i.e. (r, q) form a section of the Mobius line bundle on $\mathbb{T}$, so they can't be real if they are smooth.

2) The $h \to 0$ asymptotics should show that the index density is $\mathrm{tr}(e^{uX^2}\sqrt{u}\,dX)\frac{i}{\sqrt{\pi}}$ where $X = \frac{q}{r}$.

3) Smoothness on the symbol level leads to
$$\frac{1}{pi\xi r^{-1} + qr^{-1}} = \frac{r}{pi\xi + q} = \frac{r}{p}\frac{1}{i\xi}\left\{1 - \frac{q}{p}\frac{1}{i\xi} + \left(\frac{q}{p}\frac{-1}{i\xi}\right)^2 + \cdots\right\}$$
requirements that $\frac{rq^n}{p^{n+1}}$ be smooth. (see 739)

We decided to try to construct a smooth example of index 1, where g is a "blip". The subspace corresponding to the C.T. is $\mathrm{Im}\begin{pmatrix} G \\ 1\end{pmatrix} + \mathbb{C}\begin{pmatrix}\text{zero mode} \\ 0\end{pmatrix}$, where G is a Green's function, see 770. The hope is that G is ΨDO on the circle.

4) We learned that we have to carry out our constructions using ΨDO methods. Formal expansions for $r\partial^{-1}$'s.

Let us consider $p\partial_x + q$ on the line where $p=0$ for $x \leq 0$ and $p>0$ for $x>0$ and where $q = -1$. Let φ be the solution of the homogeneous equation

$$p\varphi' + q\varphi = 0$$

so

$$\varphi = e^{-\int_1^x \frac{q}{p}} = e^{-\int_x^1 \frac{1}{p}}$$

~~We suppose~~ Then φ is smooth $=0$ for $x \leq 0$ and >0 increasing for $x>0$.

Next we form the Green's function for $p\partial_x + q = p\varphi\,\partial_x\,\varphi^{-1}$. The solution of

$$p\varphi\,\partial_x\varphi^{-1}u = f$$

is

$$u(x) = \begin{cases} \varphi(x)\displaystyle\int_1^x \frac{1}{(p\varphi)(y)} f(y)\,dy & \text{for } x>0 \\ -f(x) & \text{for } x \leq 0 \end{cases}$$

and I claim that u is smooth. Note that

$$p\varphi' = -q\varphi \qquad \text{so} \qquad p\varphi = -q\frac{\varphi^2}{\varphi'} = +q\left[(\varphi^{-1})'\right]^{-1}$$

So

$$u(x) = -\varphi(x)\int_1^x [\varphi^{-1}(y)]'\, q(y)^{-1} f(y)\,dy$$

$$= +\frac{\int_x^1 [\varphi^{-1}(y)]'\, f(y)\,dy}{\int_x^1 [\varphi^{-1}(y)]'\,dy} \times \underbrace{}_{\varphi(x)\left(\frac{1}{\varphi(1)} - \frac{1}{\varphi(x)}\right)}$$

Now the arguments will be that the measure (probability) given by

$$\frac{\chi_{[x,1]}(y)\,[\varphi^{-1}(y)]'\,dy}{\int_x^1 [\varphi^{-1}(y)]'\,dy}$$

is converging to the delta function $\delta(x)$. This is clear and so we see that $u(x) \longrightarrow -f(0)$ as [redacted] $x \downarrow 0$. Thus $u(x)$ is continuous, and the hope is that it is actually smooth at 0.

If this all works then what happens is that we obtain an inverse H for the operator

$$p\partial_x + q$$

on smooth functions, i.e. this operator is hypoelliptic. Notice that if you wish to show that

$$\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix} C^\infty = \begin{pmatrix} G \\ 1 \end{pmatrix} C^\infty$$

then you do have to show that $p\partial_x + q$ maps C^∞ onto C^∞, and hence that such an H ought to exist.

The basic problem is that <u>H is not a ψDO.</u> [redacted] To see this try to find an asymptotic expansion for the symbol. It has to be 1 for $x < 0$ and then jump to

$$\frac{1}{pi\xi + q} = \frac{1}{pi\xi}\left(1 - \frac{q}{pi\xi} + \cdots\right)$$

for $x > 0$.