March 4, 1987 – April 21, 1987 p 507 – 650

March 4, 1987

I want to review various symplectic facts. Let E be a real symplectic vector space of dimension $2n$. Attached to E is a Siegel UHP whose points can be conveniently described as complex subspaces $V \subset E_c$ of dimension n such that $[v, v'] = 0$ for $v, v' \in V$ and $[v, v^*] \geq 0$ with equality iff $v = 0$. In other words $E_c = V \oplus \bar{V}$ is a splitting into annihilation and creation operators.

This description is convenient because it exhibits a natural compactification of the Siegel UHP, namely those maximal isotropic subspaces V of E_c such that $[v, v^*] \geq 0$. If V is real then we have a Lagrangian subspace of E. So there is a kind of building type structure to the boundary.

Let us fix a basepoint V_0 of the Siegel UHP and choose a basis a_k for V_0 and a_k^* for $\bar{V}_0$.

I think I want V_0 to consist of the creation ops, so let's change notation.

Then any $V \subset E_c = V_0 \oplus \bar{V}_0$ is the graph of a map from $\bar{V}_0$ to V_0, call it Z. Z is symmetric because V is isotropic and $Z^*Z < 1$. Thus one gets the model of the Siegel UHP consisting of complex symmetric matrices Z with $Z^*Z < 1$. The boundary includes Z with $Z^*Z \leq 1$, and the Lagrangian Grassmannian is the space of symmetric unitary matrices. (Actually to get this description as matrices one has effectively chosen an isom. $\bar{V}_0 = V_0$, i.e.

a real structure ε on V_0, which is what you used before to identify the Lagrangian Grassmannian with symmetric unitary matrices.)

Now there is a simpler version of this whole business where one supposes fixed from the beginning a complex structure on E, denote it i which is compatible with the symplectic structures in the sense that e^{it} is a 1-parameter automorphism group. ~~Then one can look at the fixed part of the Siegel UHP. The circle action on E_c has two eigenspaces, so an invariant V has to split $V = V^+ \oplus V^-$~~

We consider E_c ~~and~~ which we have the conjugation $*$ and the bracket $[\ ,\ ]$. E_c splits into two eigenspaces $E_c^+ \oplus E_c^-$ for the circle action. Clearly conjugation reverses these and both are isotropic for $[\ ,\ ]$. Thus both are maximal isotropic and in duality. Also we have a hermitian form on E_c^+ given by $[v^*, v']$.

What does a point V in the Siegel UHP fixed by the circle action look like? It is a direct sum $V = V^+ \oplus V^-$ where $V^\pm \subset E_c^\pm$, and to be maximal isotropic we must have $V^- = (V^+)^\perp$ for the $[\ ,\ ]$ pairing. Next the ~~there~~ is a positivity condition

$$[(v^+)^* + (v^-)^*,\ v^+ + v^-] \geq 0$$
$$\|$$
$$[(v^+)^*, v^+] + [(v^-)^*, v^-]$$

This means that the hermitian form on V^+ is >0 and also for V^-. This bounds the dimensions of V^+, V^- and because the sum of their dims is n, this probably means that V^+ is a maximal subspace of E_c^+ on which the hermitian form is positive.

Let's begin again with a complex structure on a real symplectic vector space, E, where we suppose the 1-parameter group e^{it} preserves the symplectic structure.

Let's start with a complex vector space E and suppose given on the underlying real vector space a skew-symmetric nondegenerate form $\Omega(\xi, \eta)$. We suppose

$$\Omega(i\xi, \eta) + \Omega(\xi, i\eta) = 0$$

which implies that e^{it} is a 1-parameter group of symplectic automorphisms. This implies

$$\Omega(\xi, i\eta) = -\Omega(i\xi, \eta) = \Omega(\eta, i\xi)$$

so that $\Omega(\xi, i\eta) = B(\xi, \eta)$ is a symmetric bilinear form on the underlying real vector space. It is clearly nondegenerate.

Next we have the circle which is a compact Lie group acting symplectically on E, also orthogonally for the form B. Enlarge the circle to a maximal compact. This means we find a positive inner product on the underlying real vector space E such that Ω is given as $(\xi, J\eta)$ with $J^2 = -1$, and i is orthogonal relative to $(,)$. Then i and J commute, so that $J = i\varepsilon$ where ε is an involution.

Thus we get a complex Hilbert space E together with a grading ε and the symplectic form is

$$\Omega(\xi,\eta) = (\xi, J\eta) = (\xi, i\varepsilon\eta) = \mathrm{Re}\langle \xi, i\varepsilon\eta\rangle$$
$$= -\mathrm{Im}\langle \xi, \varepsilon\eta\rangle$$

Now what are the points of the Siegel UHP invariant under the circle action? But before I do this I probably want to discuss the basic representation where elements of E are hermitian operators and Ω is $\pm i[\ , \ ]$. I guess I need to introduce two sets a_i^*, a_i, b_j^*, b_j of creation + annihilation operators. The circle action assigns to a_i^*, b_j the weight 1 and their adjoints the weight -1.

For earlier work see 198-199, Feb 11, 86. and 375-376, Jan 23, 1987.

March 10, 1987

Problem: Conformal field theory on a Riemann surface.

It seems to be difficult to get started by considering the Fock spaces attached to the different circles on the surface. It seems desirable to figure out what Witten does with meromorphic sections.

However there is one interesting idea worth recording. Consider the single real fermion field on S^1 with anti-periodic boundary conditions. On the Fock space we have operators $\psi(f(x)dx^{1/2})$ for any L^2 half-density. Now put the circle in the cylinder. More accurately identify S^1 with $|z|=1$ inside $\mathbb{C}^\times$. The complex structure on the cylinder determines a kind of imaginary time-evolution. To be more specific the operator $\psi(f(x)\,dx^{1/2})$ on S^1 will extend to $\psi(f(z)\,dz^{1/2})$ which has a meaning for lots of circles, at least for analytic f.

So a holomorphic half density is consistent with the imaginary time-evolution in some sense.

Now let us concentrate on Green's functions for the fermi field. There are example given by the cylinder and the torus.

March 11, 1987

Let's take up the real fermi field on the torus. This we can handle via the temperature formalism. I want to calculate the Green's function, the normal ordering belonging to the temperature state, etc.

Recall the formulas for the real fermi field:

$$\psi(f dx^{1/2})^* = \psi(\overline{f dx^{1/2}})$$

$$\psi(f dx^{1/2})^2 = \frac{1}{2}\int f^2(x)\frac{dx}{2\pi}$$

$$\psi(f dx^{1/2}) = \sum c_k \psi_k \qquad \text{if } f = \sum c_k e^{ikx}$$

$$= \int \frac{dx}{2\pi} f(x) \sum e^{-ikx}\psi_k$$

$$= \int \frac{dx}{2\pi} \left(f(x) dx^{1/2}\right) \underbrace{\left(\sum e^{-ikx}\psi_k dx^{1/2}\right)}_{\psi(x)\, dx^{1/2}}$$

$$\psi_k = \psi(e^{ikx} dx^{1/2})$$

$$\psi_k^* = \psi_{-k} \qquad \{\psi_k, \psi_\ell\} = \delta_{k,-\ell}$$

The natural Hamiltonian is $H = L_0$ whose effect on the Clifford algebra is

$$[H, \psi(f dx^{1/2})] = \psi(\tfrac{1}{i} f' dx^{1/2})$$

$$[H, \psi_k] = k\psi_k$$

We then set

$$\psi(xt) = e^{tH}\psi(x) e^{-tH}$$

$$= \sum_k e^{-ikx+kt}\psi_k$$

Also this Hamiltonian leads to a ground state $|0\rangle$ satisfying

$$\psi_k |0\rangle = 0 \qquad k < 0$$

at least in the anti-periodic (NS) case. Thus ψ_k creates for $k>0$, destroys for $k<0$.

Now let's find the Green's functions in the zero temperature case.

$$G(xt, x't') = \langle T[\psi(xt)\psi(x't')]\rangle$$

where $\langle A\rangle = \langle 0|A|0\rangle$. Thus for $t>t'$

$$G(xt, x't') = \sum_{k,l} e^{-ikx+kt}\, e^{-ilx'+lt'} \langle 0|\psi_k\psi_l|0\rangle$$

$$= \sum_k e^{-ik\Delta x + k\Delta t} \langle 0|\psi_k\psi_{-k}|0\rangle$$

$$= \sum_{k<0} e^{k(\Delta t - i\Delta x)} = \frac{e^{-\frac{1}{2}(\Delta t - i\Delta x)}}{1 - e^{-(\Delta t - i\Delta x)}}$$

similarly for $t<t'$

$$G(xt, x't') = \sum e^{-ikx+kt}\, e^{-ilx'+lt'} \langle 0|-\psi_l\psi_k|0\rangle$$

$$= \sum e^{-ik\Delta x + k\Delta t} \langle 0|-\psi_{-k}\psi_k|0\rangle$$

$$= -\sum_{k>0} e^{-ik\Delta + k\Delta t} = -\frac{e^{\frac{1}{2}(\Delta t - i\Delta x)}}{1 - e^{\Delta t - i\Delta x}}$$

and so

$$\boxed{G(xt, x't') = \frac{1}{e^{\frac{1}{2}(\Delta t - i\Delta x)} - e^{-\frac{1}{2}(\Delta t - i\Delta x)}}}$$

To put this in a more appealing form we want to introduce complex variables.

There is a principle to be found which roughly links the imaginary time evolution to the holomorphic structure. Thus

$$\left(\partial_t + \frac{1}{i}\partial_x\right)\psi(xt) = 0$$

i.e. $\psi(xt)$ is holomorphic in the complex variable $t-ix$. Now let's put

$$z = e^{t-ix}$$

so that increasing t corresponds to increasing $|z|$.

Let's now try to figure out what to do with the $dx^{1/2}$. The principle alluded to should say that the time evolution of the field ψ, that is the equation of motion it satisfies, is the same condition as requiring ψ to be holomorphic. Now on the circle we know we are dealing with half densities, sections of a square root of T^*. And $\psi(x)dx^{1/2}$ is an operator valued half density on the circle. It should make sense to analytically continue this half density from the circle to the cylinder, or at least to an annulus containing the circle.

Let's write this in terms of $w = t-ix$. Restricting to $t=0$ we have $i\,dw \mapsto dx$ so $\psi(x)dx^{1/2}$ comes from

$$\psi(w)(i\,dw)^{1/2} = \sum_k (e^w)^k \psi_k (i\,dw)^{1/2}.$$

Here $(i\,dw)^{1/2}$ denotes what? Over the circle $t=0$, $x \in \mathbb{R}/2\pi\mathbb{Z}$ we choose a square root L

of the canonical bundle and fixed the AP or NS case which is the simplest. Then $dx^{1/2}$ is a section of the bundle pulled back to the double covering of the circle whose square is dx. Thus in $f(x)dx^{1/2}$, $f(x)$ is also defined over the double covering.

Now if we start with a Riemann surface then we suppose [redacted] given a square root L of the canonical line bundle K over the surface. So on the cylinder $\mathbb{C}/2\pi i\mathbb{Z}$ we have the section dw of [redacted] K and hence $(dw)^{1/2}$ is a section of L pulled back to the double covering.

Now go one stage further and put $z = e^w$, whence $\psi(x)dx^{1/2}$ comes from

$$\sum_k z^k \psi_k \left(i\frac{dz}{z}\right)^{1/2}$$

Lets try to describe the picture that seems to be emerging. [redacted] Given a circle on the surface one has an operator-valued half-density, or maybe you should say an operator-valued spinor (Weyl spinor) [redacted] defined on the circle. This then is to be extended holomorphically to a nbd of the circle and possibly to the whole Riemann surface.

Thus we seek an operator-valued spinor field (section of L) over the Riemann surface which is holomorphic.

The problem with this statement is that one doesn't know how to interpret the words "operator-valued". Certainly for any circle on the surface we want this field, which I might write as $\psi(z)\,dz^{1/2}$, to become the operator-valued (distributional) half-density on the Fock space associated to the circle.

Next let's go back to the Green's function $G(xt, x't')$ which we know has something to do with the inverse of the $\bar\partial$-operator. I want to discuss this from an invariant viewpoint.

Thus let L be a square root of the canonical line bundle K over M. Recall $K = T^{1,0}$. Following Atiyah-Bott set $L' = L^* \otimes T^{1,1}$. Then sections of L pair naturally with sections of L'. The definition of L' makes sense for any vector bundle.

In general if E is a holom. v.b., its $\bar\partial$ operator is a map

$$\Gamma(E) \xrightarrow{\ \bar\partial\ } \Gamma(E \otimes T^{0,1})$$

and its transpose is ~~the~~ minus map

$$\Gamma(E') \xleftarrow{\ \bar\partial^t\ } \Gamma(E^* \otimes \overbrace{T^{1,0}}^{K})$$

which is the $\bar\partial$ operator on $E^* \otimes K$. This is the origin of Serre duality.

If $L \otimes L = K$, then $L = L^* \otimes K$ and $L' = L^* \otimes T^{1,1} = L^* \otimes K \otimes T^{0,1} = L \otimes T^{0,1}$. Thus the Dirac operator of L is a map

$$\Gamma(L) \xrightarrow{\bar\partial_L} \Gamma(L')$$

and it is skew-symmetric, i.e. $\bar\partial^t = -\bar\partial$.

Now suppose further that L has no holomorphic sections, so that $\bar\partial_L$ is an isomorphism. The Green's function gives the inverse map. We can use the Schwartz kernel theorem to write G as a linear combination of rank one operators obtained by pairing $\xi \in \Gamma(L')$ with an element $\beta \in \Gamma(L)$ and then multiplying by $\alpha \in \Gamma(L)$:

$$\xi \longmapsto \alpha(\beta, \xi).$$

Thus it is clear that G is some sort of section of $L \boxtimes L$ over $M \times M$. It has to be holomorphic off the diagonal and to have a simple pole along the diagonal. It is skew-symmetric.

It seems that it is meaningful to speak of the residue of a meromorphic section of $L \boxtimes L$ with simple pole along the diagonal. The point is that the normal bundle to the diagonal is the tangent bundle to M and this is dual to $L \boxtimes L | \Delta M = L \otimes L = K$. Thus G is probably specified by requiring it to be such a merom. section of $L \boxtimes L$ with residue = some constant along the diagonal.

One of the problems with viewing the fermion field ψ as a holomorphic section of $Op \otimes L$ is that if one applies a linear fnl to Op then one would obtain a holomorphic section of L which would be zero!

So we have to find another method, perhaps something adelic.

Let us next consider the operator products and normal ordering

March 12, 1987

Let $L \otimes L = K = T^{1,0}$ be a square root of the canonical line bundle. Then we have a duality pairing $H^0(L) \otimes H^1(L) \longrightarrow H^1(K) \xrightarrow[\sim]{\int} \mathbb{C}$. However we have actually a pairing on sections of L. In effect we have

$$C^\infty(L) \xrightarrow{\bar\partial} C^\infty(\underline{L \otimes T^{0,1}})$$

$$L^* \otimes T^{1,0} \otimes T^{0,1} = L'$$

so we can form a bilinear form

$$B(s_1, s_2) = \int s_1 \bar\partial s_2$$

on $C^\infty(L)$. This form is <u>skew-symmetric</u> because we have $L \otimes L = K$ and so $s_1 s_2$ is a 1,0 form and

$$\bar\partial(s_1 s_2) = (\bar\partial s_1) s_2 + s_1 (\bar\partial s_2)$$

so $$B(s_1, s_2) + B(s_2, s_1) = \int \bar\partial (s_1 s_2) = 0$$

Now instead of using the $\bar\partial$ complex to resolve the sheaf of holomorphic sections of L one can use meromorphic sections. Thus we embed the sheaf of holomorphic sections of L into the sheaf of meromorphic sections. Given two meromorphic spinors s_1, s_2 their product is a meromorphic 1-form $s_1 s_2$ which has a global residue ??

I want now to describe the cohomology of a holomorphic vector bundle E using meromorphic + adelic ideas. First review the $C^\infty, \bar\partial$ approach.

$$\begin{array}{ccccccccc}
0 \to H^0(E) & \longrightarrow & C^\infty(E) & \longrightarrow & C^\infty(E\otimes T^{0,1}) & \longrightarrow & H^1(E) \to 0 \\
& & \times & & \times & & \\
0 \leftarrow H^1(E^*\otimes \Omega) & \longleftarrow & C^\infty(E\otimes T^{1,1}) & \longleftarrow & C^\infty(E^*\otimes T^{1,0}) & \longleftarrow & H^0(E^*\otimes\Omega) \leftarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & \mathbb{C} & & \mathbb{C} & & \Omega = T^{1,0}
\end{array}$$

In the above we have the non degenerate pairings vertically provided on one side C^∞ is enlarged to distributions. Notice the action is the canonical pairing

$$C^\infty(E) \times C^\infty(E^*\otimes\Omega) \longrightarrow \mathbb{C}$$

$$(s_1 \quad , \quad s_2) \longmapsto \int s_2 \bar\partial s_1 = -\int (\bar\partial s_2) s_1$$

and this does not induce the canonical pairing of $H^1(E)$ with $H^0(E^*\otimes\Omega)$.

Now I want a non-C^∞ version which gives the cohomology and there are two candidates. Either we use the resolution

$$0 \to \mathcal{O} \longrightarrow \underline{F} \longrightarrow \underline{F/\mathcal{O}} \longrightarrow 0$$
$$\underline{F/\mathcal{O}} = \coprod_p F_p/\mathcal{O}_p$$

or the adelic version which involves the inclusion maps

$$\begin{tikzcd} & F \arrow[d, hook] \\ \prod_p \hat{O}_p \arrow[r, hook] & \prod_p' \hat{F}_p = A \end{tikzcd}$$

Thus we have the following two sequences for computing the cohomology of E:

$$0 \to H^0(E) \to E \otimes_{\mathcal{O}} F \to E \underset{\mathcal{O}}{\otimes} F/\mathcal{O} \to H^1(E) \to 0$$

$$E \underset{\mathcal{O}}{\otimes} F/\mathcal{O} = \coprod E \otimes_{\mathcal{O}} F / E \otimes_{\mathcal{O}} \mathcal{O}_p$$

or

$$0 \to H^0(E) \to \begin{matrix} E \otimes_{\mathcal{O}} F \\ \oplus \\ \prod_p \hat{E}_p \end{matrix} \longrightarrow E \underset{\mathcal{O}}{\otimes} A \to H^1(E) \to 0$$

The reason I'm worrying about these two cases is because I need a pairing. To keep things simple suppose $E = \mathcal{O}$. Then I want pairings

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}) \arrow[r] & \square \arrow[r] \arrow[d, phantom, "\times"] & \square \arrow[r] \arrow[d, phantom, "\times"] & H^1(\mathcal{O}) \arrow[r] & 0 \\ 0 & H^1(\Omega) \arrow[l] & \square \arrow[l] \arrow[d] & \square \arrow[l] \arrow[d] & H^0(\Omega) \arrow[l] & 0 \arrow[l] \\ & & \mathbb{C} & \mathbb{C} & & \end{tikzcd}$$

where the boxes are to be filled in.

In Weil's adelic proof of RR, differentials (meromorphic) are defined as linear functionals on adeles A which vanish on F and some parallelotope

$$\Lambda(D) = \prod_p \hat{m}_p^{-n_p(D)}$$

This amounts to filling the boxes in as follows

$$0 \longrightarrow H^0(\mathcal{O}) \longrightarrow \prod \hat{\mathcal{O}}_p \longrightarrow A/F \longrightarrow H^1(\mathcal{O}) \longrightarrow 0$$

$$0 \longleftarrow H^1(\Omega) \longleftarrow \Omega \otimes_{\mathcal{O}} F/\mathcal{O} \longleftarrow \Omega \otimes_{\mathcal{O}} F \longleftarrow H^0(\Omega) \to 0.$$

So Weil tells us that A/F and $\Omega \otimes F$ are dual, the former being linearly compact and the latter discrete.

There has to be a pairing

$$A \otimes (\Omega \otimes_{\mathcal{O}} F) \longrightarrow \mathbb{C}$$

which ought to be

$$(f_p), \omega \longmapsto \sum_p \operatorname{res}_p(f_p \omega).$$

If $(f_p) = f \in F$, then $f\omega \in \Omega \otimes_{\mathcal{O}} F$ and the sum of the residues is zero, so we get an induced pairing $(A/F) \otimes (\Omega \otimes_{\mathcal{O}} F) \longrightarrow \mathbb{C}$.

More generally we can define a pairing

$$A \times (\Omega \otimes_{\mathcal{O}} A) \longrightarrow \mathbb{C}$$

$$(f_p) \quad (\omega_p) \longmapsto \sum \operatorname{res}_p(f_p \omega_p)$$

and I think has to be a duality. Locally at P the completion $\hat{F}_p$ is the sum of a linear compact space $\hat{\mathcal{O}}_p$ and a discrete one $\hat{F}_p/\hat{\mathcal{O}}_p$ and these are dual to each other. This should generalize to A by taking tensor products.

Let's formulate the problem. There is a duality between $H^0(E)$ and $H^1(E^*\otimes\Omega)$ which we obtain from the $\bar{\partial}$ operators on these two bundles. I wanted to find a version using meromorphic or adelic sections. Meromorphic sections don't seem to work because the product of a meromorphic section of E and one of $E^*\otimes\Omega$, when contracted is a meromorphic 1-form, and the ~~only~~ number one can attach is the sum of the residues, and this is zero. So one seems to want to use adelic sections.

Now the product of two adelic sections of E and $E^*\otimes\Omega$ is an adelic section of Ω when contracted, and this gives a number upon taking sum of residues. Unfortunately when you take E to be a square root L of Ω, then the pairing defined on adelic sections of L is <u>symmetric.</u> This is still not like the action pairing on sections of L which is skew-symmetric.

Suppose that L has $H^0(L)=0$. Then we know that

$$L\otimes F \oplus \prod_p \hat{L}_p = L\otimes A$$

and this is a nice splitting of the space of adelic sections of L into complementary maximal isotropic subspaces for the pairing.

January 13, 1987

I am still trying to figure out what Witten is doing about QFT on a Riemann surface. I have decided to concentrate on vacuum expectation values in the simple case of the real fermi field on the circle, using the Hamiltonian $\frac{1}{i}\partial_x$. This should lead to a QFT on $\mathbb{P}^1$.

We have the operators ψ_k for $k \in \frac{1}{2} + \mathbb{Z}$ satisfying $\{\psi_k, \psi_\ell\} = \delta_{k,-\ell}$ and Fock space is the cyclic representation of these operators have a ground state $|0\rangle$ such that

$$\psi_k |0\rangle = 0 \qquad k < 0.$$

Fock space $\mathcal{F}$ is a Hilbert space and these operators satisfy

$$\psi_k^* = \psi_{-k}.$$

Somehow this is the $t = 0$ picture and when I work with the little Fock space $\check{\mathcal{F}}$ and the big Fock space $\hat{\mathcal{F}}$ I am trying to keep track of all times $-\infty < t < \infty$. For each time t there is a Hilbert space $\mathcal{F}_t$ sandwiched between $\check{\mathcal{F}}$ and $\hat{\mathcal{F}}$. I should be able to describe it using the basis e_S which should be ~~orthonormal~~ orthogonal in any of these spaces. Let's defer this.

The basic operator is the field operator

$$\psi(z) = \sum z^{k-\frac{1}{2}} \psi_k$$

where $z \in \mathbb{C}^\times$. This makes sense as a transformation from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$ and probably makes sense as a transformation from $\mathcal{F}_{t'}$ to $\mathcal{F}_t$ provided $e^{t'} < |z| < e^{t}$.

I forgot to mention normal ordering for the Clifford algebra generated by the ψ_k. We have

$$\psi_k \psi_\ell = :\psi_k \psi_\ell: + \begin{cases} 1 & \text{if } k = -\ell < 0 \\ 0 & \text{otherwise} \end{cases}$$

This is the normal ordering belonging to the ground state $|0\rangle$. The map $\psi_I \longrightarrow :\psi_I:$ goes from the exterior algebra to the Clifford algebra generated by the ψ_k.

Now we have established before the operator product expansion

$$\psi(z)\psi(w) = :\psi(z)\psi(w): + \frac{1}{z-w} \qquad |z| > |w|$$

$$\parallel$$

$$\sum z^{k-\frac{1}{2}} w^{\ell-\frac{1}{2}} :\psi_k \psi_\ell:$$

and I want to discuss generalizations of this.

Thus I want to consider certain kinds of infinite linear combinations of the monomials $:\psi_I:$. Examples are $:\psi(z)\psi(w):$, $:\partial_z \psi(z)\psi(w):$ I want to generalize the operator product expansion.

Thus I want to define, or specify, some kind of exterior algebra generated by ~~[illegible]~~ symbols $\psi(z)$ for $z \in \mathbb{C}^\times$ as well as their derivatives. Then on this exterior algebra I hope to define a twisted product by a Wick process. Perhaps one can't define this for coincident points: $z = w$

But the first step appears to be to ~~define~~ specify the underlying vector space of which to take the exterior algebra. It will consist of ~~[illegible]~~ linear combinations of operators

$$\sum z^k \psi_k \qquad \text{for} \quad z \in \mathbb{C}^\times$$

as well as derivatives

$$\sum k z^k \psi_k$$

etc. Thus we are looking at $\sum c_k \psi_k$ where (c_k) is a sequence which is exponential polynomial.

Recall that an exponential polynomial fn. of x is a linear combination of $x^n e^{ax}$ where $n \in \mathbb{N}$, $a \in \mathbb{C}$. Its a function killed by $P(\partial_x)$ where P is a $\neq 0$ polynomial. On sequences one has the shift operator T and an exponential polynomial sequence is one killed by $P(T)$ where $0 \neq P \in \mathbb{C}[z, z^{-1}]$. For example

$$\left[(T-\lambda)(c)\right](k) = c_{k+1} - \lambda c_k = 0$$

$$\Longleftrightarrow \quad c_k = \lambda^k c_0$$

and

$$(T-\lambda)(k\lambda^{k-1}) = (k+1)\lambda^k - k\lambda^k = \lambda^k$$

$$\Longrightarrow \quad (T-\lambda)^2 (k\lambda^{k-1}) = 0 .$$

Now the classification of exponential polynomial sequences is immediate. There is a natural basis indexed by (n, λ), $n \in \mathbb{N}$, $\lambda \in \mathbb{C}^\times$. Thus the vector space out of which we make the exterior algebra is isomorphic to $\mathbb{C}(z)/\mathbb{C}[z, z^{-1}]$.

We learned above that we probably want to start with the vector space of principal parts, then form the exterior algebra, which will be the space of normal product "operators". Maybe I should say space of symbols. Then the deformed algebra is defined by Wick rules and the Green's function. An obvious thing to try to do is to see how the Green's function is to be defined for principal parts. Thus if I take two principal parts which are disjoint then the Green's function is defined between them. I am being imprecise; the idea is that the Green's function is usually written

$$G(z, w) = \langle \psi(z)\, \psi(w) \rangle$$

but it extends to where $\psi(z)$ is replaced by a linear combination, i.e. by a principal part.

So an obvious question is whether we can make sense out of the Green's function pairing between disjoint principal parts on a Riemann surface.

Let us fix a spin structure L: $L \otimes L = \Omega^1$

and suppose $H^0(L)=0$, so that a Green's function for the $\bar\partial$ operator on L is defined. On the other hand the fact that $H^0(L)=H^1(L)=0$ tells us that one can prescribe the [redacted] singular part or [redacted] principal part of a meromorphic section of L arbitrarily.

$$0 \longrightarrow \overset{0}{\overset{\|}{H^0(L)}} \longrightarrow L\otimes_{\mathcal{O}} F \xrightarrow{\ \sim\ } \bigoplus_p L\otimes F/L\otimes\mathcal{O}_p \longrightarrow \overset{0}{\overset{\|}{H^1(L)}} \to 0$$

Put another [redacted] way

$$L\otimes_{\mathcal{O}} A = L\otimes_{\mathcal{O}} F \oplus \prod_p \hat{L}_p .$$

By principal parts of meromorphic sections of L we mean $L\otimes_{\mathcal{O}} A/\prod \hat{L}_p = \bigoplus_p L\otimes\hat{F}_p/\hat{L}_p = \bigoplus_p L\otimes F/L\otimes\mathcal{O}_p$.

So we [redacted] proceed as follows. Take two [redacted] "singular parts of sections" of L, call them ξ and η. We suppose they have disjoint supports. Now we can lift these to meromorphic sections if we wish, call them f and g. Now fg is a meromorphic 1-form so the sum of its residues is zero. But the thing we have to do is [redacted] probably to ignore the polar part of f and just sum over the residues at the poles of g. Then this is indeed skew-symmetric.

March 14, 1987

Complex fermi field. Start over S^1

$$\psi^*(f) = \sum c_k \psi_k^* \qquad c_k = \int \frac{dx}{2\pi} f(x) e^{-ikx}$$

$$= \int \frac{dx}{2\pi} f(x) \underbrace{\sum e^{-ikx} \psi_k^*}_{\psi^*(x)}$$

$$\psi(f) = \sum \bar{c}_k \psi_k = \int \frac{dx}{2\pi} \bar{f}(x) \sum e^{ikx} \psi_k$$

If $H = \frac{1}{i}\partial_x$ extended to $\mathcal{F}$ $= :\sum k \psi_k^* \psi_k:$, then

$$e^{tH} \psi^*(f) e^{-tH} = \int \frac{dx}{2\pi} f(x) \underbrace{\sum e^{-ikx+kt} \psi_k^*}_{\psi^*(xt)}$$

$$e^{tH} \psi(f) e^{-tH} = \int \frac{dx}{2\pi} \bar{f}(x) \underbrace{\sum e^{ikx-kt} \psi_k}_{\psi(xt)}.$$

~~When it comes to doing this over a R.S. it is awkward that $\psi(f)$ is conjugate linear in f.~~

Consider now a R.S. M with a spin structure $L \otimes L = T^{1,0}$. Then for any circle C on M, $\Gamma(C,L)$ has quadratic form $\xi, \eta \mapsto \int \xi\eta$ and real structure $\xi \longmapsto \bar{\xi}$ making it into a complex vector space with hermitian inner product. Then the Fock space to be attached to C is a suitable exterior tensor space of $\Gamma(C,L)$ with the operators

$$\psi^*(\xi) = \xi \wedge$$
$$\psi(\xi) = \langle \xi | \lrcorner$$

satisfying $\{\psi(\xi), \psi(\eta)\} = \{\psi^*(\xi), \psi^*(\eta)\} = 0, \quad \{\psi(\xi), \psi^*(\eta)\} = \int \bar{\xi}\eta.$

As we want to let C move it is awkward that $\psi(\xi)$ depends conjugate linearly on ξ, since the conjugation depends on C. Therefore we change the notation and require

$$\psi^*(\xi) = \xi \wedge \quad \text{as before}$$
$$\psi(\xi) = \xi \lrcorner \quad \text{defined by the pairing } \int \xi\eta$$

Thus we get the Clifford algebra ~~structure~~

$$\{\psi(\xi), \psi(\eta)\} = \{\psi^*(\xi), \psi^*(\eta)\} = 0$$
$$\{\psi(\xi), \psi^*(\eta)\} = \int_C \xi\eta$$

which means we can let the circle move. In fact it can even be a 1-cycle. Now when C is a circle we can get a $*$ algebra by defining

$$\psi^*(\xi)^\dagger = \psi(\bar{\xi}), \quad \psi(\xi)^\dagger = \psi^*(\bar{\xi})$$

Thus with this new notation in the circle case we have

$$e^{tH}\psi^*(f)e^{-tH} = \int \frac{dx}{2\pi} f(x) \underbrace{\sum e^{-ikx+kt}\psi_k^*}_{\psi^*(xt)}$$

$$e^{tH}\psi(f)e^{-tH} = \int \frac{dx}{2\pi} f(x) \underbrace{\sum e^{ikx-kt}\psi_k}_{\psi(xt)}$$

I next want to discuss consistency with imaginary time evolution. Recall that Fock space contains lines corresponding to certain subspaces of L^2. Consider two times say $0, t$ where $0<t$.

Given a subspace of L^2 of the circle at time 0 one obtains a subspace at the later time t by using holomorphic functions on $(\mathbb{R}/2\pi\mathbb{Z}) \times [0,t]$ whose boundary values at time 0 belong to the given subspace.

I want to discuss imaginary time-evolution more carefully. Let us take a fairly concrete viewpoint starting with the Fock space associated to AP half densities $f(x)\,dx^{1/2}$ on S^1. Then as we have seen above this Fock space has operators

$$\psi^*(f\,dx^{1/2}) = \sum c_k \psi_k^* \qquad c_k = \int \frac{dx}{2\pi} f(x)\, e^{-ikx}$$

$$\psi(f\,dx^{1/2}) = \sum c_k \psi_{-k}$$

and a vacuum $|0\rangle$ satisfying

$$\psi_k^* |0\rangle = 0 \qquad k<0$$

$$\psi_k |0\rangle = 0 \qquad k>0$$

The Hamiltonian $H = \sum k :\psi_k^* \psi_k:$ has eigenvalues ≥ 0 so the operator e^{-tH} is well defined.

Recall there is a basis e_S for the Fock space consisting of eigenvectors of H.

Now suppose we have two operators $\psi^*(f_0\,dx^{1/2})$ and $\psi^*(f_t\,dx^{1/2})$ which are consistent with e^{-tH}:

$$\psi^*(f_t\,dx^{1/2})\, e^{-tH} \alpha = e^{-tH} \psi^*(f_0\,dx^{1/2})\, \alpha$$

for all α. Then if $f_t = \sum c_k^t e^{ikx}$ we have

$$\sum c_k^t \psi_k^* e^{-tH}\alpha = \sum c_k^\circ \underbrace{e^{-tH}\psi_k^* e^{tH}}_{e^{-kt}\psi_k^*} e^{-tH}\alpha$$

whence $\quad c_k^t = e^{-kt} c_k^\circ$

and this means that

$$f_t(x) = \sum c_k^t e^{ikx} = \sum c_k^\circ e^{ikx-kt}$$

is an analytic function of $x+it$.

Note this means that the time evolution given by

$$\psi^*(f_t\, dx^{1/2}) = e^{-tH}\psi^*(f_\circ dx^{1/2})e^{tH}$$

corresponds to taking the half density $f_\circ dx^{1/2}$ and analytically continuing it in the imaginary direction.

Let's now consider the cylinder of $(x,t) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ equipped with the complex ~~structure~~ such that $x+it$ is analytic. Put on this cylinder the spin structure anti-periodic in the x direction. For each time t we have a Fock space $\mathcal{F}_t$, so we have a bundle of Fock spaces over the t line. One has imaginary-time translation $\quad e^{-tH} : \mathcal{F}_s \longrightarrow \mathcal{F}_{s+t}$. $\qquad t \geq 0$

It would be better to say we have trivialized this bundle one way since all $\mathcal{F}_t$ are obviously isomorphic to $\mathcal{F}$ in an obvious way. But then we don't want this trivialization, rather we want to use time evolution. But then because e^{-tH} is a contraction operator we obtain a tower of Fock spaces.

$$\check{\mathcal{F}} \subset \mathcal{F}_t \subset \hat{\mathcal{F}}$$

We can think of $\mathcal{F}_t$ as the completion of $\check{\mathcal{F}} = \bigoplus \mathbb{C}e_S$ in a suitable norm which weights the line $\mathbb{C}e_S$ according to the energy and t.

One would like to $\check{\mathcal{F}}$ and $\hat{\mathcal{F}}$ as being replaceable by $\bigcap \mathcal{F}_t$ and $\bigcup \mathcal{F}_t$.

Now what kind of Clifford operators do we have? We have agreed to refer operators on $\mathcal{F}_t$ or from $\mathcal{F}_s$ to $\mathcal{F}_t$ back to $\mathcal{F}_0$ where they are unbounded. It [redacted] might be better to say that [redacted] our operators go from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$, or maybe to concentrate on ones which go from $\mathcal{F}_s$ to $\mathcal{F}_t$.

The first question is to worry about a Clifford operator $\sum c_k \psi_k^*$ on $\mathcal{F}_0$. These are linear combinations such that $\sum |c_k|^2 < \infty$. We can transport this to $\mathcal{F}_t$ by

$$e^{-tH}\left(\sum c_k \psi_k^*\right) e^{tH} = \sum c_k e^{-kt} \psi_k^*$$

and so therefore we can recognize Clifford operators on $\mathcal{F}_t$.

So what are the Clifford operators [redacted] $\sum c_k \psi_k^*$ which map $\mathcal{F}_s$ to $\mathcal{F}_t$?

Question: What linear combinations $\sum c_k \psi_k^*$ make sense as operators from $\mathcal{F}_s$ to $\mathcal{F}_t$?

We can split into creation + destruction parts the former being $\sum_{k>0} c_k \psi_k^*$. If c_k is an ℓ^2 sequence this makes sense on $\mathcal{F}_0$ and hence as an operator from $\mathcal{F}_s$ to $\mathcal{F}_t$ for $s \leq 0 \leq t$.

Now formally $\sum c_k \psi_k^*$ corresponds to

$$\sum_{k>0} c_k e^{ikw} \qquad w = x + it$$

which ought to be analytic for $\operatorname{Im}(w) > a - \varepsilon$. Thus suppose that $c_k e^{-ka}$ is an ℓ^2 sequence. Then

$$e^{-aH}\left(\sum{}' c_k \psi_k^*\right) e^{+aH} = \sum c_k e^{-ka} \psi_k^*$$

is well defined on $\mathcal{F}_0$ so that $\sum' c_k \psi_k^*$ is well defined on $e^{+aH}\mathcal{F}_0 = \mathcal{F}_a$.

Similarly $\sum_{k<0} c_k e^{ikw}$ ought to be analytic for $\operatorname{Im}(w) < b + \varepsilon$ (some strip). Thus $c_k e^{-kb}$ will be an ℓ^2 sequence so

$$\sum_{k<0} c_k \psi_k^* = e^{bH}\left(\sum_{k<0} c_k e^{-kb} \psi_k^*\right) e^{-bH}$$

will be defined on $e^{bH}\mathcal{F}_0 = \mathcal{F}_b$.

Thus a typical series $\sum c_k e^{ikw}$ will when split converge in regions $\operatorname{Im} w < b + \varepsilon$, $\operatorname{Im} w > a - \varepsilon$ and will make sense as an operator from $\mathcal{F}_s$ to $\mathcal{F}_t$ for $s \leq b$, $a \leq t$.

I am primarily interested in the operator

$$\sum e^{-ikx+kt}\psi_k^*$$

which will make sense from $\mathcal{F}_{t-\varepsilon}$ to $\mathcal{F}_{t+\varepsilon}$. Now we have

$$\psi^*(f(x)\,dx^{1/2}) = \int \frac{dx}{2\pi} f(x) \sum e^{-ikx}\psi_k^*$$

Let us write $z = e^{t-ix}$ so that

$$\frac{dz}{z} = dt - i\,dx$$

Notice that as x goes from 0 to 2π and $t=0$ that z traces out the unit circle backwards. On this circle

$$dx = i\frac{dz}{z}$$

so the above integral is

$$\int \frac{dz}{2\pi i\, z} f(z) \sum z^k \psi_k^*$$

This suggests the naturality of putting

$$\psi^*(z) = \sum z^{k-\frac{1}{2}}\psi_k^*$$

So where are we? We know now that $\psi^*(z)$ and probably also $\psi(z)$ are well defined operators from $\mathcal{F}_s$ to $\mathcal{F}_t$ provided $e^t > |z| > e^s$.

March 15, 1987

The problem is to specify a vector space of operators of the form $\sum c_k \psi_k^*$ which are associated in some way to rational functions of z. By this I mean that the power series

$$\sum_{k>0} c_k z^{-k-\frac{1}{2}} \qquad \sum_{k<0} c_k z^{-k-\frac{1}{2}}$$

converge near ∞ and near 0 resp. to rational functions. To simplify let's shift from k ~~half~~ integral to k integral. When is a power series

$$\sum_{n\geq 0} c_n z^n$$

the power series of a rational function regular at $z=0$? Use partial fractions

$$\frac{1}{1-\lambda z} = \sum_{n\geq 0} \lambda^n z^n$$

$$\frac{r!\,\lambda^r}{(1-\lambda z)^{r+1}} = \sum_{n\geq 0} n(n-1)\cdots(n-r+1)\,\lambda^n z^n$$

Partial fractions so that the space of rational functions regular at zero ~~has~~ the basis

$$z^n \qquad n\geq 0$$

$$\frac{1}{(1-\lambda z)^n} \qquad n\geq 0, \quad \lambda\in\mathbb{C}^\times$$

What seems to be emerging is that one requires the sequence c_k to be exponential polynomial modulo finite sequences. There is a non-zero polynomial in the shift operator which when applied to c_k gives a finite sequences.

Let's describe the sequences (c_k) such that $(T-1)c_k = c_{k+1} - c_k$ is a finite sequence. Let W be the space of these sequences. Then W contains the finite sequence space F and W/F is 2 dimensional. In general what we get for a the sequences killed up to finite sequences by a polynomial in the shift operator is an extension of the finite sequences by two copies of the sequences killed by the polynomial.

I might try to generalize and look at an annulus $s \leq |z| \leq t$. Then the finite sequences would be replaced by the analytic functions on the annulus. Then the space of K^* operators is an extension of the meromorphic functions on the annulus with poles in the inside by the space of singular parts of such functions.

March 17, 1987

The problem is still to understand, better to describe in a good way, the kind of operators $\sum c_k \psi_k$ to be considered. At the moment I am considering the space of Laurent series $\sum c_k z^{-k}$ such that there exists a nonzero polynomial in z, $p(z)$ such that

$$p(z) \sum c_k z^{-k}$$

is a polynomial. This maps onto the space of rational functions and the kernel is the space of exponential polynomial sequences, i.e. singular parts of merom. functions regular at 0 and ∞.

I want to obtain a nicer description possibly without the $0, \infty$ restriction. It seems to be a good idea to reformulate in terms of spinors. As $\mathbb{P}^1$ is simply-connected it is not possible to have anti-periodic half-densities on a circle arising from a spin structure.

On $\mathbb{P}^1$ the canonical line bundle has the meromorphic section dz which has no zeroes and has a double pole at $z = \infty$ (since if $u = \frac{1}{z}$, $dz = d(\frac{1}{u}) = -\frac{1}{u^2} du$). Thus $\Omega^1 \simeq \mathcal{O}(-2)$ i.e. any holom diffl on an open set U can be described in the form $f(z)dz$ where f is holom. on U and vanishes twice at ∞ if $\infty \in U$.

Now the unique spin structure on $\mathbb{P}^1$ is given by the line bundle $\mathcal{O}(-1)$ with the isomorphism $\mathcal{O}(-1)^{\otimes 2} = \mathcal{O}(-2) \simeq \Omega^1$. It is more suggestive to say there is a unique line bundle L with an isom $L^{\otimes 2} = \Omega^1$, and that L has a unique meromorphic section, nonvanishing with simple pole at ∞, denote it $dz^{1/2}$, such that $(dz^{1/2})^2 = dz$. Then holom.

sections of L on U are of the form $f(dz)^{1/2}$ where f is holom on U and vanishes at ∞ if $\infty \in U$.

Suppose we have two meromorphic sections of L, $f\,dz^{1/2}$ and $g\,dz^{1/2}$ which have disjoint poles. Then f, g have disjoint poles and one must have a zero at ∞. Then we can form their product $fg\,dz$ and take the sum of the residues at the poles of one of them. For example

$$f = \frac{1}{\zeta - z} \qquad g = \frac{1}{\zeta - w}$$

then

$$\sum_{f(P)=0} \operatorname*{Res}_{\zeta = P} \left\{ \frac{1}{\zeta - z} \frac{1}{\zeta - w} d\zeta \right\} = \operatorname*{Res}_{\zeta = z} \left\{ \frac{1}{\zeta - z} \frac{1}{\zeta - w} d\zeta \right\}$$

$$= \frac{1}{z - w}$$

March 18, 1987:

Remark 1. If I think of spinors over $\mathbb{P}^1$ as sections f of $\mathcal{O}(-1)$, identifying f with $f\,dz^{1/2}$, then the quadratic form defined by taking the product and integrating over $|z|=1$ is

$$\left(z^m, z^n\right) = \frac{1}{2\pi i}\int z^{m+n}\,dz = \delta_{m+n,-1}$$

This sort of explains why one might prefer the notation
$$\psi(z) = \sum_{k\in\frac{1}{2}+\mathbb{Z}} z^{k-(1/2)}\psi_k$$
because then

$$\frac{1}{2\pi i}\int z^{k-\frac{1}{2}}(dz)^{1/2}\; z^{l-\frac{1}{2}}(dz)^{1/2} = \delta_{k+l,0}$$

Remark 2. The space spanned by the ψ_k is the space of meromorphic spinors on $\mathbb{P}^1$ with poles at $0, \infty$. In general it might make sense on a Riemann surface to fix a finite set S of points on the surface and to consider the meromorphic spinors with these poles. Is there some way these would act on the tensor product of the little Fock spaces at each point of S?

Take little circles around each point of S. Let the point of S be $P_1, \dots, P_r$, and the circles be $C_1, \dots, C_r$, and they should be oriented positively. Then we have the space of spinors on each circle C_i and it comes equipped with quadratic form. Next we can form the ~~vector space~~ Clifford algebra for each circle and their tensor product as well as the tensor product of the corresponding Fock spaces.

There should be in each Fock space $\mathcal{F}_i$ a "vacuum" line corresponding to the interior of the circle C_i. More precisely the space of spinors on C_i contains the subspace W_i of spinors extending holomorphically to the disk inside C_i. The vacuum line in $\mathcal{F}_i$ is killed by the operators $\psi(\xi)$, $\xi \in W_i$. Thus in $\otimes \mathcal{F}_i$ we have a vacuum ~~state~~ $|0\rangle = \otimes(|0\rangle_i)$.

On the other hand the Riemann surface outside of the circles C_i should give a linear functional $\langle 0|$ on the space $\otimes \mathcal{F}_i$

Now before going on it would be nice to know what the Fock space $\mathcal{F}_i$ associated to the circle C_i becomes as C_i shrinks to the point P_i. I'm not sure how to think about $\check{\mathcal{F}}$ in the single fermion case.

Let's then return to the circle $\mathbb{R}/2\pi\mathbb{Z}$

where
$$\psi(f(x)dx^{1/2}) = \sum c_k \psi_k \qquad f = \sum c_k e^{ikx}$$

$$= \int \frac{dx}{2\pi} f(x) \sum e^{-ikx} \psi_k$$

When is $\psi(f_t(x)dx^{1/2})$ consistent with e^{-tH} i.e.

$$\psi(f_t\, dx^{1/2})\, e^{-tH} = e^{-tH}\, \psi(f_0\, dx^{1/2})$$

$$c_k(t)\psi_k = e^{-tH}\, c_k(0)\psi_k\, e^{tH} = c_k(0)e^{-kt}$$

that is

$$f_t(x) = \sum c_k(0)\, e^{ikx-kt}$$

is holomorphic in $x+it$. Recall that we put $z = e^{t-ix}$ whence

$$f_t(x) = \sum c_k(0)\, z^{-k}$$

and on any of the Fock space $\mathcal{F}_t$ one has

$$\psi(f) = \sum c_k(0)\, \psi_k$$

$$c_k(0) = \frac{1}{2\pi i}\int \frac{dz}{z}\, f(z)\, z^{+k}$$

The above isn't very clear, but what I am aiming for is to identify all the Fock spaces under time evolution, so that any holomorphic spinor over $\mathbb{C}^\times$ will act. Also I want the intersection of the Fock spaces to be essentially the little Fock space.

Now f is holomorphic in $\mathbb{C}$ means that the coefficients $c_k(0) = 0$ for $k > 0$ which means that $\psi(f)$ is a linear combination of ψ_k for $k < 0$. Thus $\psi(f)|0\rangle = 0$, as it should.

Conclude: The Fock space attached to a point on a surface is the exterior algebra of $F/\mathcal{O}_p$ at that point, more precisely it is the exterior algebra of the singular parts of meromorphic spinors. Thus ~~there~~ are lines in Fock space attached to subspaces of F containing $\mathcal{O}_p$.

INACCURATE
see p. 547

I want to return to the problem of describing the space of operators $\sum c_k \psi_k$ where $\sum c_k z^{-k}$ is such that

$$p(z) \sum c_k z^{-k} = q(z)$$

for Laurent polys. p, q with $p \neq 0$. Recall that this space is an extension of the space of rational functions by the space of principal parts on $\mathbb{C}^\times$.

I feel that the above operators are well-defined from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$ and hence to each $|e_S\rangle \in \check{\mathcal{F}}$ and $\langle e_{S'}|$ in $\hat{\mathcal{F}}^*$ we get a number. Note that one uses the time evolution embedding of $\check{\mathcal{F}}$ in $\hat{\mathcal{F}}$. Thus it seems reasonable to expect these operators to act on the tensor product of the Fock spaces $\otimes$ $\mathcal{F}_0$ and $\mathcal{F}_\infty$. One then applies $\sum c_k \psi_k$ to $\langle 0|$ representing the time evolution and $|e_S\rangle \otimes \langle e_{S'}|$ and gets a number.

I think I am beginning to get somewhere on the operators from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$. Think of having a small circle and a large circle and time evolution in between. Suppose we have two meromorphic spinors f_{in} and f_{out}. We can then consider $\psi(f_{in})$ on the small circle Fock space, followed by time evolution to the large Fock space, followed by $-\psi(f_{out})$. Notice that if $f_{in} = f_{out}$ this doesn't give zero although it does if $f_{in} = f_{out}$ is holomorphic in the region between the circles. So we we are looking at the space of pairs of meromorphic spinors modulo the diagonal subspace of spinors analytic over the complement of 0 and ∞. There is an obvious map from this space to meromorphic spinors given by $(f_{in}, f_{out}) \longmapsto f_{in} - f_{out}$. The kernel is the space of meromorphic spinors modulo those regular outside of $0, \infty$, and this is the space of principal parts on the complement of $\{0, \infty\}$.

Now I want to understand many points: $S = \{P_i\}$ What I can do is to give a merom. spinor f_i for each point, and then let $\sum \psi(f_i)$ act on $|0\rangle$ in $\otimes \mathcal{F}_i$. Then we apply $\langle 0|$ representing the surface outside the P_i. This gives a number which is zero if all f_i are equal and are holomorphic except at the P_i.

The next problem is how do principal parts enter? Suppose all the f_i are equal, then this space is the space of meromorphic spinors, and the number being obtained is zero when the f_i are holomorphic. Thus we have a linear function on the space of principal parts over the complement of S.

Consider the case of a single point on the surface and a small circle around it.

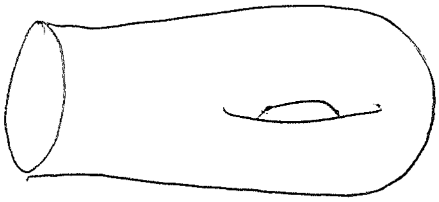
In fact it might be better to take a Riemann surface bounded by a circle and to consider the Fock space of the circle; it contains a line corresponding to the surface. In effect the space of sections of the spinors on the circle contains the subspace of spinors extending holomorphically.

This is isotropic for the quadratic form, and in good cases it would be maximal isotropic.

So let $|0\rangle$ denote the ~~state~~ corresponding to the isotropic subspace of holomorphic spinors. If $f_1, \dots, f_n$ are spinors on the circle extending to meromorphic spinors inside then we have the state

$$\psi(f_1) \cdots \psi(f_n) |0\rangle .$$

Let's start again with a Riemann surface bounded by a circle



Suppose that there is a spin structure. Then the space of spinors on the circle has the isotropic subspace of holom. spinors on the surface. The reason the subspace is isotropic is because the circle is trivial in homology: it is ∂[surface]. The rest is Stokes thm and the fact that holom. 1-forms are closed. The spinor bundle over the circle is trivial.

Suppose we now fill in the circle and that the resulting spin structure has no holom. sections. Then the space of spinors on the circle splits into the isotropic spaces of sections extending holomorphically on either side and so we will have vectors $\langle 0|$, $|0\rangle$ on the Fock space.

Then for any set of spinors on the circle we can form

$$\langle 0 | \psi(f_1) \cdots \psi(f_r) | 0 \rangle$$

For example this should make sense for any family of meromorphic spinors on the complete surface with poles off the circle.

Now arises the problem of what happened to the condition that the poles be disjoint. Secondly there is the ~~question~~ problem of evaluating the above expectation value to see if this might be interesting.

The first remark is that the operator $\psi(f)$ for f a meromorphic spinor is meaningless unless you say what circle is given.

The next remark is that the above vacuum expectation values can be evaluated by Wick's theorem. One has taken the space of spinors over the circle and split it into maximal isotropic subspaces. This defines a normal ordering.

March 19, 1987

I have to clear up some inaccurasies concerning the local Fock space at a point P. This is the intersection of the Fock spaces of small circles shrinking to P, the inclusions being defined by "time evolution". There is also an algebraic version $\check{\mathcal{F}}_P \subset \mathcal{F}_P$, but I won't worry about the distinction for the present.

If f is a meromorphic spinor near P it determines an operator $\psi(f)$ on $\mathcal{F}_P$ and these satisfy

1) $$\{\psi(f), \psi(g)\} = \mathrm{res}_P(fg).$$

We probably should work with the space $L \otimes F$ of germs of merom. spinors at P with the residue quadratic form. The germs of holom. spinors form an isotropic subspace L_P, and it is clearly a maximal isotropic subspace – this is a local power series calculation. Thus there is a unique state $|0\rangle$ in $\mathcal{F}_P$ annihilated by the $\psi(f)$ for f a holom. germ.

Now consider

2) $$\psi(f_1) \cdots \psi(f_n)|0\rangle$$

It is not true that this depends on $f_i \in L \otimes F / L_P$ because we can't move the $\psi(f_i)$ around without encountering residues 1). However suppose we are in the situation where there are no nonzero holom. spinors: $H^0(L) = 0$. Then we have the space W of germs which extend holomorphically to the complement of P. This space is isotropic and complementary to L_P in $L \otimes F$. Thus if we restrict the $f_i \in W$ we see 2) is skew-symmetric and that we get an isomorphism

$$\Lambda W \xrightarrow{\sim} \mathcal{F}_P$$

Let us now do this with a finite set of points $S = \{P_1, \cdots, P_n\}$. Then we can form the tensor product $\otimes \mathcal{F}_{P_i}$ and on this we have operators $\psi(f)$ where $f = (f_i) \in \prod_1^n (L\otimes F)$, satisfying

$$\{\psi(f), \psi(g)\} = \sum \operatorname{res}_p(fg).$$

Actually I should be doing something adelic i.e. I should probably replace $L\otimes F$ by $L\otimes \hat{F}_p$.

Again the holomorphic germs at the P_i determine states $|0\rangle_i$ in $\mathcal{F}_{P_i}$, which then can be tensored to get $|0\rangle$ in $\otimes \mathcal{F}_{P_i}$. Next you have the space W of meromorphic spinors whose poles are in S. So we have

$$\prod_i L\otimes \hat{F}_{P_i} = \prod_i L\otimes \hat{\mathcal{O}}_{P_i} \oplus W$$

where the two subspaces are maximal isotropic. As for one point this means that

$$\Lambda W \xrightarrow{\sim} \otimes \mathcal{F}_{P_i}$$

Now let the finite set go to infinity. We should get a Fock space with operators $\psi(f)$ for any adelic section of L, i.e., a representation of the Clifford algebra of the space of adelic spinors with its natural quadratic form. Corresponding to the break up into isotropic subspaces

$$L\otimes_{\mathcal{O}} A = L\otimes \prod \hat{\mathcal{O}}_p \oplus L\otimes F$$

one obtains an isomorphism of $\mathcal{F}$ with $\Lambda(L\otimes_{\mathcal{O}} F)$.

Now where does the Green's function enter? We know that any VEV

$$\langle 0|\psi(f_1)\cdots\psi(f_n)|0\rangle$$

where the $f_i \in L\otimes A$ can be evaluated by Wick's formula in terms of the case $n=2$.

Thus we want $\langle 0|\psi(f)\psi(g)|0\rangle$. Now $\psi(g)|0\rangle$ depends upon the singular part of g. Let us choose a meromorphic spinor ω with $g-\omega \in \prod \hat{L}_p$, so that $\psi(g)|0\rangle = \psi(\omega)|0\rangle$.

Then
$$\langle 0|\psi(f)\psi(g)|0\rangle = \langle 0|\psi(f)\psi(\omega)|0\rangle$$
$$= \sum_p \operatorname{res}_p(f\omega) - \underbrace{\langle 0|\psi(\omega)\psi(f)|0\rangle}_{0}$$

Maybe we should think of $\langle \psi(f)\psi(g)\rangle$ as bilinear in $f, g \in L\otimes A$. It doesn't change if g is moved by a regular adelic spinor, hence one can suppose g is a merom. spinor. It doesn't change if f is moved by a merom. spinor, hence one can suppose f regular.

One has the following choice in the evaluation
$$\langle 0|\psi(f)\psi(g)|0\rangle = \sum \operatorname{res}(f\omega)$$
$$= \sum \operatorname{res}(r\, g)$$

where $\omega \in L\otimes F$ has the same singular part as f and where r is a regular adelic spinor congruent to f modulo $L\otimes F$.

We still have a picture missing in order to understand what $\phi(z)$ is. I still feel that we must start with the space of singular parts of meromorphic spinors and associate operators somewhere to these.

Let me go over what I told Bott. I take a Riemann surface M with spin structure L such that $H^0(L)=0$, and two points P, Q. Then I have the product $\mathcal{F}_P \otimes \mathcal{F}_Q$ which is an irreducible representation of the Clifford algebra associated to

$$V = L\otimes \hat{F}_P \oplus L\otimes \hat{F}_Q$$

with a cyclic vector belonging to the max. isot. subspace

$$V_{reg} = L\otimes \hat{\mathcal{O}}_P \oplus L\otimes \hat{\mathcal{O}}_Q$$

Let W be the space of merom. sections of L which are regular outside $\{P, Q\}$. Then W is isotropic and complementary to V_{reg} so we have

$$\Lambda W \xrightarrow{\sim} \mathcal{F}_P \otimes \mathcal{F}_Q .$$

This defines a dual state $\langle 0|$ namely projection ~~onto~~ $\Lambda^0 W$. Then given f a meromorphic spinor we consider the dual vector

$$\langle 0|\ \psi(f)\otimes 1 - 1\otimes \psi(f) \qquad \text{on } \mathcal{F}_P \otimes \mathcal{F}_Q .$$

It depends only on f mod W. But $L\otimes F/W$ is the space of singular parts of merom. sections of L over $M-\{P, Q\}$. Finally we interpret a dual vector on $\mathcal{F}_P \otimes \mathcal{F}_Q$ as a map from $\mathcal{F}_P$ to $\mathcal{F}_Q^*$.

Now can we define composition?

Suppose that f and g are meromorphic in $a \leq t \leq c$ and that the poles of f are in $a<t<b$ and the poles of g are in $b<t<c$

Recall that we have defined the operator belonging to the singular part of f to be $\psi(f_a)$ on $\mathcal{F}_a$ followed by time evolution to $\mathcal{F}_c$ minus time evolution followed by $\psi(f_c)$ on $\mathcal{F}_c$. Thus it is

$$e^{-(c-a)H}\psi(f_a) - \psi(f_c)\, e^{-(c-a)H}$$

Since f is holom. for $b<t<c$, this is the same as

$$e^{-(c-b)H}\left\{ e^{-(b-a)H}\psi(f_a) - \psi(f_b)\,e^{-(b-a)H}\right\}$$

So now we compose

$$\left\{e^{-(c-b)H}\psi(g_b) - \psi(g_c)e^{-(c-b)H}\right\}\left\{e^{-(b-a)H}\psi(f_a) - \psi(f_b)e^{-(b-a)H}\right\}$$

$$\begin{aligned}
= \;& e^{-(c-b)H}\psi(g_b)e^{-(b-a)H}\psi(f_a) && = e^{-(c-a)H}\psi(g_a)\psi(f_a) \\
& - e^{-(c-b)H}\psi(g_b)\psi(f_b)e^{-(b-a)H} && = \left(-\int g_b f_b\right)e^{-(c-a)H} + \psi(f_c)e^{-(c-b)H}\psi(g_a) \\
& - \psi(g_c)e^{-(c-a)H}\psi(f_a) && = -\psi(g_c)e^{-(c-a)H}\psi(f_a) \\
& + \psi(g_c)e^{-(c-b)H}\psi(f_b)e^{-(b-a)H} && = \psi(g_c)\psi(f_c)\,e^{-(c-a)H}
\end{aligned}$$

Thus the normal ordered part is the operator from $\mathcal{F}_a$ to $\mathcal{F}_c$ which corresponds to the ~~...~~ linear fnl on $\mathcal{F}_c^* \otimes \mathcal{F}_a$ given by

$$* \qquad 1\otimes\psi(g)\psi(f) + \psi(f)\otimes\psi(g) - \psi(g)\otimes\psi(f) + \psi(g)\psi(f)\otimes 1$$

followed by pairing with the time evolution from $\mathcal{F}_a$ to $\mathcal{F}_c$. Note that $*$ is the same as

$$(1\otimes\psi(g) - \psi(g)\otimes 1)(1\otimes\psi(f) - \psi(f)\otimes 1)$$

The contraction term is

$$-\int g_b f_b = -\int g_a f_a - \sum_{f(P)=\infty} \mathrm{res}_P(gf)$$

$$= -\int g_c f_c + \sum_{g(P)=\infty} \mathrm{res}_P(gf)$$

Anyway let's let a shrink to 0 and c expand to infinity. Let's require f, g to be regular at 0 and ∞. Then the contraction term is

$$-\int g_b f_b = \sum_{g(P)=\infty} \mathrm{res}_P(gf)$$

and $*$ is skew symmetric in g and f as $\mathrm{res}_{0,\infty}(fg)=0$.

We can now pose a conjecture namely that by mapping the exterior algebra on the space of meromorphic spinors regular at $0, \infty$ into $(\mathcal{F}_0 \otimes \mathcal{F}_\infty)^*$, we get a map compatible with composition. Actually this isn't well posed as I have no way to compose these operators $\mathcal{F}_0 \to \mathcal{F}_\infty^*$ anyway.

March 20, 1987

Improvement to yesterday's calculation.

Let's work between the circles $|z| = a$ and $|z| = c$ where $a < c$. Let f be meromorphic on $a \leq |z| \leq c$ with poles inside and set

$$\tilde{\psi}(\text{"princ part of" } f) = T(c,a)\psi(f_a) - \psi(f_c)T(c,a)$$

where $T(c,a) = e^{-(c-a)H}$. Then we found a composition formula as follows. Suppose $a < b < c$, and that the poles of f are in (a,b), while the poles of another spinor g are in (b,c). Then

$$\begin{aligned}\tilde{\psi}(g)\tilde{\psi}(f) = \left(-\int g_b f_b\right)T(c,a) &+ T(c,a)\psi(g_a)\psi(f_a) \\ &- \psi(g_c)T(c,a)\psi(f_a) \\ &+ \psi(f_c)T(c,a)\psi(g_a) \\ &+ \psi(g_c)\psi(f_c)T(c,a)\end{aligned}$$

I originally thought the four terms at the right should be $:\tilde{\psi}(g)\tilde{\psi}(f):$, but it's not skew-symm. in g, f. To remedy this we use

$$\begin{aligned}\tfrac{1}{2}[\psi(g),\psi(f)] &= \tfrac{1}{2}\left(\psi(g)\psi(f) - \psi(f)\psi(g)\right) \\ &= \psi(g)\psi(f) - \tfrac{1}{2}\int gf\end{aligned}$$

Thus we set

$$\begin{aligned}:\tilde{\psi}(g)\tilde{\psi}(f): \;=\; & T(c,a)\tfrac{1}{2}[\psi(g),\psi(f)]_a \\ &+ \psi(f_c)T(c,a)\psi(g_a) \\ &- \psi(g_c)T(c,a)\psi(f_a) \\ &+ \tfrac{1}{2}[\psi(g),\psi(f)]_c\, T(c,a)\end{aligned}$$

Then the contraction term becomes $T(c,a)$ times the scalar

$$-\int g_b f_b + \frac{1}{2}\int g_a f_a + \frac{1}{2}\int g_b f_b$$

$$= -\sum_{\substack{\text{poles}\\ \text{of } f}} \operatorname{res}(gf) \underbrace{- \int g_a f_a + \frac{1}{2}\int g_a f_a + \frac{1}{2}\int g_b f_b}_{\frac{1}{2}\left(\int g_b f_b - \int g_a f_a\right) = \frac{1}{2}\sum\limits_{\substack{\text{poles}\\ \text{of } f \text{ and of } g}} \operatorname{res}(gf)}$$

$$= \frac{1}{2}\left(\sum_{\substack{\text{poles}\\ \text{of } g}} \operatorname{res}(gf) - \sum_{\substack{\text{poles}\\ \text{of } f}} \operatorname{res}(gf)\right)$$

Thus we end up with the formula

$$\boxed{\psi(g)\,\psi(f) = \,:\!\psi(g)\,\psi(f)\!:\, + \frac{1}{2}\left(\sum_{\substack{\text{poles}\\ \text{of } g}} \operatorname{res}(gf) - \sum_{\substack{\text{poles}\\ \text{of } f}} \operatorname{res}(fg)\right)}$$

March 21, 1987

Problem: Suppose we have a Riemann surface which is compact and has a single boundary component.

Show how one obtains a line in the Fock space of the circle. We have to suppose a spinor structure L is given on the surface. Then $\Gamma(C,L)$ contains the subspace W_{sm} of sections which extend holomorphically to the surface. W_{sm} is isotropic in $\Gamma(C,L)$.

I think there is no difficulty in closing up W_{sm} to get an isotropic subspace W of $L^2(C,L)$. which is isotropic. Then $W \cap \overline{W} = 0$, in fact $W, \overline{W}$ are clearly perpendicular. The issue is whether $L^2(C,L) = W \oplus \overline{W}$ in which case we obtain a "polarization" and this defines a line in the Fock space.

Certainly it is enough to show that

$$L^2(C,L)_{real} \simeq W$$

i.e. that there is ~~for~~ any real spinor over C a <u>unique</u> holomorphic spinor on the surface whose restriction to the boundary has real part equal to the given real spinor.

The method I might use to establish this is to set up a self-adjoint boundary value problem on the surface. Specifically we can consider the problem of constructing a holomorphic spinor in the interior with given real part on the boundary. Or we might try to solve $\bar\partial u = f$ on the surface with the boundary condition that the real part of u is zero on the boundary. (The latter seems wrong?)

Now the natural candidate is the Dirac operator on the surface which is a skew- or self-adjoint operator on a 2 dimensional vector bundle which is graded. This requires a metric.

The cotangent bundle has a complex structure, and when complexified it splits into $T^{1,0} \oplus T^{0,1}$ and we get obvious raising and lower operators. The spinor bundle must have the form $S = S^+ \oplus S^-$ where $S^\pm$ are line bundles such that

$$T^{0,1} \otimes S^+ \simeq S^-$$

$$T^{1,0} \otimes S^- \simeq S^+.$$

So far what we have said holds for any Dirac type operator. Something special happens in the case of a spinor bundle.

Let's proceed in general and try to get the adjointness straight.

Consider $\Gamma_c(M, L)$, where $L \otimes L = T^{1,0}$. On this space we have a skew-symmetric form given by $\xi, \eta \longmapsto \int \xi \cdot \bar\partial \eta$. Suppose a Riemannian metric given on M compatible with the complex structure. Then L acquires a unique hermitian metric consistent with the one on $T^{1,0}$ and $\Gamma(M,L)$ has an inner product allowing us to form $L^2(M,L)$. Thus we a complex Hilbert space with a skew form which is partially defined.

Let's go back to $\Gamma_c(M,L)$ the sections vanishing on the boundary. This is a complex v.s. with hermitian inner product and a skew symm. form. Forget the complex structure, i.e. look at L as a 2 diml real vector bundle. Then the skew form ~~shows~~ that we have a formally skew-adjoint operator on sections of L.

Recall that in 2 dimensions there are two Clifford algebras over $\mathbb{R}$ one might consider relative to Dirac operators. One is generated by e_1, e_2 with $e_i^2 = -1$ and it is the quaternions $\mathbb{H}$, whose unique irred real repn has dim 4. The other is generated by γ^1, γ^2 with $(\gamma^i)^2 = 1$ and is $\simeq M_2(\mathbb{R})$, so its unique irred. real repns has dimension 2. The former gives rise to a self-adjoint operator and the latter to a skew adjoint operator. (The other possibility is a Lorentz signature $\gamma^2 = 1$, $e^2 = -1$ which somehow is related to Majorana spinors.)

So now I know that the $\bar\partial$ operator on L in the presence of a metric will be formally skew-adjoint. Now I want appropriate boundary conditions and these should be half way between requiring no condition at the boundary and requiring the section to vanish at the boundary. I have the condition that the real part of the section vanish. Let's check the skew-symmetry of the form on the space of smooth sections satisfying this boundary condition. Thus I I take a section s of L and I look at

$$\int s \cdot \bar\partial s .$$

Now $s \cdot \bar\partial s$ is a section of $L \otimes L \otimes T^{0,1} = T^{1,1}$
and s^2 is a section of $T^{1,0}$ so

$$d(\tfrac{1}{2}s^2) = \bar\partial(\tfrac{1}{2}s^2) = s\bar\partial s$$

Thus

$$\int_M s\bar\partial s = \int_{\partial M} \tfrac{1}{2}s^2$$

Now there is something wrong because if s is purely imaginary along ∂M, then $s^2 \leq 0$.

One problem is that have been considering a complex valued skew-form $\int \eta\bar\partial\xi$ and we need a real-valued form. Thus we ought to consider

$$\mathrm{Re}\int \eta\bar\partial\xi = \frac{1}{2}\int(\eta\bar\partial\xi + \bar\eta\partial\bar\xi)$$

and represent this in terms of the real part of the inner product.

Thus if we are dealing with the skew-form

$$\mathrm{Im}\int \eta\bar\partial\xi$$

we see that

$$\mathrm{Im}\int_M s\bar\partial s = \mathrm{Im}\int_{\partial M} \tfrac{1}{2}s^2 = 0$$

where s is purely imaginary on the boundary. Thus the boundary conditions I have in mind seem to produce a skew-adjoint boundary value problem.

Let's compute in the plane. Typical spinor sections are $f(z)\,dz^{1/2}$. Our basic form is

$$\iint \left(g(z)dz^{1/2}\right)\,\bar\partial\left(f(z)dz^{1/2}\right) = \iint g\,\partial_{\bar z} f\,\underbrace{dz\,d\bar z}_{-2i\,dxdy}$$

Set $g = u' + iv'$, $f = u + iv$ real + imag parts.

$$(-2i)(\partial_{\bar z}) = -i(\partial_x + i\partial_y) = -i\partial_x + \partial_y$$

$$\mathrm{Re}\iint g\,\partial_{\bar z} f\,dz\,d\bar z = \iint \mathrm{Re}\left\{(u'+iv')(-i\partial_x + \partial_y)(u+iv)\right\} dxdy$$

$$= \iint \left[u'(\partial_y u + \partial_x v) + v'(\partial_x u - \partial_y v)\right] dx\,dy$$

$$= \iint \begin{pmatrix} u' \\ v' \end{pmatrix}^t \begin{pmatrix} \partial_y & \partial_x \\ \partial_x & -\partial_y \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} dxdy$$

Thus the skew-adjoint operator is

$$\begin{pmatrix} \partial_y & \partial_x \\ \partial_x & -\partial_y \end{pmatrix} = \gamma'\partial_x + \varepsilon\,\partial_y$$

Let's also compute the imaginary part

$$\mathrm{Im}\left(\iint g\,\partial_{\bar z} f\ dz\,d\bar z\right) = \iint \mathrm{Im}\left\{ \qquad\qquad \right\} dx\,dy$$

$$= \iint u'(-\partial_x u + \partial_y v) + v'(\partial_y u + \partial_x v)\,dxdy$$

so the operator is $\begin{pmatrix} -\partial_x & \partial_y \\ \partial_y & \partial_x \end{pmatrix} = -\varepsilon\partial_x + \gamma'\partial_y$

Observe this calculation is metric independent and means that I can setup everything working with skew forms on $\Gamma_c(M, L)$. It might be useful to write

$$\operatorname{Re} \iint (g\, dz^{1/2})\, \bar{\partial}\, (f\, dz^{1/2}) = \iint g^t (\gamma' \partial_x + \varepsilon \partial_y) f \; dx\, dy$$

where we treat g, f as the two vectors of real functions.

call it $\Phi(g,f)$

Now we want to show this form is skew-symmetric when we impose the correct boundary condition. Thus we consider $\Phi(g,f) + \Phi(f,g)$.
But first write it

$$\iint g^t (\gamma' \partial_x + \varepsilon \partial_y) f\, dx\, dy = \iint g^t \big(\gamma' d(f\, dy) + \varepsilon\, d(-f\, dx) \big)$$

$$= \iint g^t \{ -\gamma' dy + \varepsilon\, dx \}\, df$$

Then

[redacted] $$\Phi(f,g) = \iint f^t \{ -\gamma' dy + \varepsilon\, dx \}\, dg$$

$$= -\iint dg^t \{ -\gamma' dy + \varepsilon\, dx \} f$$

So

$$\Phi(g,f) + \Phi(f,g) = -\iint d\left[g^t (-\gamma' dy + \varepsilon\, dx) f \right]$$

$$= -\int_{\partial M} g^t (\varepsilon\, dx - \gamma' dy) f$$

I should have done this with the imaginary part

$$\Psi(g,f) = \operatorname{Im} \iint g\, dz^{1/2}\, \bar{\partial}(f\, dz^{1/2}) = \iint g^t (-\varepsilon \partial_x + \gamma' \partial_y) f\; dx\, dy$$

$$= \iint g^t (-\varepsilon\, df\, dy + \gamma'\, dx\, df) = \iint g^t (\varepsilon\, dy + \gamma' dx)\, df$$

$$\Psi(f,g) = \iint f^t(\varepsilon dy + \gamma' dx)\, dg = \iint -dg^t(\varepsilon dy + \gamma' dx) f$$

$$\Psi(g,f) + \Psi(f,g) = -\iint d\left[g^t(\varepsilon dy + \gamma' dx) f\right]$$

$$= -\int_{\partial M} g^t(\varepsilon dy + \gamma' dx) f$$

Now suppose f real on ∂M, i.e. that

$$\left(f(z)\, dz^{1/2}\right)^2 = f(z)^2(dx + i\,dy) \geq 0.$$

To simplify suppose the boundary is the x axis oriented so that $dx > 0$. Then

$$f^2 = (u + iv)^2 = u^2 - v^2 + 2iuv \geq 0$$

so $uv = 0$ and $u^2 \geq v^2 \implies v = 0$. Thus if f, g are both real we have

$$g^t(\varepsilon dy + \gamma' dx) f = \begin{pmatrix} u' \\ 0 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ 0 \end{pmatrix} dx = 0.$$

What would be the general proof? You have a quadratic form given by the matrix $\varepsilon b + \gamma' a$ where the tangent vector to the curve is $a\partial_x + b\partial_y$. Then you have the condition $f^2(a + ib) \geq 0$, and you have to see this defines an isotropic line for the form. The set of f's satisfying this condition is a real line so we just have to check that the form vanishes on any point of the line

$$(u \;\; v)\begin{pmatrix} b & a \\ a & -b \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = b(u^2 - v^2) + 2auv$$

$$\operatorname{Im}\,(u^2 - v^2 + 2iuv)(a + ib) = b(u^2 - v^2) + 2auv.$$

So it's clear.

Thus we have checked in an ugly way the following argument.

We consider the surface M with boundary ∂M, and we consider the space of smooth sections of L over M which are real when restricted to ∂M. This is a real vector space on which we have the bilinear form

$$\Psi(g,f) = \operatorname{Im} \int_M g\, \bar\partial f$$

We claim this form is skew-symmetric. But

$$\Psi(f,f) = \operatorname{Im} \int_M f\, \bar\partial f = \operatorname{Im} \int_M d\left(\tfrac{1}{2} f^2\right)$$

$$= \operatorname{Im} \int_{\partial M} \tfrac{1}{2} f^2$$

and for f to be real on ∂M means that f^2 is a non-negative 1-form on ∂M relative to its orientation. Thus $\int \frac{1}{2} f^2$ is real so $\Psi(f,f) = 0$.

Now introduce a Riemannian metric on the surface consistent with its conformal structure. This puts a inner product on $\Gamma(M,L)$ considered as a real vector space and permits one to represent the above skew symmetric form as a skew-adjoint differential operator on L; this is the Dirac operator. The boundary condition is a self adjoint boundary condition. Since we have taken exactly half of the boundary values, it follows, or should be the case, that this operator + boundary condition is essentially self-adjoint.

Here I have to appeal to elliptic boundary value theory. This theory will give us a Fredholm operator. In fact since we have a skew-adjoint problem there will be a sequence of eigenvalues + eigenvectors. In particular there will be a Green's function provided there are no zero eigenvalues.

Suppose $f \in \Gamma(M,L)$ has real boundary values and is killed by the operator, i.e.

$$\Psi(g,f) = \mathrm{Im}\int g\,\bar\partial f = 0$$

for all $g \in \Gamma(M,L)$ having real boundary values. Then taking g to have zero boundary values and using the complex structure one sees that $\bar\partial f = 0$, so f is holom. But then as it is real on the bdry it has to be zero, assuming M connected. NO

Next I want to look at the homogeneous equation. The metric on the surface puts an inner product on $\Gamma(L)$ which is hermitian. The herm. inner product on L gives a conjugation-linear isom. $L \simeq L^*$, and one has a trivialization of $T^{1,1}$ given by the volume form. Thus one has a conjugate linear isom

$$L \simeq L^* \simeq L^* \otimes T^{1,1}$$

which allows us to pair two sections of L, i.e. define the hermitian product $\langle \xi | \eta \rangle$.

Similarly $L \otimes T^{0,1} \simeq L^* \otimes T^{1,0} = L^* \otimes L^{\otimes 2} = L$, is a conjugate linear isomorphism. Thus $\bar\partial : \Gamma(L) \to \Gamma(L \otimes T^{0,1})$ can be viewed as going from $\Gamma(L)$ to itself, but as a conjugate linear operator. Still not clear.

Let's go on and take $\alpha \in \Gamma(\partial M, L)$ which is imaginary and extend it smoothly inside.

Set $h = \bar{\partial}\alpha \in \Gamma(M, L \otimes T^{0,1})$. Because we can solve the inhomogeneous equation for any h, there is an $f \in \Gamma(M, L)$ with real boundary values such that $\bar{\partial} f = h$. So taking $h = \bar{\partial}\alpha$ we see that

$$\bar{\partial}(\alpha - f) = 0.$$

Thus $\alpha - f$ is a holomorphic section of L such that on the boundary it has the same imaginary part as α.

This proves that the space W of boundary values of holom. sections of L over M is such that $W \oplus \overline{W} = \Gamma(\partial M, L)$.

Spin fields. Let's take an analytic function with a quadratic branch point say

$$(z-1)^{-1/2}$$

and draw the cut from 1 to ∞ say. Then for $|z| < 1$ this function is periodic and for $|z| > 1$ it is anti-periodic. In fact we can easily work out the Laurent series in these regions. For $0 \leq |z| < 1$

$$(z-1)^{-1/2} = (-1)^{-1/2}(1-z)^{-1/2}$$

$$= (-1)^{-1/2} \sum_{n \geq 0} \frac{(1/2)(3/2)\cdots(2n-1/2)}{n!} z^n$$

and for $|z| > 1$ one has

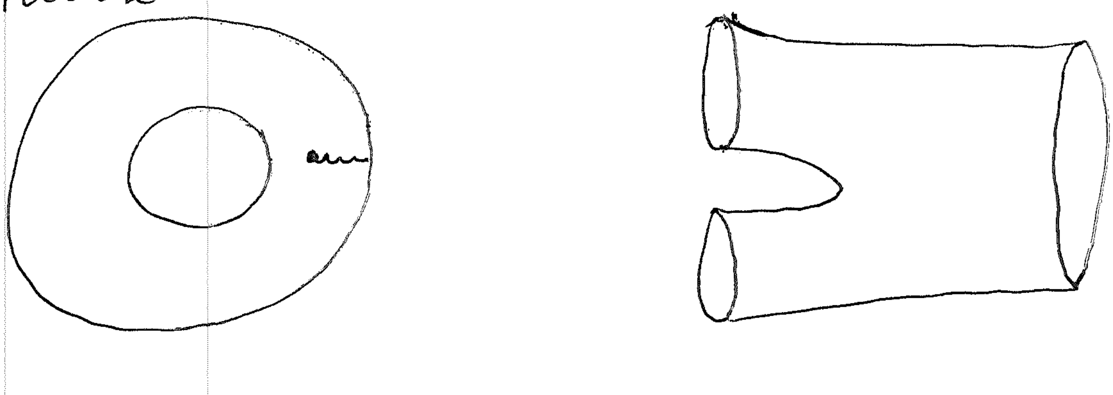
$$(z-1)^{-1/2} = z^{-1/2}\left(1-\frac{1}{z}\right)^{-1/2}$$
$$= \sum_{n\geq 0} \frac{(1/2)\cdots(2n-1/2)}{n!}\, z^{-n-\frac{1}{2}}$$

Now recall that [redacted] a meromorphic function gives rise to operators consistent with time evolution except at the poles where the operators change by something like $\psi(\xi)$. In the same spirit we recognize the two series as giving operators on the NS Fock space and the R Fock space respectively compatible with time translation. What happens at the singularity is an [redacted] operator going from the NS space to the R space. This has to be the spin field associated to the point 1.

Ultimately we have to take the spinors over two circles before an after the singular and construct [redacted] a suitable space of boundary values of holomorphic spinors in between. The problem is then to concoct a bundle in between.

One idea is to introduce the branched covering which has [redacted] a $\mathbb{Z}/2$-Galois group.

Picture



March 22, 1987 (Carl is 22)

Ramond case: It seems we have to correct our earlier impression about there being one irreducible representation of the Clifford algebra. There are two, probably related to some K-theory invariant, such as the index mod 2 of the Dirac operator. Let's try to understand this.

Actually I should probably go over what I learned about spin fields. The 'idea' is to replace a spin structure $L \otimes L = \Omega^1$ with a line bundle L such that $L \otimes L = \Omega^1(P)$. Any circle not passing through P then has a real structure on the sections of L over this circle, but as the circle crosses P the AP or P boundary conditions change. If we have an annulus with P inside, then the holomorphic sections of L over the annulus should define some sort of operator between the Fock spaces, and this is the spin field associated to the point P.

Let's look at the simple case where we have the circle bounding the disk and P inside. I'm going to try to find the state in the Fock space of the circle which is the spin field applied to the ground state of the Fock space at 0. (Except as we have said there are two Fock spaces apparently.)

On the disk or plane dz trivializes Ω^1. Let $\zeta \in \mathbb{C}$ and let $L \otimes L \simeq \Omega^1(\zeta)$. Then $\frac{dz}{z-\zeta}$ is a non-vanishing holom section of $\Omega^1(\zeta)$ over $\mathbb{C}$ and so there is a unique up to sign holom section $\left(\frac{dz}{z-\zeta}\right)^{1/2}$ of L, and any other holom section over $\mathbb{C}$ or the disk $|z| \leq 1$ can be written

$$f(z)\left(\frac{dz}{z-\zeta}\right)^{1/2}$$

with $f(z)$ holomorphic.

Let's take $\zeta = 0$ and look at $|z| = 1$. Then we have over the circle the space of

$$f(z)\left(\frac{dz}{z}\right)^{1/2} \qquad f(z) = \sum c_n z^n$$

with the quadratic form

$$\frac{1}{2\pi i}\int f(z)^2 \frac{dz}{z} \quad = \quad \sum c_n c_{-n}$$

Thus we are dealing with periodic half-densities on the circle. This is the Ramond case.

Fock space is then a representation of the Clifford algebra with defining relations

$$\{\psi_k, \psi_\ell\} = \delta_{k,-\ell}$$
$$\psi_k^* = \psi_{-k}$$

If we use the Dirac operator $\frac{1}{i}\partial_x$ on the periodic half densities on the circle, then the ground state in Fock space has to satisfy

$$\psi_k |0\rangle \qquad \text{for} \qquad k<0.$$

Then there are two possibilities

$$\psi_0 |0\rangle = \pm |0\rangle$$

leading to two Fock spaces.

A natural problem is to show these two Fock spaces are inequivalent representations of the Clifford algebra. The situation should be analogous to odd degree Clifford algebra; here one distinguishes between the irreducible representations by means of the central element $\gamma^1 \cdots \gamma^{2m+1}$. Recall that we proposed to define the polarization in the space of spinors on the circle by means of any Dirac operator. In the real setting however there is a mod 2 index namely the dimension of $\operatorname{Ker} D$ mod 2. Thus one will not be able to find an invertible real skew-adjoint Dirac in the Ramond case.

I guess we know that Gaussian states on the Clifford algebra of a real Hilbert space E are determined by skew-adjoint operators K such that $-K^2 \leq 1$. Polarizations are complex structures i.e. $J = -J^*$ such that $J^2 = -1$. Atiyah + Singer considered skew-adjoint Fredholm operators; this space is h. equiv. to those K such that $K^2 \equiv -1$ mod compacts. The latter is h. equiv. via $K \mapsto -\exp(\pi K)$ to the orthogonal gp $\equiv 1$ mod compacts, so has the homotopy type O. Thus $\pi_0(\text{skew adjoint Fred}) = \mathbb{Z}/2$.

Let's return to our line bundle L such that $L \otimes L = \Omega^1(\zeta)$ where $|\zeta| < 1$. The space W of holomorphic sections of L over the disk $|z| \leq 1$ consists of

$$f(z) \left(\frac{dz}{z-\zeta}\right)^{1/2}$$

with f holomorphic on the disk. We want to use W to define states on the Clifford algebra of spinors on the circle.

The quadratic form on W is

$$\frac{1}{2\pi i} \int_{|z|=1} f(z)^2 \frac{dz}{z-\zeta} = f(\zeta)^2$$

so W is not isotropic. W contains a unique maximal isotropic subspace, namely those $f(z)\left(\frac{dz}{z-\zeta}\right)^{1/2}$ with $f(\zeta) = 0$.

Let's review what we've learned about spin fields. Fix an annulus $a \leq |z| \leq b$ and a point ζ in the interior. We suppose L to be a holomorphic line bundle with $L \otimes L \simeq \Omega^1(\zeta)$ over the annulus. Now $\Omega^1(\zeta)$ has the non-vanishing section $\dfrac{dz}{z-\zeta}$ so there are two choices for L. Let's look at the one such that if $(z-\zeta)^{1/2}$ is made single valued by a cut radially outward from ζ, and , then L is spanned by $\dfrac{(dz)^{1/2}}{(z-\zeta)^{1/2}}$.
This isn't very clear.

Let's begin again. Over $\mathbb{C}$, Ω^1 is trivialized by dz. Let $(z-\zeta)^{1/2}$ be made single-valued via a cut outward from ζ. Let L be the holomorphic line bundle of $\mathbb{C}$ together with cut on,

$$L \otimes L = \Omega^1(-\zeta)$$

whose holom. sections are of the form

$$f(z)\ (z-\zeta)^{1/2}\, dz^{1/2}$$

with f holomorphic.

If L is restricted to $|z| = a$ where $a < |\zeta|$ then the sections of L over this circle can be identified with AP half densities on the circle, whereas if $a > |\zeta|$ the sections are P half densities. Let us call these spaces of sections over $|z| = a, b$ resp. Γ_a, Γ_b. Let W be the space of holomorphic sections of L over the annulus. Then

$$W \subset \Gamma_a \times \Gamma_b$$

in a natural way. We equip $\Gamma_a \times \Gamma_b$ with the quadratic form $\operatorname{res}_a(\zeta^2) - \operatorname{res}_b(\zeta^2)$ where res_a stands for $\frac{1}{2\pi i}\oint_{|z|=a}$. Given $f(z)(z-\zeta)^{1/2} dz^{1/2}$ in W, then $\left(f(z)(z-\zeta)^{1/2} dz^{1/2}\right)^2 = f(z)^2(z-\zeta)\,dz$ is a holom. diff'l on the annulus so W is isotropic for this quadratic form. It is probably maximal isotropic. Let's assume this, whence we obtain a Gaussian state on the Clifford algebra of $\Gamma_a \times \Gamma_b$.

It seems likely that this Gaussian state is equivalent to giving lines in

$$\mathcal{F}_a^* \otimes \mathcal{F}_b^{\pm}$$

Here we know that the Cliff algebra of Γ_a has a canonical irreducible repn. $\mathcal{F}_a$, whereas the Cliff. algebra of Γ_b has two irreducible repns. $\mathcal{F}_b^{\pm}$.

Thus I expect there to be operators

$$S_{\pm}(\zeta) : \mathcal{F}_a \longrightarrow \mathcal{F}_b^{\pm}$$

defined at least up to scalars. These are the spin fields.

But an important idea that has been ignored is that these spin fields are related to the analytic continuation map from Γ_a to Γ_b. W should be viewed as this analytic continuation map, and the fact that W is isotropic means that the map preserves the quadratic form, hence induces a "map" on Clifford algebras.

Practically what this means is that if $f \in W$

Then we have

$$S_\pm(J)\,\psi(f_a) = \psi(f_b)\,S_\pm(J)$$

<u>March 23, 1987</u>

Graeme has preferred to think of $\psi(z)$ as a subspace of spinors with simple pole at z. Now certainly the operator $\psi(z)$ can be identified with a dual vector on $\mathcal{F}_0 \otimes \mathcal{F}_\infty$. [redacted] So it's necessary to check that this dual [redacted] vector corresponds to a maximal isotropic subspace. But the dual vector is obtained by applying to $\langle 0|$ a Clifford operator $\psi(f_0) - \psi(f_\infty)$, so we should check that Clifford multiplication preserves the lines corresponding to maximal isotropic subspaces.

[redacted] Let $V = W \oplus W'$ ~~be~~ a decomposition into maximal isotropic subspaces, and [redacted] let $|W'\rangle$ [redacted] be [redacted] a vector in Fock space spanning the line corresponding to W'. If $v \in V$, write $v = w + w'$. Then

$$\psi(v)|W'\rangle = \psi(w)|W'\rangle$$

Now assume $w \neq 0$, then there is an obvious maximal isotropic subspace, namely

$$\mathbb{C}w \oplus \text{annihilator of } w. \text{ on } W'.$$

This max isotropic subspace clearly annihilates $\psi(w)|W'\rangle$

In the odd case we have

$$V = \mathbb{C}v_0 \oplus (W \oplus W')$$

where [redacted] v_0 is $\perp$ to $W \oplus W'$ and $v_0 \cdot v_0 = 1$. There are two Fock spaces; each is generated by a vector $|W'\rangle$ killed by $\psi(W')$ and the two possibilities are

$$\psi(v_0)|W'\rangle = \pm |W'\rangle$$

Now given $v \in V$ we write it

$$v = cv_0 + w + w'$$

and we have (taking the + case)

$$\psi(v)|W'\rangle = c|W'\rangle + \psi(w)|W'\rangle$$

This vector ~~space~~ is killed by the annihilator of w on W', that is by $(w)^{\perp} \cap W'$. This is not maximal isotropic and the question is whether an extra element can be added to it. ~~Certainly any vector with non-zero v_0 component will have non-zero inner~~ ~~We can suppose~~ Suppose W, W' are one dimensional, let w, w' span these spaces with $w \cdot w' = 1$. Then look for $\alpha, \beta, \gamma \ni$

$$\psi(\alpha v_0 + \beta w + \gamma w')\big(c|W'\rangle + \psi(w)|W'\rangle\big) = 0$$

$$\alpha c\,|W'\rangle - \alpha\psi(w)|W'\rangle + \beta c\,\psi(w)|W'\rangle + \gamma\underbrace{(w \cdot w')}_{1}|W'\rangle = 0$$

$$\alpha = \beta c \quad , \quad \alpha c + \gamma = 0.$$

So there is no problem.

Let V be a vector space (over $\mathbb{C}$) with non-degenerate quadratic form, and let $C(V)$ be its Clifford algebra. The orthogonal group $O(V)$ acts on $C(V)$ and can be identified with the auto. group of $C(V)$ preserving V.

If V is even dimensional the Clifford algebra is simple and so any automorphism is inner. Thus if we put

$$(*) \qquad G(V) = \{ g \in C(V)^* \mid gVg^{-1} = V \}$$

we have an exact sequence of groups.

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow G(V) \longrightarrow O(V) \longrightarrow 1$$

$G(V)$ is the Clifford group.

Let $v \in V$ be such that $v^2 \neq 0$. Then $\mathbb{C}v^\perp$ is a hyperplane in V complementary to $\mathbb{C}v$. One has

$$v x v^{-1} = \begin{cases} x & \text{if } x \in \mathbb{C}v \\ -x & \text{if } x \in \mathbb{C}v^\perp \end{cases}$$

so $v \in G(V)$ and the resulting orthogonal transf. is the reflection preserving $\mathbb{C}v$. This reflection has determinant -1 since $\mathbb{C}v^\perp$ has odd dimension.

If V is odd dimensional, then the orthogonal transformation -1 induces an autom of $C(V)$ which is -1 on the central element $\gamma^1 \cdots \gamma^{2m+1}$, and so it is not an inner automorphism. Thus one proceeds differently to define the Clifford group. Let $G(V)$ be defined by $(*)$. This time we have an exact sequence

$$1 \longrightarrow (\mathbb{C} \times \mathbb{C})^\times \longrightarrow G(V) \longrightarrow SO(V) \longrightarrow 1$$

Check $G(V)$ maps into $SO(V)$: First notice that $O(V) = SO(V) \times \{\pm 1\}$ and the orthogonal transf $v \longrightarrow -v$ gives the $\mathbb{Z}/2$ grading on $C(V)$. The Clifford algebra is a direct product of two matrix algebras which are the ideals generated by the two central ~~idempotents~~ obtained from $\gamma^1 \dots \gamma^{2m+1}$. Any automorphism acting trivially on the center will be inner, hence the autos. of $C(V)$ coming from elements of $SO(V)$ are all inner.

Notice that in the even case the group $G(V)$ splits into two cosets according to the determinant of the associated complex orthogonal transformation. The identity coset is the subgroup of even elements. And as the other coset contains the unit vectors of V the other coset is the odd elements. Thus

$$G(V) = \left[G(V) \cap C^{ev}(V)\right] \amalg \left[G(V) \cap C^{odd}(V)\right]$$

Also $G(V)$ appears to be generated by the unit vectors of V and the scalars. This is because $O(V)$ is generated by reflections associated to unit vectors.

In the odd case I think the Clifford group $\Gamma(V)$ can be defined similarly as the subgroup of $C(V)$ generated by unit vectors and scalars, and it will be the union of the two cosets consisting of even and odd elements.

Definition of the Clifford group from ABS. Let $\alpha = \pm 1$ on $C^{\pm}(V)$. Then

$$\Gamma(V) = \{ g \in C(V)^{\times} \mid \alpha(g) V g^{-1} = V \}$$

Then $\Gamma(V)$ is clearly a subgroup of $C(V)^{\times}$. Examples of elements of $\Gamma(V)$ are elements $v \in V$ with $v^2 \neq 0$. I want to prove now that $\Gamma(V)$ is generated by these elements. In particular

$$\Gamma(V) = \Gamma^{+}(V) \cup \Gamma^{-}(V)$$

where $\Gamma^{\pm}(V) = \Gamma(V) \cap C^{\pm}(V)$.

Let $g \in \Gamma(V)$ and write $g = g^{+} + g^{-}$. Then for any $v \in V$ we have $Tv \in V \ni$

$$(g^{+} - g^{-})\, v = (Tv)(g^{+} + g^{-})$$

so separating into homogeneous components

$$g^{+} v = (Tv)\, g^{+}$$
$$g^{-} v = -(Tv)\, g^{-}.$$

One of g^{+}, g^{-} is $\neq 0$. From

$$g^{-}(v^2) = g^{-} v v = -(Tv) g^{-} v = (Tv)^2 g^{-}$$

and similarly $g^{+} v^2 = (Tv)^2 g^{+}$ we see that T is orthogonal. Thus we have a homomorphism

$$\Gamma(V) \longrightarrow O(V)$$

by associating to g the transf. T such that

$$Tv = \alpha(g)\, v\, g^{-1}.$$

Next note that if $w^2 \neq 0$, then

$$\alpha(w)\, v\, w^{-1} = -w v w^{-1} = \begin{cases} -1 & \text{on } \mathbb{C} w \\ +1 & \text{on } \mathbb{C} w^{\perp} \end{cases}$$

is the reflection through the hyperplane $\perp$ to $\mathbb{C} w$. These generate the orthogonal group $O(V)$.

If we want to show that $\Gamma(V)$ is generated by the elements w with $w^2 \neq 0$, it is enough therefore to any g inducing $T=1$ lies in this subgroup. But

$$g^+ v = v g^+ \quad \text{for all } v \Rightarrow g^+ \in \text{center of } C(V).$$

In the even case the center reduces to the scalars; in the odd case it is $\mathbb{C} + \mathbb{C}\,\gamma^1 \cdots \gamma^{2m+1}$ and so its even part is again $\mathbb{C}$. Thus g^+ is a scalar.

Now consider $g^- v = -v g^-$. If we write g^- out in terms of the basis of monomials γ_I and take v to be one of the γ's, we see that the only possibility for g^- is a scalar times $\gamma^1 \cdots \gamma^{2m}$ in the even case. (The point is that $\gamma^j \gamma_I \gamma^j = \begin{cases} (-1)^{|I|}\gamma_I & \text{if } j \notin I \\ -(-1)^{|I|}\gamma_I & \text{if } j \in I \end{cases}$ so if

$hv = -vh$ for all v, then only $I = \phi$, $\{1, \ldots, n\}$ can appear.)

But g^- is odd so $g^- = 0$.

Thus $g = g^+$ is a scalar and one is done.

Moral: The way to think of the Clifford group $\Gamma(V)$ is as the group generated by the $v \in V$ such that $v^2 \neq 0$, in the same way that $O(V)$ is generated by reflections.

March 26, 1987

I would like to describe in general terms the structure of the arguments used in the Ising model. The Ising model is a classical statistical mechanical system; it is described by a probability measure on a space of configurations. The configurations are 2 dimensional fields which take the values ± 1 on lattice points in the plane.

Because of the nearest-neighbor type of the interaction the model can also be described in quantum-mechanical terms. (Here quantum-mechanical means something non-commutative.) In the quantum description the two axes of the lattice are viewed as space and (imaginary) time. There is a vector space spanned by the classical configurations on a row; this is the tensor product of 2-diml spin spaces for each site in the row. The observables are operators on this space. There is a transfer matrix giving (imaginary) time evolution.

The link between the 2D classical description and the 1D quantum description is analogous to the usual link discovered by Feynman between path integrals over space time (the Lagrangian approach) and operators on Hilbert space (the Hamiltonian approach.

Thus we have the Ising model described in terms of a vector space $\bigotimes_1^N \mathbb{C}^2$, and an algebra of observables $\bigotimes_1^N M_2(\mathbb{C})$ which contains the field operators $s(n)$ for each site n as well as the transfer matrix. If we pass to the infinite lattice, then we forget the vector space and work just with the algebra of observables which is

$$\bigotimes_{\mathbb{Z}} M_2(\mathbb{C}).$$

The reason we can solve the Ising model at zero magnetic field is because the algebra of observables can be identified with a Clifford algebra in such a way that the field operators and transfer matrix become elements of the Clifford group (in fact maybe even the spinor group). Actually one perhaps only has the identification over the even parts of the algebras.

The key point then in the model is the fact that it admits this fermion description. It should be possible to understand this transformation not only in the Hamiltonian picture but also in the path integral picture. Thus the partition function is a sum of 2^N terms where N is the number of lattice sites. A fermion integral in N Grassman variables also is a ~~sum~~ of 2^N terms. It should be possible to introduce signs in such a way that the Ising partition function becomes a a fermion integral, hopefully Gaussian.

~~Next~~

In order to proceed further I have to come to terms with this so-called Jordan-Wigner transformation. (Are there other examples of these, + just what does the name include?)

March 27, 1987

The Ising model. I want to ignore boundary conditions to begin with; I could probably fix this up by using boundary conditions. Configurations for the Ising model are ~~ ~~ ~~ ~~ spin assignments $\sigma = (\sigma_{mn})$, where $\sigma_{mn} = \pm 1$. The energy is

$$-E(\sigma) = J_1 \sum \sigma_{mn}\sigma_{m+1,n} + J_2 \sum \sigma_{mn}\sigma_{m,n+1}$$

Think of σ as the collection of its rows $\sigma = (\sigma^n)$, where $\sigma^n = (\sigma_{mn})$. Then

$$-\beta E(\sigma) = \sum_n f(\sigma^n) + \sum_n g(\sigma^n, \sigma^{n+1})$$

This means that the partition function has the form

$$Z = \sum_\sigma e^{-\beta E(\sigma)}$$

$$= \sum_{\dots \sigma^{n-1}, \sigma^n, \sigma^{n+1} \dots} \prod_n \left(e^{f(\sigma^n)} e^{g(\sigma^n, \sigma^{n-1})} \right)$$

Recall the formula for matrix multiplication

$$\cdots T \cdot T \cdot T \cdots = \sum \cdots T_{i_{n-1} i_n} T_{i_n i_{n+1}} T_{i_{n+1} i_{n+1}} \cdots$$

Thus the partition function is the trace, or a suitable expectation, of powers of the transfer matrix

$$T(\sigma, \sigma') = e^{f(\sigma)} e^{g(\sigma, \sigma')}$$

This operates on the vector space having as basis the set of row configurations $\sigma = (\sigma_m)$. It is a product of two matrices one representing row couplings and the other column couplings.

What is the space on which the transfer matrix acts? It has as basis the set of row spin assignments (σ_m) so is naturally $\bigotimes_m \mathbb{C}^2$. Look at the parts of the transfer matrix on this space. As $g(\sigma,\sigma') = \beta J_2 \sum_m \sigma_m \sigma'_m$ the matrix $e^{g(\sigma,\sigma')}$ is the $\otimes$ product over m of the operator which in the mth $\mathbb{C}^2$ is

$$\begin{pmatrix} e^{\beta J_2} & e^{-\beta J_2} \\ e^{-\beta J_2} & e^{\beta J_2} \end{pmatrix} = e^{\beta J_2} \exp\begin{pmatrix} 0 & -K \\ -K & 0 \end{pmatrix}$$

for a suitable constant $K = K(\beta J_2)$. Thus

$$e^{g(\sigma,\sigma')} = (\text{const})\; e^{-K \sum_m \sigma_x(m)}$$

where $\sigma_x(m) = \cdots \otimes 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1 \otimes \cdots$

Now the other factor of T, namely $e^{f(\sigma)}$, is diagonal in the natural basis, that is it multiplication by a function on the rows sites. There is the function $\sigma \mapsto \sigma_m$ giving the spin at site m and this gives the operator

$$\sigma_z(m) = \cdots \otimes 1 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 \otimes \cdots$$

Then we clearly have

$$e^{f(\sigma)} = e^{\sum \beta J_1 \sigma_z(m) \sigma_z(m+1)}$$

Thus we see the transfer matrix is built up out of the operators $\sigma_x(m)$ which mutually commute, and the $\sigma_z(m)\sigma_z(m+1)$ which also mutually commute (recall they are diagonal).

Now the strategy for solving the Ising model is to construct fermions, better to construct a vector space V of operators such that squaring any element of V gives a scalar. One wants the operators $\sigma_x(m)$, $\sigma_z(m)\sigma_z(m+1)$ to be quadratic in the elements of V.

March 28, 1987:

Fermionization in the Ising model. We saw above that the Ising model can be viewed as a 1 diml quantum spin system. Assume we work on the lattice where the sites are $m = 1, \dots, N$. Later we will discuss boundary conditions. Then on the tensor product of the 2D spin spaces at each site

$$S^{\otimes N} = S \otimes \cdots \otimes S$$

we have the operators

$$\sigma^x(m) = 1 \otimes \cdots \otimes \underset{m}{\sigma^x} \otimes \cdots \otimes 1$$

and similarly $\sigma^z(m)$. The operators on $S^{\otimes N}$ form the algebra

$$\mathrm{End}\left(S^{\otimes N}\right) = \bigotimes_1^m \mathrm{End}(S)$$

If we use a $\mathbb{Z}_2$ grading on S, then we also have

$$\mathrm{End}\left(S^{\otimes N}\right) = \mathrm{End}(S) \,\hat{\otimes} \cdots \hat{\otimes}\, \mathrm{End}(S)$$

and the latter is isomorphic to $C_2 \hat{\otimes} \cdots \hat{\otimes} C_2 = C_{2N}$. This is where the fermions come from.

We want to arrange things so that the transfer matrix ends up in the spinor group. We saw that the transfer matrix is the product of two operators which are exponentials of the operators

$$\sum_1^N \sigma^x(m) \qquad \sum_1^N \sigma^z(m)\,\sigma^z(m+1)$$

where $\sigma^z(m+1) = \sigma^z(1)$. We want these operators to be in the Lie algebra $so(2N)$, and so they must be even. Thus we use σ^x to grade S which means we use the representation

$$\sigma^x = \varepsilon = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \qquad \sigma^z = \gamma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

The raising and lowering operators are

$$\sigma^\pm = \left(\frac{1\pm\varepsilon}{2}\right)\gamma^1 = \begin{cases} \begin{pmatrix}0&1\\0&0\end{pmatrix} & + \\ \begin{pmatrix}0&0\\1&0\end{pmatrix} & - \end{cases}$$

Now that the grading has been introduced we define creation and annihilation operators

$$c^\pm(m) = 1\hat\otimes\cdots\hat\otimes \underset{m}{\sigma^\pm}\hat\otimes\cdots\hat\otimes 1$$

$$= \varepsilon\otimes\ \otimes\varepsilon\otimes\sigma^\pm\otimes\cdots\otimes 1.$$

~~This now allows us to represent the Clifford algebra C_{2N} on the Hilbert space of our quantum system.~~

What we are doing is to define a $2N$ subspace V of operators on $S^{\otimes N}$ such that σ^2 is a scalar form any $\sigma \in V$ and $C(V) \xrightarrow{\sim} \mathrm{End}(S^{\otimes N})$. We want this representation to be such that the operators $\sum \sigma^x(m)$, $\sum \sigma^z(m)\sigma^z(m+1)$ are in $so(2N)$. Now

$$\sigma^x(m) = 1\otimes\cdots\otimes\varepsilon\otimes 1\otimes\cdots = 1\hat\otimes 1\hat\otimes\cdots\hat\otimes\varepsilon\hat\otimes 1\hat\otimes\cdots$$

and $\varepsilon = [\sigma^+, \sigma^-]$, so that

$$\sigma^x(m) = [c^+(m), c^-(m)]$$

lies in $so(2N)$.

Next

$$\sigma^z(m) = 1^{\otimes(m-1)} \otimes (\sigma^+ + \sigma^-) \otimes 1 \otimes \cdots$$

$$\sigma^z(m)\,\sigma^z(m+1) = 1^{\otimes(m-1)} \otimes (\sigma^+ + \sigma^-) \otimes (\sigma^+ + \sigma^-) \otimes 1 \otimes \cdots$$

$$= 1^{\otimes(m-1)} \otimes \left[\sigma^+ \otimes \sigma^+ + \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ + \sigma^- \otimes \sigma^-\right] \otimes 1 \otimes \cdots$$

Let's simplify by taking $m = 1, N=2$ first.

$$c^+(1)\, c^+(2) = (\sigma^+ \hat{\otimes} 1)(1 \hat{\otimes} \sigma^+) = \sigma^+ \varepsilon \otimes \sigma^+$$

$$= -\sigma^+ \otimes \sigma^+$$

In general for $1 \leq m < N$

$$c^a(m)\, c^b(m+1) = \overbrace{1 \hat{\otimes} \cdots \hat{\otimes} 1}^{m-1} \hat{\otimes} \left[\sigma^a \hat{\otimes} \sigma^b\right] \hat{\otimes} 1 \hat{\otimes} \cdots$$

$$= \overbrace{1 \otimes \cdots \otimes 1}^{m-1} \otimes \underbrace{(\sigma^a \varepsilon \otimes \sigma^b)}_{(-a)\sigma^a \otimes \sigma^b} \otimes 1 \otimes \cdots$$

$$= (-a)\, \sigma^a(m)\, \sigma^b(m+1)$$

where $a, b = \pm 1$ and $\sigma^a(m) = 1 \otimes \cdots \otimes \underset{m\text{th}}{\sigma^a} \otimes \cdots 1$. Thus

$$\sigma^z(m)\,\sigma^z(m+1) = \begin{pmatrix} \sigma^+(m)\,\sigma^+(m+1) \\ \sigma^+(m)\,\sigma^-(m+1) \\ \sigma^-(m)\,\sigma^+(m+1) \\ \sigma^-(m)\,\sigma^-(m+1) \end{pmatrix} = \begin{pmatrix} -\,c^+(m)\,c^+(m+1) \\ -\,c^+(m)\,c^-(m+1) \\ c^-(m)\,c^+(m+1) \\ +\,c^-(m)\,c^-(m+1) \end{pmatrix}$$

$$= -\left(c^+(m) - c^-(m)\right)\left(c^+(m+1) + c^-(m+1)\right)$$

So far this works for $1 \leq m < N$. We next look at $m = N$.

It seems preferable to avoid the tensor notation and to simply define the operators $\varepsilon(m) = \sigma^x(m)$ and $\sigma^z(m)$ directly. Then the creation and annihilator operators are

$$c^{\pm}(m) = \varepsilon(1)\cdots\cdots\varepsilon(m-1)\,\sigma^{\pm}(m)$$

Then

$$\sigma^z(m)\,\sigma^z(m+1) = \varepsilon(1)\cdots\varepsilon(m-1)\big(c^+(m)+c^-(m)\big)$$
$$\times\ \varepsilon(1)\cdots\varepsilon(m)\big(c^+(m+1)+c^-(m+1)\big)$$
$$= \big(-c^+(m)+c^-(m)\big)\big(c^+(m+1)+c^-(m+1)\big)$$

Now let us look at the case $m=N$.

$$\sigma^z(N)\,\sigma^z(1) = \varepsilon(1)\cdots\varepsilon(N-1)\big(c^+(N)+c^-(N)\big)\big(c^+(1)+c^-(1)\big)$$
$$= \varepsilon(1)\cdots\varepsilon(N)\big(c^+(N)-c^-(N)\big)\big(c^+(1)+c^-(1)\big).$$

Notice that $\varepsilon(1)\cdots\varepsilon(N)$ gives the total grading mod 2 and it commutes with with the $\varepsilon(m)$ and anti-commutes with the $\sigma^{\pm}(m)$.

If we restrict to $(S\otimes\cdots\otimes S)^-$, then we see that $\sigma^z(N)\sigma^z(1) = -\big(c^+(N)-c^-(N)\big)\big(c^+(N+1)+c^-(N+1)\big)$ provided $c^{\pm}(N+1) = c^{\pm}(1)$. Thus on $(S\otimes\cdots\otimes S)^-$ the operator $\sum \sigma^z(m)\,\sigma^z(m+1)$ coincide with the operator $-\sum_m \big(c^+(m)-c^-(m)\big)\big(c^+(m+1)+c^-(m-1)\big)$ where $c^{\pm}(N+1) = c^{\pm}(1)$. Thus the transfer matrix will be a product of operators e^{α}, where $\alpha \in SO(2N)$ is cyclically invariant. Thus think of α as an skew-symmetric transformation of $\mathbb{C}^n \oplus (\mathbb{C}^n)^*$, and it commutes with the cyclic transformation.

On the other hand if we restrict to $(S\otimes\cdots\otimes S)^+$ then $\sigma^z(N)\sigma^z(1) = -\big(c^+(N)-c^-(N)\big)\big(c^+(N+1)+c^-(N+1)\big)$ provided we define $c^{\pm}(N+1) = -c^{\pm}(1)$. Thus the transfer matrix on $(S\otimes\cdots\otimes S)^+$ will be a product of e^α where is invariant under a different cyclic automorphism

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Here is a way this might be understood. Let S be two-diml spinor space graded by ε and let us form the N-fold tensor product

$$S^{\otimes N} = S \otimes \cdots\cdots \otimes S.$$

Now there are two actions of Σ_N on this tensor product depending on whether we view the tensor product in the category of vector spaces or in the category of super vector spaces.

Let us consider cyclic permutation (foward shift) in the super sense. Then on $(S^{\otimes N})^-$ this coincides with the usual cyclic permutation.

$$C_{(s)}(v_1 \otimes \cdots \otimes v_N) = (-1)^{(\deg v_N)(\deg v_1 + \cdots + \deg v_{N-1})} \times$$
$$v_N \otimes v_1 \otimes \cdots \otimes v_{N-1}$$

and if the total degree $\sum_1^N \deg v_i$ is odd, then one of the factors $(\deg v_N)$ or $\sum_1^{N-1} \deg v_i$ is even. (Recall these degrees are in $\mathbb{Z}/2\mathbb{Z}$)

On the even space $(S^{\otimes N})^+$ then the super cyclic permutation $C_{(s)}$ has no simple relation to the cyclic C.

~~However if we consider $C_{(s)}$ then~~

Now let's go back to operators on $S^{\otimes N}$. Using the super formalism, we can identify

$$\mathrm{End}(S^{\otimes N}) = \mathrm{End}(S) \hat{\otimes} \cdots \hat{\otimes} \mathrm{End}(S).$$

In the latter we can consider

$$V = \sum_{m=1}^{N} 1 \hat{\otimes} \cdots \hat{\otimes} \underset{m}{\mathrm{End}^-(S)} \hat{\otimes} 1 \hat{\otimes} \cdots \hat{\otimes} 1$$

which is the space of creation and annihalation operators. This defines the V intrinsically and shows it is invariant under the super cyclic permutation conjugation.

To handle $(S^{\otimes N})^+$, let us consider the automorphism of $S^{\otimes N}$ which is the super cyclic permutation following by $\varepsilon(1)$.

$$v_1 \otimes \cdots \otimes v_N \longmapsto (-1)^{|v_N| \sum_1^{N-1} |v_i|} v_N \otimes v_1 \otimes \cdots \otimes v_{N-1}$$

$$\longmapsto (-1)^{\left(|v_N| \cdot \sum_1^{N-1} |v_i|\right) + |v_N|} v_N \otimes v_1 \otimes \cdots \otimes v_{N-1}$$

But $|v_N| \cdot \sum_1^{N-1} |v_i| + |v_N| = |v_N| \sum_1^{N} |v_i| \equiv 0$ in $\mathbb{Z}/2\mathbb{Z}$ when $v_1 \otimes \cdots \otimes v_N \in (S^{\otimes N})^+$. Thus

$$\varepsilon(1)\, C_{(s)} = C \quad \text{on} \quad (S^{\otimes N})^+$$

$$C_{(s)} = C \quad \text{on} \quad (S^{\otimes N})^-$$

which explains to [redacted] some extent the trick above of how to define $c^{\pm}(N+1)$.

[redacted]

March 29, 1987

Review: Regarding the Ising model as a quantum system over a 1-dim'l lattice we obtain a transfer matrix acting on the vector space $H = S^{\otimes N}$. Working in the supercategory the operators on H have a Clifford algebra structure with the generating space

$$V = \sum 1 \hat{\otimes} \cdots \hat{\otimes} \operatorname{End}^-(S) \hat{\otimes} \cdots \hat{\otimes} 1 \quad \subset \quad \hat{\bigotimes}_1^N \operatorname{End}(S).$$

The transfer matrix doesn't belong to the spinor group. However if we restrict to H^+, or to H^-, then the restriction is given by an element of the spinor group.

To be more precise the transfer matrix is a product of Gaussians with exponents $\varepsilon(m)$, $\sigma^z(m)\sigma^z(m+1)$, for $m = 1, \ldots, N$ where $\sigma^z(N+1) = \sigma^z(1)$. All of these are quadratic in the creation and annihilation operators except

$$\sigma^z(N)\,\sigma^z(1) = \varepsilon(1)\cdots\varepsilon(N)\underbrace{\left[\varepsilon(N)\left(c^+(N)+c^-(N)\right)\left(c^+(N)+c^-(N)\right)\right]}_{\text{quadratic, call this } g.}$$

Thus the transfer matrix is an element of the spinor group $\times$ times e^{+g} on H^+, and times e^{-g} on H^-.

Another way of saying this is that we have identified the state space of the Ising model with the sum of the two irreducible spinor representations of $\mathrm{Spin}(V)$. The transfer matrix on $H^\pm$ is given by different elements of the spinor group.

So far we haven't gotten to the heart of the periodicity versus anti-periodicity.

Let's now consider the scaling limit for the Ising model. The idea will be to subdivide the lattice d times. Recall that the transfer matrix from one row to the next is a product

$$T = e^{\overbrace{\beta J_1}^{L} \sum \sigma^z(m)\,\sigma^z(m+1)} \; e^{+K \sum \sigma^x(m)}$$

up to the constant factor δ^N where

$$\begin{pmatrix} e^{\beta J_2} & e^{-\beta J_2} \\ e^{-\beta J_2} & e^{\beta J_2} \end{pmatrix} = \left(e^{2\beta J_2} - e^{-2\beta J_2}\right)^{1/2} \underbrace{\exp\left[+K\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]}_{\begin{pmatrix} \cosh K & +\sinh K \\ +\sinh K & \cosh K \end{pmatrix}}$$

so that $+\tanh(K) = e^{-2\beta J_2}$.

When we subdivide d times, then we obtain a transfer matrix

$$T_d = e^{L(d) \sum \sigma^z(m)\,\sigma^z(m+1/d)} \; e^{+K(d) \sum \sigma^x(m)}$$

and the tranfer matrix in time t is

$$\left(T_d\right)^{td} \longrightarrow \left(\lim_d \left(T_d\right)^d\right)^t$$

Now recall that if $\alpha(d) \sim \frac{a}{d}$ $\quad \beta(d) \sim \frac{b}{d}$ then

$$\left(e^{\alpha(d)} e^{\beta(d)}\right)^d \longrightarrow e^{a+b}.$$

The transfer matrix in the limit as $d \to \infty$ in time t is $e^{-t\Gamma}$ where

$$\Gamma = \lim \left(-d L(d) \sum \sigma^z(m)\,\sigma^z(m+1/d) - d K(d) \sum \sigma^x(m)\right)$$

Actually the above is heuristic as the operators $\alpha(d)$, $\beta(d)$ operate on different spaces. Perhaps this is where the renormalization group enters.

Let's now describe Γ using fermions.

Recall that

$$-\sigma^z(m)\,\sigma^z(m+\overset{1/d}{\overset{\parallel}{a}}) = \big(c^+(m)-c^-(m)\big)\big(c^+(m+a)+c^-(m+a)\big)$$

$$= \big(c^+(m)-c^-(m)\big)\,\Delta(c^++c^-)(m) + \underbrace{\underbrace{\big(c^+(m)-c^-(m)\big)\big(c^+(m)+c^-(m)\big)}_{c^+(m)\,c^-(m)-c^-(m)\,c^+(m)}}_{\sigma^x(m)}$$

~~Thus we have that Γ is the limit of~~

$$+\,a\,d\,L(d)\sum\big(c^+(m)-c^-(m)\big)\,\frac{\Delta(c^++c^-)(m)}{a} + d\big(K(d)+L(d)\big)\cdot\sum(c^+-c^-)(c^++c^-)(m)$$

~~Somewhere I have gotten the wrong sign.~~ ~~I want $L(d) = O(a)$, which is OKAY if $J_1(d) \sim J_1 a$. Similarly we want~~

$$d\big(K(d)+L(d)\big) = O(a)$$

$$-K(d)+L(d) = O(a^2)$$

~~which means that K, L should have opposite signs.~~

Thus Γ is the limit of

$$d\,L(d)\,a\sum(c^+-c^-)(m)\,\frac{\Delta(c^++c^-)(m)}{a} + d\big(L(d)-K(d)\big)\sum(c^+-c^-)(c^++c^-)(m)$$

Now this will approach

$$\lambda \int (c^+ - c^-)\, \partial_x (c^+ + c^-)\, dx + \mu \int (c^+ - c^-)(c^+ + c^-)\, dx$$

provided $\quad dL(d) \longrightarrow \lambda \quad$ or $\quad L(d) \sim \lambda a$

$\qquad\qquad d^2(L(d) - K(d)) \longrightarrow \mu \qquad L(d) - K(d) \sim \mu a^2$

We have $\quad L = \beta J_1 \quad$ and $\quad \tanh(K) = e^{-2\beta J_2} \quad$?

I have not taken the appropriate continuum limit of the fermion field. Thus we have

$$\circledast \quad \left\{ \sum f(m)\, c^+(m),\ \sum g(m)\, c^-(m) \right\} = \sum f(m)\, g(m)$$

and we want this to have the limit $\quad \int f(x)\, g(x)\, dx$

Thus we rewrite the above $\circledast$ in the form

$$\left\{ a \sum f(m) \frac{c^+(m)}{a^{1/2}},\ a \sum g(m) \frac{c^-(m)}{a^{1/2}} \right\} = a \sum f(m)\, g(m).$$

From this we conclude that $\psi^{\pm}(x) = \dfrac{c^{\pm}(x)}{a^{1/2}}$, which is ~~consistent~~ with $c^{\pm}(x) = \psi^{\pm}(x)\, dx^{1/2}$.

Thus Γ is the limit of

$$a\, dL(d)\ a \sum{}' \frac{c^+ - c^-}{\sqrt{a}}(m)\, \frac{\Delta(c^+ + c^-)}{a\sqrt{a}}(m) + d(L(d) - K(d))\, a \sum{}' \frac{(c^+ - c^-)}{\sqrt{a}} \frac{(c^+ + c^-)}{\sqrt{a}}(m)$$

which will be

$$\lambda \int (\psi^+ - \psi^-)\, \partial_x (\psi^+ + \psi^-)\, dx + \mu \int (\psi^+ - \psi^-)(\psi^+ + \psi^-)\, dx$$

provided $\quad L(d) \longrightarrow \lambda \quad$ and $\quad d(L(d) - K(d)) \longrightarrow \mu$

Recall that $\quad L = \beta J_1 \quad$ and $\quad \tanh(K) = e^{-2\beta J_2}$

Look at the second equation more carefully. Suppose we have $\tanh(a) = \dfrac{e^{a}-e^{-a}}{e^{a}+e^{-a}} = e^{-2b}$

or
$$e^{-2b} = \frac{1-e^{-2a}}{1+e^{-2a}} = -1 + \frac{2}{1+e^{-2a}}$$

or $\quad (1+e^{-2a})(1+e^{-2b}) = 2$

It would seem that the critical temperature is the condition $L = K$ i.e.
$$\tanh(\beta J_1) = e^{-2\beta J_2}$$

or
$$\boxed{\left(1+e^{-2\beta J_1}\right)\left(1+e^{-2\beta J_2}\right) = 2}$$

Actually $\quad \dfrac{e^{a}-e^{-a}}{e^{a}+e^{-a}} = e^{-2b}$

$$\Rightarrow e^{2b} = \frac{e^{a}+e^{-a}}{e^{a}-e^{-a}} = \frac{e^{2a}+1}{e^{2a}-1} = 1 + \frac{2}{e^{2a}-1}$$

so that
$$\left(e^{2a}-1\right)\left(e^{2b}-1\right) = 2$$

so the critical condition can be written also as
$$\boxed{\left(e^{2\beta J_1}-1\right)\left(e^{2\beta J_2}-1\right) = 2}$$