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April 27, 1987

Consider the torus example

$$f(x,y) = \sum_n e^{iny}\, u(x-n)$$

$$\nabla_x = \partial_x \longleftrightarrow \partial_t$$

$$\nabla_y = \partial_y - ix \longleftrightarrow -it$$

and the $\bar\partial$ operator $a\nabla_x + ib\nabla_y \longleftrightarrow a\partial_t + bt$ on $L^2(\mathbb{R})$

Question: Can one take a suitable limit and see that the graph of the $\bar\partial$ operator on $L^2(\mathbb{R})$ approaches the graph of an operator on $L^2(S^1, \mathcal{M})$, where $\mathcal{M}$ is the family of low energy states for ∇_y?

Recall that $L^2(S^1, \mathcal{M}) \longleftrightarrow L^2(-\frac{1}{2}, \frac{1}{2})$. On $L^2(-\frac{1}{2}, \frac{1}{2})$ we have the operator

$$(*) \qquad h\partial_t + \tan(\pi t).$$

Here $g = e^{2\pi i x}$ is the loop and

$$\frac{1}{i}\frac{g-1}{g+1} = \frac{1}{i}\frac{e^{2\pi i x}-1}{e^{2\pi i x}+1} = \tan(\pi x),$$

The ground state for $(*)$ is found by solving

$$\big(h\partial_t + \tan(\pi t)\big)\, u(t) = 0$$

$$-\int \frac{1}{h}\tan(\pi t)\,dt = \frac{1}{h\pi}\log(\cos(\pi t))$$

$$\therefore \quad u(t) = (\cos \pi t)^{1/h\pi}$$

So there is perhaps a possibility, but in the end we have to find a precise link, before we can claim to have an approximation.

Let's begin by getting the constants straight. I want to take $a = 1$ and to let $b \to \infty$. Thus I am considering the graph of the operator $\partial_t + bt$ on $L^2(\mathbb{R})$. Now I change $t \to x$, so I have $\partial_x + bx$, and the ground state is $e^{-\frac{1}{2}bx^2}$.

On the other side we have the operator $\partial_x + c\pi \tan(\pi x)$ which has the ground state

$$(\cos(\pi x))^c \sim \left(1 - \frac{(\pi x)^2}{2}\right)^c \sim e^{-\frac{c\pi^2}{2}x^2}.$$

Thus we want $c = b/\pi^2$ and so we want to consider

$$\partial_x + b\frac{1}{\pi}\tan(\pi x)$$

acting on $L^2(-\frac{1}{2}, \frac{1}{2})$.

In general how might we hope to proceed? In the above example it is sort of clear that the two ground states have the same shape for large b.

$$\cos(\pi x)^{b/\pi^2} \sim e^{-\frac{1}{2}\pi^2 x^2 \, b/\pi^2} = e^{-\frac{1}{2}bx^2}$$

April 29, 1987

Instead of $\partial_x + bx$ we can consider

$$e^{-bf}\,\partial_x\, e^{bf} = \partial_x + bf'$$

which kills e^{-bf}. Thus we want to study the graph of this operator for large b. This is related to the approach to Morse theory described by Witten, where he modifies the deRham complex in a similar way.

I want to try to discuss the sort of questions to be answered. Basically as I vary f and b I get different subspaces, namely the graphs of the operators. If I insist that f has only a critical point $x=0$ and $f(x) = \frac{1}{2}x^2 + O(x^3)$ there, then for large b these subspaces are equivalent.

In Witten's theory one looks at the spectrum and sees that it is divides into zero and high energy, i.e going to ∞ with b. I ought to understand what is happening on the level of the graphs.

It might be useful to understand exactly how Melrose and others prove the Witten result. How does one show that the local critical point model gives an upper bound for cohomology?

Witten's idea is to use the modified DR complex with $d \longrightarrow e^{-bf} d\, e^{bf}$ which has the same cohomology. Then he looks at the low energy complex: $\Delta_b < 1$ for large b and shows that in dim q it has dim $=$ no. of critical points of index q.

April 30, 1987

Review of some of the ideas about the problem of coupling a loop $g: S^1 \to U(V)$ to the Dirac ∂_x on S^1.

The first idea is that the result is to be a subspace of $\mathbb{C}^2 \otimes L^2(S^1, V)$, more precisely an involution on this Hilbert space congruent to $-\varepsilon$ mod $\mathcal{L}^p$ $p > 1$. I want above all to have a smooth mapping from the loop group to the restricted Grassmannian.

The original candidate was to form the sum of unbounded operators on $L^2(S^1, V)$

$$\partial_x + i\,\frac{g-1}{g+1}$$

and take its graph. This operator is not densely defined; its domain is contained in the image (closed) of $g+1$ acting on $L^2(S^1, V)$. The problem with this construction is that already when g is a constant loop, the associated subspace doesn't depend smoothly on g at the point $g = -1$. Formulas: $g \in S^1$

$$z = \xi + \frac{g-1}{g+1} \qquad \frac{1}{z} = \frac{g+1}{\xi(g+1) + g - 1}$$

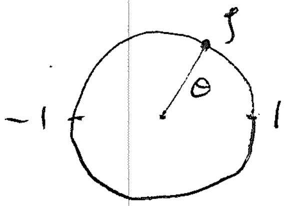
$$\partial_g\left(\frac{1}{z}\right)\Big|_{g=-1} = \frac{1}{-2} \qquad \text{not an } \ell^p \text{ sequence in } \xi.$$

Another idea is to follow Kasparov's construction of the cup product. Here we choose a function $\rho(\xi)$ which is zero at -1 and 1 at $\xi = 1$, and we define an operator on the image of $\rho(g)$ on $L^2(S^1, V)$.

What is the function of this operator $p(g)$?

It removes from our consideration the eigenvalues near $\varsigma = -1$, and what this means is that the operator has been defined to be infinite not just where $g = -1$ but near $g = -1$.

The simplest continuous choice for p, q is



$$p(\varsigma) = \cos\frac{\theta}{2} \qquad \theta \in (-\pi, \pi)$$

$$q(\varsigma) = \sin\frac{\theta}{2} \qquad \theta \in (-\pi, \pi)$$

Then

$$\frac{q(\varsigma)}{p(\varsigma)} = \tan\frac{\theta}{2} = \frac{1}{i}\,\frac{e^{i\theta}-1}{e^{i\theta}+1} = \frac{1}{i}\,\frac{\varsigma-1}{\varsigma+1}$$

Thus we have

$$\left(\partial_x + \frac{1}{i}\,\frac{g-1}{g+1}\right) p(g) = \partial_x\, p(g) + \frac{1}{i}\, q(g).$$

Also we have

$$p(\varsigma) = \frac{\varsigma^{1/2} + \varsigma^{-1/2}}{2} = \varsigma^{-1/2}\left(\frac{\varsigma+1}{2}\right)$$

so

$$p(g) = \frac{g+1}{2}\, g^{-1/2} \implies \operatorname{Im} p(g) = \operatorname{Im} (g+1)$$

where one chooses a branch of $\varsigma^{1/2}$.

This is nice because it suggests generalizations. I can hope to find a construction based on certain kinds of p, q.

The next idea comes from the example of the operators

$$\partial_x + bx$$

$$\partial_x + bf' = e^{-bf}\partial_x e^{bf}$$

where $f(x) = \frac{1}{2}x^2 + O(x^3)$. For b large Witten's approach to Morse theory shows that the low energy part of the Dirac constructed from this operator and its adjoint are the same.

So it seems that what I need is an approximation theorem. It should say something about $\partial_x p \pm iq$ as the cutoff is removed.

Think as follows: A choice of p, q allow me to replace $\partial_x \pm \frac{1}{i}\frac{g-1}{g+1}$ by a cut-off operator $\partial_x p \pm iq$, which operates on a subspace. I should be able to remove the cutoff, at least when $g = \frac{1+x}{1-x}$, and have an approximation theorem for the graphs.

May 1, 1987

I propose to try to find a formula such as $\partial_x \cdot p \pm q$ where p, q are functions of the loop g. This operator will be defined on the image of p and we take its graph to obtain an elt of the restricted Grassmannian representing the pairing of g with the fundamental K-homology class of the circle.

In order to make any of this work I have problems of analysis to solve. I have to be able to describe, or at least control, an operator defined on the image of p. I feel that a prototype for the sort of question I am concerned with is the following. Consider the torus $S^1 \times S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ the Poincaré line bundle over it whose smooth sections are

$$f(x,y) = \sum_n e^{iny} u(x-n)$$

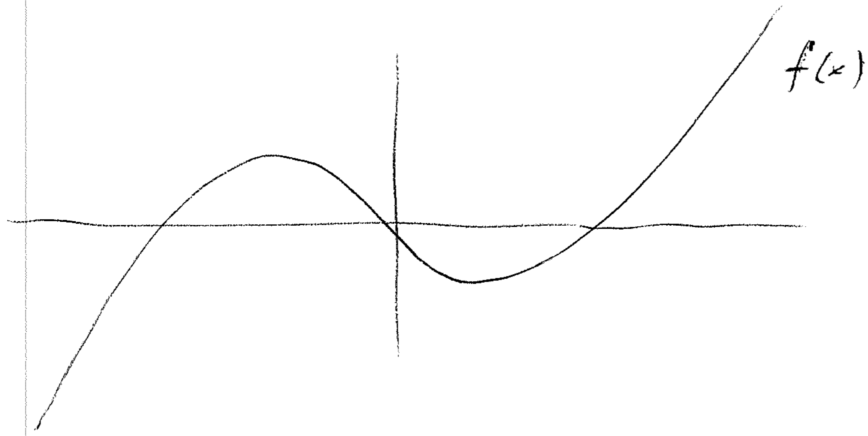
where $u \in \mathcal{S}(\mathbb{R})$. Then

$$\nabla_x = \partial_x \quad \longleftrightarrow \quad \partial_t$$

$$\nabla_y = \partial_y - ix \quad \longleftrightarrow \quad -it$$

The idea here is to look at the low energy ~~states~~ for ∇_y in the fibre over $x \in \mathbb{R}/\mathbb{Z}$. This is a well-defined line bundle except at $x = \frac{1}{2}$. One should be able to study the relation between the Dirac on the torus $D_x \pm bi\nabla_y$ and the subspace of sections of the Poincaré line bundle which are low energy in the y direction.

So we know this means we want to look at $\partial_t \pm bt$ on $\mathcal{S}(\mathbb{R})$ and the subspace of sections supported in $(-\frac{1}{2}, \frac{1}{2})$. But this is part of a more general question, namely suppose we were to take $\partial_x \pm b f(x)$ operating on some interval where $f(x) \to \pm\infty$ fast at the end points. Picture



Then we want to link the operator with the subspace of sections supported where $|f(x)| < 1$.

Witten's theory tells us that things are nice for large b in the following sense. The spectrum splits into a low-lying part which is concentrated near the zeroes of f, and which can be described approximately by sections supported where $|f(x)| < 1$.

At this point it appears that we are getting close to the situation encountered in quantum mechanics, where you perturb from a situation consisting of light + heavy objects, where the mass of the heavy particle goes to infinity. Thus we want to look at $i^* \frac{1}{\lambda - H} i$ where i is the inclusion of the light (low energy) particles.

May 2, 1987

Yesterday I decided that before any progress could be made it is necessary to understand how the operator $\partial_x \pm x$ on the line is to be approximated by an operator on the interval $(-\delta, \delta)$. I want to try to explain why.

The problem is to assign to a loop $g: S^1 \to U(V)$ a point ~~in~~ a restricted Grassmannian, in such a way that we get the graph of $\partial_x \pm \frac{1}{i} X$ when $g = \dfrac{1+X}{1-X}$.

It seems one has to proceed as follows. When $g = -1$ we have to remove the -1 eigenspaces at least. This gives a subspace of $L^2(S^1, V)$. On this subspace we can hope to define an operator whose graph will then be extended suitably to obtain the required subspace of $\mathbb{C}^2 \otimes L^2(S^1, V)$.

My intuitive idea is that there should be a meaningful way to define something like $\partial_x \pm \frac{1}{i} \frac{g-1}{g+1}$ on functions $s: S^1 \to V$ which stay well out of the -1 eigenspaces for g.

If I take $V = \mathbb{C}$, then I want to consider a differential operator on intervals where g lies in an interval near 1 i.e. $Re(g) > 0$. This means that the first case to understand is $\partial_x + x$ on a small interval around zero. Is there a natural self-adjoint boundary value problem?

May 3, 1987

Yesterday I concluded that I need to understand the sort of operators on $L^2(-\delta,\delta)$ which are relevant to my problem. I know from the Atiyah-Singer index theory that there is a natural class of Fredholm operators all having index 1. These are ΨDO's of order 0 whose symbol is the canonical generator in K-theory of the cotangent bundle. But I can't even write one of these down at the moment.

Generically any of these operators has a one dimensional kernel. This "ground state" should be more accessible than the whole spectrum of the operator plus its adjoint. So it seems worthwhile to actual exhibit in a concrete way an example of the operators of interest along with its ground state.

What I propose to do is to start with a loop of winding number 1, something like the blip used by Graeme to define the vertex operators. This is a loop $g: S^1 \longrightarrow U(1)$ which is 1 except near $0 \in S^1$ where it winds rapidly around the circle. I can use this loop to define the symbol of a ΨDO of order zero on $L^2(S^1)$; it essentially the Toeplitz operator T_g. Then the problem becomes how to localize it near 0.

Thus the problem is to take a loop g which is 1 everywhere except a small interval near zero, and then somehow relate the Toeplitz operator T_g to an operator which is supported in this interval. This means that it should be the direct sum of an operator on L^2(interval) and the identity on the orthogonal complement.

Because the Hardy projection isn't local it is likely that T_g is not such a direct sum. As mentioned above we should focus our attention first on the ground state.

Recall that the ΨDO of order 0 associated g has the symbol given by $\begin{cases} g & \xi = 1 \\ \pm 1 & \xi = -1 \end{cases}$

and an explicit way to realize this by an operator is $\pm(1-P) + gP$, where P is projection on the Hardy space H_+. The kernel of this operator consists of $f = f_- + f_+$ such that

$$\pm(f_-) + g f_+ = 0$$

(It seems we perhaps should choose the $-$ sign, so the kernel consists of $f = f_+ + f_-$ such that

$$g f_+ = f_- .$$

Moreover the operator is -1 for $g = -1$.)

The important point is that the "ground state" is related to the factorization of Birkhoff. Normally this is done by taking the logarithm of the loop and splitting the Fourier series into the positive and negative frequency parts. Even though the loop has small support, the positive and negative frequency parts tend to be supported everywhere.

Recall how Graeme described the projections onto the holom. inside and outside pieces. Given f on S^1, or more generally a simple closed curve γ one forms the Cauchy integral

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi$$

This defines a holom. function $f^+(z)$ for z inside of γ and another $f^-(z)$ for z outside γ. The latter vanishes at $z=\infty$. Cauchy's thm. says the operators $f \to f^+, f^-$ are projectors.

So starting with the loop g on S^1, suppose it is of degree 0, we apply the Cauchy integral as above to obtain

$$h^\pm(z) = \frac{1}{2\pi i}\int \frac{\log g(\xi)}{\xi - z}\, d\xi$$

We can see from this how non-localized the factorization of g will be.

Now we know that the symbol of this operator is supported in a nbd of where $g \neq 1$. There has to be a different realization of this operator which is the identity for functions on S^1 which are supported outside of a nbd of where $g \neq 1$.

Let's try to describe operators using kernels. P the Hardy projector has the kernel

$$P(z,\xi) = \sum_{n \geq 0} z^n \xi^{-n}$$

so $1-P+gP$ has the kernel

$$\delta(z,\xi) + (g-1)(z)\,P(z,\xi)$$

As long as $z \neq \xi$ we have

$$P(z, \zeta) = \frac{1}{1 - z/\zeta}$$

however one of the first examples of the theory of distributions shows that there are different ways of interpreting this function defined for $z \neq \zeta$ as a distribution. We recall that $\frac{1}{x}$ on the line gives rise to distributions

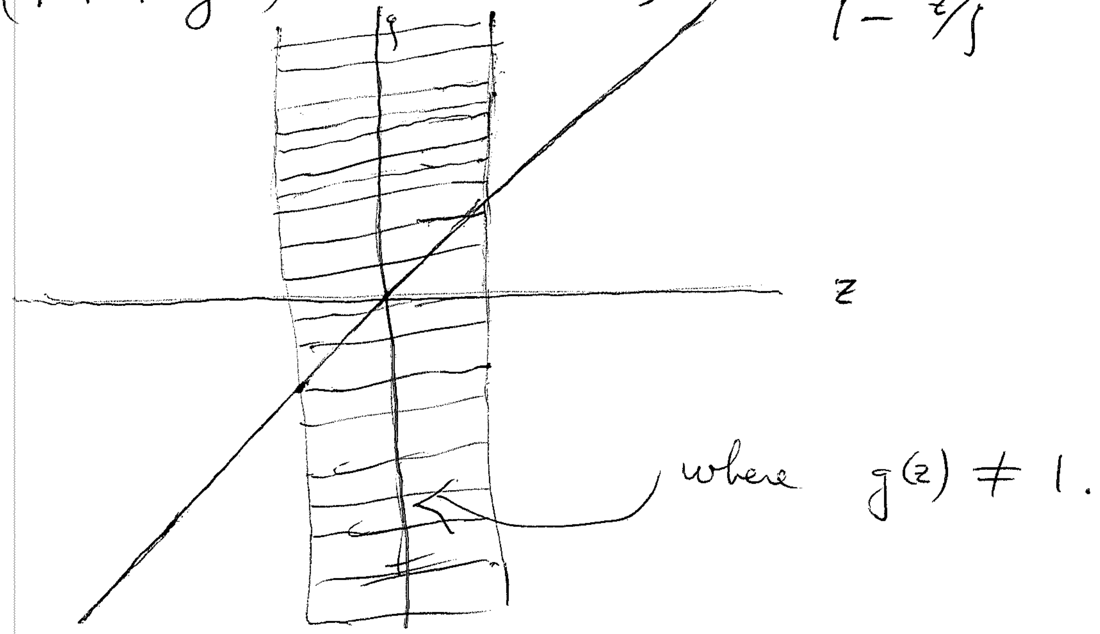
$$\frac{1}{x + i0^+} \qquad \frac{1}{x - i0^+}$$

whose average is the Cauchy PV and whose difference is a multiply of $\delta(x)$.

So we have to be a bit careful about treating these kernels as functions.

Draw a picture of the kernel

$$(1 - P + gP)(x, z) = \delta(z, \zeta) + \frac{g(z) - 1}{1 - z/\zeta}$$



This picture shows that the operator $1 - P + gP$ is not the direct sum of an operator on the support of $g - 1$ and the identity on the complement.

Now, I think it might be worthwhile

to see if it is possible to construct a ΨDO with the desired symbol such that it is entirely supported in an interval around O and such that the ground state can be found. I might even be able to get calculated.

$pF \pm iq$ or $p^{1/2}Fp^{1/2} \pm iq$

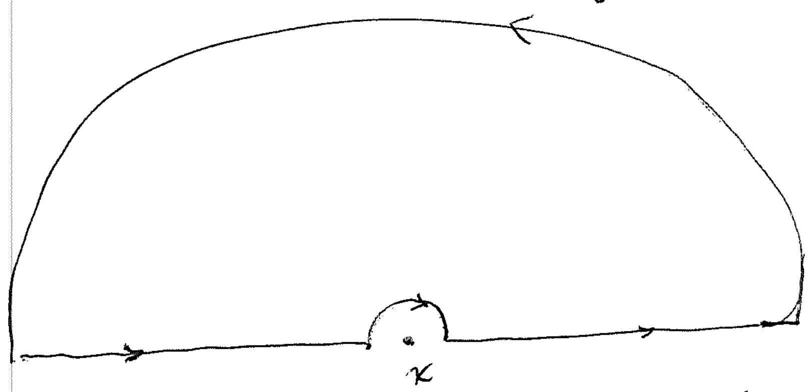
May 4, 1987

Problem: Construct explicitly a ΨDO (of index 1) on the line or circle which is the direct sum of an operator on an interval with the identity outside and such that the ground state can be found. You might want to start with a candidate for the ground state and then construct the operator.

Let us work on the line. Recall the Cauchy P.V operator

$$PV\,\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1}{x-y}\,u(y)\,dy$$

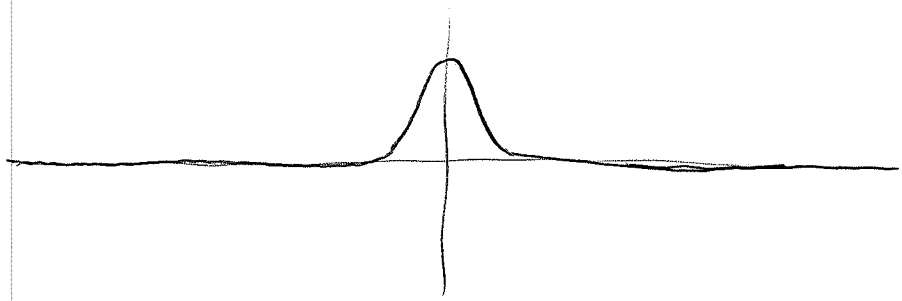
If $u \in H^+$ i.e. analytic in the UHP and decaying, then we can evaluate this by contour integration



and we see that the above integral $-$ residue $=0$, where the residue is $u(x)$. Similarly if $u \in H^-$ we see the integral $+$ residue $(u(x)) = 0$. Thus we see that

$$Fu(x) = PV \frac{1}{\pi i} \int \frac{1}{x-y} u(y)\,dy$$

Let us now take $u(x)$ to be a smooth function with compact support looking like



and symmetric $u(x) = u(-x)$. We know from the Fourier transform picture that Fu is smooth. We have

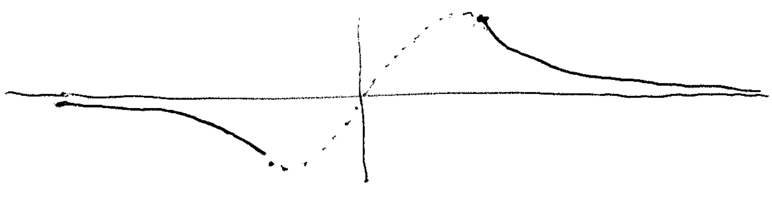
$$i(Fu)(x) = PV \frac{1}{\pi} \int \frac{1}{y} \overbrace{u(x-y)}^{u(y-x)}\,dy$$

First note that for $|x|$ large we have

$$(iFu)(x) = \frac{1}{\pi} \int \left[\frac{1}{x} + \frac{y}{x^2} + \frac{y^2}{x^3} + \cdots \right] u(y)\,dy$$

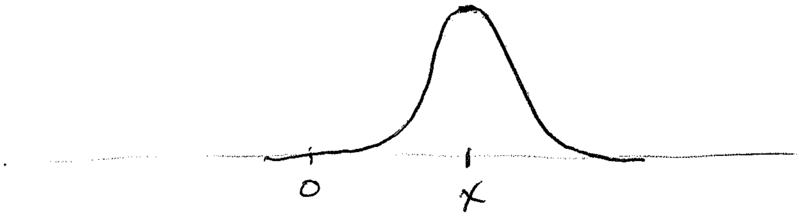
$$= \sum_{k \geq 0} \frac{a_k}{x^{k+1}} \qquad a_k = \frac{1}{\pi} \int y^k u(y)\,dy = \begin{cases} 0 & k \text{ odd} \\ >0 & k \text{ even} \end{cases}$$

This power series converges for $|x|$ outside the support of u. It shows that iFu has the shape

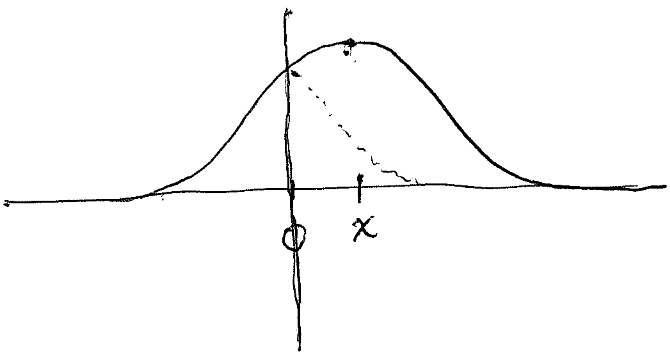


except for the dotted region.

Next note that $u(y-x)$ has the graph obtained by shifting the graph of u a distance x.



Better picture



The important thing is that for $x>0$, if we reflect $u(y-x)$ for $y<0$ around $y=0$, then we obtain something below $u(y-x)$ for $y>0$. This shows that

$$(iFu)(x)\begin{cases} >0 & \text{for } x>0 \\ 0 & \text{for } x=0 \\ <0 & \text{for } x<0 \end{cases}$$

So now we want to find p, q such that $(ipF - q)u = 0$ and $p^2+q^2=1$. This equation implies

$$-p(iFu) + qu = 0$$

$$\frac{p}{q} = \frac{u}{iFu}$$

(Think of $-iFu$ as something like the derivative of u).

$$p = \frac{u}{\sqrt{u^2+(iFu)^2}} \qquad q = \frac{iFu}{\sqrt{u^2+(iFu)^2}}$$

It's clear we also get the ground state for $p^{1/2}(-iF)p^{1/2}+q$ namely $v = p^{-1/2}u$.

It is perhaps useful to view $-iF$ as an analogue of ∂_x and to emphasize the general fact that $\partial_x u$ vanishes to a lower order than u does. Thus as u goes to zero $\frac{u'}{u}$ can be expected to become infinite.

I wish to continue trying to produce a simple example of an index 1 operator on a finite interval. My idea is to consider an operator whose graph is the image of something like

$$\begin{pmatrix} r \\ p\partial_x + q \end{pmatrix}$$

where p, q, r are real functions of x. Where r is non-zero we can divide by it and reduce to

$$\begin{pmatrix} 1 \\ p_1\partial_x + q_1 \end{pmatrix}.$$

There is here an extra function to play with, since before we were concerned only with graphs of operators of the form $\partial_x + q$ with q real. The latter are related to transmission lines.

We have some feeling for the spectrum of an operator of the form

(*) $$\begin{pmatrix} 0 & +\partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$$

with f real. But now we are interested in a more general operator, namely

(+) $$\begin{pmatrix} 0 & \partial_x a - b \\ a\partial_x + b & 0 \end{pmatrix}.$$

It would be better to change $a\partial_x$ to $a^{1/2}\partial_x a^{1/2}$ $= a\partial_x + \frac{1}{2}a'$ which is skew-adjoint.

It seems necessary to think intrinsically. What is the Dirac operator on the circle? It is obtained by starting from a metric. However the Dirac is supposed to have some kind of conformal invariance?

The question is whether (+) is equivalent to an operator (*) after reparametrization.

May 5, 1987

Let's consider half-densities $f(x)\,dx^{1/2}$ on the line. A vector field $a\partial_x$ acts on a half density by the rule

$$\mathcal{L}_{a\partial_x}\left(f\,dx^{1/2}\right) = \left(a\partial_x + \tfrac{1}{2}a'\right)f \quad dx^{1/2}$$

Note that $\quad a\partial_x + \frac{1}{2}a' = a^{1/2}\,\partial_x\, a^{1/2} \quad$ (when $a>0$) showing this operator is (formally) skew-adjoint on functions for the usual L^2 norm.

Now suppose one is given a vector field $a\partial_x$ and a function b on the line. This function determines a multiplication operator on half-densities which is self-adjoint when b is real. We can form the operator

$$\begin{pmatrix} 0 & a\partial_x + \frac{1}{2}a' - \bar{b} \\ a\partial_x + \frac{1}{2}a' + b & 0 \end{pmatrix}$$

which is skew-adjoint.

Let's suppose $a>0$. Then we can define a new coordinate y on the line such that

$$dy = \frac{dx}{a} \qquad \text{or} \qquad \partial_y = a\,\partial_x$$

Let's also put $\quad f(x)\,dx^{1/2} = g(y)\,dy^{1/2} \quad$ so that

$$\left(a^{1/2}f\right)(x) = g(y). \qquad \text{Then}$$

$$a^{1/2}\left(a^{1/2}\partial_x\, a^{1/2}\right)f = (a\,\partial_x)\left(a^{1/2}f\right) = \partial_y g$$

and $\quad \int f(x)^2 dx = \int g^2\, dy. \quad$ Thus it is clear that in the new coordinate we have the operator

$$\begin{pmatrix} 0 & \partial_y - \bar{b} \\ \partial_y + b & 0 \end{pmatrix}$$

The conclusion is therefore that we don't get anything really new provided $a > 0$.

Recall that I still want to exhibit a simple example of an index one operator on a small interval. I am looking at the possibilities as differential operators. I see that these are closely related to transmission lines and Krein strings.

I have just seen how to treat these intrinsically namely you have a vector field and a function and you combine them to act on half-densities. Notice that right on the surface one has the "ground state", by which I mean the kernel of the operator

$$a\partial_x + \tfrac{1}{2}a' + b.$$

Now the program has to be to find a clean description of what happens for these operators.

First observation: suppose the vector field $a\partial_x$ has a simple zero at $x=0$. Say $a(x) = cx + O(x^2)$ where $c > 0$. Then $y = \int^x \frac{dx}{a}$ will go to $-\infty$ as $x \downarrow 0$. In fact this is true even when a vanishes to higher order. This means that we probably don't have to consider a finite interval, better, a finite length string.

Second observation is that the modified DR operator considered by Hormander + Witten is

$$\begin{pmatrix} 0 & \partial_x - tf' \\ \partial_x + tf' & 0 \end{pmatrix} \qquad e^{-tf}\partial_x e^{tf} = \partial_x + tf'.$$

This probably means that we want b to have at least linear growth at ∞.

Next I wish to make a systematic study of the operator

$$\begin{pmatrix} 0 & \partial_x - b \\ \partial_x + b & 0 \end{pmatrix}$$

acting on $L^2(\mathbb{R})^{\oplus 2}$. I want to study the graph of $\partial_x + b$ to see what restricted Grass it belows to. And I want to work out all of the analysis completely. This means to show the operator is essentially skew-adjoint with discrete spectrum, etc. The way to approach this is to produce a parametrix, even better, a Green's function for $\partial_x + b$.

So then, let's consider trying to invert the operator $\partial_x + b$. Let

$$F(x) = e^{-\int_0^x b\,dx}$$

be the solution of the homogeneous equation. I want to assume, in order to fix the ideas, that b is an odd function, real-valued, and is shaped roughly like x, so that $b'(x) > 0$ and $b \to \pm\infty$ as $x \to \pm\infty$.

Then $F(x)$ belongs to $L^2(\mathbb{R})$ and so $\partial_x + b$ will not be uniquely invertible. However from the example $b = x$ we can expect it to be onto. In fact one can solve

$$(\partial_x + b)\, u = f$$

by elementary means: $u(x) = F(x) \int^x F(y)^{-1} f(y)\, dy$

We have the Green's function

$$G(x,y) = \begin{cases} \dfrac{F(x)}{F(y)} & x > y \\ 0 & x < y. \end{cases}$$

Any other Green's function differs from this one by $F(x) \cdot$ function of y. So we can take

$$G(x,y) - \frac{F(x)}{F(y)} \underbrace{\left(1 - \theta(y)\right)}_{\substack{0 \quad y>0 \\ 1 \quad y<0}} = \begin{cases} \dfrac{F(x)}{F(y)} & x > y > 0 \\ 0 & x > y < 0 \\ & x < y > 0 \\ -\dfrac{F(x)}{F(y)} & x < y < 0 \end{cases}$$

which is the Green's function vanishing at $x = 0$. Thus this Green's function for $y > 0$ is

$$\begin{cases} \dfrac{F(x)}{F(y)} & x > y \\ 0 & x < y \end{cases}$$

and for $y < 0$ it is

$$\begin{cases} 0 & x > y \\ -\dfrac{F(x)}{F(y)} & x < y \end{cases}$$

What do I want to do with this Green's function? I would like to see it is Hilbert-Schmidt perhaps. Given an operator T we are interested in its graph $\mathrm{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$

When T is invertible this graph is the same as

$$\operatorname{Im}\begin{pmatrix} T^{-1} \\ 1 \end{pmatrix}$$

If I wish to show that $\operatorname{Im}\begin{pmatrix} 1 \\ T \end{pmatrix}$ belongs to the restricted Grassmannian consisting of subspaces congruent to $\operatorname{Im}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ mod $\mathcal{L}^p$, then its enough to see that $T^{-1} \in \mathcal{L}^p$.

In order to keep things simple, let us work on $0 \leq x < \infty$ with the vanishing at 0 boundary condition. Then

$$G(x,y) = \begin{cases} \dfrac{F(x)}{F(y)} & x > y > 0 \\[2ex] 0 & 0 < x < y \end{cases}$$

and the Hilbert Schmidt condition is

$$\iint G(x,y)^2 \, dx\, dy = \int_0^\infty dx \int_0^x \frac{F(x)^2}{F(y)^2}\, dy < \infty$$

$$= \int_0^\infty dx \; F(x)^2 \int_0^x \frac{dy}{F(y)^2}$$

I recall ~~seeing~~ this integral is logarithmically divergent for the harmonic oscillator case

Well now this is all very subtle, maybe too subtle for my purposes.

May 6, 1987

I eventually want to couple the Dirac on the circle to a unitary loop by means of a formula:

$$\mathrm{Im}\begin{pmatrix} r(g) \\ p(g)\partial_x + q(g) \end{pmatrix}$$

where p, q, r are functions on S^1 extended to unitaries via the spectral theorem. The natural question is whether this prescription can furnish for constant g a smooth map to the Grassmannian. Look at the map from $S^1 \times S^1$ to S^2 given by

$$(\xi, \zeta) \to \qquad z = \frac{p(\zeta)\, i\xi + q(\zeta)}{r(\zeta)}$$

and ask whether it is smooth. Let's worry about ξ near ∞

$$\frac{1}{z} = \frac{r}{pi\xi + q} = \frac{r(i\xi)^{-1}}{p + (i\xi)^{-1}q}$$

In order that this be smooth at $\xi = \infty$ I must be able to expand in a power series in ξ^{-1}. This has to be

$$r(i\xi)^{-1} \sum_{k \geq 0} \frac{1}{p}\left(\frac{-i\xi^{-1}q}{p}\right)^k$$

Now I know more or less that p will be >0 near $\zeta = 1$ and then zero in an interval containing $\zeta = -1$; also q will be $\neq 0$ where p is zero with a discontinuity at $\zeta = -1$. So smooth map from S^1 to S^1. The other condition is that

$$\frac{r(\zeta)}{p(\zeta)} \cdot \left(\frac{q(\zeta)}{p(\zeta)}\right)^k$$

will be smooth and vanish for all k at $\zeta = -1$. So we seem to be in exactly the same situation as studied before.

Next I want to gain a little more insight by considering the projector constructed by Loring. This is a 2×2 idempotent of continuous functions (or smooth fns.) on the torus $S^1 \times S$. which we think of as $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. Set $U = e^{iy}$. Lorings projector is $e =$

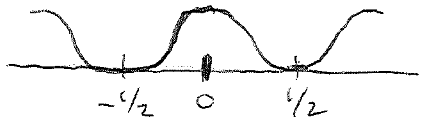
$$e \cdot e = \begin{pmatrix} f & g+hU^* \\ g+hU & 1-f \end{pmatrix}\begin{pmatrix} f & g+hU^* \\ g+hU & 1-f \end{pmatrix}$$

$$= \begin{pmatrix} f^2 + (g+hU^*)(g+hU) & g+hU^* \\ g+hU & (1-f)^2 + (g+hU)(g+hU^*) \end{pmatrix}$$

which gives an idempotent when

$$f^2 + g^2 + h^2 + ghU + hgU^* = f$$

i.e. when $\begin{cases} gh = 0 \\ f - f^2 = g^2 + h^2 \end{cases}$

He takes f :  and $g = \begin{cases} \sqrt{f-f^2} & x \in [0, \frac{1}{2}] \\ 0 & x \in [-\frac{1}{2}, 0] \end{cases}$

it is clear that in order to make sense of the above power as a smooth function of ξ that r must be divisible by p^N for all N.

Let's think of p as having a double zero at $\xi = -1$ and positive elsewhere. Also r will have an infinite order zero at $\xi = -1$ and be positive elsewhere. Then everything is smooth for $\xi \neq -1$. I seem to have treated the case ξ near infinity. Assuming this is OKAY we need only worry about $|\xi| \leq R$ and we may use the series

$$\frac{1}{z} = r \sum_{k \geq 0} \frac{(-pi\xi)^k}{q^{k+1}}$$

which will converge for ξ close enough to -1 so that $pR < 1$.

Actually I previously considered the map

$$(x, y) \longmapsto (x + iy)\, e^{x^2} = z$$

and showed this was smooth from $S^1 \times S^1 \longrightarrow S^2$. The above is

$$z = \frac{q(\xi) + p(\xi)\, i\xi}{r(\xi)}$$

$$= \left(\frac{q(\xi)}{p(\xi)} + i\xi \right) \cdot \frac{p(\xi)}{r(\xi)}$$

Now we agree that $\xi \longmapsto x = \dfrac{q(\xi)}{p(\xi)}$ should be a and $h = \begin{cases} 0 & [0, \frac{1}{2}] \\ \sqrt{f - f^2} & x \in [\frac{1}{2}, 1] \end{cases}$

We see this is a standard way of constructing a Bott map. One uses the normal Bott map $[0,\infty] \times \mathbb{T} \longrightarrow S^2$ to go from 0 to ∞ and returns by a constant (in $\mathbb{T}$) map.

We can like it to the above by the formula

$$z = \frac{g + hU}{f} = \frac{1-f}{g + hU^*}$$

But this is ~~[illegible]~~ apparently not in the same form.

May 7, 1987

I want now to tackle papers on bosonization on Riemann surfaces. References

Bost-Nelson: Spin 1/2 boson. on comp. surfaces. Phys. Rev. Lett 57 (1986) 795

Alv.-G., Moore, Vafa: θ fns, mod inv., strings CMP 106 (1986) 1

UGM {Alv.-G., Gomez, Reina; (CERN 87)
Ishibashi, Matsuo, Ooguri.

First point to observe is that "spin-1/2" refers to two fields c, b which are sections of a holomorphic line bundle ξ and $K\otimes\xi^{-1}$ respectively. The action is

$$\int b\bar{\partial}c \;+\; c.c.$$

which means that one is considering all the anti-holomorphic theory at the same time.

Apparently twisted spin bundle means any line bundle of degree $g-1$. Such a twisted spin bundle can be parametrized by theta characters (θ,ϕ) with respect to a choice of canonical homology basis. There is a θ function with characteristics

$$\vartheta\begin{bmatrix}\theta\\ \phi\end{bmatrix}\left(\int_y^t \omega \;\middle|\; \tau\right)$$

τ = period matrix

ω some basis of holom. 1-forms

Let's try now to go over Vafa's paper "Operator Formulation on Riemann Surfaces". The first paper is concerned with fermions, the second with bosons. He focuses on spin-$\frac{1}{2}$ fermions and spin-0 bosons.

Σ closed Riemann surface

p point of Σ

t analytic coordinate vanishing at p

D disk $|t| < 1$.

Next we suppose given a "twisted spin bundle" ξ over Σ i.e. a line bundle of degree $g-1$. One supposes that $\bar{\partial} : \xi \longrightarrow \xi \otimes T^{0,1}$ is invertible for simplicity.

The quantum field theory is supposed to be defined by a fermion functional integral over a space of classical fields c, b which are sections of ξ and $K \otimes \xi^{-1} = T^{1,0} \otimes \xi^{-1}$. Given a circle γ on the surface, oriented, one gets a pairing of sections of these line bundles over the circle γ by

$$\int_\gamma bc$$

This defines a Clifford algebra and leads to a Fock space which is a suitable semi-infinite exterior tensor space of the sections of ξ over γ. Thus we should think of c as giving exterior multiplication operators and b as giving interior mult. ~~contraction~~ operators.

Now one must describe the line in the

Fock space $\mathcal{F}$ belonging to ∂D, which corresponds to the subspace of boundary values of holomorphic sections of ξ over $\Sigma - D$. Call this subspace W. We also have the space of boundary values of holomorphic sections of ξ over D call this H_+. Since we are assuming the $\bar{\partial}$-operator is invertible we know that W is complementary to H_+.

The rest of Vaffa's discussion is clear up to conventions. Specifically he considers the transformation $B: H_- \longrightarrow H_+$ of which W is the graph, then implements the endomorphism

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} : H_+ \oplus H_- \longrightarrow H_+ \oplus H_-$$

by a Bogoliubov transformation on Fock space.

Next move to the bosonic theory. He considers meromorphic functions on Σ which are allowed to have poles only at $t = 0$. These are not enough of these and so one must bring in anti-holomorphic functions.

In this boson setup the fundamental problem is to find the operator context in which we can understand things. Things are confused because we have both holomorphic and anti-holomorphic degrees of freedom.

On the disk the field $X(t)$ is written

$$X(t) = x + ip \log t + \sum_{n=1}^{\infty} \left(t^{-n} \frac{a_n^+}{\sqrt{n}} + t^n \frac{a_n}{\sqrt{n}} + (\bar{t})^{-n} \frac{\bar{a}_n^+}{\sqrt{n}} + (\bar{t})^n \frac{\bar{a}_n}{\sqrt{n}} \right)$$

where the non-zero comm. relations are

$$-i[x, p] = [a_n, a_n^+] = [\bar{a}_n, \bar{a}_n^+] = 1$$

Clearly $e^{iX(t)}$ is a vertex operator of some sort.

Let us see if we can get a better picture of the basic objects in the boson theory.

What is X? Classically it is the exponent in e^{iX} which is a well-defined $\mathbb{T}$-valued function on Σ or part of Σ. This isn't quite right in the same way that when dealing with a "real fermion field" we expect the fields satisfying the imaginary time equation of motion to remain real-valued.

In Vafa's paper he considers the functions

$$X_n(t) = \int^t \eta_n - \pi A_n \frac{1}{\operatorname{Im}\tau}(\omega - \bar{\omega})$$

where η_n is a meromorphic differential on Σ with only a single pole of order $n+1$ at $t=0$ and residue $-n$ and zero a-periods. Here ω is a basis for holom 1-forms adapted to the a,b cycles. and the A_n are coefficients

$$\omega = \sum_1^\infty A_n t^{n-1} dt$$

Vafa claims X_n single valued function on the R.S. It has an expansion

$$X_n = X_n^h + X_n^a \qquad \text{holom. + anti holom.}$$

$$X_n^h = t^{-n} + \sum_{m=1}^\infty (\qquad) t^m$$

$$X_n^h = \sum_{m=1}^\infty (\qquad) \bar{t}^m$$

A first approximation as to what X is then [redacted] a harmonic function with values [redacted] in $\mathbb{R}/2\pi\mathbb{Z}$. Evidently instead of a meromorphic **spinor** ψ [redacted] whose only poles are at p one wants such functions X and one would like [redacted] meromorphic functions instead.

What are instantons in this context? A map $\Sigma - \{p\} \longrightarrow S^1$ is described up to homotopy by a class in $H^1(\Sigma - \{p\}, \mathbb{Z}) = H^1(\Sigma)$, and so any X is shifted from a periodic one by one with windings about the a, b cycles. Apparently one must also make the path integral [redacted] over all fields with a given set of windings and then combine them.

Let's next try to shift to the holomorphic case. In this case we are concerned with holom. maps $f : \Sigma - \{p\} \longrightarrow \mathbb{G}_m = \mathbb{C}^\times$. Then we can look at $\frac{df}{f}$ which is a holomorphic 1-form on $\Sigma - \{p\}$.

May 8, 1987

Let us consider on the interval containing 0 ($I = [-1,1]$) a positive smooth function φ and a smooth function $r \geq 0$ vanishing to infinite order at the ends and >0 in the interior. Typical example is $\varphi = e^{-x^2/2}$, $r(x) = \exp\left(\frac{-1}{1-x^2}\right)$. We then consider the subspace of $L^2(I)^{\oplus 2}$ obtained by closing the space of pairs

$$\begin{pmatrix} r \\ \partial_x - \frac{\varphi'}{\varphi} \end{pmatrix} f$$

where f is smooth on the interval but not necessarily vanishing at the endpoints.

I want to identify this subspace, call it Γ. First notice that Γ contains

$$\begin{pmatrix} r\varphi \\ 0 \end{pmatrix}$$

Next consider the following Green's function for $\left(\partial_x - \frac{\varphi'}{\varphi}\right)$ with vanishing condition to the left

$$G(x,y) = \begin{cases} \dfrac{\varphi(x)}{\varphi(y)} & x > y \\[2ex] 0 & x < y \end{cases}$$

~~This is just a Green's function for $\partial_x - \frac{\varphi'}{\varphi}$ multiplied by r. (The only problem I see with this is that it is not compactly supported in the y direction as $y \to -1$.)~~

We observe that Γ contains

pairs $\begin{pmatrix} rGu \\ u \end{pmatrix}$ $\quad u \in C^\infty([-1,1])$

since

$$\begin{pmatrix} r \\ \partial_x - \frac{\varphi'}{\varphi} \end{pmatrix} Gu = \begin{pmatrix} rGu \\ u \end{pmatrix}.$$

Now take any $f \in C^\infty([-1,1])$, put $u = \overbrace{\left(\partial_x - \frac{\varphi'}{\varphi}\right)}^{D} f$.

Then $f - Gu$ is a multiple of φ, so

$$\begin{pmatrix} r \\ D \end{pmatrix} f = \begin{pmatrix} rG \\ 1 \end{pmatrix} Df + c \begin{pmatrix} r\varphi \\ 0 \end{pmatrix}.$$

Thus we see immediately that Γ is the (transpose) closed graph of rG, which is a bounded operator on $L^2(-1,1)$, direct sum with the line generated by $r\varphi$.

Now we should describe $\Gamma^\perp$.

First note that

$$\Gamma \supset \operatorname{Im} \begin{pmatrix} rG \\ 1 \end{pmatrix} \text{ on } L^2$$

$$\therefore \Gamma^\perp \subset \operatorname{Im} \begin{pmatrix} 1 \\ -G^* r \end{pmatrix} \text{ on } L^2$$

In fact, $\Gamma^\perp$ consists of $\begin{pmatrix} u \\ -G^* r u \end{pmatrix}$ where $u \perp r\varphi$

~~To summarize Γ is the closure of the graph of the operator Dr^{-1} defined on smooth functions g such that $Dr^{-1}g$ is in L^2. We know that this operator has a right~~

To summarize, we start with the operator Dr^{-1} defined on rC^∞. This is a densely defined operator on L^2 and can be closed. This gives a closed densely defined operator $\overline{Dr^{-1}}$ whose graph is Γ.

Now Dr^{-1} has the right inverse rG defined on C^∞. This extends to a bounded operator $\overline{rG}$ on L^2 which is right inverse to $\overline{Dr^{-1}}$.

Since the graph of Dr^{-1} is the sum $\begin{pmatrix} rG \\ 1 \end{pmatrix} C^\infty + \mathbb{C}\begin{pmatrix} r\varphi \\ 0 \end{pmatrix}$, it follows that Γ is the sum $\begin{pmatrix} \overline{rG} \\ 1 \end{pmatrix} L^2 + \mathbb{C}\begin{pmatrix} r\varphi \\ 0 \end{pmatrix}$. Thus we have a Fredholm operator of index 1.

Comments on the general case. In the general case we have $g: S^1 \longrightarrow U(V)$ and we propose to define a subspace of $L^2(S^1, V^{\oplus 2})$ by closing something like

$$\begin{pmatrix} r(g) \\ p(g)\partial_x + q(g) \end{pmatrix} C^\infty(S^1, V)$$

The important point which has to be controlled in some way is that this will not be the graph of an operator defined on rC^∞. Whatever we are after involves indeterminacy. One has to keep in mind that a correspondence can differ from an operator in two ways: It can be defined only on a subspace and it can be indeterminate i.e. infinite valued.

To be more specific a correspondence from V^0 to V^1

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr] & \\ V^0 & & V^1 \end{tikzcd}$$

can fail to be an operator from V^0 to V^1 because p fails to be surjective or injective. It seems that I need to develop a method, some sort of parametrix method, to handle this problem.

Remark: We want $\Gamma \subset V^0 \times V^1$ to be congruent modulo $\mathcal{L}^2$ to V^1. This should mean that p the projection of Γ on V^1 is Fredholm. So therefore I must really look for some sort parametrix for $p(g)\partial_x + q(g)$.

The idea will be to use the symbol calculus. We ~~=~~ starting with

$$\mathrm{Im}\begin{pmatrix} r(g) \\ \partial_x + i\,\frac{g-1}{g+1} \end{pmatrix}$$

Recall that on the symbol level this is the map

$$x, \xi \longmapsto z = i\left(\xi + \frac{g-1}{g+1}\right)\frac{1}{r}$$

and its parametrix has the symbol

$$\frac{1}{z} = \frac{1}{i}\,\frac{r}{\xi + \frac{g-1}{g+1}}$$

which indeed has the desired asymptotic expansion

$$\frac{i}{z} \sim r\left\{\frac{1}{\xi} - \frac{1}{\xi^2}\left(\frac{g-1}{g+1}\right) + \frac{1}{\xi^3}\left(\frac{g-1}{g+1}\right)^2 - \cdots\right\}$$

provided that $r(g)$ is divisible by $(g+1)^N$ for all N. So far I haven't allowed for the fact that ∂_x and g don't commute. We can handle this by writing the perturbation expansion

$$\frac{i}{z} \sim r\left\{\frac{1}{\xi} - \frac{1}{\xi}\left(\frac{g-1}{g+1}\right)\frac{1}{\xi} + \frac{1}{\xi}\left(\frac{g-1}{g+1}\right)\frac{1}{\xi}\left(\frac{g-1}{g+1}\right)\frac{1}{\xi} - \cdots\right\}$$

and using

$$\cancel{\frac{1}{A}B = B\frac{1}{A} - \left[B, \frac{1}{A}\right]}$$

$$\cancel{= B\frac{1}{A} - \underbrace{\frac{1}{A}[A,B]\frac{1}{A}}}$$

$$\cancel{[B,A]\frac{1}{A} + \frac{1}{A}[B,[B,A]]\frac{1}{A}}$$

$$\cancel{= B\frac{1}{A} + [B,A]\frac{1}{A} + }$$

$$\frac{1}{A}B = B\frac{1}{A} + \left[\frac{1}{A}, B\right]$$

$$= B\frac{1}{A} - \frac{1}{A}B'\frac{1}{A} \qquad B' = [A,B]$$

$$= B\frac{1}{A} - B'\frac{1}{A^2} + \frac{1}{A}B''\frac{1}{A^2}$$

$$= B\frac{1}{A} - B'\frac{1}{A^2} + B''\frac{1}{A^3} - \cdots$$

I guess it's clear that one can actually produce a complete symbol for a parametrix. Once I have this parametrix it is probably clear that I have defined a subspace modulo smooth kernel variations. It then remains to pin down the subspace exactly and to do the asymptotics necessary to evaluate the index.

Let's try to pin down the definition exactly. I would like to start with

$$\begin{pmatrix} r \\ \partial_x + i\frac{g-1}{g+1} \end{pmatrix} C^\infty$$

except that I don't know how to define $\frac{g-1}{g+1}$ on C^∞. So the next best thing is

$$\begin{pmatrix} r\cdot(g+1) \\ \partial_x(g+1) + i(g-1) \end{pmatrix} C^\infty$$

This is a perfectly well-defined subspace of $C^\infty(S^1, V^{\oplus 2})$. Next we suppose that we can produce a parametrix for $r(g+1)\left(\partial_x(g+1)+i(g-1)\right)^{-1}$, call it G. By construction G will be a ψDO of order -1 such that

$$G\left(\partial_x(g+1)+i(g-1)\right) = r(g+1) + \text{smoothing}$$

In any case ~~we~~ get~~s~~ a ψDO ~~[illegible]~~ of order -1 and we can consider the subspace

$\binom{G}{1}$ C^∞

There are obviously technical problems involved with the fact that we can only construct an approximate Green's function. However the details involving the choice of r are irrelevant in the case $g = \frac{1+X}{1-X}$ and one still has to produce an asymptotic analysis in this case. So the key question becomes how to establish a local index theorem on S^1 in this case.

Suppose we review the methods. We start with the Dirac operator $D = \gamma^1 h\partial_x + \gamma^2 X$ acting on $L^2(S,V)^{\oplus 2}$ and calculate the index by a suitable function of its square.

$$\text{Index} = tr_\varepsilon\left(\phi(D^2)\right)$$

$\phi(0) = 1$ and $\phi(x^2)$ decays fast enough so the trace is defined. Standard candidates for ϕ.

$$\phi(-x^2) = e^{+ux^2}$$

$$\phi(-x^2) = \frac{\lambda}{\lambda - (-x^2)}$$

Let's go over the steps using the heat operator e^{uD^2} where $D = \gamma h\partial_x + \sigma X$, $\gamma = \gamma^1, \sigma = \gamma^2$.

$$D^2 = h^2\partial^2 + X^2 + h\gamma(\partial_x X)\sigma$$

$$tr_s\, e^{uD^2} = tr_s\left(e^{u(h^2\partial^2 + X^2 + h\gamma\sigma(\partial_x X))}\right)$$

Now expand using perturbation series

$$\operatorname{tr}_s(e^{uD^2}) = \operatorname{tr}_s\left(e^{u(h^2\partial^2+X^2)} uh\gamma\sigma(\partial_x X)\right) + \cdots$$

where we have used the symmetry of tr_s. Now the next step is to understand the asymptotics as $h \downarrow 0$. This means we use Getzler's calculus and it leads to a formula

$$\sim \int dx \int \frac{d\xi}{2\pi} \operatorname{tr}\left(e^{u(-\xi^2+X^2)} u\partial_x X\right) \cdot \underset{2i}{\text{const.}}$$

What sort of quantities go into the constant? One expects to use

$$\operatorname{tr}_\varepsilon(\gamma\sigma) = 2i$$

$$\int \frac{dk}{2\pi} e^{-uh^2k^2} = \frac{\sqrt{\pi}}{2\pi h\sqrt{u}}$$

so maybe the constant is $2i$ and then index is

$$\frac{1}{2\pi}\frac{\sqrt{\pi}}{} 2i$$

$$\int dx \operatorname{tr}\left(e^{uX^2}\sqrt{u}\,\partial_x X\right) \frac{i}{\sqrt{\pi}}$$

Check taking $X = ix$ on the line and we get

$$\int_{-\infty}^{\infty} dx \left(e^{-ux^2}\sqrt{u}\, i\right)\frac{i}{\sqrt{\pi}} = -1$$

This seems OK.

Conclusion is that if we use $\phi(D^2) = e^{uD^2}$, then the local thm. leads ~~to~~ the 1-form

$$\operatorname{tr}\left(e^{uX^2}\sqrt{u}\, dX\right)\frac{i}{\sqrt{\pi}}$$

We might also do the calculation using $\phi(D^2) = \dfrac{1}{\lambda - D^2}$. Then

$$\operatorname{tr}_s\left(\frac{1}{\lambda - h^2\partial^2 - X^2 - h\gamma\sigma\,\partial_x X}\right) = \operatorname{tr}_s\left(\frac{1}{(\lambda - h^2\partial^2 - X^2)^2}\, h\gamma\sigma\,\partial_x X\right) + \cdots$$

$$\sim \int dx \int \frac{d\xi}{2\pi}\operatorname{tr}\left(\frac{1}{(\lambda + \xi^2 - X^2)^2}\,\partial_x X\right) 2i$$

$$= \int dx\, \operatorname{tr}\left[\left(\int \frac{d\xi}{2\pi}\frac{1}{(\lambda + \xi^2 - X^2)^2}\right)\partial_x X\right] 2i$$

Now

$$\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{1}{(\xi^2 + a^2)^2} = \frac{1}{a^3}\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{1}{(\xi^2+1)^2}$$

$$\parallel$$

$$\frac{1}{2a}(-\partial_a)\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\frac{1}{\xi^2 + a^2} = \frac{1}{2a}(-\partial_a)\frac{1}{2\pi a}\left[\arctan\xi\right]_{-\infty}^{\infty}$$

$$= \frac{1}{4a^3}$$

Thus

$$\operatorname{tr}_s\left(\frac{1}{\lambda - h^2\partial^2 - X^2 - h\gamma\sigma\,\partial_x X}\right) \sim \int dx\, \operatorname{tr}\left(\frac{1}{(\lambda - X^2)^{3/2}}\,\partial_x X\right)\frac{i}{2}$$

Note that this checks with

$$\operatorname{tr}\left(e^{uX^2}\sqrt{u}\,dX\right)\frac{i}{\sqrt{\pi}} \xrightarrow{\text{L.T.}} \operatorname{tr}\left(\frac{\Gamma(3/2)}{(\lambda - X^2)^{3/2}}\,\partial_x X\right)\frac{i}{\sqrt{\pi}}$$

$$\Gamma(3/2)\frac{i}{\sqrt{\pi}} = \frac{1}{2}\sqrt{\pi}\,\frac{i}{\sqrt{\pi}} = \frac{i}{2}$$

Actually this computation is rather instructive and potentially important. It shows ~~some~~ of the limitations of our present approach, maybe [redacted]. Let's try to see why:

[redacted] What is strange in the above is the resolvent $\frac{1}{\lambda - X^2}$ to the $3/2$ power. This was obtained from the asymptotic limit of the s-trace of the resolvent of the Dirac.

Discuss ideas: First of all even in the case of the circle we don't understand the index theorem. Starting from the Dirac operator $i\partial_x + \sigma X$ one first has to show analytically that it is a Fredholm operator, and hence has an index. This can be done by [redacted] constructing a parametrix. The other method is to produce the heat operator. Again one may do this by an asymptotic expansion at the diagonal. This [redacted] gives something approximate which can be [redacted] refined via Volterra iteration to the exact heat kernel.

2nd idea: This concerns the feeling that the index should be computable by integration over $T^*(S^1)$ and this has got something to do with the superconnection character form of the symbol map $T^*(S^1) \to S^2$.

3rd idea: Gaussian supercon. forms are compatible with cup products and maybe index integration.

May 9, 1987

Some analysis. Suppose we have an elliptic DO $D : E \to F$ over a compact manifold. Suppose metric given, so that Hilbert spaces $L^2(E)$, $L^2(F)$ are defined. Then by closing the graph of $D : C^\infty(E) \to C^\infty(F)$ in $L^2(E) \oplus L^2(F)$ we obtain a densely-defined closed operator $\overline{D}$ from $L^2(E)$ to $L^2(F)$.

I can do the same thing for the adjoint $D^* : F \longrightarrow E$ defined using the metrics, thus obtaining a closed operator $\overline{D^*} : L^2(F) \longrightarrow L^2(E)$. ~~The problem is to show that~~

$$\cancel{\overline{D}^* = \overline{D^*}}$$

~~I should have said that it is obvious that~~

$$\cancel{\overline{D^*} \subset \overline{D}^*}$$

~~In fact what we have~~

Now by definition of D^* we have

$$\langle v \mid Du \rangle = \langle D^* v \mid u \rangle \qquad u \in C^\infty(E), v \in C^\infty(F)$$

which shows

$$\text{graph}(D) \perp \text{graph}(-D^*) \qquad \text{in } L^2(E) \oplus L^2(F).$$

Recall that the adjoint $\overline{D}^*$ is defined by

$$\text{graph}(-\overline{D}^*) = (\text{graph } \overline{D})^\perp.$$

Thus we have

$$\overline{\text{graph}(-D^*)} \subset \text{graph}(-\overline{D}^*) \qquad \text{or}$$

$$\overline{D^*} \subset \overline{D}^*$$

The problem is to show these are the same when

D is elliptic.

Recall that $\bar{D}$ is the minimal closed operator defined by D. There is also a maximal operator $\bar{\bar{D}}$ defined by $\bar{\bar{D}}u = v$ when

$$\langle v' | v \rangle = \langle D^* v' | u \rangle \qquad \forall v' \in C^\infty(F)$$

Thus
$$\begin{aligned} \mathrm{graph}(\bar{\bar{D}}) &= \mathrm{graph}(-D^*)^\perp \\ &= \mathrm{graph}(-\overline{D^*})^\perp \\ &= \mathrm{graph}(\overline{D^*}^*) \end{aligned}$$

or
$$\bar{\bar{D}} = \overline{D^*}^*$$

so
$$\begin{aligned} \overline{D^*} = \bar{D}^* &\iff \mathrm{graph}(-\overline{D^*}) = \mathrm{graph}(\bar{D})^\perp \\ &\iff \underbrace{\mathrm{graph}(-\overline{D^*})^\perp}_{\mathrm{graph}(\bar{\bar{D}})} = \mathrm{graph}(\bar{D}) \end{aligned}$$

So
$$\overline{D^*} = \bar{D}^* \iff \bar{D} = \bar{\bar{D}}$$

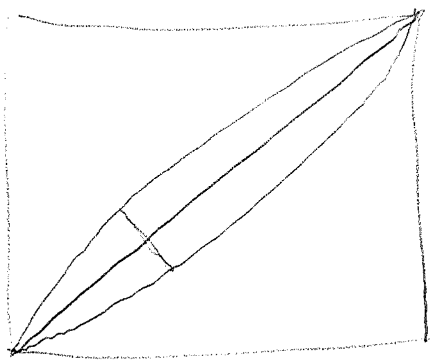
Notice that if $D = D^*$, this implies that $\bar{D}$ is self-adjoint as a densely-defined operator.

Problem: By ψDO calculus we can construct a parametrix P for D, which gives a bounded operator $P: L^2(F) \longrightarrow L^2(E)$ such that DP, PD differ from the identity by smooth kernel operators. Does the existence of P imply that $\bar{D} = \bar{\bar{D}}$?

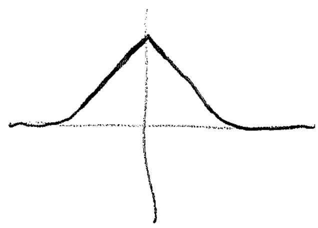
Certainly there is no abstract theorem. We can consider $D = \frac{d}{dx}$ on $C_c^\infty(0,1)$. The minimal operator $\bar{D}$ is $\frac{d}{dx}$ on functions with L^2-deriv. which vanish at the ends. The maximal operator $\bar{\bar{D}}$ is the same without the boundary condition. We [redacted] obtain a parametrix for $\bar{D}$ by integrating and subtracting the linear fn [redacted] that makes the boundary values vanish.

Similarly in the self-adjoint case we can take $D = -\frac{d^2}{dx^2}$ on $(0,1)$. We can construct a parametrix by making a continuous kernel $K(x,y)$ [redacted] which is smooth on the product [redacted] except on the diagonal where the slope jumps by -1.

NOT CLEAR



K is symmetric and supported in the lens region, normal to the diagonal looks



Then one gets a self-adjoint parametrix whose image [redacted] is contained in the domain of the minim[redacted]al operator $\bar{D}$.

Now it is clear that there is a problem involved with constructing what we want via a parametrix. NO

NO It is not clear that the above counterexample works without inequalities to justify it.

There are still some fundamental mysteries in the whole index theory business which have to be explained. Some of the mystery involves the Seeley type analysis. Seeley uses ΨDO type methods for constructing the heat operator. ΨDO type methods involve constructing a type of singularity, i.e. a parametrix for the heat operator. The actual heat operator then results by Volterra iterating the parametrix.

It seems that this whole process can be carried on a compact manifold and yields a proof that the Laplacean operator is essentially self-adjoint.

Thus there appears to be a ΨDO proof of the essential self-adjointness of a Laplacean. Recall there are also finite propogation speed methods for <u>Dirac</u> operators.

Review the Volterra iteration process. We want to solve the equation

$$(\partial_t + H) G(t) = \delta(t)$$

$$G(t) = 0 \qquad t < 0$$

starting from the fact that we have a parametrix:

$$(\partial_t + H)\, G_0(t) = \delta(t) - K(t)$$

$$G_0(t) = 0 \qquad t < 0$$

The solution is then given by a Neumann series

$$G = G_0 (1-K)^{-1} = G_0 + G_0 K + G_0 K K + \cdots$$

Here we interpret a function of t as a convolution operator:

$$(Kf)(t) = \int_0^\infty K(t-t') f(t')\, dt'$$

Note that $f(t)=0$ for $t<0 \implies$ same for Kf.
Note that $K(t-t')=0$ for $t<t'$ so that the convolution operator K can be expected to be topologically nilpotent and the Neumann series will converge.

Since we are dealing with a constant coefficient operator $\partial_t + H$, it is useful to use the L.T. to simplify things, i.e. to replace convolution by multiplication. Thus

$$\int_0^\infty e^{-\lambda t} (\partial_t + H) G_0(t)\, dt = 1 - \hat{K}(\lambda)$$

$$(\lambda + H)\hat{G}_0(\lambda) = 1 - \hat{K}(\lambda)$$

and so the issue is when we can invert $1-\hat{K}$ and take the inverse L.T. Naively we want $\|K(\lambda)\| \leq 1-\varepsilon$ for $Re(\lambda) \geq a$.

Lesson: It seems that working with the resolvent as a function of λ holomorphic in a right half plane ~~corresponds to the m~~ avoids difficulties connected with a parametrix not being a true inverse to the operator.

Let's see if I can formulate a program. What I want to do is to start with a

Dirac operator, say $\not{D} = h\gamma^1\partial_x + \gamma^2 X$ over the circle, and to prove that it is essentially skew-adjoint. In order to do this, it should suffice to establish that $(\lambda - \not{D})^{-1}$ exists for one value of λ in each half plane $\mathrm{Re}(\lambda) > 0$ or < 0.

To construct $(\lambda - \not{D})^{-1}$ in one of the half planes it might be enough to construct a parametrix and then show the Neumann series converges.

Once we have the fact that X is essentially skew-adjoint, then we have a map from X's to unitaries. Presumably these should be unitaries congruent to -1 modulo a certain Schatten ideal.

In order to make any progress in this direction I have to be able to treat the case of

$$\not{D} = \gamma^1\partial_x + \gamma^2 X = \begin{pmatrix} 0 & \partial_x - g \\ \partial_x + g & 0 \end{pmatrix}$$

where g is a real-valued function and we are over the circle. Thus I now want to see how to construct $(\lambda - \not{D})^{-1}$.

Let's begin by considering g constant. I want to use ψDO methods, which means asymptotic expansions in powers of ξ. At the same time I have to keep control of λ.

So let us start with the exact Fourier series solution. This means I work with $\partial_x = i\xi$.

and then

$$\frac{1}{\lambda - \not{D}} \longleftrightarrow \frac{1}{\lambda - \gamma^1 i\xi - \gamma^2 X}$$

Now I guess we should know from hyperbolic theory or somewhere that its OK to think in terms of an asymptotic expansion with λ, ξ both considered to be ~~of~~ of degree 1. Thus we don't want to expand $\frac{1}{\lambda - \gamma^1 i\xi}$ in powers of ξ^{-1}, but rather we want to keep λ, ξ together. Put another way I have to know what to expect from the constant hyperbolic Dirac.

May 10, 1987

On the circle let us consider

$$\not{D} = \gamma^1 \partial_x + \gamma^2 X = \begin{pmatrix} 0 & \partial_x - q \\ \partial_x + q & 0 \end{pmatrix} \qquad X = \begin{pmatrix} \\ & \pm q \end{pmatrix}$$

where q is real-valued. I would like to prove that $s - \not{D}$ is invertible on L^2 for $|\mathrm{Re}(s)|$ large and I would like to evaluate the index.

First of all we have to understand the case $X = 0$. We can conjugate to change $\gamma^1 \longrightarrow \varepsilon$ and $\gamma^2 \longrightarrow \gamma^1$ so that

$$s - \not{D}_0 \longrightarrow \begin{pmatrix} s - \partial_x & 0 \\ 0 & s + \partial_x \end{pmatrix}$$

The inverses for $s \pm \partial_x$ on $L^2(S^1)$ are easily constructed for $\mathrm{Re}(s) \neq 0$; the corresponding kernels $G(x,y)$ have ± 1 jumps as x crosses y and otherwise are smooth. On the Fourier transform side

$$\frac{1}{s \mp \partial_x} \longleftrightarrow \frac{1}{s \mp i\xi}$$

and as

$$\int \frac{1}{|s \mp i\xi|^p} d\xi = \frac{1}{|\mathrm{Re}(s)|^{p-1}} \int \frac{1}{(1+\xi^2)^{p/2}} d\xi$$

we see that $\frac{1}{s \pm \partial_x} \in \mathcal{L}^p$ for $p > 1$, and

$$\left\| \frac{1}{s \pm \partial_x} \right\|_p = O\left(\frac{1}{|\mathrm{Re}(s)|^{1-1/p}} \right) \qquad \text{as } |\mathrm{Re}(s)| \to \infty$$

[redacted] Next we consider $\not{D} = \gamma^1\partial_x + \gamma^2 X$. We have

$$\left(s-\not{D}\right)\frac{1}{s-\not{D}_0} = \left(s-\gamma^1\partial_x - \gamma^2 X\right)\left(\frac{1}{s-\gamma^1\partial_x}\right)$$

$$= 1 - \underbrace{\gamma^2 X\frac{1}{s-\gamma^1\partial_x}}_{K}$$

Then K is the composition of the bdd operator $\gamma^2 X$ on L^2 with $\frac{1}{s-\not{D}_0}$ which we have seen is in $\mathcal{L}^p$ for $p>1$ and whose $\mathcal{L}^p$ norm decreases as $|Re(s)| \to \infty$. Thus for $|Re(s)|$ large $1-K$ is invertible and we have a [redacted] right inverse for $s-\not{D}$. This should be enough to show that $\not{D}$ with domain the first Sobolev space is skew-adjoint.

Now let's turn to computing the index. The index turns out to be zero but our goal is to find an analytical proof which gives an integral expression for the index. It's an integral of a 1-form which turns out to be [redacted] exact in the present case.

Let's get the link straight between heat kernel and parametric approaches to the index. In general given $\not{D} = \begin{pmatrix} 0 & -D^* \\ D & 0 \end{pmatrix}$ we can calculate the index by means of a parametrix P for D. This means

$$PD = 1 - K_+$$
$$DP = 1 - K_-$$

where $K_\pm$ are of trace class. The index is

$$\mathrm{Ind}(D) = \mathrm{tr}(K_+) - \mathrm{tr}(K_-)$$

An example of a parametrix is

$$P = \frac{1-e^{-uD^*D}}{D^*D} D^* = D^* \frac{1-e^{-uDD^*}}{DD^*}$$

since

$$PD = \frac{1-e^{-uD^*D}}{D^*D} D^*D = 1-e^{-uD^*D}$$

$$DP = 1-e^{-uDD^*}$$

Then

$$\mathrm{Index} = \mathrm{tr}\left(e^{-uD^*D}\right) - \mathrm{tr}\left(e^{-uDD^*}\right)$$

Another approach would be to set

$$\not{P} = \frac{1-e^{u\not{D}^2}}{\not{D}} = \begin{pmatrix} 0 & +\frac{1-e^{-uD^*D}}{D^*D} D^* \\ -\left(\frac{1-e^{-uDD^*}}{D^*D}\right) D & 0 \end{pmatrix} = \begin{pmatrix} 0 & P \\ -P^* & 0 \end{pmatrix}$$

Then this is a parametrix for $\not{D}$ as

$$\not{P}\not{D} = \not{D}\not{P} = 1-e^{u\not{D}^2}$$

However if $\not{Q}$ is any parametrix for $\not{D}$ which is odd, we have

$$-\mathrm{tr}_s\left\{\tfrac{1}{2}[\not{D}, Q]_+ - 1\right\} = \mathrm{tr}_s\left\{\underbrace{1-\tfrac{1}{2}[\not{D},\not{P}]_+}_{e^{u\not{D}^2}} + \underbrace{\tfrac{1}{2}[\not{D},\not{P}-\not{Q}]_+}_{\text{gives } 0 \ \mathrm{tr}_s}\right\}$$

Thus

$$\boxed{\mathrm{Index} = \mathrm{tr}_s\left\{1-\tfrac{1}{2}[\not{D},\not{Q}]_+\right\}}$$

Let's try to use this formula to compute the index of $\not{D} = \gamma^1 \partial_x + \gamma^2 X$. Let's try

$$\not{Q} = \frac{1 - e^{u\not{D}_0^2}}{\not{D}_0^2} \not{D}_0 = \frac{1 - e^{u\partial_x^2}}{\partial_x^2} \gamma^1 \partial_x$$

for the parametrix. Set $\Phi = 1 - e^{u\partial_x^2}$. Then

$$\not{D}\not{Q} = (\gamma^1 \partial_x + \gamma^2 X) \frac{\Phi}{\partial_x} \gamma^1 = \Phi - \gamma^1\gamma^2 X \frac{\Phi}{\partial_x}$$

$$\not{Q}\not{D} = \frac{\Phi}{\partial_x} \gamma^1 (\gamma^1 \partial_x + \gamma^2 X) = \Phi + \gamma^1\gamma^2 \frac{\Phi}{\partial_x} X$$

$$1 - \tfrac{1}{2}[\not{D}, \not{Q}]_+ = 1 - \Phi + \tfrac{1}{2}\gamma^1\gamma^2 \left(X \frac{\Phi}{\partial_x} - \frac{\Phi}{\partial_x} X \right)$$

$$\circledast \quad \boxed{\mathrm{Tr}_s\left(1 - \tfrac{1}{2}[\not{D}, \not{Q}]_+\right) = i\, \mathrm{Tr}\left(X \frac{\Phi}{\partial_x} - \frac{\Phi}{\partial_x} X \right)}$$

Just to simplify suppose that we fix boundary conditions on the circle so that ∂_x is invertible. Then we can let $u \to \infty$ and we get

$$\text{Index} = i\, \mathrm{Tr}\left(X \frac{1}{\partial_x} - \frac{1}{\partial_x} X \right)$$

This expression gives the index zero, since if we use the basis $|m\rangle = e^{imx}$, then

$$\langle m | X \tfrac{1}{\partial_x} | n \rangle = \left(\int e^{-i(m-n)x} X(x) \frac{dx}{2\pi} \right) \frac{1}{in}$$

$$= \frac{f(m-n)}{in}$$

$$\langle m | \tfrac{1}{\partial_x} X | n \rangle = \frac{f(m-n)}{im}$$

so the matrix of $X \frac{1}{\partial_x} - \frac{1}{\partial_x} X$ vanishes on the diagonal. Actually this works for $\circledast$ also

Thus I am interested in the other limit, namely, as $u \downarrow 0$. In this case we expect roughly

$$\frac{\Phi}{\partial_x} = \frac{1-e^{u\partial_x^2}}{\partial_x} \sim -u\partial_x$$

Thus $\left[X, \frac{\Phi}{\partial_x}\right] \sim u\,\partial_x X$ and $\partial_x X$ is not a trace class operator so that we expect the trace to become an integral of $\partial_x X$.

Now let's do the same calculation with $D_0 = h\gamma^1\partial_x$, $D = h\gamma^1\partial_x + \gamma^2 X$, and the parametrix for D_0:

$$Q = \frac{1-e^{+uD_0^2}}{D_0} = \frac{1-e^{uh^2\partial_x^2}}{h\partial_x}\gamma^1 = \frac{\Phi}{h\partial_x}\gamma^1$$

Then we get

$$\text{Index} = i\,\mathrm{Tr}\left(X\frac{\Phi}{h\partial_x} - \frac{\Phi}{h\partial_x}X\right)$$

Now the problem is to let $h \downarrow 0$ and adjust u suitably so this has a limit.

Let's suppose that we have twisted boundary conditions on the circle so that $(\partial_x)^{-1}$ is defined. Then I can take $\Phi = 1$. Then the above expression for the index blows up as $h \downarrow 0$ unless the index is zero. Thus we want to use that Φ depends on h.

But it should have been clear earlier that the above approach can't possibly yield the desired local index formula. What you want is to obtain the superconnection answer

$$\mathrm{tr}_s\left(e^{-u\not{D}^2}\right) = \mathrm{tr}_s\left(e^{-u(h^2\partial_x^2 + X^2 + h\gamma^1\gamma^2\partial_x X)}\right)$$

$$\xrightarrow[\text{limit}]{\text{local}} \int \mathrm{tr}\left(e^{-uX^2}dX\right)\cdot \text{const}$$

It's clear that you are not going to get e^{-uX^2} for X a matrix out of Φ.

So we learn that today's attempt to handle things via perturbing from a parametrix for $\not{D}_0$ is not going to lead to the desired index formula. In a similar way we can't expect large λ asymptotics to tell us the local index formula.

The only hope is to set up everything properly as an asymptotic expansion in h.

May 11, 1987

Recall that the problem is to pair the Dirac on the circle with a loop g. On the symbol level this means taking a Bott type map

$$S^1 \times U(V) \longrightarrow Gr_m(\mathbb{C}^2 \otimes V) \qquad m = \dim V$$

Formula I have been using is

$$\xi, g \longmapsto \text{graph of } \xi + \frac{g-1}{g+1}$$

$$\Longleftrightarrow \text{C.T. of } \gamma^1 i\xi + \gamma^2 X.$$

Interesting is the fact that

$$S^1 \times U(1) \longrightarrow \mathbb{P}(\mathbb{C}^2) = S^2$$

is a degree 1 map. Something similar happens for the next map

$$S^1 \times \mathbb{P}(\mathbb{C}^2) \longrightarrow SU(2)$$

$$\xi, z \longmapsto \text{C.T. of } \begin{pmatrix} i\xi & -z^* \\ z & -i\xi \end{pmatrix}$$

In general I can take a g inverted by ε and form the C.T. of

$$i\xi\varepsilon + \frac{g-1}{g+1}.$$

It's important to note that previous work with the graph of $\left(\partial_x + \frac{1}{i}\frac{g-1}{g+1}\right)r^{-1}$ is not suitable for generalizing. You need to ~~[illegible]~~ write down a skew-adjoint operator, e.g.

$$r^{-1/2}\left(\varepsilon\partial_x + \frac{g-1}{g+1}\right)r^{-1/2}$$

From now on let us take the "operator" $r^{-1/2}\left(\varepsilon\partial_x + \frac{g-1}{g+1}\right)r^{-1/2}$ where r is a function on S^1 vanishing to infinite order at -1, and it is applied to the unitary operator g. I want to show this "operator" has a well-defined Cayley transform which is a unitary operator on $L^2(S^1, \mathbb{C}^2)$. Here g is a smooth map $S^1 \longrightarrow S^2$ where S^2 is identified with elements of $SU(2)$ inverted by ε.

The first thing to check carefully is that things work perfectly on the symbol level. So I want to see that there is a smooth map from the compactified cotangent bundle to $SU(2)$.

Now I propose to first look at the map

$$S^1 \times S^2 \longrightarrow SU(2)$$

$$\xi,\ z \longmapsto \text{C.T. of } i\underbrace{(\varepsilon\xi + \gamma^1 x + \gamma^2 y)}_{\begin{pmatrix} \xi & x-iy \\ x+iy & -\xi \end{pmatrix}}$$

and to understand the singularities. Notice that this map behaves nicely with respect to rotation in the z plane and conjugation by the maximal torus of $SU(2)$. Notice that the eigenvalues of an element of $SU(2)$ are conjugate points on the circle. The eigenvalues of $\begin{pmatrix} \xi & x-iy \\ x+iy & -\xi \end{pmatrix}$ are $\pm\sqrt{\xi^2+x^2+y^2}$

What I want is a natural way to analyze the singularities of this map over a nbd of $-1 \in SU(2)$. In the present case a skew-adjoint 2×2 matrix X of trace 0 is singular iff it's zero. Thus to study $g = \frac{1+X}{1-X}$ I can write it

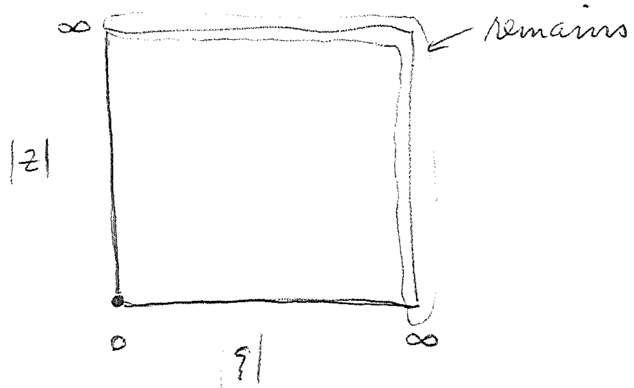
$$g = \frac{X^{-1}+1}{X^{-1}-1}.$$

In other words I have a covering of $SU(2)$ by two charts the complements of 1 and -1.

Thus I can ask about the singularities of the map

$$\xi, z \longmapsto \frac{1}{\varepsilon\xi + \gamma^1 x + \gamma^2 y}$$

Obviously this is smooth as it stands for $\xi \neq 0, \infty$ and $z \neq \infty$, and for $\xi \neq \infty$, $z \neq 0, \infty$. We can eliminate (ξ, z) near $(0,0)$.



Suppose $|z|$ near ∞. Then

$$\frac{1}{\varepsilon\xi + \gamma^1 x + \gamma^2 y} = \frac{1}{\gamma^1 x + \gamma^2 y} \; \frac{1}{1 + (\varepsilon\xi)\frac{1}{\gamma^1 x + \gamma^2 y}}$$

$$\left(= \begin{pmatrix} 0 & \bar{z} \\ z & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & z^{-1} \\ (\bar{z})^{-1} & 0 \end{pmatrix} \right)$$

shows things are smooth for $|z|$ near ∞ and $|\xi|$ away from ∞. Similarly we can handle $|\xi|$ near ∞ and $|z|$ away from ∞. Then when $|z|, |\xi|$ are both near ∞ we have

$$(\gamma^1 x + \gamma^2 y)^{-1} \frac{1}{\xi^{-1} + \varepsilon(\gamma^1 x + \gamma^2 y)^{-1}} \xi^{-1}$$

which leads to the old problem of non smoothness for a homogeneous function.

Question: Is the Clifford multiplication map

$$S^n = \mathbb{R}^n \cup \{\infty\} \longrightarrow U(S_n) \text{ or } Gr(S_n)$$

$$\xi \longmapsto \frac{1 + i\gamma^\mu \xi_\mu}{1 - i\gamma^\mu \xi_\mu} = g$$

smooth? Here S_n is an irreducible module over $\mathrm{Cliff}(\mathbb{R}^n)$, so it's graded for n even and ungraded for n odd.

This obviously depends on whether the map is smooth at ∞. To get coordinates on S^n near infinity use inversion $\xi_\mu \longmapsto \frac{\xi_\mu}{|\xi|^2} = y_\mu$. Then

$$g = \frac{1 + i\gamma^\mu \xi_\mu}{1 - i\gamma^\mu \xi_\mu} = \frac{\gamma^\mu \xi_\mu + i|\xi|^2}{\gamma^\mu \xi_\mu - i|\xi|^2} = \frac{+i\gamma^\mu \frac{\xi_\mu}{|\xi|^2} - 1}{i\gamma^\mu \frac{\xi_\mu}{|\xi|^2} + 1}$$

$$= - \frac{1 - i\gamma^\mu \frac{\xi_\mu}{|\xi|^2}}{1 + i\gamma^\mu \frac{\xi_\mu}{|\xi|^2}}$$

Clearly it's OKAY

May 12, 1987

Let us now consider some inequalities.

To fix the ideas consider $\not{D} = \gamma\partial_x + \sigma\frac{g-1}{g+1}$ on the circle. What I am proposing to do is to replace this by $r^{-1/2}\not{D}\, r^{-1/2}$ where r is a suitable function of g which vanishes to infinite order at $g=-1$. I want to show that $r^{-1/2}\not{D}\, r^{-1/2}$ determines a unitary operator on $L^2(S^1, \mathbb{C}^2)$. One way to try to do this is by constructing the resolvent

$$\frac{1}{\lambda - r^{-1/2}\not{D}\, r^{-1/2}} = \left(\lambda - r^{-1/2}\not{D}\, r^{-1/2}\right)^{-1}$$
$$= r^{1/2}\left(r\lambda - \not{D}\right)^{-1} r^{1/2}$$

I might hope to do this by ΨDO techniques. This means that we first consider the symbol

$$\frac{r}{r\lambda - \left(\gamma i\xi + \sigma\frac{g-1}{g+1}\right)}$$

and we understand this well-enough to do a perturbation series around this symbol.

Thus on the symbol level I have a map from the cotangent bundle of the circle to the 2-sphere = 2×2 special unitaries inverted by ε. I believe this map extends smoothly from the torus $S^1 \times S^1$ = (ξ circle) × (x-circle) to S^2.

Natural question: What does the superconnection form look like? There is only a 2-form as we are in the component of $Gr(\mathbb{C}^2)$ containing the graphs of invertible operators on $\mathbb{C}$

To fix ideas, let $g = e^{2\pi i x}$ be the identity map (essentially) of the circle. Then we have the map

$$\xi, x \longmapsto i\left(\gamma\xi + \sigma \tan(\pi x)\right)/r(x)$$

where $r(x)$ vanishes to infinite order at the ends of $[-\frac{1}{2}, \frac{1}{2}]$. Let's reparametrize $(-\frac{1}{2}, \frac{1}{2})$ using $\tan(\pi x)$. Then we have the map

$$(\xi, x) \longmapsto \left(\frac{\xi}{r}, \frac{x}{r}\right) \longmapsto i\left(\gamma^1 \frac{\xi}{r} + \gamma^2 \frac{x}{r}\right)$$

Now we calculated the superconnection form for the Bott map on the plane: $(x^1, x^2) \longrightarrow i\gamma^\mu x_\mu$. and found the Gaussian 2-form.

$$2i u^2 e^{-u(x_1^2 + x_2^2)} dx_1 dx_2 \qquad \xrightarrow{\int} (2i\pi)u$$

Pulling this back gives

$$2iu^2 e^{-u(\xi^2+x^2)/r^2} \underbrace{d(\xi/r)\, d(x/r)}_{= \frac{d\xi}{r} d\left(\frac{x}{r}\right)}$$

Now if we ~~[illegible]~~ do the ξ integral we get the 1-form

$$2i\sqrt{\pi}\cdot\sqrt{u}\, u\, e^{-u(x/r)^2} d(x/r)$$

Now we (maybe) ought to do this calculation for a general loop $g: S^1 \longrightarrow U(V)$.

May 12, 1987

It's time to learn Weil's Acta paper. Apparently there is an adelic version of the Θ function and a proof of quadratic reciprocity having something to do with $K_2(\mathbb{Q})$.

Start by reconstructing the idea that $\mathbb{Q}$ is a maximal isotropic subgroup of the adeles A. This is supposed to generalize $\mathbb{Z}$ being maximal isotropic for the pairing $x, y \longmapsto e^{2\pi i xy}$, $\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{T}$. This pairing identifies $\mathbb{R}$ with its dual $\hat{\mathbb{R}}$ in the sense of Pontryagin duality.

$\mathbb{Q}_p$ is also self-dual by a map

$$\mathbb{Q}_p \times \mathbb{Q}_p \xrightarrow{\text{mult}} \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xhookrightarrow{\chi_p} \mathbb{T}$$

Here χ_p is any character of $\mathbb{Q}_p/\mathbb{Z}_p$ which is injective on $p^{-1}\mathbb{Z}/\mathbb{Z}$, and then it has to be injective on all of $\mathbb{Q}_p/\mathbb{Z}_p$. The obvious choice for χ_p is to use the isomorphism

$$\mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\sim} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \xrightarrow{\exp(2\pi i\, ?)} \mathbb{T}$$

Putting together these local pairings should give a self-duality of the adeles $A = \left(\prod_p{}' \mathbb{Q}_p\right) \times \mathbb{R}$.

Notice that this self-pairing is a composite

$$A \times A \xrightarrow{\text{mult}} A \xrightarrow{\chi} \mathbb{T}$$

where χ is some character on A. Notice also that because $\mathbb{R}, \mathbb{Q}_p, A$ act on themselves as abelian groups, their duals are modules over themselves, and so we are just showing they are free modules of rank 1. Thus we get a self-duality of $\mathbb{R}, \mathbb{Q}_p$ ~~starting~~ from any non-zero character.

Also notice that the pairings are symmetric and completely analogous the pairings on spinors. Unfortunately it is not clear what a Clifford algebra ought to be for one of these abelian group with quadratic form.

Let's see how close we can come to setting up an analogue of meromorphic spinors, i.e. where we have complementary isotropic subspaces.

First consider $\mathbb{Q}_p$ which is analogous to $\hat{F}_p \otimes L$. The latter in the case the spin structure has no cohomology is the sum of complementary isotropic subspaces W_p, $\hat{O}_p \otimes L$; here W_p is the space of meromorphic spinors whose only poles are at p. The best we can do in the case of $\mathbb{Q}_p$ is the two subgroups

$$\mathbb{Z}[\tfrac{1}{p}] \quad , \quad \mathbb{Z}_p$$

which satisfy

$$\mathbb{Z}[\tfrac{1}{p}] + \mathbb{Z}_p = \mathbb{Q}_p$$

$$\mathbb{Z}[\tfrac{1}{p}] \cap \mathbb{Z}_p = \mathbb{Z}$$

This last $\mathbb{Z}$ is annoying - it means that we haven't imposed a condition at ∞. If we also restrict $\mathbb{Z}[\frac{1}{p}]$ by the condition $|x| \leq \frac{1}{2}$, then we do get something resembling a complement to $\mathbb{Z}_p$ in $\mathbb{Q}_p$.

So now let's turn to adeles. The first thing is to put together the correct local pairings.

Ultimately we want a map

$$\prod{}' \mathbb{Q}_p \Big/ \prod \mathbb{Z}_p \longrightarrow \mathbb{T}$$

$$\parallel$$

$$\bigoplus \mathbb{Q}_p/\mathbb{Z}_p \;=\; \mathbb{Q}/\mathbb{Z}$$

so it is clear that we want to use the map

$$\mathbb{Q}/\mathbb{Z} \xrightarrow[\sim]{\exp(2\pi i\,?)} \mu_\infty \subset \mathbb{T}$$

which extends the previous maps $\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{T}$

Inside $\prod' \mathbb{Q}_p$ we have the subgroups $\mathbb{Q}$, $\prod \mathbb{Z}_p$ which satisfy

$$\mathbb{Q} + \prod \mathbb{Z}_p = \prod{}' \mathbb{Q}_p$$

$$\mathbb{Q} \cap \prod \mathbb{Z}_p = \mathbb{Z}$$

So what is done I guess is to add on $\mathbb{Q}_\infty = \mathbb{R}$ and to use the character

$$\chi_\infty : \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \xrightarrow{\exp(-2\pi i\,?)} \mathbb{T}$$

It then follows that for the pairing defined by multiplication and the different χ_p including $p = \infty$, that $\mathbb{Q}$ is isotropic.

The situation then is that we have the adeles A with a symmetric nondegenerate self-pairing, and two isotropic subgroups

$$\mathbb{Q} \qquad \prod \mathbb{Z}_p \times \mathbb{Z}$$

which are not complementary. Instead we have a picture

$$
\begin{tikzcd}
 & A \arrow[d, dash, "\Pi"] & \\
 & \operatorname{Ker}\chi \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q} \arrow[dr, dash] & & \prod \mathbb{Z}_p \times \mathbb{Z} \arrow[dl, dash] \\
 & \mathbb{Z} \arrow[d, dash] & \\
 & e &
\end{tikzcd}
$$

χ collection of χ_p, χ_∞.

Something analogous happens for functions on a curve with the skew-symmetric pairing $\operatorname{Res}(f\,dg)$. In this case one has

$$
\begin{tikzcd}
 & A \arrow[d, dash, "H^1(\mathcal{O})"] & \\
 & B & \\
F \arrow[ur] & & \prod \hat{\mathcal{O}}_p \arrow[ul] \\
 & k \arrow[ul, leftrightarrow] \arrow[ur] & \\
 & 0 \arrow[u, dash] &
\end{tikzcd}
$$

It's not exactly the same as the top $A/B = H^1(\mathcal{O})$ and the bottom $k = H^0(\mathcal{O})$ are not paired nicely. The residue pairing is also singular on A. ?