February 13, 1987

Real fermion field. You start with the real Hilbert space of real L^2 functions on S^1 and then form the complex Clifford algebra. Thus you are forming the Clifford algebra of the complex $L^2(S^1)$ with the quadratic form

$$\frac{1}{2}\int f^2 \frac{dx}{2\pi} = \frac{1}{2}\sum f_k f_{-k} \qquad \text{if} \qquad f = \sum f_k e^{ikx}$$

Thus the Clifford algebra is generated by ϕ_k such that

$$\begin{array}{ccc} \left(\sum c_k \phi_k\right)^2 & = & \frac{1}{2}\sum c_k c_{-k} \\ \| & & \| \\ \sum_{k,l} c_k c_l \phi_k \phi_l & & \frac{1}{2}\sum c_k c_l \delta_{k+l,0} \\ \| & & \\ \sum_{k,l} c_k c_l \frac{1}{2}\{\phi_k, \phi_l\} & & \end{array}$$

so the commutation relations are

$$\{\phi_k, \phi_l\} = \delta_{k+l,0}$$

In addition we have the reality condition

$$\phi_k^* = \phi_{-k}$$

Thus if $f = \sum f_k e^{ikx}$ is real, i.e. $\overline{f_k} = f_{-k}$, then $\sum f_k \phi_k$ is hermitian.

Now a complex fermion field is equivalent to two real fermion fields.

$$\psi^*(x) = \phi_1(x) + i\phi_2(x)$$
$$\psi(x) = \phi_1(x) - i\phi_2(x).$$

and one has the current

$$\psi^*(x)\psi(x) = -i(\phi_1(x)\phi_2(x) - \phi_2(x)\phi_1(x))$$

In general given real fermion fields ϕ_i and an orthogonal representation M^a_{ij} of a Lie algebra, better a Lie subalgebra of $so(n)$ with basis M^a, one can form the family of currents

$$M^a_{ij}\phi_i\phi_j .$$

Let's now compare vacuum states for ψ^* and ϕ_1, ϕ_2. We have

$$\phi(x) = \sum e^{-ikx}\phi_k$$

because the operator associated to $f(x) = \sum c_k e^{ikx}$ is to be

$$\int f(x)\phi(x)dx = \sum c_k\phi_k = \sum\left(\int f(x)e^{-ikx}dx\right)\phi_k$$

Thus we have

$$\phi^1(x) = \tfrac{1}{2}(\psi^*(x) + \psi(x)) = \tfrac{1}{2}\sum e^{-ikx}(\psi^*_k + \psi_{-k})$$

$$\phi^2(x) = \frac{1}{2i}(\psi^*(x) - \psi(x)) = \frac{1}{2i}\sum e^{-ikx}(\psi^*_k - \psi_{-k})$$

so $\phi^1_k = (\psi^*_k + \psi_{-k})/2$, $\phi^2_k = (\psi^*_k - \psi_{-k})/2i$. Thus the natural ground state condition is

$$(*) \qquad \phi_k|0\rangle = 0 \qquad k < 0$$

It's best to avoid problems with $k=0$ by starting with anti-periodic functions on S^1, i.e. sections of the Möbius line bundle. Then $k \in \tfrac{1}{2} + \mathbb{Z}$.

The condition $(*)$ fits with the Hamiltonian $\frac{1}{i}\partial_x$ and also our previous conventions, i.e. that J_k creates for $k > 0$.

Let's now go on to the Virasoro algebra. Recall that corresponding to the (ex.) vector field $\frac{1}{i}e^{iqx}\partial_x$ on S^1 is the hermitian operator $e^{iqx}\left(\frac{1}{i}\partial_x + \frac{1}{2}q\right)$ on $L^2(S^1)$ and this becomes the operator

$$L_q = \sum_k \left(k+\tfrac{1}{2}q\right) :\psi^*_{q+k}\psi_k:$$

on Fock space. I now wish to work out commutation relations for the L's J's and ψ's.

$$[L_q, \psi^*_l] = \sum_k \left(k+\tfrac{1}{2}q\right)\underbrace{[\psi^*_{q+k}\psi_k, \psi^*_l]}_{\psi^*_{q+k}\delta_{kl}} = \left(l+\tfrac{1}{2}q\right)\psi^*_{q+l}$$

$$[L_q, \psi^*(z)] = \sum z^l [L_q, \psi^*_l] = \sum z^l \left(l+\tfrac{1}{2}q\right)\psi^*_{q+l}$$

$$= z^{-q}\sum_l \left(l+q-\tfrac{1}{2}q\right) z^{q+l}\psi^*_{q+l} = z^{-q}\left(z\partial_z - \tfrac{1}{2}q\right)\psi^*(z)$$

$$[L_q, \psi_l] = \sum_k \left(k+\tfrac{1}{2}q\right)\underbrace{[\psi^*_{q+k}\psi_k, \psi_l]}_{-\delta_{q+k,l}\psi_k} = -\left(l-q+\tfrac{1}{2}q\right)\psi_{l-q}$$

$$[L_q, \psi(z)] = -\sum \overset{\cancel{q+l-q}}{z^{-l}}\left(l-q+\tfrac{1}{2}q\right)\psi_{l-q}$$

$$= +z^{-q}\sum z^{-(l-q)}\left(-(l-q)-\tfrac{1}{2}q\right)\psi_{l-q}$$

$$= z^{-q}\left(z\partial_z - \tfrac{1}{2}q\right)\psi(z)$$

Here's how to deduce the commutation relations. I begin with the commutation relations among the J_n.

$$[J_{-p}, :\psi^*(z)\psi(\zeta):] = [J_{-p}, \psi^*(z)\psi(\zeta)]$$

$$= \left(z^p - \zeta^p\right)\psi^*(z)\psi(\zeta)$$

$$= \left(z^p - \zeta^p\right)\left(:\psi^*(z)\psi(\zeta): + \frac{z}{z-\zeta} \right)$$

Now let $z \longrightarrow \zeta$ and you get

$$\left[J_{-p}, J(\zeta)\right] = \lim_{z\to\zeta} \frac{z^p - \zeta^p}{z-\zeta}\, z = p\,\zeta^p$$

which gives $\quad [J_{-p}, J_k] = p\,\delta_{p,k}$

Apply this method next to the L's.

$$\left[L_{-p}, :\psi^*(z)\psi(\zeta):\right] = \left[L_{-p}, \psi^*(z)\psi(\zeta)\right]$$

$$= \underbrace{\left[z^p\left(z\partial_z + \tfrac{1}{2}p\right) + \zeta^p\left(\zeta\partial_\zeta + \tfrac{1}{2}p\right)\right]\left(:\psi^*(z)\psi(\zeta): + \frac{z}{z-\zeta}\right)}$$

Thus $\quad \left[L_{-p}, J(\zeta)\right] = \lim_{z\to\zeta} \left[\qquad \right] \cancel{\frac{z}{z-\zeta}}$

Note $\quad \dfrac{z}{z-\zeta} - \dfrac{\zeta}{z-\zeta} = \text{const.}$ So

$$\left(z^{p+1}\partial_z + \zeta^{p+1}\partial_\zeta\right)\left(\frac{z}{z-\zeta}\right) = z^{p+1}\partial_z\left(\frac{\zeta}{z-\zeta}\right) + \zeta^{p+1}\partial_\zeta\left(\frac{z}{z-\zeta}\right)$$

$$= z^{p+1}\frac{-\zeta}{(z-\zeta)^2} + \zeta^{p+1}\frac{z}{(z-\zeta)^2}$$

So

$$\left[z^p\left(z\partial_z + \tfrac{1}{2}p\right) + \zeta^p\left(\zeta\partial_\zeta + \tfrac{1}{2}p\right)\right]\left(\frac{z}{z-\zeta}\right) =$$

$$\frac{-z^{p+1}\zeta + \zeta^{p+1}z}{(z-\zeta)^2} + \frac{1}{2}p\,\frac{(z^p + \zeta^p)}{(z-\zeta)}\, z$$

~~Work modulo regular functions of z near $z=\zeta$.~~

$$\cancel{= \frac{-z^{p+1}\zeta + \zeta^{p+1}z}{(z-\zeta)^2} + \frac{1}{2}p\,\frac{2\zeta^p}{z-\zeta}\,\zeta = \frac{-z^{p+1}\zeta + \zeta^{p+1}z + p\,\zeta^{p+1}(z-\zeta)}{(z-\zeta)^2}}$$

$$\cancel{\frac{p\,\zeta^{p+1}(z-\zeta)}{(z-\zeta)^2}}$$

$$\frac{\zeta^{p+1} z - z^{p+1}\zeta}{(z-\zeta)^2} = \frac{\cancel{\zeta^{p+2}} + \zeta^{p+1}(z-\zeta) - \zeta\left(\cancel{\zeta^{p+1}} + (p+1)\zeta^{p}(z-\zeta) + \frac{(p+1)p}{2}\zeta^{p-1}(z-\zeta)^2\right)}{(z-\zeta)^2}$$

$$= \frac{-p\,\zeta^{p+1}}{z-\zeta} - \frac{(p+1)p}{2}\,\zeta^{p}$$

$$\frac{1}{2}p\,\frac{(z^p+\zeta^p)z}{z-\zeta} = \frac{1}{2}p\,2\,\zeta^p + \frac{1}{2}p\,\frac{\zeta\left(\zeta^p + \zeta^p + p\,\zeta^{p-1}(z-\zeta)\right)}{z-\zeta}$$

$$= \frac{p\,\zeta^{p+1}}{z-\zeta} + p\,\zeta^p + \cancel{\frac{1}{2}p^2\zeta^p}$$

so
$$\lim_{z\to\zeta}\left[z^p\left(z\partial_z + \tfrac{p}{2}\right) + \zeta^p\left(\zeta\partial_\zeta + \tfrac{p}{2}\right)\right]\left(\frac{z}{z-\zeta}\right) = \frac{p}{2}\zeta^p$$

On the other hand

$$\lim_{z\to\zeta}\left[z^p\left(z\partial_z + \tfrac{p}{2}\right) + \zeta^p\left(\zeta\partial_\zeta + \tfrac{p}{2}\right)\right] :\psi^\dagger(z)\psi(\zeta):$$
$$= \zeta^p\left(\zeta\partial_\zeta + p\right)J(\zeta)$$

because if the operator in [ ] is applied to $z^k\zeta^l$ it gives

$$k\,z^{p+k}\zeta^l + \frac{p}{2}z^{p+k}\zeta^l + z^k\left(l+\frac{p}{2}\right)\zeta^{p+l}$$
$$\longrightarrow \left(k+\frac{p}{2}+l+\frac{p}{2}\right)\zeta^{p+k+l} = \zeta^p\left(\zeta\partial_\zeta + p\right)\zeta^{k+l}$$

What's going on here is $\frac{d}{dx}F(x,x) = (\partial_x+\partial_y)F(x,y)\big|_{x=y}$.

Thus we get the formula

$$\boxed{\begin{aligned}[L_{-p}, J(z)] &= z^p(z\partial_z + p)J(z) + \frac{p}{2}\zeta^p \\ [L_{-p}, J_m] &= m\,J_{m-p} + \frac{p}{2}\delta_{p,m}\end{aligned}}$$

Next let's check the leading term by computing the bracket on $L^2(S^1)$.

$$[L_{-p}, J_m] = \left[e^{-ipx}\left(\tfrac{1}{i}\partial_x - \tfrac{p}{2}\right), e^{imx}\right] = e^{-ipx}\tfrac{1}{i}\partial_x\left(e^{imx}\right)$$

$$= m\, e^{i(m-p)x} = m\, J_{m-p}.$$

Next we want to derive the commutation relation for the L's. Therefore we need the appropriate generating function for the L_q which is

$$L(z) = \sum z^q L_q.$$

Reason: $L_q|0\rangle = 0$ for $q \le 0$, so $L(z)|0\rangle$ should involve positive powers of z like $\psi^*(z)|0\rangle$, $\psi(z)|0\rangle$, $J(z)|0\rangle$.

Note that if we write

$$L_q = \sum_{k\in\mathbb{Z}} \left(k + \tfrac{q}{2}\right) :\psi^*_{k+q}\psi_k: = \sum_{l \in \frac{q}{2}+\mathbb{Z}} l :\psi^*_{l+\frac{q}{2}}\psi_{l-\frac{q}{2}}:$$

then it is obvious that

$$L_q^* = L_{-q}$$

Next we need a formula for $L(z)$ in terms of $\psi^*(z), \psi(z)$.

$$z\partial_z\psi^*(z)\cdot\psi(\zeta) = \sum_{k,l} k z^k \zeta^{-l}\psi^*_k\psi_l$$

$$(z\partial_z\psi^*(z))\psi(z) = \sum k z^{k-l}\psi^*_k\psi_l$$

$$= \sum_q z^q\left(\sum_l (l+q)\psi^*_{l+q}\psi_l\right)$$

$$\psi^*(z)\, z\partial_z \psi(\zeta) = \sum_{k,\ell} (-\ell) z^k \zeta^{-\ell} \psi_k^* \psi_\ell$$

$$\psi^*(z)\left(z\partial_z \psi(z)\right) = \sum -\ell\, z^{k-\ell} \psi_k^* \psi_\ell$$

$$= \sum_q{}' z^q \sum_\ell (-\ell)\, \psi_{\ell+q}^* \psi_\ell$$

Thus

$$\tfrac{1}{2}\left(z\partial_z \psi^*(z)\right)\psi(z) - \tfrac{1}{2}\psi^*(z)\left(z\partial_z\psi(z)\right) = \sum z^q \Big(\sum_\ell{}' (\ell + \tfrac{q}{2}) \psi_{\ell+q}^* \psi_\ell\Big)$$

$$= \sum z^q L_q$$

and so we have found

$$\boxed{L(z) = \tfrac{1}{2}\Big\{\left(z\partial_z\psi^*(z)\right)\psi(z) - \psi^*(z)\left(z\partial_z\psi(z)\right)\Big\}}$$

There is another derivation which is a little less manipulative. Start with the fact that to each $f(x)$ on S^1 we have the operator $\frac{1}{i}\left(f\partial_x + \frac{1}{2}f'\right)$ on $L^2(S^1)$ and its extension to Fock space:

$$L(f) = \sum{}' \langle k | \tfrac{1}{i}\left(f\partial_x + \tfrac{1}{2}f'\right) | \ell \rangle\, \psi_k^* \psi_\ell$$

$$= \int dx\; \psi^*(x)\, \tfrac{1}{i}\left(f\partial_x + \tfrac{1}{2}f'\right) \psi(x)$$

$$= \tfrac{1}{i}\int dx\; f(x) \left\{ \psi^*(x)\, \partial_x\psi(x) - \tfrac{1}{2}\partial_x\left(\psi^*(x)\psi(x)\right)\right\}$$

$$= i\int dx\; f(x) \left\{ \tfrac{1}{2}\left(\partial_x\psi^*(x)\right)\psi(x) - \tfrac{1}{2}\psi^*(x)\left(\partial_x\psi(x)\right)\right\}$$

whence $\boxed{L(x) = \frac{1}{2}\left\{(i\partial_x \psi^*(x))\psi(x) - \psi^*(x)(i\partial_x\psi(x))\right\}}$

Now just ~~substitute~~ $z = e^{-ix}$, $\frac{dz}{z} = -i dx$,

$i\partial_x = z\partial_z$.

The next step in our program is to derive commutation relation among the L's. It seems interesting to proceed via operator products. The idea is that

$$L(z) = \frac{1}{2}\left(\nabla_z - \nabla_\zeta\right) :\psi^*(z)\psi(\zeta): \Big|_{z=\zeta}$$

where $\nabla_z = z\partial_z$. Thus we ought to be able to spot the singularities of $L(z)\, L(\zeta)$ starting from Wick's formula for $(:\psi^*_{z_1}\psi_{z_2}:)(:\psi^*_{z_3}\psi_{z_4}:)$ and differentiating, then setting $z_1 = z_2 = z$, $z_3 = z_4 = \zeta$. Wick's formula is

$$(:\psi^*_{z_1}\psi_{z_2}:)(:\psi^*_{z_3}\psi_{z_4}:) = :\psi^*_{z_1}\psi_{z_2}\psi^*_{z_3}\psi_{z_4}:$$

$$+ \frac{z_1}{z_1 - z_4} :\psi_{z_2}\psi^*_{z_3}: \quad + \frac{z_3}{z_2 - z_3} :\psi^*_{z_1}\psi_{z_4}:$$

$$+ \frac{z_1}{z_1 - z_4}\frac{z_3}{z_2 - z_3}$$

As a check set $z_1 = z_2 = z$, $z_3 = z_4 = w$ and look at the singularities as $z \longrightarrow w$

$$J_z J_w \equiv \frac{z}{z-w}(-J_w) + \frac{w}{z-w} J_w + \frac{zw}{(z-w)^2}$$

$$\equiv \frac{zw}{(z-w)^2}$$

As a check

$$[J_p, J(w)] = \operatorname*{Res}_{z=w} z^{-p-1}\frac{zw}{(z-w)^2} = -pz^{-p-1}w\Big|_{z=w}$$

or $$[J_p, J(w)] = -p\, w^{-p}$$

or $$[J_p, J_q] = -p\, \delta_{q,-p}$$

Next lets apply $\frac{1}{2}(\nabla_{z_1} - \nabla_{z_2})$ to Wicks formula and set $z_1 = z_2 = z$, $z_3 = z_4 = w$. I am only concerned with the singularities.

$$\frac{1}{2} \times \quad \nabla_{z_1}\left(\frac{z_1}{z_1 - z_4}\right) :\psi_{z_2}\psi^*_{z_3}: \; - \left(\frac{z_1}{z_1 - z_4}\right) :\left(\nabla_{z_2}\psi_{z_2}\right)\psi^*_{z_3}:$$

$$+ \left(\frac{z_3}{z_2 - z_3}\right) : \nabla_{z_1}\psi^*_{z_1}\psi_{z_4}: \; - \nabla_{z_2}\left(\frac{z_3}{z_2 - z_3}\right) :\psi^*_{z_1}\psi_{z_4}:$$

$$+ \nabla_{z_1}\left(\frac{z_1}{z_1 - z_4}\right)\frac{z_3}{z_2 - z_3} - \nabla_{z_2}\left(\frac{z_3}{z_2 - z_3}\right)\frac{z_1}{z_1 - z_4}$$

~~Let $z_1 = z_2 = z$, $z_3 = z_4 = w$.~~

$$\|$$

$$\frac{-z_1 z_4}{(z_1 - z_4)^2} :\psi_{z_2}\psi^*_{z_3}: + \frac{z_2 z_3}{(z_2 - z_3)^2} :\psi^*_{z_1}\psi_{z_4}:$$

$$+ \frac{z_3}{z_2 - z_3} :\nabla_{z_1}\psi^*_{z_1}\psi_{z_4}: - \frac{z_1}{z_1 - z_4} :\left(\nabla_{z_2}\psi_{z_2}\right)\psi^*_{z_3}:$$

$$+ \frac{-z_1 z_4}{(z_1 - z_4)^2}\frac{z_3}{z_2 - z_3} + \frac{z_2 z_3}{(z_2 - z_3)^2}\frac{z_1}{z_1 - z_4}$$

Let $z_1 = z_2 = z$, $z_3 = z_4 = w$

$$\downarrow$$

$$-\frac{zw}{(z-w)^2} :\psi_z\psi^*_w: + \frac{zw}{(z-w)^2} :\psi^*_z\psi_w:$$

$$+ \frac{w}{z-w} :\nabla_z\psi^*_z\psi_w: - \frac{z}{z-w} :\nabla_z\psi_z\psi^*_w:$$

$$-\frac{zw^2}{(z-w)^3} + \frac{z^2 w}{(z-w)^3} = \frac{zw}{(z-w)^2}$$

Note that modulo regular functions of z at w we have

$$\frac{w}{z-w} : \nabla_z \psi^*_z \psi_w : - \frac{z}{z-w} : \nabla_z \psi_z \psi^*_w :$$

$$\equiv \frac{w}{z-w} : \nabla_w \psi^*_w \psi_w : - \frac{w}{z-w} : \nabla_w \psi_w \psi^*_w :$$

Also

$$\frac{zw}{(z-w)^2} : \psi^*_z \psi_w : \equiv \frac{zw}{(z-w)^2} : \psi^*_w \psi_w : + \frac{zw}{(z-w)^2} : (z-w) \partial_w \psi^*_w \psi_w :$$

$$\equiv \frac{zw}{(z-w)^2} : \psi^*_w \psi_w : + \frac{w}{(z-w)} : \nabla_w \psi^*_w \psi_w :$$

$$-\left(\begin{aligned} \frac{zw}{(z-w)^2} : \psi_z \psi^*_w : &= \frac{zw}{(z-w)^2} : \psi_w \psi^*_w : + \frac{zw}{(z-w)^2} : (z-w) \partial_w \psi_w \psi^*_w : \\ &\equiv \frac{zw}{(z-w)^2} : \psi_w \psi^*_w : + \frac{w}{z-w} : \nabla_w \psi_w \psi^*_w : \end{aligned} \right)$$

So the sum of the normal product terms is

$$2 \frac{zw}{(z-w)^2} J(w) + 2 \frac{w}{(z-w)} \underbrace{\left(: \nabla_w \psi^*_w \psi_w : - : \nabla_w \psi_w \psi^*_w : \right)}_{\nabla_w J(w)}$$

~~So we seem to get something like~~

$$\cancel{L(z) J(w) \equiv \frac{zw}{(z-w)^2} J(z) + \frac{1}{2} \frac{zw}{(z-w)^2}}$$

Thus remembering the $\times \frac{1}{2}$ factor we get

$$\boxed{L(z) J(w) \equiv \frac{zw}{(z-w)^2} \left(J(w) + \frac{1}{2} \right) + \frac{w}{z-w} w \partial_w J(w)}$$

So

$$[L_p, J(w)] = \operatorname*{Res}_{z=w} z^{-p-1}\left\{\frac{zw}{(z-w)^2}\left(J(w)+\tfrac{1}{2}\right) + \frac{w}{z-w}\, w\partial_w J(w)\right\}$$

$$= \frac{d}{dz}\Big|_{z=w} z^{-p} w \left(J(w)+\tfrac{1}{2}\right) \quad + \quad w^{-p} w \partial_w J(w)$$

$$= -p\, w^{-p}\left(J(w)+\tfrac{1}{2}\right) \quad + \quad \text{———}$$

$$\boxed{[L_p, J(w)] = w^{-p}(w\partial_w - p)\, J(w) - \frac{p}{2} w^{-p}}$$

which agrees with page 442.

So now we are ready to tackle the commutation relations for the L's.

February 14, 1987:

I want to work out the operator product expansion for the stress energy operator

$$L(z) = :\tfrac{1}{2}\left[(z\partial_z \psi^*(z))\psi(z) - \psi^*(z)(z\partial_z \psi(z))\right]:$$

using the above sort of methods. This means we compute the singularities in $L(z)\,L(w)$ starting from Wick's formula

$1 = \psi^*_{z_1}$, $2 = \psi_{z_2}$, $3 = \psi^*_{z_3}$, $4 = \psi_{z_4}$

$$(:12:)(:34:) = :1234: + \langle 14\rangle :23: + \langle 23\rangle :14: + \langle 14\rangle\langle 23\rangle$$

~~and~~ applying $\frac{1}{4}(\nabla_1 - \nabla_2)(\nabla_3 - \nabla_4)$ $\qquad \nabla_i = z_i \partial_{z_i}$

and then setting $z_1 = z_2 = z$, $\quad z_3 = z_4 = w$

I'm now going to work out the constant term

$$\frac{1}{4}(\nabla_1 - \nabla_2)(\nabla_3 - \nabla_4)\ \frac{z_1}{z_1 - z_4}\ \frac{z_3}{z_2 - z_3}$$

February 14, 1987 (cont)

$$\nabla_1\nabla_3 \left(\frac{z_1}{z_1-z_4}\right)\left(\frac{z_3}{z_2-z_3}\right) = \frac{z_1 z_4}{-(z_1-z_4)^2}\ \frac{z_3 z_2}{(z_2-z_3)^2}$$

$$\longrightarrow \frac{-z^2 w^2}{(z-w)^4} = -\frac{w^2}{(z-w)^4}\left(w^2+2w(z-w)+(z-w)^2\right)$$

$$= \frac{(-w^4)}{(z-w)^4} + \frac{(-2w^3)}{(z-w)^3} + \frac{(-w^2)}{(z-w)^2}$$

$$\nabla_2\nabla_4 \left(\frac{z_1}{z_1-z_4}\right)\left(\frac{z_3}{z_2-z_3}\right) = \frac{z_4 z_1}{(z_1-z_4)^2}\ \frac{z_2 z_3}{-(z_2-z_3)^2}$$

$$\longrightarrow -\frac{z^2 w^2}{(z-w)^4} = \frac{(-w^4)}{(z-w)^4} + \frac{(-2w^3)}{(z-w)^3} + \frac{(-w^2)}{(z-w)^2}$$

$$-\nabla_2\nabla_3 \left(\frac{z_1}{z_1-z_4}\right)\left(\frac{z_3}{z_2-z_3}\right) = \left(\frac{-z_1}{z_1-z_4}\right) \times \nabla_2 \frac{z_3 z_2}{(z_2-z_3)^2}$$

$$= \left(\frac{-z_1}{z_1-z_4}\right)\left[\frac{z_2 z_3}{(z_2-z_3)^2} + z_2(z_3 z_2)\frac{(-2)}{(z_2-z_3)^3}\right]$$

$$\longrightarrow \frac{-z^2 w}{(z-w)^3} + \frac{2z^3 w}{(z-w)^4}$$

$$= \frac{2w^4}{(z-w)^4} + \frac{w^3(-1+6)}{(z-w)^3} + \frac{w^2(-2+6)}{(z-w)^2} + \frac{w(-1+2)}{(z-w)}$$

$$-\nabla_1\nabla_4 \left(\frac{z_1}{z_1-z_4}\right)\left(\frac{z_3}{z_2-z_3}\right) = \left(\frac{-z_3}{z_2-z_3}\right) \nabla_1 \left(\frac{z_4 z_1}{(z_1-z_4)^2}\right)$$

$$= \left(\frac{-z_3}{z_2-z_3}\right)\left[\frac{z_1 z_4}{(z_1-z_4)^2} + z_1(z_4 z_1)\frac{(-2)}{(z_1-z_4)^3}\right]$$

$$\longrightarrow \frac{-w}{z-w}\left[\frac{zw}{(z-w)^2} + \frac{-2z^2 w}{(z-w)^3}\right] = \frac{2z^2w^2}{(z-w)^4} - \frac{zw^2}{(z-w)^3}$$

$$= \frac{2w^4}{(z-w)^4} + \frac{w^3(4-1)}{(z-w)^3} + \frac{w^2(2-1)}{(z-w)^2} + \frac{w(0)}{(z-w)}$$

So the total singularity is

$$\frac{1}{4}\times\left\{\frac{2w^4}{(z-w)^4}+\frac{4w^3}{(z-w)^3}+\frac{3w^2}{(z-w)^2}+\frac{w}{(z-w)}\right\}$$

Now calculate the residue at $z=w$ of z^{-p-1} and we get

$$\left(\frac{2}{4}\,\frac{(-p-1)(-p-2)(-p-3)}{3!}+\frac{4}{4}\,\frac{(-p-1)(p-2)}{2!}+\frac{3}{4}(-p-1)+\frac{1}{4}\right)w^{-p}$$

$$=\left(-\frac{p^3}{12}-\frac{p}{6}\right)w^{-p}$$

Thus the operators L_p seem to satisfy

$$\boxed{[L_p, L_m] = (m-p)L_{p+m}+\left(-\frac{p^3}{12}-\frac{p}{6}\right)\delta_{-p,m}}$$

Observe that if we change ~~xxx~~ L_0 by a suitable constant, then we can alter the $-\frac{p}{6}$ to $\frac{p}{12}$. In this case L_{-1}, L_0, L_1 form a sl_2 algebra.

As I try to work with these question it seems that the simplest method might be the original one based on the series for J_p, L_p etc.

Let's try this out and so get ~~xxx~~ over the point of the commutation relation.

$$[L_p, J_q] = \lim_{I,I'}\sum_{(k,l)\in I\times I'}\left(k+\frac{p}{2}\right)\underbrace{[:\psi^*_{k+p}\psi_k:;:\psi^*_{l+q}\psi_l:]}_{\delta_{k,l+q}\psi^*_{k+p}\psi_l-\delta_{k+p,l}\psi^*_{l+q}\psi_k}$$

$$=\lim_{I,I'}\sum_{\substack{l\in I'\\ l+q\in I}}\left(l+q+\frac{p}{2}\right)\psi^*_{l+p+q}\psi_l-\sum_{\substack{k\in I\\ k+p\in I'}}\left(k+\frac{p}{2}\right)\psi^*_{k+p+q}\psi_k$$

$$= \lim_{I,I'} \left\{ \sum_{\substack{k \in I' \\ k \in I-q}} \left(k+q+\frac{p}{2}\right) \psi^*_{k+p+q} \psi_k - \sum_{\substack{k \in I \\ k \in I'-p}} \left(k+\frac{p}{2}\right) \psi^*_{k+p+q} \psi_k \right\}$$

If $p+q \neq 0$, then these monomials are already in normal order and so we can take the limit for each sum obtaining $q J_{p+q}$. If $p+q=0$, then we obtained the normal ordered part which is $q J_0$ plus the constant

$$\lim_{I,I'} \left\{ \sum_{\substack{k \in I' \cap (I+p) \\ k \leq 0}} \left(k + \overbrace{q+\frac{p}{2}}^{-\frac{p}{2}}\right) - \sum_{\substack{k \in I \cap (I'-p) \\ k \leq 0}} \left(k+\frac{p}{2}\right) \right\}$$

To evaluate this we shift

$$\lim_{I,I'} \left\{ \sum_{\substack{k \in (I'-\frac{p}{2}) \cap (I+\frac{p}{2}) \\ k \leq -p/2}} k - \sum_{\substack{k \in (I+\frac{p}{2}) \cap (I'-\frac{p}{2}) \\ k \leq p/2}} k \right\}$$

$$= \lim_{J} \left\{ \sum_{\substack{k \leq -p/2 \\ k \in J}} k - \sum_{\substack{k \leq p/2 \\ k \in J}} k \right\}$$

Note k is a half-integer if p is odd

It is clear this last sum is independent of J once J contains the interval $-p/2 < k \leq p/2$ if $p \geq 0$ or $p/2 < k \leq -p/2$ if $p \leq 0$. By symmetry, i.e. $k \longleftrightarrow -k$, there is cancellation except for one term. Take $J = [-p/2, p/2]$, say $p > 0$. One gets $-\frac{p}{2}$. Take $J = [p/2, -p/2]$ for $p < 0$. One gets $-p/2$. Thus

$$\boxed{[L_p, J_q] = q J_{p+q} + \frac{q}{2} \delta_{p+q,0}}$$

Now

$$[L_p, L_q] = \lim \sum (k+\tfrac{p}{2})(\ell+\tfrac{q}{2}) [:\psi^*_{k+p}\psi_k:, :\psi^*_{\ell+q}\psi_\ell:]$$

$$= \lim_{I,I'} \Big\{ \sum_{\substack{k\in I' \\ k+q\in I}} (k+q+\tfrac{p}{2})(k+\tfrac{q}{2}) \psi^*_{k+p+q}\psi_k$$

$$- \sum_{\substack{k\in I \\ k+p\in I'}} (k+\tfrac{p}{2})(k+p+\tfrac{q}{2}) \psi^*_{k+p+q}\psi_k \Big\}$$

Now $\quad (k+q+\tfrac{p}{2})(k+\tfrac{q}{2}) - (k+\tfrac{p}{2})(k+p+q/2)$

$$= q(k+\tfrac{q}{2}) - (k+\tfrac{p}{2})p \quad = (q-p)k + (q-p)\left(\tfrac{q+p}{2}\right)$$

$$= (q-p)\left(k + \tfrac{p+q}{2}\right)$$

Thus when $p+q \neq 0$ so that the monomials are already normally ordered we get $[L_p, L_q] = (q-p)L_{p+q}$. Suppose then that $p+q=0$. Then we have the extra constant

$$\lim_{I,I'} \Big\{ \sum_{\substack{k\in I' \\ k+q\in I \\ k\leq 0}} (k+\tfrac{q}{2})^2 - \sum_{\substack{k\in I \\ k+p\in I' \\ k\leq 0}} (k-\tfrac{q}{2})^2 \Big\}$$

$$= \lim_{I,I'} \sum_{\substack{k\in(\frac{q}{2}+I')\cap(-\frac{q}{2}+I) \\ k\leq \frac{q}{2}}} (k)^2 - \sum_{\substack{k\in(-\frac{q}{2}+I)\cap(\frac{q}{2}+I') \\ k\leq -\frac{q}{2}}} (k)^2$$

$$= \sum_{-\frac{q}{2} < k \leq \frac{q}{2}} k^2 \qquad \text{if } q>0$$

Note if q is odd then k is over half-integers.

$$= \frac{q^3}{12} + \frac{q}{6}$$

(Use $f(n) = 2\frac{(n+2)(n+1)n}{6} - \frac{n(n+1)}{2}$ satisfies $f(n) - f(n-1) = n^2$)

Amazingly, I still am not yet done with the Virasoro algebra, because I found in Shenker's intro to CFT that the physicists divide my $L(z)$ by z^2 to get $T(z)$. Then the OPE is

$$T(z)\,T(w) \sim \frac{1}{2(z-w)^4} + \text{terms in } T(w)$$

instead of the mess I got on page 450. Also the way they do things leads to the simple formula

$$[L_p, J_m] = m J_{p+m}$$

instead of my formula on p. 451. I think this just means adjusting J_0 by a constant, perhaps using a different normal ordering.

It's clearly desirable to understand their approach, which must be based on invariant considerations. For example, the theory we are looking at has operators $L(f\partial_x)$ associated to any vector field $f\partial_x$ on S^1. Writing

$$L(f\partial_x) = \int (f\partial_x)\cdot \underbrace{T(x)(dx)^2}$$

we see that the natural $L(x)$ is a quadratic differential. So if $z = e^{-ix}$, $\left(\frac{dz}{z}\right)^2 = (-idx)^2 = -(dx)^2$, which perhaps indicates why one wants to divide by z^2.

My idea roughly is this. I want set things up invariantly so that I have a Clifford algebra acted on by Diff(S^1), and I am interested in the induced projective representation on the Fock space representation of the Clifford algebra. Next I consider the "maximal torus" in Diff(S^1) given by the rotations and decompose everything in sight into weight spaces for the action of this circle.

February 15, 1987.

I thought a bit about $Z = h\gamma^\mu\partial_\mu + \sigma X$, $X = \frac{g-1}{g+1}$, without getting anywhere. However I'd like to list some approaches worth trying to study the problem of the existence of the resolvent $\frac{1}{\lambda - Z}$.

1) $$\frac{1}{\lambda - Z^2} = \frac{(g+1)^2}{(g+1)(\lambda - h^2\partial_\mu^2)(g+1) - (g-1)^2 - 2h\gamma^\mu\partial_\mu g\,\sigma}$$

becomes invertible in the limit as $h \to 0$. I should have said that the denominator on the right is in Getzler's algebra and is invertible at $h=0$. Thus one can probably write down a formal power series in h for $\frac{1}{\lambda - Z^2}$ whose coefficients are operators defined by the F.T. by functions $f(x,p)$ having the denominator factors $(\lambda+p^2)(g+1)^2 - (g-1)^2$. Maybe one can construct a parametrix.

2) By doubling one can deform g so that it doesn't have the eigenvalue -1. You are trying to produce a unitary operator $\equiv -1$ mod some Schatten class depending on g, and so what you are after is an extension of something which you know exists.

3) Modify Hörmander's paper using $(\lambda+p^2)(g+1)^2 - (g-1)^2$ and maybe it fits his hypoelliptic theory. The difficulty is that it is only bounded below and doesn't grow at all in p.

4) Consider the case of the circle and $\begin{pmatrix} 0 & h\partial_x - p \\ h\partial_x + p & 0 \end{pmatrix} = Z$. Here you can construct a parametrix for Z and verify it is Hilbert-Schmidt for small h.

5) Bismut forms on LM generalized to superconnections

February 15, 1987: (continued)

Today I want to straighten out the formulas for the Virasoro generators using the principle of first setting things up invariantly and then decomposing into weight spaces for the rotation of the circle.

Let's consider a real fermion field over the circle. For this I need to have a real vector space E with positive inner product, because I want operators $\psi(\xi)$, $\xi \in E$, which are hermitian satisfying $\psi(\xi)^2 = (\xi, \xi)$. Now the only way I can see to obtain such an E which is acted on naturally by $Diff(S^1)$ is to take a square root of T^*. There are two real line bundles L over S^1 with $L^{\otimes 2} = T^*$ namely those whose sections are $\frac{1}{2}$ densities $f(x)\,dx^{1/2}$ where f is periodic or anti-periodic.

Given $\varphi \in Diff(S^1)$ we have $\varphi^*(L)^{\otimes 2} = \varphi^*(T^*) = T^*$ and there are two possible isomorphisms $\varphi^*(L) \simeq L$. Thus a central extension of $Diff(S^1)$ by $\mathbb{Z}/2$ acts on L. When L is trivial, this extension is $Diff(S^1) \times \mathbb{Z}/2$, but when L is Mobius, then we get a non-trivial double covering as one sees by looking at rotations.

So for each $f\,dx^{1/2}$ we have an operator, or say, an element $\psi(f\,dx^{1/2})$ in the Clifford algebra which is hermitian for f real such that

$$\psi(f\,dx^{1/2})^2 = \int f^2 \frac{dx}{2\pi}$$

E_c has the basis $e^{ikx}\,dx^{1/2}$ of eigenfunctions for the translation circle action. Here $k \in \mathbb{Z}$ or $k \in \frac{1}{2}+\mathbb{Z}$ in the periodic or anti periodic cases. So if $\psi_k = \psi(e^{ikx}\,dx^{1/2})$

$$\psi(f\,dx^{1/2})^2 = \left(\sum c_k \psi_k\right)^2 = \sum c_k c_{-k}$$

where $f = \sum c_k e^{ikx}$. Thus we have the relations

$$\psi_k^* = \psi_{-k}, \qquad \psi_k \psi_\ell + \psi_\ell \psi_k = 2\delta_{k,-\ell}$$

We have

$$\psi(f\,dx^{1/2}) = \sum \left(\int e^{-ikx} f(x) \frac{dx}{2\pi} \right) \psi_k$$

$$= \int \frac{dx}{2\pi} f(x) \sum e^{-ikx} \psi_k$$

$$= \frac{1}{2\pi} \int \left(f(x)\, dx^{1/2} \right) \underbrace{\left(\sum e^{-ikx} \psi_k \, dx^{1/2} \right)}_{\psi(x)}$$

The field $\psi(x)$ is an operator-valued distribution on the space of half densities.

Next we consider a vector field $a(x)\partial_x$ on S^1. This acts naturally on the space E_c preserving the bilinear form, and so it can be extended to a derivation of the Clifford algebra which is inner. Thus we have an operator $T(a(x)\partial_x)$ determined up to a scalar by the formula

$$\left[T(a(x)\partial_x), \psi(f(x)dx^{1/2}) \right] = \psi\left(\mathcal{L}_{a\partial_x} (f(x)dx^{1/2}) \right)$$

Let's start again with a slightly different notation. Again let us consider the space of $\frac{1}{2}$ densities $f(x)dx^{1/2}$ on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ where f is anti-periodic, and equip this space with the obvious real structure $f(x)dx^{1/2} \longmapsto \overline{f(x)}\, dx^{1/2}$ and the quadratic form

$$\frac{1}{2}\int \frac{dx}{2\pi} f(x)^2 = \frac{1}{2}\sum c_k c_{-k} \qquad f = \sum c_k e^{ikx}$$

We then form the Clifford algebra generated by elements $\psi(f(x)dx^{1/2})$ satisfying

$$\psi(f(x)dx^{1/2})^* = \psi(\overline{f(x)}\, dx^{1/2})$$

$$\psi(f(x)dx^{1/2})^2 = \frac{1}{2}\int \frac{dx}{2\pi} f(x)^2$$

Thus if $\psi_k = \psi(e^{ikx}dx^{1/2})$, then we have

$$\boxed{\psi_k^* = \psi_{-k}}$$

and

$$\Big(\sum c_k \psi_k\Big)^2 = \frac{1}{2}\sum c_k c_{-k}$$

$$\sum_{k,\ell} c_k c_\ell \{\psi_k, \psi_\ell\}/2 = \frac{1}{2}\sum c_k c_\ell\, \delta_{k+\ell,0}$$

whence

$$\boxed{\{\psi_k, \psi_\ell\} = \delta_{k+\ell,0}}$$

We can now form the Fock space representation of this Clifford algebra having ground state $|0\rangle$ such that ψ_k for $k>0$ are creation and for $k<0$ are destruction operators. I think I am mainly interested in the normal ordering associated to this ground state. Thus all products $\psi_k\psi_\ell$ are in normal order except $\psi_k\psi_{-k}$ for $k<0$:

$$\psi_k\psi_\ell = :\psi_k\psi_\ell: + \begin{cases} 1 & \text{if } k=\ell<0 \\ 0 & \text{otherwise} \end{cases}$$

Check: $\psi_k\psi_{-k} = -\psi_{-k}\psi_k + 1 = :\psi_k\psi_{-k}: + 1$ for $k<0$.

Thus setting $\psi(z) = \sum z^k \psi_k$

we have

$$\psi(z)\psi(w) = \sum z^k w^l :\psi_k\psi_l: + \sum_{k<0} z^k w^{-k}$$

$$\sum_{k<0} (z/w)^k = \sum_{n\geq 0} (w/z)^{\frac{1}{2}+n} = \frac{(w/z)^{1/2}}{1-(w/z)} = \frac{z^{1/2}w^{1/2}}{z-w}$$

So we have

$$\boxed{\psi(z)\psi(w) = :\psi(z)\psi(w): + \frac{\sqrt{zw}}{z-w} \qquad |z|>|w|}$$

(It is better to use $\psi(z) = \sum z^{k-1/2}\psi_k$) (see below)

Now one thing I want to understand is why it ~~is~~ better to work with $z^{-1/2}\psi(z)$. In the present case, as opposed to $L(z)$, it's clear because one wants single-valued functions.

Let's go onto vector fields. Given a vector field $f(x)\partial_x$ we want to find an element $L(f\partial_x)$ in the Clifford algebra such that

$$[L(f\partial_x), \psi(g\,dx^{1/2})] = \psi(\mathcal{L}_{f\partial_x}(g\,dx^{1/2}))$$

$$= \psi((fg' + \tfrac{1}{2}f'g)\,dx^{1/2})$$

Set $L_k = L(e^{ikx}\frac{1}{i}\partial_x)$. As

$$\frac{1}{i}e^{ikx}(e^{imx})' + \frac{1}{2}\left(\frac{1}{i}e^{ikx}\right)' e^{imx} = e^{i(k+m)x}\left(m+\frac{k}{2}\right)$$

we have

$$\boxed{[L_k, \psi_m] = \left(m+\frac{k}{2}\right)\psi_{k+m}}$$

Now observe

$$\Big[\sum_{k+l=p} \frac{k-l}{4}\psi_k\psi_l, \psi_m\Big] = \sum_{k+l=p}\left(\frac{k-l}{4}\right)(\psi_k\delta_{l,-m} - \delta_{k,-m}\psi_l)$$

$$= \frac{(p+m-(-m))}{4}\psi_{p+m} - \frac{(-m-(p+m))}{4}\psi_{p+m} = \left(\frac{p}{2}+m\right)\psi_{p+m}$$

Thus we have in the Clifford algebra

$$\boxed{L_p = \sum_{k+l=p} \frac{k-l}{4} \psi_k \psi_l}$$

normal ordered for $p=0$.

$$\sum z^p L_p = \sum_{k,l}' \frac{k-l}{4} :z^k \psi_k \, z^l \psi_l:$$

Now use that $:\psi_k \psi_l:$ is skew-symmetric in k, l, whence

$$\sum z^p L_p = \sum \frac{k}{2} : z^k \psi_k \, z^l \psi_l :$$

$$\sum z^{p-1} L_p = \sum \frac{k-\frac{1}{2}}{2} : z^{k-\frac{1}{2}} \psi_k \, z^{l-\frac{1}{2}} \psi_l :$$

$$= \frac{1}{2} : z(\partial_z \tilde{\psi}_z) \tilde{\psi}_z :$$

where $\boxed{\tilde{\psi}_z = z^{-1/2} \psi_z}$. Thus

$$\boxed{T(z) = \sum' z^{p-2} L_p = \frac{1}{2} :(\partial_z \tilde{\psi}_z) \tilde{\psi}_z :}$$

~~and now we are able to work out the operator product expansion starting from the Wick formula~~

$$\cancel{(:\tilde{\psi}_{z_1} \tilde{\psi}_{z_2}:)(:\tilde{\psi}_{z_3} \tilde{\psi}_{z_4}:) = :\tilde{\psi}_{z_1} \tilde{\psi}_{z_2} \tilde{\psi}_{z_3} \tilde{\psi}_{z_4}: +}$$

$$\cancel{+ \frac{1}{z_1 - z_4} :\tilde{\psi}_{z_2} \tilde{\psi}_{z_3}: + \frac{1}{z_2 - z_3} :\tilde{\psi}_{z_1} \tilde{\psi}_{z_4}: + \frac{1}{z_1 - z_4} \frac{1}{z_2 - z_3}}$$

$$\cancel{\frac{1}{4}(:(\partial_{z_1} \tilde{\psi}_{z_1}) \tilde{\psi}_{z_2}:)(:(\partial_{z_3} \tilde{\psi}_{z_3}) \tilde{\psi}_{z_4}:) = \text{regular} +}$$

$$\cancel{\frac{-1}{4(z_1 - z_4)^2} :\tilde{\psi}_{z_2} \partial_{z_3} \tilde{\psi}_{z_3}: + \frac{1}{4(z_2 - z_3)^2} :(\partial_{z_1} \tilde{\psi}_{z_1}) \tilde{\psi}_{z_4}: + \frac{-1}{4(z_1 - z_4)^2 (z_2 - z_3)^2}}$$

We want now to calculate the operator product expansion for $T(z)T(w)$. Change notation and put

$$\boxed{\psi_z = \sum_k z^{k-1/2}\psi_k}$$

so that

$$\boxed{\psi_z \psi_w = :\psi_z\psi_w: + \frac{1}{z-w}}$$

$$\boxed{T(z) = \frac{1}{2}:\partial\psi_z\psi_z:}$$

Wick's thm. gives with $\psi_j = \psi_{z_j}$:

$$(:\psi_1\psi_2:)(:\psi_3\psi_4:) = :\psi_1\psi_2\psi_3\psi_4: +$$

$$+\frac{-1}{z_1-z_3}:\psi_2\psi_4: + \frac{1}{z_1-z_4}:\psi_2\psi_3: + \frac{1}{z_2-z_3}:\psi_1\psi_4: + \frac{-1}{z_2-z_4}:\psi_1\psi_3:$$

$$+\frac{-1}{z_1-z_3}\frac{1}{z_2-z_4} + \frac{1}{z_1-z_4}\frac{1}{z_2-z_3}$$

So $(:\partial\psi_1\psi_2:)(:\partial\psi_3\psi_4:) = \text{reg.} +$

$$+\frac{+2}{(z_1-z_3)^2}:\psi_2\psi_4: + \frac{-1}{(z_1-z_4)^2}:\psi_2\partial\psi_3: + \frac{1}{(z_2-z_3)^2}:\partial\psi_1\psi_4: + \frac{-1}{(z_2-z_4)}:\partial\psi_1\partial\psi_3:$$

$$+\frac{2}{(z_1-z_3)^3}\frac{1}{z_2-z_4} + \frac{-1}{(z_1-z_4)^2}\frac{1}{(z_2-z_3)^2}$$

Now set $z_1=z_2=z$, $z_3=z_4=w$.

$$4\,T(z)T(w) \equiv \frac{1}{(z-w)^4} + \frac{2}{(z-w)^3}:\psi_z\psi_w: + \left(\frac{-1}{(z-w)^2}\right):\psi_z\partial\psi_w:$$

$$+\frac{1}{(z-w)^2}:\partial\psi_z\psi_w: + \frac{-1}{(z-w)}:\partial\psi_z\partial\psi_w:$$

$$\equiv \frac{1}{(z-w)^4} + \frac{2}{(z-w)^3}\left\{:\cancel{\psi_w\psi_w}: + (z-w):\partial\psi_w\psi_w: + \frac{(z-w)^2}{2}:\partial^2\psi_w\psi_w:\right\}$$

$$\frac{-1}{(z-w)^2}\left\{:\psi_w\partial\psi_w: + (z-w):\cancel{\partial\psi_w\partial\psi_w}:\right\}$$

$$+\frac{1}{(z-w)^2}\left\{:\partial\psi_w\psi_w: + (z-w):\partial^2\psi_w\psi_w\right\}$$

$$4T(z)T(w) \equiv \frac{1}{(z-w)^4} + \frac{4}{(z-w)^2} :\partial\psi_w\psi_w: + \frac{2}{(z-w)} :\partial^2\psi_w\psi_w:$$

Now $\quad \partial_w T_w = \frac{1}{2}\left(:\partial^2_w\psi_w\psi_w: + \cancel{:\partial\psi_w\partial\psi_w:}\right)$

So

$$\boxed{T(z)T(w) \equiv \frac{1}{4(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{(z-w)}\partial_w T(w)}$$

Now from this we obtain

$$[L_p, T(w)] = \operatorname*{Res}_{z=w} z^{-p+2-1}\{\quad\}$$

$$= \frac{1}{4}\left.\frac{\partial^3_z(z^{-p+1})}{3!}\right|_{z=w} + 2\,\partial_z(z^{-p+1})\Big|_{z=w} T(w) + w^{-p+1}\partial_w T(w)$$

$$= \frac{1}{24}(-p+1)(-p)(-p-1)w^{-p-2} + 2(-p+1)w^{-p}T(w) + \underline{\qquad}.$$

$$\boxed{[L_p, T(w)] = w^{-p}(w\partial_w - 2p+2)T(w) - \frac{p^3-p}{24}w^{-p-2}}$$

Take coeff of w^{-m-2} $\qquad w^{m+p-2}$

$$\boxed{[L_p, L_m] = (m-p)L_{p+m} - \frac{p^3-p}{24}\delta_{-p,m}}$$

Now the physicists prefer reversing signs as if we ~~take~~ $p \mapsto -p$, $m \mapsto -m$, then we get

$$[L_{-p}, L_{-m}] = (p-m)L_{-p-m} + \frac{p^3-p}{24}\delta_{p,-m}$$

Similarly we ~~know~~ compute

$$\boxed{T(z)\psi(w) \equiv \frac{1}{2(z-w)^2}\psi(w) + \frac{1}{z-w}\partial_w\psi(w)}$$

$$[L_p, \psi(w)] = w^{-p}\left(w\partial_w + \frac{-p+1}{2}\right)\psi(w)$$

So $\psi(z)$ is a *primary field*

February 16, 1987

Return to $J_p = \sum :\psi^*_{p+k}\psi_k:$, $L_p = \sum (k+\frac{p}{2}):\psi^*_{p+k}\psi_k:$

On p. 451 we found

$$* \qquad [L_p, J_m] = m J_{p+m} + \frac{m}{2}\delta_{p,-m}$$

So $[L_{-m}, J_m] = m J_0 + \frac{m}{2} = m\left(J_0 + \frac{1}{2}\right)$. Thus if we alter J_0 to $J_0 + \frac{1}{2}$ the commutation relations * simplify. Notice that

$$J_0 + \frac{1}{2} = \sum_{k\neq 0} :\psi^*_k\psi_k: + \overbrace{-\psi_0\psi^*_0 + \frac{1}{2}}^{\psi^*_0\psi_0 - \frac{1}{2}}$$

so if we were to change our normal ordering so that

$$\psi^*_0\psi_0 = :\psi^*_0\psi_0: + \frac{1}{2}$$

then the new J_0 has the simple commutation relations.

Next from 450 we have

$$[L_p, L_m] = (m-p)L_{p+m} + \left(\frac{-p^3}{12} + \frac{-p}{6}\right)\delta_{-p,m}$$

so

$$[L_{-m}, L_m] = 2m L_0 + \frac{m^3+2m}{12}$$
$$= 2m\left(L_0 + \frac{1}{8}\right) + \frac{m^3 - m}{12}$$

Now we can't arrange to get $L_0 + \frac{1}{8}$ by changing the normal ordering in the $k=0$ term for $L_0 = :\sum k\psi^*_k\psi_k:$ The reason for the $\frac{1}{8}$ is apparently due to the fact we are using Ramond (or periodic) fermions. In general one gets the number of real fermions $\times \frac{1}{16}$ ~ see Goddard + Olive 5.4.1.

February 17, 1987

One more Virasoro calculation so as to see the Ramond or periodic case. Recall

$$L_p = \sum_{k+l=p}' \frac{k-l}{4} :\psi_k\psi_l: = \sum_{k+l=p}' \frac{k}{2} :\psi_k\psi_l:$$

$$[\psi_k\psi_l, \psi_m\psi_n] = [\psi_k\psi_l, \psi_m]\psi_n + \psi_m[\psi_k\psi_l, \psi_n]$$

$$= (\psi_k \delta_{l+m}\psi_n - \delta_{k+m}\psi_l\psi_n) + \psi_m(\psi_k\delta_{l+n} - \delta_{k+n}\psi_l)$$

$$= \underbrace{\delta_{l+m}\psi_k\psi_n - \delta_{k+m}\psi_l\psi_n + \delta_{l+n}\psi_m\psi_k - \delta_{k+n}\psi_m\psi_l}$$

$$[L_p, L_q] = \lim_N \sum_{\substack{k+l=p \\ m+n=q \\ |k|,|l|,|m|,|n| \le N}} \frac{km}{4}[:\psi_k\psi_l:, :\psi_m\psi_n:]$$

$$\begin{matrix} a+l=p \\ m+b=q \end{matrix} \qquad \begin{matrix} k+a=p \\ m+b=q \end{matrix} \qquad \begin{matrix} b+l=p \\ a+n=q \end{matrix} \qquad \begin{matrix} k+b=p \\ a+n=q \end{matrix}$$

$$= \lim_N \Bigg(\sum_{\substack{|a|,|p-a|, \\ |q-b|,|b| \le N \\ a+b=p+q}} \frac{a(q-b)}{4}\psi_a\psi_b - \sum_{\substack{|p-a|,|a| \\ |q-b|,|b| \le N \\ a+b=p+q}} \frac{(p-a)(q-b)}{4}\psi_a\psi_b$$

$$+ \sum_{\substack{|b|,|p-b| \\ |a|,|q-a| \le N \\ a+b=p+q}} \frac{ba}{4}\psi_a\psi_b - \sum_{\substack{|p-b|,|b| \\ |a|,|q-a| \le N \\ a+b=p+q}} \frac{(p-b)a}{4}\psi_a\psi_b \Bigg)$$

Now take the normal ordered part of this and take the limit as $N \to \infty$. We can take this limit separately over the four pieces and we obtain

$$\sum_{a+b=p+q} \frac{1}{4}\left(a(q-b) - (p-a)(q-b) + ba - (p-b)a \right) :\psi_a\psi_b:$$

$$= \sum_{a+b=p+q}' \frac{1}{4}\left(aq + aq + p\overset{(p+q-a)}{b} - pa \right) :\psi_a\psi_b:$$

$$= (q-p) \sum_{a+b=p+q} \frac{a}{2} :\psi_a\psi_b: = (q-p) L_{p+q}$$

Next we have to worry about the contraction terms $\langle \psi_a\psi_b \rangle$. These are non-zero only when ~~p+q=…~~ $a+b=0$, so we suppose $p=-q$. We get for the contraction part the sum

$$\lim_N \left(\sum_{|a|,|q+a|\le N}' \frac{a(q+a)}{4} \langle \psi_a\psi_{-a} \rangle + \sum_{|a|,|q+a|\le N} \frac{(q+a)^2}{4} \langle \psi_a\psi_{-a} \rangle \right.$$

$$\left. \sum_{|a|,|q-a|\le N} \frac{-a^2}{4} \langle \psi_a\psi_{-a} \rangle + \sum_{|q-a|,|a|\le N}' \frac{(q-a)a}{4} \langle \psi_a\psi_{-a} \rangle \right)$$

$$= \lim_N \left\{ \sum_{|a|,|q+a|\le N} \frac{(q+2a)(q+a)}{4} \langle \psi_a\psi_{-a} \rangle + \sum_{|a|,|q-a|\le N}' \frac{\overset{a(q-2a)}{qa-2a^2}}{4} \langle \psi_a\psi_{-a} \rangle \right\}$$

Now suppose $q>0$ so the first sum is over $-N \le a \le N-q$ and the second sum is over $-N+q \le a \le N$. Better approach: Without assuming ~~the above~~ that $q>0$, rewrite the above

$$\lim_N \sum_{|a|,|q+a|\le N}' \frac{1}{2}(a+q)\left(a+\frac{q}{2}\right) \langle \psi_a\psi_{-a} \rangle - \sum_{|a|,|a-q|\le N}' \frac{1}{2} a\left(a-\frac{q}{2}\right) \langle \psi_a\psi_{-a} \rangle$$

$$= \lim_N \left\{ \sum_{|a-\frac{q}{2}|,\ |a+\frac{q}{2}|\le N}' \frac{1}{2}\left(a+\frac{q}{2}\right) a \langle \psi_{a-q/2}\,\psi_{-a+q/2}\rangle - \sum_{|a+\frac{q}{2}|,\ |a-\frac{q}{2}|\le N}' \frac{1}{2}\left(a+\frac{q}{2}\right) a \langle \psi_{a+q/2}\,\psi_{-a-q/2}\rangle \right\}$$

$$= \sum_a' \frac{1}{2}\left(a+\frac{q}{2}\right) a \left\{ \langle \psi_{a-q/2}\,\psi_{-a+q/2}\rangle - \langle \psi_{a+q/2}\,\psi_{-a-q/2}\rangle \right\}$$

$$= \sum_a' \frac{1}{2}\, a\left(a-\frac{q}{2}\right) \left\{ \langle \psi_{a-q}\,\psi_{-a+q}\rangle - \langle \psi_a\,\psi_{-a}\rangle \right\}$$

Let's now try to evaluate this in the AP case where

$$\langle \psi_a \psi_{-a} \rangle = \begin{cases} 0 & a > 0 \\ 1 & a < 0 \end{cases}$$

Thus assuming $q > 0$ we have

$$* \qquad \sum_{0<a<q}' \frac{1}{2}\, a\left(a-\frac{q}{2}\right)$$

To evaluate use that with $(\Delta f)(x) = f(x) - f(x-1)$

$$\Delta\, \frac{2n^3+3n^2+n}{6} = n^2$$

$$\Delta\, \frac{n^2+n}{2} = n.$$

Thus since a ranges over half integers we want

$$\left[\frac{1}{2}\,\frac{2n^3+3n^2+n}{6} - \frac{q}{4}\,\frac{n^2+n}{2} \right]_{-1/2}^{q-1/2}$$

$$= \frac{1}{12}\left\{ 2\left[\left(q-\tfrac{1}{2}\right)^3 - \left(-\tfrac{1}{2}\right)^3\right] + 3\left[\left(q-\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2\right] + \left[\left(q-\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}\right)\right] \right\}$$

$$- \frac{q}{8}\left\{ \left[\left(q-\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2\right] + \left[\left(q-\tfrac{1}{2}\right) - \left(-\tfrac{1}{2}\right)\right] \right\}$$

$$= \frac{1}{12}\left\{ 2q^3 + q^2\left(\cancel{2\cdot 3\cdot\left(-\tfrac{1}{2}\right) + 3}\right) + q\overbrace{\left(2\cdot 3\cdot \tfrac{1}{4} + 3\cdot 2\cdot\left(-\tfrac{1}{2}\right) + 1\right)}^{-1/2} \right\}$$

$$- \frac{1}{8}\, q \left\{ q^2 + q\left(\cancel{2\cdot\left(-\tfrac{1}{2}\right) + 1}\right) \right\}. \qquad = \frac{1}{24}\left(q^3 - q\right)$$

Thus in the AP = NS case we have

$$[L_p, L_q] = (q-p)L_{p+q} + \frac{1}{24}(q^3-q)\,\delta_{p,-q}$$

as we learned already

Now for the periodic = Ramond case where $a \in \mathbb{Z}$ and

$$\langle \psi_a \psi_{-a} \rangle = \begin{cases} 0 & a>0 \\ \frac{1}{2} & a=0 \\ 1 & a<0 \end{cases}$$

Then the constant we want

$$\sum_a \tfrac{1}{2}a\left(a-\tfrac{q}{2}\right)\left\{\langle \psi_{a-q}\psi_{-a+q}\rangle - \langle \psi_a \psi_{-a}\rangle\right\}$$

is $\displaystyle \frac{1}{2}q\left(\frac{q}{2}\right)\frac{1}{2} + \sum_{0<a<q} \frac{1}{2}a\left(a-\frac{q}{2}\right) + 0$ for $q>0$

$$= \frac{q^2}{8} + \left[\frac{1}{12}\left(2n^3+3n^2+n\right) - \frac{q}{8}\left(n^2+n\right)\right]_0^{q-1}$$

$$= \frac{q^2}{8} + \frac{1}{12}\left[2(q-1)^3 + 3(q-1)^2 + (q-1)\right] - \frac{q}{8}\left[(q-1)^2 + (q-1)\right]$$

$$= \frac{q^2}{8} + \frac{1}{12}\left[2q^3 + q^2\overbrace{(2\cdot 3\cdot(-1)+3)}^{-3} + q\underbrace{(2\cdot 3+3\cdot 2(-1)+1)}_{1} + \cancel{(-2+3-1)}\right]$$

$$- \frac{q}{8}\left[q^2 - 2q \cancel{+1} + q \cancel{-1}\right]$$

$$= \frac{1}{24}\left[(3q^2) + (4q^3 - 6q^2 + 2q) - (3q^3 - 3q^2)\right]$$

$$= \frac{1}{24}\left[q^3 + 2q\right].$$

So

$$[L_{-q}, L_q] = 2qL_0 + \frac{1}{24}\underbrace{(q^3+2q)}_{q^3-q+3q}$$

$$= 2qL_0 + \frac{q}{8} + \frac{1}{24}(q^3-q)$$

$$\boxed{[L_{-q}, L_q] = 2q\left(L_0 + \frac{1}{16}\right) + \frac{1}{24}(q^3-q)}$$

February 18, 1987

It is necessary now to discuss quantum field theory invariantly on a Riemann surface M. We suppose given the holomorphic structure on M but not a metric. We also suppose given a spin structure, that is, a square root of the canonical line bundle.

Now suppose we have a circle embedded in M, and suppose it is oriented. If L is the given square root of $K = T^{1,0}$, then we have a canonical isomorphism $L \otimes L \simeq K$. Over the circle K has a real structure. Why? We have $(T^*)_r \simeq T^{1,0}$ and over the circle $T(M)$ splits into the tangent bundle to the circle and i times it in the sense of the complex structure.

Put another way $T(M)$ is a complex line bundle over M which upon restriction to the circle has a canonical real sub line bundle given by the tangent space to the circle. Thus $T(M)$ along the circle has a real structure and so does its dual $T^{1,0} = K$. Then from the isom $L \otimes L \simeq K$, we see L has a real structure, namely $\xi \in L$ is real iff $\xi \otimes \xi$ is real.? This is not quite right. Notice that because of the orientation of the circle it makes sense for an element of K to be real and ≥ 0, so $\xi \in L$ is real iff $\xi \otimes \xi \in K$ is real and ≥ 0.

Than the space of real sections of L over the circle form a real inner product space and we obtain a Clifford algebra attached to any oriented embedded circle. This works for immersed circles.

Now it occurs to me that there is a link between what I am doing with a surface and what Witten does for a spin manifold, except that in Witten's case one considers all loops in M, not just immersed ones.

(Witten's construction: He wants spinors on $\mathcal{L}M$, M Riemannian, so he takes for each loop the tangent space to the loop which is the tangent fields along the loop in M. This has an inner product, whence one has a Clifford algebra. Also one has the covariant derivative operator relative to the Levi-Civita connection on the tangent fields along the loop. This gives the polarization except for the 0 modes which lead to the $p_1 = 0$ condition on M.)

Let's return then to the case where M is a real surface with complex structure but without metric and L is a square root of $T^*(M)$. Then over an immersed circle L acquires a real structure and we have a Clifford algebra attached to sections of L over the circle. This Clifford algebra has a canonical irreducible representation which depends only on the orientation of the circle. If you parametrize the circle you get a metric on its cotangent bundle and hence a unique connection preserving the metric, and similarly for L.

~~[illegible]~~ Suppose M is Riemannian, then the connection on $T^*(M)$ and metric induce one on L, and if we have an immersed curve we get a metric on it too. Maybe there is some link between the Clifford algebras.

It seems that we don't obtain anything except for immersed circles.

The next question is how to compare different circles. What are the field operators $\psi(z)$?

More Virasoro computations.

February 19, 1987:

The goal is to describe the action of the Virasoro algebra in the boson picture. We start with the commutation relations

$$[L_p, J_q] = q\, J_{p+q} \qquad\qquad [J_p, J_q] = q\,\delta_{-p,q}$$

which we established for

$$L_p = \sum \left(\tfrac{p}{2}+k\right) :\psi^*_{p+k}\psi_k:$$

$$J_q = \sum :\psi^*_{q+k}\psi_k:$$

provided we work either with AP conditions or with periodic conditions and the convention

$$\langle \psi^*_k \psi_k \rangle = \begin{cases} 0 & k>0 \\ \frac{1}{2} & k=0 \\ 1 & k<0 \end{cases}$$

(Actually I did this for the P case - see 442, 448, 451 where I obtained the more complicated formulas $[L_p, L_q] = qJ_{p+q} + \frac{q}{2}$, and then 462 where the simpler formulas are obtained by changing the normal ordering.)

Now we want to work entirely in the boson situation, and to construct the Virasoro generators directly in terms of the J_k. We have

$$\sum_{k+l=p} [J_k J_l, J_q] = \sum_{k+l=p} (J_k q \delta_{-l,q} + q \delta_{-k,q} J_l) = 2q J_{p+q}$$

so that

$$L_p = \frac{1}{2} \sum_{k+l=p} :J_k J_l: = \frac{1}{2} \sum :J_k J_{p-k}:$$

for a suitable normal ordering. Recall that normal ordering does not apply to operators, but rather it produces operators out of polynomials in the field variables, i.e. it is a quantization process. Thus I have to have a separate definition for $:J^\alpha:$ than the one I have for the $\psi^\dagger\psi$ fields. The obvious choice is

$$\cancel{:J_k J_{-k}: = \begin{cases} J_k J_{-k} & k \geq 0 \\ J_{-k} J_k & k < 0 \end{cases}}$$

~~Then the contraction is~~

$$\cancel{\langle J_k J_{-k} \rangle = 0 \qquad k \geq 0}$$

$$\cancel{\langle J_k J_{-k} \rangle = J_k J_{-k} - J_{-k} J_k =}$$

$$\cancel{J_{+k} J_l = :J_{-k} J_l:} \quad \text{~~unless~~}$$

~~If $k<0$, then $:J_{-k} J_k: = J_k J_{-k}$. Thus if $k<0$~~

$$\cancel{\langle J_k J_{-k} \rangle = J_k J_{-k} - J_{-k} J_k = -k}$$

~~and if $k>0$ the contraction is 0. $\therefore \langle J_0 J_0 \rangle = 0$~~

$$J_k J_\ell = :J_k J_\ell: \quad \text{unless } k=-\ell<0.$$

In this case

$$J_{-\ell} J_\ell = \underbrace{J_\ell J_{-\ell}}_{:J_{-\ell} J_\ell:} + \ell$$

Thus we want

$$J_k J_\ell = :J_k J_\ell: + \begin{cases} \ell & \text{if } \ell=-k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and we have the following operator product formula with $J(z) = \sum z^k J_k$

$$J(z)\, J(w) = \sum z^k \cancel{w} w^\ell :J_k J_\ell: + \sum_{\ell \geq 0} z^{-\ell} w^\ell \ell$$

Now $\displaystyle\sum_{n\geq 0} n t^n = t\partial_t \left(\sum t^n\right) = t\partial_t \frac{1}{1-t} = \frac{t}{(1-t)^2}$

so

$$\boxed{J(z)J(w) = :J(z)J(w): + \frac{zw}{(z-w)^2}}$$

Now put $L(z) = \sum z^p L_p = \sum z^p :\frac{1}{2}\sum_{k+\ell=p} J_k J_\ell:$

$$= :\sum_{k,\ell} \frac{1}{2} z^{k+\ell} J_k J_\ell: = :\frac{1}{2} J(z) J(z):$$

and let's work out the O.P. Expansion for $L(z)L(w)$.

$$(:J_{z_1} J_{z_2}:)(:J_{z_3} J_{z_4}:) = :1234: +$$

$$\langle 13\rangle(:24:) + \langle 14\rangle(:23:) + \langle 23\rangle(:14:)$$

$$+ \langle 24\rangle(:13:) + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle$$

Let $z_1 = z_2 = z,\quad z_3 = z_4 = w.$

$$4\,L(z)L(w) \equiv \frac{4zw}{(z-w)^2}\left(:J_z J_w:\right) + \frac{2(zw)^2}{(z-w)^4}$$

Things simplify if we put $\tilde{J}_z = z^{-1} J_z$, $T(z) = z^{-2}L(z)$.
We get

$$T(z)\,T(w) \equiv \frac{1}{2}\frac{1}{(z-w)^4} + \frac{1}{(z-w)^2}\,:\tilde{J}_z \tilde{J}_w:$$

$$\underbrace{:\tilde{J}_w \tilde{J}_w:}_{2T(w)} + (z-w)\underbrace{:\left(\partial_w \tilde{J}_w\right)\tilde{J}_w:}_{\frac{1}{2}\partial_w\left(\underbrace{:\tilde{J}_w \tilde{J}_w:}_{2T(w)}\right)}$$

Thus

$$\boxed{T(z)\,T(w) \equiv \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}}$$

and we know this leads to the standard Virasoro relations with $c = 1$.

$$[L_p, L_q] = (q-p)L_{p+q} + \frac{1}{12}(q^3 - q)\,\delta_{-p,q}$$

February 22, 1987:

Apparently the continuum limit of the Ising model is a free fermion theory where the mass is zero at the critical temperature.

Recall the Ising model has configurations given by assignments of spins ± 1 at each point of a lattice. Take the lattice to be a discrete torus $\mathbb{Z}/N \times \mathbb{Z}/N = I \times J$. We propose to write the partition fn. as $\mathrm{tr}(T^N)$ where T is a $2^{|J|} \times 2^{|J|}$ matrix called the transfer matrix.

To do this let us write a configuration $s : I \times J \longrightarrow \{\pm 1\}$ as a map $\sigma : I \longrightarrow \{\pm 1\}^J$, so that σ_i gives the column of s over i. We then want to show the energy can be written

$$(*) \qquad E(s) = \sum_{i=0}^{N-1} F(\sigma_i, \sigma_{i+1})$$

If this is so then

$$Z = \sum_{\sigma_0, \dots, \sigma_{N-1}} T_{\sigma_0 \sigma_1} T_{\sigma_1 \sigma_2} \cdots T_{\sigma_{N-1} \sigma_0} = \mathrm{tr}(T^N)$$

where $T_{\sigma\sigma'} = e^{-\beta F(\sigma, \sigma')}$.

Now $(*)$ is clear from the assumption that the energy depends on nearest neighbor spins. In fact suppose

$$-E(s) = \underbrace{\lambda \sum_{i,j \in I \times J}' s_{i,j}\, s_{i,j+1}} + \mu \sum_{I \times J}' s_{ij}\, s_{i+1,j}.$$

Then

$$-E(s) = \sum_i f(\sigma) + \sum_i F(\sigma_i, \sigma_{i+1})$$

and so the transfer matrix $T_{\sigma\sigma'}$ is the product of the diagonal matrix with entries $e^{\beta f(\sigma)}$ and the matrix $e^{\beta F(\sigma,\sigma')}$ which is a tensor product of 2×2 matrices over the rows, better indexed by the rows.

I want to think of the possible σ as being a basis for the tensor product of 2diml subspaces, one for each $j \in J$. Ultimately I would like the transfer matrix T to turn out to be a recognizable element of a Clifford algebra.

Feb 23:

A Clifford algebra has two natural generating sets. One can work either with the 2 group generated by the γ^i or with the vector space spanned by the γ^i. The first question is whether we can find ~~the~~ analogue of this space. Thus the transfer matrix T operates on a 2^m-diml space, and we need to find $2m$ operators whose span is stable under conjugation (or bracketing?) with T.

Let's recall what T looks like. Recall that it operators on $S\otimes\cdots\otimes S$ where $S =$ 2diml spin span and there is a factor for each column site. Then T is the product of an operator T^h coming from the horizontal bonds and a operator T^v coming from the vertical bonds: $T = T^v T^h$. We have

$$T^h = P\otimes\cdots\otimes P$$

where P operates on S. T^v is the product of

February 23, 1987 (cont.)

the commuting operators $e^{\beta\sigma_i\sigma_{i+1}}$. Let's look at this operator $\sigma_1\sigma_2$ on $S_{(1)}\otimes S_{(2)}$. This is just the operator $\varepsilon_1\otimes\varepsilon_2$ which is diagonal in the standard basis. Now ε_i is quadratic in γ_i^1, γ_i^2, so it appears that $\varepsilon_1\otimes\varepsilon_2$ is quartic in the obvious Clifford generators. What I would like is something quadratic in Clifford generators. Note that $\varepsilon_1\otimes\varepsilon_2$ is the total grading on $S_{(1)}\otimes S_{(2)}$.

Now let's look at P on S. Relative to the standard basis it is

$$\begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} \qquad e^{2a} = (e^{\beta})^2 - (e^{-\beta})^2$$

and this is of the form $e^{a+b\gamma^1}$. Thus T^h is essentially the exponential of a linear operator in the standard γ's and T^v is the exponential of a quartic operator in the standard γ's.

Is it possible to change variables in our standard model for 2-diml spin space S so as to change γ^1 to $\varepsilon = -i\gamma^1\gamma^2$ so it becomes quadratic and to change ε to γ^1? This certainly seems alright and it means that we have an operator

$$T = T^v T^h \quad \text{where} \quad T^h = e^{\sum_j (a + b(-i\gamma_j^1\gamma_j^2))}$$

and $T^v = e^{c\sum \gamma_j^1 \otimes \gamma_{j+1}^1}$. But now I see there is a problem with signs.

February 25, 1987

I want to consider again the case of Dirac operators over the circle. The problem is to define these with coefficients to be any superconnection. There is an odd and even case. In the odd case one is given a loop $g : S^1 \longrightarrow U(V)$ and one wants to give some meaning to the Cayley transform of the operator $\gamma^1 \partial_x + \gamma^2 X$, where $X = \frac{g-1}{g+1}$. Since this operator anti-commutes with ε, the C.T. should represent a closed subspace of $L^2(S^1, V \oplus V)$, hopefully in a Hilbert-Schmidt Grassmannian.

Now it seems desirable to study the problem of Hilbert-Schmidt estimates.

In the even case one is given a loop $g : S^1 \longrightarrow G(V \oplus V)$ and one wants to define the Cayley transform of $\varepsilon \partial_x + X$, where $X = \frac{g-1}{g+1}$. I feel that analytically these two cases are the same, ~~the problem concerns~~ as one can pass from one to the other by doubling. The nice thing about the even case is that the singular hypersurface is of real codim ≥ 2, so we can move any g off it.

So let's look carefully at

$$h\varepsilon\partial_x + X = \begin{pmatrix} h\partial_x & -\bar{p} \\ p & -h\partial_x \end{pmatrix}$$

where $p(x)$ has values in $\mathbb{C}$. I'd like to know about the eigenvalues of the operator. We know

$$\|(h\varepsilon\partial_x + X)f\|^2 = (f, -(h\varepsilon\partial_x + X)^2 f)$$

$$= +\, h^2\|\partial_x f\|^2 + \|Xf\|^2 - \langle f, h\varepsilon \cdot \partial_x(X) f\rangle$$

General discussion. Set $Y = h\varepsilon\partial_x + X$. I'd like to show that the Cayley transform of Y

$$\frac{1+Y}{1-Y} = -1 + \frac{2}{1-Y}$$

is congruent to -1 modulo the Hilbert-Schmidt class. Thus I want to show $\frac{1}{1-Y}$ is Hilbert-Schmidt, i.e.

$$\left(\frac{1}{1-Y}\right)^*\left(\frac{1}{1-Y}\right) = \frac{1}{1-Y^2}$$

is of trace class. Now the graph norm is

$$\|f\|^2 + \|Yf\|^2 = \langle f, (1-Y^2)f\rangle$$

and we want to bound this below. From the above we have

$$\|Yf\|^2 = h^2\|\partial_x f\|^2 + \|Xf\|^2 - \langle f, h\varepsilon X' f\rangle$$

We want therefore to control this last term, and we've learned from Hörmander that this might be possible if h is small enough. Put in now that $X = \frac{g-1}{g+1}$, when we wish to compare

$$\|Xf\|^2 \qquad -\langle f, h\varepsilon X' f\rangle \qquad \|f\|^2$$

$$\frac{(g^{-1}-1)(g-1)}{(g^{-1}+1)(g+1)} \qquad -h\underbrace{\varepsilon\frac{1}{g+1}}_{\frac{1}{g^{-1}+1}\varepsilon} 2g'\frac{1}{g+1} \qquad 1$$

Now
$$1-X^2 = 1+\frac{(g^{-1}-1)(g-1)}{(g^{-1}+1)(g+1)} = \frac{(g^{-1}+1)(g+1)+(g^{-1}-1)(g-1)}{(g^{-1}+1)(g+1)}$$
$$= \frac{\cancel{2}4}{(g^{-1}+1)(g+1)}$$

Thus
$$\|f\|^2+\|Xf\|^2-\langle f, h\varepsilon X'f\rangle$$
$$= \left\langle \frac{1}{g+1}f, (\cancel{2}4 - h\varepsilon 2g')\frac{1}{(g+1)}f\right\rangle$$

So if $h\|g'\| < 1^2$, we get the estimate

$$(*) \qquad \|f\|^2+\|Yf\|^2 \geq h^2\|\partial_x f\|^2+\delta\left\|\frac{1}{(g+1)}f\right\|^2$$

with $\delta>0$.

Suppose we take a sequence g_n of unitaries without the eigenvalue -1 which converge to a g. We consider the graph of $Y_n = h\varepsilon\partial_x + X_n$, $X_n = \frac{g_n-1}{g_n+1}$ and take a sequence $(f_n, Y_n f_n)$ of points from these graphs which converges in $L^2\oplus L^2$ to (f, f'). Then the inequality $(*)$ shows that $f_n \to f$ in the H'-norm and also that $\frac{1}{(g_n+1)} f_n$ converges in L^2. One has to assume that $\|g_n'\| \longrightarrow \|g'\| < \frac{1}{n}$. If $\lambda =$ $\lim \frac{1}{(g_n+1)} f_n$, then $(g+1)\lambda = f$ so we see that $\left(\frac{1}{g+1}\right) f \in L^2$.

February 26, 1987

Let's review what we learned yesterday, but in higher dimensions. Take M to be an even dim'l torus $\mathbb{R}^n/\Gamma$ and take the generalized super-connection over M to be given by the trivial bundle $\tilde{V}$, where V is graded, the canonical connection d, and the unitary autom. g inverted by ε_V. First we look at the case where g is the C.T. of X, where X is a skew-adjoint odd endom. of $\tilde{V}$.

Let $S = S_n$ be the vector space of n-dim'l spinors, graded as usual by $\varepsilon_S = i^{-m}\gamma^1\dots\gamma^n$, $m = n/2$. The Dirac operator with coefficients $(\tilde{V}, d, X)$ is

$$Y = h\gamma^\mu\partial_\mu + \varepsilon_S X$$

operating on sections of $\tilde{S}\otimes\tilde{V}$. It is (formally) skew-adjoint ~~[illegible]~~ and odd relative to the total grading $\varepsilon_S\otimes\varepsilon_V$.

We have (put $\sigma = \varepsilon_S$)

$$\begin{aligned} Y^2 &= (h\gamma^\mu\partial_\mu + \sigma X)^2 \\ &= h^2\partial_\mu^2 + (h\gamma^\mu\partial_\mu\sigma X + \sigma X h\gamma^\mu\partial_\mu) + X^2 \\ &= h^2\partial_\mu^2 + h\gamma^\mu\sigma\,\partial_\mu(X) + X^2 \end{aligned}$$

which gives the formula

$$\begin{aligned} \|f\|^2 + \|Yf\|^2 &= \langle f,f\rangle - \langle f, Y^2 f\rangle \\ &= \langle f,f\rangle - \langle f, (h^2\partial_\mu^2 + h\gamma^\mu\sigma(\partial_\mu X) + X^2)f\rangle \\ &= h^2\|\partial_\mu f\|^2 + \langle f, (1 - X^2 - h\gamma^\mu\sigma(\partial_\mu X))f\rangle \end{aligned}$$

Lets put in $X = \dfrac{g-1}{g+1} = 1 - \dfrac{2}{g+1}$.

$$1-\left(\frac{g-1}{g+1}\right)^2 - h\gamma^\mu\sigma \frac{1}{g+1}\, 2\partial_\mu g \,\frac{1}{g+1}$$

$$= \frac{(g+1)^2-(g-1)^2}{(g+1)^2} - h\gamma^\mu\sigma \frac{1}{g+1}\, 2\partial_\mu g\, \frac{1}{g+1}$$

$$= \frac{4g}{(g+1)^2} - h\gamma^\mu\sigma \frac{g}{g+1}\, 2g^{-1}\partial_\mu g\, \frac{1}{g+1}$$

$$= \frac{1}{(g^{-1}+1)}\, 4\, \frac{1}{g+1} + \frac{1}{g^{-1}+1}\left(-2h\gamma^\mu\sigma\, g^{-1}\partial_\mu g\right)\frac{1}{g+1}$$

$$\therefore \quad \|f\|^2+\|Xf\|^2 = h^2\|\partial_\mu f\|^2 + \left\langle \frac{1}{g+1}f, \left(4-2h\gamma^\mu\sigma g^{-1}\partial_\mu g\right)\frac{1}{g+1}f\right\rangle$$

Now $\|h\gamma^\mu\sigma\, g^{-1}\partial_\mu g\| \le h\sum_\mu \|\gamma^\mu\sigma g^{-1}\partial_\mu g\|$

$$\le h\sum_\mu \|\partial_\mu g\|$$

But we might be able to do better, e.g. $g^{-1}\partial_\mu g$ is skew adjoint, so $\gamma^\mu\sigma\, g^{-1}\partial_\mu g$ is self adjoint. No, the elements $g^{-1}\partial_\mu g$ don't necessarily commute, so $(\gamma^\mu\sigma\, g^{-1}\partial_\mu g)^2$ is more than just "$|\text{grad} \log g|^2$".

I need to understand "doubling". What I am trying to do is to make sense out of the Dirac operator

$$h\gamma^\mu\partial_\mu + \sigma X$$

where $X=\frac{g-1}{g+1}$ and the unitary g_n can have the eigenvalue -1. This is an analytical question which should be independent of whether we are in the graded case or not, or if we replace g by $g \oplus g^{-1}$. If we perform either the process of forgetting ε in the graded case, or replasing g by $g \oplus g^{-1}$ in the ungraded case, then we kill the obstruction to deforming g away from unitaries having the eigenvalue -1.

For past work see Nov 5, 86 p. 263-270.

The idea is that the natural embeddings

$$U(V) \hookrightarrow Gr(V \oplus V) \quad , \quad Gr(V) \hookrightarrow U(V)$$

extend to the suspensions and are the periodicity maps. The deformation moves in the suspension direction.

Consider the map $Gr(V) \hookrightarrow U(V)$ which takes a g inverted by ε and sends it to g, i.e. we forget ε. The relevant Bott map take F to the path $e^{i\theta F} = (\cos\theta) + (i\sin\theta)F$, so if I put $F = g^{-1}\varepsilon$ I get path in $U(V)$

$$\cos\theta + (i\sin\theta)g^{-1}\varepsilon$$

and hence the path in $U(V)$:

$$(\cos\theta)g + (i\sin\theta)\varepsilon .$$

Check:

$$\left((\cos\theta)g + (i\sin\theta)\varepsilon\right)^* \left((\cos\theta)g + (i\sin\theta)\varepsilon\right)$$

$$= \left((\cos\theta)g^{-1} - i\sin\theta\,\varepsilon\right)\left((\cos\theta)g + (i\sin\theta)\varepsilon\right)$$

$$= (\cos^2\theta) + \left[\cancel{\cos\theta(i\sin\theta)g^{-1}\varepsilon + (-i\sin\theta)(\cos\theta)\varepsilon g}\right] + \sin^2\theta = 1$$

since $g^{-1}\varepsilon = \varepsilon g$.

Next we need the eigenvalues of $u = (\cos\theta) g + (i \sin\theta)\varepsilon$. We have

$$\frac{1}{2}(u + u^{-1}) = (\cos\theta)\frac{g+g^{-1}}{2}$$

so the real parts of the eigenvalues of u are in absolute value $\le |\cos\theta|$. Thus for θ small + positive u doesn't have the eigenvalue ± 1.

Next we can consider the map sending g to $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$. The latter is inverted both by γ^1 and γ^2 $(= i\varepsilon\gamma^1)$. So by the above we have a path of unitaries

$$u_\theta = (\cos\theta)\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which for θ small $\neq 0$ don't have eigenvalues ± 1. Notice that

$$\gamma^1 u_\theta \gamma^1 = \cos\theta \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} + \sin\theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = u_\theta^* = u_\theta^{-1}$$

so that $\gamma^1 u_\theta$ represents a point in $Gr(V \oplus V)$. Thus we really have the path

$$\gamma^1 u_\theta = \cos\theta \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} + \sin\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defining the Bott map $\Sigma\, U(V) \longrightarrow Gr(V \oplus V)$.

February 27, 1987:

Notes for Roe (continued).

The basic formula which is valid for all g is

$$\|(g+1)u\|^2 + \|[h\gamma^\mu\partial_\mu(g+1) + (g-1)\sigma]u\|^2$$

$$= h^2 \sum_\mu \|\partial_\mu(g+1)u\|^2 + \langle u, (4 - 2h\gamma^\mu\sigma g^{-1}\partial_\mu g)u\rangle$$

Now I propose to define a unitary operator U which in the case $g = \frac{1+X}{1-X}$ is the C.T. $\frac{1+Q}{1-Q}$, $Q = h\gamma^\mu\partial_\mu + X\sigma$. In order to do this I look at the subspace of $L^2 \oplus L^2$ consisting of

$$\begin{pmatrix} g+1 \\ h\gamma^\mu\partial_\mu(g+1) + (g-1)\sigma \end{pmatrix} u$$

where u is in the Sobolev space W^1 initially, (or smooth will do). Then I close this up to get a closed subspace $\Gamma_g \subset L^2 \oplus L^2$.

Let's now check that $\Gamma_g \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_g$.

$$\langle [h\gamma^\mu\partial_\mu(g+1) + (g-1)\sigma]v, (g+1)u\rangle$$

$$= -\langle (g+1)v, h\gamma^\mu\partial_\mu(g+1)u\rangle + \langle \sigma v, \underbrace{(g^{-1}-1)(g+1)}_{g^{-1}(g+1)(1-g)}u\rangle$$

$$= -\langle (g+1)v, h\gamma^\mu\partial_\mu(g+1)u\rangle - \langle (g+1)v, (g-1)\sigma u\rangle$$

In doing this I used that u, v are in L^2 and that $(g+1)u, (g+1)v$ are in W^1.

The problem is to see that Γ_g and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_g$ are complementary.

Suppose we now assume that h is small enough so that

$$\| h \gamma^\mu g^{-1} \partial_\mu g \| \leq 2 - \delta < 2$$

at each point of M. Then we see that the norms

$$\| (g+1)u \|^2 + \| [h\gamma^\mu \partial_\mu (g+1) + (g-1)\sigma] u \|^2$$

$$h^2 \sum_\mu \| \partial_\mu (g+1) u \|^2 + \| u \|^2$$

on W^1 are equivalent. We can conclude that Γ_g consists exactly of

$$\begin{pmatrix} g+1 \\ h\gamma^\mu \partial_\mu (g+1) + (g-1)\sigma \end{pmatrix} u$$

with $u \in L^2$ such that $(g+1) u \in W^1$.

I have the feeling I am missing a compactness argument. I should be using that $\frac{1}{1-Q^2}$ is in a certain Schatten class.

Notes for Roe:

A. The setting: Let M be an even diml torus $\mathbb{R}^n/\text{lattice}$, $n = 2m$. The Dirac operator on M is $\gamma^\mu \partial_\mu$ where the γ^μ are anti-commuting self-adjoint involutions acting on the space S of n-diml spinors. S is graded by the involution $\sigma = i^{-m}\gamma^1 \cdots \gamma^n$.

Let (E, D, X) be a complex vector bundle over M with inner product, a connection preserving the inner product, and a skew-adjoint endomorphism of E. One can then form the Dirac operator with coefficients in (E, D, X). It is

$$Q = h\gamma^\mu D_\mu + \sigma X$$

acting on sections of $S \otimes E$. It is skew-adjoint. In the graded case (E equipped with a self-adjoint involution ε such that $\varepsilon D = D\varepsilon$, $\varepsilon X = -X\varepsilon$) the Dirac operator is odd relative to the total grading $\sigma\varepsilon$, and an index is defined.

Here h is a constant (Planck's constant$/2\pi$ in physics) which one lets go to zero to evaluate the index. One does not use the parameter t in the operator e^{+tQ^2}. One has the Weitzenbock formula

$$Q^2 = h^2 D_\mu^2 + \tfrac{1}{2} h^2 \gamma^\mu \gamma^\nu [D_\mu, D_\nu] + h\gamma^\mu [D_\mu, X]\sigma + X^2.$$

This shows that Q^2 can be treated by Getzler's symbol calculus, where both D_μ, γ^μ are considered to be of first order. Thus in the classical limit $h \to 0$, where $hD_\mu \sim ip_\mu$ $h\gamma^\mu \sim dx^\mu$, we have

$$Q^2 \sim -p_\mu^2 + \frac{1}{2} dx^\mu dx^\nu [D_\mu, D_\nu] + dx^\mu [D_\mu, X]\sigma + X^2$$

$$= -p_\mu^2 + D^2 + [D,X]\sigma + X^2$$

and by the Weyl thm. in Getzler's setting

$$\text{Index} = \lim_{h \to 0} \mathrm{Tr}_s\left(e^{Q^2}\right)$$

$$= \int_M \int \frac{d^n p}{(2\pi)^n} e^{-p_\mu^2} (2i)^m \, tr_s\left(e^{D^2 + [D,X]\sigma + X^2}\right)_{[n]}$$

$$= \left(\frac{i}{2\pi}\right)^m \int_M tr_s\left(e^{D^2 + [D,X]\sigma + X^2}\right)_{[n]}$$

(The above is intended as a sketch of a possible proof.)

B. The problem. The problem is to generalize the above so as to allow the skew-adjoint endomorphism X to be replaced by an arbitrary unitary automorphism g inverted by ε, and where the above is to correspond to the case $g = \frac{1+X}{1-X}$.

The generalization of the superconnection character form

$$tr_s\left(e^{D^2 + [D,X]\sigma + X^2}\right)$$

was carried out in my paper using the resolvent

$$\frac{1}{\lambda - X^2 - [D,X]\sigma - D^2}.$$

Upon

c

substituting $X = \frac{g-1}{g+1}$, this becomes

$$\circledast \qquad (g+1)\,\frac{1}{\lambda(g+1)^2-(g-1)^2 - 2[D,g]\sigma - (g+1)D^2(g+1)}\,(g+1)$$

and this expression was shown to make sense for an arbitrary unitary automorphism g, provided $\lambda \notin \mathbb{R}_{\leq 0}$. Similarly one can consider the resolvent $\frac{1}{\lambda - Q^2}$ and make the same substitution. One obtains

$$\circledast\!\circledast \qquad (g+1)\,\frac{1}{(g+1)(\lambda - h^2 D_\mu^2)(g+1) - (g-1)^2 - 2h\gamma^\mu[D_\mu,g]\sigma - (g+1)\frac{h^2}{2}\gamma^\mu\gamma^\nu[D_\mu,D_\nu](g+1)}\,(g+1)$$

On letting $h \to 0$ this expression goes into $\circledast$ above with λ replaced by $\lambda + p_\mu^2$. As $\lambda \notin \mathbb{R}_{\leq 0} \Rightarrow \lambda + p_\mu^2 \notin \mathbb{R}_{\leq 0}$, one sees that the $h \to 0$ limit of $\frac{1}{\lambda - Q^2}$ is defined for an arbitrary unitary automorphisms.

This suggests that $\circledast\!\circledast$ might be meaningful for small h.

C. Before going on let's simplify and take $(E, D) = (\tilde{V}, d)$, where $\tilde{V}$ = trivial bundle with fibre V. I suspect that by using the Narasimhan-Ramanan theorem and extending g by -1, one can reduce to the trivial bundle case as in §4 of my paper.

In the trivial bundle case $(\tilde{V}, d, g)$ g is a smooth map $g : M \longrightarrow U(V)$ such that $\varepsilon g \varepsilon = g^{-1}$. We are trying to make sense of

$$Q = h\gamma^\mu\partial_\mu + \sigma X \qquad X = \frac{g-1}{g+1}$$

for a general g having possibly -1 as an eigenvalue.

Now it might be useful to note that if we drop the requirement that our unitary automorphisms be inverted by ε, then g is a limit of unitary automorphisms not having the eigenvalue -1 anywhere. Set

$$g_\theta = (\cos\theta)g + (i\sin\theta)\varepsilon$$

Then

$$\begin{aligned} g_\theta^* g_\theta &= \left[(\cos\theta)g^{-1} + (-i\sin\theta)\varepsilon\right]\left[(\cos\theta)g + (i\sin\theta)\varepsilon\right] \\ &= \cos^2\theta + (i\sin\theta\cos\theta)\left[g^{-1}\varepsilon - \varepsilon g\right] + \sin^2\theta \\ &= 1 \end{aligned}$$

if $\varepsilon g \varepsilon = g^{-1}$. Thus g_θ is unitary and

$$\tfrac{1}{2}\left(g_\theta + g_\theta^{-1}\right) = (\cos\theta)\frac{g+g^{-1}}{2}$$

which shows that $-|\cos\theta| \leq \frac{1}{2}(g_\theta + g_\theta^{-1}) \leq |\cos\theta|$, hence g_θ doesn't have the eigenvalue ± 1 for $\theta \notin \mathbb{Z}$.

Similarly if we were to consider the ungraded case, where V is ungraded and g is an arbitrary unitary autom. of $\tilde{V}$, then by "doubling", that is, by considering $g \oplus g^{-1}$ on $\tilde{V} \oplus \tilde{V}$, we reach a situation where our unitary automorphism is a limit of ones without the eigenvalue -1. In effect $\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$ is inverted by $\gamma^2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$, so

$$g_\theta = \cos\theta \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} + i\sin\theta \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}$$

is unitary and has all eigenvalues $\neq \pm 1$ for $\theta \notin \mathbb{Z}$.

D. Formulas & Inequalities. Let g be a unitary autom. of $\tilde{V}$ not having the eigenvalue -1 and let

$$Q = h\gamma^\mu \partial_\mu + \sigma X \qquad X = \frac{g-1}{g+1}$$

Then

$$Q^2 = h^2 \partial_\mu^2 + h\gamma^\mu \sigma (\partial_\mu X) + X^2 \qquad \text{so}$$

$$\|f\|^2 + \|Qf\|^2 = \langle f, (1 - h^2\partial_\mu^2 - h\gamma^\mu\sigma(\partial_\mu X) - X^2) f \rangle$$

$$= \sum_\mu h^2 \|\partial_\mu f\|^2 + \langle f, \underbrace{(1 - h\gamma^\mu\sigma(\partial_\mu X) - X^2)} f \rangle$$

$$= 1 - h\gamma^\mu\sigma \frac{1}{g+1} 2(\partial_\mu g) \frac{1}{g+1} - \frac{(g-1)^2}{(g+1)^2}$$

$$= \frac{1}{g+1} \Big(\underbrace{(g+1)^2 - (g-1)^2}_{4g} - h\gamma^\mu\sigma\, 2\,\partial_\mu g \Big) \frac{1}{g+1}$$

$$= \frac{1}{g^{-1}+1} \left(4 - 2h\gamma^\mu\sigma g^{-1}\partial_\mu g \right) \frac{1}{g+1}$$

$$\boxed{\|f\|^2 + \|Qf\|^2 = h^2 \sum \|\partial_\mu f\|^2 + \left\langle \frac{1}{(g+1)} f, \left(4 - 2h\gamma^\mu\sigma g^{-1}\partial_\mu g\right) \frac{1}{(g+1)} f \right\rangle}$$

Putting $f = (g+1)u$ gives

f

$$\left\| \begin{pmatrix} g+1 \\ h\gamma^\mu \partial_\mu (g+1) + (g-1)\sigma \end{pmatrix} u \right\|^2 = \begin{array}{l} h^2 \sum_\mu \|\partial_\mu (g+1) u\|^2 \\ + \langle u, (4 - 2h\gamma^\mu \sigma g^{-1}\partial_\mu g) u \rangle \end{array}$$

This formula is valid for any g, as one sees by taking the limit as $g_0 \to g$ as in §C.

Suppose now that g arbitrary. The subspace of $L^2 \oplus L^2$ consisting of

$$\begin{pmatrix} g+1 \\ h\gamma^\mu \partial_\mu (g+1) + (g-1)\sigma \end{pmatrix} u$$

with u smooth can be closed. Let Γ_g denote the closure. When g is the Cayley transform of X, then Γ_g is just the closed graph of the Dirac operator Q.

One can check that

$$\Gamma_g \perp \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_g$$

for a general g. When $g = \frac{1+X}{1-X}$ these two subspaces are complementary – this is just the fact that Q is essentially skew-adjoint.

A natural <u>problem</u> is to prove that Γ_g and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Gamma_g$ are complements for small h. If this is the case, then one obtains a unitary operator u_g on L^2 which coincides with the Cayley transform $\frac{1+Q}{1-Q}$ in the case $g = \frac{1+X}{1-X}$.

Suppose now that h is small enough

that the norm of

$$h\gamma^\mu \sigma\, g^{-1}\partial_\mu g$$

is ≤ 1 at each point of M. Then we have an equivalence of norms

$$\left\| \begin{pmatrix} g+1 \\ h\gamma^\mu\partial_\mu(g+1) + (g-1)\sigma \end{pmatrix} u \right\|^2 \sim h^2 \sum \|\partial_\mu (g+1)u\|^2 + \|u\|^2 .$$

This shows that Γ_g is isomorphic to the space of u in L^2 such that $(g+1)u \in$ the Sobolev space W^1.

~~Apparently, Γ_g is the graph of the unbounded not necessarily densely defined operator which is the sum of $h\gamma^\mu\partial_\mu$ defined on W^1 and $\frac{g-1}{g+1}\sigma$ defined on the space of $f \in L^2$ such that $\frac{1}{g+1} f$ is defined a.e. and is in L^2. ?~~

(end of notes for Roe)

February 27, 1987 (continued)

Let's formulate the problem as follows. We work over a torus $M = \mathbb{R}^n/\Gamma$, $n = 2m$. Given a skew-hermitian $X : M \longrightarrow u(V)$ we can form the Dirac operator

$$Q_X = h\gamma^\mu \partial_\mu + \sigma X$$

acting on $C^\infty(M$[redacted]$, S \otimes V)$, where S = spinors and $\sigma =$ [redacted] $i^{-m}\gamma^1 \dots \gamma^n$. Here h is real $\neq 0$. Now it is known that Q_X is essentially [redacted] skew-adjoint, [redacted] so that its Cayley transform

$$U_X = \frac{1+Q_X}{1-Q_X} = -1 + \frac{2}{1-Q_X}$$

$$U_X^{-1} = \frac{1+Q_X}{1-Q_X} = -1 + \frac{2}{1+Q_X}$$

is a unitary operator on $L^2(M, S \otimes V)$.

[redacted] It is also known that the eigenvalues of Q_X have a certain growth which is the same for all first order elliptic operators. The way I probably want to use this is to say that $\frac{1}{1 \pm Q_X} \in \mathcal{L}^p$ for all $p > n$

I think it is probably pretty clear from known results that $X \longmapsto \frac{1}{1 \pm Q_X}$ is smooth. The problem is now to extend this map $X \mapsto U_X$ [redacted] to all unitary maps $g : M \longrightarrow U(V)$, i.e., to define U_g so as to coincide with U_X when $g = \frac{1+X}{1-X}$.

Ultimately I would like to understand how to prove the required properties of $\frac{1}{1 \pm Q_X}$. I can start with the case $X = 0$, which can be handled explicitly with Fourier series. Then I can treat X as a perturbation. This leads to a geometric series which converges only for small X. For some reason the skew-adjointness rescues you.

Suppose that Q is a densely-defined closed operator on H such that $Q \subset -Q^*$. Then it has deficiency indices. These are defined as follows. It turns out that $\mathrm{Ker}(Q^* - \lambda)$ has constant dimension for λ in either the RHP or LHP. These two dimensions are the deficiency indices. I think another definition is to consider $\Gamma_{Q^*} \ominus \Gamma_Q$ which is the orthogonal complement of $\Gamma_Q \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma_Q$. The involution $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts on this and the deficiency indices are the dimensions of the two eigenspaces.

February 28, 1987

Consider a closed subspace Γ of $H \oplus H$ such that

$$\Gamma^\perp = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma$$

e.g. the graph $\Gamma_Q = \left\{ \begin{pmatrix} x \\ Qx \end{pmatrix} \right\}$ of a skew-adjoint operator. Then we have isometries

$$(*) \qquad \Gamma \longrightarrow H \qquad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x \pm y$$

In effect $\begin{pmatrix} x \\ y \end{pmatrix} \perp \begin{pmatrix} y \\ x \end{pmatrix}$ for $\begin{pmatrix} x \\ y \end{pmatrix} \in \Gamma$ so

$$\|x+y\|^2 = \|x\|^2 + \cancel{\langle x,y\rangle + \langle y,x\rangle} + \|y\|^2 .$$

so the map $x+y$ is isometric. To see it is onto, let h be $\perp$ to the image. Then $\begin{pmatrix} h \\ h \end{pmatrix} \in \Gamma^\perp = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma$

so
$$\begin{pmatrix} h \\ h \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h \\ h \end{pmatrix} \in \Gamma \cap \Gamma^\perp = 0 .$$

When $\Gamma = \Gamma_Q$ the isometry $(*)$ can be identified with

$$\mathcal{D}_Q \xrightarrow{1 \pm Q} H$$

and so one gets the unitary operator $(1+Q)(1-Q)^{-1}$

Thus in general the maps

$$\begin{array}{ccccc} H & \xleftarrow{\sim} & \Gamma & \rightrightarrows & H \\ x-y & \longleftarrow & \begin{pmatrix} x \\ y \end{pmatrix} & \begin{array}{c} \longmapsto x \\ \longmapsto y \end{array} & \end{array}$$

are the analogues of $(1-Q)^{-1}$ and $Q(1-Q)^{-1}$.

An alternative + more complicated approach is to introduce the involution F and unitary $g = F\varepsilon$ corresponding to Γ. The condition $\Gamma^\perp = \gamma'\Gamma$ is equivalent to $\gamma' F \gamma' = -F$ which is equivalent to $\gamma' g \gamma' = g$.

Thus a subspace Γ with $\Gamma^\perp = \gamma'\Gamma$ is the same as a g unitary inverted by ε and commuting with γ'. If we conjugate we can arrange $\varepsilon \mapsto -\gamma'$ and $\gamma' \mapsto \varepsilon$, whence g would have the form $\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}$ with a unitary. The conjugation required is given by $1 + \gamma'\varepsilon = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ for

$$\varepsilon(1+\gamma'\varepsilon) = (1+\gamma'\varepsilon)(-\gamma')$$
$$\gamma'(1+\gamma'\varepsilon) = (1+\gamma'\varepsilon)\varepsilon$$

Thus

$$g = (1+\gamma'\varepsilon)\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\underbrace{(1+\gamma'\varepsilon)^{-1}}_{(1+\varepsilon\gamma')/2}.$$

This furnishes a unitary operator a on H associated to Γ.

It's clear from this approach that there is an equivalence between unitaries on H and closed subspaces Γ of $H \oplus H$ such that $\Gamma^\perp = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\Gamma$.

Now the next point project should be related to the condition that a be $\equiv -1$ modulo a Schatten ideal. This is equivalent to $g \equiv -1$ or that $F \equiv \varepsilon$.

The program: I propose to define a map from the space of $g: M \to U(V)$ to the manifold of unitaries on $L^2(M, S \otimes V)$ which are congruent to -1 modulo the appropriate Schatten ideal (i.e. $\mathcal{L}^p$ where $p > \dim M$). Now this isn't quite correct because of the parameter h. It is probably necessary to restrict to the open subset of g with $h \, \| \gamma^\mu \sigma \, g^{-1} \partial_\mu g \| \leq 1$.

Now this will not be useful unless I get a smooth map $g \mapsto U_g$. Thus I ought to understand the tangent map.

Let's discuss the issues more carefully. There appear to be two different approaches to constructing U_g — perturbation + parametrix. I am going to work on the perturbation approach first as it seems easier.

In the perturbation approach one supposes that U_{g_0} has been constructed and then one wants to treat a point g_1 near g_0. I guess the sensible way to proceed is to work with the tangent space at U_{g_0} to the unitaries congruent to -1 modulo the Schatten ideal.

Suppose $\quad U = \dfrac{1+Q}{1-Q}, \quad Q = h\partial\!\!\!/ + \sigma X, \quad X = \dfrac{g-1}{g+1}$

Then

$$\delta U = \delta\left(-1 + \frac{2}{1-Q}\right) = \frac{2}{1-Q}\,\delta Q\,\frac{1}{1-Q}$$

$$U^{-1}\delta U = \frac{2}{1+Q}\,\delta Q\,\frac{1}{1-Q}$$

Recall that $\delta U \to U^{-1}\delta U$ is the natural way to identify the ~~sp~~ tangent space to the space of unitary operator at U with the skew-adjoint operators. Now in our case

$$\delta Q = \sigma\,\delta X = \frac{1}{g+1}\,2\,\delta g\,\sigma\,\frac{1}{g+1}$$

so that

$$U^{-1}\delta U = 4\left(\frac{1}{1+Q}\,\frac{1}{g^{-1}+1}\right)(g^{-1}\delta g)\,\sigma\left(\frac{1}{g+1}\,\frac{1}{1-Q}\right)$$

$$= 4\left(\frac{1}{g+1}\,\frac{1}{1-Q}\right)^{*}(g^{-1}\delta g)\,\sigma\left(\frac{1}{g+1}\,\frac{1}{1-Q}\right)$$

$$\frac{1}{g+1}\,\frac{1}{1-Q} = \frac{1}{(1-h\partial\!\!\!/-\sigma X)(g+1)} = \frac{1}{(g+1)-[h\partial\!\!\!/(g+1)+\sigma(g-1)]}$$

Now recall our integration by parts formula

$$\|((g+1)-[h\partial\!\!\!/(g+1)+\sigma(g-1)])u\|^2$$

$$= \|(g+1)u\|^2 + \|(h\partial\!\!\!/(g+1)+\sigma(g-1))u\|^2$$

$$\sim \quad h^2\sum_\mu \|\partial_\mu(g+1)u\|^2 + \|u\|^2$$

This shows that $\frac{1}{g+1}\,\frac{1}{1-Q}$ is <u>not</u> in in the Schatten ideal we want it to be in for $g = -1$. I need to check this carefully before going on. Notice that the problem occurs already when g is constant.

Let's then return to the circle and the operator $h\gamma^1\partial_x + \gamma^2 \frac{1}{i} a$ where a is a real-valued function. For each $\frac{1}{i}a = X$ we get a skew-adjoint operator Q_X on $L^2(S^1, \mathbb{C}^2) = H$, such that $\frac{1}{1 \pm Q_X}$ is in $\mathcal{L}^p$ for $p > 1$. Thus I have a map from X's to unitaries $U_X = \frac{1+Q_X}{1-Q_X}$ which are congruent to -1 mod $\mathcal{L}^p$. The problem was to extend this to any map $g : S^1 \to U(1)$ and define U_g so that $U_g = Q_X$ when g is the C.T. of X.

Now I wanted U_g to depend smoothly on g, but it seems this isn't the case. Let's look carefully at the case where a is constant. What are the eigenvalues of the operator $Q_a = h\gamma^1\partial_x + \gamma^2\frac{1}{i}a$. We have $Q_a^2 = h^2\partial_x^2 - a^2$, hence the eigenvalues of Q_a^2 are $-(h^2n^2 + a^2)$ for $n \in \mathbb{Z}$. I really want to describe the C.T. U_a of Q_a carefully so as to see what's happening.

March 1, 1987

Finish example $Q_a = h\gamma^1\partial_x + \gamma^2(-ia)$, where $a \in \mathbb{R}$. For each a, the C.T. $u_a = \dfrac{1+Q_a}{1-Q_a}$ is $\equiv -1 \mod \mathcal{L}^p$, $p > 1$. Decompose $L^2(S^1, \mathbb{C}^2)$ as $\bigoplus_{n\in\mathbb{Z}} e^{inx}\mathbb{C}^2$. On the degree n piece

$$Q_a = \begin{pmatrix} 0 & ihn-a \\ ihn+a & 0 \end{pmatrix}$$

and

$$\frac{1}{2}(u_a+1) = \frac{1}{1-Q_a} = \frac{1+Q_a}{1-Q_a^2}$$

$$= \begin{pmatrix} \dfrac{1}{1+h^2n^2+a^2} & \dfrac{ihn-a}{1+h^2n^2+a^2} \\[2ex] \dfrac{ihn+a}{1+h^2n^2+a^2} & \dfrac{1}{1+h^2n^2+a^2} \end{pmatrix}$$

Since we want to study this as $a \to \infty$, put $a = 1/t$ and we get in degree n.

$$* \qquad \begin{pmatrix} \dfrac{t^2}{1+t^2(1+h^2n^2)} & \dfrac{t(-1+tihn)}{1+t^2(1+h^2n^2)} \\[2ex] \dfrac{t(1+tihn)}{1+t^2(1+h^2n^2)} & \dfrac{t^2}{1+t^2(1+h^2n^2)} \end{pmatrix}$$

If $t \neq 0$ this is asymptotic to

$$\begin{pmatrix} 0 & \dfrac{i}{hn} \\[2ex] \dfrac{i}{hn} & 0 \end{pmatrix}$$

and it vanishes for $t = 0$. The derivative at $t = 0$ of $*$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

which is not Hilbert-Schmidt.

It seems to be all right to work with $\frac{1}{Q_a}$ instead of $\frac{1}{1-Q_a}$, in order to get the essential behavior. In degree n Q_a^{-1} is

$$\begin{pmatrix} 0 & \frac{1}{ihn+a} \\ \frac{1}{ihn-a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{t}{1+tihn} \\ \frac{t}{tihn-1} & 0 \end{pmatrix}$$

and

$$\sum \left| \frac{t}{1+tihn} \right|^p = \sum \underbrace{\left(\frac{t^2h^2n^2}{1+t^2h^2n^2} \right)^{p/2}}_{\leq 1} \frac{1}{h^p n^p}$$

Dominated convergence says this goes to zero as $t \to 0$. Thus $a \mapsto \frac{1}{Q_a}$ is a continuous map to $\mathcal{L}^p$ at $t = \frac{1}{a} \to 0$. But it's not differentiable.

Now it is necessary to discuss the implications of the above discovery, in order to see if something can be salvaged of the original program.

Let's start at the beginning with the K-theory. Take $M = S^1$. Then the Dirac operator on the circle represents the fundamental K-homology class, whereas maps $g : M \to U(V)$ represent K-cohomology.

Capping with the fundamental class defines a map from odd K-cohomology of S^1 to even K-cohomology of a point. I wanted to have a concrete realization of this map which would take a map $g: M \to U$ and produce a map $pt \longrightarrow \mathbb{Z} \times BU$. The Toeplitz construction does this, but the hope is to something with the Dirac operator.

Now somehow the problems I have encountered have to do with the lack of smoothness of the map

$$\Sigma U(V) \longrightarrow Gr(V \oplus V)$$

defined by sending a pair $(it, X) \in i\mathbb{R} \times \mathfrak{u}(V)$ to $\gamma^1(it) + \gamma^2(X)$ and then compactifying. This is a Bott map of some sort, but it is not smooth when t and X are both infinite. In the example considered above, this map is the symbol of the operator $\gamma^1\partial_x + \gamma^2 X$. I take this cup product up on T^*, then I try to interpret it as an operator on $L^2(S^1, \mathbb{C}^2 \otimes V)$.

One thing I might do is to make explicit the quantization process which assigns to $\gamma^1(i\xi) + \gamma^2(X)$ an operator on $L^2(S^1, \mathbb{C}^2 \otimes V)$.

h

More notes for Roe:

E. Review of von Neumann's approach to (unbounded) skew-adjoint operators. Let H be a Hilbert space and let Γ be a closed subspace of $H \oplus H$ such that

(1) $$\Gamma^{\perp} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma$$

For example Γ could be the graph $\Gamma_Q = \{ \binom{x}{Qx} \mid x \in \mathcal{D}_Q \}$ of a skew-adjoint operator Q. Then we have isometries

$$\Gamma \xrightarrow{\sim} H \qquad \binom{x}{y} \longmapsto x \pm y .$$

We obtain a unitary operator U on H by composing:

$$H \xrightarrow{\sim} \Gamma \xrightarrow{\sim} H$$

$$x-y \longleftarrow \binom{x}{y} \longmapsto x+y$$

This gives the Cayley transform $U = (1+Q)(1-Q)^{-1}$ when $\Gamma = \Gamma_Q$.

The map $\Gamma \mapsto U$ from closed Γ satisfying (1) to unitaries is a 1-1 correspondence. To see this associate to Γ the involution $F = +1$ on Γ and $F = -1$ on $\Gamma^{\perp}$, and then the unitary $g = F\varepsilon$. This gives a 1-1 corresp. between Γ satisfying (1) and F such that $\gamma' F \gamma' = -F$. (Here $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) Similarly we get a 1-1 corresp. between Γ satisfying (1) and g such that $\varepsilon g \varepsilon = g^{-1}$ and $\gamma' g = g \gamma'$.

Now conjugation by $1 + \gamma' \varepsilon$ carries ε, γ' to $-\gamma', \varepsilon$ so the latter set of g's is in (1-1)-corresp.

with the set of unitaries g' which commute with ε and are inverted by γ', and these are clearly in the form

$$g' = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

with u unitary. This gives a 1-1 corresp. between Γ and unitaries on H, and the rest involves some checking.

F. <u>A program:</u> To each $X: M \longrightarrow \mathfrak{u}(V)$ we can associate the Dirac operator

$$Q_X = h\gamma^\mu \partial_\mu + \sigma X$$

acting on $C^\infty(M, S \otimes V)$. This is essentially skew-adjoint, so it Cayley transform

$$U_X = \frac{1+Q_X}{1-Q_X} = -1 + \frac{2}{1-Q_X}$$

$$U_X^{-1} = \frac{1-Q_X}{1+Q_X} = -1 + \frac{2}{1+Q_X}$$

is unitary on $\overbrace{L^2(M, S\otimes V)}^{H=}$. Also $\dfrac{1}{1 \pm Q_X} \in \mathcal{L}^p(H)$ for $p > n = \dim M$.

Thus we have a nice map $X \longmapsto U_X$

~~and the problem is to ex~~

from $C^\infty(M, \mathfrak{u}(V))$ to the manifold $-\mathcal{U}^p(H)$ of unitaries $\equiv -1 \mod \mathcal{L}^p(H)$. The problem is to extend this to all $g \in C^\infty(M, U(V))$, i.e. to define U_g so that $U_g = U_X$ where $g = \dfrac{1+X}{1-X}$.

j

Ideally $g \longmapsto U_g$ should be a smooth map

$$C^\infty(M, \mathfrak{u}(V)) \longrightarrow -U^P(H).$$

One can think of it as an explicit way to go from odd K-cohomology on M to odd K-cohomology of a point. (The even case is handled by using a grading on V.)

G. <u>A difficulty</u>: Unfortunately the map $g \longmapsto U_g$, obtained by continuity from $X \longmapsto U_X$, assuming it is well-defined, can't be smooth.

<u>Example</u>: Let's shift from an even torus M to the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Given $X : S^1 \longrightarrow \mathfrak{u}(V)$ the associated Dirac is

$$Q_X = h\gamma^1\partial_x + \gamma^2 X \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and this is odd relative to $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; Q_X acts on $C^\infty(S^1, \mathbb{C}^2 \otimes V)$. Take X to be constant, say $X = \frac{1}{i}a$ with $a \in \mathbb{R}$.

Although one can work with $\frac{1}{1 \pm Q_X}$ it is simpler to calculate the inverse $\frac{1}{Q_X}$. Using $H = L^2(S^1, \mathbb{C}^2 \otimes V) = \widehat{\bigoplus_{n\in\mathbb{Z}}}^{\ell^2} e^{inx}\mathbb{C}^2$, one has in degree n

$$Q_X = \begin{pmatrix} 0 & inh - a \\ inh + a & 0 \end{pmatrix}$$

$$Q_X^{-1} = \begin{pmatrix} 0 & \frac{1}{ihn+a} \\ \frac{1}{ihn-a} & 0 \end{pmatrix}$$

Let $a = \frac{1}{t}$ and let $t \to 0$.

$$Q_X^{-1} = \begin{pmatrix} 0 & \frac{t}{1+tihn} \\ \frac{t}{tihn-1} & 0 \end{pmatrix}$$

For $p > 1$ this is in $\mathcal{L}^p$ and is continuous in t, as

$$\sum_{n\neq 0} \left| \frac{t}{1+tihn} \right|^p = \sum_{n \neq 0} \underbrace{\left(\frac{t^2h^2n^2}{1+t^2h^2n^2} \right)^{p/2}}_{\leq 1} \frac{1}{h^p n^p}$$

But the derivative at $t=0$

$$\left.\frac{d}{dt} Q_X^{-1}\right|_{t=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which is not in $\mathcal{L}^p$.

Conclusion: It seems that any attempt to make sense of $h\gamma^\mu\partial_\mu + \sigma\frac{g-1}{g+1}$ as it stands is too naive. One needs a more subtle mixing of the two unbounded operators. Two approaches are

1) Kasparov: $M(h\gamma^\mu\partial_\mu) + N(\sigma\frac{g-1}{g+1})$

2) Hörmander: His deformation of the Gaussian DR complex.