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Let's review the problem. I've noticed from my work on the operators ~~over~~ the circle that interesting characteristic numbers for elements of $K_0(A)$ can be obtained from a trace on an extension of A by a nilpotent ideal. Thus if the extension is

$$0 \rightarrow I \longrightarrow \tilde{A} \longrightarrow A \longrightarrow 0$$

we have $\quad K_0\tilde{A} \xrightarrow{\sim} K_0 A \quad$ and so a trace on $\tilde{A}$ induces a linear functional on K_0A.

Now I believe that Goodwillie has shown that the localized cyclic cohomology for $\tilde{A}$ and A are the same. Thus there should be a mechanism whereby a trace on $\tilde{A}$ when suspended enough times becomes equivalent to a cyclic cocycle on A.

Let's consider the case where $I^2 = 0$. Suppose $\tilde{A}, A$ unital. Let's start with an involution F_0 over A. Replacing $\tilde{A}$ by $M_N(\tilde{A})$ is necessary, we can suppose F_0 lies in A. Choose a linear cross-section $s: A \longrightarrow \tilde{A}$ and let $\varphi: A \otimes A \longrightarrow I$ be the associated Hochschild 2-cocycle:

$$s(a)s(b) = s(ab) + \varphi(a,b)$$

$$a\varphi(b,c) - \varphi(ab,c) + \varphi(a,bc) - \varphi(a,b)c = 0$$

We lift F_0 to an involution $\quad F = s(F_0) + \alpha$ with $\alpha \in I$. Then

$$1 = s(F_0)^2 + F_0\alpha + \alpha F_0 = s(F_0^2) + \varphi(F_0,F_0) + F_0\alpha + \alpha F_0$$

Since $s(\underline{1}) = 1$ can be supposed, we need α to satisfy

$$\varphi(F_0, F_0) + F_0\alpha + \alpha F_0 = 0$$

But the cocycle condition implies

$$F_0\,\varphi(F_0, F_0) - \cancel{\varphi(1, F_0)} + \cancel{\varphi(F_0, 1)} - \varphi(F_0, F_0)F_0 = 0$$

(these are zero as $s(1) = 1$)

and so we have

$$\alpha = -\frac{1}{2}F_0\,\varphi(F_0, F_0) + \text{something anti-commuting with } F_0.$$

Thus given a trace τ on $\tilde{A}$, the effect of τ on $[F_0] \in K_0(A)$ is

$$\boxed{\tau(F) = \tau(s(F_0)) - \frac{1}{2}\tau(F_0\,\varphi(F_0, F_0))}$$

Look at the last term, and generalize to a cochain $\tau(a\,\varphi(b,c))$. We have

$$\psi(a^0, a^1, a^2) = \tau(a^0\varphi(a^1, a^2))$$

[redacted]

$$\left.\begin{aligned}&\tau(a^0a^1\,\varphi(a^2, a^3))\\ &-\tau(a^0\,\varphi(a^1a^2, a^3))\\ &+\tau(a^0\,\varphi(a^1, a^2a^3))\\ &-\tau(a^3a^0\,\varphi(a^1, a^2))\end{aligned}\right\} = (\delta\psi)(a^0, a^1, a^2, a^3)$$

$$\|$$

$$\tau(a^0(\delta\varphi)(a^1, a^2, a^3))$$

so ψ should be a Hochschild 2-cocycle with values in A^*.

Consider $\Theta(a^0, a^1, a^2) = \psi(a^2, a^0, a^1) = \tau(a^2\varphi(a^0, a^1))$

Then $\theta(a,b,c) = \tau(\varphi(a,b)c)$

$$\begin{aligned}(\delta\theta)(a^0,\ldots,a^3) = {} & \tau(\varphi(a^0a^1,a^2)a^3) \\ & -\tau(\varphi(a^0,a^1a^2)a^3) \\ & +\tau(\varphi(a^0,a^1)a^2a^3) \\ & -\tau(\varphi(a^3a^0,a^1)a^2) \\ = {} & \tau(a^0\varphi(a^1,a^2)a^3) - \tau(\varphi(a^3a^0,a^1)a^2) \qquad ?\end{aligned}$$

It might help to understand what a a trace on $\tilde{A}$ looks like. We have

$$\Lambda^2\tilde{A} \xrightarrow{[\,,\,]} \tilde{A} \longrightarrow \tilde{A}/[\tilde{A},\tilde{A}] \longrightarrow 0$$

and $\quad gr\,\Lambda^2\tilde{A} = \Lambda^2 I \oplus A\otimes I \oplus \Lambda^2 A$

Thus we have

$$\begin{tikzcd} 0 \arrow[r] & \tilde{A}\otimes I/S^2I \arrow[r] \arrow[d] & \Lambda^2\tilde{A} \arrow[r] \arrow[d] & \Lambda^2 A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I \arrow[r] & \tilde{A} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

which by serpent should give an exact sequence

$$\begin{tikzcd} \mathrm{Ker}\{\Lambda^2\tilde{A}\to\tilde{A}\} \arrow[r] \arrow[d] & \mathrm{Ker}\{\Lambda^2A\to A\} \arrow[r] \arrow[d] & I/[A,I] \arrow[r] \arrow[d, equal] & \tilde{A}/[\tilde{A},\tilde{A}] \arrow[r] \arrow[d, equal] & A/[A,A] \arrow[r] \arrow[d, equal] & 0 \\ HC_1(\tilde{A}) \arrow[r] & HC_1(A) \arrow[r] & H_0(A,I) \arrow[r] & HC_0(\tilde{A}) \arrow[r] & HC_0(A) \arrow[r] & 0 \end{tikzcd}$$

Let's consider the example where $\tilde{A} = a/h^2a$ In this case A is abelian and $H_0(A,I) = A$ and the image of $HC_1(A)$ in $H_0(A,I)$ is $\{A,A\} =$

the space spanned by Poisson brackets.

I guess I have to make clear what the goals are. In the example of the extension

$$0 \longrightarrow h\mathcal{A}/h^2\mathcal{A} \longrightarrow \mathcal{A}/h^2\mathcal{A} \longrightarrow \mathcal{A}/h\mathcal{A} \longrightarrow 0$$

we have two infinitesimal methods for getting a non-trivial functional on $K_0(\mathcal{A}/h\mathcal{A})$. The central problem is to relate them.

Natural question: Let us consider all nilpotent extensions of A. Consider the category of algebras B mapping onto A with nilpotent kernels.

$$I \longrightarrow B \twoheadrightarrow A \qquad\qquad I^n = 0$$

and take $\varprojlim B/[B,B]$ over this category.

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[redacted] We've seen that given a nilpotent extension $B \twoheadrightarrow A$ and a trace τ on B we [redacted] get an induced map on $K_0 A$:

$$\begin{tikzcd} K_0 A & K_0 B \arrow[l, "\sim"'] \arrow[r, "\text{[redacted]}"] \arrow[rr, bend left, "\tau_*"] & B/[B,B] \arrow[r] & \mathbb{C} \end{tikzcd}$$

Thus we [redacted] have a natural map

① $$K_0 A \longrightarrow \varprojlim_{B} B/[B,B]$$

where the inverse limit is taken over the category of nilpotent extensions of A.

Similar given any epimorphism $B \twoheadrightarrow A$, let I be the kernel. Then given a trace τ on I^n, we get map on $K_1 A$:

$$\begin{tikzcd} K_1 A \arrow[r, "\partial"] & K_0 I \arrow[dr, dotted] & K_0 I^n \arrow[l, two heads] \arrow[d, "\tau_*"] \\ & & \mathbb{C} \end{tikzcd}$$

hence we have a natural map

② $$K_1 A \longrightarrow \varprojlim_{B,n} (\underbrace{\text{Ker } B \to A}_{I_B})^n / [I_B^n, I_B^n]$$

where B ranges over all extensions of A.

A natural conjecture is whether the above inverse limits are the same as the localized cyclic homology of A in odd + even degrees, resp.

In both cases we can consider the category of epimorphisms $B \twoheadrightarrow A$ and we have a functor F from this category to sets (or vector spaces). In the first case it is

$$F(B) = \varprojlim_{n} B/I_B^n + [B,B]$$

and the second case it is

$$F(B) = \varprojlim_{n} I_B^n/[I_B^n, I_B^n].$$

Lemma: Let F be a covariant functor to sets ~~from~~ from the category of epimorphisms $B \twoheadrightarrow A$ with A fixed, (the category of extensions of A), and let $P \twoheadrightarrow B$ ~~be~~ an extension with P projective (e.g. P a free algebra or tensor algebra). Then

$$\varprojlim_{B \twoheadrightarrow A} F(B) \xrightarrow{\sim} \operatorname{Ker}\{F(P) \rightrightarrows F(P * P)\}$$

Proof: Injectivity: An element of the inverse limit is an assignment $B \mapsto \xi(B) \in F(B)$ compatible with morphisms. If ξ, η are elts of the inverse limit with $\xi(P) = \eta(P)$, then $\xi(B) = \eta(B)$ for any B since there is a map $P \to B$ in the category.

Surjectivity: Suppose given $\alpha \in F(P)$ equalized by the two maps $P \overset{\iota}{\underset{\bar{\iota}}{\rightrightarrows}} P * P$. Then for any pairs of maps $P \overset{u}{\underset{v}{\rightrightarrows}} B$, α is equalized by them, since u, v come from $\iota, \bar{\iota}$ via a map $P * P \to B$. Thus we can define $\xi \in \varprojlim F(B)$ as follows. Given B choose a map $P \xrightarrow{u} B$, and let $\xi(B) =$ image of α. This is well-defined + functorial.

Let I be an ideal in an algebra A. We are interested in

$$\varprojlim_n I^n/[I^n, I^n]$$

as a way to detect elements of $K_1(A/I)$. In Connes work, or better, in analytical situations the typical trace on I^n will vanish on commutators in $[I^p, I^q]$ with $p+q=n$.

Proposition: The inverse systems

$$I^n/[A, I^n] \quad , \quad I^n/[I^n, I^n] \quad , \quad I^n/\sum_{p+q=n} [I^p, I^q]$$

are isomorphic as pro-objects.

The first thing to get straight is the relative size of $[I^p, I^q]$ for $p+q=n$. We have the identity

$$xym - mxy = x(ym)-(ym)x$$
$$+ y(mx)-(mx)y$$
$$[xy, m] = [x, ym] + [y, mx]$$

for x, y in an algebra and m in a bimodule. This implies

$$[I^{p+q}, I^r] \subset [I^p, I^{q+r}] + [I^q, I^{p+r}]$$

from which one sees that

$$[I^p, I^{n-p}] \subset [I, I^{n-1}] + [I^{p-1}, I^{n-p+1}].$$

This shows that $[I, I^{n-1}]$ is the largest among the ideals $[I^p, I^{n-p}]$. On the other hand we have

$$[A, I^n] = [I^{p+(n-p)}, A]$$

$$\subset [I^p, I^{n-p}] + [I^{n-p}, I^p] = [I^p, I^{n-p}]$$

showing that $[A, I^n]$ is the smallest. Thus we have

$$* \quad \boxed{[A, I^n] \subset [I^p, I^{n-p}] \subset [I, I^{n-1}]}$$

So next return to inverse systems. We have

$$\begin{tikzcd}
I^{n+1}/[A, I^{n+1}] \arrow[r, two heads] \arrow[d] & I^n/[A, I^n] \arrow[d] \\
I^{n+1}/[I, I^{n}] \arrow[r, two heads] \arrow[ur, dashed] & I^n/[I, I^{n-1}]
\end{tikzcd}$$

showing that $\{I^n/[A, I^n]\} \xrightarrow{\sim} \{I^n/[I, I^{n-1}]\}$. We also have

$$\begin{tikzcd}
I^{2n}/[I^{2n}, I^{2n}] \arrow[r, two heads] \arrow[d] & I^n/[I^n, I^n] \arrow[d] \\
I^{2n}/[A, I^{2n}] \arrow[r, two heads] \arrow[ur, dashed] & I^n/[A, I^n]
\end{tikzcd}$$

showing $\{I^n/[I^n, I^n]\} \xrightarrow{\sim} \{I^n/[A, I^n]\}$. QED

Remark: It's better to prove $*$ above by thinking of a linear fnl τ on I^n, and when $\tau(xy) = \tau(yx)$.

The next project will be to check the "Goodwillie theorem" for these inverse limits of trace spaces. Thus suppose $B \twoheadrightarrow A$ with nilpotent kernel. Let P be a free algebra mapping onto B and set $I = \mathrm{Ker}\{P \to A\}$ and $J = \mathrm{Ker}\{P \to B\}$, so that $J \subset I$ and $I/J = \mathrm{Ker}\{B \to A\}$ is nilpotent, so $I^k \subset J$ for some k.

$$\begin{tikzcd} P \arrow[r, two heads] & P/J \arrow[r, two heads] \arrow[d, phantom, "\shortparallel"] & P/I \arrow[d, phantom, "\shortparallel"] \\ & B & A \end{tikzcd}$$

Then certainly the inverse systems

$$\{P/J^n\} \xrightarrow{\sim} \{P/I^n\}$$

are isomorphic as proobjects, hence it follows that their commutator quotients are also isomorphic. The inverse limits are just the completions of $P/[P,P]$

$$\varprojlim P/J^n + [P,P] \qquad \varprojlim P/I^n + [P,P]$$

for the I-adic and J-adic topologies, and these are equivalent.

Next we have to consider the two maps $\iota, \bar{\iota} : P \to P * P$. The kernel of the epim. $P * P \longrightarrow B$ is clearly $qP + \iota J$, since this epimorphism factors

$$P * P \xrightarrow{\text{fold}} P \longrightarrow P/J = B$$

Now the inverse limit of trace spaces we are after for B is the inverse limit of the system

$$\mathrm{Ker}\left\{ P/J^n + [P,P] \rightrightarrows P*P/(\iota J + qP)^n + [P*P, P*P] \right\}$$

and it gives the same result as for A.

Similarly we consider

$$\{J^n/[P,J^n]\} \longrightarrow \{I^n/[P,I^n]\}$$

which is again the result of applying a functor $(/[P,?])$ to the systems $\{J^n\}$. $\{I^n\}$ which are isomorphic as pro-objects. I can also look at the corresponding quotients for $P*P$:

$$\left\{ (\iota J + qP)^n / [P*P, (\iota J + qP)^n] \right\}$$

and similarly for I. It seems to be OK.

November 6, 1987

Square zero extensions:

$$0 \to M \to Q \to A \to 0$$

According to Grothendieck these form a cofibred category over the category of A-bimodules which is left exact. In any case let us fix a square zero extension and try to determine the subgroup of $Q/[Q,Q]$, which equalizes all maps from Q to another extension E. It's enough to consider the two canonical maps $Q \rightrightarrows Q \amalg Q$, where $Q \amalg Q$ denotes the direct sum in the category of square zero extensions.

This we can control as follows. In general given two maps from Q to E

$$\begin{tikzcd}
M \arrow[r] & Q \arrow[r] \arrow[d, shift right, "u"'] \arrow[d, shift left, "v"] & A \arrow[d, equal] \\
N \arrow[r] & E \arrow[r] & A
\end{tikzcd}$$

their difference $v-u$ is a map $Q \to N$ which is a derivation of Q with values in N considered as an A-bimodule via the map $Q \to A$. In effect

$$\begin{aligned} v(xy) - u(xy) &= v(x)[v(y)-u(y)] + [v(x)-u(x)]u(y) \\ &= x\cdot(v-u)(y) + (v-u)(x)\cdot y \end{aligned}$$

and once u is given the set of maps v is in 1-1 correspondence with such derivations:

$$[u(x)+\delta(x)][u(y)+\delta(y)] = u(xy) + x\delta(y) + \delta(x)\cdot y$$

$$\therefore \ u+\delta \text{ homom.} \iff \delta(xy) = x\delta(y) + \delta(x)y$$

This tells us that

$$Q \amalg Q = Q \oplus D$$

where D represents derivations of Q with values in A-bimodules:

$$D = A^+ \otimes_{Q^+} \Omega^1_Q \otimes_{Q^+} A^+ = \Omega^1 / M\Omega^1_Q + \Omega^1_Q M$$

So far we haven't used that $M^2 = 0$, except to identify the Q in $Q \amalg Q$.

Next we know that the commutator ~~quotient~~ of a semi-direct product $Q \oplus D$ is

$$Q \oplus D / [Q \oplus D, Q \oplus D] = (Q/[Q,Q]) \oplus (D/[A,D]).$$

Thus the trace space we are after is

$$\operatorname{Ker}\left\{ Q/[Q,Q] \xrightarrow{d} D/[A,D] \right\}$$

$$D/[A,D] = \Omega^1_Q \otimes_{Q \otimes Q^{op}} A$$

Before going on I ought to understand the non-commutative Ω^1 a lot better. First of all there is supposed to be an exact sequence of A-bimodules

$$(*) \qquad 0 \longrightarrow \Omega^1_A \longrightarrow A^+ \otimes A^+ \longrightarrow A^+ \longrightarrow 0$$

Check: A homomorphism $\Omega^1_A \longrightarrow M$ of A-bimodules is the same as a derivation $\delta : A \longrightarrow M$, which is the same as a derivation $\delta : A^+ \longrightarrow M$ such that $\delta(1) = 0$.

Let $K = \operatorname{Ker}\{A^+ \otimes A^+ \longrightarrow A^+\}$. Let $\delta : A \to K$ be $d(x) = x \otimes 1 - 1 \otimes x$. Then

$$\begin{aligned} d(xy) &= xy \otimes 1 - 1 \otimes xy \\ &= x(y \otimes 1 - 1 \otimes y) + (x \otimes 1 - 1 \otimes x)y = x(dy) + (dx)y \end{aligned}$$

so we get an A-bimodule map $\Omega^1_A \longrightarrow K$. Given a derivation $\delta : A \longrightarrow M$ extend it to $\tilde\delta$ on A^+ with $\tilde\delta(1) = 0$, and define a map $\varphi : A^+ \otimes A^+ \longrightarrow M$ by $\varphi(x\otimes y) = \tilde\delta(x)y$. Then we claim $\varphi | K$ is a homomorphism. K is generated by elements $x \otimes y - 1 \otimes xy = (dx)y$, and φ is defined so as to be a right A-module map. So we have to check it's a left A-module map.

$$\varphi\big(z(x\otimes y - 1\otimes xy)\big) = \tilde\delta(zx)y - \tilde\delta(z)xy = z\tilde\delta(x)y$$

$$z\,\varphi(x\otimes y - 1\otimes xy) = z\big(\tilde\delta(x)y - \tilde\delta(1)xy\big) = z\tilde\delta(x)y.$$

The rest is clear.

The trouble with the above exact sequence is that it doesn't exhibit Ω^1_A in the most natural way, namely as the ideal in a split square zero extension. This we consider instead the universal split extension with extra ~~splitting~~

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r] & A * A \arrow[r] & A \arrow[l, bend right] \arrow[l, bend right=50] \arrow[r] & 0 \end{tikzcd}$$

and we find

$$\begin{tikzcd} 0 \arrow[r] & qA/(qA)^2 \arrow[r] \arrow[d, equal] & A*A/(qA)^2 \arrow[r] \arrow[d, equal] & A \arrow[l, bend right=50, "\bar\iota"'] \arrow[l, bend right, "\iota"] \arrow[r] & 0 \\ & \Omega^1_A & A \oplus \Omega^1_A & & \end{tikzcd}$$

and here $d = \bar\iota - \iota$

Base change with non-unital rings. Given a homomorphism $P \longrightarrow A$ of non-unital rings the left adjoint to the forgetful functor $\mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(P)$ is

$$N \longmapsto A^+ \otimes_{P^+} N = A^+ \otimes_P N$$

If $A = P/I$, where I is an ideal, then

$$A^+ \otimes_{P^+} N = P^+/I \otimes_{P^+} N = N/IN$$

as usual.

<u>Example</u>: Consider the morphism $0 \longrightarrow \mathbb{C}$ of non-unital rings. A 0-module is a vector space W, a $\mathbb{C}$-module is a pair of vector spaces V_0, V_1 such that 1 acts as projection on V_0 in $V_0 \oplus V_1$. A map $W \to V_0 \oplus V_1$ of 0-modules extends uniquely to a map $(W, W) \longrightarrow (V_0, V_1)$ of $\mathbb{C}$-modules.

Let us now try to make sense of the trace group

$$\mathrm{Ker}\left\{ Q/[Q,Q] \xrightarrow{d} D/[A,D] \right\}$$

where

$$D = A^+ \otimes_Q \Omega^1_Q \otimes_Q A^+ = \Omega^1_Q / M\Omega^1_Q + \Omega^1_Q M$$

$$= \mathrm{Ker}\left\{ Q * Q \longrightarrow A \right\}$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_Q \arrow[r] \arrow[d] & Q * Q/(qQ)^2 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & D \arrow[r] \arrow[d] & Q * Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_A \arrow[r] & A * A/qA^2 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Notice that in the square of square zero extensions of A, $A \longrightarrow A$ is the final object and its direct sum with itself is $A + A = A * A/(qA)^2$. So if we take Q to be this extension we get

$$H^0_{DR}(A) = \operatorname{Ker} \left\{ A/[A,A] \longrightarrow \Omega^1_A/[A, \Omega^1_A] \right\}$$

which agrees with Connes theorem that

$$H^0_{DR}(A) = \operatorname{Im} \left\{ S : HC_2(A) \to HC_0(A) = A/[A,A] \right\}$$

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We continue with the category of square zero extensions of the algebra A. Let's fix such an extension Q:

$$0 \longrightarrow M \longrightarrow Q \xrightarrow{\pi} A \longrightarrow 0$$

and try to determine the direct sum $Q \amalg A$ in this category.

A map $Q \amalg A \longrightarrow E$ is the same thing as a pair $u : Q \to E$, $v : A \to E$. Once v is given one has the map $v\pi : Q \to A \to E$, and the difference $u - v\pi$ is a derivation δ of Q with values in the A-bi-module $\mathrm{Ker}(E \to A)$. The pair (u,v) is equivalent to the pair (v, δ). Recall that

$$D = A^+ \otimes_{Q^+} \Omega^1_Q \otimes_{Q} A^+ = \Omega^1_Q / M\Omega^1_Q + \Omega^1_Q M$$

represents derivations of Q with values in A-bimodules. A map

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L \oplus A \arrow[r] \arrow[d] & A \arrow[l, dashed, bend right] \arrow[d, equal] \\ N \arrow[r] & E \arrow[r] & A \end{tikzcd}$$

semi-direct product extension of A by the A-bimodule L

is the same thing as a map $A \to E$ in the category (i.e. section of $E \to A$) together with an A-bimodule map $L \to N$. Thus we conclude that $Q \amalg A$ is the semi direct product of A with the A-bimodule D.

$$Q \amalg A \simeq D \oplus A$$

Next consider the map of extensions

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d] & Q \amalg A \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_A \arrow[r] & A \amalg A \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Recall that there is an exact sequence of A-bimodules

$$M \longrightarrow D \longrightarrow \Omega^1_A \longrightarrow 0$$

which comes from the exact sequence

$$0 \longrightarrow \mathrm{Der}(A, N) \longrightarrow \mathrm{Der}(Q, N) \longrightarrow \mathrm{Hom}_{A\otimes A^{op}}(M, N)$$

In effect let $D : Q \longrightarrow N$ be a derivation with values in the A-bimodule N. Then if $\pi(q) = a$

$$D(am) = D(qm) = \underbrace{(Dq)}_{0} m + q\,Dm = a(Dm)$$

and similarly $D(ma) = (Dm)a$, so $D|M$ is an A-bimodule homomorphism.

<u>Proposition</u>: One has a short exact sequence

$$0 \longrightarrow M \longrightarrow D \longrightarrow \Omega^1_A \longrightarrow 0$$

<u>Proof</u>: It suffices to show for any injective A-bimodule N, that $\mathrm{Der}(Q, N) \twoheadrightarrow \mathrm{Hom}_{A\otimes A^{o}}(M, N)$. But given $u : M \longrightarrow N$, there is a map of extensions

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "u"] & Q \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & u_* Q \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

One knows that the extensions of A by N vanish when N is injective. This comes from the theory:

$$\text{Extensions}(A,N) = H^2(A,N) = \text{Ext}^2_{A\otimes A^\circ}(A,N).$$

Choose a section of u_*A one gets another map from Q to u_*Q (via A), and the difference of these two maps is a derivation from Q to N restricting to u.

<u>Proposition</u>: One has an exact sequence

$$0 \longrightarrow M \xrightarrow{in_1} Q \amalg A \longrightarrow A \amalg A \longrightarrow 0$$

~~The injectivity follows because of the~~

This follows by combining the previous proposition with the serpent lemma.

~~Here's a simpler proof of the injectivity of $M \to D$. Form diagram:~~

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, equal] & Q \amalg Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] & Q \amalg A \arrow[r] & A \arrow[r] & {}
\end{tikzcd}$$

From the diagram

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, equal] & Q \amalg Q \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D \arrow[r] & Q \amalg A \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

we see that the kernel of $Q \amalg Q \longrightarrow Q \amalg A$ is $in_2(M)$, whence the kernel of $Q \amalg Q \to A \amalg A$ is $M \oplus M$.

Let's go over the group analogue as a check. Suppose we have an extension

$$1 \longrightarrow M \longrightarrow Q \longrightarrow G \longrightarrow 1$$

of G by an abelian group M. Then we claim that there is an exact sequence

$$* \qquad 0 \to M \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[Q]} I[Q] \longrightarrow I[G] \longrightarrow 0.$$

Here $I[G] = \operatorname{Ker}\{\mathbb{Z}[G] \to \mathbb{Z}\}$ is the augmentation ideal; the map $d : G \to I(G)$, $dg = g - 1$ is a universal derivation with values in a G-module.

$$d(xy) = xy - 1 = x(y-1) + x - 1$$
$$= x\,dy + dx$$

(Given $D : G \to M$ a derivation, define $u : I[G] \to M$ by $u(g-1) = Dg$. Well-defined as $D1 = 0$. Then

$$u(x(g-1)) = u(xg - 1 - x + 1) = D(xg) - Dx = x\,Dg$$
$$= x\,u(g-1)$$

showing u is a G-module morphism.)

To show $*$ is exact it suffices to show it gives an exact sequence upon applying $\operatorname{Hom}_G(\ ,N)$ for any injective module N. Again the essential point is whether a G-module map $u : M \to N$ extends to a derivation of Q to N, and this follows because the induced extension $u_* Q$ splits.

But we can make things more explicit by using coinduced modules, i.e. $C(G,\Lambda)$ = all set maps from G to the abelian group Λ. Thus Let G act on $C(G,\Lambda)$ by $(g\alpha)(x) = \alpha(xg)$. Then there is an embedding

$$M \overset{\alpha}{\hookrightarrow} C(G,M)$$

$$m \longmapsto \alpha_m \qquad \alpha_m(x) = {}^{x}m$$

which is a G-module map since

$$(g\alpha_m)(x) = \alpha_m(xg) = xgm = \alpha_{gm}(x).$$

Next we show explicitly how the induced extension by α splits.

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook, "\alpha"'] & Q \arrow[r] \arrow[d] & G \arrow[d, equal] \\
C(G,M) \arrow[r] & \alpha_* Q \arrow[r] & G
\end{tikzcd}$$

Choose a cross-section $s: G \longrightarrow Q$ and let $f: G\times G \to M$ by the associated cocycle:

$$s(x)\, s(y) = f(x,y)\, s(xy)$$

$$x f(y,z) - f(xy, z) + f(x,yz) - f(x,y) = 0.$$

Define $t: G \longrightarrow C(G,M)$ by $t(g)(x) = f(x,g)$.

Then
$$\big((\delta t)(g,h)\big)(x) = \big[g\, t(h) - t(gh) + t(g)\big](x)$$
$$= t(h)(xg) - f(x, gh) + f(x,g)$$
$$= f(xg, h) - f(x, gh) + f(x,g)$$
$$= x f(g,h) = \left(\alpha_{f(g,h)}\right)(x)$$

Thus under the embedding α, f becomes δt which means $\alpha_* Q$ splits.

Question: Is the preceding construction related to the Stinespring theorem?

Here, one is given a linear map $A \xrightarrow{u} B$ between algebras (say unital $+ u(1)=1$), and one forms the algebra $R = \mathrm{End}_{B^o}(A \otimes B) = \mathrm{Hom}_{\mathbb{C}}(A, A\otimes B)$. We then have ~~[illegible]~~ an algebra map $A \longrightarrow B$ sending a to $a \otimes 1$. We also have B^o-module maps

$$\begin{tikzcd} A\otimes B \arrow[r, "a\otimes b \mapsto u(a)b"] & B \arrow[l, bend left, "1\otimes"] \end{tikzcd}$$

which determine a projector e in $A \otimes B$ with image $B = 1 \otimes B$, so allow us to identify $B = \mathrm{End}_{B^o}(B)$ with eRe, and $u : A \to B$ with the map $A \longrightarrow R \xrightarrow{e?e} B$. ?

~~[illegible]~~

Consequence of the exact sequence

$$0 \longrightarrow N/[N,N] \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[P]} I[P] \longrightarrow I[G] \longrightarrow 0$$

associated to any extension of groups

$$1 \longrightarrow N \longrightarrow P \longrightarrow G \longrightarrow 1 \quad :$$

Suppose $P_\bullet$ is a free simplicial resolution of G and $N_\bullet = \mathrm{Ker}\{P_\bullet \to G_\bullet\}$ where $G_\bullet = G$ in each degree. Then $N_\bullet$ is a simplicial group free in each degree, and we ought to know that N_{ab} gives the reduced homology of the associated classifying space. But $\pi_*(N_\bullet) = 0$, so $\therefore N_{ab}$ is acyclic and we conclude that the "cotangent complex" $\mathbb{Z}[G.] \otimes_{\mathbb{Z}[P.]} I[P.]$ is quis to $I[G.]$. The same conclusion can be drawn from

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[P.]} I[P.] = \mathbb{Z}[G) \overset{L}{\otimes}_{\mathbb{Z}[P.]} I[P.]$$
$$\sim \mathbb{Z}[G] \overset{L}{\underset{\mathbb{Z}[G]}{\otimes}} I[G] \simeq I[G].$$

Now that we understand something about $Q \amalg Q$ we should return to the problem of determining

$$1) \qquad \mathrm{Ker}\left\{ Q/[Q,Q] \rightrightarrows Q \amalg Q/[\ ,\ ] \right\}$$

We show that $Q \amalg Q$ is the semi-direct product of Q and D. Hence

$$Q \amalg Q/[\ ,\ ] = Q/[\ ,\ ] \oplus D/[A,D]$$

Thus 1) is isomorphic to

$$2) \qquad \mathrm{Ker}\left\{ Q/[Q,Q] \xrightarrow{d} D/[A,D] \right\}$$

Another thing we can say is that because $Q \amalg A = A \oplus D$, the same kernel occurs for the two maps $Q \longrightarrow Q \amalg A$. I guess a good way to see this is to consider the three maps

$$Q \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} Q \amalg Q \amalg A$$

call them a, b, c. If $\xi \in Q/[Q,Q]$ is equalized by a, c and by b, c, then it is equalized by a, b. No, this doesn't work because we would need the injectivity on $?/[\ ,\ ]$ of $Q \amalg Q \longrightarrow Q \amalg Q \amalg A$.

From the exact sequence $0 \to M \to D \to \Omega^1_A \to 0$ we get [redacted]

$$\begin{tikzcd}
 & & HC_1(A) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{A^+\otimes A^{+o}}(\Omega^1_A, A^+) \arrow[d] \\
 & & M/[A,M] \arrow[r, "="] \arrow[d] & M/[A,M] \arrow[d] \\
0 \arrow[r] & \mathrm{Ker} \arrow[r] \arrow[d] & Q/[Q,Q] \arrow[r, "d"] \arrow[d] & D/[A,D] \arrow[d] \\
0 \arrow[r] & H^0_{DR}(A) \arrow[r] & A/[A,A] \arrow[r, "d"] \arrow[d] & \Omega^1_A/[A,\Omega^1_A] \arrow[d] \\
 & & 0 & 0
\end{tikzcd}$$

for the case of HC_1 see p. 262

Diagram chasing shows that $\mathrm{Ker} \twoheadrightarrow H^0_{DR}(A)$ in fact that we have an exact sequence

$$\mathrm{Tor}_1^{A^+\otimes A^{+o}}(\Omega^1_A, A^+) \to \mathrm{Ker} \to H^0_{DR}(A) \to 0$$

If we take $Q = P/I^2$ where $P/I \cong A$ and P is free, then Ω^1_P is a free P-bimodule, and so $D = A^+ \otimes_{P^+} \Omega^1_P \otimes_{P^+} A^+$ should be a free A-bimodule, and thus the Tor_1 should inject, giving perhaps an exact sequence

$$HC_1(A) \to \underbrace{\mathrm{Tor}_1^{A^+\otimes A^{+o}}(\Omega^1_A, A^+)}_{= \mathrm{Tor}_2^{A^+\otimes A^{+o}}(A^+, A^+) \;=\; H_2(A^+) = H_2(A)} \to \mathrm{Ker} \to H^0_{DR}(A) \to 0$$

Thus it looks very [redacted] much as if Ker is $HC_2(A)$.

Our next project will be to go over the case of $HC_1(A)$. We write $A = P/I$ with P free, and then we consider the "first" trace group $I/[P,I]$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & HC_1(P) \arrow[r] \arrow[d] & HC_1(A) \arrow[d] & \\
 & P\otimes I \arrow[r] \arrow[d] & \Lambda^2 P/bP^{\otimes 3} \arrow[r] \arrow[d] & \Lambda^2 A/bA^{\otimes 3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Here the middle row is exact by

$$\begin{tikzcd}
 & P^{\otimes 3} \arrow[r] \arrow[d] & A^{\otimes 3} \arrow[r] \arrow[d] & 0 \\
P\otimes I \arrow[r] \arrow[d, equal] & \Lambda^2 P \arrow[r] \arrow[d] & \Lambda^2 A \arrow[r] \arrow[d] & 0 \\
P\otimes I \arrow[r] & \Lambda^2 P/bP^{\otimes 3} \arrow[r] \arrow[d] & \Lambda^3 A/bA^{\otimes 3} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

Applying serpent to first diagram gives

$$\begin{tikzcd}
HC_1(P) \arrow[r] \arrow[d, equal] & HC_1(A) \arrow[r] & I/[P,I] \arrow[r] & P/[P,P] \arrow[r] & A/[A,A] \arrow[r] & 0 \\
0 & & & & &
\end{tikzcd}$$

because P is free.

So we see that $HC_1(A) \hookrightarrow I/[P,I]$. Next we would like to see if the naturality conditions cut down $I/[P,I]$ exactly to $HC_1(A)$.

Let's try the following argument. We need two maps from the extension P to an extension Q of A by J, such that the equalizer of the maps $I/[P,I] \rightrightarrows J/[Q,J]$ is $HC_1(A)$. Take Q to be

$$0 \longrightarrow J \longrightarrow P*A \longrightarrow A \longrightarrow 0$$

and consider the map $P*A \longrightarrow P$ sending A to zero. This induces a map

$$\begin{tikzcd}
J/[P*A, J] \arrow[r] & P*A/[\ ,\ ] \arrow[r] & P/[P,P] \\
I/[P,I] \arrow[u, shift left] \arrow[u, shift right] \arrow[r] & P/(P,P] \arrow[u, shift left, "in_1"] \arrow[u, shift right, "in_2"'] \arrow[ur, shift left, "id"] \arrow[ur, shift right, "0"'] &
\end{tikzcd}$$

and so the equalizer of $I/[P,I] \rightrightarrows J/[P*A,J]$ is contained in the kernel of $I/[P,I] \longrightarrow P/[P,P]$ which is $HC_1(A)$. This proves

<u>Prop:</u> $HC_1(A) \xrightarrow{\sim} \varprojlim_{P \twoheadrightarrow A} I/[P,I]$.

Note that

$$0 \longrightarrow \Omega^1_A \longrightarrow A^+ \otimes A^+ \longrightarrow A^+ \longrightarrow 0$$

implies

$$0 \longrightarrow \mathrm{Tor}_1^{A^+\otimes A^+}(A^+, A^+) \longrightarrow \Omega^1_A/[A,\Omega^1_A] \longrightarrow A^+ \longrightarrow A^+/[,] \longrightarrow 0$$

so $$H_1(A) = H_1(A^+) = \mathrm{Ker}\{\Omega^1_A/[A,\Omega^1_A] \longrightarrow [A,A]\}$$

November 10, 1987 (Jeanie's birthday)

Consider A = a vector space V with 0 multiplication. Then we know

$$HC_n(V) = V^{\otimes(n+1)}/1-\tau$$

where τ is [redacted] $(-1)^n\sigma$, [redacted] and σ is the cyclic permutation $\sigma(v_0\otimes\cdots\otimes v_n) = (v_n\otimes v_0\otimes\cdots\otimes v_{n-1})$. I would like to see how this fits with my scheme for cyclic homology.

Let $P = \bar{T}(V) = \bigoplus_{k\geq 1} V^{\otimes k}$, $I = \bigoplus_{k\geq 2} V^{\otimes k}$

so that P is free and $P/I = A$. [redacted]

[redacted]

$$0 \longrightarrow HC_1(A_n) \longrightarrow I^n/[P,I^n] \longrightarrow P/[P,P] \longrightarrow A_n/[A_n,A_n] \to 0$$

Now [redacted] recall that

$$P/[P,P] = \bar{T}(V)/[V,\bar{T}(V)] = \bigoplus_{k\geq 1} V^{\otimes k}/(1-\sigma)$$

Notice that relative to the grading I^n starts with $V^{\otimes 2n}$, in fact $I^n = \bigoplus_{k\geq 2n} V^{\otimes k}$. In degree $2n$ $[P,I^n]$ is [redacted] zero, but after that degree is the same as [redacted] $[P,P]$:

$$[P,I^n] = \bigoplus_{k\geq 2n} [V, V^{\otimes k}] = \bigoplus_{k>2n} V^{\otimes k}/(1-\sigma)$$

Thus we see that $HC_1(A_n)$ is concentrated in degree $2n$

$HC_1(A_n) =$ [redacted] $\mathrm{Ker}(V^{\otimes 2n} \longrightarrow V^{\otimes 2n}/1-\sigma)$

$$= (1-\sigma)V^{\otimes 2n}$$

[redacted] Our candidate for $HC_{2n-1}(A)$ is contained in $HC_1(A_n)$. Thus we should [redacted] be able to find $V^{\otimes 2n}/(1-\tau)$ inside $HC_1(A_n)$, assuming our ideas are correct. In even degrees $\tau = -\sigma$, so we want an injection

$$\circledast \quad V^{\otimes 2n}/(1+\sigma) \longrightarrow (1-\sigma)V^{\otimes 2n}$$

Let's consider the map

$$\underbrace{(1-\sigma)(1+\sigma^2+\dots+\sigma^{2(n-1)})}_{-\prod_{\substack{\zeta^{2n}=1\\ \zeta\neq -1}}(\sigma-\zeta)} : V^{\otimes 2n} \longrightarrow V^{\otimes 2n}$$

Using the decomposition into characters, we see that the kernel of this mapping is the sum of the eigenspaces for all $\zeta \neq -1$. This is the same as the image of $\sigma+1$. Thus we do get the required injective map. Notice that

$$(1-\sigma)(1+\sigma^2+\dots+\sigma^{2n-2}) = 1-\sigma+\sigma^2-\dots-\sigma^{2n-1} = \sum_{k=0}^{2n-1}\tau^k$$

So the image of $\circledast$ once $n>1$ is smaller than $(1-\sigma)V^{\otimes n}$. In fact the image is exactly the kernel of $1-\tau$.

Thus to finish off the identification of $HC_{2n-1}(A)$ with our candidate we need to produce two maps from P to another extension, whose effect on $V^{\otimes 2n}$ in I^n give essentially $1, \tau$.

Let's now try the two maps $P \overset{\iota}{\underset{\bar{\iota}}{\rightrightarrows}} P*A$. Let $J = \mathrm{Ker}\{P*A \longrightarrow A\}$, so that both ι and $\bar{\iota}$ carry $I = \mathrm{Ker}\{P \longrightarrow A\}$ into J. In fact $\bar{\iota}(I) = 0$. We are interested is the kernel of $V^{\otimes 2n}$

$$\left(I^n/[P,I^n]\right)_{2n} \xrightarrow{\iota} \left(J^n/[P*A,J^n]\right)_{2n}$$

since $\bar{\iota}\,|\,I = 0$.

Now we have

$$P * A = (P + \bar{A}) + (P \otimes \bar{A} + \bar{A} \otimes P) + (P \otimes \bar{A} \otimes P + \bar{A} \otimes P \otimes \bar{A})$$

where we write $\bar{A} = \bar{\iota}(A)$. We have

$$J = (\delta A + I) + (P \otimes \bar{A} + \bar{A} \otimes P) + (\qquad) +$$

where $\delta = \iota - \bar{\iota}$. Our first problem will be to determine J/J^2.

First note that J^2 contains all words of length ≥ 4.

Next let $a \in A$, $y \in I$, and let's compute

$$(\delta a + y) \cdot (\bar{a}_1, x) \qquad \text{where} \quad a_1 \in A,\ x \in P.$$

$$= (\iota a - \bar{a} + y)(a_1 x) = (\iota a + y)\bar{a}_1\, x$$

This gives a general element of $P \otimes \bar{A} \otimes P$. Next consider the products in $(\delta A + I)(P \otimes \bar{A})$. This contains

$$(\delta a + y)(x \otimes \bar{a}_1) = \iota(a) x \otimes \bar{a}_1 + (yx) \otimes \bar{a}_1 - \bar{\iota} a \otimes x \otimes \bar{a}_1$$

This is contained in

$$(I) \otimes \bar{A} + \bar{A} \otimes P \otimes \bar{A} \qquad ?$$

What we have shown is that

$$(\delta A + I)(\bar{A} \otimes P) = P \otimes \bar{A} \otimes P$$

$$(\delta A + I)(P \otimes \bar{A}) \subset I \otimes \bar{A} + \bar{A} \otimes P \otimes \bar{A}$$

$$(\bar{A} \otimes P)(\delta A + I) \subset \bar{A} \otimes I + \bar{A} \otimes P \otimes \bar{A}$$

Too hard.

Let's instead return to the derivation viewpoint. Thus we have a split square zero

$$\begin{tikzcd} 0 \arrow[r] & J/J^2 \arrow[r] & P*A/J^2 \arrow[r] & A \arrow[l, dashed, bend right] \arrow[r] & 0 \end{tikzcd}$$

which identifies J/J^2 with a universal A-bimod for a derivation of P with values in it. ∴

$$J/J^2 = A^+ \otimes_{P^+} \Omega^1_P \otimes_{P^+} A^+$$

But we know [redacted] because P is free that

$$\Omega^1_P = P^+ \otimes A \otimes P^+$$

whence $$J/J^2 = A^+ \otimes A \otimes A^+ .$$ Thus additively J/J^2 is the direct sum of A, $A^{\otimes 2}$, $A^{\otimes 2}$, $A^{\otimes 3}$. And this also checks with the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & \underbrace{A^+ \otimes_{P^+} \Omega^1_P \otimes_{P^+} A^+}_{J/J^2} \arrow[r] & \Omega^1_A \arrow[r] \arrow[d, equal] & 0 \\ & A^{\otimes 2} \oplus A^{\otimes 3} & & A \oplus A^{\otimes 2} & \end{tikzcd}$$

Let's return to the theory of [redacted] square zero extensions:

$$0 \longrightarrow M \longrightarrow Q \longrightarrow A \longrightarrow 0$$

Let's consider the problem of computing $\varprojlim Q/[\,,]$ over this category.

Let's generalize a bit. Suppose we give a quotient alg $A/\mathfrak{a}$ of A, and require that $\mathfrak{a}M = M\mathfrak{a} = 0$. Thus we consider extensions of A by $A/\mathfrak{a}$-bimodules.

November 11, 1987

Consider the category of non-unital algebras over k of characteristic zero. We consider the functor $A \longmapsto A/[A,A]$ from this category to vector spaces. It is the forgetful functor to Lie algebras followed by abelianization. We can take the left-derived functors of this functor is the sense of non-abelian homological algebra. Thus we construct a free (semi-)simplicial resolution $P_\cdot$ of A, apply the functor dimension-wise to obtain a simplicial vector space $P_\cdot/[P_\cdot,P_\cdot]$, and then take homology. We claim this gives the cyclic homology:

$$H_q\left(P_\cdot/[P_\cdot,P_\cdot]\right) \simeq HC_q(A)$$

Proof: Let $C_\cdot(A)$ be the chain complex giving the cyclic homology of A, i.e. $C_p(A) = A^{\otimes(p+1)}/(1-\tau)$ with differential b. Apply this functor to $P_\cdot$ degree-wise to obtain a simplicial chain complex $C_\cdot(P_\cdot)$. Notice that $C_p(A)$ depends only on the underlying vector space of A. Since $P_\cdot \to A$ is a resolution, its a simplicial homotopy equivalence ~~on~~ the level of simplicial vector spaces, so the same is true of $C_p(P_\cdot) \longrightarrow C_p(A)$ for each p. Thus ~~standard~~ spectral sequence theory shows the ~~total~~ homology of $C_\cdot(P_\cdot)$ is $HC(A)$. On the other hand because P_q is free, one knows that its cyclic homology is trivial, i.e.

$$HC_p(P_q) = \begin{cases} P_q/[P_q,P_q] & p=0 \\ 0 & p>0 \end{cases}$$

(This uses $\mathrm{char}(k)=0$). $\therefore$ Other spectral sequence degenerates QED.

Idea: Recall that the abelianization functor $\mathfrak{g} \longmapsto \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ from Lie algebras to vector spaces leads to the Lie algebra homology upon taking left derived functors. Check: let $\mathfrak{p}_\cdot \longrightarrow \mathfrak{g}$ be a free Lie algebra resolution of $\mathfrak{g}$. We can then apply dimension-wise the functor giving the standard chain complex for the Lie algebra homology: Taking reduced homology gives

$$\cdots\cdots \longrightarrow \Lambda^3\mathfrak{g} \xrightarrow{\partial} \Lambda^2\mathfrak{g} \xrightarrow{\partial} \mathfrak{g}$$

The same sort of double cx ideas reduces us to checking that for a free Lie algebra one has
$$H_p(\Lambda^\cdot\mathfrak{g}, \partial) = \begin{cases} k & p=0 \\ \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] & p=1. \end{cases}$$

But from the theory of Lie algebra cohomology we know that $H^2(\mathfrak{g}, M) = 0$ for $\mathfrak{g}$ free and all $\mathfrak{g}$-modules M. This is because H^2 classifies extensions. Then by derived functor theory $H^p(\mathfrak{g}, M) = 0$ for all $p \geq 2$. Taking $M = k$ this shows $H_p(\mathfrak{g}, k) = 0$ for $p \geq 2$. Also one can use
$$H_p(\mathfrak{g}, M) = \operatorname{Tor}_p^{U(\mathfrak{g})}(k, M)$$
and the fact that $U(\mathfrak{g})$ is a free algebra.

So what's interesting is the different functors.

$$\begin{tikzcd} (\text{algs}) \arrow[r, "M_n \text{ or } M_\infty", "\text{exact}"'] & (\text{Lie algs}) \arrow[loop above] \arrow[r, "\text{ab}"] & \begin{pmatrix}\text{vector} \\ \text{spaces}\end{pmatrix} \end{tikzcd}$$

One wants to argue by composite derived functor techniques. But it is all very mysterious, because, the first functor is exact, but probably doesn't take projectives into acyclics for the second functor. Somehow it's important that one deal with matrices on one hand and take primitive elements in the Lie algebra [illegible] homology on the other.

Idea: Instead of simplicial algebras use differential graded algebras. It would seem that one can define cyclic homology for differential graded algebras using the usual Connes complex but in the super framework. Then provided one can show the cyclic homology reduces to the commutator quotient for a free algebra, it would follow that cyclic homology can be calculated by taking a free DG resolution and looking at traces.

November 12, 1987

Interesting fact. It appears that by taking a free differential graded algebra resolution of A, call it $\to R_2 \to R_1 \to R_0$, one can obtain the cyclic homology as the homology of the chain complex $R/[R,R]$. On the other hand Connes proves that the cyclic homology can essentially be obtained by taking the non-commutative diff'l cochain algebra $\Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots$ and taking the homology of the cochain complex $\Omega/[\Omega,\Omega]$. Perhaps the natural way to reconcile these approaches would be to work in a superalgebra context.

I think we have to begin with a careful study of HC_1 and HC_2. The main mystery is how to handle an extension of A, which is always the first step in a DG algebra resolution.

Let's try to derive the exact sequence

$$0 \longrightarrow HC_1(A) \longrightarrow I/[P,I] \longrightarrow P/[P,P] \longrightarrow A/[A,A] \to 0$$

when P is a free algebra mapping onto A with kernel I. Let $P_\bullet$ be a free simplicial resolution of A with $P_0 = P$. Then we have a surjection

$$\begin{tikzcd}
\cdots P_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & P_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P_0 \arrow[d] \\
\cdots P_0 \times_A P_0 \times_A P_0 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & P_0 \times_A P_0 \arrow[r, shift left] \arrow[r, shift right] & P_0
\end{tikzcd}$$

of simplicial algebras. The reason is that if we normalize the bottom we obtain the DG algebra $\to 0 \to 0 \xrightarrow{d} I \xrightarrow{d} P_0$, and N is an equivalence on the level of simplicial + DG vector spaces.

Set $Q_k = P_0 \times_A \cdots \times_A P_0$ (k+1)-times so that $Q_\cdot$ is the semi-direct product extension of the constant simplicial algebra P_0 by the simplicial bimodule $N^{-1}(I[1]) = K(I,1)$, K in the sense of Eilenberg-Maclane. Then the kernel of $P_\cdot \longrightarrow Q_\cdot$ is a simplicial ideal $J_\cdot$ which is acyclic $H_*(J_\cdot) = 0$ and such that $J_0 = 0$. It should be true that

$$\mathbf{J}_\cdot / [P_\cdot, J_\cdot] \quad = \quad \mathbf{J}_\cdot \otimes_{(P_\cdot \otimes P_\cdot^{op})} P_\cdot$$

agrees with $J_\cdot \overset{\mathbb{L}}{\otimes}_{(P_\cdot \otimes P_\cdot^{op})} P_\cdot$ up to a kernel whose homology begins in degree 2. Thus we have $H_1(J_\cdot/[P_\cdot, J_\cdot]) = 0$. (Proof. Choose a flat $(P_\cdot \otimes P_\cdot^{op})$ resolution $K_\cdot$ of $J_\cdot$; we can suppose $K_0 = 0$. Then if $L_\cdot = \mathrm{Ker}(K_\cdot \to J_\cdot)$ we have

$$\begin{tikzcd}
L_\cdot \otimes_{(P_\cdot \otimes P_\cdot)} P_\cdot \arrow[rr] \arrow[dr, two heads] & & K_\cdot \otimes_{(P_\cdot \otimes P_\cdot^{op})} P^\cdot \arrow[r] \arrow[d, equal] & J_\cdot \otimes_{P_\cdot \otimes P_\cdot^{op}} P^\cdot \arrow[r] & 0 \\
& \mathrm{Im} \arrow[ur, hook] & J_\cdot \otimes^{\mathbb{L}}_{P_\cdot \otimes P_\cdot} P^\cdot & &
\end{tikzcd}$$

where Im starts in degree 1. Thus

$$H_1\left(J_\cdot \otimes^{\mathbb{L}}_{P_\cdot \otimes P_\cdot} P^\cdot\right) \longrightarrow\!\!\!\!\!\!\rightarrow H_1\left(J_\cdot \otimes_{P_\cdot \otimes P_\cdot} P^\cdot\right)$$

and the former is zero as $H_*(J_\cdot) = 0$.)

Because
$$\begin{tikzcd}
J_\cdot/[P_\cdot, J_\cdot] \arrow[rr] \arrow[dr, two heads] & & P_\cdot/[P_\cdot, P_\cdot] \arrow[r] & Q_\cdot/[Q_\cdot, Q_\cdot] \arrow[r] & 0 \\
& \mathrm{Im} \arrow[ur, hook] & & &
\end{tikzcd}$$
is exact

and $J_\cdot$ begins in degree we conclude that

$$H_1(\mathrm{Im}) = 0 \text{ , so}$$

$$H_1(P_0/[P_0,P_0]) = H_1(Q_\bullet/[Q_\bullet,Q_\bullet])$$

But the latter is very easy to compute using $Q_\bullet = K(I,1) \tilde{\times} P_0$ and we have

$$Q_\bullet/[Q_\bullet,Q_\bullet] = K(I/[P_0,I], 1) \oplus P_0/[P_0,P_0]$$

Actually these isomorphisms are not compatible with the first (or last) face operator, so the above ignores the inclusion map $I \longrightarrow P_0$. However it is clear that we do get an isomorphism of $N(Q_\bullet/[Q_\bullet,Q_\bullet])$ with

$$[redacted] \rightarrow 0 \longrightarrow I/[P_0,I] \longrightarrow P_0/[P_0,P_0]$$

which then identifies the kernel of this map with $HC_1(A)$.

[redacted]

Next let's consider HC_2. Let us consider a DG resolution R of A. Then we can consider the simplicial [redacted] vector space $N^{-1}(R)$ with product defined by the Alexander-Whitney map

$$N^{-1}R \otimes N^{-1}R \xrightarrow{(AW)} N^{-1}(R \otimes R) \longrightarrow N^{-1}R$$

and the product in R. (Recall that the shuffle map for simplicial vector spaces V, W goes ~~from~~ $NV \otimes NW \longrightarrow N(V \otimes W)$, and the AW map goes the other way $N(V \otimes W) \longrightarrow NV \otimes NW$ or $V \otimes W \longrightarrow N^{-1}(NV \otimes NW)$.) Thus $N^{-1}R$ is a simplicial algebra resolution of A.

Let P be a free simplicial resolution of A. Then we have a map [illegible] $P \longrightarrow N^{-1}R$ of simplicial algebras unique up to homotopy, hence a map of DG algebras $NP \longrightarrow N(N^{-1}R)$.

Question: Is the composition

$$NV \otimes NW \xrightarrow{\text{shuffle}} N(V \otimes W) \xrightarrow{AW} NV \otimes NW$$

the identity?

Let's assume this is so, as it is OKAY for a $V = K(\mathbb{C}, q)$, $W = K(\mathbb{C}, r)$. Then we know that the isomorphism $R = N(N^{-1}R)$ is compatible with products. Thus

$$\begin{tikzcd}
R \otimes R \arrow[r, "\text{shuffle}"] \arrow[d, phantom, "\shortparallel"] \arrow[rrr, bend left, "id"] & N(N^{-1}R \otimes N^{-1}R) \arrow[r, "AW"] \arrow[rd, "N\mu_{(N^{-1}R)}"] & N(N^{-1}(R \otimes R)) \arrow[r, phantom, "="] \arrow[d, "NN^{-1}(\mu_R)"] & R \otimes R \arrow[d, "\mu_R"] \\
N(N^{-1}R) \otimes N(N^{-1}R) \arrow[ru, "\text{shuffle}"] \arrow[rr, "\text{product in } N(N^{-1}R)"'] & & NN^{-1}(R) \arrow[r, phantom, "="] & R
\end{tikzcd}$$

If all this works, then we get a map $N(P/[P,P]) \longrightarrow NP/_{[NP,NP]} \longrightarrow R/[R,R]$, which should therefore give us a canonical map

$$\textcircled{*} \qquad HC_*(A) \longrightarrow H_*(R/[R,R]).$$

Let's then adopt as a <u>working hypothesis</u> the assertion that for any DG algebra resolution R of A we have a canonical map $\textcircled{*}$. This follows either from the above partial argument or from the triviality of higher cyclic cohomology for free superalgebras.

It seems that one can't prove anything about cyclic homology starting from the derived functors of $A \mapsto A/[A,A]$. For example given a free DG algebra resolution R of A, I was unable to construct a map $H_2(R/[R,R]) \longrightarrow A/[A,A]$. The best I could do using the standard complex for cyclic homology was to define a map

$$H_2(R/[R,R]) \longrightarrow HC_2(A)$$

Let's return now to HC_2 and square zero extensions. However it seems better to first understand HC_1. For example we have two quite different formulas

$$HC_1(A) = \mathrm{Ker}\left\{ I/[P,I] \longrightarrow P/[P,P] \right\} \qquad \begin{matrix} P \text{ free} \\ P/I = A \end{matrix}$$

$$= \mathrm{Ker}\left\{ \Omega^1_A / dA + [A, \Omega^1_A] \longrightarrow [A,A] \right\}$$

which should be linked as efficiently as possible.

~~In fact given any extension whatever $0 \to I \to P \to A \to 0$ P not necessarily free, we should be able to define a map α:~~

~~$$\Omega^1_A / dA + [A, \Omega^1_A] \xrightarrow{\alpha} I/[P,I]$$~~

~~such that α maps onto the kernel of the map from $I/[P,I]$ to $P/[P,P]$.~~

Let's notice that if A is free, better if the extension splits

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & P \arrow[r] & A \arrow[r] \arrow[l, bend right, dashed] & 0 \end{tikzcd}$$

then the kernel in question is zero. In effect, P is then the semi-direct product of A and I, and so

$$[P,P] = \text{[redacted]}\big([I,I]+[A,I]\big)+[A,A]\ \text{[redacted]}$$
$$= [P,I] \oplus [A,A] \subset I \oplus A$$

whence $\quad P/[P,P] = I/[P,I] \oplus A/[A,A]$.

November 13, 1987

It seems that an important point in studying extensions $I \to P \to A$ is the fact that one can find an injection into a split extension. Look at the universal situation:

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & P \arrow[r] \arrow[d, hook, "in_1"] & A \arrow[d, equal] \\ J \arrow[r] & P * A \arrow[r] & A \end{tikzcd}$$

The map in_1 is injective because of the map $P*A \to P$ sending A to zero.

But if we work with square zero extensions

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & D \arrow[r] & Q \amalg A \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

$$D = \Omega^1_Q / M\Omega^1_Q + \Omega^1_Q M$$

then we have to [redacted] show $M \to D$ is injective by some method. We have seen how to do this by showing that there is an embedding of A-bimodules of M into an "induced" bimodule which splits the extension.

I think it is necessary to make a careful analysis of $HC_1(A)$. In particular it would be nice to understand the formulas

$$HC_1(A) = \operatorname{Ker}\left(I/[P,I] \longrightarrow P/[P,P] \right) \qquad \begin{array}{c} P/I = A \\ P \text{ free} \end{array}$$

$$= \operatorname{Ker}\left\{ \Omega^1_A/dA + [A,\Omega^1_A] \twoheadrightarrow [A,A] \right\}$$
$$x\,dy \longmapsto [x,y]$$

and the links between them.

First of all we have to start with the A-bimodule exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^1_A & \longrightarrow & A^+\otimes A^+ & \xrightarrow{\text{mult}} & A^+ & \longrightarrow & 0 \\ & & dy & \longmapsto & 1\otimes y - y\otimes 1 & & & & \\ & & x\,dy & \longmapsto & x\otimes y - xy\otimes 1 & & & & \end{array}$$

Let's take this as the defn. of Ω^1_A, and forget about the universal property relative to derivations for the moment. If we right tensor over $A^+\otimes(A^+)^{op}$ with A^+, then we get

$$0 \to \underbrace{\operatorname{Tor}_1^{A^+\otimes(A^+)^{op}}(A^+,A^+)}_{H_1(A^+)} \to \Omega^1_A/[A,\Omega^1_A] \to A^+ \to A^+/[A,A] \to 0$$
$$x\,dy \longmapsto [y,x]$$
$$dy \longmapsto 0$$

$$\left(A^+ \otimes_{(A^+\otimes A^{+op})} (A^+\otimes A^{+op}) \longrightarrow A^+ \right.$$
$$\left. a \otimes (x\otimes y) \longmapsto yax \right)$$

Now recall that $\operatorname{Tor}_*^{A^+\otimes(A^+)^{op}}(A^+,A^+)$ is computed via a chain complex

$$\xrightarrow{b} A^+\otimes A\otimes A \xrightarrow{b} A^+\otimes A \xrightarrow{b} A^+.$$

Notice that this is not the same as the complex $(A^{\otimes(*+1)}, b)$. In fact if we take the reduced Hochschild complex for A^+, that is, change A^+ in degree zero to A, then we obtain the first two columns in the double complex for the non-unital ring A:

$$\begin{tikzcd}
 \arrow[d] & & \\
A^{\otimes 3} \arrow[d, "b"'] & \arrow[l, "1-t"'] \arrow[d, "b'"] & \\
A^{\otimes 2} \arrow[d, "b"'] & A^{\otimes 2} \arrow[l, "1-t"'] \arrow[d, "b'"] & \arrow[l, "N"'] \\
A & A \arrow[l, "1-t"'] & \arrow[l, "N"']
\end{tikzcd}$$

see the paper with Loday. The important thing to remember is that the complex $(A^{\otimes(*+1)}, b)$ does not give the groups $H_*(A)$, or what amounts to the same thing, the complex $(A^{\otimes(*+1)}, b')$ is not necessarily acyclic.

Recall also that $H_1(A^+) = H_1(A)$.

Notice that the ~~sequence~~ chain complex

$$\longrightarrow A^+ \otimes A \otimes A \xrightarrow{b} \underset{\Omega^1_A}{\underset{\parallel}{A^+ \otimes A}} \xrightarrow{b} A$$

$$x \otimes y \otimes z \xmapsto{b} \underbrace{xy\,dz - x\,d(yz)}_{-x(dy)z} + zx\,dy$$

~~so~~ that $b(A^+ \otimes A \otimes A) = [A, \Omega^1_A]$. Thus it's clear we get the [redacted] exact sequence

$$(+) \qquad 0 \longrightarrow H_1(A) \longrightarrow \Omega^1_A / [A, \Omega^1_A] \longrightarrow A \longrightarrow A/[A,A] \longrightarrow 0$$

$$x\,dy \longmapsto [x, y]$$

The cyclic cohomology of A is obtained by taking a quotient of the reduced Hochschild complex

$$\begin{tikzcd}
{} \arrow[r] & A^+\otimes A^{\otimes 2} \arrow[r, "b"] \arrow[d] & A^+\otimes A \arrow[r, "b"] \arrow[d] & A \arrow[d] \\
{} \arrow[r] & A^{\otimes 3}/(1-t) \arrow[r, "b"] & \Lambda^2 A \arrow[r] & A
\end{tikzcd}$$

Notice that

$$\Omega^1_A / dA + [A, \Omega^1_A] = A^+\otimes A / 1\otimes A + b(A^+\otimes A^{\otimes 2})$$

$$= \Lambda^2 A / bA^{\otimes 3}$$

because $x\,dy + y\,dx \equiv x\,dy + dx\,y$

$$= d(xy) \equiv 0 \qquad \text{mod } dA + [A, \Omega^1_A].$$

So we also have an exact sequence

$$(++) \qquad 0 \longrightarrow HC_1(A) \longrightarrow \Omega^1_A / dA + [A, \Omega^1_A] \longrightarrow A \longrightarrow A/[A,A] \longrightarrow 0$$

and also by mapping (+) onto (++).

$$A/[A,A] \longrightarrow H_1(A) \longrightarrow HC_1(A) \longrightarrow 0$$

Notice that elements of

$$\left(\Omega^1_A / [A, \Omega^1_A]\right)^* \qquad \left(\Omega^1_A / dA + [A, \Omega^1_A]\right)^*$$

are respectively Hochschild and cyclic 1-cocycles.

Next let's consider an extension

$$0 \longrightarrow I \longrightarrow P \longrightarrow A \longrightarrow 0$$

Then we can use the diagram

$$\begin{tikzcd}
 & & bP^{\otimes 3} \arrow[r, two heads] \arrow[d] & bA^{\otimes 3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^2 I \arrow[r] & P \otimes I \arrow[r] \arrow[d] & \Lambda^2 P \arrow[r] \arrow[d] & \Lambda^2 A \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & I \arrow[r] & P \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

rows exact

to obtain an exact sequence

$$HC_1(P) \rightarrow HC_1(A) \xrightarrow{\partial} I/[P,I] \rightarrow P/[P,P] \rightarrow A/[A,A] \rightarrow 0$$

Here's how ∂ is defined. Given $\sum x_i \wedge y_i \in \Lambda^2 A$ such that $\sum [x_i, y_i] = 0$, we lift this element of $\sum \tilde{x}_i \wedge \tilde{y}_i \in \Lambda^2 P$. Then $\sum [\tilde{x}_i, \tilde{y}_i] \in I$ and we take its image in $I/[P,I]$.

Here's how to proceed on the level of cocycles. Fix a linear lifting $s : A \rightarrow P$. Then by sending $x, y \in A$ to $[s(x), s(y)] - s[x,y]$ we get a map from $\Lambda^2 A$ to I. Actually what we are doing is to use $s, s^{\otimes 2}$, etc to construct a lifting of the ~~quotient~~ in

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P^{\otimes 3}/(1-t) \arrow[r] \arrow[d] & A^{\otimes 3}/(1-t) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & P \otimes I / S^2 I \arrow[r] \arrow[d] & \Lambda^2 P \arrow[r] \arrow[d] & \Lambda^2 A \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & I \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0
\end{tikzcd}$$

(this is an exact sequence of complexes). Thus we get a map of degree -1 from

$$\begin{tikzcd}
 & & \arrow[d] \\
C_{\cdot}(A) \arrow[r] & \text{complex} & K \arrow[d] \\
 & & P \otimes I / S^2 I \arrow[d] \\
 & & I
\end{tikzcd}$$

Passing to homology gives a map $HC_1(A) \longrightarrow I/[P,I]$. In fact we even get a map

$$\begin{aligned} \Lambda^2 A / bA^{\otimes 3} &\longrightarrow I/[P,I] \\ x \wedge y &\longmapsto [s(x),s(y)] - s([x,y]) \end{aligned}$$

which means that a linear map $\tau : I/[P,I] \to \mathbb{C}$ gives rise to a cyclic 1-cocycle φ given by the formula

$$\varphi(x,y) = \tau\big([s(x), s(y)] - s[x,y] \big)$$

November 14, 1987

Goal: To define a canonical map $HC_2(A) \longrightarrow Q/[Q,Q]$ associated to any square zero extension $0 \to M \to Q \to A \to 0$. Recall the diagram with exact columns

$$\begin{tikzcd}
HC_1(A) \arrow[r] \arrow[d] & \mathrm{Tor}_2^{A^+\otimes A^{+op}}(A^+, \Omega^1_A) = H_2(A) \arrow[d] \\
M/[A,M] \arrow[r, "="] \arrow[d] & M/[A,M] \arrow[d] \\
Q/[Q,Q] \arrow[r] \arrow[d] & D/[A,D] \arrow[d] \\
A/[A,A] \arrow[r] \arrow[d] & \Omega^1_A/[A,\Omega^1_A] \arrow[d] \\
0 & 0
\end{tikzcd}$$

This gives rise to a map $\alpha : H_2(A) \longrightarrow Q/[Q,Q]$ which vanishes on the image of $HC_1(A) \longrightarrow H_2(A)$. Hopefully by understanding why, we can extend α to a map $HC_2(A) \longrightarrow Q/[Q,Q]$.

Nov. 15: It is possible to give another proof of the injectivity of $M \longrightarrow D = A^+ \otimes_Q \Omega^1_Q \otimes_Q A^+$ as follows. Start with the exact sequence of Q^+-bimods

$$0 \longrightarrow \Omega^1_Q \longrightarrow Q^+ \otimes Q^+ \longrightarrow Q^+ \longrightarrow 0$$

and base change via $Q^+ \otimes Q^{+op} \longrightarrow A^+ \otimes A^{+op}$:

$$0 \longrightarrow \mathrm{Tor}_1^{Q^+\otimes Q^{+op}}(A^+ \otimes A^+, Q^+) \longrightarrow D \longrightarrow A^+ \otimes A^+ \longrightarrow A^+ \to 0$$

Thus we have to identify the Tor_1 with M. We have the exact sequences

November 15, 1987 (cont.)

$$0 \longrightarrow Q^+ \otimes M + M \otimes Q^+ \longrightarrow Q^+ \otimes Q^+ \longrightarrow A^+ \otimes A^+ \longrightarrow 0$$

$$0 \longrightarrow M \otimes M \longrightarrow (Q^+ \otimes M) \oplus (M \otimes Q^+) \longrightarrow Q^+ \otimes M + M \otimes Q^+ \longrightarrow 0$$

If $P_\cdot \to M$ is a free left Q^+-module resln. of M, then

$$\mathrm{Tor}_q^{Q^+ \otimes Q^{+op}}(Q^+ \otimes M, Q^+) = H_q\left\{ (Q^+ \otimes P_\cdot) \otimes_{Q^+ \otimes Q^{+op}} Q^+ \right\}$$

$$= H_q\left\{ P_\cdot \otimes_{(Q^+)^{op}} Q^+ \right\} = M[0]$$

so we get

$$0 \longrightarrow \mathrm{Tor}_1^{Q^+ \otimes Q^{+op}}(A^+ \otimes A^+, Q^+) \longrightarrow (Q^+ \otimes M + M \otimes Q^+) \otimes_{Q^+ \otimes Q^{+op}} Q^+ \longrightarrow Q^+ \longrightarrow A^+ \longrightarrow 0$$

$$(M \otimes M) \otimes_{Q^+ \otimes Q^{+op}} Q^+ \xrightarrow{0} M \oplus M \longrightarrow (Q^+ \otimes M + M \otimes Q^+) \longrightarrow 0$$

so it's clear more or less.

Next let us return to defining $HC_2(A) \to Q/[Q,Q]$. We first construct a map $H_2(A) \to M/[A,M]$. For this we compare the resolution

$$0 \longrightarrow M \longrightarrow \underbrace{A^+ \otimes_Q \Omega^1_Q \otimes_Q A^+}_{D} \longrightarrow \Omega^1_A \longrightarrow 0$$

with the Hochschild resolution

$$1 \otimes a \otimes 1 \longmapsto da = a \otimes 1 - 1 \otimes a$$

$$\longrightarrow A^+ \otimes A^{\otimes 2} \otimes A^+ \xrightarrow{b'} A^+ \otimes A \otimes A^+ \xrightarrow{b'} \Omega^1_A \longrightarrow 0$$

which is a free $A^+ \otimes A^{+op}$-module resolution. Thus there has to be a map from the latter to the former unique up to homotopy. If $s : A \to Q$ is a linear lifting, then we have a map of complexes:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & D \arrow[r] & \Omega^1_A \arrow[r] & 0 \\
\cdots \arrow[r] & A^+\otimes A^{\otimes 2}\otimes A^+ \arrow[r, "b'"] \arrow[u, "\varphi"] & A^+\otimes A\otimes A^+ \arrow[r, "b'"] \arrow[u, "\tilde\varphi"] & \Omega^1_A \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $\tilde\varphi(x\otimes a\otimes y) = x\otimes ds(a)\otimes y$. Thus

$$\tilde\varphi\, b'(x\otimes a^1\otimes a^2\otimes y) = \tilde\varphi\left[xa^1\otimes a^2\otimes y - x\otimes a^1a^2\otimes y + x\otimes a^1\otimes a^2 y\right]$$

$$= x\left\{a^1\otimes ds(a_2)\otimes 1 - 1\otimes ds(a^1a^2)\otimes 1 + 1\otimes ds(a^1)\otimes a^2\right\}y$$

$$= x\left\{1\otimes\Big[\underbrace{s(a^1)\,ds(a_2) - d\,s(a^1a^2) + ds(a^1)\,s(a^2)}_{d\left(\underbrace{s(a^1)s(a^2) - s(a^1a^2)}_{f(a^1,a^2)}\right)}\Big]\otimes 1\right\}y$$

the cocycle

$\therefore$ $$\varphi(x\otimes a^1\otimes a^2\otimes y) = x\, f(a^1,a^2)\,y$$

Now tensor with A^+ over $A^+\otimes A^{+\,op}$ and we get a map of complexes

$$\begin{tikzcd}
 & 0 \arrow[r] & M/[A,M] \arrow[r] & D/[A,D] \arrow[r] & \Omega^1_A/[A,\Omega^1_A] \arrow[r] & 0 \\
A^+\otimes A^{\otimes 3} \arrow[rr, "b"] & & A^+\otimes A^{\otimes 2} \arrow[r, "b"] \arrow[u, "\bar\varphi"] & A^+\otimes A \arrow[r] \arrow[u, "\bar\varphi"] & \Omega^1_A/[A,\Omega^1_A] \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $\bar\varphi(x\otimes a^1\otimes a^2) = x f(a^1,a^2)$.

Thus we have obtained a map

(*) $$A^+\otimes A^{\otimes 2}/b(A^+\otimes A^{\otimes 3}) \xrightarrow{\bar\varphi} M/[A,M]$$

i.e. Hochschild 2-cocycle with values in $M/[A,M]$. The cohomology class is a well-defined map

$$H_2(A) \longrightarrow M/[A,M]$$

independent of the choice of s.

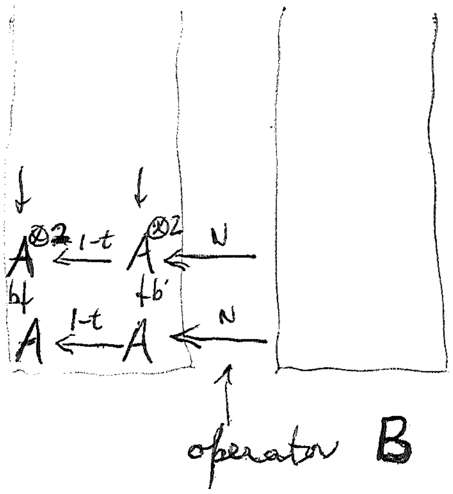
Next we wish to use this (*) to go from

$HC_2(A)$ to $Q/[Q,Q]$. Let's consider the maps of complexes

$$\begin{tikzcd}
& 0 \arrow[r] & Q/[Q,Q] \arrow[r, "d"] & D/[A,D] \arrow[r] & 0 \arrow[r] & {} \\
{} \arrow[r] & A^+\otimes A^{\otimes 3} \arrow[u] \arrow[r, "b"] & A^+\otimes A^{\otimes 2} \arrow[u, "\bar{\varphi}"'] \arrow[r, "b"] & A^+\otimes A \arrow[u, "\bar{\tilde{\varphi}}"'] \arrow[r] & A \arrow[r] & {} \\
{} \arrow[r, "b"] & A^{\otimes 3}/(1-t) \arrow[u, hook, "B"'] \arrow[r, "-b"] & \Lambda^2 A \arrow[u, hook, "B"'] \arrow[r, "-b"] & A \arrow[u, hook, "B"'] \arrow[r] & 0
\end{tikzcd}$$

In the middle is the Hochschild complex, and B is the composite map $N: A^{\otimes(*+1)}/(1-t) \longrightarrow A^{\otimes(*+1)} \xrightarrow{1\otimes} A^+\otimes A^{\otimes(*+1)}$.

(Think of the Hochschild complex as the first two columns of the double complex



Let Φ be the map of complexes defined by $\bar{\varphi}$ and $\bar{\tilde{\varphi}}$. When restricted to $B(C.(A))$ it is not zero but is homotopic to zero. In effect

$$(\bar{\varphi}B)(a^1\wedge a^2) = \bar{\varphi}(1\otimes a^1\otimes a^2 - 1\otimes a^2\otimes a^1)$$

$$= f(a^1,a^2) - f(a^2,a^1) \quad \text{in } M/[A,M]$$

$$= \cancel{[s(a^1),s(a^2)]} - s[a^1,a^2] \quad \text{in } Q/[Q,Q]$$

$$= (+s)(-b)(a^1\wedge a^2)$$

$$(\bar{\tilde{\varphi}}B)(a) = \bar{\tilde{\varphi}}(1\otimes a) = 1\otimes da\otimes 1 = d(s(a))$$

Thus (+5) gives a chain homotopy of $\Phi \circ B$ to 0. At this point it follows that we have a well-defined map

$$HC_2(A) \longrightarrow \mathrm{Ker}\{Q/[Q,Q] \longrightarrow D/[A,D]\}.$$

Next we would like to realize it by an explicit cocycle, i.e. a map from $C_\cdot(A)$ to this two stage complex.

Improvements. Starting with a map of resolutions

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] & M \arrow[r] & D \arrow[r, Rightarrow] & A^+ \otimes A^+ \arrow[r] & A^+ \arrow[r] & 0 \\
& \cdots \arrow[r] & A^+ \otimes A^{\otimes 2} \otimes A^+ \arrow[r] \arrow[u] & A^+ \otimes A \otimes A^+ \arrow[r] \arrow[u] & A^+ \otimes A^+ \arrow[r] \arrow[u, "\|"] & A^+ \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

we get a map of complexes

$$\begin{tikzcd}
{} \arrow[r] & 0 \arrow[r] & M/[A,M] \arrow[r] & D/[A,D] \arrow[r] & A^+ \\
& \arrow[r] & A^+ \otimes A^{\otimes 2} \arrow[r] \arrow[u] & A^+ \otimes A \arrow[r] \arrow[u] & A^+ . \arrow[u, equal]
\end{tikzcd}$$

Then we can remove k in degree zero to get a map of complexes

$$\begin{tikzcd}
& & (2) & (1) & (0) \\
{} \arrow[r] & 0 \arrow[r] & M/[A,M] \arrow[r] & D/[A,D] \arrow[r] & A \\
{} \arrow[r] & A^+ \otimes A^{\otimes 3} \arrow[r] \arrow[u] & A^+ \otimes A^{\otimes 2} \arrow[r, "b"] \arrow[u] & A^+ \otimes A \arrow[r, "b"] \arrow[u] & A \arrow[u, equal]
\end{tikzcd}$$

In the case $Q = P/I^2$, P free we know that D is a free $A^+ \otimes (A^+)^{op}$ module, then we know that the above map of complexes induced an isoms on H_0, H_1, H_2.

Return to our basic map from the Hochschild complex to complex of length 3 constructed from Q, which we want to use to map the cyclic complex to this 3 step complex.

$$\begin{tikzcd}
\arrow[r] & 0 \arrow[r] & Q/[Q,Q] \arrow[r, "d"] & D/[A,D] \arrow[r] & A \\
\arrow[r] & A^+\otimes A^{\otimes 3} \arrow[r, "b"] & A^+\otimes A^{\otimes 2} \arrow[r, "b"] \arrow[u, "\psi"] & A^+\otimes A \arrow[r, "b"] \arrow[u, "\tilde{\psi}"] \arrow[ul, dashed, "h"] & A \arrow[u, equal] \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}$$

Here $D = \underbrace{(A\otimes_Q \Omega^1_Q)_{\otimes_Q}}_{} = \Omega^1_Q/[Q,\Omega^1_Q] + M\Omega^1_Q + \Omega^1_Q M$

$$\psi(x,a^1,a^2) = s(x)\left[s(a^1)s(a^2) - s(a^1a^2)\right]$$

$$\tilde{\psi}(x,a) = s(x)\,ds(a)$$

$$d\psi(x,a^1,a^2) = ds(x)\,[\underbrace{s(a^1)s(a^2) - s(a^1a^2)}_{f(a^1,a^2)\,\in M}] + s(x)\,df(a^1,a^2)$$

$$= s(x)\,df(a^1,a^2)$$

$$\tilde{\psi}\,b(x,a^1,a^2) = \tilde{\psi}\left[xa^1\otimes a^2 - x\otimes a^1a^2 + a^2x\otimes a^1\right]$$

$$= s(xa^1)\,ds(a^2) - s(x)\,ds(a^1a^2) + s(a^2x)\,ds(a^1)$$

$$= s(x)s(a^1)\,ds(a^2) - s(x)\,ds(a^1a^2) + \underbrace{s(a^2)s(x)\,ds(a^1)}_{}$$

$$= s(x)\,df(a^1,a^2).$$

Now we need a homotopy $(h,\tilde{h})$ deforming $\underline{\Phi}$ $\Phi = (\psi,\tilde{\psi})$ to a map Φ from the Hochschild complex which factors through $C.(A)$. ~~The new map on $A^+\otimes A$ has to vanish on $1\otimes A + S^2(A) \subset A^+\otimes A$. One has~~

~~$\tilde{\psi}(a\otimes a' + a'\otimes a) = s(a)\,ds(a') + s(a')\,ds(a)$~~

~~$= d(s(a)s(a'))$~~

~~$\tilde{\psi}(1\otimes a) = ds(a)$~~

Let's write down the conditions that $(h, \tilde{h})$ must satisfy. They are really not uniquely defined, because once we've deformed Φ into a map $\Phi : C.(A) \to \{Q \otimes \Omega \to \cdots\}$ then we are free to deform Φ within such maps. This means that $\tilde{h}$ can be varied arbitrarily and that h can be varied by any map $\Lambda^2 A \longrightarrow Q/[Q,Q]$. So we suppose $\tilde{h} = 0$.

~~We want $\tilde{\psi} - dh$ to vanish on $1\otimes A + S^2 A$ so~~

~~$0 = (\tilde{\psi} - dh)(1 \otimes a) = ds(a) - d\,h(1\otimes a)$.~~

~~$0 = (\tilde{\psi} - dh)(a^1 \otimes a^2 + a^2 \otimes a^1) = s(a^1)\,ds(a^2) + s(a^2)\,ds(a^1) - dh(a^1 \otimes a^2 + a^2 \otimes a^1)$~~

~~$= d\,s(a^1)s(a^2)$~~

~~But D.A~~

We want h to extend the contracting homotopy we alreary have for Φ restricted to $(1\otimes N)(C.(A))$. Thus

$$\circledast \qquad \boxed{h(1\otimes a) = s(a)}$$

We want $\tilde{\psi} - dh$ to vanish on $1 \otimes A$ and $S^2(A)$ and $\psi - hb$ to vanish on $1 \otimes A^{\otimes 2}$ and to factor through $A^{\otimes 3}/(1-t)$.

$$(\tilde{\psi} - dh)(1\otimes a) = ds(a) - d\,h(1\otimes a) = 0 \quad \text{by } \circledast$$

$$0 = (\psi - hb)(1\otimes a^1 \otimes a^2) = f(a^1, a^2) - h(a^1 \otimes a^2 - 1 \otimes a^1 a^2 + a^2 \otimes a^1)$$

$$= f(a^1, a^2) + s(a^1 a^2) - h(a^1 \otimes a^2 + a^2 \otimes a^1)$$

$$= s(a^1)s(a^2) - h(a^1 \otimes a^2 + a^2 \otimes a^1)$$

$$\therefore \quad \boxed{h(a^1 \otimes a^2 + a^2 \otimes a^1) = s(a^1)\,s(a^2)}$$

These conditions determine h up to a linear map on $\Lambda^2 A$, so effectively things are now determined. We have to check that

$$(\tilde{\psi} - dh)(a^1 \otimes a^2 + a^2 \otimes a^1) = s(a^1)\,ds(a^2) + s(a^2)\,ds(a^1) - dh(a^1 \otimes a^2 + a^2 \otimes a^1)$$

$$= d\big(s(a^1)\,s(a^2)\big) - dh(a^1 \otimes a^2 + a^2 \otimes a^1)$$

vanishes, which is clear. We also want

$$(\psi - hb)(a^0 \otimes a^1 \otimes a^2) = s(a^0) f(a^1, a^2) - h\left(a^0 a^1 \otimes a^2 - a^0 \otimes a^1 a^2 + a^2 a^0 \otimes a^1\right)$$

to be fixed under cyclic permutations. But ~~f~~

$$h(a^0 \otimes a^1 a^2) = -h(a^1 a^2 \otimes a^0) + s(a^0)\,s(a^1 a^2) \quad \text{so}$$

$$\boxed{(\psi - hb)(a^0 \otimes a^1 \otimes a^2) = s(a^0)\,s(a^1)\,s(a^2) - h\left(a^0 a^1 \otimes a^2 + a^1 a^2 \otimes a^0 + a^2 a^0 \otimes a^1\right)}$$

which makes the cyclic symmetry clear.

All that remains to produce a cyclic 2-cocycle on A with values in $Q/[Q,Q]$ is to choose an appropriate formula for h, for example

$$\boxed{h(a^1 \otimes a^2) = \tfrac{1}{2}\, s(a^1)\, s(a^2)}$$

is the unique choice which vanishes on the skew-symmetric tensors in $A^{\otimes 2} \subset A^+ \otimes A$.

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Let $A = C(S^1)$ act on $H = L^2(S^1)$ as usual, let e be the Hardy-Hilbert projection. Then we have the Toeplitz extensions

$$\begin{tikzcd} K(eH) \arrow[r] & T_+ \arrow[r] & A \end{tikzcd}$$

(1)

$$\begin{tikzcd} K((1-e)H) \arrow[r] & T_- \arrow[r] & A \end{tikzcd}$$

These can be added using the Whitney sum $K(eH) \oplus K((1-e)H) \longrightarrow K(H)$ and the diagonal $A \longrightarrow A \times A$. First one can form the product of the two extensions and push forward by the Whitney sum map

$$\begin{tikzcd} K(eH) \times K((1-e)H) \arrow[r] \arrow[d] & T_+ \times T_- \arrow[r] \arrow[d] & A \times A \arrow[d, equal] \\ K(H) \arrow[r] & \Psi \arrow[r] & A \times A . \end{tikzcd}$$

Ψ is (essentially?) the singular integral operators on $L^2(S^1)$, and $A \times A$ = functions on the cosphere bundle. Next pull-back by the diagonal

$$\begin{tikzcd} K(H) \arrow[r] \arrow[d, equal] & R \arrow[r] \arrow[d] & A \arrow[d, "\Delta"] \\ K(H) \arrow[r] & \Psi \arrow[r] & A \times A . \end{tikzcd}$$

Since $\Delta(A)$ lifts to Ψ, this extension of A by K splits.

What we have described is an example of dilating an invertible extension. The reason we are interested in this is it's a process for converting an extension into an (odd or ungraded) Fredholm module. This process lies behind the introduction of non-commutative differential forms. For me the main mystery is to understand why the non-commutative Chern-Weil formalism appears in cyclic cohomology.

Let's concentrate on the extension

$$\begin{tikzcd} 0 \arrow[r] & K(H) \arrow[r] & \mathcal{E} \arrow[r] & A\times A \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$

which we can think of in terms of singular integral operators and symbols. Actually we want to take a smooth version as I have discussed previously, so that there's a trace on the kernel. This trace gives rise to a cyclic 1-cocycle on $\mathcal{E}$ by the formula

$$f,g \longmapsto \text{trace}\{[s(f),s(g)] - s[f,g]\}$$

where s is a lifting of $A\times A$ into $\mathcal{E}$. (Actually this cyclic cocycle depends on s and the trace, but its class depends only on the trace.) So we need the lifting s, and it seems natural to extend the obvious lifting of the diagonal subalgebra $\Delta A \subset A\times A$.

The idea perhaps is to write $A\times A = A \oplus A(\varepsilon)$ where $\varepsilon = (1,-1)$, and define $s(f+g\varepsilon) = f+gF$.

$$\begin{aligned}[f^0+g^0F,\ f'+g'F] &= [g^0F,f'] + [f^0,g'F] + [g^0F,g'F] \\ &= g^0[F,f'] + g'[f^0,F] + [g^0F,g']F + g'[g^0F,F] \\ &= g^0[F,f'] + g'[f^0,F] + g^0[F,g']F + g'[g^0,F]F\end{aligned}$$

When we take the trace the last two terms cancel:

$$\left.\begin{aligned} \tau(g^0[F,g^1]F) &= \tau(Fg^0[F,g^1]) \\ &= -\tau(g^0F[F,g^1]) \\ &= \tfrac{1}{2}\tau([F,g^0][F,g^1]) \\ \tau\left(g^1[g^0,F]F\right) &= \tau(Fg^1[g^0,F]) \\ &= -\tau(g^1F[g^0,F]) \\ &= \tfrac{1}{2}\tau\left([F,g^1][g^0,F]\right) \end{aligned}\right\} \text{opposite sign.}$$

So the cyclic cocycle is

$$\varphi(f^0+g^0F,\ f^1+g^1F) = \tau\left(g^0[F,f^1] - g^1[F,f^0]\right).$$

In today's lecture I started the theory of cyclic homology using as motivation the question: One has associated to an extension $I \to P \twoheadrightarrow A$ an exact sequence in K-theory

$$\begin{tikzcd} K_1(P) \arrow[r] \arrow[d, dashed] & K_1(A) \arrow[r, "\partial"] \arrow[d, dashed] & K_0(I) \arrow[r] \arrow[d] & K_0(P) \arrow[r] \arrow[d] & K_0(A) \arrow[d] \\ HC_1(P) \arrow[r, dashed] & HC_1(A) \arrow[r, dashed] & I/[P,I] \arrow[r] & P/[P,P] \arrow[r] & A/[A,A] \arrow[r] & 0 \end{tikzcd}$$

Can one define HC_1 and fill in the dotted arrows?

But it remains to define the map $K_1(A) \to HC_1(A)$ explicitly and check the commutativity. Notice that if we use $HC_1(P) = 0$ for P free there can only be one map compatible with the connecting homomorphism (This is somehow satellite theory).

Let's be explicit. Let $u \in (1+A)^\times \subset (A^+)^\times$ represent and elt of $K_1(A)$; more generally we could consider a matrix. Calculate $\partial[u]$. Lift u to $p \in 1+P$, u^{-1} [redacted] to $q \in 1+P$ such that $1 -$ [redacted] $qp = \alpha\beta$ with $\alpha, \beta \in I$. Then one has

$$(q \;\; \alpha)\begin{pmatrix} p \\ \beta \end{pmatrix} = 1$$

so we get a projector

$$e = \begin{pmatrix} p \\ \beta \end{pmatrix}(q \;\; \alpha) = \begin{pmatrix} pq & p\alpha \\ \beta q & \beta\alpha \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \bmod I$$

and $\tau\,\partial[u] \in I/[P,I]$ is

$$\begin{aligned} \tau\,\partial[u] &= \tau(pq-1) + \tau(\beta\alpha) \\ &= \tau(pq-1) + \tau(\underbrace{\alpha\beta}_{1-qp}) \\ &= \tau[p,q] \end{aligned}$$

This obviously (because τ kills $[P,I]$) depends only on u, u^{-1}. If $s : A \to P$ is a linear lifting, we can take

$$\begin{aligned} p &= 1 + s(u-1) \\ q &= 1 + s(u^{-1}-1) \end{aligned}$$

and then

$$\tau\,\partial[u] = \tau[s(u-1), s(u^{-1}-1)] \in I/[P,I]$$

[redacted] Recall that the map $HC_1(A) \to I/[P,I]$ is induced by the 1-cocycle

$$\begin{aligned} \Lambda^2 A / bA^{\otimes 3} &\longrightarrow I/[P,I] \\ x \wedge y &\xmapsto{\varphi} [s(x), s(y)] - s[x,y] \end{aligned}$$

Then it's clear that we want to define

$$\boxed{\begin{aligned} K_1(A) &\longrightarrow HC_1(A) \\ [u] &\longmapsto (u-1)\wedge(u^{-1}-1) \end{aligned}}$$

By [redacted] general nonsense using $HC_1(\text{free}) = 0$, this is well-defined + natural.

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The big mystery is still why non-comm. differential forms enter into cyclic homology. Apparently the reason is related to the fact that extensions (in good cases) are invertible, that is, that one can find another extension so that the sum is split. This is formally analogous to a vector bundle being a direct summand of a trivial bundle.

Example 1. Consider the Toeplitz extension

$$0 \longrightarrow K(H_+) \longrightarrow T_+ \longrightarrow C(S^1) \longrightarrow 0$$

which is "classified" by a homomorphism

$$C(S^1) \xrightarrow{f} Q(H_+) = B(H_+)/K(H_+)$$

If we add this extension to the corresponding H_- extension, then we obtain a split extension

$$\begin{tikzcd} 0 \arrow[r] & K(H) \arrow[r] & K(H)\oplus C(S^1) \arrow[r] & C(S^1) \arrow[l, dashed, bend right, "\phi"'] \arrow[r] & 0 \end{tikzcd}$$

This means that we have a homomorphism

$$\phi : C(S^1) \longrightarrow B(H_+ \oplus H_-)$$

such that $\phi_{11} : C(S^1) \longrightarrow B(H_+)$ is a linear lifting of $C(S^1)$ into T_+.

Somehow there are special features of the lifting $\phi_{11} : f \longmapsto efe$ (e = proj on H_+) which enable one to do higher trace calculations associated with the Toeplitz extension via non-commutative differential forms.

What does this last statement mean?

Let's review some K-theory, specifically ideas starting with the BDF theory of extensions and later extended by Kasparov.

I recall that odd K cohomology is represented by the space of Fredholm operators, or better, the space of invertible [illegible] (or unitary) elements in the Calkin algebra.

$$\begin{tikzcd} K(H) \arrow[r] & \mathrm{Fred}(H) \arrow[r] & \mathcal{Q}(H)^{\times} \\ & & U(H) \arrow[u, "\sim"'] \end{tikzcd}$$

fibration
contractible fibre

It might be better to use the group $GL(\mathcal{Q}(H))$, but one can work with $GL_1(\mathcal{Q}(H))$ by Kuiper's thm.

Given an C^*-alg morphism $A \xrightarrow{\rho} \mathcal{Q}(H)$ it gives rise to map from the [illegible] [illegible] K-theory of A to the K-theory of a point of odd degree. Better: it gives rise to a map from $K_1(A)$ to $K_0(pt) = \mathbb{Z}$. In other words such an algebra map behaves like an odd-dimensional K-homology class on A. BDF showed that when $A = C(X)$, that the isom. classes of such ρ (for conjugation [illegible] in $\mathcal{Q}(H)$) are in 1-1 correspondence with elements of $K_1(X)$. (X separable?)

It turns out that $\rho : A \to \mathcal{Q}(H)$ is equivalent to the extension it defines by pull-back

$$\begin{tikzcd} K(H) \arrow[r] & \mathcal{B}(H) \arrow[r, two heads] & \mathcal{Q}(H) \\ K(H) \arrow[r] \arrow[u, equal] & P \arrow[r] \arrow[u] & A \arrow[u, "\rho"'] \end{tikzcd}$$

In fact ρ is the *Busby invariant* of the extension.

Suppose we now consider an extension

$$K \longrightarrow P \longrightarrow A \qquad\qquad K = K(H)$$

In the case of nuclear C^*-algebras ($C(X)$ is nuclear), it turns out (existence of a completely positive lifting $s : A \to P$ together with the Stinespring theorem) that this extension is invertible is the semi-group of extensions. This means there is an extension

$$K \longrightarrow Q \longrightarrow A$$

such that the sum extension defined by

$$\begin{tikzcd}
K \oplus K \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & P \times_A Q \arrow[r] \arrow[d] & A \arrow[d, phantom, "\parallel"] \\
M_2(K) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & R \arrow[r] & A \\
K & &
\end{tikzcd}$$

is split. (I guess if one wants to use isomorphism classes of $\rho : A \to \mathcal{Q}(H)$, one must use *injective* ρ — these correspond to essential extensions. Otherwise one uses a homotopy equivalence relation.)

So in the nuclear case given $\rho : A \to \mathcal{Q}(H)$ we can dilate it to an alg. map

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} : A \longrightarrow M_2(\mathcal{B}(H)) = \mathcal{B}(H \oplus H)$$

such that $\phi_{11} : A \longrightarrow \mathcal{B}(H)$ is a linear lifting (completely positive in fact) of ρ, and such that $\phi_{21}, \phi_{12} : A \longrightarrow K(H)$. Thus one has refined $\rho : A \rightarrow \mathcal{Q}(H)$ to a homomorphism $\phi : A \longrightarrow \mathcal{B}(H^2)$ together with a projector e on H^2 such that

$$[\phi(a), e] \in K(H^2)$$

$$\rho(a) \equiv e\phi(a) \equiv \phi(a)e \mod K(H^2).$$

I guess the point of all this is that we refine an extension to a situation consisting of $\phi : A \longrightarrow \mathcal{B}(H)$ and an F on $\mathcal{B}(H)$ such that $[F, \phi(a)] \in K(H)$. This is an ungraded [illegible] Fredholm module.

Formally I perhaps can view this refining process as analogous to [illegible] expressing a vector bundle as a direct factor of a trivial bundle. Such a choice defines a connection on the bundle and leads to differential forms.

~~Next consider the graded case. Let's start with the space of nontrivial involutions (or projectors) $\mathcal{I}(\mathcal{Q})$ which we know represents K^1 = odd K-cohomology.~~

~~Let's start again.~~

Next let's consider the case of even K-homology. We want a way to map $K_0 A$ to $\mathbb{Z}$, i.e. a way to go from projectors over A to invertible elements over $\mathcal{Q}$. Probably we want to use a periodicity map which would convert an element of $K_0(A)$ to an element of $K_1(A\otimes C_1)$. These means we have to come to grips with K-theory for a graded algebra such as $A \otimes C_1$.

Let's consider a graded algebra morphism

$$C_1 \longrightarrow \mathcal{Q}(H_+ \oplus H_-) \cong M_2(\mathcal{Q}(H_+))$$

I think I want this to be a unital homomorphism. The generator γ of C_1 becomes an odd involution in $\mathcal{Q}(H_+\oplus H_-)$ i.e. maps $H_+ \overset{P}{\underset{Q}{\rightleftarrows}} H_-$ defined mod compacts which are inverse modulo compacts.

Now it's confusing how to think of a graded morphism

$$A \otimes C_1 \xrightarrow{\rho} \mathcal{Q}(H_+ \oplus H_-)$$

First of all we have two morphisms

$$\rho_\pm : A \longrightarrow \mathcal{Q}(H_\pm)$$

and we have an isomorphism $H_+ \overset{P}{\underset{Q}{\rightleftarrows}} H_-$ mod compacts which intertwines them.

What is the cyclic homology of the superalgebra C_1? First calculate the low-dimensional cases

$$HC_0(C_1) = C_1/[C_1,C_1] = \mathbb{C}+\mathbb{C}\gamma/\mathbb{C}[\gamma,\gamma] = \mathbb{C}\gamma$$

$C_1^{\otimes 2}$ has basis $1\otimes 1,\ \gamma\otimes\gamma,\ 1\otimes\gamma,\ \gamma\otimes 1$

and $t(1\otimes 1) = -1\otimes 1$ $\quad t(1\otimes\gamma) = -\gamma\otimes 1$

$t(\gamma\otimes\gamma) = \gamma\otimes\gamma$ $\quad t(\gamma\otimes 1) = -1\otimes\gamma$

Thus $(1-t)\,C_1^{\otimes 2} = \mathbb{C}(1\otimes 1) + \mathbb{C}(1\otimes\gamma+\gamma\otimes 1)$

and $C_1^{\otimes 2}/(1-t)C_1^{\otimes 2}$ has basis $\gamma\otimes\gamma$, $1\otimes\gamma$

$$b(\gamma\otimes\gamma) = \gamma^2+\gamma^2 = 2, \qquad b(1\otimes\gamma)=0.$$

Thus $\operatorname{Ker}\{b : C_1^{\otimes 2}/(1-t)C_1^{\otimes 2} \longrightarrow C_1\} \simeq \mathbb{C}(1\otimes\gamma)$ is entirely odd. But $b(1\otimes 1\otimes\gamma) = \cancel{1\otimes\gamma - 1\otimes\gamma} + \gamma\otimes 1$ $= -1\otimes\gamma$, so we see

$$\boxed{HC_1(C_1) = 0}.$$

Let's check via Hochschild homology. The category of graded C_1-bimodules is the [redacted] category of graded vector spaces $V = V^+\oplus V^-$ together with [redacted] two odd involutions L, R which commute. Thus if we use L to identify $V^+ = V^-$ so that $L = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ then $R = \begin{pmatrix}0&u^{-1}\\u&0\end{pmatrix}$ in order to commute with L must have $u^2=1$. Thus the category of graded C_1-bimodules is the same as the category of $\mathbb{C}[\mathbb{Z}/2]$-modules, all exact sequences split, and the Hochschild homology is trivial. It follows that the cyclic homology is $\simeq \mathbb{C}\gamma$ in even degrees, and is 0 in odd degrees.

Thus [redacted]

$$HC_n(C_1) = \begin{cases}\mathbb{C}\gamma & n \text{ even}\\ 0 & n \text{ odd}\end{cases}$$

and similar formulas hold for the other Clifford algebras. For example

graded C_2-modules $\sim$ C_3-modules

$\sim$ C_1-modules

so K_0(graded C_2-modules) $= \mathbb{Z}\oplus\mathbb{Z}$. Also

HC[redacted]$_{2n}(C_2) = \mathbb{C}(\gamma^1\gamma^2\dots\gamma^{2n})$

Somehow we are missing the appropriate homotopies. For example the good cyclic theory for C_1 should have the odd groups $= \mathbb{C}$ and the even groups should be trivial.

I conclude that when we move to the graded case not only must K_0 change but also the cyclic theory.

The problem should now be to understand the Kasparov theory in the graded case and to develop an appropriate cyclic theory.

November 19, 1987

We first remark that instead of looking for equivalence classes of traces on [redacted] $I^n/[P, I^n]$, where $I \to P \to A$ is an extension of A, we can [redacted] restrict to P free and use

$$0 \longrightarrow HC_1(P/I^n) \longrightarrow I^n/[P, I^n] \longrightarrow P/[P,P]$$

and the fact that two traces on $I^n/[P, I^n]$ with the same restriction to $HC_1(P/I^n)$ are equivalent. Thus our candidate for the periodic cyclic homology is

$$\varprojlim HC_1(Q)$$

where Q runs over the category of nilpotent extensions of A.

The first thing ~~to~~ check is to see how $HC_3(A)$ compares with the inverse limit of $HC_1(Q)$ taken over the category of square zero extensions. This inverse limit is contained in $HC_1(P/I^2)$ where $P/I = A$ and P is free. [redacted] In fact

$$0 \to \varprojlim_{\substack{Q \text{ square} \\ \text{zero ext}}} HC_1(Q) \longrightarrow HC_1(P/I^2) \rightrightarrows HC_1(P/I^2 \amalg P/I^2)$$

$\amalg$ = direct sum in category of square 0 extensions

If we want to [redacted] work with this as we did for HC_0, we will need to know something about HC_1 for [redacted] semi-direct products such as

$$P/I^2 \amalg P/I^2 = P/I^2 \oplus [redacted] D \quad \text{and}$$

$$P/I^2 \amalg A = A \oplus D.$$

So let's consider a semi-direct product

$Q = A \oplus M$, where M is an A-bimodule. Notice that this is a ($\mathbb{Z}$-)graded algebra, so $HC_*(Q)$ is also graded. We want to compute $HC_1(Q)$ or more precisely the relative HC_1 of the map $Q \to A$.

We have

$$\Lambda^2 Q = \Lambda^2 A \oplus M \otimes A \oplus \Lambda^2 M$$

$$Q^{\otimes 3}/(1-t) = A^{\otimes 3}/(1-t) \oplus (M \otimes A^{\otimes 2}) \oplus M \otimes A \otimes M \oplus M^{\otimes 3}/(1-t)$$

$$b(m \otimes a^1 \otimes a^2) = ma^1 \otimes a^2 - m \otimes a^1 a^2 + a^2 m \otimes a^1$$

$$b(m \otimes a \otimes m') = ma \wedge m' - m \wedge am' \qquad \text{as } m'm = 0$$

$$b(m \otimes a) = ma - am$$

$$b(m \wedge m') = 0$$

So we have the following

$$HC_1(Q) = \cancel{HC_1(A) \oplus H_1(A,M) \oplus}$$

$$= \underbrace{HC_1(Q)_{(0)}}_{HC_1(A)} \oplus \underbrace{HC_1(Q)_{(1)}}_{H_1(A,M)} \oplus HC(Q)_{(2)}$$

where

$$HC_1(Q)_{(2)} = \Lambda^2 M / b(M \otimes A \otimes M)$$

$$= M \otimes M / \text{relations} \quad \begin{aligned} ma \otimes m' &= m \otimes am' \\ m \otimes m' &= -m' \otimes m \end{aligned}$$

Recall that given an A bimodule M, we can form the cyclic tensor product

$$\underbrace{\circlearrowright M \otimes_A M \otimes_A \cdots \otimes_A M \otimes_A}$$

It's clear we can write

$$HC(Q)_{(2)} = \left(M\otimes_A M \otimes_A\right)/(1-t).$$

I should have said that the cyclic tensor product is acted on by the cyclic group, and so can be decomposed according to its characters. So we have established

Prop: If $Q = A\oplus M$ is the semi-direct [redacted] product square zero extension of A with the bimodule M [redacted] [redacted], then

$$HC_0(Q) = HC_0(A) \oplus \underbrace{M/[A,M]}_{H_0(A,M)}$$

$$HC_1(Q) = HC_1(A) \oplus H_1(A,M) \oplus \left(M\otimes_A M\otimes_A\right)/(1-t)$$

Now we want to apply this to the case of the square zero semi-direct product extensions

$$Q \amalg Q = Q\oplus D$$
$$Q \amalg A = A \oplus D$$

So we have to [redacted] describe the map

$$H_1(Q,D) \longrightarrow H_1(A,D).$$

Let's consider this problem in general. Thus suppose $M\to Q\to A$ is a square-zero extension and N is an A-bimodule. Consider the maps of Hochschild cxs.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & N\otimes(Q\otimes M + M\otimes Q) \arrow[r] \arrow[d] & N\otimes M \arrow[r] \arrow[d] & 0 \arrow[d] \\
\arrow[r] & N\otimes Q^{\otimes 2} \arrow[r] \arrow[d] & N\otimes Q \arrow[r] \arrow[d] & N \arrow[d, equal] \\
\arrow[r] & N\otimes A^{\otimes 2} \arrow[r] \arrow[d] & N\otimes A \arrow[r] \arrow[d] & N \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

This will give rise to a long exact sequence in homology. Note that

$$b(n\otimes q\otimes m) = nq\otimes m - n\otimes qm$$
$$b(n\otimes m\otimes q) = -n\otimes mq + qn\otimes m$$

so that $$N\otimes M \Big/ b\begin{pmatrix} N\otimes Q\otimes M \\ +N\otimes M\otimes Q \end{pmatrix} = \underbrace{N\otimes_A M\otimes_A}$$

Thus we get an exact sequence

$$\boxed{H_2(Q,N) \longrightarrow H_2(A,N) \longrightarrow \underbrace{N\otimes_A M_A} \rightarrow H_1(Q,N) \rightarrow H_1(A,N) \rightarrow 0}$$

More generally we have

<u>Prop</u>: If N is a Q/J-bimodule, then

$$H_2(Q,N) \rightarrow H_2(Q/J,N) \longrightarrow \overbrace{J\otimes_Q N\otimes_Q} \longrightarrow H_1(Q,N) \rightarrow H_1(Q/J,N) \rightarrow 0$$

Now let us return to our problem of computing $\varprojlim HC_1(Q)$ over the category of square zero extensions of A. Fix $P/I = A$ with P free and set $Q = P/I^2$, $M = I/I^2$ and

$$D = A^+\otimes_{Q^+}\Omega^1_Q\otimes_{Q^+}A^+ = A^+\otimes_{P^+}\Omega^1_P\otimes_{P^+}A^+$$

so that D is a free A-bimodule and

$$0 \longrightarrow M \longrightarrow D \longrightarrow \Omega^1_A \longrightarrow 0$$

is an exact sequence of A-bimodules.

Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & H_2(A) \arrow[r] \arrow[d] & HC_2(A) \arrow[d] & \\
HC_1(Q) \arrow[r] & HC_1(A) \arrow[r] & M/[A,M] \arrow[r] \arrow[d] & Q/[Q,Q] \arrow[r] \arrow[d] & A/[A,A] \arrow[r] & 0 \\
 & & D/[A,D] \arrow[r, shift left] \arrow[r, shift right] & D/[A,D] & &
\end{tikzcd}$$

From this diagram we get an exact sequence

$$HC_1(Q) \longrightarrow HC_1(A) \longrightarrow H_2(A) \longrightarrow HC_2(A) \longrightarrow HC_0(A) \longrightarrow 0$$

which suggests that $HC_1(Q)$ is closely linked to $HC_3(A)$. Now I believe Connes constructs a map $HC_3(A) \longrightarrow HC_1(Q)$, which confirms the suggestion.

I want the inverse limit of HC_1 over the category of square zero extensions of A. Since $Q = P/I^2$, this is the same as

$$\operatorname{Ker}\left\{ HC_1(Q) \rightrightarrows HC_1(\underbrace{Q \sqcup Q}_{Q \oplus D}) \right\}$$

We know $HC_1(Q \oplus D) = HC_1(Q) \oplus H_1(Q,D) \oplus (D \otimes_A D \otimes_A)/(1-t)$

This kernel maps to

$$\operatorname{Ker}\left\{ HC_1(Q) \rightrightarrows HC_1(\underbrace{Q \sqcup A}_{A \oplus D}) \right\}$$

and $$HC_1(A \oplus D) = HC_1(A) \oplus H_1(A,D) \oplus (D \otimes_A D \otimes_A)/(1-t)$$

We have an exact sequence

$$H_2(A,D) \longrightarrow \overbrace{D \otimes_A M \otimes_A} \longrightarrow H_1(Q,D) \longrightarrow H_1(A,D) \rightarrow 0$$

In the present situation D is a free A-bimodule so that $$\overbrace{D \otimes_A M \otimes_A} = H_1(Q,D).$$

Thus it appears that we have two

canonical maps

$$HC_1(Q) \longrightarrow (D\otimes_A \otimes D\otimes_A)/(1-t)$$

$$HC_1(Q) \longrightarrow (D\otimes_A M\otimes_A)$$

to be understood, whose kernel gives us the inverse limit.

It is perhaps a good idea to understand the deformation theory of $HC_*(A)$ to the first order at least. For example given a derivation on A how does it act on $HC_*(A)$? This is in Goodwillie's paper.

~~Better approach.~~

Analysis of a deformation. Let $\delta : A \to A$ be a derivation, whence one gets a formal 1-param. group of automorphisms $e^{t\delta}$. By making base change to $\mathbb{C}[[t]]$, one should get a formal 1-param. group of automorphisms on $HC_*(A)$. One wants to understand the various coefficients of powers of t, in particular the first order term. This means one works over the ring $\mathbb{C}[t]/(t^2)$ of dual numbers. One has a semi-direct product extension

$$0 \longrightarrow tA \longrightarrow \overbrace{A\otimes \mathbb{C}[t]/(t^2)}^{A+tA} \longrightarrow A \longrightarrow 0$$

and one has two liftings of A into the extension, namely $a \longmapsto a$ and $a \longmapsto a + t\delta a$

Now
$$HC_1(A+tA) = HC_1(A) \oplus H_1(A,tA) \oplus (tA\otimes_A tA \otimes_A)/1-\tau$$
(τ = cyclic perm. with sign.). The latter term is 0

$$\overbrace{A\otimes_A A\otimes_A}^{\text{because}} = A\otimes_A = A/[A,A]$$

$$x\otimes y \longmapsto xy$$

(Check $ax\otimes y - x\otimes ya \longmapsto axy - xya = 0$). Note that the cyclic group acts trivially on this, since $xy = yx$ in $A/[A,A]$, hence τ acts as -1, so $1-\tau$ acts as 2.) So

$$\boxed{HC_1(A+tA) = HC_1(A) \oplus t\, H_1(A,A)}$$

In this formula t keeps track of the grading, and could be omitted.

Let's consider $HC_k(A\oplus M)$ where M is a bimodule and $M^2=0$. $A\oplus M$ is $\mathbb{Z}$-graded, hence so is the cyclic complex $C_\cdot(A\oplus M)$. We have

$$(A\oplus M)^{\otimes k} = \underbrace{A^{\otimes k}}_{\deg 0} \oplus \underbrace{\bigoplus_{i=1}^{k} A^{\otimes(i-1)}\otimes M\otimes A^{\otimes(k-i)}}_{\deg 1} \oplus \cdots \oplus \underbrace{M^{\otimes k}}_{\deg k}$$

and similarly for $C_{k-1}(A\oplus M) = (A\oplus M)^{\otimes k}/(1-\tau)$

Thus we have a grading

$$HC_k(A\oplus M) = HC_k(A) \oplus \underbrace{\quad}_{\deg 1} \oplus \cdots\cdots \oplus \underbrace{\quad}_{\deg(k+1)}$$

Let's compute deg 1 and deg$(k+1)$ pieces.

$$C_k(A\oplus M)_{(1)} = M\otimes A^{\otimes(k)}$$

$$b(m\otimes a_1\otimes\cdots\otimes a_k) = ma_1\otimes\cdots\otimes a_k + \sum_1^{k-1}(-1)^i m\otimes\cdots a_ia_{i+1}\cdots\otimes a_k$$
$$+ (-1)^k a_1 m\otimes\cdots a_k$$

Thus the degree 1 part is the Hochschild homology $H_k(A, M)$. For the degree $(k+1)$ pieces, we are looking for the quotient of $M^{\otimes(k+1)}$ by the relations

$$(m_0, \dots, m_k) \equiv (-1)^k (m_k, m_0, \dots, m_{k-1})$$

and

$$\begin{aligned} 0 \equiv b(a, m_0, \dots, m_k) = & (am_0, m_1, \dots, m_k) \\ & + (-1)^{k+1}(m_k a, m_0, \dots, m_{k-1}) \end{aligned}$$

Putting these together gives

$$(am_0, m_1, \dots, m_k) - (m_0, m_1, \dots, m_k a) \equiv 0$$

$$(m_k, am_0, m_1, \dots, m_{k-1}) - (m_k a, m_0, \dots, m_{k-1}) \equiv 0$$

$$(m_{k-1}, m_k, am_0, m_1, \dots) - (m_{k-1}, m_k a, m_0, \dots, m_{k-2}) \equiv 0$$

Therefore

$$\boxed{\begin{aligned} HC_k(A\oplus M)_{(k+1)} &= \Big(\overbrace{M\otimes_A M\otimes_A \cdots \otimes_A M\otimes_A}^{(k+1)\text{ times}}\Big)\Big/(1-\tau) \\ HC_k(A\oplus M)_{(1)} &= H_k(A, M) \end{aligned}}$$

Let us consider automorphisms of the algebra $A\oplus M$. First of all there is the multiplicative group $\mathbb{C}^\times$ acting trivially on A and by multiplication on M. Secondly we have the group of automorphism of the extension $0 \to M \to A\oplus M \to A \to 0$ inducing the identity on M and on A. This group can be identified with $\mathrm{Der}(A, M)$. So we can form the semi-direct product

$$\mathbb{C}^\times \ltimes \mathrm{Der}(A, M).$$

Now consider the induced action on $HC_k(A\oplus M)$.

[redacted] It might be better to view the Lie algebra semi-direct product as acting in $HC_k(A\oplus M)$. Then to each $\delta \in Der(A,M)$ we have an operator on $HC_k(A\oplus M)$ raising the grading degree by 1. The operator is obtained by extending δ to a derivation on $A\oplus M$ which is 0 on M, and then extending in the obvious way to the cyclic complex. The various operators $\tilde{\delta}$ on $HC_k(A\oplus M)$ commute, so that $HC_k(A\oplus M)$ becomes a graded module over $S[Der(A,M)]$.

Now take the [redacted] universal case, i.e. where $M =$ [redacted] Ω^1_A, and $A\oplus\Omega^1_A = A \amalg A$ in the category of square zero extensions. We then have a canonical derivation $d: A \longrightarrow \Omega^1_A$ [redacted] which then gives [redacted] us maps

$$\begin{tikzcd} HC_k(Q)_{(0)} \arrow[r, "\rho(d)"] \arrow[d, equal] & HC_k(Q)_{(1)} \arrow[r, "\rho(d)"] \arrow[d, equal] & HC_k(Q)_{(2)} \arrow[r, dashed] & HC_k(Q)_{(k+1)} \\ HC_k(A) \arrow[r] & H_k(A,\Omega^1_A) & & \end{tikzcd}$$

[redacted]

and the effect of the homomorphism $A \xrightarrow{1+td} A\oplus\Omega^1_A$ is

$$1 + t\rho(d) + \frac{t^2}{2!}\rho(d)^2 + \cdots\cdots$$

Hence an element $\xi \in HC_k(A)$ is equalized by the maps $HC_k(A) \rightrightarrows HC_k(A\oplus\Omega^1_A)$ induced by $A \underset{1+td}{\overset{1}{\rightrightarrows}} \Omega^1_A$ iff [redacted] ξ is killed by

$$\rho(d): HC_k(A) \longrightarrow H_k(A,\Omega^1_A) = H_{k+1}(A,A^+) = H_{k+1}(A)$$

It seems we have found a new proof of Goodwillie's thm. that derivations act trivially on $Im\{S: HC_*(A) \to HC_*(A)\} = H^{DR}_*(A)$.

It seems we have proved

Proposition: Let $\delta : A \longrightarrow M$ be a derivation. Then an elt $\xi \in HC_k(A)$ is equalized by the two maps $HC_k(A) \rightrightarrows HC_k(A \oplus M)$ induced by $A \overset{1}{\underset{1+\delta}{\rightrightarrows}} A \oplus M$ iff $\xi \in \mathrm{Ker}\{\rho(\delta) : HC_k(A) \longrightarrow H_k(A,M)\}$.

Now we want to apply this to the case of $Q = P/I^2$ and $Q \amalg Q = Q \oplus D$. This proposition tells us that

$$\mathrm{Ker}\{HC_1(Q) \rightrightarrows HC_1(Q \amalg Q)\} = \mathrm{Ker}\{HC_1(Q) \xrightarrow{\rho(\delta)} H_1(Q,D)\}$$

where $\delta : Q \longrightarrow D$ is the universal derivation with values in A-bimodules.

November 20, 1987

Consider an extension $0 \to I \to P \to A \to 0$. Let's view $0 \to I \to P$ as a DG algebra and form its cyclic complex which is a double complex:

$$C_1(I \to P) = I \to P$$

$$(I \to P)^{\otimes 2} = I^{\otimes 2} \longrightarrow I\otimes P + P\otimes I \longrightarrow P\otimes P$$

$$(I \to P)^{\otimes 3} = I^{\otimes 3} \longrightarrow I^{\otimes 2}\otimes P \oplus I\otimes P\otimes I \oplus P\otimes I^{\otimes 2} \longrightarrow I\otimes P^{\otimes 2} \oplus P\otimes I\otimes P \oplus P^{\otimes 2}\otimes I \longrightarrow P^{\otimes 3}$$

Double complex is

$$\begin{tikzcd}
0 \arrow[r] & I^{\otimes 3}/(1-\sigma) \arrow[r] & P\otimes I^{\otimes 2} \arrow[r] \arrow[d] & P^2\otimes I \arrow[r] \arrow[d] & P^{\otimes 3}/(1-t) \arrow[d] & A^{\otimes 3}/(1-t) \\
 & & S^2 I \arrow[r] & P\otimes I \arrow[r] \arrow[d] & \Lambda^2 P \arrow[d] & \Lambda^2 A \\
 & & & I \arrow[r] & P & A
\end{tikzcd}$$

This double complex gives a spectral sequence

$$E^1 = HC_g(P \oplus I)_{(p)} \Longrightarrow HC(A)$$

where $P \oplus I$ is considered as a graded algebra and the cyclic homology is taken in the super sense with I of odd degree.

Let's begin by using first quadrant pictures. The above double complex appears

$$\begin{tikzcd}
A^{\otimes 3}/(1-t) \arrow[d] & P^{\otimes 3}/(1-t) \arrow[l] \arrow[d] & P^{\otimes 2}\otimes I \arrow[l] \arrow[d] & P\otimes I^{\otimes 2} \arrow[l] \arrow[d] & I^{\otimes 3}/(1-\sigma) \arrow[l] \\
\Lambda^2 A \arrow[d] & \Lambda^2 P \arrow[l] \arrow[d] & P\otimes I \arrow[l] \arrow[d] & S^2 I \arrow[l] & \\
A & P \arrow[l] & I \arrow[l] & &
\end{tikzcd}$$

This gives a spectral sequence with E^1 term

$$\begin{tikzcd}
HC_2(A) & HC_2(P) & H_2(P,I) \arrow[l] & * & * \\
HC_1(A) & HC_1(P) & H_1(P,I) \arrow[l] & S^2I/b(P\otimes I^{\otimes 2}) \arrow[l] \arrow[ull, dotted, "d_2"'] & 0 \\
HC_0(A) & HC_0(P) & H_0(P,I) \arrow[l] & 0 & 0
\end{tikzcd}$$

from which we deduce an exact sequence

$$\begin{tikzcd}
0 & HC_0(A) \arrow[l] & HC_0(P) \arrow[l] & H_0(P,I) \arrow[l] \\
& HC_1(A) \arrow[urr] & HC_1(P) \arrow[l] & H_1(P,I)/\operatorname{Im} S^2I \arrow[l] \\
& HC_2(A) \arrow[urr] & &
\end{tikzcd}$$

Special cases.

Suppose P free. Then the spec. sequence looks as follows:

$$\begin{array}{c|ccccc}
 & 0 & 0 & & & \\
 & 0 & 0 & * & * & 0 \\
 & 0 & H_1(P,I) & S^2I/b(P\otimes I^{\otimes 2}) & 0 & 0 \\
 & HC_0(P) & H_0(P,I) & 0 & 0 & 0 \\
\hline
\end{array}$$

and we conclude

$$\boxed{\underset{P\ \text{free}}{HC_2(P/I)} = H_1(P,I)/\mathrm{Im}(S^2 I)}$$

$$\boxed{\underset{P\ \text{free}}{HC_3(P/I)} = \mathrm{Ker}\left\{S^2 I/b(P\otimes I^{\otimes 2}) \longrightarrow H_1(P,I)\right\}}$$

Notice the map $P\otimes I^{\otimes 2} \longrightarrow S^2 I$ in the double complex sends $x\otimes y\otimes z \longmapsto xy\otimes z + \overset{y\otimes zx}{\overset{\parallel}{zx\otimes y}}$ since I is a square zero ideal in $P\oplus I$.

Let's be more precese. Let $u\in P$; $x, y\in I$.

Then $$b(u\otimes x\otimes y) = ux\otimes y - u\otimes(\overset{0}{\cancel{xy}}) \pm yu\otimes x$$

where the sign should normally be +, except that in our case y, x are considered of odd degree.

Thus $$\begin{aligned} b(u\otimes x\otimes y) &= ux\otimes y - yu\otimes x \\ &= ux\otimes y - x\otimes yu \\ \text{also} &= y\otimes ux - yu\otimes x \end{aligned}$$

Thus $$\boxed{S^2 I/b(P\otimes I^{\otimes 2}) = (I\otimes_P I\otimes_P)/1-\sigma}$$

~~[illegible]~~

Next consider a square zero split extension $P = A\oplus I$, $I^2 = 0$.

$$\begin{tikzcd} {} \arrow[r] & H_2(A,I) \arrow[r] \arrow[l, dashed, bend right] & (I\otimes_A I\otimes_A) \arrow[r] & H_1(P,I) \arrow[r] & H_1(A,I) \arrow[r] \arrow[l, dashed, bend right] & 0 \end{tikzcd}$$

$$\therefore \quad H_1(P,I) = H_1(A,I) \oplus (I\otimes_A I\otimes_A)$$

Clearly $(I\otimes_A I\otimes_A)/(1-\sigma)$ goes in the second summand so one gets

$$HC_1(A\oplus I) = HC_1(A) \oplus H_1(A,I) \oplus (I\otimes_A I\otimes_A)/(1-t)$$

Edge homomorphisms for the spectral sequence of an extension. First let us calculate the $E_1^{n,n-1}$ term. This is

$$\mathrm{Coker}\left\{P\otimes I^{\otimes n} \xrightarrow{\quad b \quad} I^{\otimes(n)}/(1-\sigma)\right\}$$

But actually we can think of it as first taking the cokernel in the Hochschild complex

$$\bigoplus_{i=0}^{n} I^{\otimes i}\otimes P\otimes I^{\otimes(n-i)} \xrightarrow{\quad b \quad} I^{\otimes(n)}$$

and then dividing out by by the action of $\mathbb{Z}/n$. Recall that I has zero multiplication, so that

~~$b(x_0,\dots,x_{i-1},p,x_i,\dots,x_n) = (-1)^{i-1}(x_0,\dots,x_{i-1}p,x_i,\dots,x_n)$~~
~~$+(-1)^i(x_0,\dots,x_{i-1},px_i,\dots,x_n)$~~
~~for~~ [illegible] ~~$1\le i<n$. For $i=0$ we have $b(p,x$~~

$$b(x_1,\dots,x_i,p,x_{i+1},\dots,x_n) = (-1)^{i-1}(x_1,\dots,x_ip,\dots x_n)$$
$$+(-1)^i(x_1,\dots,x_i,px_{i+1},\dots,x_n)$$

for $1\le i\le n-1$. For $i=0,n$

$$b(p,x_1,\dots,x_n) = (px_1,x_2,\dots,x_n) + (-1)^{\overset{\text{normal sign}\ \downarrow\quad\ \swarrow\ \text{because } p \text{ even}, x_i \text{ odd}}{n+(n-1)}}(x_np,x_1,\dots,x_{n-1})$$

$$b(x_1,\dots,x_n,p) = (-1)^{n-1}(x_1,\dots,x_{n-1},x_np) + (-1)^n(px_1,x_2,\dots,x_n)$$

Dividing out by the images for $i=1,\dots,n$ gives us

$$\overbrace{I\otimes_P I\otimes_P\cdots\otimes_P I\otimes_P}^{n\text{-factors}}$$

then the $i=0$ relation modulo cyclic permutations (no signs as the normal sign is cancelled by the fact I is odd) is taken care of. $\therefore$ We have

Proposition: $$\boxed{E_1^{n,n-1} = \overbrace{I\otimes_P \cdots \otimes_P I\otimes_P}^{n\text{-times}}/1-\sigma}$$

and hence there is a canonical edge homom.

$$HC_{2n-1}(A) \longrightarrow (\overbrace{I\otimes_P \cdots I\otimes_P}^{n\text{-times}})/(1-\sigma)$$

Application: Suppose we have a trace

$$\tau : I^n/[P,I^n] \longrightarrow \mathbb{C}$$

Then we get an induced map

$$HC_{2n-1}(A) \longrightarrow (I\otimes_P \overset{n\text{-times}}{\cdots})/(1-\sigma) \longrightarrow I^n/\overset{\cancel{[P,I^n]}}{[I,I^{n-1}]} \longrightarrow \mathbb{C}$$

Better we get a canonical homomorphism

$$\boxed{HC_{2n-1}(A) \longrightarrow I^n/[I,I^{n-1}]}$$

which must be the one constructed by Connes.

Question: Is the above map injective when P is free? When P is free we have seen that (see bottom p 333) that E_1 has at most one non-zero group in each degree ≤ 4, so it's natural to ask if this generalizes.

There is a problem in that the map

$$(I\otimes_P \cdots I\otimes_P)/(1-\sigma) \longrightarrow I^n/[I,I^{n-1}]$$

off hand doesn't look injective once $n>1$.

In any case we have some more control over $HC_3(A)$, $HC_4(A)$ and so we should persist with the investigation.

[redacted]

Let A be a unital algebra, and let X be a right A-module and Y a left A-module. We have a canonical isomorphism

$$X \otimes_A Y = A \otimes_{(A\otimes A^{op})} (Y \otimes X)$$

In effect define

$$\Phi : x \otimes y \longmapsto 1 \otimes (y \otimes x)$$

$$\Psi : a \otimes (y \otimes x) \longmapsto x \otimes ay = xa \otimes y.$$

Φ is well-defined because

$$\Phi(xa \otimes y) = 1 \otimes (y \otimes xa) = 1 \otimes (y\otimes x)\cdot(1\otimes a)$$
$$= a1 \otimes (y \otimes x)$$

$$\Phi(x \otimes ay) = 1 \otimes (ay \otimes x) = 1 \otimes (a\otimes 1)\cdot(y \otimes x)$$
$$= 1\cdot a \otimes (y \otimes x.$$

Ψ is well defined because

$$\Psi(a \otimes (by \otimes xc)) = xca \otimes by = xcab \otimes y$$
$$= \Phi(cab \otimes (y \otimes x)).$$

The fact Φ + Ψ are inverses is clear.

Now let $P_\cdot \longrightarrow X$, $Q_\cdot \longrightarrow Y$ be free resolutions of right + left A-modules respectively. Then

$$\mathrm{Tor}^A_*(X,Y) = H_*(P \otimes_A Q)$$
$$= H_*(A \otimes_{(A\otimes A^{op})} (Q_\cdot \otimes P_\cdot))$$

$$\boxed{\mathrm{Tor}^A_*(X,Y) = \mathrm{Tor}^{A\otimes A^{op}}_*(A, Y\otimes X) = H_*(A, Y \otimes X)}$$

because $Q_\circ \otimes P_\circ$ is a resolution of $Y \otimes X$ by free $A \otimes A^{op}$ modules.

Now if A is free, i.e. $T(V)^+ = \bigoplus_{n \geq 0} V^{\otimes n}$ then we know that Hochschild homology vanishes in degrees ≥ 2. $\therefore$

$$\boxed{\mathrm{Tor}_n^A(X,Y) = 0 \qquad n \geq 2 \qquad A \text{ free}}$$

Similarly if X, Y are left A-modules we can consider $\mathrm{Hom}_A(X,Y) \subset \mathrm{Hom}_{\mathbb{C}}(X,Y)$. The latter is an $A \otimes A^{op}$-module with

$$\big((a \otimes b) f\big)(x) = a f(bx)$$

Claim

$$\mathrm{Hom}_A(X,Y) = \mathrm{Hom}_{A \otimes A^{op}}\big(A, \mathrm{Hom}(X,Y)\big)$$

$$f \longmapsto \big(a \mapsto (x \mapsto af(x) = f(ax))\big)$$

In effect suppose given $g(a)(x)$ in the right side so that $g(bac)(x) = \big((b \otimes c) g(a)\big)(x) = b g(a)(cx)$. Then

$$g(1)(cx) = g(c)(x) = c g(1)(x)$$

so $g(1) \in \mathrm{Hom}_A(X,Y)$. The rest is clear.

More generally.

$$\mathrm{Hom}_A(M \otimes_A X, Y) = \mathrm{Hom}_{A \otimes A^{op}}\big(M, \mathrm{Hom}(X,Y)\big)$$

This shows that if X is A-flat and if Y is A-injective, then $\mathrm{Hom}(X,Y)$ is injective over $A \otimes A^{op}$. Thus we can argue using resolutions that

$$\boxed{\mathrm{Ext}_A^n(X,Y) = \mathrm{Ext}^n_{A \otimes A^{op}}\big(A, \mathrm{Hom}(X,Y)\big) = H^n(A, \mathrm{Hom}(X,Y))}$$

In particular if A is free we know $\mathrm{Ext}^n(X,Y)$ vanishes for $n \geq 2$, which implies that submodules of projective A-modules are projective: $0 \to X \to \underset{\text{proj}}{P} \to Q \to 0$

$$\underset{\overset{\|}{0}}{\cancel{\mathrm{Ext}_A^1(P,Y)}} \longrightarrow \mathrm{Ext}_A^1(X,Y) \longrightarrow \underset{\overset{\|}{0}}{\cancel{\mathrm{Ext}_A^2(Q,Y)}}$$

So $\mathrm{Ext}_A^1(X,Y)=0$ for all Y which implies X is projective.

Let's now consider two ideals I, J in a unital algebra A. Take

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

and apply $A/I \otimes_A$ to get

~~$0 \to \mathrm{Tor}_1^A(A/I, J) \to A/I \to A/I \otimes_A A/J \to 0$~~

$$0 \to \mathrm{Tor}_1^A(A/I, A/J) \longrightarrow \underset{\underset{J/IJ}{\|}}{A/I \otimes_A J} \longrightarrow A/I$$

so $\boxed{\mathrm{Tor}_1^A(A/I, A/J) = I \cap J/IJ}$

But I rather ~~tensor with the~~ apply $I \otimes_A$ to get

$$0 \longrightarrow \mathrm{Tor}_1^A(I, A/J) \to I \otimes_A J \longrightarrow I \longrightarrow I/JI \to 0$$

or $\boxed{\mathrm{Tor}_1^A(I, A/J) = \mathrm{Ker}\{I \otimes_A J \longrightarrow IJ\}}$

But from $0 \to I \to A \to A/I \to 0$ we have

$$0 \to \mathrm{Tor}_2^A(A/I, M) \xrightarrow{\sim} \mathrm{Tor}_1^A(I, M) \to 0$$

so $\boxed{\mathrm{Tor}_2^A(A/I, A/J) = \mathrm{Ker}\{I \otimes_A J \longrightarrow IJ\}}$

In the above I can be a right ideal + J a left ideal