March 30, 1987

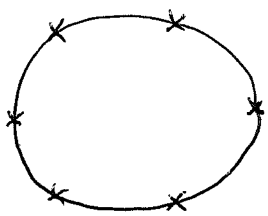
Let us consider discrete analogues of spinors over the circle. Consider $V = \mathbb{R}^N$ with the obvious action of $\mathbb{Z}/N\mathbb{Z}$, or the twisted action:

$$(x_1, \ldots, x_N) \longmapsto (x_N, x_1, \ldots, x_{N-1})$$

$$\longmapsto (-x_N, x_1, \ldots, x_{N-1})$$

Actually the latter is an action of $\mathbb{Z}/2N$.

If we complexify V, then in the former case we have the regular representation of $\mathbb{Z}/N\mathbb{Z}$. This breaks up into [redacted] 1-diml representations according to the characters $\tau \longmapsto (e^{2\pi i/N})^k$. [redacted] These are just the Nth roots of 1.



If N is odd, then the regular repn. V_c splits into $W \oplus \overline{W} \oplus \mathbb{C}$, where W is the sum of the eigenspaces belonging to $\zeta \in \mu_N$ with $\mathrm{Im}(\zeta) > 0$, and where $\mathbb{C}$ is the trivial representation. Thus we have the familiar R situation.

In the twisted action case [redacted] V_c breaks up according to the characters where τ has the values $\zeta = e^{2\pi i k/N}$, $k = \frac{1}{2}, \ldots, \frac{2N-1}{2}$. If N is even we get $V = W \oplus \overline{W}$ where W is the sum of the eigenspaces of τ with eigenvalues ζ such that $\mathrm{Im}(\zeta) > 0$. Thus we have the NS situation.

For some reason $\zeta = -1$ if it occurs is not so important as it doesn't survive in the $N \to \infty$ limit.

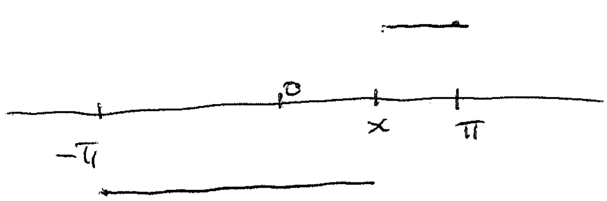
We can put both cases together by considering the regular representation of the cyclic group of order $2N$. If τ denotes the generator, then $(\tau^N)^2 = 1$ and so the representation breaks up into the two eigenspaces for τ^N. These are respectively

$$V^P = \{(x_1, \ldots, x_N, x_1, \ldots, x_N)\}$$

$$V^{AP} = \{(x_1, \ldots, x_N, -x_1, \ldots, -x_N)\}$$

Next we come to the question of spin fields. These are supposed ~~to go~~ between the NS and R setups. Let's try to define them in the circle case using the translation group rather than the infinitesimal generator.

Over the circle $\mathbb{R}/2\pi\mathbb{Z}$ we have two line bundles whose sections are the periodic (resp. anti-periodic) functions. Let's denote the spaces of L^2 sections by V^P and V^{AP}. Given a point x on the circle there ~~is~~ are two anti-periodic functions which are constant and have the value ± 1 except at x.



Actually these functions are defined for points ~~$\neq x$~~. There is a unique up to sign section of L^{AP} on $S^1 - \{x\}$ which is flat and has absolute value 1.

Let's denote by θ_x the unique anti-periodic function which is constant except at points $y \equiv x \ (2\pi\mathbb{Z})$

and which jumps from -1 to $+1$ as y crosses x. Then $\theta_x^2 = 1$ and $\theta_{x+2\pi} = -\theta_x$.

Now we can define an involution on $V^P \oplus V^{AP}$ by multiplying by θ_x. This interchanges the two summands.

Finally we consider the effect of translation:

$$e^{t\partial_x} f(x) = f(x+t).$$

Then

$$e^{t\partial_x}\,\theta_y(x)\,e^{-t\partial_x} = \theta_y(x+t) = \theta_{y-t}(x)$$

The operator θ_x is an orthogonal involution on $V^P \oplus V^{AP}$ and induces an automorphism of the Clifford algebra. The question arises as to what takes place on the associated Fock space.

Suppose we identify $V^P \oplus V^{AP}$ with the space of real L^2 functions on the double covering $\mathbb{R}/4\pi\mathbb{Z}$. θ_y is simply the function on the double covering such that θ_y is $+1$ between y and $y+2\pi$ and -1 between $y-2\pi$ and y. Thus we are looking at a discontinuous multiplication operator. This probably doesn't preserve the polarization and so it is not unitarily implementable on the Fock space, however it might have the same nature as a vertex operator. In other words we might be able to make sense of it after composing with imaginary time evolution.

I now want to connect up the above picture with the Ising model. I start with $V = \mathbb{R}^{2n}$ with the cyclic group $\mathbb{Z}/2N$ acting in the usual way. Then we have the splitting

$V = V^P \oplus V^{AP}$, the eigenspaces wrt τ^N. In a natural way V is the direct sum of two dimensional spaces V_k $k = 1, \dots, N$, where V_k is spanned by the kth and $(N+k)$th axes in $\mathbb{R}^{2N}$. So the Clifford algebra of V is a (super) tensor product

$$C(V) = C(V_1) \hat{\otimes} \cdots \hat{\otimes} C(V_k)$$

I want to draw elements of V in the form

$$\begin{pmatrix} v_1, \dots, v_N \\ v_{N+1}, \dots, v_{2N} \end{pmatrix}$$

so that elements of V_m are supported in the mth column, while V^P has equal rows, and V^{AP} has rows of opposite sign. Can I see the spin operator $\sigma^z(m)$?

I recall that the fermion structure in the Ising model comes from the operators $\sigma^x(m)$ which are used to grade the spin spaces: $\varepsilon(m) = \sigma^x(m)$. Once these are fixed, then the mth space of operators V_m is just $\varepsilon(1)\varepsilon(2)\cdots\varepsilon(m-1)\,\mathrm{End}^{odd}(S)$. Conjugation by $\varepsilon(j)$ acts on V by -1 on V_j and 1 on the V_m, $m \neq j$.

What I am trying to do is to see clearly the divergence between the pictures. Before one started with $S \otimes \cdots \otimes S$, N-times, and the operators $\varepsilon(1), \dots, \varepsilon(N)$. This determined the V_m and V. Next one wants $\sigma^z(m)$. Recall that $\sigma^z(m) = \varepsilon(1) \cdots \varepsilon(m-1)\big(c^+(m) + c^-(m)\big)$. This shows that $\sigma^z(m)$ belongs to the Clifford group. What is the orthogonal transformation corresponding to $\sigma^{(z)}(m)$? It is trivial on V_j $j < m$, a reflection in V_m, and -1 on V_j $j > m$.

Therefore the spin fields have no obvious relation to the $\sigma^z(m)$.

Let's review what we've learned. The correct discrete picture of spin fields comes by looking at the obvious action of $\mathbb{Z}/2N$ on $\mathbb{R}^{2N} = V$. Think of a double covering of the circle. Then write elements of V as $2 \times N$ matrices

$$\begin{pmatrix} v_1 & v_N \\ v_{N+1} & v_{2N} \end{pmatrix}$$

where τ shifts to the right and moves N to $N+1$ and $2N$ to 1. Then one has $V = V^P \oplus V^{AP} = \bigoplus_1^N V_m$. In this picture we can see the spin field as an orthogonal transformation diagonal in the natural basis:

$$\left(\begin{array}{cc|ccc} -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \end{array}\right)$$

If one starts with the diagonal matrix with entries $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}$ then the other spin fields are obtained by translating this one.

April 8, 1987

Getzler and Toeplitz operators. Toeplitz operators in a general setting were studied by Boutet de Monvelle. He proved an index theorem using the AS index thm. However the Toeplitz case is more general in a certain sense.

The setting is a compact contact manifold M. This is an odd dimensional manifold, $\dim = 2n+1$, with a 1-form α such that $\alpha(d\alpha)^{2n} \neq 0$ everywhere. Examples are 1) cosphere bundle of a manifold of dim n, 2) boundary of a smooth pseudo convex domain in $\mathbb{C}^n$.

~~One has T^*M splits into the line bundle~~ The structural group of T^*M can be reduced to $U(n)$. In effect $d\alpha$ defines a skew form on T^*M, whose radical is the line generated by α. Thus one has a reduction to $Sp(2n, \mathbb{R})$ which has maximal compact subgroup $U(n)$.

One uses the coordinates ξ, τ on T^*M. It would be better to say we have $T^*M = VM + \tilde{\mathbb{R}}\alpha$ and $\xi \in VM$, $\tau\alpha \in \tilde{\mathbb{R}}\alpha$.

Once the metric on M is chosen one gets a Dirac operator $\not{D}$. In the ψ convex domain boundary case one gets the operator $\bar{\partial}_b + \bar{\partial}_b^*$ of Kohn. This is subelliptic and has to be treated using a refined symbol calculus based on the Heisenberg group. One knows that the null space, i.e. the cohomology of the $\bar{\partial}_b$ operator is infinite diml for $q=0,n$ and finite in the middle. ~~But there~~ The two pieces $q=0,n$ lead to two Toeplitz algebras. Getzler would like to separate them, but can't in the general case.

To get a Toeplitz operator one takes a ψDO P on M and forms $T_P = SPS$ where S is the projection on the kernel of $\not{D}$. This gives a

Fredholm operator, and it has an index.

How is the index calculated? The symbol of P is a function $P(x,\xi,\tau)$ on T^*M- zero section which is homogeneous. I've forgotten to require P to be elliptic.

It turns out that the index of P depends only on $P(x,0,\tau)$ which must be two invertible functions

$$P(x,0,\tau) = \begin{cases} f_+(x)\,|\tau|^k & \tau>0 \\ f_-(x)\,|\tau|^k & \tau<0 \end{cases}$$

on M, where $k =$ order of P. I would like to understand the K-theory side of the situation. ~~The rough idea is that the~~

The first question is this. P is a ψDO on ~~the~~ odd diml M, so it has an index. This index is zero if P is a differential operator, but can be non-zero for ψDO's. So the question is <u>what does the index of P have to do with the index of T_P?</u>

Let's look at the circle $M=S^1$. Here $n=0$ so there are no ξ variables. The cosphere bundle S^*M is two copies of M, + so the symbol of P is just two invertible functions $f_\pm(x)$ on the circle. As an operator on functions on the circle

$$u(x) = \sum u_n e^{inx}$$

one has

$$(Pu)(x) = \sum f_{\text{sign}(n)}(x)\,|n|^k e^{inx} u_n + \text{lower order}$$

Let's take the order to be zero whence

$$(Pu)(x) = \sum f_{\text{sign}(n)}(x)\, e^{inx} u_n \quad + \text{lower order}$$

I will ignore the problems with $n=0$ so by working with anti periodic fns. I want to understand the index of this operator, so let's examine some simpler cases.

Suppose $f_-(x) = 1$ and $f_+(x) = e^{ikx}$. It's clear that the index is $-k$. On the other hand [redacted] if $f_-(x) = e^{ikx}$ and $f_+(x) = 1$, then the index is k. In general the index should be multiplicative with respect to composition so that we expect

$$\text{Index}(P) = -\text{degree}(f_+) + \deg(f_-).$$

Notice that if $f_+ = f_-$ which means we have ~~a~~ differential operator, then the index is zero.

What is now the Toeplitz operator T_P? First it involves spinors, [redacted] and $\not{D} = \bar{\partial}_b + $ [redacted] $\bar{\partial}_b^*$, and S. Since $n=0$ one has only $q=0$, so $\bar{\partial}_b = 0$ and it would appear that $S = 1$. Thus $T_P = P$.

Let's examine the general $\bar{\partial}$ Neumann problem. Let's consider a domain X in $\mathbb{C}^n$ defined by a smooth real function f, so that $X = \{z \mid f(z) < 0\}$, and $M = f^{-1}(0)$. Suppose $df \neq 0$ along M. Then $df(m)$ vanishes on a hyperplane in [redacted] $T_m(X)$, and we can multiply by the complex structure J to get another hyperplane and then intersect. This gives the largest complex subspace of $T_m(X)$ contained in $T_m(M)$. So the tangent bundle to M has a codim 1 subbundle which is complex and such that the ~~quotient~~ is the trivial [redacted] real line bundle, the trivialization being given by $J \cdot df$.

Consider the Dirac operator $\frac{1}{i}\partial_x$ on S^1 and a loop $g : S^1 \longrightarrow U(V)$. The former represents an odd K homology class of S^1, the latter an odd K-cohomology class. The problem is to construct an even K-coh. class of a point, that is a Fredholm operator or point in a Grassmannian.

The Kasparov cup product or composition product is supposed to do this. However to use this one must replace g by a family of self adjoint Fredholm operators. Thus we must replace $\tilde{V}$ by an infinite-dimensional Hilbert bundle and g by a self-adjoint Fredholm operator on the fibres. Yes.

Now we might try to proceed by first looking at the symbols. What we are perhaps trying to do is to form a ΨDO on the circle. The symbol of this ΨDO is an even K-homology class of the cotangent bundle $T^*(S^1) = S^1 \times \mathbb{R}$. One might try to define it as ~~the~~ cup product of $[g] \in K^1(S^1)$ and $[\xi] \in K^1_c(\mathbb{R})$

April 9, 1987

Let's consider the problem of mixing the Dirac ∂_x on S^1 with a loop $g: S^1 \to U(V)$. If I want to proceed by KK cup products then these have to be modified. The Dirac operator is to be replaced by an F such as $\frac{1}{i}\partial_x / \sqrt{1-\partial_x^2}$. It operates on a Hilbert space $L^2(S^1)$ which is as module over $C(S^1)$, and one has F^2-1, $[F,a]$ compact for all $a \in C(S^1)$.

Similarly g must be replaced by an F' operating on a Hilbert C^*-module M over $C(S^1)$. Such an M would be given by the continuous sections of a Hilbert space bundle over the circle, call it $\mathcal{E}$. F' would have to be a family of self-adjoint Fredholm operators on the fibres such that $(F')^2-1$ is compact. Thus F' results by using the AS homotopy equivalence between quasi-involutions (these are s.a. A's such that $-1 \le A \le 1$ and $A^2-1 \in \mathcal{K}$) and unitaries $\equiv -1 \mod \mathcal{K}$.

Then one forms

$$\mathbb{C}^2 \otimes L^2(S^1) \otimes_{C(S^1)} C(S^1, \mathcal{E}) = \mathbb{C}^2 \otimes L^2(S^1, \mathcal{E})$$

where the $\mathbb{C}^2$ is needed because we are starting with two odd classes. On this Hilbert space we construct an odd degree quasi-involution by combining $\gamma^1 F$ and $\gamma^2 F'$ in the Kasparov way

$$M\gamma^1 F + N\gamma^2 F' \qquad M^2+N^2=1$$

~~Suppose~~ There is a way to do the construction of $\mathcal{E}$ and F' explicitly. Let's recall that there is a canonical family of Dirac operators on the circle parametrized by $U(V)$. There is a canonical vector bundle E over $U(V) \times S^1$ which is obtained by taking the trivial bundle $\tilde{V}$ over $U(V) \times [0,1]$ and identifying the ends via the tautological automorphism of $\tilde{V}$ over $U(V)$. Another description is to take the trivial bundle $\tilde{V}$ over $U(V) \times \mathbb{R}$ and introduce a $\mathbb{Z}$-action. On a section ~~$g(x)$~~ $f: U(V) \times \mathbb{R} \longrightarrow V$ the action is

$$(n * f)(g, x) = g^{+n} f(g, x - 2\pi n)$$

Then one divides out by this action to get E over $U(V) \times S^1$. Sections of E are maps $f(g,x)$ from $U(V) \times \mathbb{R}$ to V such that

$$f(g, x + 2\pi n) = g^n f(g, x)$$

~~We have the family of Diracs on the fibres of~~

The operator ∂_x is well defined on sections of E and is linear over the functions on $U(V)$, so it is a family of Diracs on the fibres of $U(V) \times S^1$ over $U(V)$. Then F' can be obtained by taking the phase.

Next we can form

$$L^2(S^1, \mathcal{E}) = L^2(S^1 \times S^1, (g \times id)^* E).$$

where $g : S^1 \to U(V)$ is the given loop.

Let's review the situation. I am trying to construct the cap product of the K homology class on S^1 represented by ∂_x and the K coh. class represented by a loop g. I wish to use the Kasparov process, which means I have to invert the AS equivalence and replace the values of g which are unitaries by skew adjoint Fredholms. This I can do in a nice way using the fact that unitaries parametrize connections over the circle.

Now it is necessary to choose a metric on the circle in order to get a Dirac operator. Excluding singular metrics it would appear that only the length matters. But this point has to be checked later.

I think it is clear that one can describe the ultimate operator obtained by KK methods as follows. We have given ∂_x on S^1 and $g : S^1 \to U(V)$. We then consider

$$\begin{tikzcd} \tilde{E} \arrow[d] & E \arrow[d] \\ S^1 \times S^1 \arrow[r, "\check{g} = g \times id"] & U(V) \times S^1 \end{tikzcd}$$

and pull back to get $\tilde{E}$ over $S^1 \times S^1$. Sections of $\tilde{E}$ are smooth maps $f = f(x,y) : \mathbb{R} \times \mathbb{R} \longrightarrow V$ such that

$$\begin{cases} f(x+1, y) = f(x,y) \\ f(x, y+1) = g(x) f(x,y) \end{cases}$$

On sections of $\tilde{E}$ we have the operator ∂_y which is a family parametrized by the first circle. Now we have to make ∂_x act on sections of $\tilde{E}$. This means choosing some sort of connection in the x direction.

I should probably take the case where $\dim V = 1$ since in this case there is a rather natural connection defined already in E over $U_1 \times S^1$. This choice will be valid independent of g

In fact, once we fix a connection on E extending the canonical one in the circle direction, then as g varies we have a nice family of connections on the bundles $\tilde{E}_g = (g \times id)^* E$.

Now it's clear that Kasparov's construction yields the "phase" of the Dirac operator over $S^1 \times S^1$ with coefficients in $\tilde{E}_g$ with its connection. This depends on the metric chosen on the torus.

What sort of program is to be undertaken? The sort of thing I wanted to do originally is to obtain a point of a restricted Grassmannian for the Hilbert space $\mathbb{C}^2 \otimes L^2(S^1, V)$, which is approximately the graph assoc. to $h\gamma^1\partial_x + \gamma^2 X$ where $g = \frac{1+X}{1-X}$. Here h probably has to be taken small.

What I can probably do is the following. Over the torus with the Dirac $h\gamma^1 D_x + \gamma^2 D_y$ and h small, the zero modes are perhaps going to be essentially sections over the x circle with values in low lying states for the Dirac in the y direction.

I can get fairly explicit about this in the line bundle case. This ought to lead to some sort of approximation between the operator $h\gamma^1\partial_x + \gamma^2 X$ and the 2 diml Dirac operator in the low energy range.

April 10, 1987

Background. Yesterday I carried out in a fairly explicit way the cap product of the class of the Dirac operator $\frac{1}{i}\partial_x$ and the class of a loop $g: S^1 \longrightarrow U(V)$. I obtained the Dirac operator on the torus with coefficients the bundle $(g \times id)^*(E)$, where E is the Poincaré bundle over $U(V) \times S^1$. The idea now is to change the metric on the torus and to look at the low energy states. If I make the second circle very small, then I might be able to compare the Dirac on the torus to a suitable Dirac on the circle twisted by g.

It is essential to calculate the simplest example in order to see what is going on. This means I want to take g to be $x \longmapsto e^{ix}$ from S^1 to U_1, in which case I know that the Dirac operator on the torus is closely related to the harmonic oscillator. This has to be calculated explicitly.

Let's recall the map which assigns to $u(x) \in \mathcal{S}(\mathbb{R})$ a section of a line bundle over $S^1 \times S^1 = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$

$$f(x,y) = \sum_n g(x)^n \, u(y-n) \qquad \qquad g: \mathbb{R}/2\pi\mathbb{Z} \to U_1 \text{ given}$$

Then

$$\begin{cases} f(x+2\pi, y) = f(x,y) \\ f(x, y+1) = g(x) f(x,y) \end{cases}$$

Call this line bundle L. Let's now define a connection on it. $\nabla_y = \partial_y$ works. Let $\nabla_x = \partial_x + a(x)$. Then we want ∇_x to commute with

$$\left(g^{-1} e^{\partial_y}\right) f(x,y) = g^{-1}(x) f(x, y+1)$$

So

$$g^{-1}e^{\partial_y}\left(\partial_x + a(x,y)\right) = g^{-1}\left(\partial_x + a(x,y+1)\right)e^{\partial_y}$$

$$= \left(\partial_x + \partial_x(\log g) + a(x,y+1)\right)\, g^{-1}e^{\partial_y}$$

Thus we want $\quad a(x,y) = a(x,y+1) + \partial_x \log(g)$, so

$$a(x,y) = -y\,\partial_x \log(g)$$

is the simplest solution. Thus the connection is

$$\begin{cases} \nabla_x = \partial_x - y\,\partial_x(\log g) \\ \nabla_y = \partial_y \end{cases}$$

What are these operators back on $\mathcal{S}(\mathbb{R})$? Do for $g = e^{iqx}$.

$$f(x,y) = \sum e^{inqx}\, u(y-n)$$

$$\nabla_x f(x,y) = \underbrace{\sum e^{inqx}\{inq - y\,iq\}\,u(y-n)}_{\text{transform of } -iqt\,u(t).}$$

Thus we have the correspondence

$$\nabla_x \longleftrightarrow -iqt$$

$$\nabla_y \longleftrightarrow \partial_t$$

Check $\quad [\nabla_x, \nabla_y] = [\partial_x - iqy, \partial_y] = iq$

$$[-iqt, \partial_t] = iq.$$

The degree of the line bundle is

$$\int \frac{i}{2\pi}\, iq\, dx\,dy = -q$$

and this check because the $\partial_{\bar z}$ operator is

$$\nabla_x + i\nabla_y = i(\partial_t - qt)$$

which has elements in $\mathcal{S}(\mathbb{R})$ in its kernel for $q<0$.

I'm not sure that $u \longrightarrow f$ gives all sections of L unless $q = \pm 1$, so let's take $q=-1$.

At this point I have to worry about the metric on the torus and the spectrum of the Dirac operator on the torus. I will want to keep the x, y directions perpendicular.

The metric on the torus is used to define the Clifford algebra. We use a constant coefficient metrix; then the Clifford algebra at each point will be generated by $\gamma(dx)$, $\gamma(dy)$ which anti commute (assuming $dx \perp dy$) and whose squares are scalars. Also these are to be self-adjoint. Thus

$$\not{D} = a\gamma^1\nabla_x + b\gamma^2\nabla_y = \begin{pmatrix} 0 & a\nabla_x - ib\nabla_y \\ a\nabla_x + ib\nabla_y & 0 \end{pmatrix}$$

To find the spectrum we consider

$$\not{D}^2 = a^2\nabla_x^2 + b^2\nabla_y^2 + ab\,\underbrace{\gamma^1\gamma^2}_{(i\varepsilon)}\underbrace{[\nabla_x, \nabla_y]}_{(+iq)}$$

$$(i\varepsilon)(+iq) = \begin{pmatrix} -q & 0 \\ 0 & +q \end{pmatrix}$$

Now we know that $a^2\nabla_x^2 + b^2\nabla_y^2$ is supposed to be a harmonic oscillator type of operator and it is ≤ 0, so the spectrum is of the form $-(n+\frac{1}{2})\omega$, $n \in \mathbb{N}$ with $\omega > 0$. The signs check.

Now I should proceed carefully so as to learn what I might prove in general.

What we learn first is that the spectrum of the Dirac operator $\not{D}$ depends on the product ab.

I should begin by trying to figure out what I would like to find. The rough idea is that if b is large relative to a, then the vertical Dirac operator $b\nabla_y$ is large. Now we know its spectrum is a coset for $i2\pi\mathbb{Z}b \subset i\mathbb{R}$. The goal I am after is somehow to see the Dirac operator of the horizontal circle with coefficients in the ground line bundle for the family of vertical Dirac operators.

Let us see how this works, but using the model $S(\mathbb{R})$. Then

$$a\gamma^1\nabla_x + b\gamma^2\nabla_y = \begin{pmatrix} 0 & a(-igt)-ib\partial_t \\ a(-igt)+ib\partial_t & 0 \end{pmatrix}$$

$$= i\begin{pmatrix} 0 & -b\partial_t + at \\ b\partial_t + at & 0 \end{pmatrix}$$

Now multiplication by the function e^{ix} on the torus corresponds to the unit shift $u(t) \to u(t+1)$, and multiplication by $\varphi(y)$, φ periodic, corresponds to multiplication by $\varphi(t)$.

I think what I want to do is to keep ab fixed and let b become large. This way the spectrum of the Dirac on the torus is not changing.

I have learned that the Dirac operator on the torus can be identified with the operator

$$(*) \qquad i\begin{pmatrix} 0 & -b\partial_t + at \\ b\partial_t + at & 0 \end{pmatrix}$$

studied by Witten and Hörmander. The Dirac operator on the torus is the sum the vertical Dirac, here represented by ∂_t, and the horizontal Dirac, here represented by t. I want to look at the situation over the functions on the first circle. These are linear combinations of the unit shift e^{∂_t}.

The hope is that when b is large the index question can be reduced to something of multiplicity one over the functions on the first circle. The rough idea would be to find some subspace of $L^2(\mathbb{R})$ stable under e^{∂_t} which would be a very good approximation as far as the index is concerned.

Let us consider the above operator $(*)$ in the limit as $a \longrightarrow \infty$. I would like to consider $S(\mathbb{R})$ as a module over periodic functions. (Thus I take the Fourier transform)

April 11, 1987

We consider over the torus $M = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ a [redacted] line bundle with connection having constant curvature and degree 1. Then we consider the Dirac operator on the torus with coefficients this line bundle + connection relative to a metric on the torus. This Dirac operator is

$$\gamma^1 a \nabla_x + \gamma^2 b \nabla_y$$

assuming that the x, y directions remain $\perp$. [redacted] We now want to look at this operator when $b/a \longrightarrow \infty$. I want to find a suitable limit; the idea is that for b large a low energy eigenfn for the Dirac when restricted to a vertical circle must be nearly constant, which means that it must be nearly zero when the monodromy along the vertical circle is non trivial.

In the simplest example we know that the Dirac operator ~~is~~ related to a harmonic oscillator [redacted], so [redacted] we know the spectrum exactly. In fact the Dirac operator is a scalar times the pair of creation + annihilation operators for the oscillator. If we keep ab fixed, so the harmonic oscillator has the spectrum $(n+\frac{1}{2})$, $n \in \mathbb{N}$, then the different complex structures are just the different possible points in the [redacted] UHP. The limit we are taking is $\tau \longrightarrow i\infty$ along the imaginary axis.

So we are looking at [redacted] the limiting case of a complex structure on the torus. This is a foliation; apparently the leaves are the circles in the y direction.

Let's recall the formulas whereby we identify sections of the Poincaré line bundle over $S^1 \times S^1$ with Schwartz functions on the line

$$f(x,y) = \sum_{n\in\mathbb{Z}} e^{iny}\, u(x-n) \quad \longleftrightarrow \quad u(t)$$

$$\begin{cases} f(x, y+2\pi) = f(x,y) \\ f(x+1, y) = e^{iy} f(x,y) \end{cases}$$

$$\nabla_x = \partial_x \quad \longleftrightarrow \quad \partial_t$$

$$\nabla_y = \partial_y - ix \quad \longleftrightarrow \quad -it$$

$$[\partial_x, \partial_y - ix] = -i = [\partial_t, -it]$$

The Dirac operator is

$$a\gamma^1\nabla_x + b\gamma^2\nabla_y \iff \begin{pmatrix} 0 & +a\partial_t - bt \\ a\partial_t + bt & 0 \end{pmatrix}$$

It thus consists of $a\partial_t + bt$ and its adjoint essentially. Note that

$$[a\partial_t + bt, -a\partial_t + bt] = 2ab$$

so that if $ab = \frac{1}{2}$, then $a\partial_t + bt$ is an annihilation operator. The spectrum of the Dirac is 0 and $\pm i\sqrt{2n+1}$, $n \in \mathbb{N}$.

Now what we want to do is to let $b \to \infty$ while keeping ab fixed. The goal is to see if we can approximate the small energy behavior of the Dirac with some sort of operator over the x circle. Notice that multiplication by $e^{2\pi i x}$ corresponds to multiplying $u(t)$ by $e^{2\pi i t}$.

Now as $b \to \infty$ the ground ~~state~~, which is $(\text{const}) \times \left(e^{-\frac{1}{2}\frac{b}{a}x^2} \right)$ peaks sharply around $x = 0$. If we rescale $t \to t\sqrt{\frac{a}{b}}$ then this becomes $\text{const} \times e^{-\frac{1}{2}t^2}$, however the operator $e^{2\pi i x}$ appears to become $e^{it\sqrt{\frac{a}{b}}}$. We can check this as follows. Look at $e^{2\pi i x} \longleftrightarrow e^{2\pi i t}$ acting on the ground state $e^{-(1/2)\lambda t^2}$, $\lambda >> 0$. Then this operator is very nearly equal to 1 on this ~~state~~. Similarly $e^{2\pi i t (1/\lambda)^{1/2}}$ is **slowly** varying, hence it is **very** nearly 1 on $e^{-\frac{1}{2}t^2}$.

So what we can say is that we have the fixed Dirac operator

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \partial_t - t \\ \partial_t + t & 0 \end{pmatrix}$$

acting on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, but we have the periodic functions of x acting so that $e^{2\pi i x} = e^{2\pi i \varepsilon t}$ with $\varepsilon \to 0$.

April 12, 1987

It seems desirable to return to the Kasparov Hilbert C^* module point of view. Recall that a Hilbert C^* module over the algebra $C(S^1)$ is more or less the same as the space of sections of a continuous family of Hilbert spaces over S^1, for example, the space of continuous sections of a Hilbert space bundle.

Example. Take the space to be $U(V)$ and consider over it the trivial bundle $\tilde{V}$. Consider the family of subspaces $\{W_g \subset V,\ g \in U(V)\}$, where W_g is the orthogonal complement of $\mathrm{Ker}(g+1)$. The Hilbert C^*-module is probably the space of continuous $f: U(V) \to V$ which is the uniform closure of the image of $g+1$.

If $V = \mathbb{C}$, then we are looking at $f: U_1 \longrightarrow \mathbb{C}$ such that $f(g) = 0$ when $g = -1$, so if $g = e^{2\pi i x}$ then we have continuous functions $f(x)$ such that $f(-1) = 0$.

Example. Take the space to be $U(V)$ and consider over $U(V) \times S^1$ the canonical vector bundle E whose sections are smooth $\hat{f}(g,y)$ such that

$$\hat{f}(g, y+2\pi) = g^{-1}\hat{f}(g,y)$$

Then we get a Hilbert bundle over $U(V)$ with fibre at g the L^2 function on S^1 which are g-periodic in the above sense. Moreover we have a family of Dirac operators $\frac{1}{i}\partial_y$ on the fibres.

I a Hilbert module over $C(U(V))$ by considering $\hat{f}(g,y)$ which are continuous in g and L^2 in y. Thus we use the L^2 norm in y and the sup norm in g.

In order to do calculations, let us now pass to $V = \mathbb{C}$ and write $g = e^{2\pi i x}$. Then

$$\hat{f}(x+1, y) = \hat{f}(x, y)$$
$$\hat{f}(x, y+2\pi) = e^{-2\pi i x}\hat{f}(x,y)$$

so if we put $f(x,y) = e^{ixy}\hat{f}(x,y)$ we have

$$f(x, y+2\pi) = e^{ixy}e^{ix2\pi}\underbrace{\hat{f}(x, y+2\pi)}_{e^{-2\pi i x}\hat{f}(x,y)} = f(x,y)$$

$$f(x+1, y) = e^{i(x+1)y}\hat{f}(x+1,y) = e^{iy}f(x,y)$$

which are the equations used on p. 612.

The Dirac operator in the y-direction is $\frac{1}{i}(\partial_y - ix)$.

Now let us consider the family of Hilbert spaces W_g' over ~~[illegible]~~ $U(V)$, where W_g' is the sum of eigenspaces for the Dirac belonging to eigenvalues in $(-\frac{1}{2}, \frac{1}{2})$. Recall that $\frac{1}{i}\partial_y$ on the g-periodic functions: $u(y+2\pi) = g^{-1}u(y)$ has the eigenvalues λ where $e^{2\pi i\lambda}$ = an eigenvalue of g^{-1}. Thus the set of eigenvalues ~~[illegible]~~ is a union of cosets of $\mathbb{Z}$ in $\mathbb{R}$, and $\lambda \in \frac{1}{2} + \mathbb{Z}$ corresponds to $g = -1$.

Now I know that evaluating at the basepoint gives an isomorphism between the space W_g where $g \neq -1$ with the space W_g' where $|\frac{1}{i}\partial_y| < \frac{1}{2}$. I should be thinking ~~[illegible]~~ of $\{W_g\}$ and $\{W_g'\}$ as being Hilbert C^*-module over $C(U(V))$ equipped with F's, quasi-involutions. I guess the question now is whether I can mix this Hilbert C^* module with the operator ∂_x, where we have now restricted to the x circle using the loop $\mathbb{R}/\mathbb{Z} \to U(V)$ which is supposed given.

It seems that I am doing something very close to the Kasparov construction, and so I should try to understand his method.

The C^* algebra is $C(S^1)$, where S^1 here is $\mathbb{R}/\mathbb{Z}$ with coordinate x. We have two Hilbert C^*-modules over $A = C(S^1)$. The first consists the ideal of functions vanishing at $x = 1/2$. This is sections of the family $\{W_x \subset \mathbb{C}^q\}_{x\in S^1}$, where W_g the sum of the eigenspaces of $g = e^{2\pi i x}$ which are $\neq -1$. Thus $W_x = \mathbb{C}$ for $x \neq \frac{1}{2}$ and 0 for $x = 1/2$.

The other Hilbert module is the trivial module consisting of continuous sections of the trivial bundle with fibre $L^2(S^1)$, where here $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ has coordinate y. At x we have the Dirac operator $\frac{1}{i}(\partial_y - ix)$ on the fibre $L^2(S^1)$. There is a natural embedding of the first Hilbert module inside the second which is compatible with the operators $-x$ and $\frac{1}{i}(\partial_y - ix)$ on the two modules.

Now letting M denote either Hilbert module we want to form the tensor product

$$L^2(S^1) \otimes_{C(S^1)} M$$

and then complete it. In the first can we obtain $L^2(S^1)$ and in the second we obtain $L^2(S^1 \times S^1, E)$.

Next we wish to take the Dirac $\frac{1}{i}\partial_x$ on the first circle and combine it somehow with the F on M in order to obtain a Fredholm. Now I sort of see what must be done in the case of the torus, but it would be important to see what happens already on the circle. This assumes that one can in fact construct the cup product operator in the smaller situation.

What might this operator be? Suppose we look for a ψDO of order 0. It might not be a ψDO because we must pay attention both to the compact operators on $L^2(S^1)$ and the compact operators in the Hilbert module which are the continuous functions $\varphi(x) \ni \varphi(\frac{1}{2}) = 0$.

To get some feeling for this situation suppose we look at the Schwartz space model:

$$f(x,y) = \sum e^{iny} u(x-n)$$

$$u(x) = \frac{1}{2\pi}\int f(x,y)\, dy$$

The small Hilbert module under discussion consists of continuous $f(x,y)$ which are constant in the y direction and which vanish for $x = \frac{1}{2}$. Thus the corresponding u must satisfy

$$u(t) = 0 \qquad \text{for} \qquad t \notin (-\tfrac{1}{2}, \tfrac{1}{2})$$

Note that the vertical Dirac $\frac{1}{i}\nabla_y$ is $-t$, so we are in fact looking at where $\left|\frac{1}{i}\nabla_y\right| < \frac{1}{2}$.

So what do we have? We recognize that the Hilbert space $\left(L^2(S^1) \otimes_{C(S^1)} M\right)^1$ is $L^2(-\frac{1}{2}, \frac{1}{2})$, and now we want to define some kind of Fredholm operator of odd degree on two copies of this Hilbert space. I can move things around a bit and shrink $(-\frac{1}{2},\frac{1}{2})$ to $(-\varepsilon,\varepsilon)$. It looks to me like we are exactly in the situation studied by Hörmander.

It is now necessary to understand ΨDO's on the circle S^1. In general given a manifold X say compact, [redacted] the symbol of a ΨDO on X determines a pair E,F of bundles on X and an isomorphism $\sigma: \pi^*E \xrightarrow{\sim} \pi^*F$, where $\pi: S^*X \to X$ is the cosphere bundle. K-theoretically one has a K-class on X or the disk [redacted] bundle D^*X [redacted] represented by $E-F$ together with a trivialization over S^*X. Thus the [redacted] symbol gives an element of

$$K^0(D^*X, S^*X) = K_c(T^*X)$$

Now where X is the circle $S^*(S^1) = S^1 \times \{\pm 1\}$ so we have two isomorphisms $\sigma_\pm : E \Rightarrow F$. Thus we can suppose $E=F$ that $\sigma_- = id$ and $\sigma_+ = g$ a given invertible function on S^1. We have seen that the ΨDO associated to g is, or can be realized by

$$\sum c_n e^{inx} \longmapsto \sum_{n<0} c_n e^{inx} + g(x) \sum_{n\geq 0} c_n e^{inx}.$$

Thus one is dealing essentially with the Toeplitz operator defined by g. (Notice that the kernel consists of $f \in H_+$ such that $gf \in H_-$, and this is the kernel of the Toeplitz map [redacted] $f \mapsto (gf)_+$.)

[redacted] In the AS proof they show that is one has $j: U \hookrightarrow X$ open with X compact, and an [redacted]

$a \in K_c(TU)$, then the analytical index of $j_*(a) \in K_c(TX)$ is independent of the choice of j, X. What this means in the case of the circle that if we know that $g = 1$ outside an open interval U, then the index problem is supported inside U. Question: How does one see this?

A natural method might be to take the Fredholm ψDO which we think of as going from one ~~copy~~ of $L^2(S^1)$ to another and to produce a subspace of the first $L^2(S^1)$ which is complementary to the kernel and perhaps attached to the complementary interval to U.

April 13, 1987

I want to discuss ψDO's on the circle.

Let's think of a ψDO on S^1 in terms of its Schwartz kernel, which is a distribution with a certain kind of singularity along the diagonal.

Let's describe the singularity. Recall that

$$\int_{-\infty}^{\infty} e^{i\xi x} \frac{d\xi}{2\pi} = \delta(x)$$

$$\int_0^{\infty} e^{i\xi x} \frac{d\xi}{2\pi} = \int_0^{\infty} e^{i\xi(x+i\varepsilon)} \frac{d\xi}{2\pi} = \frac{i}{2\pi} \frac{1}{x+i\varepsilon}$$

$$\int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi}{|\xi|} \frac{d\xi}{2\pi} = \frac{i}{2\pi}\left(\frac{1}{x+i\varepsilon} + \frac{1}{x-i\varepsilon}\right) = \frac{i}{\pi} P\left(\frac{1}{x}\right)$$

Thus

$$\delta(x) + \frac{i}{\pi} P\left(\frac{1}{x}\right) = \frac{i}{\pi} \frac{1}{x+i\varepsilon} \qquad \text{or}$$

$$\frac{1}{x+i\varepsilon} = P\left(\frac{1}{x}\right) - i\pi\,\delta(x)$$

Next let's justify using the Fourier transform instead of Fourier series even when we want to discuss the circle. We are going to be looking at distributions $K(x,y)$ on the product $S^1 \times S^1$ with singularities along the diagonal. The natural thing is to use F.S.

$$K(x,y) = \sum_n K(x,n)\, e^{in(x-y)}$$

and to specify the singularity by asymptotic conditions on $K(x,n)$ as $n \to \infty$. Fix x, say $x=0$. Since the only singularity of $K(y) = K(0,y)$ is at $y=0$, we can use $1 = \varphi + (1-\varphi)$ where $\varphi \equiv 1$ near 0 and φ is supported near zero, to replace K by φK.

As $K(y)$ is supported near zero, we can transport it to the line and take its F.T.

$$\int_{y\in(-\delta,\delta)} e^{+i\xi y} K(y)\, dy = K(\xi).$$

Now

$$K(y) = \int e^{-i\xi y} K(\xi) \frac{d\xi}{2\pi}$$

is K transported to $\mathbb{R}$ and so if we want the periodic $K(y)$ we sum over $y + 2\pi n$ and obtain

$$K(y) = \sum_n K(n) e^{-i n y}$$

up to signs and 2π factors. The point is that the singularity of $K(y)$ near $y=0$ is coded in the asymptotics of the transform.

Anyway, we see that for ΨDO's of order zero on S^1 the basic singularity types are $P(\frac{1}{x})$ and $\delta(x)$. What's next are order -1 operators which are in $\mathcal{L}^p$ for $p>1$.

Next we want to mix these Hilbert transform operators with multiplication operators. To be specific we want to combine $\frac{\xi}{|\xi|}$ with $\frac{x}{|x|}$. We expect an operator with index ± 1.

Let's try to obtain a ΨDO on the line but which is in some sense supported near $x=0$. We know that any ΨDO has a symbol consisting of two invertible functions on the line, namely the symbol for $\xi = \pm 1$. For the support to be near $x=0$ means roughly the following. In general the symol is an isomorphism $\sigma : \pi^* E \longrightarrow \pi^* F$ over S^*X. I think you want σ to extend to an isomorphism on $T^*X \,|\, X-K$, where K is the support. And then we can deform so that σ agrees with a fixed isom. of E, F over $X - K$. The picture of the kernel would then be a δ function on the diagonal + a kernel supported in $K \times K$.

We are after something like the phase

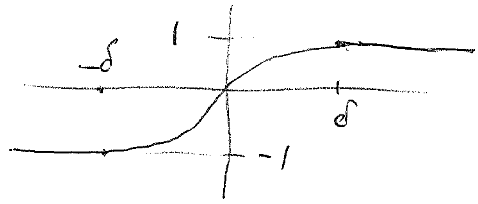
$$\frac{x + i\xi}{|x + i\xi|}$$

smoothed out at the origin. Find the leading symbol

$$\lim_{|\xi| \to \infty} \frac{x+i\xi}{|x+i\xi|} = \begin{cases} i & \xi > 0 \\ -i & \xi < 0 \end{cases}$$

So this isn't right.

Let $\rho(x)$:



The sort of symbol we are after is

$$\rho(x) + i\sqrt{1-\rho(x)^2}\,\frac{\xi}{|\xi|}$$

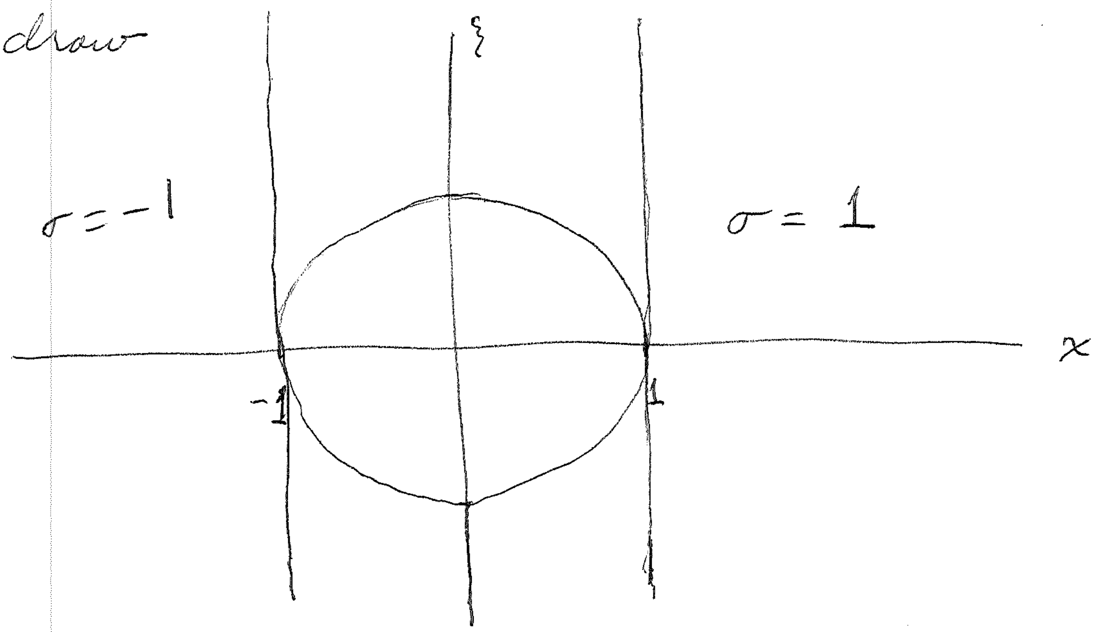
This is $+1$ for $x \geq \delta$
-1 —— $x \leq -\delta$

and for $\xi > 0$ it moves over the top half of the unit circle (resp for $\xi < 0$ over the bottom half)

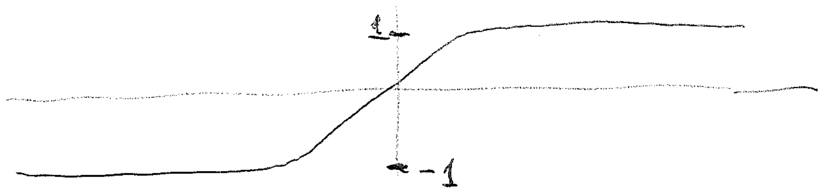
But I would still like to understand Kasparov's M, N. It's possible that I should take $\mathrm{sgn}(x)$ out of $\rho(x)$, i.e. $\rho(x) = M\,\mathrm{sgn}(x)$.

April 14, 1987

Here is the problem: Let's go over yesterday's calculation. We work on $\mathbb{R}$ with the Hilbert involution $\frac{\xi}{|\xi|}$ and the involution $\frac{x}{|x|}$. We propose to define a ΨDO of order zero. We want the symbol to depend on x only for $|x| > 1$, and to be related to the phase of $x + i\xi$. So we draw



On the top semi-circle one has the phase going on the top half of the unit circle and similarly for the bottom circle. Thus one chooses $\rho(x)$



and considers the symbol

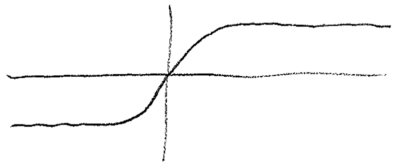
$$\rho(x) + i\sqrt{1-\rho^2(x)}\,\frac{\xi}{|\xi|}$$

It's clear that this expression only uses $\frac{\xi}{|\xi|}$ when $|\rho(x)| < 1$ i.e. near $x = 0$.

The problem is unfortunately that the symbol is -1 for $x \ll 0$ and $+1$ for $x \gg 0$, and I want an operator which depends only on $e^{i\pi x}$. Put another way, how can I transform this ψDO near $x=0$ to the circle?

April 15, 1987

Let $\rho(x)$ be as above:



Think of the circle as the 1-point compactification of the line. Consider the trivial line bundle over the circle; its continuous sections form of Hilbert module $C(S^1)$ over $C(S^1)$. Because $\rho(x)$ is discontinuous at ∞, multiplication by ρ is not an operator on $C(S^1)$. However it is an operator on $C_0(\mathbb{R})$, the space of functions vanishing at ∞. Moreover $\rho^2 - 1$ is compact on $C_0(\mathbb{R})$ considered as a Hilbert module over $C(S^1)$ or over $C_0(\mathbb{R})$. This is because we can write ρ^2-1 as a rank 1 operator, namely $\rho^2 - 1 = \varphi^2 =$ the composition of the vector φ with inner product by φ.

Remark: In the case of a general loop $g : S^1 \longrightarrow U(V)$ we will not be able to adjust the algebra of functions on the circle to fit the places where g acquires the eigenvalue -1. Thus we must work with $A = C(S^1)$.

Thus we must consider $C_0(\mathbb{R})$ as a Hilbert module over $C(S^1)$. I think then it is more or less clear that

$$L^2(S^1) \otimes_{C(S^1)} C_0(\mathbb{R}) \quad \text{becomes} \quad L^2(S^1)$$

upon completion.

Now comes the question of whether we can form the Kasparov product of $\xi/|\xi|$ with $\rho(x)$. What this means is to find an operator of the form

$$F = \gamma^1 M \rho(x) + \gamma^2 N \frac{\xi}{|\xi|}$$

with $M^2 + N^2 = 1$. This operator F is to satisfy $F^2 - 1 \in$ compact operators. It's construction should involve showing that

$$F^2 - 1 \in \left\{ \begin{matrix} \text{ideal generated by} \\ \mathcal{K} \text{ and } C_0(\mathbb{R}) \end{matrix} \right\}$$

In this example things can be done because the operator F will go between the trivial line bundle and the Mobius line bundle. It seems to be hard to make this work in general. And I feel that it's not a good way to proceed. It's naive to expect to find an F in general; what you want is a graph.

Work over $C(U(V))$ with the Hilbert module $\mathcal{M}$ of continuous maps $f: U(V) \longrightarrow V$ such that $f(g) \in \operatorname{Im}(g+1)$. Define F on $\mathcal{M}$ so that

$$g = \left(iF + \sqrt{1-F^2} \right)^2$$

Thus F at $g \in U(V)$ is the self adjoint operator with eigenvalues in $(-1,1)$ acting on $(g+1)V$ satisfying the above. Then we have

$$g = 1 - 2F^2 + 2iF\sqrt{1-F^2}$$

$$\frac{1}{2}(g+g^{-1}) = 1-2F^2$$

$$\frac{g+g^{-1}+2}{2} = 2(1-F^2) \quad \text{or}$$

$$1-F^2 = \frac{1}{4}g^{-1}(g+1)^2$$

This shows the operator $1-F^2$ on $\mathcal{M}$ is compact as an operator on a Hilbert module, because it is

$$\frac{1}{4}\sum_{i=1}^{n} (g+1)e_i \langle (g+1)e_i |$$

where e_i is an orthonormal basis for V. Here $(g+1)e_i$ is an obvious element of $\mathcal{M}$.

Again arises the question as to whether the Kasparov cup product theory allows one to take this F on $\mathcal{M}$ and the Hilbert F on $L^2(S^1)$ and combine them. (I forgot to mention that one has to take a loop $S^1 \longrightarrow U(V)$ and pull-back, i.e. make base extension via the homomorphism $C(U(V)) \longrightarrow C(S^1)$.)

I don't see what might go wrong with the M, N game. However if the loop $S^1 \to U(V)$ is identically -1, on some interval, then the tensor product

$$L^2(S^1) \otimes_{C(S^1)} \mathcal{M}$$

will not be all of $L^2(S^1) \otimes V$.

Here is a general construction over $U(V)$.
Start with a map $g: S^1 - \{-1\} \longrightarrow [-1,1]$ which is continuous and such that

$$g(-1_+) = \lim_{\zeta \to -1_+} g(\zeta) = +1$$

$$g(-1_-) = -1.$$

For example, we can take g to be the inverse of the map

$$x \longmapsto \left(ix + \sqrt{1-x^2}\right)^2 = \zeta$$

i.e. $$g(\zeta) = \operatorname{Im}(\zeta^{1/2}) = \sin(\theta/2) \quad \text{if } \zeta = e^{i\theta}$$

Then we let $p(\zeta) = \sqrt{1-g(\zeta)^2}$, e.g.

$$p(\zeta) = \cos(\theta/2) \qquad \text{if } \zeta = e^{i\theta}$$

Now consider the free Hilbert module over $C(U(V))$ consisting of continuous maps $s: U(V) \to V$. We define a Hilbert submodule M to consist of s such that

$$s(g) \in p(g)V$$

Thus sitting inside the trivial family $g \mapsto V$ of Hilbert spaces is the family $g \mapsto p(g)V$ = sum of eigenspaces of g corresp. to eigenvalues ζ s.t. $p(\zeta) \neq 0$.

By choice of g, $p(\zeta)$ is defined on all of S^1 by $p(-1) = 0$, and this is continuous. Thus $g(g)$ is a well-defined operator on $p(g)V$ for each $g \in U(V)$. Thus we get an operator which we will call just q on $\mathcal{M}$. We have

$$1 - q^2 = p^2 = \sum_i pe_i \langle pe_i|$$

as operators on $\mathcal{M}$. Here e_i is an orth. basis for V. The second ~~inequality~~ equality holds as operators on $C(S^1, V)$. But $pe_i \in \mathcal{M}$ and so p^2 is a compact operator on the Hilbert module $\mathcal{M}$.

Now we pull back q, p over $U(V)$ via the given loop $S^1 \longrightarrow U(V)$, and we ~~[illegible]~~ try to mix with the Hilbert involution F. This means we form

$$* \qquad \gamma^1 pF + \gamma^2 q$$

acting on ~~[illegible]~~ the closure of the image of p on $L^2(S^1, V)$. (I think it might ~~[illegible]~~ be better to define $\mathcal{M}$ as the closure of the image of p on $C(S^1, V)$.) The question is whether $*$ is congruent to 1 mod compact. But

$$(\gamma^1 pF + \gamma^2 q)^2 = \underbrace{pFpF}_{p^2 + p[F,p]F} + \gamma^1\gamma^2[pF, q] + q^2$$

$p[F,p]F$ certainly compact as p is continuous.

Thus the problem is whether $[pF, q]$ is compact, and the difficulty is that q isn't everywhere defined.

One way to proceed would be to ~~[illegible]~~ take $p^{1/2}$. Then $pF \equiv p^{1/2}Fp^{1/2}$ mod compacts since the symbol of F is scalar. Then

$$[pF, q] = pFq - qpF$$
$$= p^{1/2}Fp^{1/2}q + p^{1/2}[p^{1/2}, F]q - qpF$$
$$= \underbrace{[p^{1/2}F, p^{1/2}q]}_{\text{compact}} + p^{1/2}\underbrace{[p^{1/2}, F]}_{\text{compact}}q$$

The point is that $p^{1/2}q$ is globally defined

I think the above argument shows that we do have a Fredholm operator $\gamma^1 pF + \gamma^2 q$ on

$$\mathbb{C}^2 \otimes L^2(S^1) \otimes_{C(S^1)} M = \mathbb{C}^2 \otimes p\overline{L^2(S^1, V)}$$

Notice that this conclusion is reasonable from the past viewpoint, because for the simplest q, p we have

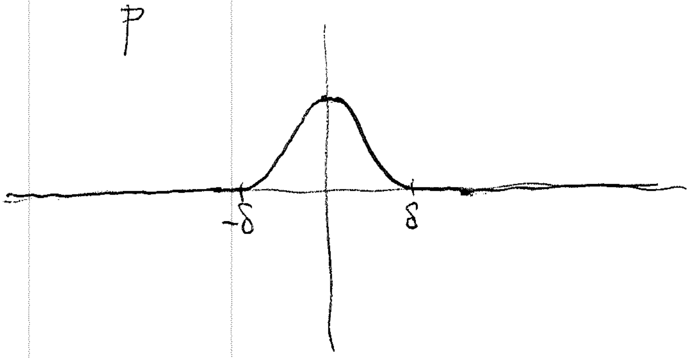
$$p(g) = \tfrac{1}{2}|g^{1/2} + g^{-1/2}| = \tfrac{1}{2}|g+1|$$

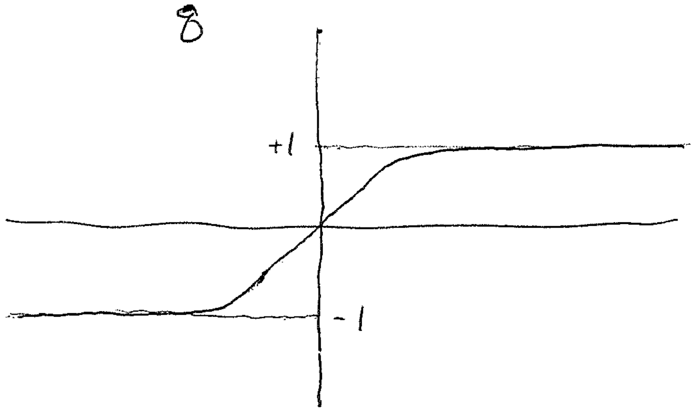
and we were trying to define an operator on the image of $g+1$.

April 16, 1987

Yesterday we looked at a Hilbert module over $C(U(V))$ and constructed operators q, p. Here the Hilbert module is a submodule of $C(U(V), V)$, and the operators p, q were defined from **certain** functions on the unit circle.

Today I want to start from the larger Hilbert module which belongs to the canonical family of Dirac operators over S^1 parametrized by $U(V)$. Thus this time p, q should be defined by applying to the Dirac operator certain functions on the line





such that $p^2 + q^2 = 1$.

In order to see what is happening let's return to the example:

$$f(x,y) = \sum e^{iny}\, u(x-n)$$

$$f(x+1,y) = e^{iy} f(x,y)$$
$$f(x,y+2\pi) = f(x,y+2\pi)$$

$$\nabla_x = \partial_x \longleftrightarrow \partial_t$$

$$\nabla_y = \partial_y - ix \longleftrightarrow -it$$

Suppose $p^{-1}(0,1] = (-\delta, \delta)$, and assume $0 < \delta < \tfrac{1}{2}$.

$$p(i\nabla_y) \longleftrightarrow \text{multiplication by } p(t)$$

Thus we see that $\overline{\mathrm{Im}\, p(i\nabla_y)} = L^2(-\delta,\delta)$ and the operator on this space defined by q is just multiplication by $q(t)$.

Finally we have multiplication by functions on the circle $\mathbb{R}/\mathbb{Z}$, i.e. periodic fns. $u(x)$ of period 1. This mult. by $u(x)$ corresponds to mult. by $u(t)$.

I now see completely the setup over the circle embedded in the setup for the torus. Over the torus we have the space $L^2(\mathbb{R}) \supset \mathcal{S}(\mathbb{R})$ and the operators ∂_x, $p(x)$, $q(x)$, maybe x. Over the circle we have the space $L^2(-\delta,\delta)$ with the same $p(x)$, $q(x)$.

At this point we have to formulate a goal. We have a good operator $\partial_x + x$ in the torus situation, to which heat operator methods can be applied. Hopefully these methods can be linked to the superconnection character forms. On the other hand we have the KK operator $pF + iq$ on $L^2(-\delta,\delta)$. The problem is to link these.

April 17, 1987

Yesterday we achieved some understanding of the KK method for attaching to a loop $g: S^1 \longrightarrow U(V)$ its pairing with the fundamental class of S^1. This was based upon a construction

$$(*) \qquad \gamma^1 q + \gamma^2 pF$$

where q, p are certain functions on the line which are interpreted suitably as operators.

Now I still have the task of linking $(*)$ to the Dirac operator over the torus. I have the following situation.

Over $U(V) \times S^1$ is a canonical vector bundle E with partial connection in the S^1 direction. One can extend it to a connection consistent with the natural trivialization over the basepoint of S^1. Once this is done one can pull back via the loop g and obtain a bundle $(g \times id)^* E$ over the torus with connection. Then upon choosing a metric on the torus one gets the Dirac operator.

One obtains a trivial Hilbert C^* module over $C(U(V))$ by taking sections of E; this uses the metric on the second circle. Inside is a sub-C^*-module where the (vertical) Dirac operator is bounded by δ. We know this submodule also sits inside $C(U(V), V)$.

The program is probably to make some

sort of sense out of the Dirac operator on the first circle with values in this Hilbert submodule.

Lesson: We've seen in the example $g = e^{2\pi i x}$ on $\mathbb{R}/\mathbb{Z}$, that the sections of the line bundle over the torus can be identified with elts. of $\mathcal{S}(\mathbb{R})$ in such a way that $\nabla_x = \partial_x$, $\nabla_y = -ix$. Moreover elements of the Hilbert C^*-module can be identified with $C_c(-\delta,\delta)$. Thus our problem is to link the Dirac operator $\gamma^1\nabla_x + \gamma^2\nabla_y$ on $L^2(\mathbb{R})$ with some operator on $L^2(-\delta,\delta)$.

Hence we may ask <u>whether $\partial_x + x$ on $L^2(\mathbb{R})$ can be deformed to something on $L^2(-\delta,\delta)$.</u>

Now we do have some kind of operator on $L^2(-\delta,\delta)$ namely $\gamma^1 p \frac{\xi}{|\xi|} + \gamma^2 q$. It is a ΨDO of order zero.

The question is now what sort of role do p, q play? Are they part of some process for constructing the cup product? We can examine this question on the symbol level. After all the symbol of the operator we seek is the element of $K_c^{ev}(T^*S^1)$ which is the cup product of the classes belonging to $\frac{\xi}{|\xi|}$ and $g : S^1 \longrightarrow \mathcal{U}(V)$.

Thus p, q ought to enter into computing the top-index class in $K_c^{ev}(T^*S^1)$.

Let's recall that we used the map

$$(i\xi, X) \longmapsto \gamma^1 i\xi + \gamma^2 X$$

to define a continuous map

$$(*) \qquad U(1) \wedge U(V) \longrightarrow \mathrm{Gr}(V \oplus V)$$

where the smash product is defined using -1 as a basepoint.

What should be the link between p, q and X? Recall the simplest choice

$$p = \left|\frac{g+1}{2}\right| \qquad\qquad g = \frac{1+X}{1-X} = -1 + \frac{2}{1-X}$$

$$p = \left|\frac{1}{1-X}\right| = \frac{1}{\sqrt{1-X^2}}$$

and then $q = \frac{1}{i}\frac{X}{\sqrt{1-X^2}}$. Thus for $|\xi| = 1$

$$\gamma^1 p \frac{\xi}{|\xi|} + \gamma^2 q = (-i)\frac{(\gamma^1 i\xi + \gamma^2 X)}{|\gamma^1 i\xi + \gamma^2 X|}.$$

So apparently one has taken the natural candidate for the symbol which is $\gamma^1 i\xi + \gamma^2 X$ and one has restricted it to $|\xi| = 1$ and made it unitary by taking the phase.

The above is the simplest case. More complicated choices of p, q might lead to different versions of the cup product. In fact we ought to see that we obtain a smooth version of $(*)$

April 18, 1987

Recall the map

$$\begin{array}{ccc} S^1 \times S^1 & \longrightarrow & S^2 \\ \shortparallel & & \shortparallel \\ (\mathbb{R}\cup\infty)\times(\mathbb{R}\cup\infty) & \longrightarrow & \mathbb{C}\cup\infty \\ (x\ ,\ y) & \longmapsto & x+iy. \end{array}$$

is smooth except at the point (∞,∞). In effect

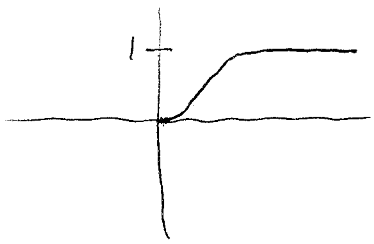
$$\frac{1}{x+iy} = \frac{x^{-1}}{1+iyx^{-1}} \qquad \text{smooth for } x^{-1}, y \in \mathbb{R}$$

$$= \frac{y^{-1}}{y^{-1}x+i} \qquad \text{smooth for } x, y^{-1} \in \mathbb{R}$$

$$= \frac{x^{-1}y^{-1}}{y^{-1}+ix^{-1}} \qquad \text{smooth for } x^{-1}, y^{-1} \in \mathbb{R} \text{ but } (x^{-1},y^{-1}) \neq (0,0)$$

The function $\dfrac{uv}{u+iv}$ is homogeneous of degree 1, and is not linear, so it can't be smooth, although it is continuous.

A nice way to smooth this singularity is to multiply by $\rho(u^2+v^2)$ where



April 19, 1987

I want to define a smooth version of the cup product map

$$(*) \qquad U(V) \times U(W) \longrightarrow Gr(\mathbb{C}^2 \otimes V \otimes W)$$

$$\left(\frac{1+X}{1-X}, \frac{1+Y}{1-Y}\right) \longmapsto \text{CT of } \gamma^1 \otimes X \otimes 1 + \gamma^2 \otimes 1 \otimes Y.$$

We want to do this first for V, W one dimensional, and then to extend via the spectral theorem.

An important observation is that the extension is always possible. In effect, suppose we produce the map when $V = W = \mathbb{C}$, so that we have a map

$$S^1 \times S^1 \longrightarrow S^2 = \mathbb{C} \cup \{\infty\}$$

$$\text{e.g.} \qquad \left(\frac{1+ix}{1-ix}, \frac{1+iy}{1-iy}\right) \longmapsto x + iy$$

Then we have a 2×2 idempotent matrix function on $S^1 \times S^1$ which is smooth or continuous depending on our construction; in the example it is just continuous. Now consider a manifold M and a pair of commuting unitary $N \times N$-matrix functions g, g' over M. Then [redacted] associated to g, g' is a map

$$C(S^1 \times S^1) \longrightarrow M_N(C(M))$$

and so if we take our 2×2 idempotent matrix over $C(S^1 \times S^1)$, we get a $2N \times 2N$ idempotent matrix over $C(M)$. $(*)$ results by taking $M = U(V) \times U(W)$ and g, g' to be $g_{univ} \otimes 1$, $1 \otimes g_{univ}$.

This obviously works in the smooth setup also. Well, you have to check that if g, g'

are smooth unitary functions, then given a smooth function $\sum a_{mn} e^{i(m\theta+n\varphi)}$ on $S^1\times S^1$, the matrix function

$$\sum a_{mn}\, g^m (g')^n$$

is smooth. If $X_1,\dots,X_k$ are vector fields then

$$X_1\cdots X_k\, g^m(g')^n \quad \text{is a sum of}$$

$(|m|+|n|)^k$ monomials involving finitely many derivatives of g, g' and their inverses. Thus one gets a bound

$$\left| X_1\cdots X_k\, g^m(g')^n \right|_K \le C\,(|m|+|n|)^k$$

where C is independent of m,n. Thus the series of derivatives converges uniformly on compacts.

Here is a simple way to smooth the singularity of the map $S^1\times S^1 \longrightarrow S^2$, $x,y \longmapsto x+iy$. Set

$$z=(x+iy)\,e^{\varepsilon(x^2+y^2)} \qquad \varepsilon>0$$

Then with $u=x^{-1}$, $v=y^{-1}$ we have

$$z^{-1} = \frac{1}{x+iy}\, e^{-\varepsilon(x^2+y^2)}$$

$$= \frac{x^{-1}}{1+ix^{-1}y}\, e^{-\varepsilon(x^{-1})^{-2}-\varepsilon y^2}$$

$$= \frac{u}{1+iuy}\, e^{-\varepsilon\frac{1}{u^2}}\, e^{-\varepsilon y^2}$$

is smooth for $u,y\in\mathbb{R}$. Similarly

$$z^{-1} = \frac{uv}{v+iu} e^{-\varepsilon\frac{1}{u^2}} e^{-\varepsilon\frac{1}{v^2}}$$

is smooth for $u, v \in \mathbb{R}$.

A more algebraic way of proceeding is to set

$$z = (x+iy)\,\rho(x)\,\rho(y)$$

where say $\rho(x) = (1+x^2)^r$. Then

$$z^{-1} = \frac{x^{-1}y^{-1}}{y^{-1}+ix^{-1}} \frac{x^{-2r}}{(1+x^{-2})^{+r}} \frac{y^{-2r}}{(1+y^{-2})^{+r}}$$

$$= \frac{u^{2r+1}v^{2r+1}}{v+iu} \frac{1}{(1+u^2)^r} \frac{1}{(1+v^2)^r}$$

so the singularity has been [redacted] partially removed.

[redacted] Can we smooth on one side? Consider the map

$$z = (x+iy)\,e^{x^2}$$

Then

$$z^{-1} = \frac{1}{x+iy} e^{-x^2}$$

$$= \frac{u}{1+iuy} e^{-(u^{-2})} \qquad \text{smooth for } u, y \in \mathbb{R}$$

$$= \frac{v}{vx+i} e^{-x^2} \qquad \text{smooth for } v, x \in R$$

$$= \frac{uv}{v+iu} e^{-(u^{-2})}$$

Consider $\partial_u^a \partial_v^b (z^{-1})$. This is a finite sum of terms of the form $\frac{1}{(v+iu)^k} \frac{1}{u^l} e^{-(u^{-2})} \times \{c u^a v^b$

and

$$|v+iu| = \sqrt{u^2+v^2} \text{ [redacted] } \geq |u|$$

so each of these terms is

$$\text{[redacted]} \quad O\left(|u|^{-N} e^{-(u^{-2})}\right)$$

which goes to zero as $u \to 0$. Thus it appears that it is enough to smooth one side.

Summarizing we seem to have understood the cup product of odd K-classes represented as smooth maps to the unitary group. The product is defined [redacted] via a choice of map $S^1 \times S^1 \longrightarrow S^2$ and then using the spectral theorem in a suitable way. The natural choice for this map is $(x,y) \longmapsto x+iy$, but it is only continuous. However it can be smoothed [redacted] to

$$(x,y) \longmapsto (x+iy)e^{-\varepsilon x^2} \qquad \varepsilon > 0$$

Next we want to look again at Dirac operators on the circle. We start with $g: S^1 \to U(V)$ and wish to couple it to ∂_x. [redacted]

The problem seems to be the following. We have been analyzing the symbol and have constructed a map from $T^*(S^1)$ to $Gr(\mathbb{C}^2 \otimes V)$. We have then a unitary matrix function on $T^*(S^1)$ which is inverted by ε and we want to produce a unitary operator on $\mathbb{C}^2 \otimes L^2(S^1, V)$ also inverted by ε. The problem in a nutshell is how a unitary valued symbol leads to a unitary operator. [redacted] One can modify this and work with involutions if required.

The only idea I have is [redacted] the one [redacted] mentioned by Getzler — realize the symbol by an operator, then take the phase. This assumes the realization is invertible self-adjoint, which seems wrong cohomologically.

So it seems necessary to study an example carefully.

Notes for J. Roe:

I propose to give a Kasparov type construction of the pairing of odd K-cohomology classes of S^1 with the fundamental K-homology class.

1. Background & Conventions. The fundamental class is represented by the Hilbert transform involution, i.e. 'the' ΨDO in $L^2(S^1)$ of order 0 with symbol $\frac{\xi}{|\xi|}$. Classes in $K^1(S^1)$ will be represented by loops $g : S^1 \longrightarrow U(V)$. One way to construct the pairing of $[g] \in K^1(S^1)$ with the fundamental class is to form 'the' ΨDO from $L^2(S^1, V)$ to itself with the symbol

$$(x,\xi) \longmapsto \begin{cases} 1 & \xi < 0 \\ g(x) & \xi > 0 \end{cases}$$

and take the index. This process is equivalent to taking the Toeplitz operator associated to g on the Hardy space.

In the sequel we give a different construction.

2. We start with a continuous map $q(e^{i\theta})$ from the unit circle with -1 removed to $[-1,1]$ such that

$$\lim_{\theta \uparrow \pi} q(e^{i\theta}) = +1$$

$$\lim_{\theta \downarrow -\pi} q(e^{i\theta}) = -1$$

For example $q(e^{i\theta}) = \frac{\theta}{\pi} \quad -\pi < \theta < \pi$ or

$q(e^{i\theta}) = \sin(\theta/2)$.

Let $p(e^{i\theta}) = \sqrt{1 - q(e^{i\theta})^2}$ so that p is continuous, ≥ 0, $p^2 + q^2 = 1$, and moreover p extends continuously to the whole unit circle such that

$$p(-1) = 0.$$

Next we consider a f.d. vector space V with inner product, and form the "free" Hilbert C^*-module over $C(\mathcal{U}(V))$ consisting of continuous maps $s : \mathcal{U}(V) \to V$. The function $p(e^{i\theta})$ defines an endomorphism of this Hilbert C^*-module $C(\mathcal{U}(V), V)$ by

$$(ps)(g) = p(g)\, s(g)$$

where $p(g)$ is defined via the spectral thm. Let $\mathcal{M}$ be the Hilbert C^*-module over $C(\mathcal{U}(V))$ which is the closure of the image of p in $C(\mathcal{U}(V), V)$.

If $s \in \mathcal{M}$, then $s(g) \in p(g)V$ for all g, so $\mathcal{M}$ is contained in (and perhaps equal to) the space of continuous sections of the family of spaces $p(g)V$, $g \in \mathcal{U}(V)\}$.

Now because $q(e^{i\theta})$ is not defined at $e^{i\theta} = -1$, this function doesn't define an endomorphism of $C(\mathcal{U}(V), V)$. But because $p(-1) = 0$, one does obtain an endomorphism q of $\mathcal{M}$ given by

$$(qs)(g) = q(g)\, s(g)$$

where $q(g)$ is defined by picking any value for $q(-1)$. Then we have

$$1 - q^2 = p^2$$

and if $e_1, \dots, e_n$ is an orthonormal basis for V we have

$$p^2 = \sum_1^n (pe_i)(pe_i)^*$$

as endomorphisms of $C(U(V), V)$. Here $(pe_i)^*$: $C(U(V), V) \longrightarrow C(U(V))$ is inner product with pe_i. Since $pe_i \in \mathcal{M}$, this shows that $1 - g^2 = p^2$ is a compact operator on the Hilbert C^*-module $\mathcal{M}$.

3. Let $g : S^1 \longrightarrow U(V)$ be a continuous map. Pulling back by g, i.e. tensoring with $C(S^1)$ over $C(U(V))$ we obtain a Hilbert module $g^*\mathcal{M}$ over $C(S^1)$ equipped with operators g, p as above. Here $g^*\mathcal{M}$ is contained in (perhaps equal to) the closure of the image of multiplication by $p(g(x))$ on $C(S^1, V)$.

Since $L^2(S^1) \otimes_{C(S^1)} C(S^1, V) = L^2(S^1, V)$ we have a map

$$L^2(S^1) \otimes_{C(S^1)} \mathcal{M} \longrightarrow \overline{p(g) L^2(S^1, V)}$$

and perhaps the latter is the completion of the former. In any case we now construct a Fredholm operator on the latter Hilbert space.

Let F be the Hilbert involution on $L^2(S^1, V)$. and consider $p^{1/2} F p^{1/2} + i g$ on $\overline{p(g) L^2(S^1, V)} = \mathcal{H}$. Then combining with the adjoint

$$\begin{pmatrix} 0 & p^{1/2} F p^{1/2} - i g \\ p^{1/2} F p^{1/2} + i g & 0 \end{pmatrix} = \gamma^1 p^{1/2} F p^{1/2} + \gamma^2 g$$

we have

$$\left(\gamma^1 p^{1/2}Fp^{1/2} + \gamma^2 q\right)^2 = p^{1/2}\overbrace{Fp}^{pF+[F,p]}Fp^{1/2} + q^2 + \gamma^1\gamma^2[p^{1/2}Fp^{1/2}, q]$$

$$[p^{1/2}Fp^{1/2}, q] = p^{1/2}Fp^{1/2}q - qp^{1/2}Fp^{1/2}$$

$$= [p^{1/2}F, p^{1/2}q] - qp^{1/2}[F, p^{1/2}]$$

Now $p^{1/2}q$ is defined on $L^2(S^1, V)$ as a multiplication operator. All the brackets are ψDO's of order -1, hence compact. Thus we see that $p^{1/2}Fp^{1/2} + iq$ on $\mathcal{H}$ is unitary modulo compacts.

---

4. Link with the Toeplitz construction. We take $pF + iq$ and combine it with the inverse of the isomorphism obtained when $\xi = -1$. Thus we look at (working with symbols)

$$(-p+iq)^{-1}(p\xi + iq) = \begin{cases} 1 & \xi = -1 \\ -(p+iq)^2 & \xi = 1 \end{cases}$$

However $(p+iq)^2$ is homotopic to the identity map on the unit circle, e.g.

$$(\cos\theta/2 + i\sin\theta/2)^2 = e^{i\theta}.$$

(But what does this mean when q is only defined on the image of p?)

April 21, 1987

Idea: Look at constant coefficient operators as in the counterexample contained in your notes to Roe.

Let's go over the problem. What we are trying to do is to associate to any $g: S^1 \to U(V)$ an element of a restricted Grassmannian belonging to the Hilbert space $L^2(S^1, \mathbb{C}^2 \otimes V)$. This subspace is the graph of an "operator" from $L^2(S^1, V)$ to itself. We want the map from smooth loops g to the restricted Grassmannian to be smooth.

Suppose we now restrict attention to g which are constant and we require the associated operator to have constant coeffs, i.e. to commute with translation on the circle. We think of subspaces of $L^2(S^1, \mathbb{C}^2 \otimes V)$ as unitary operators inverted by ε. Thus to any $g \in U(V)$ we want an operator $\varphi(g)$ on $L^2(S^1, \mathbb{C}^2 \otimes V)$ which is unitary inverted by ε and which is translation invariant. Now we use the FT to decompose $L^2(S^1, \mathbb{C}^2 \otimes V)$ into copies of $\mathbb{C}^2 \otimes V$, one for each $\xi \in \mathbb{Z}$. Then $\varphi(g)$ becomes a multiplication operator $\varphi(\xi, g)$ acting on l^2-functions from $\mathbb{Z}$ to $\mathbb{C}^2 \otimes V$. Clearly $\varphi(\xi, g)$ is for each $\xi \in \mathbb{Z}$ a smooth map from $U(V)$ to unitaries on $\mathbb{C}^2 \otimes V$ reversed by ε.

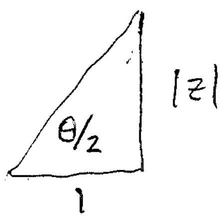
As I want to construct examples using the spectral theorem, it is probably enough to worry about the case $V = \mathbb{C}$, in which case we are after a 2×2 unitary matrix valued functions $\varphi(\xi, \zeta)$ smooth in ζ which is reversed by ε.

So far I havent discussed the topology on the Grassmannian. If we want $\varphi(g)$ to be congruent to -1 mod compacts, then we must have

$$\varphi(\xi, g) \longrightarrow -1 \qquad \text{as} \qquad |\xi| \longrightarrow \infty.$$

To see what it means for $\varphi(g) \in -1 + \mathcal{L}^p$ we need to know the eigenvalues of $\varphi(\xi, g)$.

In general we can consider the eigenvalues of the 2×2 unitary $g = \frac{1+X}{1-X}$ where $X = \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix}$. The eigenvalues of X are $\pm i|z|$ so the eigenvalues of $g = \frac{1+X}{1-X}$ are $\frac{1+i|z|}{1-i|z|} = e^{i\theta}$ and $e^{-i\theta}$ where $\theta/2 = \arctan |z|$.



The eigenvalues of

$$\frac{g+1}{2} = \frac{1}{1-X} \quad \text{are} \quad \frac{1}{1 \pm |z| i} = \frac{1 \pm |z| i}{1 + |z|^2}$$

and so both have absolute value $\frac{1}{\sqrt{1+|z|^2}} \doteq \frac{1}{|z|}\left(1 + O\left(\frac{1}{|z|^2}\right)\right)$

Thus the rule of thumb is that if $g = \frac{1+X}{1-X}$ corresp. to $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$, then the Schatten class of $\frac{g+1}{2} = \frac{1}{1-X}$ is the same as the Schatten class of $\frac{1}{|T|}$.

Now I am ready to discuss the Schatten class. I suppose given maps $\rho \longmapsto z(\xi, \rho)$ from S^1 to S^2 for each $\xi \in \mathbb{Z}$. These are to be smooth in ρ and to take -1 to ∞. For example

$$\circledast \qquad z(\xi, \rho) = \xi + \frac{\rho - 1}{\rho + 1}.$$

The following seems clear. For each ξ we obtain a smooth map $S^1 \longrightarrow S^2$ such that $\rho = -1$ goes to $z = \infty$. Moreover as ξ increases these loops shrink to $z = \infty$. What is significant for us is the rate of the shrinking. Once $|\xi|$ is suff large the loop doesn't pass through $z = 0$, so we can replace $z(\xi, \rho)$ with $w(\xi, \rho) = \dfrac{1}{z(\xi, \rho)}$. Thus we have a family $w(\xi, \rho)$ of loops in $\mathbb{C}$ which sends $\rho = -1$ to 0 and which shrink to zero as $|\xi| \longrightarrow \infty$.

Now we should replace ξ by $m \in \mathbb{Z}$ and think of having a $\rho \longmapsto \{w(m, \rho)\}$ from S^1 to sequences in $\mathbb{C}$. What we are interested in is whether we have a smooth map from S^1 to ℓ^p. So for example we can consider $\circledast$ above. This gives the sequence

$$w_m(\zeta) = \frac{1}{m + \frac{\zeta-1}{\zeta+1}} = \frac{\zeta+1}{m(\zeta+1)+\zeta-1}$$

It is clear that this sequence is in ℓ^p for $p>1$ and any $\zeta \neq -1$, and it is 0 for $\zeta = -1$. Thus [redacted] it has values in ℓ^p, but it is not smooth at $\zeta = -1$, since

$$\partial_\zeta \, w_m(\zeta)\Big|_{\zeta=-1} = \frac{1}{-2}$$

Thus we recover our previous calculation.

The natural question now is when a sequence $w_m(\zeta)$ of smooth complex functions on the circle constitutes a smooth map from the unit circle to ℓ^p. Let's use Fourier series. Suppose $v(\zeta)$ [redacted] is a map from the unit circle to a Banach space V. If this map is continuous, then the Fourier coeffs

$$v_n = \frac{1}{2\pi i}\int v(\zeta)\, \zeta^{-n-1}\, d\zeta$$

are defined and [redacted] they form a bounded sequence. If $v(\zeta)$ is C^1, then $n v_n$ is bdd. and continuing we see v smooth $\Rightarrow$ v_n rapidly decreasing. The converse is also true, e.g.

$$\left\| \sum v_n \zeta^n \right\| \leq \sum \|v_n\| < \infty$$

if $n^2 \|v_n\|$ is bounded.

Thus in order to see that $\zeta \to (w_m(\zeta))$ is a smooth map with values in ℓ^p it is enough to look at the Fourier coefficients

$$w_{m,n} = \frac{1}{2\pi i}\int w_m(\zeta)\,\zeta^{-n-1}\,d\zeta$$

and see that for each n the sequence $w_{m,n}$ is in ℓ^p and its ℓ^p norm is rapidly decreasing in n. It actually might be easier to show that the sequence

$$m \longmapsto \partial_\zeta^n w_m(\zeta) \ .$$

lies in ℓ^p for each ζ and is bounded.

Now let's <u>summarize</u>. I still haven't figured out how to combine ∂_x and g, but I suspect now that all I need to do is to produce a suitable cup product map

$$S^1 \times S^1 \longrightarrow S^2$$

$$\xi \quad \zeta \longmapsto z(\xi, \zeta)$$

It is important that it be smooth in ζ for $|\zeta| = 1$, and that the loop for fixed ξ shrink to $z = \infty$ as $|\xi| \to \infty$.

Suppose we now look at the ξ dependence for fixed ζ. If $z(\xi, \zeta)^{-1}$ is smooth in ξ at $\xi = \infty$, then one has

$$z(\xi, \zeta)^{-1} = \frac{a_1(\zeta)}{\xi} + \frac{a_2(\zeta)}{\xi^2} + \cdots$$

and then automatically one will know that one obtains an ℓ^p sequence for $p > 1$ by restricting ξ to be integral. It seems enough to ask that the map $S^1 \times S^1 \longrightarrow S^2$ be smooth in ζ and C^1 in ξ.

Now let's see if we can get the whole business together. We should think of $S^1 \times S^1$ as a compactification of $T^*(S^1)$. The first circle is the ξ line $\cup\, \infty$, and the second is the unit circle: $|\zeta| = 1$. We are after a map

$$\xi, \zeta \longrightarrow z(\xi, \zeta) \in S^2 = \mathbb{C} \cup \{\infty\};$$

actually we are probably after many such maps, but they should all be homotopic.

It seems we are after a map $z(\xi, \zeta)$. In addition we ought to be able to interpret this map as a point in the restricted Grass. of $L^2(S^1, \mathbb{C}^2)$. It is clear that we might want the graph of an operator on one part of the circle and the graph of the inverse operator on the other part of the circle.

Now you do have certain operators around, e.g. the ψDO discussed in §1. of your latest notes to Roe, as well as $p^{1/2} F p^{1/2} + iq$. Can you see the functions or maps $S^1 \times S \longrightarrow S^2$ belonging to these operators.