Conformal field theory on a Riemann surface
principal parts + operator product expansion 557
Renormal case 566 Spin fields 570
Igusa model 578

tetrad + Torsion operators 577

Diracs on the circle + Dirac coupled to superconnections.
Dirac op. on torus 610, Kasparov Hilbert modules 616, calculation of a cup product of Dirac on circle with a loop (see 628 ).

Cup product in K-theory + non-smoothness 635
Notes for Roe on Kasparov cup product example 641
March 9, 1987

I want to review various symplectic facts. Let $E$ be a real symplectic vector space of dimension $2n$. Attached to $E$ is a Siegel UHP whose points can be conveniently described as subspaces $V \subset E_c$ of dimension $n$, such that $[v, v^*] = 0$ for $v, v^* \in V$ and $[v, v^*] > 0$ with equality iff $v = 0$. In other words, $E_c = V \oplus \overline{V}$ is a splitting into annihilation and creation operators.

This description is convenient because it exhibits a natural compactification of the Siegel UHP, namely those maximal isotropic subspaces $V$ of $E_c$ such that $[v, v^*] > 0$. If $V$ is real then we have a Lagrangian subspace of $E$. So there is a kind of building type structure to the boundary. Let us fix a basepoint $V_0$ of the Siegel UHP and choose a basis $a_k$ for $V_0$ and $a_k^*$ for $V_0^*$.

I think I want $V_0$ to consist of the creation ops, so let's change notation.

Then any $V \subset E_c = V_0 \oplus \overline{V}_0$ is the graph of a map from $V_0$ to $V_0^*$, call it $Z$. $Z$ is symmetric because $V$ is isotropic and $Z^* Z < 1$. Thus one gets the model of the Siegel UHP consisting of complex symmetric matrices $Z$ with $Z^* Z < 1$.

The boundary includes $Z$ with $Z^* Z = 1$, and the Lagrangian Grassmannian is the space of symmetric unitary matrices. (Actually to get this description as matrices one has effectively chosen an isom. $V_0 = V_0^*$, i.e.
a real structure \( \varepsilon \) on \( V_0 \), which
is what you used before to identify the
Lagrangian Grassmannian with symmetric
unitary matrices.

Now there is a simpler version of this
whole business where one supposes fixed from
the beginning a complex structure on \( E \); denote
it \( \iota \) which is compatible with the symplectic
structure in the sense that \( e^{it} \) is a 1-parameter
automorphism group. Then one can look at the
fixed part of the Siegel UHP. The circle action
on \( E_c \) has two eigenspaces \( E^+_c \oplus E^-_c \) for the
circle action. Clearly conjugation reverses these
and both are \( \mathbb{R} \)-isotropic for \( [\ , \] \). Thus
both are maximal isotropic and in duality. Also
we have a \( \mathbb{K} \)-hermitian form on \( E^+_c \) given by
\[
[u^*, v^*].
\]

What does a point \( V \) in the Siegel UHP fixed
by the circle action look like? It is a direct
sum \( V = V^+ \oplus V^- \) where \( V^\pm \subset E^\pm_c \), and to
be maximal isotropic we must have \( V^- = (V^+)\) for
the \( [\ , \] \) pairing. Next the there is a positivity
condition
\[
[(v^+)^* + (v^-)^*, v^+ + v^-] > 0
\]

\[
[(v^+)^*, v^+] + [(v^-)^*, v^-] = 0.
\]
This means that the hermitian form on \( V^+ \) is positive and also for \( V^- \). This bounds the dimensions of \( V^+, V^- \) and because the sum of their dimensions is \( n \), this probably means that \( V^+ \) is a maximal subspace of \( \mathbb{C}^n \) in which the hermitian form is positive.

Let's begin again with a complex structure on a real symplectic vector space \( E \), where we suppose the 1-parameter group preserves the symplectic structure.

Let's start with a complex vector space \( E \) and suppose given in the underlying real vector space a skew-symmetric nondegenerate form \( \Omega (\dot{x}, \eta) \). We suppose

\[
\Omega (i \dot{x}, \eta) + \Omega (\dot{x}, i \eta) = 0
\]

which implies that \( e^{it} \) is a 1-parameter group of symplectic automorphisms. This implies

\[
\Omega (\dot{x}, i \eta) = -\Omega (i \dot{x}, \eta) = \Omega (\eta, i \dot{x})
\]

so that \( \Omega (\dot{x}, i \eta) = \Omega (\eta, \dot{x}) \) is a symmetric bilinear form in the underlying real vector space. It is clearly non-degenerate.

Next we have the circle which is a compact Lie group acting symplectically on \( E \), also orthogonally for the form \( B \). Enlarge the circle to a maximal compact. This means we find a positive inner product on the underlying real
vector space $E$ such that $\Omega$ is given as $(\xi, J\eta)$ with $J^2 = -1$, and $i$ is orthogonal relative to $(\cdot, \cdot)$. Then $i$ and $J$ commute, so that $J = i^2$ where $i$ is an involution.

Thus we get a complex Hilbert space $E$ together with a grading $E$ and the symplectic form is

$$\Omega(\xi, \eta) = (\xi, J\eta) = (\xi, i\eta) = \text{Re} \langle \xi, i\eta \rangle$$

$$= -\text{Im} \langle \xi, i\eta \rangle$$

Now what are the points of the Siegel UHP invariant under the circle action? But before I do this I probably want to discuss the basic representation where elements of $E$ are hermitian operators and $\Omega$ is $\pm i [\cdot, \cdot]$. I guess I need to introduce two sets $a^*_i, a_i, b^*_j, b_j$ of creation and annihilation operators. The circle action assigns to $a^*_i, b_j$ the weight $1$ and their adjoints the weight $-1$.

For earlier work see 198-199, Feb 11, 86, and 375-376, Jan 23, 1987.
March 10, 1987

Problem: Conformal field theory on a Riemann surface.

It seems to be difficult to get started by considering the Fock spaces attached to the different circles on the surface. It seems desirable to figure out what Witten does with meromorphic sections.

However there is one interesting idea worth recording. Consider the single fermion field on $S^1$ with anti-periodic boundary conditions. In the Fock space we have operators $\psi(f(x) dx^{1/2})$ for any $L^2$ half-density. Now put the circle in the cylinder. More accurately, identify $S^1$ with $|z|=1$ inside $\mathbb{C}^\times$. The complex structure on the cylinder determines a kind of imaginary time-evolution. To be more specific the operator $\psi(f(x) dx^{1/2})$ on $S^1$ will extend to $\psi(f(z) dz^{1/2})$ which has a meaning for lots of circles, at least for analytic $f$.

So a holomorphic half-density is consistent with the imaginary time-evolution in some sense.

Now let us concentrate on Greens functions for the fermi field. There are example given by the cylinder and the torus.
March 11, 1987

Let's take up the real fermi field on the torus. This we can handle via temperature formalism. I want to calculate the Greens function, the normal ordering belonging to the temperature state, etc.

Recall the formulas for the real fermi field:

\[ \psi (f dx^{1/2})^* = \psi (f dx^{1/2}) \]
\[ \psi (f dx^{1/2})^2 = \frac{1}{2} \int f^2(x) \frac{dx}{2\pi} \]
\[ \psi (f dx^{1/2}) = \sum c_k \psi_k \quad \text{if} \quad f = \sum c_k e^{ikx} \]

\[ = \int \frac{dx}{2\pi} f(x) \sum e^{-ikx} \psi_k \]
\[ = \int \frac{dx}{2\pi} (f(x) dx^{1/2}) \left( \sum e^{-ikx} \psi_k dx^{1/2} \right) \]

\[ \psi_k^* = \psi(e^{ikx} dx^{1/2}) \]
\[ \psi_k^* = \psi_{-k} \]
\[ \{ \psi_k, \psi_l \} = \delta_{k,-l} \]

The natural Hamiltonian is \( H = L_0 \) whose effect on the clifford algebra is

\[ \left[ H, \psi(f dx^{1/2}) \right] = \psi(f dx^{1/2}) + \psi(f dx^{1/2}) \]
\[ \left[ H, \psi_k \right] = k \psi_k \]

We then set

\[ \psi(x,t) = e^{tH} \psi(x) e^{-tH} \]
\[ = \sum_k e^{-ikx+kt} \psi_k \]
Also this Hamiltonian leads to a ground state $|0\rangle$ satisfying
\[ \psi_k |0\rangle = 0 \quad k < 0 \]

at least in the anti-periodic (NS) case. Thus $\psi_k$ creates for $k > 0$, destroys for $k < 0$.

Now let's find the Green's functions in the zero temperature case.

\[ G(x, x') = \langle [\psi(x), \psi(x')] \rangle \]

where $\langle A \rangle = \langle 0 | A | 0 \rangle$. Thus for $t > t'$

\[ G(x, x') = \sum_{k, \ell} e^{-ikx + \ell t} e^{-i\ell x' + \ell t'} \langle 0 | \psi_k \psi_{-\ell} | 0 \rangle \]

\[ = \sum_{k < 0} e^{-ikAx + kAt} \langle 0 | \psi_k \psi_{-\ell} | 0 \rangle \]

\[ = \sum_{k < 0} e^{k(Dt - iAx)} = \frac{e^{-\frac{1}{2} (Dt - iAx)}}{1 - e^{iAx}} \]

Similarly for $t < t'$

\[ G(x, x') = \sum_{k, \ell} e^{-ikx + \ell t} e^{-i\ell x' + \ell t'} \langle 0 | -\psi_{-\ell} \psi_k | 0 \rangle \]

\[ = \sum_{k > 0} e^{-ikAx + kAt} \langle 0 | -\psi_{-\ell} \psi_k | 0 \rangle \]

\[ = \sum_{k > 0} e^{-ikAx + kAt} = -\frac{e^{\frac{1}{2} (Dt - iAx)}}{1 - e^{iAx}} \]

and so

\[ G(x, x') = \frac{1}{e^{\frac{1}{2} (Dt - iAx)} - e^{-\frac{1}{2} (Dt - iAx)}} \]

To put this in a more appealing form we want to introduce complex variables.
There is a principle to be found which roughly links the imaginary time evolution to the holomorphic structure. Thus

\[(\partial_t + i\partial_x) \psi(x,t) = 0\]

i.e. \(\psi(x,t)\) is holomorphic in the complex variable \(t - ix\). Now let's put

\[z = e^{t-ix}\]

so that increasing \(t\) corresponds to increasing \(|z|\).

Let's now try to figure out what to do with the \(dx^{1/2}\). The principle alluded to that the time evolution of the field \(\psi\) that is the equation of motion it satisfies, is the same condition as requiring \(\psi\) to be holomorphic. Now in the circle we know we are dealing with half densities, sections of a square root of \(T^*\). And \(\psi(x)dx^{1/2}\) is an operator valued half density on the circle, it should make sense to analytically continue this half density from the circle to the cylinder, or at least an annulus containing the circle.

Let's write this in terms of \(w = t - ix\). Restricting to \(t = 0\) we have \(idw \mapsto dx\) so \(\psi(w)(idw)^{1/2}\) comes from

\[\psi(w)(idw)^{1/2} = \sum_k (e^w)^k \psi_k (idw)^{1/2}\]

Here \((idw)^{1/2}\) denotes what? Over the circle \(t = 0\), \(x \in R/2\pi Z\) we choose a square root \(L\)
of the canonical bundle and fixed the AP or NS case which is the simplest. Then \( dx^{1/2} \) is a section of the bundle pulled back to the double covering of the circle whose square is \( dx \). Thus in \( f(x)dx^{1/2} \), \( f(x) \) is also defined over the double covering.

Now if we start with a Riemann surface then we suppose \( L \) given a square root of the canonical line bundle \( K \) over the surface. So on the cylinder \( \mathbb{C}/2\pi i \mathbb{Z} \) we have the section \( dw \) of \( K \) and hence \( (dw)^{1/2} \) is a section of \( L \) pulled back to the double covering.

Now go one stage further and put \( z = e^w \), whence \( f(x)dx^{1/2} \) comes from

\[
\sum_k z^k \psi_k \left( i \frac{dz}{z} \right)^{1/2}
\]

Let's try to describe the picture that seems to be emerging. Given a circle in the surface one has an operator-valued half-density, or maybe you should say an operator-valued spinor (Weyl spinor) defined on the circle. This then is to be extended holomorphically to a mod of the circle and possibly to the whole Riemann surface.

These we seek an operator-valued spinor field (section of \( L \)) over the Riemann surface which is holomorphic.

The problem with this statement is that one
doesn't know how to interpret the
words "operator-valued". Certainly for
any circle on the surface we want this
field, which I might write as \( \psi(z) dz^{1/2} \), to
become the operator-valued (distributional) half-
density on the Fock space associated to the
circle.

Next let's go back to the Green's function
\( G(x,t, x,t') \) which we know has something to
do with the inverse of the \( \bar{\nabla} \)-operator. I
want to discuss this from an invariant
viewpoint.

Thus let \( L \) be a square root of the
canonical line bundle \( K \) over \( M \). Recall \( K = T^{0,1}_0 \).
Following Atiyah-Bott set \( L' = L^* \otimes T^{0,1} \). Then
sections of \( L' \) pair naturally with sections of \( L \).
The definition of \( L' \) makes sense for any vector
bundle.

In general if \( E \) is a holm. v.b., its \( \bar{nabla} \)
operator is a map

\[
\Gamma(E) \xrightarrow{\bar{nabla}} \Gamma(E \otimes T^{0,1})
\]

and its transpose is the map

\[
\Gamma(E') \xleftarrow{\bar{nabla}^t} \Gamma(E^* \otimes T^{0,1})
\]

which is the \( \bar{nabla} \)-operator on \( E^* \otimes K \). This is
the origin of Serre duality.
If $L \otimes L = K$, then $L = L^* \otimes K$ and $L' = L^* \otimes T^{0,1} = L \otimes T^{0,1}$. Thus the Dirac operator of $L$ is a map

$$\Gamma(L) \xrightarrow{\delta} \Gamma(L')$$

and it is skew-symmetric, i.e., $\delta^* = -\delta$.

Now suppose further that $L$ has no holomorphic sections so that $\overline{\delta}_L$ is an isomorphism. The Green's function gives the inverse map. We can use the Schwartz kernel theorem to write $G$ as a linear combination of rank one operators obtained by pairing with an element $\beta \in \Gamma(L)$ and then multiplying by $\alpha \in \Gamma(L)$:

$$\xi \mapsto \alpha(\beta, \xi).$$

Thus it is clear that $G$ is some sort of section of $L \otimes L$ over $M \times M$. It has to be holomorphic off the diagonal, and to have a simple pole along the diagonal. It is skew-symmetric.

It seems that it is meaningful to speak of the residue of a meromorphic section of $L \otimes L$ along the diagonal. The point is that the normal bundle to the diagonal is the tangent bundle to $M$ and this is dual to $L \otimes L |_{\Delta M} = L \otimes L = K$. Thus $G$ is probably specified by requiring it to be such a meromorphic section of $L \otimes L$ with residue = some constant along the diagonal.
One of the problems with viewing the fermion field $\psi$ as a holomorphic section of $\mathcal{O}_{\mathbb{P}^1} \otimes L$ is that if one applies a linear field to $\mathcal{O}_{\mathbb{P}^1}$ then one would obtain a holomorphic section of $L$ which would be zero!

So we have to find another method, perhaps something adelic.

Let us next consider the operator products and normal ordering.
March 12, 1987

Let \( L \otimes L = K = \Omega^{1,0} \) be a square root of the canonical line bundle. Then we have a duality pairing \( H^0(L) \otimes H^1(L) \rightarrow H^0(K) \rightarrow \mathbb{C} \).

However, we have actually a pairing on sections of \( L \).

In effect, we have

\[
\mathcal{C}^\infty(L) \xrightarrow{\bar{\partial}} \mathcal{C}^\infty(\underline{L \otimes \Omega^{1,0}})
\]

\[
L \otimes \Omega^{1,0} \otimes \Omega^{1,0} = \mathcal{L}
\]

so we can form a bilinear form

\[
B(s_1, s_2) = \int s_1 \bar{\partial}s_2
\]

on \( \mathcal{C}^\infty(L) \). This form is skew-symmetric because we have \( L \otimes L = K \) and so

\[
\bar{\partial}(s_1s_2) = (\bar{\partial}s_2)s_2 + s_1(\bar{\partial}s_2)
\]

so

\[
B(s_1, s_2) + B(s_2, s_1) = \int \bar{\partial}(s_1s_2) = 0
\]

Now, instead of using the \( \bar{\partial} \) complex to resolve

one can use the sheaf of holomorphic sections of \( L \)

and resolve the sheaf of meromorphic sections. Thus we

embed the sheaf of holomorphic sections of \( L \)

into the sheaf of meromorphic sections.

Given two meromorphic spinors \( s_1, s_2 \),

their product is a meromorphic 1-form \( s_1s_2 \)

which has a global residue ??
I want now to describe the cohomology of a holomorphic vector bundle $E$ using meromorphic $+$ adelic ideas. First review the $C^\infty,$ $\mathfrak{F}$ approach.

$$0 \to H^0(E) \to C^\infty(E) \to C^\infty(E \otimes \Omega_1) \to H^1(E) \to 0$$

$$0 \to H^1(E^* \otimes C) \to C^\infty(E^* \otimes T^\vee) \to C^\infty(E^* \otimes T_1^\vee) \to H^0(E^* \otimes \Omega) \to 0$$

In the above we have the non-degenerate pairings vertically provided on one side $C^\infty$ is enlarged to distributions. Notice the action is the canonical pairing

$$C^\infty(E) \times C^\infty(E^* \otimes \Omega) \to C$$

$$(s_1, s_2) \mapsto \int s_2 \overline{\partial} s_1 = -\int (\overline{\partial} s_2) s_1$$

and this does not induce the canonical pairing of $H^1(E)$ with $H^0(E^* \otimes \Omega).$

Now I want a non-$C^\infty$ version which gives the cohomology and there are two candidates. Either we use the resolution

$$0 \to \mathcal{O} \to F \to F/\mathcal{O} \to 0$$

$$\xrightarrow{\text{inclusion}}$$

or the adelic version which involves the maps $p$.
Thus we have the following two sequences for computing the cohomology of $E$:

$$0 \to H^0(E) \to E \otimes \mathcal{O} \to E \otimes F / \mathcal{O} \to H^1(E) \to 0$$

or

$$0 \to H^0(E) \to E \otimes \mathcal{O} \otimes F / \mathcal{O} \to E \otimes \mathcal{O} A \to H^1(E) \to 0$$

The reason I'm worrying about these two cases is because I need a pairing. To keep things simple, suppose $E = \mathcal{O}$. Then I want pairings

$$0 \to H^0(\mathcal{O}) \to \square \to \square \to H^1(\mathcal{O}) \to 0$$

$$0 \leftarrow H^1(\mathcal{O}) \leftarrow \square \leftarrow \square \leftarrow H^0(\mathcal{O}) \leftarrow 0$$

where the boxes are to be filled in.

In Weil's adelic proof of RR, differentials (meromorphic) are defined as linear functionals on adeles $A$ which vanish in $F$ and some parallelopiped.
\[ \Lambda(\mathfrak{O}) = \prod \hat{m}_p \]

This amounts to filling in as follows

\[ 0 \rightarrow H^0(\mathfrak{O}) \rightarrow \prod \hat{\mathfrak{O}}_p \rightarrow A/F \rightarrow H^1(\mathfrak{O}) \rightarrow 0 \]

\[ 0 \leftarrow H^1(\Omega) \leftarrow \Omega \otimes F/\mathfrak{O} \leftarrow \Omega \otimes F \leftarrow H^0(\Omega) \rightarrow 0. \]

So Weil tells us that \( A/F \) and \( \Omega \otimes F \) are dual, the former being linearly compact and the latter discrete.

There has to be a pairing

\[ A \otimes (\Omega \otimes F) \rightarrow \mathbb{C} \]

which ought to be

\[ (f_p), \omega \mapsto \sum_p \text{res}_p(f_p \omega). \]

If \( (f_p) = f \in F \), then \( f\omega \in \Omega \otimes F \) and the sum of the residues is zero, so we get an induced pairing

\[ (A/F) \otimes (\Omega \otimes F) \rightarrow \mathbb{C}. \]

More generally we can define a pairing

\[ A \times (\Omega \otimes A) \rightarrow \mathbb{C} \]

\[ (f_p)(\omega_p) \mapsto \sum \text{res}_p(f_p \omega_p) \]

and I think has to be a duality. Locally at \( p \) the completion \( \hat{F}_p \) is the sum of a linearly compact space \( \hat{\mathfrak{O}}_p \) and a discrete on \( \Omega \hat{F}_p/\hat{\mathfrak{O}}_p \), and these are dual to each other. This should generalize...
to A by taking tensor products.

Let's formulate the problem. There is a duality between $H^0(E)$ and $H^1(E^* \otimes \Omega)$ which we obtain from the 3 operators on these two bundles. I wanted to find a version using meromorphic or adelic sections. Meromorphic sections don't seem to work because the product of a meromorphic section of $E$ and one of $E^* \otimes \Omega$, when contracted is a meromorphic 1-form, and the only number one can attach is the sum of the residues, and this is zero. So one seems to want to use adelic sections.

Now the product of two adelic sections of $E$ and $E^* \otimes \Omega$ is an adelic section of 2 when contracted, and this gives a number upon taking sum of residues. Unfortunately when you take $E$ to be a square root of $\Omega$, then the pairing defined on adelic sections of $\Omega$ is symmetric. This is still not like the action pairing on sections of $\Omega$ which is skew-symmetric.

Suppose that $\Omega$ has $H^0(\Omega) = 0$. Then we know that

$$L \otimes F \oplus \prod_{\mathbf{p}} \hat{E}_\mathbf{p} = L \otimes A$$

and this is a nice splitting of the space of adelic sections of $\Omega$ into complementary maximal isotropic subspaces for the pairing.
January 13, 1987

I am still trying to figure out what Witten is doing about QFT on a Riemann surface. I have decided to concentrate on vacuum expectation values in the simple case of the real fermi field on the circle, using the Hamiltonian $\frac{1}{i} \partial_x$. This should lead to a QFT on $\mathbb{R}$.

We have the operators $\Psi_k$ for $k \in \frac{1}{2} + \mathbb{Z}$ satisfying $\{ \Psi_k, \Psi_{k'} \} = 2\delta_{k+k',0}$, and Fock space $F_0$ is the cyclic representation of these operators. They have a ground state $\left| 0 \right>$ such that $\Psi_k \left| 0 \right> = 0$ for $k < 0$.

Fock space $F_0$ is a Hilbert space and these operators satisfy $\Psi_k = \Psi_{-k}$.

Somehow this is the $t=0$ picture and when I work with the little Fock space $F_0$ and the big Fock space $F_t$ I am trying to keep track of all times $-\infty < t < \infty$. For each time $t$ there is a Hilbert space $F_t$ sandwiched between $F_0$ and $F_t$. I should be able to describe it using the basis $e_s$ which should be orthogonal in any of these spaces. Let's defer this.

The basic operator is the field operator.
\[ \psi(z) = \sum z^{k-\frac{1}{2}} \psi_k \]

where \( z \in \mathbb{C}^n \). This makes sense as a transformation from \( \hat{F} \) to \( \hat{F}^* \) and probably makes sense as a transformation from \( \hat{F}^* \) to \( \hat{F} \) provided \( e^{\frac{1}{2}} \leq |z| \leq e^{\frac{1}{2}} \).

I forgot to mention normal ordering for the Clifford algebra generated by the \( \psi_k \). We have

\[ \psi_k \psi_e = :\psi_k \psi_e: + \left\{ \begin{array}{ll} 1 & \text{if } k = -k < 0 \\ 0 & \text{otherwise} \end{array} \right. \]

This is the normal ordering belonging to the ground state \( |0\rangle \). The map \( \psi_0 \rightarrow :\psi_0: \) goes from the exterior algebra to the Clifford algebra generated by the \( \psi_k \).

Now we have established before the operator product expansion

\[ \psi(z) \psi(w) = :\psi(z) \psi(w): + \frac{1}{z - w} \left| z > \right| w \left| \right. \]

\[ \sum z^{k-\frac{1}{2}} w^{\frac{1}{2}} \right| \psi_k \psi_e \]

and I want to discuss generalizations of this.

Thus I want to consider certain kinds of infinite linear combinations of the monomials \( :\psi_0: \). Examples are \( :\psi(z) \psi(w): \), \( \partial_z \psi(z) \psi(w): \). I want to generalize the operator product expansion.
Thus I want to define, or specify, some kind of exterior algebra generated by \( \psi(z) \) for \( z \in \mathbb{C}^x \) as well as their derivatives. Then in this exterior algebra I hope to define a twisted product by a Wick process. Perhaps one can't define this for coincident points: \( z = \omega \).

But the first step appears to be to specify the underlying vector space of which to take the exterior algebra. It will consist of linear combinations of operators

\[
\sum z^k \psi_a
\]

for \( z \in \mathbb{C}^x \)
as well as derivatives

\[
\sum k z^k \psi_a
\]

etc.

\( (c_k) \) is a sequence which is exponential polynomial.

Recall that an exponential polynomial fn. of \( x \) is a linear combination of \( x^k e^{\alpha x} \) where \( k \in \mathbb{N}, \alpha \in \mathbb{C} \). If it is a function killed by \( \partial(x) \) where \( P \) is a \( \pm 0 \) polynomial. On sequences one has the shift operator \( T \) and an exponential polynomial sequence is one killed by \( P(T) \) where \( 0 \neq P \in \mathbb{C}[z, z^{-1}] \). For example

\[
[(T-\lambda)(c)](k) = c_{k+1} - \lambda c_k = 0
\]

\[
\iff c_k = \lambda^k c_0
\]

and

\[
(T-\lambda)(k \lambda^{k-1}) = (k+1) \lambda^k - k \lambda^k = \lambda^k
\]

\[
(T-\lambda)^2 (k \lambda^{k-1}) = 0.
\]
Now the classification of exponential polynomial sequences is immediate. There is a natural basis indexed by \((n, \lambda), n \in \mathbb{N}, \lambda \in X\). Thus the vector space out of which we make the exterior algebra is isomorphic to \(C(\mathcal{E})/C[\mathcal{E}, \mathcal{E}^{-1}]\).

We learned above that we probably want to start with the vector space of principal parts, then form the exterior algebra, which will be the space of normal product operators. Maybe I should say space of symbols. Then the deformed algebra is defined by Wick rules and the Green's function. An obvious thing to try to do is to see how the Green's function is to be defined for principal parts. Thus if I take two principal parts which are disjoint then the Green's function is defined between them. I am being imprecise; the idea is that the Green's function is usually written

\[ G(\mathcal{E}, \mathcal{E}') = \langle \psi(\mathcal{E}) \psi(\mathcal{E}') \rangle \]

but it extends to where \(\psi(\mathcal{E})\) is replaced by a linear combination, i.e. by a principal part.

So an obvious question is whether we can make sense out of the Green's function pairing between disjoint principal parts on a Riemann surface.

Let us fix a spin structure \(L : \mathcal{E} \otimes L = \mathcal{E} \).
and suppose $H^0(L) = 0$, so that a Green’s function for the $\overline{\partial}$ operator on $L$ is defined. On the other hand the fact that $H^0(L) = H^1(L) = 0$ tells us that one can prescribe the singular part or principal part of a meromorphic section of $L$ arbitrarily.

$$0 \to H^0(L) \to L \otimes F \sim \bigoplus_p L \otimes F / L \otimes \mathcal{O}_p \to H^1(L).$$

Put another way

$$L \otimes A = L \otimes F \oplus \bigoplus_p T \hat{L}_p.$$

By principal parts of meromorphic sections of $L$ we mean

$$L \otimes A / T \hat{L}_p = \bigoplus_p L \otimes \hat{F}_p / \hat{L}_p = \bigoplus_p L \otimes F / L \otimes \mathcal{O}_p.$$

So we proceed as follows. Take two "singular parts of sections" of $L$, call them $\xi$ and $\eta$. We suppose they have disjoint supports. Now we can lift these to meromorphic sections if we wish, call them $f$ and $g$. Now $fg$ is a meromorphic 1-form, so the sum of its residues is zero. But the thing we have to probably ignore the polar part of $f$ and just sum over the residues at the poles of $g$. Then this is indeed skew-symmetric.
March 14, 1987

Complex Fermi field. Start over $S^1$
\[
\psi^*(t) = \sum c_k \psi^*_k \\
c_k = \int \frac{dx}{2\pi} f(x) e^{-ikx} \\
\psi(t) = \sum \xi_k \psi_k = \int \frac{dx}{2\pi} \overline{F(x)} \sum e^{ikx} \psi_k
\]
\[\mathcal{H} = \frac{i}{\hbar} \mathcal{D}_x \text{ extended to } \mathcal{F} = \sum k \psi^*_k \psi_k; \text{ then}
\]
\[e^{it\mathcal{H}} \psi^*(t) e^{-it\mathcal{H}} = \int \frac{dx}{2\pi} f(x) \sum e^{-ikx + it} \psi^*_k \psi_k
\]
\[e^{it\mathcal{H}} \psi(t) e^{-it\mathcal{H}} = \int \frac{dx}{2\pi} \overline{F(x)} \sum e^{ikx - it} \psi_k
\]

When it comes to doing this, one needs a R.S. in a complex vector space with conjugation linear in $\psi$. Consider now a R.S. $\mathcal{F}$ with a spin structure $L \otimes L = \mathbb{R}^0$. Then for any circle $\mathcal{C}$ on $\mathcal{F}$, $\mathcal{F}(\mathcal{C}, L)$ has quadratic form $\xi, \eta \mapsto \xi \eta^*$ and real structure $\xi \mapsto \xi$ making it into a complex vector space with hermitian inner product. Then the Fock space to be attached to $\mathcal{C}$ is a tensor space of $\mathcal{F}(\mathcal{C}, L)$ with the operators
\[\psi^*(\xi) = \xi \wedge
\]
\[\psi(\xi) = |\xi|\wedge
\]
satisfying $\{\psi(\xi), \psi(\eta)\} = \{\psi^*(\xi), \psi^*(\eta)\} = 0$, $\{\psi(\xi), \psi^*(\eta)\} = (\overline{\xi} \eta)$. 


As we want to let $C$ move, it is awkward that $\psi(\xi)$ depends conjugate linearly on $\xi$, since the conjugation depends on $C$. Therefore we change the notation and require

$\psi^*(\xi) = \xi^\dagger$ as before

$\psi(\xi) = \xi^\dagger$ defined by the pairing $\langle \xi \rangle$

Thus we get the Clifford algebra structure

$\{ \psi(\xi), \psi(\eta) \} = \{ \psi^*(\xi), \psi^*(\eta) \} = 0$

$\{ \psi(\xi), \psi^*(\eta) \} = \int_C \xi \eta$

which means we can let the circle move.

In fact it can even be a 1-cycle. Now, when $C$ is a circle we can get a $\times$ algebra by defining

$\psi^*(\xi)\dagger = \psi(\xi), \quad \psi(\xi)^\dagger = \psi^*(\xi)$

Thus with this new notation in the circle case we have

$e^{tH} \psi^*(\xi) e^{-tH} = \int \frac{dx}{2\pi} f(x) \sum_{k} e^{-ikx+kt} \psi^*_k \psi^*(\xi t)$

$e^{tH} \psi(\xi) e^{-tH} = \int \frac{dx}{2\pi} f(x) \sum_{k} e^{ikx-kt} \psi_k \psi(\xi t)$

I next want to discuss consistency with imaginary time evolution. Recall that Fock space contains lines corresponding to certain subspaces of $L^2$. Consider two times say $0, t$ where $0 < t$. 


Given a subspace of $L^2$ of the circle at time 0, one obtains a subspace at the later time $t$ by using holomorphic functions on $(R/2\pi Z) \times [0,t]$ whose boundary values at time 0 belong to the given subspace.

I want to discuss imaginary time-evolution more carefully. Let us take a fairly concrete viewpoint starting with the Fock space associated to AP half densities $f(x) \, dx^{1/2}$ on $S^1$. Then as we have seen above, this Fock space has operators

$$\psi^*(f \, dx^{1/2}) = \sum c_k \psi_k^*$$
$$\psi(f \, dx^{1/2}) = \sum c_k \psi_k$$

and a vacuum $1_0 >$ satisfying

$$\psi_k^* 1_0 = 0 \quad k < 0$$
$$\psi_k 1_0 = 0 \quad k > 0$$

The Hamiltonian $H = \sum k \psi_k^* \psi_k$ has eigenvalues $>0$ so the operator $e^{-tH}$ is well defined. Recall there is a basis $e_\alpha$ for the Fock space consisting of eigenvectors of $H$.

Now suppose we have two operators $\psi^*(f \, dx^{1/2})$ and $\psi^*(f_t \, dx^{1/2})$ which are consistent with $e^{-tH}$:

$$\psi^*(f_t \, dx^{1/2}) e^{-tH} \alpha = e^{-tH} \psi^*(f_0 \, dx^{1/2}) \alpha$$

for all $\alpha$. Then if $f_t = \sum c_k e^{ikx}$ we have
\[ \sum c_k^t \psi_k^* e^{-tH} \alpha = \sum c_k^0 \left( \frac{e^{-tH} \psi_k^* e^{tH}}{e^{-kt} \psi_k^*} \right) e^{-tH} \alpha \]

where \( c_k^t = e^{-kt} c_k^0 \)

and this means that
\[ f_t(x) = \sum c_k^t e^{ikx} = \sum c_k^0 e^{ikx-k} \]

is an analytic function of \( x+it \).

Note this means that the time evolution
\[ \psi^*(f_t dx^{1/2}) = e^{-tH} \psi^*(\int_0 dx^{1/2}) e^{tH} \]

corresponds to taking the half density \( f_0 dx^{1/2} \) and analytically continuing it in the imaginary direction.

Let's now consider the cylinder of \((x,t)\) \( e^{R/2\pi Z} \times R \) equipped with the complex structure such that \( x+it \) is analytic. Put on this cylinder the spin structure anti-periodic in the \( x \) direction. For each time \( t \) we have a Fock space \( F_t \), so we have a bundle of Fock spaces over the \( t \) line. One has imaginary-time translation \( e^{-tH} : F_s \to \overline{F_{s+t}} \). \( t > 0 \)

It would be better to say we have trivialized this bundle one way since all \( F_t \) are obviously isomorphic to \( F \) in an obvious way. But then we don't want this trivialization, rather we want to use time evolution. But then because \( e^{-tH} \) is
a contraction operator we obtain a tower of Fock spaces:
\[ \mathcal{F} < \mathcal{F}_t < \hat{\mathcal{F}} \]

We can think of \( \mathcal{F}_t \) as the completion of \( \mathcal{F} = \bigoplus \text{Ces} \) in a suitable norm which weights the line \( \text{Ces} \) according to the energy and \( t \).

One would like to \( \mathcal{F} \) and \( \hat{\mathcal{F}} \) as being replaceable by \( \Lambda \mathcal{F}_t \) and \( \mathcal{U} \mathcal{F}_t \).

Now what kind of Clifford operators do we have? We have agreed to refer operators on \( \mathcal{F}_t \) or from \( \mathcal{F}_s \) to \( \mathcal{F}_t \) back to \( \mathcal{F}_0 \) where they are unbounded. It might be better to say that our operators go from \( \hat{\mathcal{F}} \) to \( \hat{\mathcal{F}} \) or maybe to concentrate on ones which go from \( \mathcal{F}_s \) to \( \mathcal{F}_t \).

The first question is to worry about a Clifford operator \( \Sigma c_k \gamma_k^* \) on \( \mathcal{F}_s \). These are linear combinations such that \( \Sigma |c_k|^2 < \infty \).

We can transport this to \( \mathcal{F}_t \) by
\[ e^{-tH} \left( \Sigma c_k \gamma_k^* \right) e^{tH} = \Sigma c_k e^{-kt} \gamma_k^* \]

and so operators on \( \mathcal{F}_s \) are Clifford operators
\[ \Sigma c_k \gamma_k^* \]
which map \( \mathcal{F}_s \) to \( \mathcal{F}_t \).
Question: What linear combinations \( \sum c_k \psi_k^* \) make sense as operators from \( F_s \) to \( F_t \)?

We can split into creation + destruction parts the former being \( \sum_{k>0} c_k \psi_k^* \). If \( c_k \) is an \( l^2 \) sequence this makes sense on \( F_0 \) and hence as an operator from \( F_s \) to \( F_t \) for \( s \leq 0, t \).

Now formally \( \sum c_k \psi_k^* \) corresponds to

\[
\sum_{k>0} c_k e^{i k \omega} \quad \omega = x + i t
\]

which ought to be analytic for \( \text{Im}(\omega) > a \varepsilon \).

Thus suppose that \( c_k e^{-k a} \) is an \( l^2 \) sequence. Then

\[
e^{-a H} \left( \sum_{k>0} c_k \psi_k^* \right) e^{a H} = \sum_{k>0} c_k e^{-ka} \psi_k^*
\]

is well defined on \( F_0 \) so that \( \sum c_k \psi_k^* \) is well defined on \( e^{a H} F_0 = F_a \).

Similarly \( \sum c_k e^{i k \omega} \) ought to be analytic for \( \text{Im}(\omega) < b + \varepsilon \). Thus \( c_k e^{-k b} \) will be an \( l^2 \) sequence so

\[
\sum_{k>0} c_k \psi_k^* = e^{b H} \left( \sum_{k>0} c_k e^{-k b} \psi_k^* \right) e^{-b H}
\]

will be defined on \( e^{b H} F_0 = F_b \).

Thus a typical series \( \sum c_k e^{i k \omega} \) will when split converge in regions \( \text{Im}(\omega) < b + \varepsilon \), \( \text{Im}(\omega) > a - \varepsilon \) and will make sense as an operator from \( F_s \) to \( F_t \) for \( s < b \), \( a \leq t \).
I am primarily interested in the operator
\[ \sum e^{-ikx + kt} \psi_k^* \]
which will make sense from \( F_{t-\epsilon} \) to \( F_{t+\epsilon} \).

Now we have
\[ \psi_k^*(f(x) dx^{1/2}) = \int \frac{dx}{2\pi} f(x) \sum e^{-ikx} \psi_k^* \]

Let us write \( z = e^{t-ix} \) so that
\[ \frac{dz}{z} = dt - i dx \]

Notice that as \( x \) goes from 0 to \( 2\pi \) and \( t=0 \)
that \( z \) traces out the unit circle backwards.

On this circle
\[ dx = i \frac{dz}{z} \]
so the above integral is
\[ \int \frac{dz}{2\pi i} \frac{f(z)}{z} \sum z^k \psi_k^* \]

This suggests the naturality of putting
\[ \psi^*(z) = \sum z^{k-\frac{i}{2}} \psi_k^* \]

So where are we? We know now that \( \psi^*(z) \) and probably also \( \psi(z) \) are well defined operators from \( F_s \) to \( F_t \) provided
\[ e^t > |z| > e^s. \]
March 15, 1987

The problem is to specify a vector space of operators of the form \( \sum c_k \phi_k^* \) which are associated in some way to rational functions of \( z \). By this I mean that the power series

\[ \sum_{k>0} c_k z^{-k-\frac{1}{2}} \quad \sum_{k<0} c_k z^{-k-\frac{1}{2}} \]

converge near \( \infty \) and near 0 resp. to rational functions. To simplify let's shift from \( k \) half integer to \( k \) integer. When

\[ \sum_{n>0} c_n z^n \]

the power series of a rational function regular at \( z=0 \)? Use partial fractions

\[ \frac{1}{1-\lambda z} = \sum_{n \geq 0} \lambda^n z^n \]

\[ \frac{n! \lambda^n}{(1-\lambda z)^{n+1}} = \sum_{n \geq 0} \frac{n(n-1) \ldots (n-n+1)}{n!} \lambda^n z^n \]

Partial fractions so that the space of \( n \) rational functions regular at zero has the basis

\[ z^n, \quad n \geq 0 \]

\[ \frac{1}{(1-\lambda z)^n}, \quad n>0, \quad \lambda \in \mathbb{C}^* \]

What seems to be emerging is that one requires the sequence \( c_k \) to be exponential polynomial modulo finite sequences. There is a non-zero polynomial in the shift operator which when applied to \( c_k \) gives a finite
sequences.

Let's describe the sequences \( (c_k) \) such that \((T-1)c_k = c_{k+1} - c_k\) is a finite sequence. Let \( W \) be the space of these sequences. Then \( W \) contains the finite sequence space \( F \) and \( W/F \) is 2-dimensional. In general what we get for a the sequences killed by a polynomial in the shift operator is an extension of the finite sequences by two copies of the sequences killed by the polynomial.

I might try to generalize and look at an annulus \( s \leq |z| \leq t \). Then the finite sequences would be replaced by the analytic functions in the annulus. Then the space of \( K \) operators is an extension of the meromorphic functions on the annulus with poles in the inside by the \( K \) space of singular parts of such functions.
March 17, 1987

The problem is still to understand, better to describe, in a good way, the kind of operators $\sum c_k y_k$ to be considered. At the moment I am considering the space of Laurent series $\sum c_k z^{-k}$ such that there exist a non-zero polynomial in $z$, $p(z)$ such that $p(z) \sum c_k z^{-k}$ is a polynomial. This maps into the space of rational functions and the kernel is the space of exponential polynomial sequences, i.e. singular meromorphic functions regular at 0 and $\infty$.

I want to obtain a nicer description without the $0, \infty$ restriction. It seems a good idea to reformulate in terms of a circle arising from a spin structure on $\mathbb{P}^1$ being simply-connected it is not to have anti-periodic half-densities.

On $\mathbb{P}^1$ the canonical line bundle has the meromorphic section $dz$ which has no zeroes and has a double pole at $z = \infty$ (since $u = \frac{1}{z}$).

Thus $\Omega^1 \cong \mathcal{O}(-1)$ i.e. any holomorphic differenial on an open set $U$ can be described in the form $f(z) dz$ where $f$ is holomorphic on $U$ and vanishes twice at $\infty$ if $\infty \in U$.

Now the unique spin structure on $\mathbb{P}^1$ is given by the line bundle $\mathcal{O}(-1)$ with the isomorphism $\mathcal{O}(-1) \otimes 2 = \mathcal{O}(-2) \cong \Omega^1$. It is more suggestive to say there is a unique line bundle $\mathcal{L}$ with the isomorphism $\mathcal{L} \otimes 2 = \Omega^1$, and that $\mathcal{L}$ has a unique meromorphic section, nonvanishing with simple pole at $\infty$, denote it $dz/\sqrt{2}$, such that $(dz/\sqrt{2})^2 = dz$. Then holomorphically...
sections of \( L \) are of the form \( f(z)^{1/2} \)
where \( f \) is holomorh in \( \mathbb{U} \) and vanishes
at \( \infty \) if \( \infty \in \mathbb{U} \).

Suppose we have two meromorphic sections
\( f dz^{1/2} \) and \( g dz^{1/2} \) which have disjoint poles.

If \( f, g \) have disjoint poles and one must have
at \( \infty \). Then their product \( fg dz \) and

For example
\[
f = \frac{1}{f - z} \quad g = \frac{1}{f - \omega}
\]

\[
\sum_{j=0}^{f(\mathcal{P})} \text{Res} \left\{ \frac{1}{f - z} \frac{1}{f - \omega} dz \right\} = \text{Res} \left\{ \frac{1}{f - z} \frac{1}{f - \omega} dz \right\}
\]

\[
= \frac{1}{z - \omega}
\]
March 18, 1987

Remark 1. If I think of spinors over $\mathbb{P}^1$ as sections $f$ of $\mathcal{O}(-1)$, identifying $f$ with $f \, dz^{1/2}$, then the quadratic form defined by taking the product and integrating over $|z| = 1$ is

$$
(z^m, z^n) = \frac{1}{2\pi i} \int z^{m+n} \, dz = \delta_{m+n, -1}
$$

This sort of explains why one might prefer the notation

$$
\psi(z) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} z^{k-\frac{1}{2}} \psi_k
$$

because then

$$
\frac{1}{2\pi i} \int z^{k-\frac{1}{2}} (dz)^{1/2} \cdot z^{l-\frac{1}{2}} (dz)^{1/2} = \delta_{k+l, 0}
$$

Remark 2. The space $\psi_k$ is the space of meromorphic spinors on $\mathbb{P}^1$ with poles at $0, \infty$. In general it might make sense in a Riemann surface to fix a finite set $S$ of points on the surface and to consider the meromorphic spinors with these poles. Is there some way these would act on the tensor product of the little Fock spaces at each point of $S$?

Take circles around each point of $S$.

Let the point of $S$ be $P_1, \ldots, P_n$, and circles be $C_1, \ldots, C_n$, and they should be oriented positively. Then we have the space of spinors on each circle $C_i$ and it comes equipped with quadratic form. Next we can form the Clifford algebra
for each circle and their tensor product as well as the tensor product of the corresponding Fock spaces.

There should be in each Fock space \( F_i \) a "vacuum" line corresponding to the interior of the circle \( C_i \). More precisely the space of spinors in \( C_i \) contains the subspace \( W_i \) of spinors extending holomorphically to the disk inside \( C_i \). The vacuum line in \( F_i \) is killed by the operators \( \psi(i) \), \( i \in W_i \). Thus in \( \otimes F_i \) we have a vacuum state \( |0\rangle = \otimes (|0\rangle_i) \).

On the other hand the Riemann surface outside of the circles \( C_i \) should give a linear functional \( \langle 0 | \) on the space \( \otimes F_i \).

Now before going on it would be nice to know what the Fock space \( F_i \) associated to the circle \( C_i \) becomes as \( C_i \) shrinks to \( \gamma_i \) the point \( P_i \). I'm not sure how to think about \( F_i \) in the single fermion case.

Let's then return to the circle \( \mathbb{R}/2\pi \mathbb{Z} \) where

\[
\psi(f(x)\,dx^{1/2}) = \sum c_k \psi_k \quad f = \sum c_k e^{ikx}
\]

\[
= \int \frac{dx}{2\pi} f(x) \sum e^{-ikx} \psi_k
\]

What is \( \psi(f_t(x)\,dx^{1/2}) \) consistent with \( e^{-tH} \), i.e.

\[
\psi(f_t\,dx^{1/2}) e^{-tH} = e^{-tH} \psi(f_0\,dx^{1/2})
\]

\[
c_k(t)\psi_k = e^{-tH} c_k(0)\psi_k e^{tH} = c_k(0) e^{-kt}
\]
that is
\[ f_t(x) = \sum c_k(0) e^{ikx - kt} \]
in holomorphic in \( x + it \). Recall that we put \( z = e^{t - ix} \) whence
\[ f_t(x) = \sum c_k(0) z^{-k} \]
and on any of the Fock spaces \( F_t \) one has
\[ \psi(f) = \sum c_k(0) \psi_k \]
\[ c_k(0) = \frac{1}{2\pi i} \int \frac{dz}{z} f(z) z^{1-k} \]

The above isn't very clear, but what I am aiming for is to identify all the Fock spaces under time evolution, so that any holomorphic spinor \( \psi(f) \) will act. Also I want the intersection of the Fock spaces to be essentially the little Fock space.

Now \( f \) is holomorphic in \( C \) means that the coefficients \( c_k(0) = 0 \) for \( k > 0 \) which means that \( \psi(f) \) is a linear combination of \( \psi_k \) for \( k < 0 \). Thus \( \psi(f) |_0 = 0 \), as it should.

Conclude: The Fock space attached to a point on a surface is the exterior algebra of \( F/\Omega_p \) at that point, more precisely it is the exterior algebra of the singular parts of meromorphic spinors. Thus these are lines in Fock space attached to subspaces of \( F \) containing \( \Omega_p \).

INACCURATE see p. 547
I want to return to the problem of describing the space of operators
\[ \sum c_k \hat{\cal F}_k \] where \[ \sum c_k z^{-k} \] is such that
\[ p(z) \sum c_k z^{-k} = \varphi(z) \]
for Laurent polynomials \( p, \varphi \) with \( p \neq 0 \). Recall that this space is an extension of the space of rational functions by the space of principal parts on \( \hat{\cal F} \).

I feel that the above operators are well-defined from \( \hat{\cal F} \) to \( \hat{\cal F}^* \) and hence to each \( |e_s> \in \hat{\cal F} \) and \( <e_s| \in \hat{\cal F}^* \) we get a number. Note that one uses the time evolution embedding of \( \hat{\cal F} \) in \( \hat{\cal F}^* \).

Thus it seems reasonable to expect these operators to act on the tensor product of the Fock spaces \( \hat{\cal F}_0 \) and \( \hat{\cal F}_s \). One then applies \( \sum c_k \hat{\cal F}_k \) to \( |e_s> \otimes <e_s| \) and gets a number.

I think I am beginning to get somewhere on the operators from \( \hat{\cal F} \) to \( \hat{\cal F}^* \). Think of having a small circle and a large circle and time evolution in between. Suppose we have two meromorphic functions \( f_i \) and \( f_o \). We can then consider \( \hat{\cal F}(f_i) \) in the small circle Fock space, followed by time evolution to the large Fock space, followed by \( -\hat{\cal F}(f_o) \).

Notice that if \( f_i = f_o \) this doesn't give zero although it does if \( f_i = f_o \) is holomorphic in the region between the circles. So we
we are looking at the space of pairs of meromorphic spinors modulo the diagonal subspace of spinors analytic over the complement of 0 and \( \infty \). There is an obvious map from this space to meromorphic spinors given by \((\varphi_1, \varphi_2) \mapsto \varphi_1 - \varphi_2\). The kernel is the space of meromorphic spinors modulo those regular outside of 0, \( \infty \), and this is the space of principal parts on the complement of \([0, \infty]\).

Now I want to understand many points: \( S = \{ P_i \}\).

What I can do is to give a merom. spinor \( \varphi_i \)
for each point \( P_i \), and then let \( \sum \varphi_i(f_i) \) act on \( |0\rangle \) in \( \otimes F_i \). Then we apply \( \langle 0 | \) representing the surface outside the \( P_i \). This gives a number which is zero if all \( \varphi_i \) are equal and are holomorphic except at the \( P_i \).

The next problem is how do principal parts enter? Suppose all the \( \varphi_i \) are equal, then this space is the space of meromorphic spinors, and the number being obtained is zero when the \( \varphi_i \) are holomorphic. Thus we have a function on the space of principal parts over the complement of \( S \).

Consider the case of a single point on the surface and a small circle around it.

In fact it might be better to take a Riemann surface bounded by a circle and to consider the Ford space of the circle; it contains a line corresponding to the surface. In effect the space of sections of the spinors on the circle contains the
subspace of spinors extending holomorphically. This is isotropic for the quadratic form, and in good cases it would be maximal isotropic. So let \( |0\rangle \) denote the state corresponding to the isotropic subspace of holomorphic spinors. If \( |t_1\rangle, \ldots, |t_n\rangle \) are spinors on the circle extending to meromorphic spinors inside then we have the state

\[
|\psi(t_1) \cdots |\psi(t_n)\rangle |0\rangle.
\]

Let's start again with a Riemann surface bounded by a circle

Suppose that there is a spin structure. Then the space of spinors on the circle has the isotropic subspace of holomorphic spinors on the surface. The reason this subspace is isotropic is because the circle is trivial in homology: it is \( S^1 \). The Stokes' theorem and the fact that holomorphic 1-forms are closed make the spinor bundle over the circle trivial.

Suppose we now fill in the circle and that the resulting spin structure has no holomorphic sections. Then the spaces of sections extending holomorphically on either side and so we will have vectors \( <0| |0\rangle \) on the Fock space.

Then for any set of spinors on the circle
we can form

\[ \langle 0 \mid \psi(t_1) \cdots \psi(t_r) \mid 10 \rangle \]

For example this should make sense for any family of meromorphic spinors on the complex surface with poles off the circle.

Now arises the problem of what happened to the condition that the poles be disjoint. Secondly there is the problem of evaluating the above expectation values to see if this might be interesting.

The first remark is that the operator \( \psi(t) \) for \( t \) a meromorphic spinor is meaningless unless you say what circle is given.

The next remark is that the above vacuum expectation values can be evaluated by Wick's theorem. One has taken the space of spinors over the circle and split it into maximal isotropic subspaces. This defines a normal ordering.
March 19, 1987

I have to clear up some inaccuracies concerning the local Fock space at a point \( P \). This is the intersection of the Fock spaces of small circles shrinking to \( P \), the inclusions being defined by "time evolution". There is also an algebraic version \( \tilde{F}_P \subset \tilde{F}_P \), but I won't worry about the distinction for the present.

If \( f \) is a meromorphic spinor near \( P \) it determines an operator \( \psi(f) \) on \( \tilde{F}_P \) and these satisfy

\[
\{ \psi(f), \psi(g) \} = \text{res}_P(fg).
\]

We probably should work with the space \( L_EF \) of germs of meromorphic spinors at \( P \) with the residue form quadratic form. The germs of holomorphic spinors form an isotropic subspace \( \Lambda^1 \), and it is clearly a maximal isotropic subspace - this is a local power series calculation. Thus there is a unique state \( |0\rangle \) in \( \tilde{F}_P \) annihilated by the \( \psi(f) \) for \( f \) a holomorphic germ.

Now consider

\[
\psi(f_1) \cdots \psi(f_n) |0\rangle
\]

It is not true that this depends on \( f_i \in L_EF/\Lambda^1 \) because we can't move the \( \psi(f_i) \) around without encountering residues \( 1 \). However suppose we are in the situation where there are no nonzero holomorphic spinors: \( H^0(L) = 0 \). Then we have the space \( W \) of germs which extend holomorphically to the complement of \( P \). This space is isotropic and complementary to \( \Lambda^1 \) in \( L_EF \).

Thus if we restrict the \( f_i \in W \) we see 2) is skew-symmetric and that we get an isomorphism

\[
\wedge W \cong \tilde{F}_P
\]
Let us now do this with a finite set of points \( S = \{ P_1, \ldots, P_r \} \). Then we can form the tensor product \( \otimes \hat{F}_i \) and on this we have operators \( \psi(f) \) where \( f = (f_i) \in \prod (L \otimes F) \) satisfying
\[
\{ \psi(f), \psi(g) \} = \sum \text{res}(fg).
\]

Actually I should be doing something adelic i.e. I should probably replace \( L \otimes F \) by \( L \otimes \hat{F}_i \).

Again the holomorphic germs at the \( P_i \) determine states \( |0> \) in \( \hat{F}_i \) which then can be tensored to get \( |0> \) in \( \otimes \hat{F}_i \). Next you have the space \( W \) of meromorphic spinors whose poles are in \( S \). So we have
\[
\prod_i L \otimes \hat{F}_i = \prod_i L \otimes \hat{F}_i \oplus W
\]
where the two subspaces are maximal isotropic. As for one point this means that
\[
\Lambda W \rightarrow \otimes \hat{F}_i.
\]

Now let the finite set go to infinity. We should get a Fock space with operators \( \psi(f) \) for any adelic section of \( L \), i.e., a representation of the Clifford algebra of the space of adelic spinors with its natural quadratic form. Corresponding to the break up into isotropic subspaces
\[
L \otimes \hat{A} = L \otimes \prod \hat{F}_i \oplus L \otimes F
\]
on one obtains an isomorphism of \( F \) with \( \Lambda (L \otimes F) \).

Now where does the Green's function enter? We know that any VEV
\[
<0 | \psi(f_1) \cdots \psi(f_n) | 0>
\]
where the \( f_i \in L \otimes \hat{A} \) can be evaluated by Wick's
formula in terms of the case \( n=2 \).

Thus we want \( \langle 0 \mid \psi(f) \psi(g) \mid 0 \rangle \). Now \( \psi(g) \mid 0 \rangle \) depends upon the singular part of \( g \). Let us choose a meromorphic spinor \( \omega \) with \( g-\omega \in \prod \mathbb{Q} \), so that \( \psi(g) \mid 0 \rangle = \psi(\omega) \mid 0 \rangle \).

Then
\[
\langle 0 \mid \psi(f) \psi(g) \mid 0 \rangle = \langle 0 \mid \psi(f) \psi(\omega) \mid 0 \rangle
= \sum_P \text{res}_p (f \omega) - \langle 0 \mid \psi(\omega) \psi(f) \mid 0 \rangle
\]

Maybe we should think of \( \langle \psi(f) \psi(g) \rangle \) as bilinear in \( f, g \in L \otimes A \). It doesn't change if \( g \) is moved by a regular adelic spinor, hence one can suppose \( g \) is a meromorphic spinor. It doesn't change if \( f \) is moved by a meromorphic spinor, hence one can suppose \( f \) regular.

One has the following choice in the evaluation
\[
\langle 0 \mid \psi(f) \psi(g) \mid 0 \rangle = \sum \text{res} (f \omega)
= \sum \text{res} (r \cdot g)
\]

where \( \omega \in L \otimes F \) has the same singular part as \( f \) and where \( r \) is a regular adelic spinor congruent to \( f \) modulo \( L \otimes \mathbb{F} \).
We still have a picture missing in order to understand what \( \psi(z) \) is. I still feel that we must start with the space of singular parts of meromorphic spinors and associate operators somewhere to these.

Let me go over what I told Bott. I take a Riemann surface \( M \) with spin structure \( L \) such that \( H^0(L) = \emptyset \), and two points \( P, Q \). Then I have the product \( \hat{F}_P \otimes \hat{F}_Q \) which is an irreducible representation of the Clifford algebra associated to

\[
V = L \otimes \hat{F}_P \oplus L \otimes \hat{F}_Q
\]

with a cyclic vector belonging to the max. isot. subspace

\[
V_{reg} = L \otimes \hat{F}_P \oplus L \otimes \hat{F}_Q
\]

Let \( W \) be the space of meromorphic sections of \( L \) which are regular outside \( \{ P, Q \} \). Then \( W \) is isotropic and complementary to \( V_{reg} \) so we have

\[
\Lambda W \longrightarrow \hat{F}_P \otimes \hat{F}_Q.
\]

This defines a dual state \( \psi \) namely projection onto \( \Lambda W \).

Then given \( f \) a meromorphic spinor we consider the dual vector

\[
\langle 0 | \psi(t) \otimes 1 - 1 \otimes \psi(t) \rangle \quad \in \quad \hat{F}_P \otimes \hat{F}_Q.
\]

It depends only on \( f \mod W \). But \( L \otimes F/W \) is the space of singular parts of merom. sections of \( L \) over \( M - \{ P, Q \} \).

Finally we interpret a dual vector on \( \hat{F}_P \otimes \hat{F}_Q \) as a map from \( \hat{F}_P \) to \( \hat{F}_Q^* \).

Now can we define composition? Suppose that \( f \) and \( g \) are meromorphic in \( a < t < c \) and that the poles of \( f \) are in \( a < t < b \) and the poles of \( g \) are in \( b < t < c \).
Recall that we have defined the operator belonging to the singular part of $f$ to be

$\psi(f)$ on $F_a$ followed by time evolution to $F_c$.

Thus it is

$\psi(f) e^{-\epsilon f H} - \psi(f_c) e^{-\epsilon f_c H}$

Since $f$ is holom., for $b < t < c$, this is the same as

$\psi(f) e^{-\epsilon f H} - \psi(f_b) e^{-\epsilon f_b H}$

So now we compose

$\{ e^{-\epsilon f A} \psi(g_b) - \psi(g_c) e^{-\epsilon f H} \} \{ e^{-\epsilon f f} \psi(f_a) - \psi(f_b) e^{-\epsilon f_b H} \}$

$= e^{-\epsilon f f} \psi(g_b) e^{-\epsilon f_a H} \psi(f_a) - e^{-\epsilon f_b H} \psi(g_c) e^{-\epsilon f H} \psi(f_a) - e^{-\epsilon f_b H} \psi(g_b) \psi(f_b) e^{-\epsilon f_b H} - e^{-\epsilon f_b H} \psi(g_c) e^{-\epsilon f_b H} \psi(f_a) + \psi(g_c) e^{-\epsilon f_b H} \psi(f_b) e^{-\epsilon f_b H} - \psi(g_c) e^{-\epsilon f_b H} \psi(f_b) e^{-\epsilon f_b H}$

Thus the normal ordered part is the operator from $F_a$ to $F_c$ which corresponds to the linear froe in $\mathcal{F}_c \otimes \mathcal{F}_a$ given by

$\star = 1 \otimes \psi(g) \psi(f) + \psi(f) \otimes \psi(g) - \psi(g) \otimes \psi(f) + \psi(g) \otimes \psi(f)$

followed by pairing with the time evolution from $F_a$ to $F_c$. Note that $\star$ is the same as

$(1 \otimes \psi(g) - \psi(g) \otimes 1)(1 \otimes \psi(f) - \psi(f) \otimes 1)$

The contraction term is
and we may assume in $g$ and $f$ to $\alpha$ at $0$, $g \circ \psi$ into $h$ of the exterior.

Actually, let's let a drift to $0$ and $\sum_{\text{new}}(gf)$. Hence, expand $\Sigma + \sum_{\text{new}}(gf)$.
March 20, 1987

Improvement to yesterday's calculation.

Let's work between the circles \(|z| = a\) and \(|z| = c\)
where \(a < c\). Let \(f\) be meromorphic in \(a \leq |z| \leq c\)
with poles inside and set

\[
\tilde{f} \quad \text{"princ part of"} \quad f = T(c, a) \psi(f_a) - \psi(f_c) T(c, a)
\]

where \(T(c, a) = e^{-(c-a)H}\). Then we found a

composition formula as follows. Suppose \(a < b < c\), and

that the poles of \(f\) are in \((a, b)\) while the poles of

another function \(g\) are in \((b, c)\). Then

\[
\tilde{g}(c) \tilde{f}(f) = (-\int \tilde{g} \tilde{f}) T(c, a) + T(c, a) \psi(g_a) \psi(f_a) - \psi(g_c) T(c, a) \psi(f_a)
++ \psi(f_c) T(c, a) \psi(g_a)
++ \psi(g_c) \psi(f_a) T(c, a)
\]

I originally thought the four terms at the right

should be \(\tilde{g}(c) \tilde{f}(f)\), but it's not skew-symmetric

in \(g, f\). To remedy this we use

\[
\frac{1}{2} [\tilde{g}(c) \tilde{f}(f)] = \frac{1}{2} (\psi(g) \psi(f) - \psi(f) \psi(g))
= \psi(g) \psi(f) - \frac{1}{2} \int g \psi(f)
\]

Thus we have

\[
\tilde{g}(c) \tilde{f}(f) = T(c, a) \frac{1}{2} [\psi(g) \psi(f)]_a + \psi(f_c) T(c, a) \psi(g_a) - \psi(g_c) T(c, a) \psi(f_a) + \frac{1}{2} [\psi(g), \psi(f)]_c T(c, a)
\]

Thus the contraction term becomes \(T(c, a)\) times

the scalar.
\[- \int g_b f_b + \frac{1}{2} \int g_a f_a + \frac{1}{2} \int g_b f_b\]

\[= - \sum_{\text{poles of } f} \text{res} (gf) - \frac{1}{2} \left( \int g_b f_b - \int g_a f_a \right) = \frac{1}{2} \sum_{\text{poles of } f \text{ and } g} \text{res} (gf)\]

\[= \frac{1}{2} \left( \sum_{\text{poles of } g} \text{res} (gf) - \sum_{\text{poles of } f} \text{res} (gf) \right)\]

Thus we end up with the formula:

\[\Psi(g) \Psi(f) = \cdot \Psi(g) \Psi(f) \cdot + \frac{1}{2} \left( \sum_{\text{poles of } g} \text{res} (gf) - \sum_{\text{poles of } f} \text{res} (gf) \right)\]
March 21, 1987

Problem: Suppose we have a Riemann surface which is compact and has a single boundary component.

Show how one obtains a line in the Fock space of the circle. We have to suppose a structure \( L \) is given on the surface. Then contains the subspace \( W \), of sections which is isotropic in \( \Gamma(C,L) \).

I think there is no difficulty in closing up \( W \) to get an isotropic subspace \( W \) of \( L^2(C,L) \) which is isotropic. Then \( W \cap \overline{W} = 0 \), in fact \( W, \overline{W} \) are clearly perpendicular. The issue is whether \( L^2(C,L) = W \oplus \overline{W} \) in which case we obtain a "polarization" and this defines a line in the Fock space.

Certainly it is enough to show that \( L^2(C,L) \) real \( \sim W \)

i.e. that there is for any real spinor over \( C \) a unique holomorphic spinor in the surface whose restriction to the boundary has real part equal to the given real spinor. The method I use to establish this is to set up a self-adjoint boundary value problem on the surface. Specifically we can consider the problem of constructing a holomorphic...
spinors in the interior with given real part on the boundary. Or we might try to solve \( \overline{\partial} u = f \) on the surface with the boundary condition that the real part of \( u \) is zero on the boundary. (The latter seems wrong.)

Now the natural candidate is the Dirac operator on the surface which is a skew-or self-adjoint operator on a 2-dimensional vector bundle which is graded. This requires a metric.

The cotangent bundle has a complex structure, and when complexified it splits into \( T^{0,0} \oplus T^{0,1} \) and we get obvious raising and lowering operators. The spinor bundle must have the form \( S = S^+ \oplus S^- \) where \( S^\pm \) are line bundles such that
\[
T^{0,1} \otimes S^+ \simeq S^- \\
T^{1,0} \otimes S^- \simeq S^+.
\]

So far what we have said holds for any Dirac type operator. Something special happens in the case of a spinor bundle.

Let's proceed in general and try to get the adjointness straight.

Consider \( \Gamma(M, L) \), where \( L \otimes L = T^{1,0} \).

On this space we have a skew-symmetric form given by \( \langle \xi, \eta \rangle \mapsto \int_\gamma \xi \cdot \overline{\eta} \). Suppose a Riemannian metric given on \( M \) compatible with
the complex structure. Then $L$ acquires a unique hermitian metric consistent with the one on $T^{\mathbb{C}}$ and $\Gamma^2(M, L)$ has an inner product allowing us to form $L^2(M, L)$. Thus we a complex Hilbert space with a skew form which is partially defined.

Let's go back to $\Gamma^2(M, L)$ the sections vanishing on the boundary. This is a complex v.s. with hermitian inner product and a skew symm. form. Forget the complex structure, i.e. look at $L$ as a 2 diml real vector bundle. Then the skew form shows that we have a formally adjoint operator on sections of $L$.

Recall that in 2 dimension there are two Clifford algebras, one might consider relative to Dirac operators. One is generated by $e_1, e_2$ with $e^2 = -1$ and it is the quaternions $\mathbb{H}$, whose unique irreducible real reps has dim 4. The other is generated by $g_1 g_2$ with $(g_1)^2 = 1$ and is $\mathbb{M}_2(\mathbb{C})$, so its unique irreducible real reps has dimension 2. The former gives rise to a self-adjoint operator and the latter to a skew-adjoint operator. (The other possibility is a Lorentz signature $g^2 = 1$, $e^2 = -1$ which somehow is related to Majorana spinors.)

So now I know that the $\nabla \bar{\nabla}$ operator on $L$ in the presence of a metric will be formally skew-adjoint. Now I want appropriate boundary conditions and these should be half way between requiring no condition at the boundary and requiring the section to vanish at the boundary. I have the
condition that the real part of the section vanish. Let's check the skew-symmetry of the form on the space of smooth sections satisfying this boundary condition. Thus I take a section \( s \) of \( L \) and I look at

\[
\int s \, \bar{\delta} s
\]

Now \( s \, \bar{\delta} s \) is a section of \( L \otimes L \otimes T^0,1 = T^{1,1} \)

and \( s^2 \) is a section of \( T^{1,0} \) so

\[
d\left(\frac{1}{2} s^2\right) = \bar{\delta}\left(\frac{1}{2} s^2\right) = \frac{1}{2} s \, \bar{\delta} s
\]

Thus

\[
\int_M s \, \bar{\delta} s = \int_M \frac{1}{2} s^2 \, \bar{\delta} s = 0
\]

Now there is something wrong because if \( s \) is purely imaginary along \( \delta M \), then \( s^2 \leq 0 \).

One problem is that we have been considering a complex valued skew-form \( \int \eta \, \bar{\delta} \xi \) and we need a real-valued form. Thus we ought to consider

\[
\text{Re} \int \eta \, \bar{\delta} \xi = \frac{1}{2} \int \eta \, \bar{\delta} \xi + \eta \, \bar{\delta} \xi
\]

and represent this in terms of the real part of the inner product.

Thus if we are dealing with the skew-form

\[
\text{Im} \int \eta \, \bar{\delta} \xi
\]

we see that

\[
\text{Im} \int_M s \, \bar{\delta} s = \text{Im} \int_M \frac{1}{2} s^2 = 0
\]
where $s$ is purely imaginary on the boundary. Thus the boundary conditions I have in mind seem to produce a skew-adjoint boundary value problem.

Let's compute in the plane. Typical spinor sections are $f(z) dz''$. Our basic form is

\[
\iint (g(z) dz'') \overline{\partial} (f(z) dz'') = \iint g \partial \frac{dz}{-2i} \overline{dz} \overline{d\overline{z}}
\]

Set $g = u' + iv'$, $f = u + iv$ real + imag parts.

\[
(-2i)(\partial \overline{z}) = -i(\partial_x + i \partial_y) = -i \partial_x + \partial_y
\]

\[
\text{Re} \iint g \partial \overline{z} f \overline{dz} d\overline{z} = \iint \text{Re} \{(u' + iv')(\partial_x + i \partial_y)(u + iv)\} \overline{dx} d\overline{dy}
\]

\[
= \iint [u'(\partial_y u + \partial_x v) + v'(\partial_y u - \partial_x v)] \overline{dx} d\overline{dy}
\]

\[
= \iint (u')^t \begin{pmatrix} \partial_y & -\partial_x \\ \partial_x & \partial_y \end{pmatrix} (v) \overline{dx} d\overline{dy}
\]

Thus the skew-adjoint operator is

\[
\begin{pmatrix} \partial_y & \partial_x \\ -\partial_x & -\partial_y \end{pmatrix} = \gamma' \partial_x + \varepsilon \partial_y
\]

Let's also compute the imaginary part

\[
\text{Im} (\iint g \partial \overline{z} f \overline{dz} d\overline{z}) = \iint \text{Im} \{ f \} \overline{dx} d\overline{dy}
\]

\[
= \iint u'(-\partial_x u + \partial_y v) + v'(-\partial_y u + \partial_x v) \overline{dx} d\overline{dy}
\]

So the operator is

\[
\begin{pmatrix} -\partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} = -\varepsilon \partial_x + \gamma' \partial_y
\]
Observe this calculation is metric independent and means that I can setup everything working with skew forms on $\Gamma_c(M,L)$. It might be useful to write
\[
\text{let } \int \int (gdz^{'1/2}) \delta(fdz^{'1/2}) = \int \int g^t (x^1 \partial_x + \varepsilon \partial_y) f \; dy \; dx
\]
when we treat $g,f$ as the two vectors of real functions. Call it $\Phi(g,f)$

Now we want to show this form is skew-symmetric when we impose the correct boundary condition. Thus we consider $\Phi(g,f) + \Phi(f,g)$.

But first write it
\[
\int \int g^t (x^1 \partial_x + \varepsilon \partial_y) f \; dy \; dx = \int \int g^t (x^1 d(f dy) + \varepsilon d(-f dx))
\]
\[
= \int \int g^t \{ -x^1 dy + \varepsilon dx \} \; df
\]

Then
\[
\Phi(f,g) = \int \int f^t \{ -x^1 dy + \varepsilon dx \} \; dg
\]
\[
= -\int \int dg^t \{ -x^1 dy + \varepsilon dx \} \; f
\]

So
\[
\Phi(g,f) + \Phi(f,g) = -\int \int d \{ g^t (-x^1 dy + \varepsilon dx) f \}
\]
\[
= - \int \int g^t (\varepsilon dx \# -x^1 dy) f
dM
\]

I should have done this with the imaginary part
\[
\Psi(g,f) = \text{Im} \int \int gdz^{'1/2} \delta(fdz^{'1/2}) = \int \int g^t (-\varepsilon \partial_x + x^1 \partial_y) f \; dy \; dx
\]
\[
= \int \int g^t (\varepsilon d(f dy) + x^1 dx df) = \int \int g^t (\varepsilon dy + x^1 dx) df
\]
\[ \mathbb{E}(f,g) = \iint f^t(\varepsilon dy + \varphi' dx) \, dg = \iint -g^t(\varepsilon dy + \varphi' dx) \, df \]

\[ \mathbb{E}(g,f) + \mathbb{E}(f,g) = -\iint d[g^t(\varepsilon dy + \varphi' dx) \, f] \]

\[ = -\int_{\partial M} g^t(\varepsilon dy + \varphi' dx) \, f \]

Now suppose \( f \) real on \( \partial M \), i.e. that

\[ (f^2)(dx'^2) = f(x^2) (dx + i dy) > 0. \]

To simplify suppose the boundary is the x axis oriented so that \( dx > 0 \). Then

\[ f^2 = (u + iv)^2 = u^2 - v^2 + 2iuv > 0 \]

so \( uv = 0 \) and \( u^2 > v^2 \implies v = 0 \). Thus if \( f, g \) are both real we have

\[ g^t(\varepsilon dy + \varphi' dx) \, f = \begin{pmatrix} u' \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \end{pmatrix} \, dx = 0. \]

What would be the general proof? You have a quadratic form given by the matrix \( \varepsilon b + \varphi' a \) where the tangent vector to the curve is \( dx + b dy \). Then you have the condition \( f^2(a + ib) > 0 \), and you have to see this defines an isotropic line for the form. The set of \( f \)'s satisfying this condition is a real line so we just have to check that the form vanishes on any point of the line

\[ (u \, v) \begin{pmatrix} \frac{1}{b} & \frac{a}{b} \\ \frac{a}{b} & -\frac{1}{b} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = b(u^2 - v^2) + 2auv \]

\[ \operatorname{Im}(u^2 - v^2 + 2iuv)(a + ib) = b(u^2 - v^2) + 2auv. \]

So it's clear.
Thus we have checked in an ugly way the following argument.

We consider the surface $M$ with boundary $\partial M$, and we consider the space of smooth sections of $L$ over $M$ which are real when restricted to $\partial M$. This is a real vector space on which we have the bilinear form

$$\mathcal{F}(g, f) = \text{Im} \int g \overline{f} \, dM$$

We claim this form is skew-symmetric. But

$$\mathcal{F}(f, f) = \text{Im} \int f \overline{f} \, dM = \text{Im} \int d\left(\frac{i}{2} f^2\right) \, dM$$

$$= \text{Im} \int \frac{1}{2} f^2 \, d\partial M$$

and for $f$ to be real on $\partial M$ means that $f^2$ is a non-negative 1-form on $\partial M$ relative to its orientation. Thus $\frac{1}{2} f^2$ is real so $\mathcal{F}(f, f) = 0$.

Now introduce a Riemannian metric on the surface consistent with its conformal structure. This puts a inner product on $\Gamma(M, L)$ considered as a real vector space and permits one to represent the above skew-symmetric form as a skew-adjoint differential operator on $L$; this is the Dirac operator. The boundary condition is a self-adjoint boundary condition. Since we have taken exactly half of the boundary values, it follows, or should be the case, that this operator + boundary condition is essentially self-adjoint.
Here I have to appeal to elliptic boundary value theory. This theory will give us a Fredholm operator. In fact since we have a skew-adjoint problem there will be a sequence of eigenvalues + eigenvectors. In particular there will be a Green's function provided there are no zero eigenvalues.

Suppose \( f \in \Gamma(M,L) \) has real boundary values and is killed by the operator, i.e.

\[
\mathcal{F}(g,f) = \text{Im} \int g \overline{\partial f} = 0
\]

for all \( g \in \Gamma(M,L) \) having real boundary values. Then taking \( g \) to have zero boundary values and using the complex structure one sees that \( \overline{\partial f} = 0 \), so \( f \) is holomorphic. But then as it is real on the bdry it has to be zero, assuming \( M \) connected.

Next I want to look at the homogeneous equation. The metric on the surface puts an inner product on \( \Gamma(L) \), which is hermitian. The hermitian inner product on \( L \) gives a conjugation -linear isomorphism \( L \cong L^* \) and one has a trivialization of \( T^{1,1} \) given by the volume form. Thus one has a conjugate linear isomorphism

\[
L \cong L^* = L^* \otimes T^{1,1}
\]

which allows us to pair two sections of \( L \), i.e. define the hermitian product \( \langle \eta, \eta \rangle \).

Similarly \( L \otimes T^{0,1} \cong L^* \otimes T^{1,0} = L^* \otimes L^{\otimes 2} = L \), is a conjugate linear isomorphism. Thus \( \overline{\partial} : \Gamma(L) \to \Gamma(L \otimes \mathcal{O}^1) \) can be viewed as going from \( \Gamma(L) \) to itself, but as a conjugate linear operator. Still not clear.

Let's go on and take \( \omega \in \Gamma(\Omega^2 M, L) \) which is imaginary and extend it smoothly inside.
Let \( h = \overline{\partial} \alpha \in \Gamma(M, L^{\otimes 0,1}) \). Because we can solve the inhomogeneous equation for any \( h \), there is an \( f \in \Gamma(M, L) \) with real boundary values such that \( \overline{\partial} f = h \). So taking \( h = \overline{\partial} \alpha \) we see that

\[
\overline{\partial} (\alpha - f) = 0.
\]

Thus \( \alpha - f \) is a holomorphic section of \( L \) such that on the boundary it has the same imaginary part as \( \alpha \).

This proves that the space \( W \) of boundary values of holomorphic sections of \( L \) over \( M \) is such that \( W \oplus \overline{W} = \Gamma(\partial M, L) \).

Spin fields. Let's take an analytic function with a quadratic branch point say

\[
(\bar{z} - 1)^{-1/2}
\]

and draw the cut from 1 to \( \infty \) say. Then for \( |z| < 1 \) this function is periodic and for \( |z| > 1 \) it is anti-periodic. In fact we can easily work out the Laurent series in these regions. For \( 0 \leq |z| < 1 \)

\[
(\bar{z} - 1)^{-1/2} = (-1)^{-1/2} (1 - z)^{-1/2} = (-1)^{-1/2} \sum_{n>0} \frac{(1/2)(3/2) \ldots (2n-1/2)}{n!} z^n
\]

and for \( |z| > 1 \) one has
\[(z-1)^{-1/2} = \sum_{n=0}^{\infty} \frac{(1/2) \cdots (2n-1/2)}{n!} z^{-n-1/2}\]

Now recall that a meromorphic function gives rise to operators consistent with time evolution except at the poles where the operators change by something like \(\Psi(z)\). In the same spirit we recognize the two series as giving operators on the NS Fock space and the R Fock space respectively compatible with time translation. What happens at the singularity is an operator going from the NS space to the R space. This has to be the spin field associated to the point 1.

Ultimately we have to take the spinors over two circles before an after the singularity and construct a suitable space of boundary values of holomorphic spinors in between. The problem is then to concoct a bundle in between.

One idea is to introduce the branched covering which has a \(2/2\)-Galois group.
March 22, 1987 (Carl is 22)

Ramond case: It seems we have to correct our earlier impression about there being an irreducible representation of the Clifford algebra. There are two, probably related to some K-theory invariant such as the index mod 2 of the Dirac operator. Let's try to understand this.

Actually I should probably go over what I learned about spin fields. The idea is to replace a spin structure $\Lambda \otimes L = \Omega^1$ with a line bundle $L$ such that $L \otimes L = \Omega^1(p)$. Any circle not passing through $p$, then has a real structure on the sections of $L$ over $p$ that is, a real structure on the sections of $L$ over $p$ for any circle, but as the circle crosses $p$ the AP or P condition change. If we have an annulus with conditions change. If we have an annulus with conditions change, then the holomorphic sections of $L$ over the annulus should define some sort of operator associated to the point $p$.

Let's look at the simple case when we have the circle bounding the disk and $p$ inside. I'm going to try to find the state in the Fock space of the circle which is the spin field applied to the ground state. (Except as we have said there are two Fock spaces apparently.)

On the disk or plane $dz$ trivializes $\Omega^1$. Let $f \in \mathcal{C}$ and let $L \otimes L = \Omega^1(f)$. Then $\frac{dz}{z - f}$ is a non-vanishing holomorphic section of $\Omega^1(f)$ over $\mathcal{C}$ and so there is a unique holomorphic section $(\frac{dz}{z - f})^{1/2}$ of $L$ and any other section over $\mathcal{C}$ or the disk $|z| < 1$ can be written

$$f(z) \left(\frac{dz}{z - f}\right)^{1/2}$$

with $f(z)$ holomorphic.

Let's take $f = 0$ and look at $|z| = 1$. Then
we have over the circle the space of
\[ f(z) \left( \frac{dz}{z} \right)^{1/2} f(z) = \sum c_n z^n \]
with the quadratic form
\[ \frac{1}{2\pi i} \int f(z)^2 \frac{dz}{z} = \sum c_n c_{-n} \]
Thus we are dealing with periodic half-densities on the circle. This is the Remond case.

Fock space is then a representation of the Clifford algebra with defining relations

\[ \{ \psi_k, \psi_l \} = \delta_{k,-l} \]
\[ \psi^*_k = \psi_{-k} \]

If we use the Dirac operator \( \frac{1}{i} \partial_x \) on the half densities on the circle, then the ground state in Fock space has to satisfy

\[ \psi_k |0> \quad \text{for} \quad k < 0. \]

Then there are two possibilities

\[ \psi_0 |0> = \pm |0> \]

leading to two Fock spaces.

A natural problem is to show these two Fock spaces are inequivalent representations of the Clifford algebra. The situation should be analogous to odd degree Clifford algebra; here one distinguishes between the irreducible representations by means of the central element \( 1 \cdot z^{2n+1} \)

Recall that we proposed to define the polarization in the space of spinors on the circle by means of any Dirac operator. In the real setting however there is a mod 2 index namely the dimension of \( \ker D \) mod 2. Thus one will not be able to
find an invertible real skew-adjoint 
Dirac in the Ramond case.

I guess we know that states on the Clifford
algebra of a real Hilbert space E are determined
by skew-adjoint operators K such that -K^2 = 1.

Polarizations are complex structures i.e. \( J = -J^* \)

Atiyah + Singer
considered skew-adjoint Fredholm operators; this space
is h. equiv. to those K such that \( K^2 = -1 \) mod compacts.
The latter is h. equiv. via \( K \mapsto \exp(\pi K) \) to the
orthogonal gp \( \equiv 1 \) mod compacts, so has the homotopy
type 0. Thus \( \pi_0(\text{skew adjoint Fred}) = \mathbb{Z}/2 \).

Let's return to our line bundle L such that
\( L \otimes L = \Omega^1(\mathcal{F}) \) where \( |\mathcal{F}| < 1 \). The space \( W \) of
holomorphic sections of \( L \) over the disk \( |z| \leq 1 \)
consists of
\[
\frac{f(z)}{\left( \frac{dz}{z-\mathcal{F}} \right)^{1/2}}
\]

with \( f \) holomorphic on the disk. We want
to use \( W \) to define states on the Clifford
algebra of spinors on the circle.

The quadratic form on \( W \) is
\[
\frac{1}{2\pi i} \int f(z)^2 \frac{dz}{z-\mathcal{F}} = f(\mathcal{F})^2
\]

So \( W \) is isotropic, \( W \) contains a unique
maximal isotropic subspace, namely those \( f(z) \left( \frac{dz}{z-\mathcal{F}} \right)^{1/2} \)

with \( f(\mathcal{F}) = 0 \).
Let's review what we've learned about spin fields. Fix an annulus $a < |z| < b$ and a point $j$ in the interior. We suppose $L$ to be a holomorphic line bundle with $L \otimes L \cong \Omega^1(j)$ over the annulus. Now $\Omega^1(j)$ has the non-vanishing section $\frac{dz}{z-j}$ so there are two choices for $L$.

Let's look at the one such that if $(z-j)^{1/2}$ is made single-valued by a cut radially outward from $j$, then $L$ is spanned by $\frac{(dz)^{1/2}}{(z-j)^{1/2}}$.

This isn't very clear.

Let's begin again. Over $C$, $\Omega^1$ is trivialized by $dz$. Let $(z-j)^{1/2}$ be made single-valued via a cut outward from $j$. Let $L$ be the holomorphic line bundle of $C$ together with antiholomorph.

\[ L \otimes L = \Omega^1(-j) \]

whose sections are of the form

\[ f(z) \frac{(z-j)^{1/2}}{dz^{1/2}} \]

with $f$ holomorphic.

If $L$ is restricted to $|z| = a$ where $a < |j|$, then the sections of $L$ over this circle can be identified with AP half densities on the circle, whereas if $a > |j|$ the sections are P half densities.

Let us call these spaces of sections over $|z| = a$, $b$ resp. $\Gamma_a$, $\Gamma_b$. Let $W$ be the space of holomorphic sections of $L$ over the annulus. Then

\[ W \subset \Gamma_a \times \Gamma_b \]

in a natural way. We equip $\Gamma_a \times \Gamma_b$ with the
The quadratic form $\text{res}_a(f^2)$ where $\text{res}_a$ stands for $\frac{1}{2\pi i} \oint_{\gamma_a} f$. Given $f(z)(z-i)^{1/2} dz$ in $W$, then $\left(\frac{f(z)(z-i)^{1/2} dz}{2\pi i}\right)^2 = f(z)^2 (z-i) dz$ is a holomorphic section on the annulus $A W$ so $W$ is isotropic for this quadratic form. It is probably maximal isotropic. Let's assume this, whence we obtain a Gaussian state in the Clifford algebra $\Gamma_a \otimes \overline{\Gamma}_b$.

It seems likely that this Gaussian state is equivalent to giving lines in $\mathbb{R}^4$.

$$\mathbb{R}^4 \cong \mathbb{R}^4$$

Here we know that the Clifford algebra of $\Gamma_a$ has a canonical irreducible rep $\mathbb{F}_a^+$, whereas the Clifford algebra of $\overline{\Gamma}_b$ has two irreducible repns, $\overline{\mathbb{F}}_b^+$.

Thus I expect there to be operators

$$S_+(j): \mathbb{F}_a^+ \rightarrow \overline{\mathbb{F}}_b^+$$

defined at least up to scalars. These are the spin fields.

But an important idea that has been ignored is that these spin fields are related to the analytic continuation map from $\Gamma_a$ to $\overline{\Gamma}_b$. $W$ should be viewed as this analytic continuation map, and the fact that $W$ is isotropic means that the map preserves the quadratic form, hence induces a "map" on Clifford algebras.

Practically what this means is that if $f \in W$...
Then we have

\[ S_{\pm}(\mathcal{J}) \mathcal{Y}(\mathcal{f}_0) = \mathcal{Y}(\mathcal{f}_0) S_{\pm}(\mathcal{J}) \]

March 23, 1987

Giuseppi has preferred to think of \( \mathcal{Y}(\mathcal{f}) \) as a
subspace of spinors with simple pole at \( \mathcal{Z} \).
Now certainly the operator \( \mathcal{Y}(\mathcal{f}) \) can be identified
with a dual vector on \( F_0 \otimes F_{\mathcal{J}} \). Is it necessary to check that this dual vector corresponds to a maximal isotropic subspace? But the dual vector is obtained by applying to \( \omega | \mathcal{O} \) a Clifford operator \( \mathcal{Y}(\mathcal{f}_0) - \mathcal{Y}(\mathcal{f}_{\mathcal{J}}) \), so we should check that Clifford multiplication preserves the lines corresponding
to isotropic subspaces.

Let \( V = W \oplus W' \) be a decomposition into
maximal isotropic subspaces, and let \( |w' \rangle \) be
a vector in Fock space spanning the line corresponding
to \( W' \). If \( \psi \in V \), write \( \psi = \omega + w' \). Then

\[ \mathcal{Y}(\psi)|w' \rangle = \mathcal{Y}(\omega)|w' \rangle \]

Now assume \( \omega \neq 0 \), then there is an obvious maximal isotropic subspace, namely

\[ C\omega \oplus \text{annihilator of } \omega \text{ in } W' \]

This isotropic subspace clearly annihilates \( \mathcal{Y}(\omega)|w' \rangle \)

In the odd case we have

\[ V = W_0 \oplus (W \oplus W') \]

where \( W_0 \) is \( \perp \) to \( W \oplus W' \) and \( W_0 \cdot W_0 = 1 \). There are two Fock spaces; each is generated by a vector \( |w' \rangle \) killed by \( \mathcal{Y}(W) \) and the two possibilities are

\[ \mathcal{Y}(W_0)|w' \rangle = \pm |w' \rangle \]
Now given \( v \in V \) we write it

\[ v = c v_0 + \omega + \omega' \]

and we have (taking the + case)

\[ \psi(\omega)|w'\rangle = c|w'\rangle + \psi(\omega)|w'\rangle \]

This vector is killed by \( \text{annihilator} \) of \( \omega \) on \( W' \), that is by \( (\omega)^\dagger \) on \( W' \). This is not maximal isotropic and the question is whether an extra element can be added to it. Suppose \( W, W' \) are one dimensional, let \( \omega, \omega' \) span these spaces with \( \omega \cdot \omega' = 1 \). Then look for \( \alpha, \beta, \gamma \)

\[ \psi(\alpha v_0 + \beta \omega + \gamma \omega')(c|w'\rangle + \psi(\omega)|w'\rangle) = 0 \]

\[ \alpha c|w'\rangle - \alpha \psi(\omega)|w'\rangle + \beta c \psi(\omega)|w'\rangle + \gamma (\omega \cdot \omega')|w'\rangle = 0 \]

\[ \alpha = \beta c, \quad \alpha c + \gamma (\omega \cdot \omega') = 0. \]

So there is no problem.
Let $V$ be a vector space (over $\mathbb{C}$) with non-degenerate quadratic form, and let $C(V)$ be its Clifford algebra. The orthogonal group $O(V)$ acts on $C(V)$ and can be identified with the automorphism group of $C(V)$ preserving $V$.

If $V$ is even dimensional, the Clifford algebra is simple and so any automorphism is inner. Thus if we put

$$(*) \quad G(V) = \{ g \in C(V)^* \mid gVg^{-1} = V \}$$

we have an exact sequence of groups:

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G(V) \longrightarrow O(V) \longrightarrow 1$$

$G(V)$ is the Clifford group.

Let $v \in V$ be such that $v^2 \neq 0$. Then $C_0v^\perp$ is a hyperplane in $V$ complementary to $C_0v$. One has

$$v \times v^{-1} = \begin{cases} x & \text{if } x \in C_0v \\ -x & \text{if } x \in C_0v^\perp \end{cases}$$

so $v \in G(V)$ and the resulting orthogonal transformation $-1$ acts on the central element $g_1 \cdots g_{2m+1}$, and so it is not an inner automorphism. Thus we proceed differently to define the Clifford group. Let $G(V)$ be defined by $(*)$. This time we have an exact sequence

$$1 \longrightarrow (\mathbb{C} \times \mathbb{C})^* \longrightarrow G(V) \longrightarrow SO(V) \longrightarrow 1$$
Check $G(V)$ maps into $SO(V)$: First notice that $O(V) = SO(V) \times \{\pm 1\}$ and the orthogonal transform $v \rightarrow -v$ gives the $\mathbb{Z}_2$ grading on $C(V)$. The Clifford algebra is a direct product of two matrix algebras which are the ideals generated by the two central idempotents obtained from $g_1 \ldots g_m$. Any automorphism acting trivially on the center will be inner, hence the automorphisms coming from elements of $SO(V)$ are all inner.

Notice that in the even case the group $G(V)$ splits into two cosets according to the determinant of the associated orthogonal transformation. The identity is the subgroup of even elements. The other coset contains the unit vectors of $V$. Thus

$$G(V) = [G(V) \cap C^e(V)] \sqcup [G(V) \cap C^{\text{odd}}(V)]$$

Also $G(V)$ appears to be generated by the unit vectors of $V$ and the scalars. This is because $O(V)$ is generated by reflections associated to unit vectors.

In the odd case I think the Clifford group can be defined similarly as the subgroup of $C(V)$ generated by unit vectors and scalars, and it will be the union of the two cosets consisting of even and odd elements.

Definition of the Clifford group from ABS. Let $\alpha = \pm 1$ on $C^e(V)$. Then

$$\Gamma(V) = \{ g \in C(V) \mid \alpha(g) V g^{-1} = V^2 \}$$
Then \( \Gamma(V) \) is clearly a subgroup of \( C(V)^* \). Examples of elements of \( \Gamma(V) \) are elements \( v \in V \) with \( v^2 \neq 0 \).

I want to prove now that \( \Gamma(V) \) is generated by these elements. In particular

\[
\Gamma(V) = \Gamma^+(V) \cup \Gamma^-(V)
\]

where

\[
\Gamma^\pm(V) = \Gamma(V) \cap C^\pm(V).
\]

Let \( g \in \Gamma(V) \) and write \( g = g^+ + g^- \).

Then for any \( v \in V \) we have \( T v \in V \Rightarrow (g^+ - g^-) v = (T v) (g^+ + g^-) \)

so separating into homogeneous components

\[
g^+ v^- = (T v) g^+ \]

\[
g^- v^- = - (T v) g^-.
\]

One of \( g^+, g^- \) is \( \neq 0 \). From

\[
g^- (v^2) = g^- v v^- = -(T v) g^- v^- = (T v)^2 g^-
\]

and similarly \( g^+ v^2 = (T v)^2 g^+ \) we see that \( T \) is orthogonal. Thus we have a homomorphism

\[
\Gamma(V) \to O(V)
\]

by associating to \( g \) the transform \( T \) such that

\[
T v = \alpha(g) v g^{-1}.
\]

Next note that if \( w^2 \neq 0 \), then

\[
\alpha(w) v w^{-1} = -w v w^{-1} = \begin{cases} -1 & \text{on } C w^- \\ +1 & \text{on } C w^+ \end{cases}
\]

is the reflection through the hyperplane \( \perp \) to \( C w \).

These generate the orthogonal group \( O(V) \).
If we want to show that $\Gamma(V)$ is generated by the elements $\omega$ with $\omega^2 \neq 0$, it is enough therefore to any $g$ inducing $T=1$ lies in this subgroup. But

$$g^+ v = v g^+ \quad \text{for all } v \Rightarrow g^+ \in \text{center of } C(V).$$

In the even case the center reduces to the scalars; in the odd case it is $\mathbb{C} \oplus \mathbb{C} v_1 ... v_{2m+1}$ and so its even part is again $\mathbb{C}$. Thus $g^+$ is a scalar.

Now $g^- v = -v g^-$.

If we write $g^+$ out in terms of the basis of monomials $\gamma_I$ and take $v$ to be one of the $\gamma_I$'s, we see that the only possibility for $g^-$ is a scalar times $\gamma_1 ... \gamma_{2m}$ in the even case. (The point is that $\gamma_I \gamma_J = \begin{cases} (-1)^{|I\cap J|} \gamma_I & \text{if } J \not\subseteq I \\ (-1)^{|J\cap I|} \gamma_I & \text{if } I \not\subseteq J \end{cases}$)

$\nu v = -v \nu$ for all $v$, then only $I = \emptyset$, $\{1,...,n\}$ can appear.

But $g^-$ is odd so $g^- = 0$.

Thus $g = g^+$ is a scalar and one is done.

Moral: The way to think of the Clifford group $\Gamma(V)$ is as the group generated by the $v \in V$ such that $v^2 \neq 0$, in the same way that $O(V)$ is generated by reflections.
March 26, 1987

I would like to describe in general terms the structure of the arguments used in the Ising model. The Ising model is a classical statistical mechanical system; it is described by a probability measure on a space of configurations. The configurations are 2-dimensional fields which take the values $\pm 1$ on lattice points in the plane.

Because of the nearest-neighbor type of the interaction the model can also be described in quantum-mechanical terms. (Here quantum-mechanical means something non-commutative.) In the quantum description the two axes of the lattice are viewed as space and (imaginary) time. There is a vector space spanned by the configurations on a row; this is the tensor product of 2-dimensional spin spaces for each site in the row. The observables are operators on this space. There is a transfer matrix giving (imaginary) time evolution.

The link between the classical description and the 1D quantum description is analogous to the usual link discovered by Feynman between path integrals over space time (the Lagrangian approach) and operators in Hilbert space (the Hamiltonian approach).

Thus we have the Ising model described in terms of a vector space $\otimes \mathbb{C}^2$, and an algebra of observables $\otimes M_2(\mathbb{C})$ which contains the field operators $s(n)$ for each site $n$ as well as the transfer matrix. If we pass to the infinite
lattice, then we forget the vector space and work just with the algebra of observables which is

$$\mathcal{O} \otimes M_2(\mathbb{C}) \mathcal{M}_2.$$ 

The reason we can solve the Ising model at zero magnetic field is because the algebra of observables can be identified with a Clifford algebra in such a way that the field operators and transfer matrix become elements of the Clifford group (in fact maybe even the spinor group). Actually one perhaps only has the identification over the even parts of the algebras.

The key point then in the model is the fact that it admits this fermion description. It should be possible to understand this transformation not only in the Hamiltonian picture but also in the path integral picture. Thus the partition function is a sum of $2^N$ terms where $N$ is the number of lattice sites. A fermion integral in $N$ Grassmann variables also is a sum of $2^N$ terms. It should be possible to introduce signs in such a way that the Ising partition function becomes a fermion integral, hopefully Gaussian.

In order to proceed further I have to come to terms with this so-called Jordan-Wigner transformation. (Are there other examples of these, I just wonder what the name includes?)
March 27, 1987

The Ising model. I want to ignore boundary conditions to begin with; I could probably fix this up by using boundary conditions.

Configurations for the Ising model are spin assignments $\sigma = (\sigma_m)$, where $\sigma_m = \pm 1$. The energy is

$$-E(\sigma) = J_1 \sum \sigma_m \sigma_{m+1} + J_2 \sum \sigma_m \sigma_{m+1}$$

Think of $\sigma$ as the collection of its rows $\sigma = (\sigma^m)$, where $\sigma^m = (\sigma_m)$. Then

$$-\beta E(\sigma) = \sum \sigma^n f(\sigma^n) + \sum \sigma_{m,n} g(\sigma_{m,n}, \sigma_{m+1,n})$$

This means that the partition function has the form

$$Z = \sum_{\sigma} e^{-\beta E(\sigma)}$$

$$= \sum \prod (e^{\sigma^m} e^{g(\sigma_{m,n}, \sigma_{m+1,n})})$$

Thus the partition function is the trace, or a suitable expectation, of powers of the transfer matrix

$$T(\sigma, \sigma') = e^{f(\sigma)} e^{g(\sigma, \sigma')}$$

This operates on the vector space having as basis the set of row configurations $\sigma = (\sigma_m)$. It is a product of two matrices representing rowcouplings and the other column couplings.
What is the space in which the transfer matrix acts? It has as basis the set of row-
spin assignments \((\sigma_m)\) so is naturally \(\otimes_m \mathbb{C}^2\).

Look at the parts of the transfer matrix on this space. As \(g(\sigma, \sigma') = \beta J_2 \sum_m \sigma_m \sigma'_m\) the matrix \(g(\sigma, \sigma')\)
is the tensor product over \(m\) of the operator which in the

with \(\mathbb{C}^2\) is

\[
\begin{pmatrix}
  e^{\beta J_2} & e^{-\beta J_2} \\
  e^{-\beta J_2} & e^{\beta J_2}
\end{pmatrix}
= e^{\beta J_2} \exp \begin{pmatrix} 0 & -K \\ -K & 0 \end{pmatrix}
\]

for a suitable constant \(K = K(\beta J_2)\). Thus

\[e^g(\sigma, \sigma') = (\text{const}) \ e^{-K \sum_m \sigma_x(m)}\]

where

\[\sigma_x(m) = \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \]

Now the other factor of \(T\), namely \(e^{f(\sigma)}\), is

diagonal \hspace{1cm} in the natural basis, that is it multi-
plication by a function on the rows sites. There \(\sigma\) is

the function \(\sigma \mapsto \sigma_m\) giving the spin at site \(m\)

and this gives the operator

\[\sigma_z(m) = \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \]

Then we clearly have

\[e^{f(\sigma)} = e^{\sum_m \beta \sigma_z(m) \sigma_z(m+1)}\]

Thus we see the transfer matrix is built

up out of the operators \(\sigma_x(m)\) which mutually

commute, and the \(\sigma_z(m)\sigma_z(m+1)\) which also mutually

commute (recall they are diagonal).

Now the strategy \(\hspace{1cm} \) is \textit{to construct fermions}, better
to construct a vector space $V$ of operators such that squaring any element of $V$ gives a scalar. One wants the operators $\sigma_x(m), \sigma_y(m), \sigma_z(m+1)$ to be quadratic in the elements of $V$.

March 28, 1987:

Fermionization in the Ising model. We saw above that the Ising model can be viewed as a 1-dim quantum spin system. Assume we work on the lattice where the sites are $m = 1, \ldots, N$. Later we will discuss boundary conditions. Then on the tensor product of the 2d spin spaces at each site $s^\otimes N = s^\otimes \cdots \otimes s$

we have the operators

$\sigma^x(m) = 1 \otimes \cdots \otimes \sigma^x \otimes \cdots \otimes 1$

and similarly $\sigma^z(m)$. The operators on $s^\otimes N$ form the algebra

$$\text{End}(s^\otimes N) = \bigotimes_{m} \text{End}(s)$$

If we use a $Z_2$ grading in $s$, then we also have

$$\text{End}(s^\otimes N) = \text{End}(s) \hat{\otimes} \cdots \hat{\otimes} \text{End}(s)$$

and the latter is isomorphic to $C_2 \hat{\otimes} \cdots \hat{\otimes} C_2 = C_{2N}$. This is where the fermions come from.

We want to arrange things so that the transfer matrix ends up in the spinor group. We saw that the transfer matrix is the product of two operators which are exponentials of the operators

$$\sum_{m=1}^{N} \sigma^x(m) \quad \sum_{m=1}^{N} \sigma^z(m) \sigma^z(m+1)$$
where $\sigma^2(m+1) = \sigma^2(1)$. We want these operators to be in the Lie algebra $\mathfrak{so}(2N)$, and so they must be even. Thus we use $\sigma^x$ to grade $S$ which means we use the representation

$$\sigma^x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^z = \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

The raising and lowering operators are

$$\sigma^\pm = \begin{pmatrix} 1 \pm \gamma_1 \\ \gamma_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & - \end{cases} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Now that the grading has been introduced we define creation and annihilation operators

$$c^\pm(m) = 1 \otimes \cdots \otimes \sigma^\pm \otimes \cdots 1_m$$

$$= \varepsilon \otimes \cdots \otimes \sigma^\pm \otimes \cdots 1.$$

What we are doing is to define a $2N$ subspace $V$ of operators on $S \otimes N$ such that $\sigma^2$ is a scalar form any $\sigma \in V$ and $C(V) \rightarrow \text{End}(S \otimes N)$. We want this representation to be such that the operators $\sum \sigma^x(m)$, $\sum \sigma^2(m) \sigma^2(m+1)$ are in $\mathfrak{so}(2N)$. Now

$$\sigma^x(m) = 1 \otimes \cdots \otimes \varepsilon \otimes 1 \otimes \cdots = 1 \otimes 1 \otimes \cdots \otimes \varepsilon \otimes \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \right.$$ 

and

$$\varepsilon = [\sigma^+, \sigma^-], \text{ so that}$$

$$\sigma^x(m) = \left[ \begin{pmatrix} c^+(m) \right] \begin{pmatrix} c^-(m) \right], \text{ lies in } \mathfrak{so}(2N).$$
Next
\[ \sigma^2(m) \sigma^2(m+1) = I^{(m-1)} \otimes (\sigma^+ \sigma^-) \otimes (\sigma^+ \sigma^-) \otimes I \otimes \cdots \]
\[ = I^{(m-1)} \otimes [\sigma^+ \otimes \sigma^+ + \sigma^+ \otimes \sigma^- + \sigma^- \otimes \sigma^+ + \sigma^- \otimes \sigma^-] \otimes I \otimes \cdots \]

Let's simplify by taking \( m = 1, N = 2 \) first.
\[ c^+ (1) c^+ (2) = (\sigma^+ \otimes 1)(1 \otimes \sigma^+) = \sigma^+ \sigma \otimes \sigma^+ \]
\[ = -\sigma^+ \otimes \sigma^+ \]

In general for \( 1 \leq m < N \)
\[ c^a (m) c^b (m+1) = \bigotimes_{i=1}^{m-1} (\sigma^a \otimes \sigma^b) \otimes (\sigma^a \otimes \sigma^b) \]
\[ = (-a)^{a+b} \sigma^a \otimes \sigma^b \]
\[ = (-a)^{a+b} \sigma^a (m) \sigma^b (m+1) \]

where \( a, b = \pm 1 \) and \( \sigma^a (m) = \bigotimes_{i=m+1}^{m+k} \).

Thus
\[ \sigma^2 (m) \sigma^2 (m+1) = \begin{pmatrix} \sigma^+ (m) \sigma^+ (m+1) & -c^+ (m) c^+ (m+1) \\ \sigma^+ (m) \sigma^- (m+1) & -c^+ (m) c^- (m+1) \\ \sigma^- (m) \sigma^+ (m+1) & c^- (m) c^+ (m+1) \\ \sigma^- (m) \sigma^- (m+1) & c^- (m) c^- (m+1) \end{pmatrix} \]
\[ = - (c^+ (m) - c^- (m)) (c^+ (m+1) + c^- (m+1)) \]

So far this works for \( 1 \leq m < N \). We next look at \( m = N \).

It seems \( \epsilon (m) \), \( \sigma^x (m) \), and \( \sigma^2 (m) \) are preferable to avoid the tensor notation and to simply define the operators.
creation and annihilation operators are

\[ \sigma^+(m) = \varepsilon(1) \cdots \varepsilon(m-1) \sigma^+(m) \]

Then

\[ \sigma^+(m) \sigma^+(m+1) = \varepsilon(1) \cdots \varepsilon(m-1)(\sigma^+(m) + \sigma^-(m)) \]

\[ \times \varepsilon(1) \cdots \varepsilon(m) \left( \sigma^+(m+1) + \sigma^-(m+1) \right) \]

\[ = (-\sigma^+(m) + \sigma^-(m))(\sigma^+(m+1) + \sigma^-(m+1)) \]

Now let us look at the case \( m = N \).

\[ \sigma^+(N) \sigma^+(1) = \varepsilon(1) \cdots \varepsilon(N-1)(\sigma^+(N) + \sigma^-(N))(\sigma^+(1) + \sigma^-(1)) \]

\[ = \varepsilon(1) \cdots \varepsilon(N)(\sigma^+(N) - \sigma^-(N))(\sigma^+(1) + \sigma^-(1)) \]

Notice that \( \varepsilon(1) \cdots \varepsilon(N) \) gives the total grading mod 2 and it commutes with both \( \varepsilon(m) \) and \( \sigma^+(m) \).

If we restrict to \((S \otimes \cdots \otimes S)^{\oplus} \), then we see that

\[ \sigma^+(N) \sigma^+(1) = -(\sigma^+(N) - \sigma^-(N))(\sigma^+(N+1) + \sigma^-(N+1)) \]

provided \( \sigma^+(N+1) = \sigma^+(1) \). Thus in \((S \otimes \cdots \otimes S)^{-}\)

the operator \[ \sum \sigma^+(m) \sigma^+(m+1) \]

coincides with the operator

\[ -\sum (\sigma^+(m) - \sigma^-(m))(\sigma^+(m+1) + \sigma^-(m+1)) \]

where \( \sigma^+(N+1) = \sigma^+(1) \). Thus the transfer matrix will

be a product of operators \( e^x \) where \( x \in SO(2N) \) is cyclically

invariant. Thus think of \( x \) as an skew-symmetric transformation of \( \mathbb{C}^n \oplus (\mathbb{C}^n)^{\ast} \), and it

commutes with the cyclic transformation.

On the other hand if we restrict to \((S \otimes \cdots \otimes S)^{\ast} \)

then \[ \sigma^+(N) \sigma^+(1) = -(\sigma^+(N) - \sigma^-(N))(\sigma^+(N+1) + \sigma^-(N+1)) \]

provided we define \( \sigma^+(N+1) = -\sigma^+(1) \). Thus the

transfer matrix in \((S \otimes \cdots \otimes S)^{\ast} \) will be
A product of $e^x$ where is invariant under a different cyclic automorphism.

Here is a way this might be understood. Let $S$ be two-dual spinor space graded by $\sigma$ and let us form the $N$-fold tensor product:

$$S^{\otimes N} = S \otimes \cdots \otimes S.$$  

Now there are two actions of $\Sigma_N$ on this tensor product depending on whether we view the tensor product in the category of vector spaces or in the category of super vector spaces.

Let us consider cyclic permutation (forward shift) in the super sense. Then on $(S^{\otimes N})^+$ this coincides with the usual cyclic permutation:

$$C_{(S)} (\sigma_1 \otimes \cdots \otimes \sigma_N) = (-1)^{\sum \deg \sigma_i} (\sigma_{N+1} \otimes \cdots \otimes \sigma_{N-1}) \otimes \sigma_1 \otimes \cdots \otimes \sigma_N$$

and if the total degree $\sum \deg \sigma_i$ is odd, then one of the factors $(\deg \sigma_N)$ or $\sum \deg \sigma_i$ is even.

On the even space $(S^{\otimes N})^+$ then the super cyclic permutation $C_{(S)}$ has no simple relation to the cyclic one $C_{(S)}$.

Now let's go back to operators on $S^{\otimes N}$. Using the super formalism, we can identify:

$$\text{End}(S^{\otimes N}) = \text{End}(S) \otimes \cdots \otimes \text{End}(S).$$

In the latter we can consider

$$V = \sum_{m=1}^N 1 \otimes \cdots \otimes \text{End}^{-m}(S) \otimes 1 \otimes \cdots \otimes 1.$$
which is the space of creation and annihilation operators. This defines the intrinsically and shows it is invariant under the super cyclic permutation conjugation.

To handle \( (S^*N)^+ \), let us consider the automorphism \( \epsilon \) which is the super cyclic permutation following by \( \epsilon(1) \).

\[
\begin{align*}
\epsilon_1 \otimes \cdots \otimes \epsilon_N & \rightarrow (-1)^{\left| \nu_N \right|} \left( \sum_{i=1}^{N-1} |\nu_i| \right) \epsilon_N \otimes \epsilon_1 \otimes \cdots \otimes \epsilon_{N-1} \\
& \rightarrow (-1)^{\left| \nu_N \right|} \left( \sum_{i=1}^{N-1} |\nu_i| \right) + \left| \nu_N \right| \epsilon_N \otimes \epsilon_1 \otimes \cdots \otimes \epsilon_{N-1}
\end{align*}
\]

But \( \left| \nu_N \right| \sum_{i=1}^{N-1} |\nu_i| + \left| \nu_N \right| = \left| \nu_N \right| \sum_{i=1}^{N} |\nu_i| = 0 \) in \( \mathbb{Z}/2\mathbb{Z} \) when \( \epsilon_1 \otimes \cdots \otimes \epsilon_N \in (S^*N)^+ \). Thus

\[
\epsilon(1) C_{(S)} = C \quad \text{on} \quad (S^*N)^+
\]

\[
C_{(S)} = C \quad \text{on} \quad (S^*N)^-
\]

which explains to some extent the trick above of how to define \( e^{\pm (N+1)} \).
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Review: Regarding the Ising model as a quantum system on a 1-disk lattice we obtain a transfer matrix acting on the vector space $H = S^\otimes N$. Working in the supercategory the operators have a Clifford algebra structure with the generating space

$$V = \sum 1 \otimes \cdots \otimes \text{End}^-(S) \hat{\otimes} \cdots \hat{\otimes} 1 < \hat{\otimes}^N \text{End}(S).$$

The transfer matrix doesn't belong to the spinor group. However if we restrict to $H^+$ or to $H^-$, then the restriction is given by an element of the spinor group.

To be more precise the transfer matrix is a product of Grassmannians with exponents $\varepsilon(m)$, $\sigma^2(m)\sigma^2(m+1)$, for $m = 1, \ldots, N$ where $\sigma^2(N+1) = \sigma^2(1)$. All of these are quadratic in the creation and annihilation operators except

$$\sigma^2(N) \sigma^2(0 \cdot 1) = \varepsilon(1) \cdots \varepsilon(N) \left[ 2(N)(c^+(N) + c^-(N))(c^+(N)+c^-(N)) \right]$$

which is quadratic, call this $g$.

Thus the transfer matrix is an element of the spinor group $\propto$ times $e^+g$ on $H^+$, and times $e^-g$ on $H^-$.

Another way of saying this is that we have identified the state space of the Ising model with the sum of the two irreducible spinor representations of $\text{Spin}(N)$. The transfer matrix on $H^+$ is given by different elements of the spinor group.

So far we haven't gotten to the heart of the periodicity versus anti-periodicity.
Let's now consider the scaling limit for the Ising model. The idea will be to subdivide the lattice $d$ times. Recall that the transfer matrix from one row to the next is a product

\[ T = e^{\frac{1}{\beta J_1} \sum \sigma^2(m) \sigma^2(m+1)} + K \sum \sigma^x(m) \]

up to the constant factor $\delta^N$ where

\[
\begin{pmatrix}
  e^{\beta J_2} & e^{-\beta J_2} \\
  e^{-\beta J_2} & e^{\beta J_2}
\end{pmatrix} = \left( e^{2\beta J_2} - e^{-2\beta J_2} \right)^{1/2} \exp\left[ +K \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]
\]

\[
\begin{pmatrix}
  \cosh K + \sinh K \\
  \cosh K - \sinh K
\end{pmatrix}
\]

so that $+\tanh(K) = e^{-2\beta J_2}$.

When we subdivide $d$ times, then we obtain a transfer matrix

\[ T(d) \propto \sum \sigma^2(m) \sigma^2(m+1/d) + K(d) \sum \sigma^x(m) \]

and the transfer matrix in time $t$ is

\[ (T_d)^t \to \left( \lim_d (T_d) \right)^t \]

Now recall that if $\alpha(d) \sim \frac{a}{d}$ and $\beta(d) \sim \frac{b}{d}$ then

\[ (e^{\alpha(d)} e^{\beta(d)})^t \to e^{a+b}. \]

The transfer matrix in the limit as $d \to \infty$ in time $t$ is $e^{-t\Gamma}$ where

\[ \Gamma = \lim \left( -d \sum \sigma^2(m) \sigma^2(m+1/d) + K(d) \sum \sigma^x(m) \right) \]
Actually the above is heuristic as the operators $\alpha(d), \beta(d)$ operate on different spaces. Perhaps this is where the renormalization group enters.

Let's now describe $\Gamma$ using fermions.

Recall that

$$-\frac{1}{d} \sigma^2(m) \sigma^2(m + \frac{a}{4}) = (c^+(m) - c^-(m)) (c^+(m + a) + c^-(m + a))$$

$$= (c^+(m) - c^-(m)) \Delta(c^+c^-)(m) + (c^+(m) - c^-(m)) \left[ \frac{c^+(m) c^-(m) - c^-(m) c^+(m)}{\sigma^2(m)} \right]$$

Thus we have that $\Gamma$ is the limit of

$$a \sum (c^+c^-)(m) \Delta(c^+c^-)(m) + d(L(d) - K(d)) \sum (c^+c^-)(c^+c^-)(m)$$

Somedays I have gotten the wrong sign. I want

$$L(d) = O(a)$$

which is okay if $J_1(d) = J_1 a$.

Similarly we want

$$d(L(d) - K(d)) = O(a)$$

Which means that $K, L$ should have opposite signs.

Thus $\Gamma$ is the limit of

$$a \sum (c^+c^-)(m) \Delta(c^+c^-)(m) + d(L(d) - K(d)) \sum (c^+c^-)(c^+c^-)(m)$$

Now this will approach
\[ \lambda \int (c^- c^-) \partial_x (c^+ c^-) \, dx + \mu \int (c^- c^-)(c^+ c^-) \, dx \]

provided \[ dL(d) \rightarrow \lambda \quad \text{or} \quad L(d) \sim \lambda a \]
\[ d(L(d) - K(d)) \rightarrow \mu \quad \text{or} \quad L(d) - K(d) \sim \mu a^2 \]

We have \[ L = \beta J_1 \quad \text{and} \quad \tanh (K) = e^{-2\beta J_2} \quad ? \]

I have not taken the appropriate continuum limit of the fermion fields. Thus we have

\[ \{ \sum f(m) c^+(m), \sum g(m) c^-(m) \} = \sum f(m) g(m) \]

and we want this to have the limit

Thus we rewrite the above \( \star \) in the form

\[ \left\{ a \sum f(m) \frac{c^+(m)}{a^{1/2}}, a \sum g(m) \frac{c^-(m)}{a^{1/2}} \right\} = a \sum f(m) g(m) \]

From this we conclude that \( \Phi(x) = \frac{c^+(x)}{a^{1/2}} \), which is consistent with \( c^+(x) = \Phi^+(x) \, dx^{1/2} \).

Thus \( \Gamma \) is the limit of

\[ a dL(d) \sum \frac{c^- c^-}{\sqrt{a}} \Delta_{c^+ c^-}(m) + d(L(d) - K(d)) \sum \frac{(c^- c^-)(c^+ c^-)}{\sqrt{a}} \]

which will be

\[ \lambda \int (\Phi^+ \Phi^-) \partial_x (\Phi^+ \Phi^-) \, dx + \mu \int (\Phi^+ \Phi^-)(\Phi^+ \Phi^-) \, dx \]

provided \[ L(d) \rightarrow \lambda \quad \text{and} \quad d(L(d) - K(d)) \rightarrow \mu \]

Recall that \( L = \beta J_1 \quad \text{and} \quad \tanh (K) = e^{-2\beta J_2} \)

Look at the second equation more carefully. Suppose
we have \[ \tanh(a) = \frac{e^a - e^{-a}}{e^a + e^{-a}} = e^{-2b} \]

or

\[ e^{-2b} = \frac{1 - e^{-2a}}{1 + e^{-2a}} = -1 + \frac{2}{1 + e^{-2a}} \]

or

\[ (1 + e^{-2a})(1 + e^{-2b}) = 2 \]

It would seem that the critical temperature is the condition \( L = K \) i.e.

\[ \tanh(\beta J_1) = e^{-2\beta J_2} \]

or

\[ (1 + e^{-2\beta J_1})(1 + e^{-2\beta J_2}) = 2 \]

Actually

\[ \frac{e^a - e^{-a}}{e^a + e^{-a}} = e^{-2b} \]

\[ \Rightarrow e^{2b} = \frac{e^a + e^{-a}}{e^a + e^{-a}} = \frac{e^{2a} + 1}{e^{2a} - 1} = 1 + \frac{2}{e^{2a} - 1} \]

so that

\[ (e^{2a} - 1)(e^{2b} - 1) = 2 \]

so the critical condition can be written also as

\[ (e^{2\beta J_1} - 1)(e^{2\beta J_2} - 1) = 2 \]