November 21, 1987

Let us consider a semi-direct product $Q = A \oplus M$ where $A = T(V)$ is free unital and $M$ is a bimodule. Suppose also that

$$\mathrm{Tor}_1^A(M,M) = \mathrm{Tor}_1^{A\otimes A^{op}}(A, M\otimes M) = 0$$

e.g. if $M$ is flat either as a right or left $A$-module. Then we know the complex

$$\cdot \xrightarrow{b'} M\otimes A^{\otimes 2}\otimes M \xrightarrow{b'} M\otimes A\otimes M \xrightarrow{b'} M\otimes M$$

is a resolution of $M\otimes_A M$.

Let's now consider the acyclic Hochschild cx of $Q = A\oplus M$:

$$\xrightarrow{b'} Q\otimes Q\otimes Q \xrightarrow{b'} Q\otimes Q \xrightarrow{b'} Q \longrightarrow 0$$

I would like to replace it by a simpler resolution. In degree $n$ it is

$$Q \otimes (A\oplus M)^{\otimes n} \otimes Q$$

and this is a sum of $2^n$-parts of the form

$$Q\otimes \left(A^{\otimes k_1}\otimes M^{\otimes k_2}\otimes \cdots\cdots\right)\otimes Q$$

Let's keep track of the total number of $M$-factors.

Let's look at the Hochschild complex of $Q$

$$\xrightarrow{b} Q\otimes Q^{\otimes 2} \xrightarrow{b} Q\otimes Q \xrightarrow{b} Q$$

~~In degree $n$ we have $Q\otimes(A\oplus M)^{\otimes n}$~~ I want to follow closely the case where $Q = \mathbb{C}\oplus M$ which I have handled. There is in this case an equivalence with the normalized Hochschild complex

$$\longrightarrow Q\otimes M^{\otimes 2} \longrightarrow Q\otimes M \longrightarrow Q$$

which then contains the subcomplex $(M^{\otimes(*+1)}, b)$ and the quotient complex $(M^{\otimes(*)}, b')$.

So let's first look at the subcomplex of
(*) the Hochschild complex of $Q$:

$$\longrightarrow M\otimes Q^{\otimes 2} \xrightarrow{b} M\otimes Q \xrightarrow{b} M$$

which computes $H_*(Q,M)$. In degree $n$ this is

$$M\otimes (A\oplus M)^{\otimes n}$$

which is a sum of $2^n$-terms. The complex (*) has an obvious grading by the number of M-factors. Let's look at the subcomplex with $k+1$ M-factors. If $k=0$ it is the Hochschild complex which computes $H_*(A,M)$. Let $k=1$. Then we have in degree $n+1$

$$C_{j,n+1-j} = M\otimes A^{\otimes j}\otimes M\otimes A^{\otimes(n-j)}$$

and the differential

$$b(m, a_1, \ldots, a_j, m', a'_1, \ldots, a'_\ell)$$

$$= (ma_1, \ldots, a_j, m', a'_1, \ldots, a'_\ell) - (m, a_1a_2, \ldots, \text{———}) + (-1)^{j+1}(m, a_1, \ldots, a_j, m'a'_1, a'_2, \ldots, a'_\ell) + (-1)^{j+2}(m, a_1, \ldots, a_j, m', a'_1a'_2, \ldots, a'_\ell)$$

$$+ (-1)^j(m, a_1, \ldots, a_{j-1}, a_jm', a'_1, \ldots, a'_\ell) + (-1)^{j+\ell+1}(a'_\ell m, a_1, \ldots, a_j, m', a'_1, \ldots, a'_{\ell-1})$$

Thus the differential $b: C_{j,\ell} \longrightarrow C_{j-1,\ell}\oplus C_{j,\ell-1}$ which means we have a double complex

Take the homology with respect to the first partial differential. We have $C_{*,\ell} = M \otimes A^{\otimes *} \otimes (M \otimes A^{\otimes \ell})$ with $b'$

so
$$H_n(C_{*,\ell}) = \mathrm{Tor}_n^A(M, M) \otimes A^{\otimes \ell}$$
$$= \begin{cases} 0 & n \geq 1 \\ (M \otimes_A M) \otimes A^{\otimes \ell} \end{cases}$$

since $A$ is free and we are assuming $\mathrm{Tor}_1^A(M, M) = 0$. Thus the spectral sequence of the double complex degenerates and we find that the homology of the $M$-degree 2 part of $\circledast$ is $H_*(A, M \otimes_A M)[1]$.

It seems clear that this argument generalizes to $k \geq 2$, and shows that that the $M$-degree $k$ part of $\circledast$ has the homology $H_*(A, \underbrace{M \otimes_A M \otimes_A \cdots \otimes_A M}_{k \text{ times}})[k-1]$. One should note that we need to know
$$\mathrm{Tor}_1^A(\overbrace{M \otimes_A M \otimes_A \cdots \otimes_A M}^{k-1}, M) = 0$$
and so should assume $M$ left-flat over $A$.

Thus we have proved
$$H_*(Q, M) = \bigoplus_{k \geq 1} H_*(A, \overbrace{M \otimes_A \cdots \otimes_A M}^{k\text{-times}})[k-1]$$

Next for the ~~quotient~~ complex of the Hochschild complex of $Q$ by $\circledast$ which is
$$\longrightarrow A \otimes Q^{\otimes 2} \xrightarrow{\ b\ } A \otimes Q \xrightarrow{\ b\ } A$$

and gives the Hochschild homology $H_*(Q, A)$. Then things should be exactly the same except that the left-most $M$ factor is to replaced by $A$. Again we consider the $M$-degree $k$. If $k=0$, we have the Hochschild complex of $A$ which computes $H_*(A,A)$. If $k=1$ we have a double complex

$$C_{j,n+1-j} = A \otimes A^{\otimes j} \otimes M \otimes A^{\otimes(n-j)}$$

and the first partial homology is

$$H_n(C_{*,\ell}) = \mathrm{Tor}_n^A(A, M) \otimes A^{\otimes \ell}$$
$$= \begin{cases} M \otimes A^{\otimes \ell} & n = 0 \\ 0 & n > 0 \end{cases}$$

The rest should be the same leading to $H_*(A,M)[1]$ for the contribution of the $M$-degree $= 2$ term.
Continuing we obtain

$$H_*(Q,A) = \bigoplus_{k \geq 0} H_*(A, \overbrace{M \otimes_A \cdots \otimes_A M}^{k})[k]$$

In general we seem to be finding for any $A$-bimodules $N$ that

$$\boxed{\begin{gathered} H_*(Q,N) = \bigoplus_{k \geq 0} H_*(A, N \otimes_A \overbrace{M \otimes_A \cdots \otimes_A M}^{k})[k] \\ \text{assuming } Q = A \oplus M, \ M \text{ left-flat over } A \end{gathered}}$$

At this point we have some control over the Hochschild homology of a semi-direct product $Q = A \oplus M$ where $M^2 = 0$, and we would like to understand the cyclic homology.

Actually there is a point still to be discussed in connection with the Hochschild homology, because we eventually want to consider $Q$ as a superalgebra with $M$ of odd degree. Does this change signs in the cross-over term?

So let us consider super bimodules $N$ over $Q$, such as $Q$ itself. Then does $N/[Q,N]$ mean something different in the super-setting? Yes, this commutator quotient must equalize the two maps

$$\begin{tikzcd}
Q \otimes N \arrow[d, "\wr"'] \arrow[r, bend left, "\text{left}"] & N \arrow[r] & N/[Q,N] \\
N \otimes Q \arrow[ur, bend right, "\text{right}"'] & &
\end{tikzcd}$$

Thus we must be careful in the future. It is not likely to make any difference in the above calculations with $Q = A \oplus M$, $M^2 = 0$, since the multiplication by odd elements (i.e. those in $M$) is zero. Thus we took $N$ to be an $A = Q/M$-module.

So let's now turn to the investigation of the cyclic homology of $Q = A \oplus M$. We have the long exact sequence for Hochschild homology

$$\begin{array}{ccccc}
H_n(Q, M) & \longrightarrow & H_n(Q,Q) & \longrightarrow & H_n(Q, A) \xrightarrow{\partial} \\
\| & & & & \| \\
\bigoplus_{k \geq 0} H_{n-k}(A, M(\otimes_A M)^k) & & & & \bigoplus_{k \geq 0} H_{n-k}(A, M(\otimes_A M)^{k-1})
\end{array}$$

Because of the natural $\mathbb{Z}$-grading associated to degree in $M$, this long exact sequence in a given $M$-degree $k$ takes the form

$$\to H_n(Q,Q)_{(k)} \longrightarrow H_{n-k}(A, M(\otimes_A M)^{k-1}) \xrightarrow{\partial} H_{n-k}(A, M(\otimes_A M)^{k-1}) \longrightarrow H_{n-1}(Q,Q)_{(k)} \longrightarrow$$

One expects $\partial$ to be something like $1-t$, and in the case where $M$ is of odd degree it is reasonable to expect $\partial = 1-\sigma$. Let's assume this for the moment and we get

$$(**)\quad 0 \to H_{n-k+1}(A, (M\otimes_A M)^{k-1})_{\sigma} \to H_n(Q,Q)_{(k)} \longrightarrow H_{n-k}(A, (M\otimes_A M)^{k-1})^{\sigma} \longrightarrow 0 \quad ?$$

Next we look at the long exact sequence relating cyclic and $H_*(Q,Q) = H_*(Q)$:

$$HC_{n-1}(Q)_{(k)} \to H_n(Q)_{(k)} \longrightarrow HC_n(Q)_{(k)} \xrightarrow{S} HC_{n-2}(Q)_{(k)} \longrightarrow H_{n-1}(Q)_{(k)}$$

and let's use Goodwillie's theorem that $S$ has to be zero if $k>0$.

When $A=\mathbb{C}$ we have (with $M=V$)

$$\bar{H}_n(Q) = \frac{V^{\otimes(n+1)} \oplus (V^{\otimes n})^t}{(1-t)V^{\otimes(n+1)}}$$

$$= \underbrace{V^{\otimes(n+1)}/(1-t)}_{\bar{H}C_n(Q)} \oplus \underbrace{(V^{\otimes n})^t}_{\bar{H}C_{n-1}(Q)}$$

The exact sequence $(**)$ seems in the wrong order, so a possible guess is

$$\star\ \boxed{\underline{\text{Guess:}}\quad HC_n(Q)_{(k)} = H_{n-k+1}(A, M\otimes_A^k)/1-\sigma}$$

We can then check if this agrees with

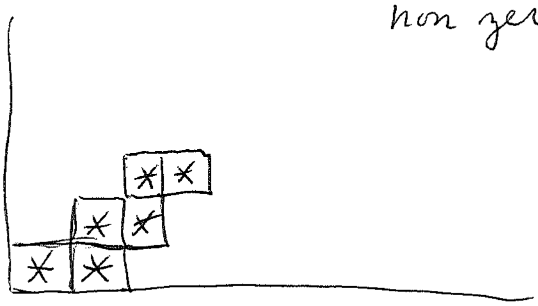
$$\begin{tikzcd} 0 \arrow[r] & HC_{n-1}(Q)_{(k)} \arrow[r] \arrow[d, equal] & H_n(Q)_{(k)} \arrow[r] & HC_n(Q)_{(k)} \arrow[r] \arrow[d, equal] & 0 \\ & H_{n-k}(A, M^{\otimes k}/1-\sigma) & & H_{n-k+1}(A, M^{\otimes k}/1-\sigma) & \end{tikzcd}$$

which agrees with $**$, at least if we assume the sequences split.

[redacted] Next suppose that our guess $\star$ is correct, and suppose $A$ is free, so that only $H_0, H_1$ are $\neq 0$. Let's return to our spectral sequence

$$E^1_{pq} = \underset{\displaystyle H_{q-p+1}(P, I^p)_{(1-\sigma)}}{\underset{\parallel}{HC_q(P\oplus I)_{(p)}}} \implies HC_{p+q}(A)$$

non zero for $q = p, p-1$



[redacted] Thus the spectral sequence should degenerate from $E^2$ on.

Notice that we have to find

$$H_*(A, M^{\otimes_A k}) \xrightarrow{\partial} H_*(A, M^{\otimes_A k})$$

It's not an obvious action of $\mathbb{Z}/k$ except in degree 0

November 22, 1987

Let $A$ be a superalgebra. A superbimodule $M$ over $A$ is [redacted] just a $\mathbb{Z}/2$-graded bimodule since there are no signs in

$$(am)a_1 = a(ma_1).$$

However brackets $[a,m] = am - (-1)^{\partial a \partial m} ma$ are different. How do we ~~[illegible]~~ reconcile this with the formula

$$A \otimes_{(A \otimes A^{op})} M = M/[A,M] \ ?$$

In the first case, $A^{op}$ is different in the supersetting; Let's write instead $A^{\hat{op}}$. It is $A$ as a vector space with multiplication

$$a_1 * a_2 = (-1)^{\partial a_1 \partial a_2} a_2 a_1$$

Then a right $A$-module $M$ becomes a left $A^{\hat{op}}$ module with $a * m = (-1)^{\partial a \partial m} ma$. Check

$$\begin{aligned} a_1 * (a_2 * m) &= a_1 * ma_2 (-1)^{\partial m \partial a_2} \\ &= ma_2 a_1 (-1)^{\partial m \partial a_2 + (\partial m + \partial a_2)\partial a_1} \\ &= (-1)^{\partial a_1 \partial a_2} (a_2 a_1) * m = (a_1 * a_2) * m \end{aligned}$$

And a bimodule $M$ becomes a left $A \hat{\otimes} A^{\hat{op}}$-module: ~~[illegible]~~

$$\begin{aligned} a_1 * (a_2 m) &= (-1)^{\partial a_1 (\partial a_2 + \partial m)} a_2 m a_1 \\ &= (-1)^{\partial a_1 \partial a_2} a_2 (a_1 * m) \end{aligned}$$

One doesn't ~~have~~ an isomorphism of $A \hat{\otimes} A^{\hat{op}}$ ~~with~~ $A \otimes A^{op}$, however because the categories of [redacted] super $A$-bimodules [redacted] over $A$ is equivalent to the category of $\mathbb{Z}/2$-graded $A$-bimodules, one expects that [redacted] after taking the cross product with $\mathbb{Z}/2$ these [redacted] algebras become isomorphic.

<u>Example.</u> $A = C_1 = \mathbb{C} \oplus \mathbb{C}\gamma$ with $\gamma^2 = 1$ of odd degree. Then $A^{\hat{op}} = \mathbb{C} \oplus \mathbb{C}\gamma^*$ where $\gamma * \gamma = -1$. So $A^{\hat{op}} \cong C_1$ and $A \hat{\otimes} A^{\hat{op}} = C_1 \hat{\otimes} C_1 = C_2 = M_2(\mathbb{C})$ as an algebra. But $A$ is commutative $\cong \mathbb{C} \times \mathbb{C}$ as an algebra, so $A^{op} = A$ and $A \otimes A^{op} = \mathbb{C}^4$.

---

Let's return to the problem of the cyclic homology of a semi-direct product $Q = A \oplus M$ where $M^2 = 0$. We wish to treat both the ordinary (non super) case and the case where $Q$ is a superalgebra with $A$ even and $M$ odd. We do the ordinary case first.

Yesterday we described the Hochschild homology of $Q$. There's an exact sequence of complexes

$$0 \to (M \otimes Q^{\otimes *}) \longrightarrow (Q \otimes Q^{\otimes *}) \longrightarrow (A \otimes Q^{\otimes *}) \to 0$$

giving rise to a long exact sequence in homology

$$\to H_n(Q, M) \longrightarrow H_n(Q, Q) \to H_n(Q, A) \to H_{n-1}(Q, M) \to$$

Moreover the Hochschild complex for $Q$ with coefficients in any graded $Q$ bimodule $N$ is naturally $\mathbb{Z}$-graded because $Q$ is. ~~Better the Hochschild~~ If $N$ is a $Q/M = A$ bimodule, then we were able (more or less) to identify the various homogeneous complexes

$$(N \otimes Q^{\otimes *}, b)_{(k)} \overset{\text{quis}}{\sim} \boxed{N \overset{\mathbb{L}}{\underset{A}{\otimes}} M \overset{\mathbb{L}}{\underset{A}{\otimes}} \cdots \overset{\mathbb{L}}{\underset{A}{\otimes}} M \overset{\mathbb{L}}{\underset{A}{\otimes}}}\,[k]$$

So we have an exact sequence of complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & (M \otimes Q^{\otimes *})_{(k)} & \longrightarrow & (Q \otimes Q^{\otimes *})_{(k)} & \longrightarrow & (A \otimes Q^{\otimes *})_{(k)} & \longrightarrow 0 \\ & \wr & & & & \wr & \\ & \left(M \overset{\mathbb{L}}{\underset{A}{\otimes}}\right)^k [k-1] & & & & \left(M \overset{\mathbb{L}}{\underset{A}{\otimes}}\right)^k [k] & \end{array}$$

[redacted] Hence the degree $k$ part of the Hochschild complex of $Q$ is the fibre of an [redacted] endomorphism of $(M \overset{L}{\otimes}_A)^k[k]$, which ought to be $1-t$.

Now we want to go on ~~to~~ treat the cyclic homology of $Q$. The idea is to set up a bicomplex analogous to

$$
\begin{array}{ccccc}
V^{\otimes 3} & \xleftarrow{1-t} & V^{\otimes 3} & \xleftarrow{N} & \\
V^{\otimes 2} & \xleftarrow{1-t} & V^{\otimes 2} & \xleftarrow{N} & \\
V & \xleftarrow{1-t} & V & \xleftarrow{N} &
\end{array}
$$

[redacted] The first candidate is to take the Connes bicomplex consisting of $(Q \otimes Q^{\otimes *}, b)$ in even columns.

$$
\begin{tikzcd}
Q^{\otimes 3} \arrow[d, "b"] & Q^{\otimes 3} \arrow[d, "b"] & \\
Q^{\otimes 2} \arrow[d, "b"] & Q^{\otimes 2} \arrow[ul, "B"'] \arrow[d, "b"] & \arrow[l, "B"'] \\
Q & Q \arrow[ul, "B"'] &
\end{tikzcd}
$$

and to see if it can be spread out into

$$
\begin{tikzcd}
\arrow[d] & \arrow[d] & \\
M\otimes Q^{\otimes 2} \arrow[d] & A\otimes Q^{\otimes 3} \arrow[d] & \arrow[d] \\
M\otimes Q \arrow[d] & A\otimes Q^{\otimes 2} \arrow[d] & M\otimes Q \arrow[d] \\
M & A\otimes Q \arrow[d] & M \\
 & A &
\end{tikzcd}
\qquad ?
$$

Let us consider carefully the complex for computing $H_*(Q,N)$ where $N$ is a $Q/M$-module. [redacted] This complex is

$$N\otimes Q^{\otimes n} = \bigoplus_{k=0}^{n} \underbrace{\bigoplus_{i_1+\cdots+i_k+k=n} N\otimes A^{\otimes i_1}\otimes M\otimes A^{\otimes i_2}\otimes M\otimes A^{\otimes i_k}}_{(N\otimes Q^{\otimes n})_{(k)}} . \quad \left(\begin{smallmatrix}\text{here } N \text{ is}\\ \text{given degree } 0\end{smallmatrix}\right)$$

[redacted]

Set
$$\tilde{N} = N\otimes_A \{A\otimes A^{\otimes *}\otimes A,\ b'\}$$
This is a resolution of $N$ by free right $A$-modules.

Similarly we set
$$\tilde{M}[1] = M\otimes_A \{A\otimes A^{\otimes *}\otimes A, b'\}[1]$$

Then it seems that we have an isomorphism of complexes
$$(N\otimes Q^{\otimes *})_{(k)} = \underbrace{\tilde{N}\otimes_A \overbrace{\tilde{M}\otimes_A\cdots\otimes\tilde{M}\otimes_A}^{k\text{-times}}}\,[k]$$

Next we want to consider $Q\otimes Q^{\otimes *}$ whose $M$-degree $= k$ piece fits into an exact sequence

$$0\to \underbrace{\overbrace{\tilde{M}\otimes_A\cdots\tilde{M}\otimes_A}^{k\text{-times}}}[k-1] \to (Q\otimes Q^{\otimes *})_{(k)} \longrightarrow \underbrace{\tilde{A}\otimes_A \overbrace{\tilde{M}\otimes_A\cdots\otimes\tilde{M}\otimes_A}^{k\text{-times}}}[k]\to 0$$

Additively $(Q\otimes Q^{\otimes *})_{(k)}$ splits into the direct sum of these two subcomplexes. Let $s$ be the inclusion of $\tilde{A}\otimes_A(\tilde{M}\otimes_A)^k[k]$ into $(Q\otimes Q^{\otimes *})_{(k)}$. Then $ds-sd$ is a map of complexes [redacted]

[redacted] $$ds-sd:\ \tilde{A}\otimes_A(\tilde{M}\otimes_A)^k[k] \longrightarrow (\tilde{M}\otimes_A)^k[k-1]$$

and $(Q\otimes Q^{\otimes *})_{(k)}$ is the mapping cone associated to this map.

Let's try to describe ds-sd in the case $k=1$.

$$(Q\otimes Q^{\otimes n})_{(1)} = \bigoplus_{i+j=n-1} A\otimes A^{\otimes i}\otimes M\otimes A^{\otimes j} \;\oplus\; M\otimes A^{\otimes n}$$

The reason $b$ does not preserve $\tilde{A}\otimes_A \tilde{M}\otimes_A$ is because when we multiply by an element of $M$ on the first $A$ factor, which sits inside $Q$, then it doesn't give zero. This happens if either $i=0$ or $j=0$:

$$b(a,m,a_1,\dots,a_j) = \underbrace{(am,a_1,\dots,a_j)}_{M\otimes A^{\otimes j}} - \underbrace{(a,ma_1,a_2,\dots)+\cdots\pm(a_ja,m,a_1,\dots)}_{\in A\otimes M\otimes A^{\otimes j-1}}$$

$$b(a,a_1,\dots,a_i,m) = \underbrace{(aa_1,a_2,\dots,a_i,m)-(a,a_1a_2,\dots,m)\cdots\cdots}_{A\otimes A^{\otimes(i-1)}\otimes M} + (-1)^{i+1}\underbrace{(ma,a_1,\dots,a_i)}_{M\otimes A^{\otimes i}}$$

Thus $\overset{b}{\overset{\shortparallel}{\text{ds-sd}}}$ is the sum of 2 maps. The first sends $\tilde{A}\otimes_A\tilde{M}\otimes_A \longrightarrow \tilde{M}\otimes_A$ and the second sends $\tilde{A}\otimes_A\tilde{M}\otimes_A \longrightarrow \tilde{A}\otimes_A M\otimes_A \simeq \underbrace{M\otimes_A}_{\tilde{M}}\tilde{A}\otimes_A$.

Thus I ought to be able to identify ds-sd with $1-t$ acting on $(\tilde{M}\otimes_A)^k[k-1]$, specifically we have two maps of complexes.

$$\begin{array}{ccc}
\tilde{A}\otimes_A(\tilde{M}\otimes_A)^k & \longrightarrow & (\tilde{M}\otimes_A)^k \\
\| & & \\
\tilde{A}\otimes_A(\tilde{M}\otimes_A)^{k-1}\tilde{M}\otimes_A & \longrightarrow & \tilde{A}\otimes_A(\tilde{M}\otimes_A)^{k-1}M\otimes_A \\
 & & \swarrow \wr \\
 & & \underbrace{M\otimes_A\tilde{A}}_{\tilde{M}}\otimes_A(\tilde{M}\otimes_A)^{k-1} = (\tilde{M}\otimes_A)^k
\end{array}$$

It would seem that these maps are qis, hence that we have an action of $\mathbb{Z}/k$ on $(\tilde{M}\otimes_A)^k$ in the derived category. But in fact there clearly is an action of $\mathbb{Z}/k$ on $(\tilde{M}\otimes_A)^k$. So this means maybe that we can shrink the Hochschild complex of $Q$.

I think the first thing to do is to see if we can actually construct an appropriate cyclic double complex. This is apparently easy:

$$\begin{array}{ccccc}
(\tilde{M}\otimes_A)^3 & \xleftarrow{1-t} & (\tilde{M}\otimes_A)^3 & \xleftarrow{N} & \\
(\tilde{M}\otimes_A)^2 & \xleftarrow{1-t} & (\tilde{M}\otimes_A)^2 & \xleftarrow{N} & \\
\tilde{M}\otimes_A & \xleftarrow{1-t} & (\tilde{M}\otimes_A) & \xleftarrow{N} &
\end{array}$$

Now the problem becomes to relate this complex to the cyclic bicomplex of $Q$. There's some difficulty with replacing the "top" of $(Q\otimes Q^{\otimes *})_{(k)}$ i.e., $\tilde{A}\otimes_A(\tilde{M}\otimes_A)^k$ with $(\tilde{M}\otimes_A)^k$. It's related to how to handle the resolutions $\tilde{A}$ and $\tilde{A} \otimes_A \tilde{A}$ of $A$ ~~by~~ free $A \otimes A^{op}$ modules.

Ideas: Let's consider an extension $I \to P \to A$ with $P$ free. Let's assume that there is no problem with identifying the $E^1$ term of the spectral sequence for cyclic homology associated to this extension. Thus we have

$$E^1_{pq} = H_{q-p+1}\big(\underbrace{I \overset{L}{\otimes}_P \cdots I \overset{L}{\otimes}_P}_{p\text{-times}} / 1-\sigma\big) \Rightarrow HC_{p+q}(A)$$

Because $P$ is free we know that this spectral sequence ~~[illegible]~~ has its only ~~—~~ non zero terms in the strip ~~[illegible]~~ $q = p$ or $p-1$, and so $E^2 = HC_*(A)$.

Also we know

$$H_*\big(I \overset{L}{\otimes}_P \cdots I \overset{L}{\otimes}_P\big) = H_*(P, \underbrace{\overbrace{I \otimes_P \cdots \otimes_P I}^{p \text{ times}}}_{= I^P})$$

~~In~~ particular $H_*(P, I^P)$ has an action of $\mathbb{Z}/p\mathbb{Z}$. ~~In~~ fact for $* = 0$ we have

$$H_0(P, I \otimes_P \cdots_P \otimes I) = I^P / [P, I^P]$$

and we know this action must be ~~[illegible]~~ trivial modulo $[I, I^{P-1}]$.

# November 23, 1987

There was a mistake made earlier. Apparently there is no canonical map
$$HC_3(A) \longrightarrow HC_1(Q)$$
when when $Q$ is a square zero extension of $A$. To understand this take $Q = P/I^2$, where $P/I = A$ and $P$ is free. Then we have an exact sequence
$$0 \longrightarrow HC_1(Q) \longrightarrow I^2/[P, I^2] \longrightarrow P/[P,P].$$
On the other hand the canonical map of Connes, which we are viewing as an edge homomorphism, goes
$$HC_{2n-1}(A) \longrightarrow H_0\left((I \overset{L}{\otimes}_P)^n/(1-\sigma)\right)$$
$$\parallel$$
$$(I \otimes_P)^n/(1-\sigma) \longrightarrow I^n/[I, I^{n-1}].$$

Thus (assuming the spectral sequence degenerates in the expected fashion for $P$ free) we only get a map
$$HC_3(A) \hookrightarrow I^2/[I,I]$$
and not a map to $I^2/[P, I^2] = I \otimes_P I \otimes_P$.

However it seems we can construct a canonical map as follows. To fix the ideas suppose $A = Q/J$. Then we have the edge homomorphism
$$HC_{2n-1}(A) \longrightarrow H_0\left((J \overset{L}{\otimes}_Q)^n/(1-\sigma)\right) = (J \otimes_Q)^n/(1-\sigma)$$
On the other hand we have
$$\tfrac{1}{n} N = \tfrac{1}{n}(1 + \sigma + \cdots + \sigma^{n-1}) : (J \otimes_Q)^n/(1-\sigma) \longrightarrow (J \otimes_Q)^n$$

and we have a canonical map

$$(J \otimes_Q)^n \longrightarrow J^n/[Q, J^n]$$

Thus we always have a canonical

map

$$\circledast \qquad \boxed{HC_{2n-1}(Q/J) \longrightarrow J^n/[Q, J^n]}$$

such that

$$\begin{tikzcd} HC_{2n-1}(Q/J) \arrow[r] \arrow[dr, "\text{Connes}"'] & J^n/[Q, J^n] \arrow[d] \\ & J^n/[J, J^{n-1}] \end{tikzcd}$$

commutes. Moreover the image of $\circledast$ is contained in the kernel of the map to $Q/[Q,Q]$, namely take $J = Q$, replacing $Q$ by a unital ring first if necessary. The question is whether there is a natural lifting

$$\begin{tikzcd} HC_{2n-1}(Q/J) \arrow[r] \arrow[dr, dotted] & J^n/[Q, J^n] \\ & HC_1(Q/J^n) \arrow[u] \end{tikzcd}$$

If $Q = P$ is free, there is a unique choice for the lifting. So for $n = 2$ we have

$$\begin{tikzcd} H_3(A) \arrow[rr, "\circledast"] \arrow[dr, "\text{unique}"'] & & I^2/[P, I^2] \\ & HC_1(P/I^2) \arrow[ur, hook] \arrow[d, shift left] \arrow[d, shift right] & \\ & HC_1(P/I^2 * P/I^2) & \end{tikzcd}$$

and the question is whether the unique map is equalized by the pair of arrows, or equivalently killed by the canonical map

$$HC_1(P/I^2) \longrightarrow H_1(P/I^2, D)$$

linked to derivations of $P/I^2$ with values in $A$-modules.

---

Let's return to the spectral sequence for the extension $I \to P \to A$ with $P$ free. Let's look what happens concerning $HC_3(A)$ and $HC_2(A)$. The spectral sequence comes from a double complex

$$\begin{array}{|llll}
P^{\otimes 3}/(1+t) & P^{\otimes 2}\otimes I & P\otimes I^{\otimes 2} & I^{\otimes 3}/(1-\sigma) \\
\Lambda^2 P & P\otimes I & S^2 I & \\
P & I & & \\
\hline
\end{array}$$

The column with one $I$ is the Hochschild complex for computing $H_*(P, I)$. Now we know this complex is quis to $V\otimes I \xrightarrow{b} I$, hence there has to be ~~an exact sequence~~ a canonical map

$$S^2(I)/b(P\otimes I^{\otimes 2}) \longrightarrow \operatorname{Ker}\{V\otimes I \xrightarrow{b} I\}$$

$$\|$$

$$(I\otimes_P I\otimes_P)/(1-\sigma)$$

$$\wr\|$$

$$I^2/[I,I]$$

whose kernel and cokernel ought to be $HC_3(A)$ and $HC_2(A)$ respectively.

Somehow this map replacing the standard resolution of $P$ as a $P\otimes P^{op}$ module by the simpler one

$$0 \longrightarrow P\otimes V\otimes P \xrightarrow{b'} P\otimes P \longrightarrow P \longrightarrow 0$$

Because of the diagram

$$\begin{tikzcd}
\arrow[r] & P\otimes P^{\otimes 2}\otimes P \arrow[r] & P\otimes P\otimes P \arrow[r, "b'"] \arrow[d, dashed, "j"] & P\otimes P \arrow[r] \arrow[d, equal] & P \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & P\otimes V\otimes P \arrow[r, "b'"] & P\otimes P \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

there's a unique map $j$. One has

$$\begin{aligned}
b'(1\otimes v_1\cdots v_k\otimes 1) &= (v_1 v_2\cdots v_k)\otimes 1 - 1\otimes(v_1v_2\cdots v_k) \\
&= (v_1\otimes 1 - 1\otimes v_1)(v_2\cdots v_k) \\
&\quad + v_1(v_2\otimes 1 - 1\otimes v_2)(v_3\cdots v_k) \\
&\quad + v_1\cdots v_{k-1}(v_k\otimes 1 - 1\otimes v_k)
\end{aligned}$$

and so

$$\begin{aligned}
j(x\otimes v_1\cdots v_k\otimes y) &= x\otimes v_1\otimes(v_2\cdots v_k y) \\
&\quad + (xv_1)\otimes v_2\otimes(v_3\cdots v_k y) \\
&\quad \cdots\cdots \\
&\quad + (xv_1\cdots v_{k-1})\otimes v_k\otimes y \\
&= \sum_{i=0}^{k}(xv_1\cdots v_{i-1})\otimes v_i\otimes(v_{i+1}\cdots v_k y)
\end{aligned}$$

Let's now start with $x\cdot y\in S^2(I)$. What does it map to in $P\otimes I$? We should be thinking of the complex

$$(P\leftarrow I)\otimes(P\leftarrow I)$$

and taking the quotient by $1-t$. Thus $x\cdot y$ in $S^2(I)$ lifts to $x\otimes y\in I^{\otimes}I$, then we apply the differential to get $x\otimes y - y\otimes x \in P\otimes I\oplus I\otimes P$. In identifying the quotient by $1-t$ with $P\otimes I$ we get $x\otimes y\in P\otimes I$ $+y\otimes x$

Next we take

$$P^{\otimes *} \otimes I = P \otimes P^{\otimes *} \otimes P \otimes_R I \otimes_P$$

and use the map $j$ in degree 1. So if $x = v_1 \cdots v_k$

$$\begin{array}{ccc} v_1 \cdots v_k \otimes y & \longleftrightarrow & 1 \otimes v_1 \cdots v_k \otimes y \\ & & \downarrow \\ & & \displaystyle\sum_{i=0}^{k} v_i \otimes v_{i+1} \cdots v_k \, y \, v_1 \cdots v_{i-1} \in V \otimes I \end{array}$$

Thus the effect of the map is take $(v_1 \cdots v_k) \cdot y$ in $S^2(I)$ differentiate $v_1 \cdots v_k$ to obtain

$$\sum v_1 \cdots v_{k-1} \otimes v_i \otimes v_{i+1} \cdots v_k \quad \in \quad P \otimes V \otimes P = \Omega^1_P$$

then multiply by $y$ to land in $\Omega^1_P \otimes_P I$. Thus we have

$$\begin{tikzcd} I \otimes I \arrow[r] \arrow[d] & P \otimes I \arrow[r, "d \otimes 1"] & \Omega^1_P \otimes I \arrow[d] \\ S^2(I) \arrow[rr, dashed] & & \Omega^1_P \otimes_P I \otimes_P = V \otimes I \end{tikzcd}$$

The dotted arrow exists because

$$\underline{dx \otimes y + dy \otimes x = \quad ?}$$

$$\left( P \xleftarrow{\partial} I \otimes P \xleftarrow{\partial} I \right) =$$

$$\begin{tikzcd} & \partial x \otimes y - x \otimes \partial y \arrow[d] & x \otimes y \arrow[l, mapsto] \\ P \otimes P \arrow[d] & P \otimes I \oplus I \otimes P \arrow[l] \arrow[d] & I \otimes I \arrow[l] \arrow[d] \\ & \partial x \otimes y + \partial y \otimes x & \\ \Lambda^2 P & P \otimes I \arrow[l] & S^2 I \arrow[l] \\ & 2 \partial x \otimes x & x^2 \arrow[l, mapsto] \end{tikzcd}$$

So review: We want to compute an explicit map which in the lowest case goes

$$\begin{array}{ccc} I\otimes_P I\otimes_P/(1-\sigma) & \longrightarrow & H_1(P,I) = \mathrm{Ker}\{V\otimes I \to I\} \\ \| & & \cap \\ I^2/[I,I] & & \Omega^1_P \otimes_P I\otimes_P \end{array}$$

Let $x, y \in I$. Then $x\cdot y \in S^2(I)$ becomes $x\otimes y + y\otimes x$ in

$$\begin{array}{ccc} P\otimes I & = & (P\otimes P\otimes P)\otimes_P I\otimes_P \\ \cup & & \\ x\otimes y & \longleftrightarrow & 1\otimes x\otimes 1\otimes y \end{array}$$

and we apply
$$d : P\otimes P\otimes P \to \Omega^1_P. \qquad p\otimes x\otimes q \mapsto p\,dx\,q$$
Thus

$$\begin{array}{ccc} x\otimes y + y\otimes x & \in & P\otimes I \\ \downarrow & & \downarrow \\ dx\otimes y + dy\otimes x & \in & \Omega^1_P\otimes_P I\otimes_P \end{array}$$

Thus our map is

$$\boxed{\begin{array}{ccc} I\otimes_P I\otimes_P/(1-\sigma) & \longrightarrow & \Omega^1_P\otimes_P I\otimes_P \\ x\otimes y & \longmapsto & dx\otimes y + dy\otimes x \end{array}}$$

Check this

$$xp\otimes y \longmapsto ((dx)p + x\,dp)\otimes y + dy\otimes xp$$

$$x\otimes py \longmapsto dx\otimes py + (dp\,y + p\,dy)\otimes x$$

Seems OKAY.

Anyway we should examine carefully the exact sequence

$$0 \to HC_3(A) \to I^2/[I,I] \to H_1(P,I) \to HC_2(A) \to 0$$

Notice that

$$HC_2(P) \to \boxed{P/[P,P] \xrightarrow{\sim} H_1(P,P)} \to HC_1(P) \to 0$$
$$HC_2(P) = 0, \qquad HC_1(P) = 0$$

And

$$0 \to H_1(P,I) \to H_1(P,P) \to H_1(P,A) \to H_0(P,I) \to H_0(P,P)$$
$$H_0(P,I) = I/[P,I] \to P/[P,P]$$

so we have an exact sequence

$$0 \to H_1(P,I) \to P/[P,P] \to H_1(P,A) \to HC_1(A) \to 0$$

So we have a diagram

truncated cyclic cx

$$\begin{tikzcd}
 & & & 0 & & \\
 & & & HC_1(A) \arrow[u] & A & \\
 & & & H_1(P,A) \arrow[u] & D/[A,D] \arrow[u] & \\
0 \arrow[r] & HC_1(P/I^2) \arrow[r] \arrow[d, dotted] & I^2/[P,I^2] \arrow[r] \arrow[d] & P/[P,P] \arrow[r] \arrow[u] & HC_0(P/I^2) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & HC_3(A) \arrow[r] & I^2/[I,I] \arrow[r] & H_1(P,I) \arrow[r] \arrow[u] & HC_2(A) \arrow[r] \arrow[u] & 0 \\
 & & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The correct way to use this diagram is that it proves

$$HC_1(P/I^2) \twoheadrightarrow HC_3(A)$$

and

$$HC_2(A) \hookrightarrow HC_0(P/I^2).$$

The latter we've already proved, but the argument might generalize.

November 24, 1987

Consider $T = T_A(M) = A \oplus M \oplus M \otimes_A M \oplus \cdots$ where $M$ is an $A$-bimodule. A bimodule $N$ over $T$ is an $A$-bimodule together with maps $M \otimes_A N \xrightarrow{\lambda} N$, $N \otimes_A M \xrightarrow{\rho} N$ such that

$$\begin{array}{ccc} M \otimes_A N \otimes_A M & \xrightarrow{1 \otimes \rho} & M \otimes_A N \\ \downarrow{\scriptstyle \lambda \otimes 1} & & \downarrow{\scriptstyle \lambda} \\ N \otimes_A M & \xrightarrow{\rho} & N \end{array}$$

commutes. Given such an $N$ we can consider

$$\mathrm{Der}(T, N) = \mathrm{Hom}_{T \otimes T^{op}}(\Omega^1_T, N).$$

Such derivations are the same as autos of the extension

$$0 \longrightarrow N \longrightarrow T \oplus N \xrightarrow[\pi]{} T \longrightarrow 0$$

which are in turn sections of $\pi$ which are alg morphisms. But an algebra map $T \to R$ is the same as an alg map $A \to R$ together with a map $M \to R$ of $A$-bimodules. Thus a section of $\pi$ which is an alg. map is given by an algebra lifting of $A$, i.e. a derivation of $A$ with values in $N$, together with an $A$-bimod. map from $M$ to $N$. $\therefore$

$$\mathrm{Der}(T, N) = \mathrm{Der}(A, N) \oplus \mathrm{Hom}_{A \otimes A^{op}}(M, N)$$

$$\Omega^1_T = T \otimes_A (\Omega^1_A \oplus M) \otimes_A T$$

Now look at the Hochschild homology.

$$H_n(T, N) = \mathrm{Tor}_n^{T \otimes T^{op}}(T, N)$$

For $n \geq 2$ we have this is the same as

$$\mathrm{Tor}_{n-1}^{T \otimes T^{op}}(\Omega^1_T, N)$$

$$\Omega^1_T = (\Omega^1_A \oplus M) \otimes_{A \otimes A^{op}} (T \otimes T^{op})$$

$$= \mathrm{Tor}_{n-1}^{A \otimes A^{op}}(\Omega^1_A \oplus M, N)$$

assuming that $T \otimes T^{op}$ is ~~right~~ left flat over $A \otimes A^{op}$. So we seem to want $T$ to be both left and right flat over $A$. (This is the same as requiring $M$ to be both left and right flat over $A$. In effect $M$ is a direct summand of $T$ as an $A$-bimodule, and also if $M$ is left $A$-flat then

$$X \longmapsto X \otimes_A (M \otimes_A M \otimes_A \cdots \otimes_A M) = (X \otimes_A M) \otimes_A M \cdots$$

is the composite of the exact functors $? \otimes_A M$.)

Thus assuming $M$ left + right flat, we have an exact sequence

$$0 \longrightarrow H_1(T,N) \longrightarrow (\Omega^1_A \oplus M) \otimes_A N \otimes_A \longrightarrow N \to H_0(T,N) \to$$

$$H_n(T,N) = H_n(A,N) \oplus \mathrm{Tor}_{n-1}^{A \otimes A^{op}}(M,N) \qquad n \geq 1.$$

---

Next let us consider a free product $C = A * B$. Then

$$\Omega^1_C = C \otimes_A \Omega^1_A \otimes_A C \oplus C \otimes_B \Omega^1_B \otimes_B C$$

The standard formula

$$C = 1 \oplus \bar{A} \oplus \bar{B} \oplus \bar{A} \otimes \bar{B} \oplus \bar{B} \otimes \bar{A} \oplus \cdots$$

shows that $C$ is free as a left or right $A$ or $B$-module. ~~Notice that~~ $\cancel{\bar{B} + A \otimes \bar{B} \simeq A^+ \otimes B}$

More precisely, to fix the ideas suppose $A, B$ augmented : $A = \mathbb{C} \oplus \bar{A}$, etc.

Then
$$C = \underset{\mathbb{C}\oplus\bar{A}}{\underset{\parallel}{A}} \oplus \underset{\bar{B}\oplus\bar{A}\otimes\bar{B}}{\underset{\parallel}{A\otimes\bar{B}}} \oplus \underset{\bar{B}\otimes\bar{A}+\bar{A}\otimes\bar{B}\otimes\bar{A}}{\underset{\parallel}{A\otimes\bar{B}\otimes\bar{A}}} \oplus \cdots$$

shows $C$ is a free left $A$-module. In general a filtration should yield the same reason.

Then we can conclude as before that

$$H_n(C, N) = H_n(A, N) \oplus H_n(B, N) \qquad n \geq 2$$

$$0 \to H_1(C, N) \longrightarrow \Omega^1_A \otimes_A N \oplus \Omega^1_B \otimes_B N \to N \to H_0(C, N) \to 0$$

In both of these cases we see a simple addition formula with some glueing taking place at the bottom.

Better process: The complex of $C$-bimods

$$\longrightarrow 0 \longrightarrow \Omega^1_C \longrightarrow C \otimes C$$

is quis to the double complex

$$\begin{array}{ccc}
C\otimes_A \Omega^1_A \otimes_A C & \longrightarrow & C\otimes C \\
\oplus & & \oplus \\
C\otimes_B \Omega^1_B \otimes_B C & \longrightarrow & C\otimes C \\
 & & \Big\uparrow{\scriptstyle (1,-1)} \\
 & & C\otimes C
\end{array}$$

which then gives using the top row as a subcomplex

$$0 \to H_1(A, N) \oplus H_1(B, N) \longrightarrow H_1(C, N) \longrightarrow N$$

$$\to H_0(A, N) \oplus H_0(B, N) \longrightarrow H_0(C, N) \longrightarrow 0$$

Take $N = C$. Note that

$$C = \underbrace{\begin{matrix}\mathbb{C}\\ \bar{A}\end{matrix}}_{A} \oplus \underbrace{\begin{matrix}\bar{B} & & \bar{B}\otimes\bar{A}\\ \bar{A}\otimes\bar{B} & \oplus & \bar{A}\otimes\bar{B}\otimes\bar{A}\end{matrix}}_{A\otimes\bar{B}\otimes A} \oplus \underbrace{\qquad\qquad}_{A\otimes\bar{B}\otimes\bar{A}\otimes\bar{B}\otimes A} \oplus$$

so that as an $A\otimes A^{op}$-module $C = A + \text{free}$
$\therefore\ H_n(A, C) = H_n(A, A)$ for $n \geq 1$, so we get

$$H_n(C, C) = H_n(A, A) \oplus H_n(B, B) \qquad n \geq 2$$

$$0 \to H_1(A,A)\oplus H_1(B,B) \to H_1(C,C) \to C \to C/[A,C] \oplus C/[B,C] \to C/[C,C] \to 0$$

Thus we have

$$0 \to H_1(A) \oplus H_1(B) \to H_1(C) \to [A,C]\cap[B,C] \to 0.$$

I don't know what to do with the last term.

~~Let's return to $T = T_A(M)$.~~

Let's return to $T = T_A(M)$. We saw that we have a $T\otimes T^{op}$ resolution of $T$

$$\to 0 \to T\otimes_A(\Omega^1_A \oplus M)\otimes_A T \to T\otimes T \dashrightarrow T \leftarrow$$

But

$$\to 0 \to T\otimes_A \Omega^1_A \otimes_A T \to T\otimes T \dashrightarrow T\otimes_A T$$

is a resolution of $T\otimes_A T$ (right exactness clear; injective by sequence) ~~assuming the flatness~~. Thus we get a $T\otimes T^{op}$ resolution of $T$

$$0 \to T\otimes_A M \otimes_A T \to T\otimes_A T \to T \to 0$$

from which one gets

$$0 \to H_1(A, N) \to H_1(T, N) \to N\otimes_A M \otimes_A \to N/[A,N] \to N/[T,N] \to 0$$

Let $Q = A \oplus M$ with $M$ an ideal of square zero. One can form the Hochschild complex $(Q \otimes Q^{\otimes *}, b)$ and its ~~quotient~~ Connes(Q). Feigin and Tsygan claim

$$\text{Connes}(Q) \sim \text{Connes}(A) \oplus \bigoplus_{k \geq 1} \overbrace{(\tilde{M} \otimes_A \tilde{M} \otimes_A \cdots \tilde{M} \otimes_A)}^{k \text{ times}} \big/ (1-t)^{[k-1]}$$

where $\tilde{M} = M \otimes_A (A \otimes A^{\otimes *} \otimes A)$. I have the impression they may have an isomorphism and not just quasi-isomorphism of complexes.

What happens if $A = \mathbb{C}$? Then $\tilde{M}$ is a complex which is $M$ in every degree.

November 25, 1987

Goodwillie's proof that if $Q = A \oplus M$, then

$$\text{Connes}(Q) = \text{Connes}(A) \bigoplus_{k \geq 1} \left(\tilde{M} \otimes_A\right)^k /(1-t)\,[k-1]$$

where $\tilde{M} = M \otimes_A (A \otimes A^{\otimes *} \otimes A)$ with $1 \otimes b'$; In degree $n$, Connes $(Q)$ is the quotient of

$$Q^{\otimes(n+1)} = \bigoplus_{k \geq 0} \left(Q^{\otimes(n+1)}\right)_{(k)}$$

by the twisted action of $\mathbb{Z}/(n+1)$, where

$$\circledast \quad Q^{\otimes(n+1)}_{(k)} = \bigoplus_{\substack{i_0, \ldots, i_k \geq 0 \\ i_0 + \cdots + i_k + k = n+1}} A^{\otimes i_0} \otimes M \otimes A^{\otimes i_1} \otimes \cdots \otimes M \otimes A^{i_k}$$

Observation. Suppose a group $G$ acts on an abelian group $N$ and

$$N = \bigoplus_{s \in S} N_s$$

is a decomposition (system of imprimitivity) permuted around by $G$. Then

$$N_G = \bigoplus_{Gs \in G\backslash S} N_{G_s}$$

---

The action of $\mathbb{Z}/(n+1)$ on $N = Q^{\otimes(n+1)}_{(k)}$ with the system of imprimitivity $\circledast$ is such that the action on the index set $S$ is not free. Goodwillie's idea is to introduce a $k$-fold cyclic covering $S \times \mathbb{Z}/k$ which distinguishes the different copies of $M$.

Let $S = \left\{ (i_0, \ldots, i_k) \;\middle|\; \begin{array}{l} i_0, \ldots, i_k \geq 0 \\ i_0 + \cdots + i_k + k = n+1 \end{array} \right\}$, so that

$\circledast$ is $\quad N = \bigoplus_{s \in S} N_s$. $\quad$ ~~Form $N = \bigoplus_{S \times \mathbb{Z}/k} N_s$~~

~~$\tilde{N} = N \otimes \oplus[\mathbb{Z}/k]$ and it is clear that $(\tilde{N})_{\mathbb{Z}/k} \neq N$. On the other hand~~

Thus an element $s$ of $S$ is a subset of cardinality $k$ inside the cyclic set $\mathbb{Z}/(n+1)$. Such a subset $s$ inherits a cyclic order. An element of $\tilde{S}$ will be a "cyclic" embedding $\mathbb{Z}/k \hookrightarrow \mathbb{Z}/(n+1)$ i.e. a subset $s$ in $S$ together with a parametrization. Denote such a "cyclic" embedding by $\omega$ and set

$$\tilde{N} = \bigoplus_{\omega \in \tilde{S}} N_{\bar{\omega}}$$

where $\bar{\omega} = \mathrm{Im}(\omega)$.

Now it's clear that $\mathbb{Z}/(n+1)$ acts freely on $\tilde{S}$, and one gets a system of representatives for the orbits given by those $\omega$ such that $\omega(1) = 0$. (Actually $\mathbb{Z}/k$ should be $\{1, \ldots, k\}$ and $\mathbb{Z}/(n+1)$ should be $\{0, \ldots, n\}$.)

Therefore

$$N_{\mathbb{Z}/(n+1)} = \left(\tilde{N}_{\mathbb{Z}/k}\right)_{\mathbb{Z}/(n+1)} = \tilde{N}_{\mathbb{Z}/k \times \mathbb{Z}/n+1}$$

$$= \left(\tilde{N}_{\mathbb{Z}/n+1}\right)_{\mathbb{Z}/k}$$

where

$$\tilde{N}_{\mathbb{Z}/n+1} = \bigoplus_{\substack{i_1, \ldots, i_k \geq 0 \\ i_1 + \ldots + i_k + k = n+1}} M \otimes A^{\otimes i_1} \otimes \cdots \otimes M \otimes A^{\otimes i_k}$$

This is the same as $\tilde{M} \otimes_A \cdots \otimes_A \tilde{M} \otimes_A$ in degree $n+1-k$. $\therefore$ At least additively we have

$$\mathrm{Connes}(Q)_{(k)} = \mathrm{Connes}(\tilde{N})/(1-t) = \left(\tilde{M} \otimes_A\right)^k/(1-t)\,[k-1]$$

I assume the differentials work.

At this point I have identified the $E^1$ term of the spectral sequence for cyclic homology of an extension $I \to P \to A$. Now to understand the edge homomorphisms.

The $E^0$ term of the spectral sequence, that is the double complex with vertical differential $b$ is the Connes complex of $P \oplus I$ where $I$ has odd degree. Thus we know by the above that the complex in the column of degree $k$ is

$$\begin{cases} \text{Connes}(P) & k=0 \\ (\tilde{I}\otimes_P)^k/(1-\sigma)^{[k-1]} & k>0. \end{cases}$$

The $d^1_0$ in the spectral sequence is induced by a map

$$\circledast \qquad \left((\tilde{I}\otimes_P)^{k+1}/(1-\sigma_{k+1})\right)[1] \longrightarrow (\tilde{I}\otimes_P)^k/(1-\sigma_k) \qquad k>0$$

depending only on $I$ as a $P$-bimodule equipped with a map of $P$-bimodules $I \to P$. For $k=0$ we have a map

$$\tilde{I}\otimes_P \longrightarrow \text{Connes}(P)$$

which is essential the canonical map from Hochschild to cyclic homology. In effect it's the composition

$$\tilde{I}\otimes_P \longrightarrow \tilde{P}\otimes_P \qquad H_*(P,I) \longrightarrow H_*(P,P)$$

of the map on Hochschild induces by $I \to P$ followed by the map $\tilde{P}\otimes_P \longrightarrow \text{Connes}(P)$.

The thing to understand is why we get a map of degree 1 in $\circledast$

Our problem is to construct a map of degree 1

$$(\tilde{I}\otimes_P)^{k+1}_\sigma\,[1] \longrightarrow (\tilde{I}\otimes_P)^{k}_\sigma$$

One way to produce such a map would be to find a map from $(\tilde{I}\otimes_P)^{k+1}_\sigma$ to $(\tilde{I}\otimes_P)^{k}_\sigma$ and give two reasons for it to be homotopic to another map. To gain insight let's look at what happens in degree 0.

Recall that we take the complex $(I\xrightarrow{u}P)^{\otimes k+1}$ and divide by the action of $t$. At the high degree end this gives

$$x_0\otimes\cdots\otimes x_k \longmapsto \sum_{i=0}^{k}(-1)^i(x_0\otimes\cdots\otimes \overset{\text{odd}}{x_{i-1}})\otimes \overset{\text{even}}{u(x_i)}\otimes(x_{i+1}\otimes\cdots\otimes x_k)$$

$$\begin{tikzcd}
(I\to P)^{\otimes k+1}: & I^{\otimes k+1} \arrow[r] \arrow[d] & \bigoplus_{i=0}^{k} I^{\otimes i}\otimes P\otimes I^{\otimes k-i} \arrow[r] \arrow[d, "\text{quotient by } t"] \arrow[dr, "b"] & {} \\
(I\to P)^{\otimes k+1}_\sigma: & I^{\otimes k+1}_\sigma \arrow[r, "\varphi"] & P\otimes I^{\otimes k} \arrow[r] \arrow[dr, "b"'] & I^{\otimes k} \arrow[d] \\
& & & I^{\otimes k}_\sigma
\end{tikzcd}$$

Formula for $b: P\otimes I^{\otimes k}\longrightarrow I^{\otimes k}_\sigma$

$$b(a\otimes x_1\otimes\cdots\otimes x_k) = ax_1\otimes\cdots\otimes x_k - x_1\otimes\cdots\otimes x_{k-1}\otimes x_k a$$

$$\varphi(x_0\otimes\cdots\otimes x_k) = \sum_{i=0}^{k} u(x_i)\otimes(x_{i+1}\otimes\cdots\otimes x_k\otimes x_0\otimes\cdots\otimes x_{i-1})$$

(signs: $t$ acting on $(I\to P)^{\otimes(k+1)}$ is $(-1)^k\sigma$, and $t^i$ moving $I^{\otimes i}$ past $P\otimes I^{\otimes(k-i)}$ produces the sign

$$\left((-1)^k(-1)^{k-1}\right)^i = (-1)^i$$

So now let us pass to the complexes. We have first of all the square

$$\begin{tikzcd}
(\tilde{I}\otimes_P)^{k+1} \arrow[r] \arrow[d] & \bigoplus_{i=0}^{k} (\tilde{I}\otimes_P)^i \tilde{P}\otimes_P (\tilde{I}\otimes_P)^{k-i} \arrow[d] \\
(\tilde{I}\otimes_P)^{k+1}_\sigma \arrow[r, "\varphi"] & \tilde{P}\otimes_P (\tilde{I}\otimes_P)^k
\end{tikzcd}$$

Here $\varphi$ will like a norm, a sum over $\mathbb{Z}/(k+1)$, together with the effect of $u: I \longrightarrow P$. Next we need to find two maps $\tilde{P}\otimes_P(\tilde{I}\otimes_P)^k \overset{v}{\underset{w}{\rightrightarrows}} (\tilde{I}\otimes_P)^k$ corresponding to the left and right multiplication of $P$ on $I$.

~~Hopefully when composed with $\varphi$, these maps become homotopic $v\varphi \simeq w\varphi$~~

Then if $\pi: (\tilde{I}\otimes_P)^k \longrightarrow (\tilde{I}\otimes_P)^k_\sigma$ is the canonical map we want to have $\pi v \varphi = \pi w \varphi$ and also a non-trivial self homotopy between them.

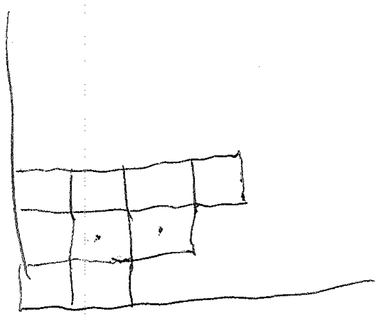
This is still too hard.

<u>November 26, 1987</u>

Consider the spectral sequence in cyclic homology for the extension $I \to P \to A$. We wish to make explicit the edge homom.

$$HC_{2n-1}(A) \longrightarrow \big(\overbrace{I\otimes_P I \otimes_P \cdots I \otimes_P}^{n}\big)_\sigma = E^1_{n,n-1}$$

$$HC_{2n}(A) \longrightarrow E^2_{n,n} \hookrightarrow \operatorname{Coker}\{E^1_{n+1,n} \xrightarrow{d^1} E^1_{nn}\}$$



Especially we need the map

$$\begin{array}{ccc} E^1_{k+1,k} & \longrightarrow & E^1_{k,k} \\ \| & & \| \\ H_0\big((\tilde{I}\otimes_P)^{k+1}_\sigma\big) & \longrightarrow & H_1\big((\tilde{I}\otimes_P)^k_\sigma\big) \end{array}$$

whose kernel and cokernel when $P$ is free are $HC_{2k+1}(A)$ and $HC_{2k}(A)$.

Recall the results so far. First of all we obtain the spectral sequence by applying the Connes functor to the differential graded ring $0 \to I \to P$.

<u>Question:</u> When is $\overset{0}{\to} M \xrightarrow{u} P$ a DGA? Clearly $P$ is an algebra and $M$ is a bimodule. Then

$$u(pm) = \partial(pm) = p\,\partial m = p\,u(m)$$

and similarly for right multiplication. Thus is a bimodule morphism. Also

$$0 = \partial(mm') = u(m)m' - m\,u(m')$$

so
$$\boxed{u(m)m' = m\,u(m')}$$

Notice this is automatic if $u$ is injective because

$$u(u(m)m') = u(m)u(m') = u(m\,u(m'))$$

once $u$ is a bimodule morphism.

The spectral sequence starts with

$$E^0_{*k} = \mathrm{Connes}_k (P \leftarrow I \hookleftarrow) = (P \leftarrow I \hookleftarrow)^{\otimes(k+1)} / (1-t)$$

where $t = (-1)^k \sigma$, $\sigma$ being the cyclic permutation. Draw as follows with arrows running $\longrightarrow$.

$$\begin{tikzcd} (I \to P)^{\otimes(k+1)} : & I^{\otimes(k+1)} \arrow[r, "\partial"] \arrow[d] & \bigoplus_{i=0}^{k} I^{\otimes i} \otimes P \otimes I^{\otimes(k-i)} \arrow[d] \\ (I \to P)^{\otimes(k+1)} / 1-t : & I^{\otimes(k+1)}_{\sigma} \arrow[r, "\varphi"] & P \otimes I^{\otimes k} \end{tikzcd}$$

We saw yesterday that

$$\partial(x_0 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} (-1)^i (x_0 \otimes \cdots \otimes x_{i-1}) \otimes u(x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k)$$

$$\varphi(x_0 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} u(x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_k \otimes x_0 \otimes \cdots \otimes x_{i-1})$$

If $b : C_k(P \leftarrow I) \longrightarrow C_{k-1}(P \leftarrow I)$ is the vertical differential in the double complex $E^0$, then

$$b : \begin{matrix} E^0_{k,k-1} & \longrightarrow & E^0_{k-1,k-1} \\ \| & & \| \\ P \otimes I^{\otimes k} & \longrightarrow & I^{\otimes (k)}_{\sigma} \end{matrix}$$

~~is given by~~ $\cancel{\boxed{[illegible]}}$

$\cancel{b(a \otimes x_1 \otimes \cdots \otimes x_k) = (a\, u(x_1)) \otimes \cdots \otimes x_k - u(x_k) a \otimes x_0 \otimes \cdots \otimes x_{k-1}}$

~~Take~~ $\cancel{k=1}$ $\cancel{\left\{ \begin{array}{l} \varphi(x \otimes y) = u(x) \otimes y + y \otimes u(x) \\ b(a \otimes x) = \boxed{[illegible]} \end{array} \right.}$

$\cancel{\boxed{b\varphi(x \otimes y) = [illegible]} = u(x)u(y) - u(y)u(x)}$

$\cancel{b(u(x) \otimes y) + b(y \otimes u(x))}$

is given by

$$b(a\otimes x_1\otimes\cdots\otimes x_k) = (ax_1)\otimes x_2\otimes\cdots\otimes x_k - (x_k a)\otimes x_0\otimes\cdots\otimes x_{k-1}.$$

Take $k=1$.

$$\varphi(x\otimes y) = u(x)\otimes y + u(y)\otimes x \in P\otimes I$$

$$b(a\otimes x) = ax - xa \in I$$

$$(b\varphi)(x\otimes y) = u(x)y - yu(x) + u(y)x - xu(y)$$

and this is zero by the identity $u(x)y = xu(y)$.

So we have the diagram near the pt $(k,k)$ in the double complex

$$\begin{tikzcd}
P\otimes I^{\otimes k+1} \arrow[r] \arrow[d, "b"] & ? \arrow[d, "b"] \\
I^{\otimes(k+1)}_\sigma \arrow[r, "\varphi"] & P\otimes I^{\otimes k} \arrow[d, "b"] \\
& I^{\otimes k}_\sigma
\end{tikzcd}$$

What is the group $?$? One starts with the summand of $(P\leftarrow I)^{\otimes(k+2)}$ of degree $k$ in $I$ which is

$$\bigoplus_{n_0+n_1+n_2=k} I^{n_0}\otimes P\otimes I^{n_1}\otimes P\otimes I^{n_2}$$

and one divides out by the action of $t = (-1)^{k+1}\sigma_{k+2}$. Thus one gets a $\mathbb{Z}/(k+2)$-module with a system of imprimitivity indexed by the 2 elt subsets of $\{0,\dots,k+1\}$. To compute the quotient one can introduce the double cover module where the 2 element subset is ordered. Then the action by $\mathbb{Z}/(k+1)$ is free and so we see

$$? = \left(\bigoplus_{i=0}^{k} P\otimes I^{\otimes i}\otimes P\otimes I^{\otimes(k-i)}\right)_{\mathbb{Z}/2\mathbb{Z}}$$

It's clear that in order to proceed any further we have to understand well the significance of

$$E^1_{k,k} = H_1\left(\left(I \overset{\mathbb{L}}{\otimes}_P\right)^k_\sigma\right)$$

and perhaps $H_1\left(\left(I \overset{\mathbb{L}}{\otimes}_P\right)^k\right)$. Especially we want to understand how to compute these groups when $P$ is free and $I$ is an ideal in $P$.

One begins with

$$H_0\left(\left(I \overset{\mathbb{L}}{\otimes}_P\right)^k\right) = \left(\overbrace{I \otimes_P I \otimes_P \cdots \otimes_P I}^{k}\right) \otimes_{P \otimes P^o} P$$

A linear functional on this is a multilinear map $\varphi(x_1, \ldots, x_k)$ on $I^k$ such that

$$(*) \quad \begin{cases} \varphi(x_1, \ldots, x_i a, x_{i+1}, \ldots, x_k) = \varphi(x_1, \ldots, x_i, a x_{i+1}, \ldots, x_n) \\ \varphi(x_1, \ldots, x_k a) = \varphi(a x_1, x_2, \ldots, x_k) \end{cases}$$

Perhaps one should think more generally of

$$M_1 \otimes_A M_2 \otimes_A \cdots \otimes_A M_k \otimes_A .$$

Then a linear functional on this would arise if one gave bimodule maps $M_i \longrightarrow \{\text{Operators}\}$, where the algebra $R$ of operators already has a map $A \to R$, such that the products $M_1 \cdots M_k$ of the operators are of trace class. In the case where all $M_i = M$, then one can view a linear functional on

$$(M \otimes_A)^k_\sigma$$

as being a multilinear map satisfying $(*)$ and cyclic symmetry.

Next we turn to $H_1$. First take $k = 1$.

Then $H_0(M \overset{\mathbb{L}}{\otimes}_A) = M \otimes_A = M/[A,M]$ and a linear functional on this is a linear functional $\varphi$ on $M$ such that $\varphi(am) = \varphi(ma)$.

$$H_1(M \overset{\mathbb{L}}{\otimes}_A) = H_1(M \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A) = \mathrm{Tor}_1^{A\otimes A^{op}}(M, A) = H_1(A,M)$$

is given by

$$0 \to H_1(A,M) \to M\otimes A/b(M\otimes A^2) \xrightarrow{b} M \to H_0(A,M) \to 0$$

$$\| $$

$$M \otimes_A \Omega^1_A \otimes_A$$

A linear functional on $H_1(A,M)$ can be extended to a linear functional on $M\otimes_A \Omega^1_A \otimes_A = M\otimes A/b(M\otimes A^2)$, which is a linear map

$$\tau : M \otimes A \longrightarrow \mathbb{C} \qquad \text{such that}$$

$$\boxed{\tau(ma_0, a_1) - \tau(m, a_0a_1) + \tau(a_1m, a_0) = 0}$$

Either we think of $\tau$ as a bilinear ~~functional~~ $\tau(m,a)$, or as a pairing

$$M \otimes \Omega^1_A \longrightarrow \mathbb{C}$$

such that
$$\tau(ma, \omega) = \tau(m, a\omega)$$
$$\tau(am, \omega) = \tau(m, \omega a).$$

Given $\lambda : M \to \mathbb{C}$ one gets such a $\tau$ by

$$\tau(m,a) = \lambda(ma - am)$$

and these are the 1-cocycles which are coboundaries (in the complex of Hochschild cochains on $A$ with values in $M^*$.)

Consider the tensor algebra

$$T_A(M) = A \oplus M \oplus M \otimes_A M \oplus \cdots$$

and let's find its cyclic homology by the method of Feigin + Tsygan. One chooses a free simplicial algebra resolution of $A$ call it $F_\cdot$, and then a free simplicial bimodule $N_\cdot$ ove $F_\cdot$ which resolves $A$. Then

$$T_F(N) = F \oplus N \oplus (N \otimes_F N) \oplus \cdots$$

is a free simplicial algebra resolution of $T_A(M)$ by virtue of Kunneth type theorems. So

$$HC_*(T_A(M)) = H_*\left(T_F(N)/[\,,\,]\right)$$

$$= H_*(F/[F,F]) \oplus \bigoplus_{k \geq 1} H_*\left((N \otimes_F)^k_\sigma\right)$$

$$= HC_*(A) \oplus \bigoplus_{k \geq 1} H_*\left((M \overset{\mathbb{L}}{\otimes}_A)^k_\sigma\right)$$

What's striking about this is that if we replace $A, M$ by $P, I$ where $P/I = A$, then $HC_*(T_P(I))$ involves the same groups as the $E^1$-term of the spectral sequence for the extension. One might hope to find another spectral sequence using $T_p(I)$ rather than the DG algebra $(I \to P)$. ▮

Remarks 1). If $M$ is free as an $A$-bimodule that is $M = A \otimes W \otimes A$ with $W$ a vector space, then the cyclic homology of $T_A(M)$ is trivial apart from the factor which is $HC_*(A)$. One has

$$\left(M \overset{\mathbb{L}}{\otimes}_A M \overset{\mathbb{L}}{\otimes}_A \cdots \overset{\mathbb{L}}{\otimes}_A M\right) \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A$$

$$= (A\otimes W\otimes A)\otimes_A (A\otimes W\otimes A)\cdots (A\otimes W\otimes A) \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A$$

$$= (W\otimes A)^{\otimes k}$$

Presumably this would also follow from the fact that $T_A(A\otimes W\otimes A) = A * T(W)$

2). Suppose $A, M$ replaced by $P, I$ where $P$ is free and $I$ is free as a right $P$-module. Then instead of a free simplicial algebra resolution of $T_P(I)$ we can construct quite easily a free DGA resolution. We start with $P$ which is a free algebra. Let $P = T(V)$, so that we have an exact sequence of $T$-bimodules

$$0 \longrightarrow P\otimes V\otimes I \longrightarrow P\otimes I \longrightarrow I \longrightarrow 0$$

Let $R = T_P(P\otimes I)$; this is free and maps onto $T_P(I)$ with kernel generated by $P\otimes V\otimes I$. It should be the case that one has an exact sequence

$$(*) \qquad 0 \longrightarrow R\otimes(P\otimes V\otimes I)\otimes R \longrightarrow R \longrightarrow T_P(I) \longrightarrow 0$$

i.e. that the kernel is a free $R$-bimodule?

---

I should really understand the cyclic homology of $T_P(I)$. According to the formula above (p 376) it is non-trivial only in degree 0, 1.

Actually Remark 2 above should have been that $T_P(I)$ is an extension $(*)$ of $R = T_P(P\otimes I)$

by a free $R$-bimodule, and that $E^1$ of the spectral sequence of this extension is very small.

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## November 28, 1987

Let us consider again an algebra $P$, a bimodule $I$, and a map $u: I \to P$ such that $u(x)y = x\,u(y)$ for $x, y \in I$. This means $0 \to I \to P$ is a DGA. Then we get double complexes by considering the Hochschild and Connes complexes of this ~~[illegible]~~ DGA. Picture of Hochschild cx:

$$\begin{array}{ccccccc}
I^{\otimes 3} & \longrightarrow & P\otimes I^{\otimes 2} \oplus I\otimes P\otimes I \oplus I^{\otimes 2}\otimes P & \longrightarrow & P^{\otimes 2}\otimes I + (P\otimes I\otimes P) + I\otimes P^{\otimes 2} & \longrightarrow & P^{\otimes 3} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & I^{\otimes 2} & \longrightarrow & P\otimes I + I\otimes P & \longrightarrow & P^{\otimes 2} \\
 & & & & \downarrow & & \downarrow \\
 & & & & I & \longrightarrow & P
\end{array}$$

Let's recall some facts about the Hochschild complex of a semi-direct product $P \oplus I[1] = Q$. Let's denote it $C(Q)$. First of all, it is graded because of the grading on $Q$; $C(Q) = \bigoplus C(Q)_{(k)}$. Next $C(Q)_{(k)}$ is an extension

$$0 \to (I\otimes_P \tilde{P})\otimes_P \cdots \otimes_P (I\otimes_P \tilde{P})\otimes_P [k-1] \longrightarrow C(Q)_{(k)} \to \tilde{P}\otimes_P (I\otimes_P \tilde{P}\otimes_P)^k [k] \to 0$$

So $C(Q)_{(k)}$ is a mapping cone of some map, probably the norm map, on $(I \overset{\mathbb{L}}{\otimes}_P)^k [k-1]$. Recall also that $\mathrm{Connes}(Q)_{(k)}$ is $(I\otimes_P \tilde{P}\otimes_P)^k_\sigma\, [k-1]$.

Now the norm map $N_k$ times $\frac{1}{k}$ is a projector, so that in the derived category we should have

$$\mathrm{Cone}\left\{\left(I\overset{\mathbb{L}}{\otimes}_P\right)^k[k-1] \xrightarrow{N} \left(I\overset{\mathbb{L}}{\otimes}_P\right)^k[k-1]\right\} = C(Q)_{(k)}$$

$$= \mathrm{Cone}\left\{\left(I\overset{\mathbb{L}}{\otimes}_P\right)^k_\sigma[k-1] \xrightarrow{0} \left(I\overset{\mathbb{L}}{\otimes}_P\right)^k_\sigma[k-1]\right\}$$

$$= \left(I\overset{\mathbb{L}}{\otimes}_P\right)^k_\sigma[k-1] \oplus \left(I\overset{\mathbb{L}}{\otimes}_P\right)^k_\sigma[k]. = \mathrm{Connes}(Q)_{(k)} \oplus \mathrm{Connes}_{(k)}(Q)[1]$$

Thus we should be able to lift $\mathrm{Connes}(Q)_{(k)}$ into $C(Q)_{(k)}$ apply the horizontal boundary operator coming from $u: I \to P$, to go into $C(Q)_{(k-1)}$, and then <u>project</u> into $\mathrm{Connes}_{(k-1)}(Q)$.

---

We really have to understand this on a concrete explicit level. For example when $k=2$ we want a map

$$(*) \qquad (I\otimes_P)^{2}_\sigma \longrightarrow H_1(P,I)$$

Recall that one has

$$\begin{array}{ccccccc} 0 \longrightarrow & H_1(P,I) & \longrightarrow & I\otimes P/b(I\otimes P^{\otimes 2}) & \xrightarrow{b} & I \\ & & & \parallel & & \\ & & & I\otimes_P \Omega^1_P \otimes_P & & \end{array}$$

and a linear functional on $I\otimes P/b(I\otimes P^{\otimes 2})$ is a bilinear functional $\tau(x,a)$ such that

$$\tau(xa,a') + \tau(a'x,a) = \tau(x,aa')$$

Given such a $\tau$ we wish to understand its composition with $(*)$.

Take then $x\otimes y \in (I\otimes_P \otimes I\otimes_P)_\sigma$, lift to $x\otimes y \in I\otimes I$, apply the horizontal boundary to get $\quad u(x)\otimes y - x\otimes u(y) \in P\otimes I + I\otimes P,$ then project into $C(Q)_{(1)}$ which we can identify with $C(P,I) = (I\otimes P^{\otimes *}, b)$. In degree 2 we have $t = -\sigma$, so that we get

$$-(y\otimes u(x) + x\otimes u(y)) \in I\otimes P$$

which indeed goes to zero under $b$:

$$b(y\otimes u(x) + x\otimes u(y)) = [y, u(x)] + [x, u(y)]$$
$$= \cancel{y\,u(x)} - \cancel{u(x)y} + \cancel{x\,u(y)} - \cancel{u(y)x}$$

Thus we find

$$(I\otimes_P)^2_\sigma \longrightarrow H_1(P,I) \hookrightarrow I\otimes_P \Omega^1_P \otimes_P$$
$$x\otimes y \longmapsto -(x\otimes du(y) + y\otimes du(x))$$

Checks

$$\underset{\checkmark}{xa\otimes d\,u(y)} + y\otimes \underbrace{d\,u(xa)}_{du(x)a + u(x)da} \overset{?}{=} x\otimes \underbrace{du(ay)}_{(da)u(y) + a\,du(y)} + ay\otimes du(x)$$

Next I need to know the homomorphism

$$P/[P,P] \longrightarrow H_1(P,P)$$

which occurs in Connes' exact sequence, and I should check it is an isomorphism when $P$ is free. The homomorphism is induced by $B$ on the Hochschild complex

$$B : P \longrightarrow P^{\otimes 2}$$
$$x \longmapsto 1\otimes x - x\otimes 1$$

Thus $\quad B = d : P \longrightarrow \Omega^1_P . \qquad x\,dy \longmapsto [x,y]$

Now $\quad H_1(P) = \operatorname{Ker}\{\Omega^1_P/[P,\Omega^1_P] \longrightarrow P\}.$

When $P = T(V)$, one has $\Omega^1_P = P\otimes V\otimes P$, so $\Omega^1_P/[P,\Omega^1_P] = P\otimes V$ and

$$(*)\quad \begin{tikzcd} 0 \arrow[r] & H_1(P) \arrow[r] & P\otimes V \arrow[r, "b"] & P \arrow[r] & H_0(P) \\ & & \cup & \cup & \\ & & V^{\otimes k}\otimes V \arrow[r, "1-\sigma"] & V^{\otimes(k+1)} & \end{tikzcd}$$

On the other hand $d: P \longrightarrow \Omega^1_P$ is

$$d(v_1\cdots v_k) = \sum_{i=1}^{k} v_1\cdots v_{i-1}\, dv_i\, v_{i+1}\cdots v_k$$

so

$$\begin{tikzcd} P \arrow[r, "d"] & \Omega^1_P \arrow[r] & \Omega^1_P/[P,\Omega^1_P] = P\otimes V \\ \cup & & \cup \\ V^{\otimes(k+1)} \arrow[rr, "N_{k+1}"] & & V^{\otimes k}\otimes V \end{tikzcd}$$ is

$$v_0\otimes\cdots\otimes v_k \longmapsto \sum_{i=0}^{k} v_{i+1}\cdots v_k v_0\cdots v_{i-1}\otimes v_i$$

which is $N_{k+1} = \sum_{i=0}^{k} \sigma_{k+1}^{i}$. Since $\operatorname{Ker} N = \operatorname{Im}(1-\sigma) = [P,P] = [P,P]$ it follows $P/[P,P] \hookrightarrow H^1(P)$.

Since $\operatorname{Ker}(1-\sigma) = H_1(P) = \operatorname{Im}(N) = \operatorname{Im}\{P \xrightarrow{d} \Omega^1_P/[P,\Omega^1_P]\}$ it follows that $P/[P,P]$ maps onto $H^1(P)$.

Finally we ought to be able to check that the diagram

$$\begin{tikzcd} x\otimes y \arrow[rr, mapsto] & & xy \arrow[r, mapsto] & d(xy) \\ I\otimes_P I\otimes_P \arrow[d] \arrow[rr] & & P/[P,P] \arrow[dr] & \\ & & & H_1(P,P) \\ (I\otimes_P I\otimes_P)_\sigma \arrow[rr] & & H_1(P,I) \arrow[ur] & \\ x\otimes y \arrow[rr, mapsto] & & dx\otimes y + dy\otimes x \in \Omega^1_P\otimes_P I\otimes_P & \end{tikzcd}$$

commutes (up to sign). This is clear.

Now we want to generalize this to higher cases. Let's first try to construct the diagram and later identify identify the map with $d'$.

$$x_0 \otimes \cdots \otimes x_k \longmapsto x_0 \cdots x_k \longmapsto$$

$$\begin{tikzcd}
(I\otimes_P)^{\otimes(k+1)} \arrow[r] \arrow[d] & P/[P,P] \arrow[dr] & & \\
 & & H_1(P,P) \subset \Omega^1_P/[P,\Omega^1_P] & \\
(I\otimes_P)^{\otimes(k+1)}_\sigma \arrow[r, "\varphi"] & H_1(P, I\otimes_P \cdots \otimes_P I) \arrow[ur] \subset \Omega^1_P \otimes_P I\otimes_P \cdots \otimes_P I \otimes_P & &
\end{tikzcd}$$

Now $\displaystyle d(x_0 \cdots x_k) = \sum_{i=0}^{k} x_0 \cdots x_{i-1}\, dx_i\, x_{i+1} \cdots x_k$

$\displaystyle \qquad = \sum (dx_i)\, x_{i+1} \cdots x_k\, x_0 \cdots x_{i-1} \in \Omega^1_P/[P,\Omega^1_P]$

so the candidate for $\varphi$ is

$$\boxed{\varphi(x_0 \otimes \cdots \otimes x_k) = \sum_{i=0}^{k} dx_i \otimes x_{i+1} \otimes \cdots \otimes x_k \otimes x_0 \otimes \cdots \otimes x_{i-1}}$$

Let's check $\varphi$ is well-defined.

$$\varphi(a x_0 \otimes \cdots \otimes x_k) \overset{?}{=} \varphi(x_0 \otimes \cdots \otimes x_k a)$$

$$\begin{array}{ccc}
\| & & \| \\
\underbrace{d(a x_0)}_{(da)x_0 + a\,dx_0} \otimes x_1 \otimes \cdots \otimes x_k & & dx_0 \otimes x_1 \otimes \cdots \otimes (x_k a) \\
+ & & \\
dx_1 \otimes x_2 \otimes \cdots \otimes x_k \otimes a x_0 & \longleftrightarrow & dx_1 \otimes x_2 \otimes \cdots \otimes (x_k a) \otimes x_0 \\
+ & \longleftrightarrow & \\
\vdots & & \\
dx_k \otimes (a x_0) \otimes x_1 \otimes \cdots \otimes x_{k-1} & & \underbrace{d(x_k a)}_{x_k\,da + (dx_k)\,a} \otimes x_0 \otimes \cdots \otimes x_{k-1}
\end{array}$$

So we have to see

$$(da)x_0 \otimes x_1 \otimes \cdots \otimes x_k \overset{?}{=} x_k\, da \otimes x_0 \otimes \cdots \otimes x_{k-1}$$

However this works because

$$(da)x_0 \otimes x_1 \otimes \cdots \otimes x_k = da \otimes x_0 x_1 \otimes x_2 \otimes \cdots$$
$$= da \otimes x_0 \otimes x_1 x_2 \otimes \cdots = da \otimes x_0 \otimes \cdots \otimes x_{k-1} x_k$$
$$= x_k\, da \otimes x_0 \otimes \cdots \otimes x_{k-1}.$$

Next we check that the image of $\varphi$ is contained in $H_1(P, I \otimes_P \cdots \otimes_P I) = \mathrm{Ker}\left\{ \begin{array}{ccc} \Omega^1_P \otimes (I \otimes_P)^k & \longrightarrow & I \otimes_P \cdots \otimes_P I \\ x\,dy \otimes m & \longmapsto & [y, mx] \end{array} \right\}$

$$\sum_{i=0}^{k} x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_k \otimes x_1 \otimes \cdots \otimes x_{i-1} \overset{?}{=} 0$$
$$- x_{i+1} \otimes \cdots \otimes x_k \otimes x_1 \otimes \cdots \otimes x_{i-1} x_i$$

But this OK because in $I \otimes_P I \otimes_P \cdots \otimes_P I$ ($k$ times)

$$x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{i-1} = x_i \otimes x_{i+1} x_{i+2} \otimes \cdots$$
$$= x_i \otimes x_{i+1} \otimes \cdots \otimes (x_{i-2}\, x_{i-1})$$

so the cyclic sums cancel.

It might be clearer to use the map $u : I \to P$, whence the above calculation should be written (for $i = 0$)

$$u(x_0)\, x_1 \otimes \cdots \otimes x_k = x_0 u(x_1) \otimes x_2 \otimes \cdots \otimes x_k$$
$$= x_0 \otimes u(x_1) x_2 \otimes \cdots$$
$$= x_0 \otimes x_1 u(x_2) \otimes \cdots$$
$$\cdots\cdots$$
$$= x_0 \otimes \cdots \otimes x_{k-2} \otimes x_{k-1} u(x_k).$$

Now let's try to relate $\varphi$ to the differential

$$\begin{array}{ccc} d^1 : E^1_{k+1,k} & \longrightarrow & E^1_{kk} \\ \| & & \| \\ (I \otimes_P)^{k+1}_\sigma & & H_1\big((I \overset{\mathbb{L}}{\otimes}_P)^k_\sigma\big) \end{array}$$

<u>November 29, 1987</u>

We suppose again $A = P/I$ with $P$ free, and we continue with the calculation of

$$d^1 : \underset{\substack{\| \\ H_0((I\overset{L}{\otimes}_P)^{k+1}_\sigma)}}{E^1_{k+1,k}} \longrightarrow \underset{\substack{\| \\ H_1((I\overset{L}{\otimes}_P)^{k}_\sigma)}}{E^1_{k,k}}$$

Now

$$\overbrace{I\overset{L}{\otimes}_P I \overset{L}{\otimes}_P \cdots \overset{L}{\otimes}_P I}^{k \text{ times}} \xrightarrow{\sim} I\otimes_P \cdots \otimes_P I \xrightarrow{\sim} I^k$$

by our assumption that $P$ is free. It should be the case that $(I\overset{L}{\otimes}_P)^k_\sigma$ is a direct factor of $(I\overset{L}{\otimes}_P)^k$, hence that

$$E^1_{kk} = H_1((I\overset{L}{\otimes}_P)^k_\sigma) \text{ is a direct factor of}$$
$$H_1((I\otimes_P)^k) = H_1(P, I^k) = \operatorname{Ker}\{\Omega^1_P \otimes_P I^k \otimes_P \longrightarrow I^k\}$$

Thus there is a mysterious action of $\mathbb{Z}/k$ on $H_1(P, I^k)$ to be understood.

It is important to get this straight now because of the exact sequences

$$\begin{array}{ccccccc} 0 \longrightarrow & HC_1(P/I^{k+1}) & \longrightarrow & I^{k+1}/[P, I^{k+1}] & \longrightarrow & P/[P,P] & \xrightarrow{\sim} H_1(P) \\ & & & \downarrow & & & \quad H_1(P,I^k) \subset H_1(P) \\ 0 \longrightarrow & HC_{2k+1}(A) & \longrightarrow & I^{k+1}/[I, I^k] & \longrightarrow & H_1(P, I^k)_\sigma & \end{array}$$

It's important how to relate $H_1(P, I^k)_\sigma$ to $H_1(P, I^k)$. If we can use the injection, then $HC_1(P/I^{k+1})$ maps onto $HC_{2k+1}(A)$.

So let us return to the double complex $\mathrm{Connes}_*(Q)$, where $Q =$ the DGA $I \longrightarrow P$. One has

$$Q^{\otimes(*)}_{(k)} : \quad \begin{array}{c} \bigoplus I^{\otimes i}\otimes P\otimes I^{\otimes j}\otimes P\otimes I^{\otimes(k-i-j)} \\ \downarrow \\ \bigoplus_{i=0}^{k} I^{\otimes i}\otimes P\otimes I^{\otimes(k-i)} \\ \downarrow \\ I^{\otimes k} \end{array}$$

and to get $(\mathrm{Connes}(Q))_{(k)}$ we have to divide by cyclic groups of different orders; thus we take the quotient by $\sigma_k$ at the bottom, then by $\tau_{k+1}$ with signs in the middle, etc. Goodwillie's arguments show how we can get the same quotient using a single cyclic group $\mathbb{Z}/k$ acting on a similar complex. His complex makes sense for bimodules $I_1, \ldots, I_k$ and appears as

$$\begin{array}{c} \bigoplus_{1\le i<j\le k} I_1\otimes\cdots\otimes(I_i\otimes P)\otimes\cdots\otimes(I_j\otimes P)\otimes\cdots\otimes I_k \oplus \bigoplus_{1\le i\le k} I_1\otimes\cdots\otimes(I_i\otimes P^{\otimes 2})\otimes\cdots\otimes I_k \\ \downarrow \\ \bigoplus_{i=0}^{k} I_1\otimes\cdots\otimes I_i\otimes P\otimes I_{i+1}\otimes\cdots\otimes I_k \\ \downarrow \\ I_1\otimes\cdots\otimes I_k \end{array} \qquad (*)$$

The vertical arrows are given by $b_P$, and they delete tensor signs on either side of a $P$ factor.

Goodwillie's complex represents $I_1\overset{L}{\otimes}_P I_2\overset{L}{\otimes}_P\cdots\overset{L}{\otimes}_P I_k\overset{L}{\otimes}_P$ which we know, when the $I_j$ are left $P$-flat to be quis to $(I_1\otimes_P I_2\otimes_P\cdots\otimes_P I_k)\overset{L}{\otimes}_P$. We want to describe the map

$$E^1_{k+1,k} = (I\otimes_P)^{k+1}_{\sigma} \xrightarrow{d^1} H_1\big((I\overset{L}{\otimes}_P)^k_{\sigma}\big) \hookrightarrow H_1\big((I\overset{L}{\otimes}_P)^k\big)$$

$$\hookrightarrow \underbrace{H_1(P, I\otimes_P\cdots\otimes_P I)}_{k\text{ times}} \hookrightarrow \Omega^1_P\otimes_P(I\otimes_P)^k$$

First we need the maps

$$H_1((I \overset{L}{\otimes}_P)^k) \longrightarrow H_1(P, I\otimes_P \cdots \otimes_P I).$$

Really one should think of this as follows

$$I_1 \overset{L}{\otimes}_P I_2 \overset{L}{\otimes}_P \cdots \overset{L}{\otimes}_P I_k \overset{L}{\otimes}_P = P \overset{L}{\otimes}_{P\otimes P^{op}} \Big( P \overset{L}{\otimes}_{P\otimes P^{op}} \big( \cdots \; (I_1 \otimes I_2 \otimes I_3 \otimes I_k)\cdots\big)\Big)$$

and one can drop all but one of the $L$'s & obtain various maps

$$I_1 \overset{L}{\otimes}_P I_2 \overset{L}{\otimes}_P \cdots I_k \overset{L}{\otimes}_P \longrightarrow P \overset{L}{\otimes}_{P\otimes P^{op}} \left( I_1 \otimes_P \cdots \otimes_P I_i \otimes I_{i+1} \otimes_P \cdots I_k \otimes_P \right)$$

The effect of this map on $H_1$ can be seen easily with $(*)$. One projects $(*)$ onto the complex

$$\begin{array}{c} I_1 \otimes_P \cdots I_i \otimes P \otimes P \otimes I_{i+1} \otimes_P \cdots I_k \otimes_P \\ \downarrow \\ I_1 \otimes_P \cdots \otimes_P I_i \otimes P \otimes I_{i+1} \otimes_P \cdots I_k \otimes_P \\ \downarrow \\ I_1 \otimes_P \cdots I_i \otimes I_{i+1} \otimes_P \cdots I_k \otimes_P \end{array}$$

Now let's start with the element $\alpha = x_0 \otimes \cdots \otimes x_k \in (I \otimes_P)^{k+1}_\sigma$. To calculate $d'\alpha$ in $H_1(\text{Connes}(Q)_{(k)})$, we can lift $\alpha$ to $x_0 \otimes \cdots \otimes x_k \in I^{\otimes k}$ apply the boundary in $(Q^{\otimes *})_{(k)}$ which gives

$$\sum_{i=0}^{k} (-1)^i \, x_0 \otimes \cdots \otimes x_{i-1} \otimes u(x_i) \otimes \cdots \otimes x_k$$

and then project into $\text{Connes}(Q)_{(k)}$ which in degree 1 we will identify with $I^{\otimes k} \otimes P$. Now the cyclic permutation for $Q^{\otimes (k+1)}$ is $t = (-1)^k \sigma$, and all the $x$'s are odd. Modulo $\text{Im}(1-t)$ the above is equivalent to

$$\sum_{i=0}^{k} u(x_i)\otimes x_{i+1}\otimes\cdots\otimes x_k\otimes x_0\otimes\cdots\otimes x_{i-1}$$

which in term is congruent to

$$\oplus \qquad (-1)^k\sum_{i=0}^{k} x_{i+1}\otimes\cdots x_k\otimes x_0\otimes\cdots\otimes x_{i-1}\otimes u(x_i)\ \in I^{\otimes k}\otimes P.$$

Now we have to lift the latter into the . Goodwillie complex. This can done in only one way if the lift is invariant under $\mathbb{Z}/k$.

In general given $y_1\otimes\cdots\otimes y_k\otimes a \in I^{\otimes k}\otimes P$ it lifts to

$$\frac{1}{k}\sum_{i=1}^{k} y_i\otimes\cdots\otimes y_k\otimes a\otimes y_1\otimes\cdots\otimes y_{i-1}$$

in Goodwillie's complex $\circledast$. So the lift of $\oplus$ will be $\frac{(-1)^k}{k}$ times a sum of $k(k+1)$ terms

$$x_j\otimes\cdots\otimes x_{i-1}\otimes u(x_i)\otimes x_{i+1}\otimes\cdots\otimes x_{j-1},$$

where the $u(x_i)$ occurs in any position except the first for $i=0,..,k$. Now this element occurs in $\bigoplus_{k=1}^{k} I^{\otimes i}\otimes P\otimes I^{\otimes(k-i)}$ = the degree 1 part of Goodwillie's complex. We then apply the projection of Goodwillie's complex (which represents $(I\overset{L}{\otimes}_P)^k$) to the Hochschild complex of $I\otimes_P\cdots\otimes_P I$ (which represents $(I\otimes_P\cdots\otimes_P I)\overset{L}{\otimes}_P$. This picks out the $i=k$ factor and gives

$$\frac{(-1)^k}{k}\sum_{i=0}^{k} x_{i+1}\otimes\cdots\otimes x_k\otimes x_0\otimes\cdots\otimes x_{i-1}\otimes u(x_i)\in (I\otimes_P\cdots\otimes_P I)\otimes P$$

Up to the factor $\frac{(-1)^k}{k}$ this is the map $\varphi$ on page 382, so we have proved

Lemma The composition

$$E^1_{k+1,k} = (I\otimes_P)^{k+1}_\sigma \xrightarrow{d^1} E^1_{k,k} = H_1\big((I\overset{L}{\otimes}_P)^k_\sigma\big) \hookrightarrow H_1\big((I\overset{L}{\otimes}_P)^k\big) \longrightarrow H_1(P, I\otimes_P\cdots\otimes_P I) \hookrightarrow (I\otimes_P\cdots\otimes_P I)\otimes_P \Omega^1_P \otimes_P$$

is given by

$$\varphi(x_0\otimes\cdots\otimes x_k) = \frac{(-1)^k}{k}\sum_{i=0}^{k} x_{i+1}\otimes\cdot\otimes x_k\otimes x_0\otimes\cdot\otimes x_{i-1}\otimes dx_i$$

Consequently the diagram

$$\begin{tikzcd}
(I\otimes_P)^{k+1} \arrow[r] \arrow[d] & P/[P,P] \arrow[r] & H_1(P,P) \arrow[r, phantom, "\subset"] & \Omega^1_P/[P,\Omega_P] \\
(I\otimes_P)^{k+1}_\sigma \arrow[r, "d^1"] & E^1_{kk} = H_1\big((I\overset{L}{\otimes}_P)^k_\sigma\big) \arrow[r] & H_1(P, I\otimes_P\cdots\otimes_P I) \arrow[u] \arrow[r, phantom, "\subset"] & (I\otimes_P\cdots\otimes_P I)\otimes_P\Omega^1_P \arrow[u]
\end{tikzcd}$$

commutes up to a $\frac{(-1)^k}{k}$ factor.

---

Assuming this is OK we can now take $P$ free and consider

$$\begin{tikzcd}
0 \arrow[r] & HC_1(P/I^{k+1}) \arrow[r] & I^{k+1}/[P, I^{k+1}] \arrow[r] \arrow[d] & P/[P,P] \arrow[r] & HC_0(P/I^{k+1}) \arrow[r] & 0 \\
0 \arrow[r] & HC_{2k+1}(A) \arrow[r] & I^{k+1}/[I,I^k] \arrow[r, "d^1"] & H_1(P,I^k)_\sigma \arrow[u, hook] \arrow[r] & HC_{2k}(A) \arrow[r] & 0
\end{tikzcd}$$

from which we deduce a canonical surjection of $HC_1(P/I^{k+1})$ onto $HC_{2k+1}(A)$ and an injection of $HC_{2k}(A)$. In fact we have

$$0 \longrightarrow HC_{2k+1}(A) \longrightarrow I^{k+1}/[I,I^k] \longrightarrow P/[P,P]$$

## November 30, 1987

Let $P/I = A$ with $P$ free: $P = T(V)$. We have exact sequences of $A \otimes A^{op}$-modules

$$0 \to \Omega^1_A \to A \otimes A \to A \to 0$$

$$0 \to I/I^2 \to \underset{\substack{\| \\ A \otimes V \otimes A}}{A \otimes_P \Omega^1_P \otimes_P A} \to \Omega^1_A \to 0$$

which are right and left $A$-flat. Thus if $M$ is an $A$-bimodule, we have an exact sequence

$$0 \to \Omega^1_A \otimes_A M \to A \otimes M \to M \to 0$$

which yields when $M$ is right $A$-flat

$$0 \to H_1(A,M) \to \Omega^1_A \otimes_A M \otimes_A \to M \to H_0(A,M) \to 0$$

$$H_{n+1}(A,M) \xrightarrow{\sim} H_n(A, \Omega^1_A \otimes_A M) \qquad n \geq 1$$

We also have an exact sequence

$$0 \to I/I^2 \otimes_A M \to A \otimes V \otimes M \to \Omega^1_A \otimes_A M \to 0$$

which yields when $M$ is right $A$-flat

$$0 \to H_1(A, \Omega^1_A \otimes M) \to (I/I^2) \otimes_A M \otimes_A \to V \otimes M \to H_0(A, \Omega^1_A \otimes M) \to 0$$

$$H_{n+1}(A, \Omega^1_A \otimes M) \xrightarrow{\sim} H_n(A, I/I^2 \otimes_A M) \qquad n \geq 1$$

Putting these together we get

$$\boxed{\begin{gathered} 0 \to H_2(A,M) \to I/I^2 \otimes_A M \otimes_A \to \overbrace{\Omega^1_P \otimes_P M \otimes_P}^{V \otimes M} \to \Omega^1_A \otimes_A M \otimes_A \to 0 \\ H_{n+2}(A,M) \xrightarrow{\sim} H_n(A, I/I^2 \otimes_A M) \qquad n \geq 1 \\ 0 \to H_1(A,M) \to \Omega^1_A \otimes_A M \otimes_A \to M \to H_0(A,M) \to 0 \end{gathered}}$$

$$\boxed{0 \to H_2(A,M) \to I/I^2 \otimes_A M \otimes_A \to H_1(P,M) \to H_1(A,M) \to 0}$$

Next note

$$I^k/I^{k+1} = P/I \otimes_P I^k = (P/I)\otimes_P I \otimes_P I^{k-1}$$

$$= (I/I^2)\otimes_P I^{k-1} = (I/I^2)\otimes_{P/I} (P/I)\otimes_P I^{k-1}$$

$$\boxed{I^k/I^{k+1} = (I/I^2)\otimes_A I^k/I^{k-1} = (I/I^2)\otimes_A \cdots \otimes_A (I/I^2) \quad k \text{ times}}$$

Thus
$$H_2(A, I^{k-1}/I^k) = H_4(A, I^{k-2}/I^{k-1}) = \cdots = H_{2k}(A,A)$$
$$H_1(A, I^{k-1}/I^k) = H_{2k-1}(A,A)$$

and we have the exact sequence

$$\boxed{0 \longrightarrow H_{2k}(A) \longrightarrow \underset{\substack{\| \\ I^k/[P,I^k]+I^{k-1}}}{(I/I^2\otimes_A)^k} \longrightarrow H_1(P, I^{k-1}/I^k) \rightarrow H_{2k-1}(A) \rightarrow 0}$$

linking the Hochschild homology of $A$ with the associated graded alg of $P$.

---

To make further progress we need to understand the action of $\mathbb{Z}/k$ on

$$H_1((I\overset{L}{\otimes}_P)^k) = H_1(P, I^k) = \operatorname{Ker}\{I^k\otimes V \xrightarrow{b} I^k\}$$

More generally, given $P$-bimodules $M, N$ which are right flat, we have

$$(N\otimes_P M)\overset{L}{\otimes}_P \xleftarrow{\sim} M\overset{L}{\otimes}_P N\overset{L}{\otimes}_P \xrightarrow{\sim} (M\overset{L}{\otimes}_P N)\otimes_P$$

which gives an isomorphism

$$\begin{array}{ccc} H_1(P, M\otimes_P N) & \xrightarrow{\sim} & H_1(P, N\otimes_P M) \\ \cap & & \cap \\ (M\otimes_P N)\otimes V & & (N\otimes_P M)\otimes V \end{array}$$

which we would like to understand.

We consider the double complex representing $M \overset{L}{\otimes}_P N \overset{L}{\otimes}_P$ which is $M \otimes_P \tilde{P} \otimes_P N \otimes_P \tilde{P} \otimes_P$, where $\tilde{P} : \overset{\circ}{\to} P \otimes V \otimes P \xrightarrow{b'} P \otimes P$. This appears:

$$\begin{tikzcd}
M\otimes V\otimes N\otimes V \arrow[r] \arrow[d] & M\otimes V\otimes N \arrow[r, dashed] \arrow[d] & (M\otimes V\otimes N)\otimes_P \arrow[d, equal] \\
 & & (N\otimes_P M)\otimes V \\
M\otimes N\otimes V \arrow[r] \arrow[d, dashed] & M\otimes N \arrow[r, dashed] \arrow[d, dashed] & (M\otimes N)\otimes_P \arrow[d, equal] \\
(M\otimes_P N)\otimes V & M\otimes_P N & N\otimes_P M
\end{tikzcd}$$

As it stands this is not much help. Perhaps when we come to discuss the action of $\mathbb{Z}/k$ on $H_1(P, I^k)$, we have to put a double (or maybe $k$ fold) complex into the picture.

---

Consider $P = T(V)$. First of all $HC_*(T(V))$ obviously contains $HC_*(k)$ as a direct summand, so it's only $\overline{HC}_n(T(V)) = 0$ for $n > 0$. So if we propose to work in the unital context it is necessary to use reduced cyclic homology. Next lets recall the appropriate "minimal" complex to use for $P = T(V)$. One has

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_P \arrow[r] \arrow[d, equal] & P\otimes P \arrow[r] & P \arrow[r] & 0 \\
 & P\otimes V\otimes P & & &
\end{tikzcd}$$

and so the replacement for the reduced Hochschild complex is

$$P \otimes V \xrightarrow{b} \overline{P} \qquad b(v_1 \cdots v_k \otimes v_{k+1}) = (1-\sigma)(v_1 \cdots v_{k+1})$$

On the other hand we have

$$\bar{P} \xrightarrow{d} \Omega^1_P = P \otimes V \otimes P \longrightarrow \Omega^1_P \otimes_P = P \otimes V$$

$$v_1 \cdots v_k \longmapsto \sum v_1 \cdots v_{i-1} \otimes v_i \otimes v_{i+1} \cdots v_k \longmapsto \sum v_{i+1} \cdots v_{i-1} v_i = N(v_1 \cdots v_k)$$

and this composite is the $B$ operator. One has the exact sequence

$$(*) \quad \xrightarrow{B} P \otimes V \xrightarrow{b} \bar{P} \xrightarrow{B} P \otimes V \xrightarrow{b} \bar{P}$$

which is just the appropriate $(b, B)$ double complex.

Now suppose $I$ is an ideal in $\bar{P}$. Then we can filter this exact sequence. Set $F_k =$

$$\longrightarrow I^{k+1} \otimes V \xrightarrow{b} I^{k+1} \xrightarrow{B} I^{k} \otimes V \xrightarrow{b} I^{k}$$

Then $gr_k = F_k / F_{k+1}$ is

$$\begin{array}{ccccccc}
\longrightarrow I^{k+1}/I^{k+2} \otimes V & \xrightarrow{b} & I^{k+1}/I^{k+2} & \xrightarrow{B} & I^k/I^{k+1} \otimes V & \xrightarrow{b} & I^k/I^{k+1} \\
 & & \downarrow & & \cup & & \\
 & & \operatorname{Coker}(b) & & \operatorname{Ker} b & & \\
 & & \| & & \| & & \\
 & & H_0(P, I^{k+1}/I^{k+2}) & \xrightarrow{\otimes} & H_1(P, I^k/I^{k+1}) & &
\end{array}$$

Unfortunately, it seems unlikely that $(*)$ is the map on p390 whose kernel + cokernel are $H_{2k}(A)$ and $H_{2k-1}(A)$ respectively.

December 1, 1987

Consider again $T_P(I) = P \oplus I \oplus I \otimes_P I \oplus \cdots$.

We have the embedding $u : I \longrightarrow P$ which is a $P$-bimodule morphism such that $u(x)y = x\,u(y)$. Let $D$ be the derivation on $T_P(I)$ which is $0$ on $P$ and $u$ on $I$. $D$ is locally nilpotent since $D(\underbrace{I \otimes_P \cdots \otimes_P I}_{k}) \subset \underbrace{I \otimes_P \cdots \otimes_P I}_{k-1}$. Hence $D$ can be exponentiated to yield a 1-parameter group $e^{tD}$ of automorphisms of $T_P(I)$.

When $P$ is free we know that $(I \otimes_P)^{k-1} I = I^k$. Viewing $u$ as an inclusion we have for $x_1, \ldots, x_k \in I$ that

$$D(x_1 \cdots x_k) = \sum_{i=1}^{k} x_1 \cdots u(x_i) \cdots x_k = k(x_1 \cdots x_k)$$

Thus $D : (I \otimes_P)^{k-1} I \longrightarrow (I \otimes_P)^{k-2} I$ is simply $k$ times the inclusion map of $I^k$ in $I^{k-1}$. Hence $\mathrm{Ker}(D) = P$.

It seems useful to view $T_P(I)$ as a subalgebra of $P \otimes \mathbb{C}[h]$:

$$T_P(I) = \bigoplus_{k \geq 0} h^k (\overbrace{I \otimes_P \cdots \otimes_P I}^{k}) \subset \bigoplus_{k \geq 0} h^k P$$

On $P \otimes \mathbb{C}[h]$ we have the derivation $\partial_h$ and this restricts to the derivation $D$. In fact $\partial_h$ is just $D$ in the case where $I = P$. As $e^{t\partial_h} f(h) = f(h+t)$, the 1-parameter group $e^{tD}$ is just the restriction of translation.

Next we can consider the largest quotient algebra on which $D$ acts trivially, i.e. the quotient by the ideal generated by $\mathrm{Im}\, D$. In this case $\mathrm{Im}(D)$ is already an ideal:

$$T_P(I)/\mathrm{Im}\, D = \bigoplus_{k \geq 0} I^k / I^{k+1}$$

Next we can consider the (reduced) cyclic homology of $T_P(I)$. According to earlier calculations this is nonzero only in degrees $0, 1$, where

$$\overline{HC}_0(T_P(I)) = \bigoplus_{k\geq 0} (I\otimes_P)^k_\sigma / \mathbb{C}$$

$$\overline{HC}_1(T_P(I)) = \bigoplus_{k\geq 1} H_1(P, I^k)_\sigma$$

How does the derivation $D$ act on this cyclic homology? How else can you derive these formulas besides the simplicial resolution method?

---

Let's consider the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & HC_{2k+1}(A) \arrow[r] \arrow[d] & I^{k+1}/[I,I^k] \arrow[r] \arrow[d] & H_1(P,I^k)_\sigma \arrow[r] \arrow[d] & HC_{2k}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_{2k-1}(A) \arrow[r] \arrow[d] & I^k/[I,I^{k-1}] \arrow[r] \arrow[d] & H_1(P,I^{k-1})_\sigma \arrow[r] \arrow[d] & HC_{2k-2}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[uurrr, bend right] & (I/I^2\otimes_A)^k_\sigma \arrow[r] \arrow[d] & H_1(P,I^{k-1}/I^k)_\sigma \arrow[r] & C \arrow[r] & 0 \\
 & & 0 & & &
\end{tikzcd}$$

By the serpent lemma & Connes exact sequence, $K$ ought to be $H_{2k}(A)$. I am not certain what $H_1(P,I^{k-1}/I^k)_\sigma$ is supposed to be, but with luck one might get $C = H_{2k-1}(A)$. The problem is that we already have an exact sequence (p390) without the $\sigma$. Thus one has a diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H_{2k}(A) \arrow[r] & (I/I^2\otimes_A)^k \arrow[r] \arrow[d] & H_1(P, I^{k-1}/I^k) \arrow[r] & H_{2k-1}(A) \arrow[r] & 0 \\
& & (I/I^2\otimes_A)^k_\sigma & H_1(P, I^{k-1}/I^k)_\sigma \arrow[u, phantom, "\cup" rotate=-90] & &
\end{tikzcd}$$

and the problem is to reconcile this.

---

I suspect that in addition to the exact sequence

$$0 \longrightarrow HC_{2k+1}(A) \longrightarrow I^{k+1}/[I, I^k] \longrightarrow H_1(P, I^k)_\sigma \longrightarrow HC_{2k}(A) \to 0$$

which results from the spectral sequence of the extension there is also an exact sequence

$$* \qquad 0 \longrightarrow HC_{2k}(A) \longrightarrow HC_0(P/I^{k+1}) \longrightarrow H_1(P, P/I^k)_\sigma \longrightarrow HC_{2k-1}(A) \to 0$$

where $H_1(P, P/I^k)_\sigma$ has to be defined suitably. It should be a subspace of

$$H_1(P, P/I^k) = \operatorname{Ker}\left\{\Omega^1_P \otimes_P (P/I^k)\otimes_P \longrightarrow P/I^k\right\}.$$

From $*$ we get (conjecturally) a diagram

$$\begin{tikzcd}[column sep=small]
& & & 0 \arrow[d] & & \\
0 \arrow[r] & (H_{2k+2}(A)) \arrow[r] \arrow[d] & H_0(P, I^{k+1}/I^{k+2})_\sigma \arrow[r] \arrow[d] & H_1(P, I^k/I^{k+1})_\sigma \arrow[r] \arrow[d] & (H_{2k+1}(A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_{2k+2}(A) \arrow[r] \arrow[d] & HC_0(P/I^{k+2}) \arrow[r] \arrow[d] & H_1(P, P/I^{k+1})_\sigma \arrow[r] \arrow[d] & HC_{2k+1}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_{2k}(A) \arrow[r] \arrow[uurrr, bend right=60] & HC_0(P/I^{k+1}) \arrow[r] \arrow[d] & H_1(P, P/I^k)_\sigma \arrow[r] & HC_{2k-1}(A) \arrow[r] & 0 \\
& & 0 & & &
\end{tikzcd}$$

Let's change $k+1$ to $k$ to get

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \phantom{X} \arrow[d, dashed] & \\
0 \arrow[r] & (H_{2k}(A)) \arrow[r] \arrow[d] & H_0(P, I^k/I^{k+1})_\sigma \arrow[r] \arrow[d] & H_1(P, I^{k-1}/I^k)_\sigma \arrow[r] \arrow[d] & (H_{2k-1}(A)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_{2k}(A) \arrow[r] \arrow[d] & HC_0(P/I^{k+1}) \arrow[r] \arrow[d] & H_1(P, P/I^k)_\sigma \arrow[r] \arrow[d] & HC_{2k-1}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_{2k-2}(A) \arrow[r] & HC_0(P/I^k) \arrow[r] \arrow[d] & H_1(P, P/I^{k-1})_\sigma \arrow[r] & HC_{2k-3}(A) \arrow[r] & 0 \\
 & & 0 & & &
\end{tikzcd}$$

Let's start by considering the kernel of

$$H_0(P, I^k/I^{k+1}) = (I/I^2 \otimes_A)^k \longrightarrow P/I^{k+1} + [P,P]$$
$$\| $$
$$I^k/I^{k+1} + [P, I^k]$$

In fact we see that $[I, I^{k-1}] \subset I^k$ goes to zero, whence this map factors through the quotient

$$I^k/I^{k+1} + [I, I^{k-1}] = (I/I^2 \otimes_A)^k_\sigma \underset{\text{def.}}{=} H_0(P, I^k/I^{k+1})_\sigma$$

---

Let's go back to

$$H_1(P, P/I^k) = \operatorname{Ker}\{ \Omega^1_P \otimes_P P/I^k \otimes_P \longrightarrow P/I^k \}$$

and the map

$$HC_0(P/I^{k+1}) \longrightarrow H_1(P, P/I^k)$$

induced by $d$. We should be able to understand why any element coming from $HC_{2k}(A)$ is killed by this map, using a naturality argument. First of all we have an exact sequence

$$0 \longrightarrow I^k/I^{k+1} \longrightarrow (P/I^k) \underset{P/I^{k+1}}{\otimes} \Omega^1_{P/I^{k+1}} \underset{P/I^{k+1}}{\otimes} (P/I^k) \longrightarrow \Omega^1_{P/I^k} \longrightarrow 0$$

but this doesn't seem to help.

We would like to understand the map $HC_0(P/I^{k+1}) \longrightarrow H_1(P, P/I^k) \subset \Omega^1_P \otimes_P P/I^k \otimes_P$ in terms of a derivation on $P/I^{k+1}$. We should begin by trying to find $\Omega^1_{P/I^{k+1}}$. This is the quotient of $\Omega^1_P = P \otimes V \otimes P$ by the ideal generated by $d(I^{k+1})$. One has

$$d(I^{k+1}) \subset \sum_{i=0}^{k} I^i \Omega^1_P I^{k-i}$$

hence we have a surjection

$$\begin{tikzcd}[column sep=small]
\Omega^1_{P/I^{k+1}} \arrow[rr, two heads] \arrow[dr] & & \Omega^1_P \Big/ \sum_{i=0}^{k} I^i \Omega^1_P I^{k-i} = W \\
& P/I^k \otimes_P \Omega^1_{P/I^{k+1}} \otimes_P P/I^k \arrow[ur] &
\end{tikzcd}$$

which means we have a derivation of $P/I^{k+1}$ with values in the latter which is a $P/I^k$-bimod. Thus we have two algebra homomorphisms

$$P/I^{k+1} \underset{1+d}{\overset{1}{\rightrightarrows}} P/I^{k+1} \oplus W$$

whose difference on the trace level is a map

$$HC_0(P/I^{k+1}) \longrightarrow W/[P,W]$$

But

$$W = P \otimes V \otimes P \Big/ \sum I^i \otimes V \otimes I^{k-i}$$

$$W/[P,W] = P \otimes V \Big/ \mathrm{Im}\left(\sum I^i \otimes V \otimes I^{k-i}\right)$$

$$= P \otimes V / I^k \otimes V = (P/I^k) \otimes V$$

Thus we conclude that the map $HC_0(P/I^{k+1}) \to H_1(P, P/I^k)$ is the effect on $HC_0$ of a derivation of $P/I^{k+1}$ with values in a $P/I^k$-bimodule.

December 3, 1987

We consider the map
$$(I \overset{L}{\otimes}_P)^k \longrightarrow (P \overset{L}{\otimes}_P)^k$$
induced by the inclusion of $I$ in $P$. This map is equivariant for the $\mathbb{Z}/k\mathbb{Z}$ action, and so there is an induced action on its mapping cone sequence. In the case where $P$ is free one has a quis of the above map with
$$I^k \overset{L}{\otimes}_P \longrightarrow P \overset{L}{\otimes}_P$$
so the mapping cone is $(P/I^k)\overset{L}{\otimes}_P$. It should be true that $\mathbb{Z}/k\mathbb{Z}$ acts trivially on $(P\overset{L}{\otimes}_P)^k$ in the derived category, since up to isomorphism it is independent of $k$ (this is true whether or not $P$ is free). We have the long exact sequence in homology

$$0 \to H_1(P, I^k) \to H_1(P,P) \to H_1(P, P/I^k) \to I^k/[P,I^k] \to \bar{P}/[P,P] \to \bar{HC}_0(P/I^k) \to 0.$$

Thus we conclude that $\sigma$ acts trivially on $H_1(P, I^k)$. Also we know that the surjection $I^k/[P,I^k] \twoheadrightarrow I^k/[I,I^{k-1}]$ can be viewed as the <u>quotient</u> by the $\mathbb{Z}/k\mathbb{Z}$ action. So applying the exact functor of taking invariants (or coinvariants which is the same) we get an exact sequence

$$\boxed{0 \to H_1(P, I^k) \to H_1(P,P) \to H_1(P, P/I^k)_\sigma \to I^k/[I,I^{k-1}] \to \bar{P}/[P,P] \to \bar{HC}_0(P/I^k) \to 0}$$

~~Now let's define the cyclic homology groups by the exact sequence~~

$$\cancel{0 \longrightarrow HC_{2k+1}(A) \longrightarrow I^{k+1}/[I,I^k] \longrightarrow \bar{P}/[P,P]}$$

Now let's piece together these sequences using the isomorphism $\bar{d} : \bar{P}/[P,P] \xrightarrow{\sim} H_1(P,P)$ induced by $d : P \longrightarrow \Omega^1_P$. One checks that $\bar{d}$ carries $I^{k+1}$ into $H_1(P, I^k)$, and hence we have a ~~map~~ diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & I^{k+1}/[I,I^k] \arrow[r] \arrow[d] & \bar{P}/[P,P] \arrow[r] \arrow[d, "\cong"] & \bar{HC}_0(P/I^{k+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1(P,I^k) \arrow[r] & H_1(P,P) \arrow[r] \arrow[d] & H_1(P,P/I^k)_\sigma \arrow[r] & I^k/[I,I^{k-1}] \\
 & & 0 & &
\end{tikzcd}$$

This shows that

$$\mathrm{Coker}\,\{ I^{k+1}/[I,I^k] \longrightarrow H_1(P,I^k) \}$$

$$= \mathrm{Ker}\,\{ \bar{HC}_0(P/I^{k+1}) \longrightarrow H_1(P, P/I^k)_\sigma \}$$

However we also have

$$\mathrm{Coker}\,\{ \bar{HC}_0(P/I^{k+1}) \longrightarrow H_1(P, P/I^k)_\sigma \}$$

$$= \mathrm{Coker}\,\{ H_1(P,P) \longrightarrow H_1(P, P/I^k)_\sigma \}$$

$$= \mathrm{Ker}\,\{ I^k/[I,I^{k-1}] \longrightarrow \bar{P}/[P,P] \}$$

$$= \mathrm{Ker}\,\{ I^k/[I,I^{k-1}] \longrightarrow H_1(P, I^{k-1}) \}$$

This shows the consistency of defining the cyclic groups by the exact sequences

$$0 \to \overline{HC}_{2k}(A) \to \overline{HC}_0(P/I^{k+1}) \to H_1(P, P/I^k)_\sigma \to \overline{HC}_{2k-1}(A) \to 0$$

$$0 \to \overline{HC}_{2k+1}(A) \to I^{k+1}/[I, I^k] \to H_1(P, I^k) \to \overline{HC}_{2k}(A) \to 0$$

Next to investigate are

1) the link with Hochschild homology (see 394-396).

2) Find another spectral sequence which has the edge homomorphism $HC_{2k}(A) \to HC_0(P/I^{k+1})$

Idea: the $E^1$ term should involve

$$\mathrm{Cone}\left((I \overset{L}{\otimes}_P)^k \to (P \overset{L}{\otimes}_P)^k\right)_\sigma$$

instead of $(I \overset{L}{\otimes}_P)^k_\sigma [k-1]$.

3) Suppose we take $P = T(\bar{A})$. Then we have

$$0 \to I/I^2 \to A \otimes_P \Omega^1_P \otimes_P A \to \Omega^1_A \to 0$$
$$A \otimes_P \Omega^1_P \otimes_P A = A \otimes \bar{A} \otimes A$$

so we have an exact sequence

$$0 \to I/I^2 \to A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \to A \to 0.$$

From this we conclude using $s(x \otimes y) = x \otimes \bar{y} \otimes 1$ that

$$I/I^2 = A \otimes \bar{A}^{\otimes 2}$$

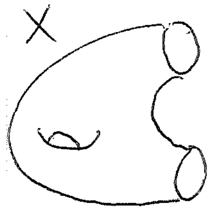
as a left $A$-module. Is $I/I^2 \simeq \Omega^2_A$ as an $A$-bimodule? If so there is a class in $H^2(A, \Omega^2_A)$ represented by the extension

$$0 \to I/I^2 \to P/I^2 \to A \to 0.$$

This fits with there being a class in $H^1(A, \Omega^1_A)$ represented by the derivation $d$.

# December 4, 1987

X



$X$ $\quad$ $g$ handles (holes)
$\qquad$ $r$ boundary circles

$$\partial X = \coprod_{i=1}^{r} S_i$$

Topology: $\hat{X} = X \cup_{\partial X} \coprod_{1}^{r} D_i$. $\qquad H^*(X) = H^*_{DR}(X) = H^*(X,\mathbb{C})$

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & H^0(\hat X, \amalg D_i) \arrow[r] \arrow[d, phantom, "\wr"] & H^0(\hat X) \arrow[r] \arrow[d, equal] & H^0(\amalg D_i) \arrow[r] \arrow[d, "\wr"] & H^1(\hat X, \amalg D_i) \arrow[r] \arrow[d, "\wr"] & H^1(\hat X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H^2(\hat X, \amalg D_i) \arrow[r, "\sim"] \arrow[d, "\wr"] & H^2(\hat X) \arrow[r] \arrow[d, phantom, "/\!/ \mathbb{C}"] & 0 \\
0 \arrow[r] & H^0(X,\partial X) \arrow[r] & H^0(X) \arrow[r] \arrow[d, equal] & H^0(\partial X) \arrow[r] \arrow[d, equal] & H^1(X,\partial X) \arrow[r] & H^1(X) \arrow[r] & H^1(\partial X) \arrow[r] \arrow[d, equal] & H^2(X,\partial X) \arrow[r] \arrow[d, equal] & \cancel{H^2(X)} \arrow[r] & 0 \\
& & \mathbb{C} & \mathbb{C}^r & & & \mathbb{C}^r & \mathbb{C} & &
\end{tikzcd}$$

$$0 \to H^1(\hat X) \to H^1(X) \to \underset{\mathbb{C}^r}{\underset{\|}{H^1(\partial X)}} \to \underset{\mathbb{C}^r}{\underset{\|}{H^2(X,\partial X)}} \to 0$$

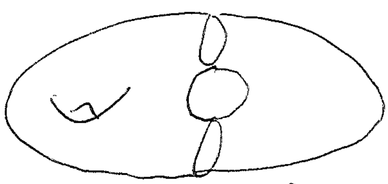
$$\therefore \quad \dim H^1(X) = 2g + r - 1.$$

$$\dim H^1(X,\partial X) = 2g + 2(r-1)$$

Have pairing $\quad \underline{H^1(X,\partial X)} \otimes H^1(X) \longrightarrow H^2(X,\partial X) = \mathbb{C}$.

Question: Is there a symplectic pairing on $H^1(X,\partial X)$?

Instead of filling in each $S_i$ by disks why not make a surface of genus $g+r-1$. Thus form



$$\tilde X = X \cup_{\partial X} \overbrace{\left(S^2 - \amalg e_i\right)}^{Y}$$

$$\begin{tikzcd}[column sep=small]
& & & \mathbb{C} \arrow[d, equal] & & & \mathbb{C}^{r-1} \arrow[d, equal] & & & \\
0 \arrow[r] & H^0(\tilde X, Y) \arrow[r] \arrow[d, equal] & H^0(\tilde X) \arrow[r, "\sim"] \arrow[d, "\wr"] & H^0(Y) \arrow[r, "0"] \arrow[d] & H^1(\tilde X, Y) \arrow[r] \arrow[d, equal] & H^1(\tilde X) \arrow[r] \arrow[d] & H^1(Y) \arrow[r, "0"] \arrow[d] & H^2(\tilde X, Y) \arrow[r, "\sim"] \arrow[d, phantom, "\wr"] & H^2(\tilde X) \arrow[r] & 0 \\
0 \arrow[r] & H^0(X,\partial X) \arrow[r] & H^0(X) \arrow[r] & H^0(\partial X) \arrow[r] \arrow[d, equal] & H^1(X,\partial X) \arrow[r] & H^1(X) \arrow[r] & H^1(\partial X) \arrow[r] \arrow[d, equal] & H^2(X,\partial X) \arrow[r] \arrow[d, equal] & 0 & \\
& & & \mathbb{C}^r & & & \mathbb{C}^r & \mathbb{C} & &
\end{tikzcd}$$

But this doesn't show $H^1(X,\partial X) = H^1(\tilde X)$, only that there's a map $\quad H^1(X,\partial X) \longrightarrow H^1(\tilde X) \quad$ with kernel + cokernel of dimension r-1. [You should probably think in terms of weights + mixed Hodge structures, and then perhaps it would be easy to see that there is no canonical ~~~~ symplectic ~~~~ product on $H^1(X, \partial X)$, certainly no canonical isomorphism of $H^1(\tilde{X})$ with $H^1(X, \partial X)$.]

Harmonic Functions and Forms.

$\mathcal{H}$ = smooth fns $f$ on $X$, harmonic in interior: $(d * d)f = 0$

$\mathcal{H}^1$ = smooth 1 forms on $X$ harmonic in interior:

$$d\omega = 0 \qquad d(*\omega) = 0.$$

$$\mathcal{H}^1 = \underset{\text{holom}}{\Omega^1} \oplus \bar{\Omega}^1$$

Dirichlet problem:

1) Given $\alpha$ smooth 2 form on $X$ there is a unique solution of $(d*d)u = \alpha$ with $u|\partial X = 0$.

2) Given $f \in C^\infty(\partial X)$ there is a unique $u \in \mathcal{H}$ with $u|\partial X = f$.

Consequences

$$0 \to \mathbb{C} \to \mathcal{H} \xrightarrow{d} \mathcal{H}^1 \to H^1(X) \to 0$$

If $\mathcal{H}^1_0 = \{\omega \in \mathcal{H}^1 \mid \omega|\partial X = 0\}$, then $\boxed{\mathcal{H}^1_0 \xrightarrow{\sim} H^1(X, \partial X)}$

Set $N = d\mathcal{H} \cap \mathcal{H}^1_0 = \left\{ df \,\middle|\, \begin{matrix} f \text{ harmonic} \\ f| \partial X \text{ loc. const.} \end{matrix} \right\}$

$$\simeq H^0(X)/\mathbb{C}.$$

Then $N^0 = \{\omega \in \mathcal{H}^1 \mid \int_{S_i} \omega = 0 \text{ all } i\}. = d\mathcal{H} + \mathcal{H}^1_0$

and $N^0/N = \underbrace{d\mathcal{H}/N}_{C^\infty(\partial X)/H^0(\partial X)} \oplus \mathcal{H}^1_0/N$

$$\mathcal{H}^1_0/N = \operatorname{Ker}\{H^1(X) \to H^1(\partial X)\}$$

December 5, 1987

Consider $A = P/I$. We have exact sequences of $A$-bimodules

$$0 \to I/I^2 \to A \otimes_P \Omega^1_P \otimes_P A \to \Omega^1_A \to 0$$

$$0 \to \Omega^1_A \to A \otimes A \to A \to 0$$

whence a length 3 resolution

$$* \quad \to 0 \to 0 \to I/I^2 \to A \otimes_P \Omega^1_P \otimes_P A \to A \otimes A$$

of the bimodule $A$. If $P$ is free, which we suppose, then $\Omega^1_P$ is a free $P \otimes P^{op}$ module, so $A \otimes_P \Omega^1_P \otimes_P A$ is a free $A \otimes A^o$-module. It follows that $I/I^2$ is projective as a left and also right $A$-module.

Let $K$ denote the complex $*$ and form

$$K \otimes_A K \otimes_A \cdots \otimes_A K \qquad k\text{-times}$$

Since $K$ is complex of projective left (or right) $A$-modules, this is quis to $K \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A K$. ~~In effect the functor~~

~~$R \overset{L}{\otimes}_A K \longrightarrow R \otimes_A K$~~

~~is a quasi-isomorphism for any $A$-bimodule complex $R$, so we have a quis~~

~~$\underbrace{K \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A K}_{k \text{ times}} \longrightarrow (\underbrace{K \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A K}_{(k-1) \text{ times}}) \otimes_A K$~~

~~$R \longmapsto R \otimes_A K$~~

~~from complexes of $A$-bimodules to itself is~~

In effect the functor

$$R \longmapsto R \otimes_A K$$

from chain complexes of $A$-bimodules to itself is exact, and it preserves quasi-isomorphisms. (This follows because it is compatible with chain homotopies, which allows one to replace any quis by its mapping cone. This is acyclic, hence admits a filtration by contractible complexes.)

Thus we have quis's

$$\underbrace{K \overset{L}{\otimes}_A K \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A K}_{k\text{-times}} \longrightarrow \underbrace{\left(K \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A K\right)}_{(k-1)\text{ times}} \otimes_A K \to \underbrace{(K \otimes_A \cdots \otimes_A K)}_{(k-1)\text{ times}} \otimes_A K$$

where we use induction.

Thus we can conclude that $K \otimes_A \cdots \otimes_A K$ is a resolution of the A-bimodule A. Next we claim that this complex consists of projective $A \otimes A^{op}$-modules except in the top degree $2k$ where it is $I/I^2 \otimes_A \cdots \otimes_A I/I^2$. In effect consider an A-bimodule of the form $X \otimes Y$, where $X$ is a projective left A-module and $Y$ is a projective right A-module. Then if $N$ is a bimodule which is left A-projective we have the tensor product

$$N \otimes_A (X \otimes Y) = \underbrace{(N \otimes_A X)}_{\text{also left } A\text{-proj}} \otimes Y$$

is also in the same form.

Thus we see that $K \otimes_A \cdots \otimes_A K$ $k$-times is a resolution of the bimodule $A$

$$0 \longrightarrow \underbrace{I/I^2 \otimes_A \cdots \otimes_A I/I^2}_{k} \longrightarrow P_{2k-1} \longrightarrow \cdots\cdots \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

by projective $A \otimes A^{op}$-modules. It can be used to compute the Hochschild homology of $A$, and yields

$$\boxed{\begin{gathered} 0 \to H_{2k}(A,A) \longrightarrow (I/I^2 \otimes_A)^k \longrightarrow \bigoplus_{i=1}^{k} (I/I^2 \otimes_A)^{i-1} (A \otimes_P \Omega^1_P \otimes_P A) \otimes_A (I/I^2 \otimes_A)^{k-i} \\ \text{or} \qquad H_{2k}(A,A) = H_{2k}\left((K \otimes_A)^k\right) \end{gathered}}$$

The next point to consider is the action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$. [redacted]

<u>Lemma</u>: The [redacted] action of $\mathbb{Z}/k\mathbb{Z}$ on the complex $(A \overset{L}{\otimes}_A)^k$ is trivial mod homotopy (i.e. $\sigma \sim id$).

<u>Proof</u>: Let's realize $(A \overset{L}{\otimes}_A)^k$ as $(R \otimes_A)^k$ where $R$ is a projective $A \otimes A^{op}$-module resolution of $A$. Take $k=2$ to see what's going on. We then have two maps

$$\begin{tikzcd} \text{[redacted]}\; R \otimes_A R \otimes_A \arrow[r] \arrow[dr] & R \otimes_A A \otimes_A = R \otimes_A \arrow[d, phantom, "\shortparallel"] \\ & A \otimes_A R \otimes_A \end{tikzcd}$$

which are related by the cyclic permutation $\sigma$ on $(R \otimes_A)^2$. Both these maps are quis's, so it suffices to show they are homotopic. But they are obtained by applying the functor $? \otimes_A$ to the two maps

$$\begin{tikzcd} & \text{[redacted]}\; R \otimes_A A \arrow[dr, phantom, "="] & \\ R \otimes_A R \arrow[ur] \arrow[dr] & & R \\ & A \otimes_A R \arrow[ur, phantom, "="] & \end{tikzcd}.$$

As $R \otimes_A R$ is also a resolution of $A$ by proj. $A \otimes A^o$-modules, these two maps (which lie[redacted] over $id_A$) have to be homotopic.

The general case is similar. One has $k$ different maps $\underbrace{R \otimes_A \cdots \otimes_A R}_{k} \longrightarrow R$ of projective $A \otimes A^o$-module resolutions of $A$, so these maps are homotopic and homotopy equivalences. These induce homotopic homotopy-equivalences $(R \otimes_A)^k \to R \otimes_A$, and as these are permuted by $\sigma$, it follows $\sigma \sim id$. QED

Now let's look at the formula for $H_{2k}(A,A)$ on p404. We have

$$H_q(A,A) = H_q\left(\left(A \overset{L}{\otimes}_A\right)^k\right) \xrightarrow[\text{for } q \le 2k]{\sim} H_q\left(\left(K \otimes_A\right)^k\right)$$

and so it follows from the lemma that $\sigma$ acts trivially on the homology of the complex $(K\otimes_A)^k$. In particular we get an exact sequence by taking invariants

$$\boxed{0 \longrightarrow H_{2k}(A) \longrightarrow \left(\left(I/I^2\otimes_A\right)^k\right)^\sigma \longrightarrow A \otimes_P \Omega^1_P \otimes_P \left(I/I^2 \otimes_A\right)^{k-1}.}$$

This can be used with the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & \bar{HC}_{2k+1}(A) \arrow[r] \arrow[d] & I^{k+1}/[I,I^k] \arrow[r] \arrow[d] & H_1(P,I^k) \arrow[r] \arrow[d] & \bar{HC}_{2k}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bar{HC}_{2k-1}(A) \arrow[r] \arrow[d] & I^k/[I,I^{k-1}] \arrow[r] \arrow[d] & H_1(P,I^{k-1}) \arrow[r] \arrow[d] & \bar{HC}_{2k-2}(A) \arrow[r] & 0 \\
0 \arrow[r] & H_{2k}(A) \arrow[r] \arrow[uurrr, bend right=70] & \left(I/I^2\otimes_A\right)^k_\sigma \arrow[r] \arrow[d] & H_1(P,I^{k-1}/I^k) & & \\
 & & 0 & & &
\end{tikzcd}$$

to derive the Connes exact sequence around $H_{ev}(A)$

---

Discussion. The exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A \otimes_P \Omega^1_P \otimes_P A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

represents an element of $\mathrm{Ext}^2_{A\otimes A^o}(A, I/I^2) = H^2(A, I/I^2)$

which presumably coincides with the class of the square zero extension

$$0 \longrightarrow I/I^2 \longrightarrow P/I^2 \longrightarrow A \longrightarrow 0$$

Actually one has the exact sequence of A-bimodules

$$0 \to \Omega^1_A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

which gives a canonical class $\chi \in \mathrm{Ext}^1_{A\otimes A^\circ}(A, \Omega^1_A) = H^1(A, \Omega^1_A)$. Cupping with $\chi$ in the sense of Yoneda defines a map

$$(*) \qquad \mathrm{Ext}^n_{A\otimes A^\circ}(\Omega^1_A, M) \longrightarrow H^{n+1}(A, M)$$

which is isomorphism for $n \geq 1$. Given a square zero extension $0 \to M \to Q \to A \to 0$, we then get an exact sequence

$$0 \to M \longrightarrow A \otimes_Q \Omega^1_Q \otimes_Q A \longrightarrow \Omega^1_A \longrightarrow 0$$

and the class of the exact sequence should correspond to the class of $Q$ under $(*)$ for $n=1$.

---

An important idea perhaps is that when dealing with the derived category of A-bimodules, one has in addition to the usual Yoneda theory also the tensor product operation $M \otimes_A N$. Since $A$ is a unit for this operation in the sense that $X \overset{L}{\otimes}_A A \simeq X$ and $A \overset{L}{\otimes}_A X \xrightarrow{\sim} X$ for any complex $X$, it follows that morphisms in the derived category from $A$ to itself (possibly shifted in degree) act on objects in the derived category.

Also have operations $R\mathrm{Hom}_A(X, Y)$, $R\mathrm{Hom}_{A^\circ}(X, Y)$ corresponding to A-bimodules $\mathrm{Hom}_A(M, N)$, $\mathrm{Hom}_{A^\circ}(M, N)$.

[redacted] I want to discuss the product structure on $H^*(A,A)$. First of all there is the Yoneda or composition product. Secondly there is a cup product

$$(1) \qquad H^*(A,X) \otimes H^*(A,Y) \longrightarrow H^*(A, X \overset{L}{\otimes}_A Y)$$

defined as follows. Given classes in $H^p(A,X)$, $H^q(A,Y)$ resp. represented by maps $u: A \to X[p]$, $v: A \to Y[q]$ in the derived category of $A$-bimodules, one obtains a map

$$A \simeq A \overset{L}{\otimes}_A A \xrightarrow{u \overset{L}{\otimes}_A v} \left(X \overset{L}{\otimes}_A Y\right)[p+q]$$

representing the cup product.

I would like to show that these two products coincide on $H^*(A,A)$. [redacted] To this end let us link (1) to the composition product. Thus consider

$$\begin{tikzcd}
A \arrow[d, "v"'] & A \overset{L}{\otimes}_A A \arrow[l, "\sim"'] \arrow[d, "1 \overset{L}{\otimes}_A v"] \\
Y[q] & A \overset{L}{\otimes}_A Y[q] \arrow[l, "\sim"'] \arrow[d, "u \overset{L}{\otimes}_A 1"] \\
 & X[p] \overset{L}{\otimes}_A Y[q]
\end{tikzcd}$$

This expresses $u \overset{L}{\otimes}_A v$ as the composition:

$$u \overset{L}{\otimes}_A v = \left(u \overset{L}{\otimes}_A 1\right)\left(1 \overset{L}{\otimes}_A v\right)$$

and when we take $X = Y = A$, it identifies the cup product $u \overset{L}{\otimes}_A v$ with the composition product $u \circ v$.

This seems to imply that $H^*(A,A)$ is commutative

The argument is quite formal: One has $A \simeq A \overset{L}{\otimes}_A A$ with $u \longleftrightarrow 1\otimes u$ or $u \otimes 1$ under this isomorphism. The rest is because $F(X,Y) = X \overset{L}{\otimes}_A Y$ is a bifunctor, so that

$$F(u,v) = F(u,1)F(1,v) = F(1,v)F(u,1).$$

It would be nice to check the conclusion that $H^*(A,A)$ is commutative, $H^0(A,A) =$ center of $A$ is a commutative ring. We should next consider $H^1(A,A) = \mathrm{Der}(A)/\text{inner derivations}$. The cup products

$$H^1(A,A) \times H^1(A,A) \longrightarrow H^2(A,A)$$

should give an interesting way to associate a first order deformation of $A$ to a pair of derivations.

Let's try to work out the cup product on the level of Hochschild cochains. A Hochschild $p$-cocycle is a map of $A$-bimodule complexes

$$\mathcal{B}(A) \xrightarrow{u} A[p]$$

Given also a $q$-cocycle $\mathcal{B}(A) \xrightarrow{v} A[q]$, we obtain the composition product by

$$\begin{tikzcd}
\mathcal{B}(A) \arrow[d, "v"] & \mathcal{B}(A)\otimes_A \mathcal{B}(A) \arrow[l, "\varepsilon\otimes 1"'] \arrow[d, "1\otimes v"] \\
A[q] & \mathcal{B}(A)\otimes_A A[q] \arrow[l, "\varepsilon\otimes 1"'] \arrow[d, "u\otimes 1"] \\
 & A[p]\otimes_A A[q]
\end{tikzcd}$$

and to get an explicit cocycle we need a map $\mathcal{B}(A) \xrightarrow{\mu} \mathcal{B}(A)\otimes_A \mathcal{B}(A)$ of resolutions of $A$. Similarly the cup product $u \otimes v$ requires such a $\mu$.

$\mu$ in degree $n$ is a collection of maps

$$A \otimes A^{\otimes n} \otimes A \longrightarrow A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes(n-i)} \otimes A$$

for $i = 0, \dots, n$. It's probably faster to try defining the cup product of cocycles directly. We have

$$C^p(A, M) = \operatorname{Hom}_{A\otimes A^o}(A \otimes A^{\otimes p} \otimes A, M) = \operatorname{Hom}_{\mathbb{C}}(A^{\otimes p}, M)$$

Given $\varphi \in C^p(A, M)$ and $\psi \in C^q(A, N)$, let's try defining $\varphi \cup \psi \in C^{p+q}(A, N)$ by

$$(\varphi \cup \psi)(a_1, \dots, a_{p+q}) = \varphi(a_1, \dots, a_p)\psi(a_{p+1}, \dots, a_{p+q})$$

We have with $\tilde{\varphi} : A \otimes A^{\otimes p} \otimes A \to M$ the bimodule map extension of $\varphi$

$$(b\varphi)(a_0, \dots, a_p) = \tilde{\varphi}\, b'(1 \otimes a_0 \otimes \dots \otimes a_p \otimes 1)$$

$$= \tilde{\varphi}(a_0, \dots, a_p, 1) + \sum_{i=1}^{p} (-1)^i \tilde{\varphi}(1, a_0, \dots, a_{i-1}a_i, \dots, a_p)$$

$$+ (-1)^{p+1} \tilde{\varphi}(1, a_0, \dots, a_{p-1})a_p$$

$$= a_0\varphi(a_1, \dots, a_p) + \sum_{i=1}^{p} (-1)^i \varphi(a_0, a_{i-1}a_i, \dots, a_p)$$

$$+ (-1)^{p+1} \varphi(a_0, \dots, a_{p-1})a_p.$$

Then
$$b(\varphi \cup \psi)(a_0, \dots, a_{p+q}) = a_0\varphi(a_1, \dots, a_p)\psi(a_{p+1}, \dots, a_{p+q})$$

$$+ \sum_{i=1}^{p} (-1)^i \varphi(a_0, \dots, a_{i-1}a_i, \dots, a_p)\psi(a_{p+1}, \dots, a_{p+q})$$

Insert

$$+ \sum_{j=1}^{q} (-1)^{p+j} \varphi(a_0, \dots, a_{p-1})\psi(a_p, \dots, a_{p+j-1}a_{p+j}, \dots, a_{p+q})$$

$$+ (-1)^{p+q+1} \varphi(a_0, \dots, a_{p-1})\psi(a_p, \dots, a_{p+q-1})a_{p+q}$$

$$\begin{cases} (-1)^{p+1} \varphi(a_0, \dots, a_{p-1})a_p\, \psi(a_{p+1}, \dots, a_{p+q}) \\ (-1)^{p} \varphi(a_0, \dots, a_{p-1})\, a_p\, \psi(a_{p+1}, \dots, a_{p+q}) \end{cases}$$

Thus we get

$$\boxed{b(\varphi \cup \psi) = (b\varphi)\cup\psi + (-1)^p \varphi \cup b\psi}$$

which means we have a pairing of cxs.

$$C^{\cdot}(A,M) \otimes C^{\cdot}(A,N) \longrightarrow C(A, M\otimes_A N)$$

Now let's work out some examples.

$p=q=0$. Then $\varphi \in M$, $\psi \in N$ and $\varphi \cup \psi = \varphi\otimes\psi \in M\otimes_A N$. We have

$$\begin{aligned} b(\varphi\cup\psi)(a) &= a(\varphi\otimes\psi) - (\varphi\otimes\psi)a \\ &= a\varphi\otimes\psi - \varphi\otimes\psi a \\ &\quad -\varphi a\otimes\psi + \varphi\otimes a\psi \\ &= [a,\varphi]\otimes\psi + \varphi\otimes[a,\psi] \\ &= (b\varphi\cup\psi + \varphi\cup b\psi)(a) \end{aligned}$$

$p=0, q=1$.

$$\begin{aligned} (b(\varphi\cup\psi))(a_0,a_1) &= a_0(\varphi\cup\psi)(a_1) - (\varphi\cup\psi)(a_0a_1) + (\varphi\cup\psi)(a_0)a_1 \\ &= a_0\varphi\otimes\psi(a_1) - \varphi\otimes\psi(a_0a_1) + \varphi\otimes\psi(a_0)a_1 \\ &\quad -\varphi a_0\otimes\psi(a_1) + \varphi\otimes a_0\psi(a_1) \\ &= (b\varphi\cup\psi)(a_0,a_1) + (\varphi\cup b\psi)(a_0,a_1) \end{aligned}$$

In particular we see that the map

$$H^0(A,A)\otimes H^1(A,A) \longrightarrow H^1(A,A)$$
$$z\ ,\quad D \longmapsto zD$$

(Check. $z[x,?] = [zx,?]$ so $D\sim 0 \Rightarrow zD\sim 0$)

$p=q=1$. $(\varphi\cup\psi)(a_1,a_2) = \varphi(a_1)\otimes\psi(a_2)$

$$\begin{aligned} (b(\varphi\cup\psi))(a_0,a_1,a_2) &= a_0\varphi(a_1)\otimes\psi(a_2) - \varphi(a_0a_1)\otimes\psi(a_2) + \varphi(a_0)a_1\otimes\psi(a_2) \\ &\quad + \varphi(a_0)\otimes\psi(a_1a_2) - \varphi(a_0)\otimes\psi(a_1)a_2 - \varphi(a_0)\otimes a_1\psi(a_2) \end{aligned}$$

$$= (b\varphi \cup \psi - \varphi \cup b\psi)(a_1, a_2)$$

Thus the cup product of 2 derivations is a 2-cocycle. We want to check the commutativity i.e. that $\varphi \cup \psi + \psi \cup \varphi$ is a 2-coboundary. It suffices to show $\varphi \cup \varphi$ is a 2-coboundary, and the only 1-cochains around are $a \mapsto \varphi(a)^2$ and possibly other polynomials. No, there are lots of possibilities, such as $a\varphi(a)$, and $\varphi(\varphi(a))$.

In fact $a \mapsto \varphi(\varphi(a))$ is what we want. To see this think of $\varphi$ as a vector field and write $Xa$. Then it's natural to look at the "Laplacean" $X^2$. We have if $f(a) = X^2 a$, that

$$\begin{aligned} f(a_1 a_2) &= X(Xa_1 \cdot a_2 + a_1 \cdot Xa_2) \\ &= (X^2 a_1)\, a_2 + 2(Xa_1)(Xa_2) + a_1 (X^2 a_2). \\ &= f(a_1) a_2 + 2\varphi(a_1)\varphi(a_2) + a_1 f(a_2) \end{aligned}$$

so $\varphi \cup \varphi = -\frac{1}{2} bf$.

We can describe the cup product

$$H^1(A,A) \otimes H^1(A,A) \longrightarrow H^2(A,A)$$

as assigning to a pair of derivations $X, Y$ of $A$ the deformation with multiplication

$$a_1 * a_2 = a_1 a_2 + h(Xa_1)(Ya_2)$$

Notice that if $X$ is inner $Xa = [c, a]$, then

$$s(a) = a + hcYa$$

is a homomorphism:

$$s(a_1 a_2) = a_1 a_2 + hc[(Ya_1)a_2 + a_1 Ya_2]$$

$$s(a_1) * s(a_2) = a_1 a_2 + h[\underbrace{Xa_1}_{ca_1 - a_1 c} Ya_2 + a_1 c Ya_2 + c(Ya_1) a_2]$$

Also if we use the section
$s(a) = a + h\,XYa$ of the trivial defm.
of $A$: $a_1 * a_2 = a_1 a_2$, we obtain the cocycle

$$s(a_1)s(a_2) - s(a_1a_2) = a_1a_2 + h[a_1 XYa_2 + (XYa_1)a_2]$$
$$- a_1a_2 - h[(XY)(a_1a_2)]$$
$$= -h(Xa_1\,Ya_2 + Ya_1\,Xa_2)$$

Thus the cocycle $Xa_1\,Ya_2 + Ya_1\,Xa_2$ is a 1-coboundary, and it means the the cocycle $Xa_1\,Ya_2$ is cohomologous to the Poisson bracket cocycle
$$\frac{1}{2}(Xa_1\,Ya_2 - Ya_1\,Xa_2).$$