October 6, 1987

Program: I have been considering various maps $T^*(S^1) \longrightarrow \mathbb{P}^1$ which represent the canonical K-class. I now want to quantize, i.e. go in the direction of non-commutative algebras. We have Connes tangent groupoid and its convolution algebra:

$$\begin{pmatrix}\text{fns of } T^* \\ \text{vanishing} \\ \text{at } \infty\end{pmatrix} \xleftarrow{\ h=0\ } \begin{pmatrix}\text{conv. alg.} \\ \text{of tgt.} \\ \text{groupoid}\end{pmatrix} \xrightarrow{\ h\neq 0\ } \begin{pmatrix}\text{alg of} \\ \text{smooth} \\ \text{kernels}\end{pmatrix}$$

which is a deformation of the functions on T^*. The problem is to construct K-classes explicitly over the convolution algebra which deform the canonical class.

A first problem is how to deal with K-classes over rings without 1. One adjoins a unit and takes finitely generated projective modules, or idempotent matrices over $\tilde{A}$. Equivalently one can consider involutions over $\tilde{A}$. One can suppose that the involution is standard. Thus modulo the augmentation ideal is standard. Thus one has an involution F over $\tilde{A}$ which agrees with a standard ε modulo ~~the~~ the augmentation ideal. Then one may also formulate things using the unitary $g = -F\varepsilon$ inverted by ε; this has to be $\equiv -1$ modulo the augmentation ideal.

Example: Let's consider the Bott class on the plane $\mathbb{R}^2$. We can represent this by the map

$$\mathbb{R}^2 \longrightarrow \mathbb{P}^1 = Gr_1(\mathbb{C}^2)$$

$$x, y \longmapsto x+iy$$

e corresponding map to unitaries inverted by ε

sends x, y to the C.T. of $\begin{pmatrix} 0 & -x+iy \\ x+iy & 0 \end{pmatrix}$:

$$g = \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 = \begin{pmatrix} 1-|z|^2 & -2\bar{z} \\ 2z & 1-|z|^2 \end{pmatrix} \begin{pmatrix} \frac{1}{1+|z|^2} & 0 \\ 0 & \frac{1}{1+|z|^2} \end{pmatrix}$$

Then $\frac{(g+1)^2}{4g} = \frac{1}{1-X^2} = \frac{1}{1+|z|^2}$. This vanishes at ∞, but is not in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, hence we don't have a projector over $\widetilde{\mathcal{S}(\mathbb{R}^2)}$, but we do have one over the space of continuous fns. vanishing at ∞.

We can obtain a projector over $\widetilde{\mathcal{S}(\mathbb{R}^2)}$, in fact over $C_0^\infty(\mathbb{R}^2)$ by using a modified map such as

$$(x,y) \longmapsto \frac{x+iy}{r(x,y)}$$

where $r(0) \neq 0$ and $r \in C_0^\infty(\mathbb{R}^2)$.

Next I'd like to quantize the above example in the following sense. I can deform $\mathcal{S}(\mathbb{R}^2)$ into the smooth Weyl algebra, and the K-theory doesn't change it seems. It should happen that the Bott class on $\mathcal{S}(\mathbb{R}^2)$ corresponds to the basic irreducible representation of the Weyl algebra. I would like to do this explicitly, exhibiting a projector (or unitary inverted by ε) depending on h.

How do we describe the smooth Weyl algebra? This depends on $V = \mathbb{R}^2$ with its symplectic structure. Either we use an explicit polarization and write elements as $K(x,p)$, where $K \in \mathcal{S}(\mathbb{R}^2)$ and $p = \frac{h}{i}\partial_x$, or we use the Weyl calculus which is symplectically invariant. This means we write elements of the algebra as

$$\int f(v)\, T_v \, dv \qquad\qquad f \in \mathcal{S}(V)$$

where T_v are translation operators satisfying the Weyl form of the CCR. Then composition of the above operators leads to a convolution production on functions

$$(f * g)(v) = \int_{v'+v''=v} f(v')\, g(v'')\, e^{iQ(v',v'')}\, dv'$$

Using the F.T. on functions, we can rewrite this as

$$(\tilde{f} * \tilde{g})(x) = \left[e^{i\tilde{Q}(\partial_{x'},\partial_{x''})}\, \tilde{f}(x')\,\tilde{g}(x'')\right]_{x'=x''=x}$$

Thus it is possible to explicitly give the deformed product on $\mathcal{S}(V)$ in the form of taking the external product $\tilde{f}(x')\tilde{g}(x'')$, applying a Gaussian operator (whose quadratic form is something over $V \times V$ obtained from the cocycle, i.e. bilinear form which has the symplectic form for its skewsymmetrization.), and then restricting to the diagonal.

Finally $\tilde{Q}$ should have h as a factor, so that if $h=0$ we get the usual product.

Instead of getting bogged down in formulas for the Weyl algebra product, it's probably better to consider the next step, which is how to describe the desired projector. There are two ideas:

1) For $h \neq 0$ we have the Heisenberg representation of the Weyl algebra. This should be the projective module which is the image of the desired projector up to isomorphism. If I pick a ground state for some oscillator Hamiltonian, (this depends on a choice of pos. quadratic form on V), then the projector P on this ground state is an idempotent in the smooth Weyl algebra, which gives the projective module (probably). (The reason it lies in the smooth Weyl algebra is

$$\mathrm{tr}\left(|0\rangle\langle 0| \cdot T_\gamma\right) = \langle 0 | \underbrace{e^{\gamma a^* - \bar{\gamma} a}}_{T_\gamma} | 0 \rangle$$

$$= e^{-\frac{1}{2}|\gamma|^2}$$

so $|0\rangle\langle 0|$ should be the Weyl transform of the Schwartz function $e^{-\frac{1}{2}|\gamma|^2}$.)

However to get something which specializes as $h \to 0$ we probably need a 2×2 idempotent matrix.

2) Cayley transform. Here the idea is to proceed by analogy with the example $\mathbb{R}^2 \longrightarrow \mathbb{P}^1$, $(x, y) \longmapsto x + iy$. This means we consider the unbounded skew-adjoint operator

$$X = \begin{pmatrix} 0 & -a^* \\ a & 0 \end{pmatrix}$$

and take its Cayley transform. Here a will be a constant ~~times~~ the annihilator operator for a quadratic form on V.

Idea: Suppose we consider an involution A_0 in the $h=0$ algebra. Then we can extend it to ~~an operator in the $h \neq 0$~~ a family $A = A(h)$ such that $A(h)^* = A(h)$. Suppose $A(h) = A_0 + hA_1 + \cdots$. Now take the phase $\dfrac{A}{|A|}$ where $|A| = \sqrt{A^2}$. This will be an involution if it is defined. Formally

$$A^2 = (A_0 + hA_1 + \cdots)^2 = 1 + h(A_0A_1 + A_1A_0) + \cdots$$

$$|A| = 1 + \frac{h}{2}(A_0A_1 + A_1A_0) + \cdots$$

Actually we should be using $*_h$ product so that $\quad A_0 * A_0 = A_0^2 + h(?)$

Thus analytically the key point is whether we can do polar decomposition. Because of Cauchy's formula

$$|A|^s = \frac{1}{2\pi i}\int \frac{\lambda^{s/2}}{\lambda - A^2}\, d\lambda$$

it may be enough to prove the existence of $\dfrac{1}{\lambda - A^2}$

October 7, 1987

Consider the circle again. I want to construct a deformation of the algebra of functions on $T^*(S^1) = S^1 \times \mathbb{R}$. Denote such a function by $f(x,p)$, where $x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $p \in \mathbb{R}$. The deformed algebra structure is obtained by interpreting p as $\frac{h}{i}\partial_x$.

Let's think of functions on S^1 as having the basis e^{inx}, $n \in \mathbb{Z}$. Then we want

$$e^{-inx}\, p\, e^{inx} = p + nh$$

and so the algebra structure is determined by the rule

$$f(p)\, e^{inx} = e^{inx} f(p+nh)$$

The algebra we are dealing with is the crossed-product of the algebra of functions of p with the integers, where the integers act by translation through multiples of h.

So far we haven't specified the type of functions being considered, but there is an obvious smooth algebra consisting of $f(x,p)$ which are smooth in x, p and Schwartz in p.

Now let us consider our basic K-class on $S^1 \times \mathbb{R}$. We take Bott representative which involves using the graph of e^{ix}. This will give a 2×2 matrix F of functions on $S^1 \times \mathbb{R}$ which is an involution. The goal will be to construct a deformation $F(h)$ of F which is an involution with the non-commutative algebra structure.

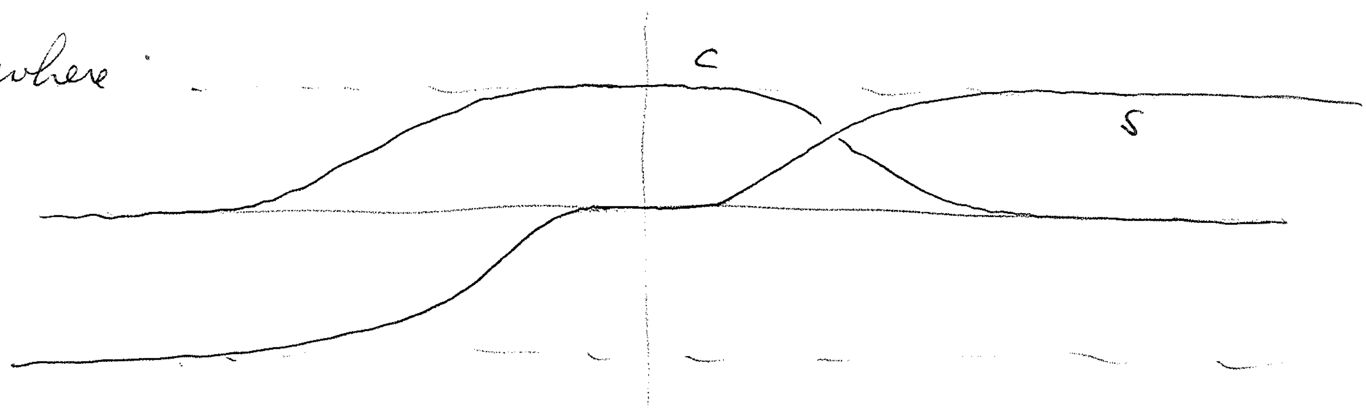
The first project must be to find the involution F. Let's begin by recalling the formula for the great circle $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$. This assigns to $\xi \in \mathbb{R}$ the Cayley transform of $\begin{pmatrix} 0 & -\xi \\ \xi & 0 \end{pmatrix}$ which is

$$g = \begin{pmatrix} \frac{1}{\sqrt{1+\xi^2}} & \frac{-\xi}{\sqrt{1+\xi^2}} \\ \frac{\xi}{\sqrt{1+\xi^2}} & \frac{1}{\sqrt{1+\xi^2}} \end{pmatrix}^2 = \begin{pmatrix} \frac{1-\xi^2}{1+\xi^2} & \frac{-2\xi}{1+\xi^2} \\ \frac{2\xi}{1+\xi^2} & \frac{1-\xi^2}{1+\xi^2} \end{pmatrix}$$

In general we want to use a smoothed version of $g^{1/2}$. Let us replace

$$\begin{pmatrix} \frac{1}{\sqrt{1+\xi^2}} \\ \frac{\xi}{\sqrt{1+\xi^2}} \end{pmatrix} \longmapsto \begin{pmatrix} c(\xi) \\ s(\xi) \end{pmatrix}$$

where



Make a choice and then the involution we want is

$$F(x,\xi) = \begin{cases} \Big(c(\xi) + \overbrace{\left(\tfrac{1}{i}\gamma^2\right)}^{J} s(\xi)\Big)^2 \varepsilon & \xi < 0 \\ \begin{pmatrix} 1 & 0 \\ 0 & e^{ix} \end{pmatrix} \Big(\quad \text{''} \quad \Big) \begin{pmatrix} 1 & 0 \\ 0 & e^{-ix} \end{pmatrix} & \xi > 0 \end{cases}$$

The important thing I guess is that one has a fixed path $F(\xi) = \big(c(\xi) + Js(\xi)\big)^2 \varepsilon$

which goes from $-\infty$ to 0 [redacted] and then from 0 to ∞, and that the loop g is used to conjugate the second part.

So now we have the formula for F and the question is whether we can [redacted] find the desired deformation. Now $F(p)$ is an involution as well as

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{ix} \end{pmatrix} F(p) \begin{pmatrix} 1 & 0 \\ 0 & e^{-ix} \end{pmatrix}$$

and maybe for small h these two involutions piece together.

Let's consider $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ for $p \leq 0$

and $\begin{pmatrix} 1 & 0 \\ 0 & e^{ix} \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-ix} \end{pmatrix}$ for $p \geq 0$

$$\begin{pmatrix} c & -s e^{-ix} \\ e^{ix} s & e^{ix} c e^{-ix} \end{pmatrix}$$

Here $s(p)$ has the shape ~~as~~ on p 178 and $c = +\sqrt{1-s^2}$. Note that we can write

$$s(p) = \underbrace{s_-(p)}_{\substack{\text{supported} \\ \text{in } p<0}} + \underbrace{s_+(p)}_{\text{supported in } p>0}$$

Let's put

$$G(x,p) = \begin{pmatrix} \overset{\alpha}{c(p)} & \overset{-\beta^*}{-s_-(p) - s_+(p)e^{-ix}} \\ \overset{\beta}{s_-(p) + e^{ix} s_+(p)} & \delta(p) \end{pmatrix}$$

where $\delta(p) = \begin{cases} c(p) & p \leq 0 \\ c(p-h) & p \geq 0 \end{cases}$

Question: Is G unitary?

$$G^*G = \begin{pmatrix} \alpha & \beta^* \\ -\beta & \delta \end{pmatrix}\begin{pmatrix} \alpha & -\beta^* \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} \alpha^2 + \beta^*\beta & -\alpha\beta^* + \beta^*\delta \\ -\beta\alpha + \delta\beta & \beta\beta^* + \delta^2 \end{pmatrix}$$

$$\alpha^2 + \beta^*\beta = c^2 + (s_- + e^{ix}s_+)^*(s_- + e^{ix}s_+)$$

$$= c^2 + \underbrace{s_-^2 + s_+^2}_{s^2} + s_- e^{ix} s_+ + s_+ e^{-ix} s_-$$

Note that $\quad s_-(p)\, e^{ix} s_+(p) = e^{ix} \underbrace{s_-(p+h)\, s_+(p)}_{0 \text{ for } |h| \ll 1.}$

$$-\beta\alpha + \delta\beta = -(s_- + e^{ix}s_+)c + \delta(s_- + e^{ix}s_+)$$

$$= (\delta s_- - s_- c) + e^{ix}(-s_+ c + \delta(p+h)s_+)$$

$\delta(p)s_-(p) = c(p)s_-(p) \quad$ since s_- is supported in [redacted] $p<0$.

$$\delta(p+h)\, s_+(p) = c(p)\, s_+(p)$$

$$\therefore \quad -\beta\alpha + \delta\beta = 0 \quad \underset{\text{take } *}{\Longrightarrow} \quad -\alpha\beta^* + [redacted]\, \beta^*\delta = 0$$

Finally we look at

$$\beta\beta^* + \delta^2 = (s_- + e^{ix} s_+)(s_- + s_+ e^{-ix})$$

$$= s_-^2(p) + s_+^2(p-h) + \cancel{e^{ix} s_+ s_-} + s_- \cancel{s_+ e^{-ix}}$$
$$+ \delta^2(p)$$

For $p \leq 0$, $\delta(p) = c(p)$ and $s_-^2 + c^2 = s^2 + c^2 = 1$.

For $p \geq 0$, $s_-(p) = 0$, $\delta(p) = c(p-h)$ and

$$s_+(p-h)^2 + c(p-h)^2 = (s^2 + c^2)(p-h) = 1$$

So it works.

Next we want to understand the meaning of this deformation. Notice that we have supposed h small because we have used

$$s_+(p-h) = s(p-h) \qquad \text{for } p \leq 0$$

The rough idea is that

$$F = G^2 \varepsilon$$

should be the involution corresponding to the graph of $(s_- + e^{ix} s_+)/c$, or more precisely

$$\mathrm{Im} \begin{pmatrix} c(p) \\ s_-(p) + e^{ix} s_+(p) \end{pmatrix}$$

This is very close to the operator

$$-P_- + e^{ix} P_+$$

Next we would like to generalize the preceding to general loops $g(x)$. The first thing one might try is to choose $s(\xi)$ suitable so that

$$\operatorname{Im}\begin{pmatrix} c(p) \\ s_-(p) + g(x)\, s_+(p) \end{pmatrix}$$

is the subspace. For this to work we would like

$$(s_- + g s_+)^*(s_- + g s_+) = (s_- + s_+ g^{-1})(s_- + g s_+)$$

$$= s_-^2 + s_+^2 + s_- g s_+ + s_+ g^{-1} s_-$$

$+ c^2$ to be 1. This can be done if g is a trignometric polynomial by separating the supports of s_+ , s_- enough. Actually this gets done by taking h small enough. But it doesn't work in general.

October 8, 1987

I think it is useful to take up the idea, explained to me by John Roe, of using K-theory exact sequences. Thus the exact sequence

$$0 \longrightarrow \Psi^{-1} \longrightarrow \Psi^{0} \longrightarrow C^\infty(S^*) \longrightarrow 0$$

leads to a boundary map

$$K_1(C^\infty(S^*)) \xrightarrow{\delta} K_0(\Psi^{-1})$$

which when composed with

$$K_0(\Psi^{-1}) \longrightarrow K_0(k) = \mathbb{Z}$$

gives the index of a symbol. Moreover the map

$$K_s^1(S^*) \longrightarrow K_c^0(T^*)$$

which I have been using is a similar sort of boundary map.

So we want to understand the map

$$K_1(A/I) \longrightarrow K_0(I)$$

in algebraic K-theory. This is discussed in Milnor's book, more generally for cartesian squares, in this case the following

$$\begin{array}{ccc} \tilde{I} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A/I \end{array}$$

Let's proceed geometrically and suppose $A = C(X)$

with I = ideal of fns. vanishing on the closed subspace Y. A vector bundle on X/Y is the same thing as a vector bundle E on X together with a trivialization over Y. We have to understand this statement on the level where vector bundles are direct summands of trivial bundles.

Let's take $E = \tilde{\mathbb{C}}$; then a trivialization of E_Y is given by a map $u : Y \to \mathbb{C}^\times$. Let $\bar{E}$ be the quotient bundle on X/Y. A section s of $\bar{E}$ is a section of E which when restricted to Y is constant relative to the trivialization. Thus

$$\Gamma(X/Y, \bar{E}) = \{ s : X \to \mathbb{C} \mid s|_Y = u\lambda \quad \text{for some } \lambda \in \mathbb{C} \}.$$

$$\Gamma(X/Y, \bar{E}^\vee) = \{ s : X \to \mathbb{C} \mid s|_Y = u^{-1}\lambda \quad \text{for some } \lambda \in \mathbb{C} \}.$$

We want to express $\bar{E}$ as a direct summand of a trivial bundle, which means we need to produce enough sections of $\bar{E}$ and $\bar{E}^\vee$. First we need a section of $\bar{E}$ which spans the fibre over the basepoint. Thus we choose $p : X \to \mathbb{C}$ with $p|_Y = u$. Similarly we choose $q : X \to \mathbb{C}$ with $q|_Y = u^{-1}$. These choices are possible because $C(X) \twoheadrightarrow C(Y)$ (~~Urysohn's lemma~~ Tietze Extension thm.).

Next we need sections to span the fibres where p, q don't. The function $1-pq$ defines a section of $\bar{E}$ vanishing at the basepoint, and it is non-vanishing where p vanishes. Thus we have two sections of $\bar{E}$, namely p and $1-pq$ which span everywhere.

We next want to write E as a direct summand of $\tilde{\mathbb{C}}^2$ in such a way that the projection onto E is

$$\circledast \qquad (p \quad 1-pq) : \tilde{\mathbb{C}}^2 \longrightarrow E$$

or something similar. In order to motivate the formula which will be given later, let's first examine what happens when the metric structure is present. Suppose then that u is unitary: $u^{-1} = u^*$. Then we can take $q = p^*$. Also its natural to require the projection $\tilde{\mathbb{C}}^2 \to E$ to be orthogonal, which means that we modify $\circledast$ above to

$$\left(p \quad (1-|p|^2)^{1/2}\right) : \tilde{\mathbb{C}}^2 \longrightarrow E$$

in which case the isometric embedding of E in $\tilde{\mathbb{C}}^2$ is

$$\begin{pmatrix} p^* \\ (1-|p|^2)^{1/2} \end{pmatrix} : E \longrightarrow \tilde{\mathbb{C}}^2 .$$

In order to do this we must arrange that $|p|^2 = pp^* \leqslant 1$.

Now in the algebraic ~~situation~~ we can't form $(1-pp^*)^{1/2}$, so one looks for something which will be a right inverse for $\circledast$. One finds:

$$\begin{pmatrix} p & 1-pq \end{pmatrix} \begin{pmatrix} q(2-pq) \\ 1-pq \end{pmatrix} = pq(2-pq) + (1-pq)^2$$
$$= 1$$

Moreover $1-pq$, $q(2-pq)$ restrict over Y to $1-uu^{-1}=0$, $u^{-1}(2-uu^{-1}) = u^{-1}$, so these are bona fide sections of $\bar{E}^{\vee}$.

We have just shown how to pass from an invertible element u over A/I to an idempotent 2×2 matrix. Namely we lift u to p, and u^{-1} to q and consider either the above row and column matrices or

$$\begin{pmatrix} (2-pq)p & 1-pq \end{pmatrix} \begin{pmatrix} q \\ 1-pq \end{pmatrix} = \begin{matrix} (2-pq)pq \\ +(1-pq)^2 \end{matrix} = 1$$

The idempotent matrix in this case is

$$e = \begin{pmatrix} q \\ 1-pq \end{pmatrix} \begin{pmatrix} (2-pq)p & 1-pq \end{pmatrix}$$

$$= \begin{pmatrix} q(2-pq)p & q(1-pq) \\ (1-pq)(2-pq)p & (1-pq)^2 \end{pmatrix}$$

which is a mess.

To proceed further we probably want

[redacted] to work in the metric situation where the construction [redacted] ought to be related to the dilation of [redacted] a contraction operator.

Here's how it goes. Instead of ρ let's use α, so that α is a contraction operator. The "projective module" we are interested in is the image of $\begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix}$

i.e. it's the graph of $\alpha(1-\alpha^*\alpha)^{-1/2}$. This column matrix is isometric:

$$\begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix}^* \begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix} = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix} \begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix} = 1$$

so one obtains the projector on this image

$$e = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} \\ \alpha \end{pmatrix} \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & \alpha^* \end{pmatrix}$$

$$= \begin{pmatrix} 1-\alpha^*\alpha & \sqrt{1-\alpha^*\alpha}\,\alpha^* \\ \alpha\sqrt{1-\alpha^*\alpha} & \alpha\alpha^* \end{pmatrix}$$

and the corresponding involution

$$F = 2e-1 = \begin{pmatrix} 1-2\alpha^*\alpha & 2\sqrt{1-\alpha^*\alpha}\,\alpha^* \\ 2\alpha\sqrt{1-\alpha^*\alpha} & 2\alpha\alpha^*-1 \end{pmatrix}$$

On the other hand one can obtain this F by taking the C.T. of $X = \begin{pmatrix} 0 & -(1-\alpha^*\alpha)^{-1/2}\alpha^* \\ \alpha(1-\alpha^*\alpha)^{-1/2} & 0 \end{pmatrix}$

Let $Y = \begin{pmatrix} 0 & -\alpha^* \\ \alpha & 0 \end{pmatrix}$, whence

$$1 + Y^2 = \begin{pmatrix} 1-\alpha^*\alpha & 0 \\ 0 & 1-\alpha\alpha^* \end{pmatrix} \quad \text{so} \quad X = \frac{Y}{\sqrt{1+Y^2}}$$

and
$$F = \frac{1+X}{1-X}\,\varepsilon = \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2 \varepsilon = \left(\sqrt{1+Y^2} + Y\right)^2 \varepsilon$$

$$= \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix}^2 \varepsilon = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix} \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix} \varepsilon$$

$$= \begin{pmatrix} 1-2\alpha^*\alpha & -\sqrt{1-\alpha^*\alpha}\,\alpha^* - \alpha^*\sqrt{1-\alpha\alpha^*} \\ \alpha\sqrt{1-\alpha^*\alpha} + \sqrt{1-\alpha\alpha^*}\,\alpha & 1-2\alpha\alpha^* \end{pmatrix} \varepsilon$$

$$= \begin{pmatrix} 1-2\alpha^*\alpha & 2\sqrt{1-\alpha^*\alpha}\,\alpha^* \\ 2\alpha\sqrt{1-\alpha^*\alpha} & 2\alpha\alpha^*-1 \end{pmatrix}$$

Finally note that

$$\sqrt{1+Y^2} + Y = \begin{pmatrix} \sqrt{1-\alpha^*\alpha} & -\alpha^* \\ \alpha & \sqrt{1-\alpha\alpha^*} \end{pmatrix}$$

is the standard unitary dilation of α in some sense, although I don't know yet what to make of this.

Next we want to link the above ideas to operators on the circle. First let's examine the map
$$K^1(S^*) \longrightarrow K^0_c(T^*)$$
in the case where the element of $K^1(S^*)$ is represented

by the automorphism

$$u(x,\xi): \begin{cases} g(x) & \xi = 1 \\ 1 & \xi = -1 \end{cases}$$

Then thinking of $S^*(S^1) = S^1 \times \{-1, 1\}$ as the boundary of $S^1 \times [-1, 1]$, we want to extend $u(x, \xi)$ on $S^1 \times \{-1, 1\}$ to α on $S^1 \times [-1, 1]$. This is exactly what the formula

$$\alpha = s_- + g s_+$$

does (see yesterday). This α is a contraction

$$1 - \alpha^* \alpha = 1 - s^2 = c^2$$

and so the involution over $S^1 \times [-1, 1]$ is

$$F = \begin{pmatrix} c & -(s_- + g^{-1} s_+) \\ s_- + g s_+ & c \end{pmatrix}^2 \varepsilon$$

$$= \begin{pmatrix} \sqrt{1 - \alpha^* \alpha} & -\alpha^* \\ \alpha & \sqrt{1 - \alpha \alpha^*} \end{pmatrix}^2 \varepsilon .$$

Now our main problem is to find this F over the algebra of operators $f(x, p)$ where $p = \frac{h}{i}\partial_x$. It might be possible to find an algebra of 0th order ψDO's with Planck's constant mapping onto functions on S^*. The problem appears to be the existence of polar decomposition in the quantized algebra.

Return to the algebra of kernels

$$k(x,p) = \sum_{n \in \mathbb{Z}} e^{inx} k_n(p)$$

where $\quad e^{-inx} k(p)\, e^{inx} = k(p+nh)$. Such a kernel operates in $L^2(S^1)$ with $p = \frac{h}{i}\partial_x$. Thus given $f(x) = \sum e^{imx} \hat{f}_m$, we have

$$k(x,p) f(x) = \sum e^{inx} k_n(p) \sum e^{imx} \hat{f}_m$$

$$= \sum_{n,m} e^{i(n+m)x}\, k_n(mh)\, \hat{f}_m$$

The trace of this operator is the sum of the diagonal entries. $k_n(p)$ is diagonal with eigenvalues $k_n(mh)$, $m \in \mathbb{Z}$, but e^{inx} is off-diagonal. Thus

$$\operatorname{tr} k(x,p) = \sum_m k_0(mh).$$

Actually we can also maybe consider twisted versions, where ∂_x is replaced by $\partial_x + ia$.

It might be interesting to go back to the projector over this algebra constructed yesterday, and to compute the trace. This should be an integer which doesn't change as h is varied. (It should be continuous, hence constant.) Finally one might be able to evaluate by letting $h \to 0$.

October 9, 1987

Let's look again at Toeplitz operators. This is the simplest situation where one has an index theorem.

Let's begin with the Hilbert space $\ell^2 \simeq H^+ \subset L^2(S^1)$ and let T be multiplication by z. We can then consider the norm closed subalgebra of $\mathcal{B}(H^+)$ generated by T, T^*. It's a C^*-algebra, call it A. We let $I = A \cap k(H^+)$. A/I is a C^*-algebra, which is commutative and generated by the image u of T which is unitary. Hence $A/I =$ continuous functions on $\mathrm{Spec}(u)$. One knows $\mathrm{Spec}(u) = \mathbb{T}$, so $A/I = C(\mathbb{T})$.

Thus we have

$$\begin{array}{ccccccc} 0 \longrightarrow & I & \longrightarrow & A & \longrightarrow & C(\mathbb{T}) & \longrightarrow 0 \\ & \cap & & & & & \\ & k(H^+) & & & & & \end{array}$$

and hence a connecting map in K-theory

$$K_1(C(\mathbb{T})) \xrightarrow{\ \delta\ } K_0(I) \longrightarrow K_0(k) = \mathbb{Z}.$$

This is the index map.

What we want to do refine this map to a map from the loop group (i.e. unitary group of $C(\mathbb{T})$) to the Grassmannian (projectors over k).

Yesterday I worked out the connecting map δ. Starting with a unitary matrix over $C(\mathbb{T})$, call it u, one lifts it to a contraction α. For example we can take α to be the Toeplitz operator:

$$\alpha = P_+ u P_+$$

Then to α we assigns the orthogonal projector onto the subspace

$$\mathrm{Im}\begin{pmatrix}\sqrt{1-\alpha^*\alpha}\\ \alpha\end{pmatrix} \subset H^+ \oplus H^+$$

Notice that because α is unitary modulo compacts, $\sqrt{1-\alpha^*\alpha} \in k$, so this subspace is in the restricted Grassmannian.

[redacted] On the other hand there is a much nicer way to map to a restricted Grassmannian, namely, one has [redacted] the Hilbert space $L^2(\mathbb{T}) = H^+ \oplus H^-$ on which $C(\mathbb{T})$ acts. One can send u to the subspace $uH^+ \subset H^+ \oplus H^-$.

If $u = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, then

$$uH^+ = \mathrm{Im}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$$

and [redacted] γ is compact, hence uH^+ is equivalent to H^+ modulo compacts.

Thus we have two different maps from the loop group to different restricted Grassmannians. In what sense are they equivalent?

Let's consider $H^+ \oplus H^+ \oplus H^-$ and the following path in the restricted Grass of subspaces congruent mod compacts to the first H^+ factor.

$$[\text{redacted}] \begin{pmatrix}\alpha\\ (\cos t)\sqrt{1-\alpha^*\alpha}\\ (\sin t)\,\gamma\end{pmatrix} H^+ \qquad 0 \le t \le \frac{\pi}{2}$$

This is an isometric embedding as

$$\begin{pmatrix}\alpha^* & \cos t\sqrt{1-\alpha^*\alpha} & \sin t\,\gamma^*\end{pmatrix}\begin{pmatrix}\alpha\\ \cos t\sqrt{1-\alpha^*\alpha}\\ \sin t\,\gamma\end{pmatrix} = \alpha^*\alpha + \cos^2 t\,(1-\alpha^*\alpha) + \sin^2 t\,(\gamma^*\gamma)$$

$$= \alpha^*\alpha + (\cos^2 t + \sin^2 t)(1-\alpha^*\alpha) = 1.$$

Here we use g unitary

$$g^*g = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\alpha^*\alpha + \gamma^*\gamma = 1$.

This deformation shows that the maps from the restricted unitary group

$$U_{res}(H,\varepsilon) = \{g \in U(H) \mid g\varepsilon g^{-1} \equiv \varepsilon \mod \mathcal{K}\}$$

to the restricted Grassmannians

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} H^+ \subset H^+ \oplus H^-$$

$$\longmapsto \begin{pmatrix} \alpha \\ \sqrt{1-\alpha^*\alpha} \end{pmatrix} H^+ \subset H^+ \oplus H^+$$

[redacted] become homotopic when embedded in the restricted Grassmannian of $H^+ \oplus H^- \oplus H^+$ relative to the first factor.

<u>Remark</u>: The first map above [redacted] namely $g \longmapsto gH^+$ is smooth in a way that the second isn't. The point is that $\sqrt{1-x^2}$ isn't smooth at $x = 1$. Also I [redacted] believe that when neither $1-\alpha^*\alpha$ nor [redacted] $1-\delta^*\delta$ has a non-zero kernel, then the[redacted] two matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \begin{pmatrix} \alpha & \sqrt{1-\alpha^*\alpha} \\ \sqrt{1-\alpha^*\alpha} & -\alpha \end{pmatrix}$$

are equivalent [redacted] canonically as they are both minimal dilations of α.

The conclusion might be to avoid situations where α is [redacted] partially unitary.

Let's look at the connecting map $K_1(A/I) \to K_0(I)$ as done in Blackadar. The point is that the matrix $\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ is a product of elementary matrices over A/I and so it can be lifted to an invertible matrix w over A. Then the formula is

$$\partial[u] = \left[w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} \right] - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Start with the identity

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & u \\ -u^{-1} & 1 \end{pmatrix}}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$$

Lift u to p, u^{-1} to q and so

$$w = \underbrace{\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1-pq & p \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-pq & 2p-pqp \\ -q & 1-qp \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So $$w = \begin{pmatrix} 2p-pqp & -(1-pq) \\ 1-pq & q \end{pmatrix}$$

Better $$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Put $$v = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \qquad \text{so} \quad w = v\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-pq & 2p-pqp \\ -q & 1-qp \end{pmatrix}$$

$$v^{-1} = \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}\begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-pq & -2p+pqp \\ q & 1-qp \end{pmatrix}$$

So

$$v\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}v^{-1} = \begin{pmatrix} 1-pq & 2p-pqp \\ -q & 1-qp \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-pq & -2p+pqp \\ q & 1-qp \end{pmatrix}$$

$$= \begin{pmatrix} 1-pq & 2p-pqp \\ -q & 1-qp \end{pmatrix}\begin{pmatrix} 0 & 0 \\ q & 1-qp \end{pmatrix}$$

$$= \begin{pmatrix} 2pq-(pq)^2 & (2p-pqp)(1-qp) \\ (1-qp)q & (1-qp)^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2p-pqp \\ 1-qp \end{pmatrix}\begin{pmatrix} q & 1-qp \end{pmatrix}$$

Since $\begin{pmatrix} & \\ q & 1-qp \end{pmatrix}\begin{pmatrix} 2p-pqp \\ 1-qp \end{pmatrix} = 2qp-(qp)^2+(1-qp)^2$
$= 1$

it works. It's clear one gets the same formula as before.

October 10, 1987

More on $K_1(A/I) \longrightarrow K_0(I)$. In alg. K-theory one starts with u invertible over A/I and lifts

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 & u \\ -u^{-1} & 0 \end{pmatrix}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

to w invertible over A and takes the projector $w\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}w^{-1}$. Changing u to $+u^{-1}$ one lifts $\begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix}$ to an invertible g over A and take $g\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}g^{-1}$.

Supposing u to be unitary, one would like the lift to be unitary, say

$$g = \begin{pmatrix} \sqrt{1-p^*p} & -p^* \\ p & \sqrt{1-pp^*} \end{pmatrix}$$

Then $g\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}g^{-1}$ is orthogonal projection on

$$\operatorname{Im} g\begin{pmatrix}0\\1\end{pmatrix} = \operatorname{Im}\begin{pmatrix}-p^*\\ \sqrt{1-pp^*}\end{pmatrix} \qquad ?$$

Start again. The original formula is to send u to the difference

$$\left[w\begin{pmatrix}1&0\\0&0\end{pmatrix}w^{-1}\right] - \left[\begin{pmatrix}1&0\\0&0\end{pmatrix}\right]$$

where w lifts $\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}$. Let's change this to

$$\left[w_1\begin{pmatrix}0&0\\0&1\end{pmatrix}w_1^{-1}\right] - \left[\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]$$

where w_1 lifts $\begin{pmatrix}u^{-1}&0\\0&u\end{pmatrix}$, and then to

$$\left[w_2\begin{pmatrix}1&0\\0&0\end{pmatrix}w_2^{-1}\right] - \left[\begin{pmatrix}0&0\\0&1\end{pmatrix}\right]$$

where w_2 lifts $\begin{pmatrix}u^{-1}&0\\0&u\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&-u^{-1}\\u&0\end{pmatrix}$

For example, we can take

$$w_2 = \begin{pmatrix}\sqrt{1-p^*p} & -p^*\\ p & \sqrt{1-pp^*}\end{pmatrix}$$

whence $w_2\begin{pmatrix}1&0\\0&0\end{pmatrix}w_2^{-1}$ is orthogonal projection on

$$\operatorname{Im} w_2\begin{pmatrix}1\\0\end{pmatrix} = \operatorname{Im}\begin{pmatrix}\sqrt{1-p^*p}\\ p\end{pmatrix}$$

Notice however that $\begin{pmatrix}0&-u^{-1}\\u&0\end{pmatrix}$ lies on a 1-parameter subgroup

$$\begin{pmatrix}\cos\theta & -\sin\theta\, u^{-1}\\ u\sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}1&0\\0&u\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}1&0\\0&u^{-1}\end{pmatrix}$$

$$= \exp \theta \begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix}$$

Therefore another way to proceed might be to lift the infinitesimal generator $\begin{pmatrix} 0 & -u^{-1} \\ u & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & -p^{*} \\ p & 0 \end{pmatrix}$ and then use the exponential map. This is the formula used by Atiyah + Singer.

Let's consider the goal of doing the index theorem over the circle by asymptotic methods, i.e. letting a Planck's constant go to zero. I want some kind of algebra of ΨDO's with Planck's constant.

One idea would use the fact that the Hilbert involution F satisfies $[F,f]$ = a smooth kernel operator where $f \in C^\infty(S^1)$. So the idea would be to adjoin F to our smooth kernel operators.

Recall that we have been looking at a deformation with parameter h of the algebra of Schwartz functions on $T^*(S^1) = S^1 \times \mathbb{R}$. The deformed algebra is the crossed product

$$C^\infty(S^1) \tilde{\otimes} S(\mathbb{R})$$

where the multiplication is such that

$$e^{-inx} g(p) e^{inx} = g(x+nh)$$

Someday it will be necessary to see this all works, i.e. that there is a really nice algebra defined in this way.)

To enlarge this algebra we enlarge $\mathcal{S}(\mathbb{R})$ by adjoining the constant functions and any smoothed version of the Heaviside function $\Theta(p)$. The enlarged algebra consists of all smooth functions $f(p)$, which on each half-line $p \geq 0$ and $p \leq 0$ differ from a constant function by a rapidly decreasing function. Call this enlarged algebra $\widetilde{\mathcal{S}(\mathbb{R})}$. We have an exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{R}) \longrightarrow \widetilde{\mathcal{S}(\mathbb{R})} \longrightarrow \mathbb{C} \times \mathbb{C} \longrightarrow 0$$

Again we can form the crossed product algebra

$$C^\infty(S^1)\,\tilde{\otimes}\,\widetilde{\mathcal{S}(\mathbb{R})}$$

and we have an exact sequence

$$0 \longrightarrow C^\infty(S^1)\tilde{\otimes}\mathcal{S}(\mathbb{R}) \longrightarrow C^\infty(S^1)\times\widetilde{\mathcal{S}(\mathbb{R})} \longrightarrow C^\infty(S^1\times\{\pm 1\}) \longrightarrow 0$$

This exact sequence should explain how to attach operators to an invertible matrix over $S^*(S^1) = S^1 \times \{\pm 1\}$.

October 11, 1987

The problem is to see if, having constructed a projector over the ~~algebra~~ algebra of the deformation, can we find the index by asymptotics as $\hbar \to 0$.

Review some formulas for the index. Let F be an involution $\equiv -\varepsilon$ modulo ~~trace~~ compact ~~class~~ operators and $g = F\varepsilon$ as usual, so that $g \equiv -1$ mod ~~trace class~~ compact operators. Recall that

$$\text{Index} \overset{\text{defn}}{=} \operatorname{tr}\{\varepsilon \text{ on } g=+1 \text{ eigenspace}\}$$

$$= \operatorname{tr}\{\varepsilon f(g)\}$$

provided f is a function on T with $f(-1)=0$ and $f(1)=1$, and such that $f(g) \in \mathcal{L}^1$. (Note that it is not necessary to suppose $f(g)=f(g^{-1})$ as

$$\operatorname{tr}\{\varepsilon f(g)\} = \operatorname{tr}\{f(g)\varepsilon\} = \operatorname{tr}\{\varepsilon f(g^{-1})\}.)$$

Thus we have when $(g+1)^n$ is of trace class

$$\text{Index} = \operatorname{tr}\left\{\varepsilon\left(\frac{g+1}{2}\right)^n\right\} = \operatorname{tr}\left(\left(\frac{g+1}{2}\right)^n\varepsilon\right) \overset{\text{NO}}{=} \operatorname{tr}\left(\frac{F+\varepsilon}{2}\right)^n. \;?$$

and if we put $F = 2e'-1$, $-\varepsilon = 2e-1$ so that $(e'-e)^n$ is of trace class, then

$$\frac{F+\varepsilon}{2} = \frac{2e'-1-2e+1}{2} = e'-e$$

and so

$$\text{Index} = \operatorname{tr}(e'-e)^n.$$

only for n odd ~~NO~~ see p. 213.

(n even $\Rightarrow (e'-e)^n \geq 0$ so it can't be true for n even)

October 12, 1987 (Becky is 21)

Let $\tilde{a}$ be the crossed product algebra of kernels $k(h,x,p)$ and suppose that I can construct a projector over this algebra associated to an invertible matrix on the circle. We have various realizations of $\tilde{a}$ for $h \neq 0$ as bounded operators. Presumably these all give the same trace. The problem is to evaluate this trace by letting $h \to 0$.

To be specific I should take

$$P = s_- + g\, s_+$$

as on p. 182. Then the projector in question corresponds to the unitary G^2 where

$$G = \begin{pmatrix} \sqrt{1-P^*P} & -P^* \\ P & \sqrt{1-PP^*} \end{pmatrix}$$

The index is $\mathrm{tr}_h\left(\varepsilon\left(G^2+1\right)\right)$ up to sign.

Maybe we can even use

$$\mathrm{tr}_h\left(\varepsilon\left(\frac{G+G^{-1}}{2}\right)\right).$$

Now our question is to somehow evaluate this by arguing it is independent of h and by using asymptotics as $h \to 0$. The problem is that the answer is roughly

$$\mathrm{tr}_h\left(\sqrt{1-P^*P} - \sqrt{1-PP^*}\right)$$

and the tr_h blows up while $\sqrt{1-P^*P} - \sqrt{1-PP^*}$ goes to zero as $h \to 0$. So it's far from there being a trace function on $\tilde{A}$ depending on h and continuous as $h \to 0$.

Somehow I have to find a simple method whereby I can take the h-supertrace of $f(g)$, where g is unitary inverted by ε and $g+1$ is a matrix over $\tilde{A}$, and know this h-supertrace is continuous in h.

Idea: Use Getzler's ideas. Recall that to study the Dirac operator he uses the operators $f(x)$, $\frac{h}{i}D_\mu$, $h\gamma^\mu$ which as $h \to 0$ becomes $f(x)$, p_μ, ω^μ. The supertrace is continuous as $h \to 0$, because the fermions contribute powers of h to kill those produced by the bosonic trace.

So it would seem that we have to augment the algebra $\tilde{A}$ of kernels $k(h, x, p)$ by adjoining ω of odd degree with $\omega^2 = 0$. We have to define a deformed product in this algebra together with an action of the deformed algebra on some Hilbert space like $L^2(S^1) \otimes \mathbb{C}^2$.

October 13, 1987

Review:

$$\mathcal{A}_h = C^\infty(S^1) \otimes \mathcal{S}(\mathbb{R}) \qquad e^{-ix} f(p)\, e^{ix} = f(p+h)$$

$$\mathcal{A} = C^\infty(S^1) \otimes \mathcal{S}(\mathbb{R} \times \underset{h}{\underset{\psi}{[0,1]}})$$

We have an extension

$$0 \longrightarrow \mathcal{S}(\mathbb{R}) \longrightarrow \widetilde{\mathcal{S}(\mathbb{R})} \longrightarrow \mathbb{C} \times \mathbb{C} \longrightarrow 0$$

giving rise to an extension of algebras

$$0 \longrightarrow \mathcal{A}_h \longrightarrow \mathcal{B}_h \longrightarrow C^\infty(\underbrace{S^1 \times \{\pm 1\}}_{S^*(S^1)}) \longrightarrow 0$$

This gives a connecting map

$$(*) \qquad K^1(S^1 \times \{\pm 1\}) \longrightarrow K_0(\mathcal{A}_h).$$

Now we have various ways to interpret elements of $\mathcal{A}_h$ as operators on $L^2(S^1)$, by letting p be the operator $\frac{h}{i}\left(\partial_x + i\left(\begin{smallmatrix}\text{real}\\ \text{const}\end{smallmatrix}\right)\right)$. Any of these gives a map

$$(**) \qquad \mathcal{A}_h \longrightarrow \mathcal{L}^1(L^2(S^1)) \subset \mathcal{K}(L^2(S^1))$$

and hence gives an index.

Combining $(*)$ and $(**)$ gives a way to assign to any element of

$$K^1(S^1 \times \{\pm 1\}) / K^1(S^1) \cong K^1(S^1)$$

various operator "Fredholm" operators. Of course one has to first make $(*)$ concrete. This means starting with φ an invertible matrix over $C^\infty(S^1)$ and constructing a suitable projector or involution.

Starting from an invertible matrix φ over $C^\infty(S^1)$ we consider element p, q of $\mathcal{B}$ which lie over φ, φ^{-1} respectively. Then we can obtain a projector e in various ways differing from a standard projector e_0.

For example starting from

$$\begin{pmatrix} 2q - qpq & 1 - qp \end{pmatrix} \begin{pmatrix} p \\ 1 - qp \end{pmatrix} = 2qp - (qp)^2 + (1-qp)^2 = 1$$

we get the projector

$$e = \begin{pmatrix} p \\ 1 - qp \end{pmatrix} \begin{pmatrix} 2q - qpq & 1 - qp \end{pmatrix}$$

which modulo $\mathcal{A}$ is congruent to

$$e_0 = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \begin{pmatrix} 2\varphi^{-1} - \varphi^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

The index is

$$\begin{aligned} tr_h(e - e_0) &= tr_h\big(p(2q - qpq) - 1\big) + tr_h(1-qp)^2 \\ &= tr_h(1-qp)^2 - tr_h(1-pq)^2 \end{aligned}$$

On the other hand suppose I can factor

$$1 - qp = xy \qquad \text{with } x, y \in \mathcal{A}$$

Then

$$\begin{pmatrix} q & x \end{pmatrix} \begin{pmatrix} p \\ y \end{pmatrix} = 1 \qquad \text{so we have}$$

the projector

$$e = \begin{pmatrix} p \\ y \end{pmatrix} \begin{pmatrix} q & x \end{pmatrix} = \begin{pmatrix} pq & px \\ yq & yx \end{pmatrix}$$

which modulo $\mathcal{a}$ is congruent to $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

The index is then

$$\operatorname{tr}_h(e - e_0) = \operatorname{tr}_h(pq - 1) + \underbrace{\operatorname{tr}_h(yx)}_{= \operatorname{tr}_h(xy)}$$

$$= \operatorname{tr}_h(1 - qp) - \operatorname{tr}_h(1 - pq).$$

For example provided we can construct the square root we can take $x = y = \sqrt{1 - qp}$.

Now our problem is to evaluate this trace as $h \to 0$. This looks reasonable since one knows that $[p, q] = O(h)$, and on the other hand the trace is

$$\operatorname{tr}_h f(x, p) = \int \frac{dx}{2\pi} \sum_{n \in \mathbb{Z}} f(x, nh).$$

$$\sim \frac{1}{h} \int \frac{dx\,dp}{2\pi} f(x, p)$$

Notice that we can suppose

$$1 - qp \in \mathcal{a}^2 = \mathcal{a}.\mathcal{a}$$

for if we write

$$1 - qp = \sum_{i=1}^{n} x_i y_i$$

then we have

$$(q \; x_1 \cdots x_n)\begin{pmatrix} p \\ y_1 \\ \vdots \\ y_n \end{pmatrix} = 1 \qquad \text{so } e = \begin{pmatrix} p \\ y_1 \\ \vdots \\ y_n \end{pmatrix}(q \; x_1 \cdots x_n)$$

and

$$\begin{aligned} tr(e - e_0) &= tr(pq - 1) + \sum_{i=1}^{n} tr(y_i x_i) \\ &= tr(pq - 1) + \sum_{i=1}^{n} tr(x_i y_i) \\ &= tr(pq - 1) + tr(1 - qp) \end{aligned}$$

as befores.

October 15, 1987

Ideas from yesterday's lecture

$$\begin{array}{ccc} Y & \subset & X \\ \downarrow & & \downarrow \\ \{\infty\} & \subset & X/Y \end{array}$$

E vector bundle on X with $\varphi : E|Y \xrightarrow{\sim} \tilde{\mathbb{C}}^n_Y$

$\bar{E}$ = ~~quotient~~:
$$\begin{array}{ccc} E|Y & \subset & E \\ \downarrow & & \downarrow \\ \mathbb{C}^n & \longrightarrow & \bar{E} \end{array}$$

To show $\bar{E}$ a vector bundle, we will show

$$\Gamma(X/Y, \bar{E}) = \{ s \in \Gamma(E) \mid \varphi(s|Y) \text{ constant} \}$$

as a $C(X/Y)$ module is a direct summand of a free module. Special case: $E = \tilde{\mathbb{C}}^n_X$.

To get sections of $\bar{E}$ spanning $\bar{E}_\infty$ we need to lift φ^{-1}. Tietze: $C(X) \twoheadrightarrow C(Y)$, so we can find $p, q \in M_n(C(X))$ such that $p|Y = \varphi$ and $q|Y = \varphi^{-1}$. Then we have maps

$$C(X/Y)^n \xrightarrow{\quad q \quad} \Gamma(X/Y, \bar{E}) \xrightarrow{\quad p \quad} C(X/Y)^n$$

($\xi \in C(X/Y)^n$, i.e. $\xi \in C(X)^n$ and $\xi|Y$ constant $\in \mathbb{C}^n$, then $q\xi \in C(X)^n$ and $\varphi(q\xi)|Y = \varphi\varphi^{-1}\xi|Y$ is const. Similarly if $s \in \Gamma(X/Y, \bar{E})$, so $s \in C(X)^n$ and $\varphi s|Y$ is const then $ps \in C(X)^n$ and $ps|Y = \varphi(s|Y)$ is constant, so $ps \in C(X/Y)^n$.)

An Other way to get a section of $\bar{E}$ or a map of $\bar{E}$ to $\tilde{\mathbb{C}}_{X/Y}$ is to use vectors over $C(X)$ which vanish along Y. So we wish to find $\alpha, \beta \in M_r(\underbrace{C_0(X/Y)}_{I})$ such that the composition

$$\Gamma(X/Y,\bar{E}) \xrightarrow{\begin{pmatrix} p \\ \beta \end{pmatrix}} C(X/Y)^{2r} \xrightarrow{(q \ \ \alpha)} \Gamma(X/Y,\bar{E})$$

is the identity: $\quad qp + \alpha\beta = 1.$

But

$$qp = 1 - v \qquad v = 1 - qp \in M_r(I)$$

~~and there's a well known way to~~, so q is an inverse of p mod I. There's a standard way to change q to an inverse mod I^2: Set

$$\tilde{q} = (1+v)q = (2-qp)q. \quad \text{Then}$$

$$\tilde{q}p = (1+v)qp = (1-v)(1+v) = 1 - v^2 \equiv 1 \quad (I^2).$$

Also $1 - \tilde{q}p = v^2$, so we can take

$$\alpha = \beta = v = 1 - qp.$$

This gives then

$$\begin{pmatrix} 2q - qpq & 1 - qp \end{pmatrix}\begin{pmatrix} p \\ 1-qp \end{pmatrix} = 2qp - (qp)^2 + (1-qp)^2 = 1$$

as desired, and so

$$\Gamma(X/Y,\bar{E}) = \operatorname{Im} \underbrace{\begin{pmatrix} p \\ 1-qp \end{pmatrix}\begin{pmatrix} 2q - qpq & 1-qp \end{pmatrix}}_{\text{projector on } C(X/Y)^{2r}}$$

Next go to the identity

$$\begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\varphi^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varphi & 0 \\ 0 & \varphi^{-1} \end{pmatrix}$$

set

$$\omega = \underbrace{\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 1-pq & p \\ -q & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-pq & 2p - pqp \\ -q & 1-qp \end{pmatrix}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then

$$\omega \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \omega^{-1} = \begin{pmatrix} 1-pq & 2p-pqp \\ -q & 1-qp \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \;\; 1)} \begin{pmatrix} 1-qp & -(2p-pqp) \\ q & 1-pq \end{pmatrix}$$

$$= \begin{pmatrix} 2p-pqp \\ 1-qp \end{pmatrix} \begin{pmatrix} q & 1-pq \end{pmatrix}$$

This is essentially the same as the preceding projector, except that before we changed q to $\tilde{q} = (1+(1-qp))q$, and here we change p to $\tilde{p} = p(1+(1-qp)) = 2p - 2pqp$.

October 16, 1987

Let's consider a 2×2 matrix $F(h,x,p)$ which is an involution and such that $F+\varepsilon$ has entries in the twisted algebra of smooth $f(h,x,p)$ which decay rapidly in p. Let

$$F(h,x,p) = F_0(x,p) + hF_1(x,p) + O(h^2)$$

~~Setting~~ $h=0$, we see that $F_0(x,p)$ is an involution over the ring of smooth functions on $T^*(S^1)$ which is $\equiv -\varepsilon$ modulo smooth functions rapidly decreasing in p. The trace of $F_0(x,p)$ as a 2×2 matrix is zero at each point (x,p), because the trace doesn't change as the involution is varied. Thus

$$\circledast \qquad \mathrm{tr}\big(F_0(x,p)+\varepsilon\big) = 0$$

(and this would be true in the case of larger matrices).

Next recall that we have for each $h\neq 0$ and $\lambda\in\mathbb{R}$ a representation of the twisted algebra on $L^2(S^1)$ such that $p\longmapsto h(\frac{1}{i}\partial_x+\lambda)$ and that when $f(h,x,p)$ is rapidly decreasing in p, the operator corresponding to f is of trace class with

$$\mathrm{Tr}_{(h,\lambda)}(f) = \int\frac{dx}{2\pi}\sum_{n\in\mathbb{Z}} f(h,x,h(n+\lambda))$$

Thus we have using $\circledast$

$$\text{Index} = \frac{1}{2}\,\mathrm{Tr}_{(h,\lambda)}\big(F(h,x,p)+\varepsilon\big) = \frac{1}{2}\int\frac{dx}{2\pi}\sum_n \mathrm{tr}\big(F(h,x,h(n+\lambda))\big)$$

$$= \frac{1}{2}\int\frac{dx}{2\pi}\sum_n h\,\mathrm{tr}\big\{F_1(x,h(n+\lambda))\big\} + \begin{matrix}\text{error}\\ \text{which}\\ \text{should be}\\ O(h)\end{matrix}$$

On the other hand $Tr_{(h,\lambda)}(F+\varepsilon)$ is independent of h and λ. To see this it would be better to introduce the homomorphism

$$\mathcal{A} \xrightarrow{\Theta_{h,\lambda}} B(L^2(S^1))$$

and to write $Tr(\Theta_{h\lambda}(F)+\varepsilon$ [redacted]$)$ instead of $Tr_{(h,\lambda)}(F+\varepsilon)$. Then the fact this trace is independent of $h\lambda$ is clear, as this sort of trace is locally constant on the restricted Grassmannian.

So we can [redacted] evaluate this trace by letting $h \to 0$ whence we obtain

$$(**) \quad \text{Index} = \frac{1}{2}\int \frac{dx\,dp}{2\pi}\ tr\{F_1(x,p)\}$$

(There should be no trouble in controlling the errors, because [redacted] the Poisson summation formula gives control on the difference between

$$\int f(x)\,dx \quad \text{and} \quad \sum_n f(x+n). \qquad f \in \mathcal{S}(\mathbb{R})$$

Also we can use

$$F(h,x,p) = F_0(x,p) + h\,F_1(x,p) + h^2\,\tilde{F}_2(h,x,p)$$

with Lagrange's formula for the remainder $\tilde{F}_2$.)

The next step will be to analyze the formula (**). Start from the [redacted] asymptotic formula for the product

$$f(x,p) \underset{h}{*} g(x,p) = (fg)(x,p) + \frac{h}{i}\partial_p f(x,p)\cdot\partial_x g(x,p) +$$
$$+ \frac{1}{2!}\,\partial_p^2 f\cdot\left(\frac{h}{i}\partial_x\right)^2 g + \cdots\cdots$$

Because $F(h,x,p)$ is an involution we have

$$1 = F_0 * F_0 + h(F_0 * F_1 + F_1 * F_0) + \cdots$$

$$= F_0^2 + \frac{h}{i}\partial_p F_0 \, \partial_x F_0 + h(F_0 F_1 + F_1 F_0) + O(h^2).$$

Thus

$$\frac{1}{i}\partial_p F_0 \, \partial_x F_0 + F_0 F_1 + F_1 F_0 = 0$$

Now because F_0 is an involution, we know that $\partial_p F_0$, $\partial_x F_0$ anti-commute with F_0, hence their product commutes with F_0. Thus we conclude that

$$F_1 = -\frac{1}{2i} F_0 \, \partial_p F_0 \, \partial_x F_0 + \left(\begin{array}{l}\text{term anti-commuting} \\ \text{with } F_0.\end{array}\right)$$

$$\mathrm{tr}\, F_1 = \frac{i}{2} \mathrm{tr}\left(F_0 \, \partial_p F_0 \, \partial_x F_0\right)$$

$$= \frac{i}{4} \mathrm{tr}\left(F_0 \left(\partial_p F_0 \, \partial_x F_0 - \partial_x F_0 \, \partial_p F_0\right)\right)$$

$$(\mathrm{tr}\, F_1)\, dx\, dp = \frac{1}{4i} \mathrm{tr}\left(F_0 (dF_0)^2\right).$$

Thus

$$\mathrm{Index} = \frac{1}{2}\int_{S^1 \times \mathbb{R}} \frac{1}{2\pi \cdot 4i} \mathrm{tr}\{F_0 (dF_0)^2\}$$

$$\boxed{\mathrm{Index} = -\frac{i}{2\pi} \int_{S^1 \times \mathbb{R} = T^*(S^1)} \frac{1}{8} \mathrm{tr}\{F_0 (dF_0)^2\}}$$

October 17, 1987

Correct error on page 200. Start with formula

$$\text{Index} = \operatorname{tr} \varepsilon f(g)$$

where $f(1) = 1$, $f(-1) = 0$, and $f(g) \in \mathcal{L}^1$. Next

$$(F+\varepsilon) = g\varepsilon + \varepsilon = (g+1)\varepsilon = \varepsilon(g^{-1}+1)$$

$$(F+\varepsilon)^2 = (g+1)(g^{-1}+1)$$

$$\therefore \circledast \quad \operatorname{tr}\left(\frac{F+\varepsilon}{2}\right)^{2n+1} = \operatorname{tr}\Big(\varepsilon \underbrace{\frac{(g^{-1}+1)}{2}\Big[\frac{(g+1)}{2}\frac{(g^{-1}+1)}{2}\Big]^n}_{f(g)}\Big) = \text{Index}$$

provided $g+1 \in \mathcal{L}^{2n+1}$. If $g+1 \in \mathcal{L}^{2n}$, then we have also

$$\text{Index} = \operatorname{tr}\left\{\varepsilon\left(\frac{(g+1)}{2}\frac{(g^{-1}+1)}{2}\right)^n\right\} = \operatorname{tr} \varepsilon \left(\frac{F+\varepsilon}{2}\right)^{2n}$$

From $\otimes$

$$2\,\text{Index} = \operatorname{tr}(F+\varepsilon)\left(\frac{F+\varepsilon}{2}\right)^{2n}$$

so also $\quad \text{Index} = \operatorname{tr} F\left(\frac{F+\varepsilon}{2}\right)^{2n}$. Thus

$$\boxed{\begin{aligned} \text{Index} &= \operatorname{tr}\left(\frac{F+\varepsilon}{2}\right)^{\text{odd}} \\ &= \operatorname{tr} \varepsilon\left(\frac{F+\varepsilon}{2}\right)^{\text{even}} = \operatorname{tr} F\left(\frac{F+\varepsilon}{2}\right)^{\text{even}} \end{aligned}}$$

provided the traces make sense.

October 19, 1987

Let's consider again an involution

$$F(h,x,p) = F_0(x,p) + hF_1(x,p) + h^2F_2(x,p) + \cdots$$

which is a matrix of the form $F = -\varepsilon + \alpha$ where α has entries in $\mathcal{A}$. For $h \neq 0$, we have $\Theta_h : \mathcal{A} \longrightarrow \mathcal{B}\, L^2(\mathbb{R}^n/\Gamma)$, and we know

$$\operatorname{tr}\left(\Theta_h(F+\varepsilon)\right) = \int \frac{d^n x}{(2\pi)^n} \sum_{n\in\hat{\Gamma}} \operatorname{tr}\left(F(h,x,nh)+\varepsilon\right)$$

is independent of h (at least it remains unchanged as h is varied). But for a Schwartz function

$$\int \frac{d^n x}{(2\pi)^n} \sum_{n\in\hat{\Gamma}} f(x,nh) = \frac{1}{h^n}\int \frac{(d^n x\, d^n p)^n}{(2\pi)^n} f(x,p) + O(h^\infty)$$

so it follows that we must have

$$\circledast \qquad \int \frac{(dx\,dp)^n}{(2\pi)^n} \operatorname{tr}\left(F_k(x,p) + (\varepsilon \text{ if } k=0)\right) = 0$$

for $k = 0, \dots, n-1$.

The problem is to explain directly why $\circledast$ is true, why it follows from the fact F is an involution. It's really a formal question, i.e. having to do with truncated series in h.

Now we've seen that

$$\operatorname{tr}\left(F_0(x,p) + \varepsilon\right) = 0$$

because the involution $F_0(x,p)$ is homotopic to $-\varepsilon$, the homotopy being given by any path from (x,p) to ∞. We also saw

$$\operatorname{tr} F_1(x,p) = \frac{1}{4i}\sum_{\mu=1}^{n} \operatorname{tr}\left(F_0\left(\partial_{x_\mu}F_0\,\partial_{p_\mu}F_0 - \partial_{p_\mu}F_0\,\partial_{x_\mu}F_0\right)\right)$$

This is definitely non zero pointwise, since for $r = 1$, it integrates to give the index essentially. If $r > 1$, then when we integrate $tr\{F_0(\partial_{x_\mu} F_0 \, \partial_{p_\mu} F_0 - \partial_{p_\mu} F_0 \, \partial_{x_\mu} F_0)\}$ for a given μ we can do ~~so~~ by first integrating over the variables x_μ, p_μ. This will give the first Chern class of the bundle over this 2-plane which will be zero, again using the homotopy given by moving the other variables to ∞.

October 20, 1987

[redacted] Idea: For $h \neq 0$ the algebra $\mathcal{Q}_h = C^\infty(S^1) \otimes \mathcal{S}(\mathbb{R})$ has a representation on $L^2(S^1)$ which is irreducible. This is analogous to the Heisenberg representation of the Weyl algebra. It should lead to a projector in $\mathcal{Q}_h$, and the class of this projector should generate the K-theory. Now the K-theory of $\mathcal{Q}_0 = C^\infty(S^1) \otimes \mathcal{S}(\mathbb{R})$ [redacted] is also $\mathbb{Z}$, but the generator is ~~the~~ difference of two projectors which are 2×2 matrices.

The idea is to express the Heisenberg representation in a form so one can see the limit as $h \to 0$. Off-hand I would expect a [redacted] length one resolution. For $C^\infty(\mathbb{R}^n/\Gamma) \otimes \mathcal{S}(\mathbb{R}^n)$ I would expect a Koszul resolution on n generators. I hope that this resolution might lead to a way of rewriting the index of an [redacted] F over $\mathcal{Q}$ so that the $h \to 0$ limit can be taken.

Let's examine carefully $S^1 \times \mathbb{R} = T^*(S^1)$. The v. bundle which is the Bott generator is described by a degree 1 clutching function.

October 22, 1987

Let A be an algebra equipped with n commuting derivations $\nabla_1, \dots, \nabla_n$. Then we can consider the algebra

$$A \otimes \Lambda \mathbb{C}^n$$

and define an operator ∇ on this algebra by

$$\nabla(a \otimes \omega) = \sum_i \nabla_i(a) \otimes (e_i \wedge \omega).$$

I claim ∇ is a derivation of degree 1 relative to the exterior algebra degree.

$$\nabla\left[(a\otimes\omega)(b\otimes\eta)\right] = \nabla(ab \otimes \omega\eta)$$

$$= \sum_i \nabla_i(ab) \otimes e_i \omega\eta$$

$$= \sum_i \left\{(\nabla_i a)b + a\nabla_i b\right\} \otimes e_i\omega\eta$$

$$= \sum_i (\nabla_i a \otimes e_i\omega)(b\otimes\eta) + (-1)^{\deg\omega}(a\otimes\omega)(\nabla_i b \otimes e_i \eta)$$

$$= \left(\nabla(a\otimes\omega)\right) b\otimes\eta + (-1)^{\deg\omega}(a\otimes\omega)\,\nabla(b\otimes\eta)$$

Moreover

$$\nabla^2(a\otimes\omega) = \nabla\sum_i(\nabla_i a)\otimes e_i\omega$$

$$= \sum_{j,i} \underbrace{(\nabla_j \nabla_i a)}_{\text{symm}} \otimes \underbrace{e_j e_i}_{\text{skew-symm.}} \omega = 0$$

Thus we have a differential graded algebra. Next let's consider the quotient by (graded) commutators:

$$A \otimes \Lambda \mathbb{C}^n / [\ ,\ ] = A/[A,A] \otimes \Lambda \mathbb{C}^n$$

since $\Lambda \mathbb{C}^n$ is already commutative. According to Connes' theory a linear functional on a differential graded algebra containing A which vanishes on super-commutators and exact "forms" determines a cyclic cocycle on A.

We can get such a linear functional by taking a trace $\tau : A \to \mathbb{C}$ (thus $\tau([A,A])=0$) such that $\tau(\nabla_i A) = 0$ for all i, and combining it with a linear functional on $\Lambda \mathbb{C}^n$.

Let's return to the algebra $\mathcal{A}$ of $f(h,x,p)$ with its various representations on $L^2(S^1)$. Then we have the derivations ∂_x, ∂_p which commute, so we can construct a de Rham complex

$$\mathcal{A} \xrightarrow{d} \mathcal{A} \otimes \mathbb{C}^2 \xrightarrow{d} \mathcal{A} \otimes \Lambda \mathbb{C}^2 \longrightarrow 0$$

which additively is the complex of rapidly decreasing differential forms on $S^1 \times \mathbb{R}$ with h as parameter. (So the complex is 1 diml in degree 2 for each h and trivial elsewhere. This cohomology before taking commutator quotient is not interesting from the cyclic cohomology viewpoint.)

Next we need a trace on $\mathcal{A}$ vanishing on the image of ∂_x, ∂_p. The only possibility is

$$\tau(f(h,x,p)) = \int dx\,dp\; f(h,x,p).$$

times a function of h.

Let e be an idempotent matrix over $\mathcal{A}^+$ which is congruent to e_0 modulo $\mathcal{a}$. I can consider the "2-form" $\operatorname{tr} e(de)^2 \in \mathcal{A} \otimes \Lambda^2 \mathbb{C}$; this is a non-commutative character form. The point is perhaps to view

$$\tau^{(h)} \operatorname{tr} e(de)^2 = \int dx\,dp\,\left(\operatorname{tr} e(de)^2\right)(h,x,p)$$

as the pairing of the class in $K_0\mathcal{A}$ represented by e with the cyclic cohomology class represented by $\tau^{(h)}$.

So it appears that we are mainly interested in the cyclic cohomology of $\mathcal{A}$. Now $H^0_\lambda(\mathcal{A})$ is the space of traces on $\mathcal{A}$, and we have quite a supply. For example, each representation of $\mathcal{A}$ on $L^2(S^1)$ (recall there is one for each assignment $p \mapsto \frac{h}{i}(\partial_x + i\lambda)$) gives a trace. Maybe it's true that all of these traces yield the same class in $H^2_\lambda(\mathcal{A})$ under the S-operator.

So we have made a first reduction, namely I have replaced idempotents by something more general. The problem is now to see if we can show that applying S to $\operatorname{Tr}(h)$, which gives the 2-cyclic-cocycle

$$\operatorname{Tr}\left(f_0 f_1 f_2 \text{ acting on } L^2(S^1)\right)$$

is cohomologous to

$$\int dx\,dp\; f_0\, df_1\, df_2 .$$

Maybe this is <u>to be true for fixed</u> h.

Let's consider the smooth Weyl algebra case. Let $\mathcal{A}$ be the deformation algebra over C^∞(h-line). $\mathcal{A}_h$ for $h \neq 0$ should be Morita equivalent to $\mathbb{C}$, so the cyclic cohomology should be $\mathbb{C}$ in even dimensions and zero in odd dimensions. $\mathcal{A}_0 = \mathcal{S}(\mathbb{R}^2)$ should have Hochschild cohomology given by closed currents with arbitrary support. The cyclic cohomology should be

$$H^0_\lambda = \text{distributions (currents of degree 0)}$$

$$H^1_\lambda = \text{closed 1-currents}$$

$$H^2_\lambda = \underbrace{\{\text{closed 2-currents}\}}_{\mathbb{C}} \oplus H^0_c(\mathbb{R}^2)$$

(given by fundamental class i.e. integration over $\mathbb{R}^2$)

$$H^3_\lambda = 0$$

$$H^4_\lambda = \mathbb{C} \qquad \text{etc.}$$

The cyclic cohomology for $\mathcal{A}$ in degree 0 is quite big: For a fixed h one has lots of traces and these can be integrated with respect to a distribution in h. However one can perhaps hope that $H^2_\lambda = \mathbb{C}$.

Idea: One has the de Rham complex

$$\mathcal{A} \longrightarrow \mathcal{A} \otimes \Lambda^1\mathbb{C}^2 \longrightarrow \mathcal{A} \otimes \Lambda^2\mathbb{C}^2 \longrightarrow 0 \longrightarrow$$

which leads to interesting cyclic 1-cocycles. These should be cohomologous to zero. Why? Similarly why is the cyclic 2-cocycle obtained from the above equivalent to S of the trace.

October 24, 1987

I have to understand very carefully the index of a pair of idempotents whose difference is compact, in particular formulas like

$$\text{index} = \operatorname{Tr}(e-e')^{\text{odd}}$$

when this is defined. I don't want to assume that e, e' are projectors, i.e. self-adjoint idempotents, as I did before.

Let's start with a Hilbert space H and two idempotents e, e' on H such that $e-e'$ is compact. <u>From the K-theory viewpoint what do we have?</u> I can consider the algebra generated by e, e' inside $B(H)$ and the ideal generated by $e-e'$; call the algebra A and ideal I. Then we have classes $[e], [e'] \in K_0 A = \operatorname{Ker}\{K_0 A^+ \to K_0 \mathbb{C}\}$; note A is non-unital, there's no reason for it to contain $1 \in B(H)$. Moreover $[e]-[e'] \in K_0 A$ goes to zero in $K_0(A/I)$, so it ought to be able to define $[e]-[e']$ as a well-defined class in $K_0 I$. If so then we can take the induced map $K_0 I \longrightarrow K_0(\text{compact})$ to obtain the index.

It might be better to work universally and let A be the universal non-unital $\mathbb{C}$-algebra generated by two idempotents. Then A^+ is the universal $\mathbb{C}$-algebra generated by 2 involutions, and hence A is the group algebra of the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$. Notation: e, e' for the idempotents and $F = 2e-1$, <u>$F' = 2e'-1$</u> for the involutions.

Let $A = \mathbb{C}e * \mathbb{C}e'$ be the universal non-unital algebra generated by two ~~idempotents~~ e, e'. This has a basis consisting of monomials

$$e, e'$$
$$ee', e'e$$
$$ee'e, e'ee'$$

etc. Let A^+ be the algebra obtained by adjoining 1 to A; then A^+ is the universal algebra generated by two involutions $F = 2e-1$, $F' = 2e'-1$, so its the group algebra of the infinite dihedral group with the generators F, F'. This gives a basis

$$1, \quad F, F', \quad FF', F'F, \quad FF'F, F'FF' \quad \text{etc}$$

for A^+. Another basis comes from describing the infinite dihedral group as $\mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ with the generators F, $FF' = g$. This gives the basis

$$1, g, g^{-1}, g^2, g^{-2}, \ldots \quad \text{and } F \text{ times these.}$$

i.e. $\quad g^n, Fg^n \quad n \in \mathbb{Z}$.

Let I be the ideal in A generated by $e - e'$. Then

$$I = \operatorname{Ker}\{\mathbb{C}e * \mathbb{C}e' \longrightarrow \mathbb{C}e\}$$

i.e. $I = q\mathbb{C}$ in Kuntz's notation. We can also describe I as

$$I = \operatorname{Ker}\{\mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] \longrightarrow \mathbb{C}[\mathbb{Z}/2]\}$$
$$= \text{ideal generated by } g-1 \text{ in } A^+.$$

There is a smaller ideal such that $A^+/$this ideal is the group ring of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$. This ideal is generated by

$$FF' - F'F = g - g^{-1}$$

A natural question is what is the K_0 of these ring? Because of split exact sequences we have

$$K_0 A = K_0 I \oplus \mathbb{Z}$$

$$K_0 A' = K_0 A \oplus \mathbb{Z} = K_0 I \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

Now the natural conjecture is that $K_0 I \simeq \mathbb{Z}$ with generator the ~~class~~ [redacted] $[e]-[e']$. We know that $K_0 I$ is at least this big, and it shouldn't be any bigger, because otherwise [redacted] there would be extra structure (primary operations [redacted]) on K_0 other than its abelian group structure.

Let's next consider the situation where we [redacted] have a homomorphism [redacted]

$$A \longrightarrow \mathcal{B}(H)$$

given by two idempotents in H such that $e-e'$ is compact, whence we have

$$I \longrightarrow \mathcal{K}(H)$$

and hence the class $[e]-[e']$ in $K_0 I$ gives rise to an element of $K_0(\mathcal{K}(H)) = \mathbb{Z}$, which is the index. Let's suppose $e-e'$ belongs to a Schatten ideal, whence I^n for large n maps to trace class operators.

Again, proceeding from the K-viewpoint, what we would like to do is show $[e]-[e']$ can be lifted to a class in $K_0(I^n)$ and then use the diagram

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(\mathcal{k}) = \mathbb{Z} \\ K_0(I^n) \arrow[u] \arrow[r] & K_0(\mathcal{L}^1) \arrow[u] \arrow[ur, "\text{trace}"'] \end{tikzcd}$$

If we want a trace formula for the index then what we must do is to explicitly lift $[e]-[e']$ in $K_0(I)$ back to $K_0(I^n)$.

We have a map of exact sequences

$$\begin{tikzcd} K_1(A^+) \arrow[r] \arrow[d, equal] & K_1(A^+/I^n) \arrow[r] \arrow[d] & K_0(I^n) \arrow[r] \arrow[d] & K_0(A^+) \arrow[r] \arrow[d, equal] & K_0(A^+/I^n) \arrow[d, "\simeq"] \\ K_1(A^+) \arrow[r] & K_1(A^+/I) \arrow[r] & K_0(I) \arrow[r] & K_0(A^+) \arrow[r] & K_0(A^+/I) \end{tikzcd}$$

which shows that $K_0(I^n) \twoheadrightarrow K_0(I)$. If we used instead

$$0 \to I^n \to I^+ \to I^+/I^n \to 0$$

we get

$$K_1(I^+) \to K_1(I^+/I^n) \to K_0(I^n) \to K_0(I^+) \to K_0(I^+/I^n)$$

or

$$K_1(I) \to K_1(I/I^n) \to K_0(I^n) \to K_0(I) \to \underbrace{K_0(I/I^n)}_{=0}$$

which gives it seems

$$0 \to GL(I^+/I^n)/\operatorname{Im} GL(I^+) \to K_0(I^n) \to K_0(I) \to 0$$

Thus the kernel of $K_0(I^n) \to K_0(I)$ should contain the obstructions to lifting invertibles mod I^n to invertibles.

Now let's be explicit. We have $[e]-[e'] \in K_0(I)$ and we want to show this class becomes zero in $K_0(A^+/I^n)$ using the fact that

$e \equiv e'$ mod I and the nilpotence of I mod I^4. This is a version of the fact that close idempotents are conjugate. Observe that we have a map

$$\operatorname{Im} e \oplus \operatorname{Im}(1-e) \xrightarrow{\begin{pmatrix} e'e & 0 \\ 0 & (1-e')(1-e) \end{pmatrix}} \operatorname{Im} e' \oplus \operatorname{Im}(1-e')$$

which becomes the identity mod I, so it will be invertible modulo I^4. Specifically

$$e'e + (1-e')(1-e') = \frac{(F'+1)(F+1)}{4} + \frac{(1-F')(1-F)}{4} = \frac{F'F+1}{2}$$

intertwines

$$F' \frac{F'F+1}{2} = \frac{F'F+1}{2} F$$

and $\dfrac{F'F+1}{2} \equiv \dfrac{F^2+1}{2} = 1$ mod I.

October 25, 1987

Multiplier algebra. Given an algebra A consider embeddings $A \hookrightarrow B$ such that A becomes a (two-sided) ideal in B. The multiplier algebra is a maximal "essential" such embedding. Let's try to figure out what this means.

Consider the case where A has a unit 1. Then it becomes an idempotent e in B which generates a 2-sided ideal. e is in the center of B ($x \in B \Rightarrow$ $\begin{matrix} xe \in A \Rightarrow exe = xe \\ ex \in A \Rightarrow exe = ex \end{matrix}$) so B is the direct product of A and the annihilator of e.

Perhaps essential means that there is no nonzero ideal I of B such that $I \cap A = 0$.

Example: Take $A = C_0(X)$ where X is a locally compact space. Let B be all bounded operators T on the Banach space $C_0(X)$ which commute with right multiplication operators:

$$T(fg) = T(f)g$$

Note that also one has

$$T(fg) = T(gf) = T(g)f = fT(g).$$

If K is a compact subspace of X, then we can choose $\chi \in A$ so that $\chi = 1$ on K. Then for f with support in K, we have

$$T(f) = T(\chi f) = T(\chi) f$$

which shows that T when restricted to $C_K(X)$ is multiplication by a continuous function. It's clear that there is a single function which works for all K, and it's bounded by the norm of T as an operator. Then approximation shows that T is multiplication by a bounded continuous fn. on X. Thus the multiplier algebra of $C_0(X)$ is $C(\beta X)$, where β is the Stone-Čech compactification.

It seems that in general the multiplier algebra is to be constructed from operators on A, and that in order to have A embedded inside one needs some sort of nondegenerateness for the multiplication.

For example if A is a vector space with the zero multiplication, i.e. an ideal of square zero in B, then the left multiplication operators and the right multiplication operators restricted to A give two commuting rings of operators on A. Call these $L, R \subset \mathrm{End}(A)$, so that one has a ring homomorphism

$$B/A \longrightarrow L \times R^{op}$$

which is injective if the embedding is essential. It is clear that by taking various maximal commuting pairs (L,R), and $B = A \tilde{\times} (L \times R^{op})$, that we can construct inequivalent embeddings.

Thus it seems that it is only reasonable to consider the multiplier algebra when A has an approximate identity, which is the case for C^* algebras.

Suppose $\mathcal{A}$ is an algebra with a trace $\tau : \mathcal{A} \longrightarrow \mathbb{C}$. Let's show τ induces a map $K_0(\mathcal{A}) \longrightarrow \mathbb{C}$. Actually we should generalize and produce a canonical map $K_0(\mathcal{A}) \longrightarrow \mathcal{A}/[\mathcal{A},\mathcal{A}]$.

Suppose $\mathcal{A}$ is unital, whence $K_0(\mathcal{A})$ is the Grothendieck group of $\mathcal{P}_{\mathcal{A}}$. Given $P \in \mathcal{P}_{\mathcal{A}}$ we can express it as a direct factor of A^n, say $P \simeq eA^n$ and define

$$\tau[P] = \tau(e) \overset{\text{defn}}{=} \sum_i \tau(e_{ii})$$

To see this is independent of the choices suppose also $P \simeq e'A^m$. Let $x : A^n \longrightarrow A^m$ be the composition $A^n \xrightarrow{e} eA^n \simeq P \simeq e'A^m \hookrightarrow A^m$, and let y be the composition the other way, namely $A^m \xrightarrow{e'} e'A^m \simeq P \simeq eA^n \hookrightarrow A^n$. Then x, y are matrices such that $yx = e$, $xy = e'$, so

$$\tau(e) = \tau(yx) = \tau(xy) = \tau(e')$$

where the middle inequality is a standard matrix calculation using that τ is a trace on $\mathcal{A}$. As $\tau([P]+[Q]) = \tau([P+Q]) = \tau([P]) + \tau([Q])$, it follows τ extends to the Grothendieck group $K_0\mathcal{A}$.

Next suppose $\mathcal{A}$ not necessarily unital, and form the unital algebra $\mathcal{A}^+$. One has

$$[\mathcal{A}^+, \mathcal{A}^+] = [\mathcal{A},\mathcal{A}] \quad \text{so}$$

$$\mathcal{A}^+/[\mathcal{A}^+,\mathcal{A}^+] = \mathcal{A}^+/[\mathcal{A},\mathcal{A}] = \mathbb{C} \oplus \mathcal{A}/[\mathcal{A},\mathcal{A}]$$

and by the above we have a canonical map

$$K_0(\mathcal{A}^+) \longrightarrow \mathcal{A}^+/[\mathcal{A}^+,\mathcal{A}^+].$$

Now $K_0(\mathcal{A}^+) = K_0(\mathbb{C}) \oplus K_0(\mathcal{A})$ where $K_0(\mathcal{A})$

is the subgroup generated by classes

$[P] - [P \otimes_{a^+} \mathbb{C} \otimes_{\mathbb{C}} a^+]$ with $P \in \mathcal{P}_{a^+}$

Let e be an idempotent matrix over a^+; then $e = e_0 + \alpha$ with e_0 idempotent over $\mathbb{C}$ and α a matrix over a. Then

$$\tau\left([P] - [P \otimes_{a^+} \mathbb{C} \otimes_{\mathbb{C}} a^+]\right) = \tau(e) - \tau(e_0) \in \mathbb{C} \oplus a/[a,a]$$

$$= \tau(e - e_0) \in a/[a,a]$$

so we get a canonical [redacted] homomorphism

$$K_0(a) \longrightarrow a/[a,a]$$

as desired.

The next step is to [redacted] consider the case where one has a trace τ on $a^2 = \operatorname{Im}(a \otimes a \longrightarrow a)$, and to see if it defines a map [redacted] on $K_0(a)$. [redacted] The idea is that one has

$$\begin{array}{ccccccccc} K_1(a^+) & \to & K_1(a^+/a^2) & \xrightarrow{\partial} & K_0(a^2) & \to & K_0(a) & \to & \underbrace{K_0(a/a^2)}_{=0} \\ & & & & \downarrow \tau & & & & \\ & & & & \mathbb{C} & & & & \end{array}$$

and there might be a reason for $\tau \partial = 0$. Or there might be an obstruction. [redacted]

Now

$$a^+/a^2 = \mathbb{C} \oplus \underbrace{a/a^2}_{I}$$

where $I^2 = 0$; this is a ring of dual numbers. We have

$$GL_n(I^+) = GL_n(\mathbb{C}) \ltimes M_n(I)$$

$$GL_n(I^+)_{ab} = GL_n(\mathbb{C})_{ab} \oplus M_n(I)_{GL_n(\mathbb{C})}$$

and

$$M_n(I)_{GL_n(\mathbb{C})} \xrightarrow[\text{trace}]{\sim} I$$

so that $K_1(I^+) = K_1(\mathbb{C}) \oplus I$. Thus it appears that we have canonical maps

$$(*) \qquad \mathcal{A}/\mathcal{A}^2 \xrightarrow{\partial} K_0(\mathcal{A}^2) \longrightarrow \mathcal{A}^2/[\mathcal{A}^2, \mathcal{A}^2]$$

If this composition is non-zero, then there is an obstruction to having a trace on $\mathcal{A}^2$ induce a map on $K_0(\mathcal{A})$.

But I notice now that the kind of traces to be used on $\mathcal{A}^2$ actually vanish on $[\mathcal{A}, \mathcal{A}]$. In any case we really ought to find what the composition $(*)$ is.

In general let's consider the composition

$$K_1(\mathcal{A}/\mathcal{A}^n) \xrightarrow{\partial} K_0(\mathcal{A}^n) \longrightarrow \mathcal{A}^n/[\mathcal{A}^n, \mathcal{A}^n]$$

An element of $K_1(\mathcal{A}/\mathcal{A}^n)$ is represented by a matrix $u = 1 - \alpha$ where α is a matrix over $\mathcal{A}/\mathcal{A}^n$. To construct $\partial[u]$ we lift u and u^{-1} to p, q over $\mathcal{A}$ such that $1 - pq \equiv \mathcal{A}^2$. For example lift α to a matrix a over $\mathcal{A}$ and take

$$p = 1 - a$$
$$q = 1 + a + \cdots + a^{2n-1}$$

so that

$$\begin{pmatrix} 1-a & a^n \end{pmatrix} \begin{pmatrix} 1 + a + \cdots + a^{2n-1} \\ a^n \end{pmatrix} = 1$$

Then $\partial[u]$ is represented by the idempotent

$$\begin{pmatrix} 1+\cdots+a^{2n-1} \\ \\ a^n \end{pmatrix} \begin{pmatrix} 1-a & a^n \end{pmatrix} = \begin{pmatrix} 1-a^{2n} & (1+\cdots+a^{2n-1})a^n \\ \\ a^n(1-a) & a^{2n} \end{pmatrix}$$

which is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ modulo $\mathcal{A}^n$. If we take the trace of the difference we get $-a^{2n}+a^{2n} = 0 \in \mathcal{A}^n/[\mathcal{A}^n, \mathcal{A}^n]$. Thus we have proved the following

Prop: Let τ be a trace on $\mathcal{A}^n$. Then it extends to a linear functional on $K_0(\mathcal{A})$. More precisely one has a unique dotted arrow:

$$\begin{tikzcd} K_0(\mathcal{A}^n) \arrow[r] \arrow[d] & K_0(\mathcal{A}) \arrow[dl, dashed] \\ \mathcal{A}^n/[\mathcal{A}^n, \mathcal{A}^n] & \end{tikzcd}$$

Next we need a formula for this kind of trace applied to an element of $K_0(\mathcal{A})$. Such a formula can be derived once and for all in the universal case $\mathcal{A} = q\mathbb{C}$. ∎

Our first problem is to describe the canonical class in $K_0(q\mathbb{C})$. This is sort of interesting because although $q\mathbb{C}$ sits inside the algebra $\mathbb{C}e * \mathbb{C}e'$ containing idempotents, $q\mathbb{C}$ nor $(q\mathbb{C})^+$ contains these idempotents. Thus we will have to find an idempotent matrix over

$q\mathbb{C}^+$ in order to represent the canonical K-class.

Consider the cartesian square

$$\begin{tikzcd}
q\mathbb{C} \arrow[r] \arrow[d, equal] & (q\mathbb{C})^+ \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\
q\mathbb{C} \arrow[r] & \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] \arrow[r] \arrow[d, equal] & \mathbb{C}[\mathbb{Z}/2] \arrow[d, equal] \\
& (\mathbb{C}e * \mathbb{C}e')^+ & (\mathbb{C}e)^+
\end{tikzcd}$$

A finite projective $(q\mathbb{C})^+$ module is a fin. proj. $\mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2]$-module equipped with a trivialization mod $q\mathbb{C}$. Over $\mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] = A^+$ we have four projectives eA^+, $e'A^+$, $(1-e)A^+$, $(1-e')A^+$. It looks like we want to take something like

$$P = eA^+ \oplus (1-e')A^+$$

together with the trivialization

$$P \otimes_{A^+} \mathbb{C}[\mathbb{Z}/2] = e\,\mathbb{C}[\mathbb{Z}/2] \oplus (1-e)\mathbb{C}[\mathbb{Z}/2]$$
$$\xrightarrow{\sim} \mathbb{C}[\mathbb{Z}/2].$$

Now we want to see explicitly that the corresponding projective $(q\mathbb{C})^+$-module $\bar{P}$ is a direct summand of a free module. Can we find an embedding of $\bar{P}$ in $((q\mathbb{C})^+)^2$?

Try to find generators of $\bar{P}$. An obvious element is $e \oplus (1-e')$. Similarly we have an obvious map from $\bar{P} \longrightarrow (q\mathbb{C})^+$ given by the sum map

$$eA^+ \oplus (1-e')A^+ \xrightarrow{+} A^+$$

Next we probably want an element of $\bar{P}$ which is zero modulo $q\mathbb{C}$.

The other way to proceed is to define the projective $\bar{Q}$ over $(q\mathbb{C})^+$ ~~...~~ by

$$Q = e'A^+ \oplus (1-e)A^+$$

~~with~~ with the trivialization

$$Q \otimes_{A^+} \mathbb{C}[\mathbb{Z}/2] = e\mathbb{C}[\mathbb{Z}/2] \oplus (1-e)\mathbb{C}[\mathbb{Z}/2] \xrightarrow{\sim} \mathbb{C}[\mathbb{Z}/2].$$

Then $\bar{P} \oplus \bar{Q}$ is defined by $P \oplus Q \simeq (A^+)^2$ together with an invertible 2×2 matrix over $\mathbb{C}[\mathbb{Z}/2]$. But $A^+ \to \mathbb{C}[\mathbb{Z}/2]$ has a section so this matrix lifts, and so $\bar{P} \oplus \bar{Q} \simeq (q\mathbb{C})^{+2}$

October 27, 1987

Let $\mathfrak{a} = q\mathbb{C} = \mathrm{Ker}\{\mathbb{C}e * \mathbb{C}e' \longrightarrow \mathbb{C}e\}$ or also

$$\mathfrak{a} = \mathrm{Ker}\{\mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] \longrightarrow \mathbb{C}[\mathbb{Z}/2]\}.$$

Call $A = \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2] = \mathbb{C}[F, F']$, where instead of F' I might write $-\varepsilon$ to make the link with earlier theory. The problem is to understand the canonical map

$$\begin{tikzcd}
K_0(\mathfrak{a}^n) \arrow[r, two heads] \arrow[d] & K_0(\mathfrak{a}) \arrow[dl, dashed] \\
\mathfrak{a}^n/[\mathfrak{a}^n, \mathfrak{a}^n] &
\end{tikzcd}$$

at least in this universal case.

Note that because $A \longrightarrow \mathbb{C}[\mathbb{Z}/2] = \mathbb{C}[F']$ has a section one has a split exact sequence

$$0 \longrightarrow K_0(\mathfrak{a}) \longrightarrow K_0(A) \overset{\dashleftarrow}{\longrightarrow} K_0(\mathbb{C}[\mathbb{Z}/2]) \longrightarrow 0$$

and so there is a canonical class in $K_0(\mathfrak{a})$ which becomes $[e] - [e']$ in $K_0(A)$. So if we wish to lift this class to $K_0(\mathfrak{a}^n)$ it will be sufficient to modify e, e' an equivalent idempotents over A which are congruent modulo $\mathfrak{a}^n$.

This we do as follows. Let's recall

$$\nu = e'e + (1-e')(1-e) = \frac{(1+F')(1+F)}{4} + \frac{(1-F')(1-F)}{4}$$

$$= \frac{1+F'F}{2} = 1 - \underbrace{\left(\frac{1-F'F}{2}\right)}_{\mu}$$

intertwines e, e':
$$e'\nu = \nu e$$
and ν is $\equiv 1 \mod \mathcal{A}$, hence invertible modulo $\mathcal{A}^n$. We can also find an invertible 2×2 matrix with ν in the upper left corner, namely
$$\begin{pmatrix} 1-\mu & -\mu^n \\ \mu^n & 1+\mu+\cdots+\mu^{2n-1} \end{pmatrix}$$
Put
$$\tilde{e} = \begin{pmatrix} 1-\mu & -\mu^n \\ \mu^n & 1+\cdots+\mu^{2n-1} \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1+\cdots+\mu^{2n-1} & \mu^n \\ -\mu^n & 1-\mu \end{pmatrix}$$
$$= \begin{pmatrix} 1-\mu & -\mu^n \\ \mu^n & 1+\cdots+\mu^{2n-1} \end{pmatrix} \begin{pmatrix} e(1+\cdots+\mu^{2n-1}) & e\mu^n \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} (1-\mu)e(1+\cdots+\mu^{2n-1}) & (1-\mu)e\mu^n \\ \mu^n e(1+\cdots+\mu^{2n-1}) & \mu^n e \mu^n \end{pmatrix}$$
$$= e'(1-\mu)(1+\cdots+\mu^{2n-1}) = e'(1-\mu^{2n})$$
Thus $\tilde{e} \equiv \begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix}$ modulo $\mathcal{A}^n$, and so the pair $\tilde{e}, \begin{pmatrix} e' & 0 \\ 0 & 0 \end{pmatrix}$ represents an element of $K_0(\mathcal{A}^n)$ which maps onto the class $[e]-[e']$ in $K_0(A)$, hence onto the class of $K_0(\mathcal{A})$ represented by the pair e, e'.

Now apply the trace $\mathcal{A}^n \xrightarrow{\tau} \mathcal{A}^n/[\mathcal{A}^n,\mathcal{A}^n]$ to this class in $K_0(\mathcal{A}^n)$ and we get

$$\tau\left(e'(1-\mu^{2n}) - e'\right) + \tau(\mu^n e \mu^n)$$

$$= \tau\left((e-e')\mu^{2n}\right).$$

We rewrite this as follows. Recall

$$\mu = \frac{1-F'F}{2} = \frac{1+\varepsilon F}{2} = \frac{1+g^{-1}}{2}$$

$$\tau(F\mu^{2n}) = \tau(\mu^n F \mu^n) = \tau\left(\left(\tfrac{1+g^{-1}}{2}\right)^n F \left(\tfrac{1+g^{-1}}{2}\right)^n\right)$$

$$= \tau\left(F\Big(\underbrace{\frac{1+g}{2}\cdot\frac{1+g^{-1}}{2}}\Big)^n\right)$$

$$\frac{(1+F\varepsilon)(1+\varepsilon F)}{4} = \frac{(\varepsilon+F)\varepsilon^2(\varepsilon+F)}{4}$$

$$= \frac{(F+\varepsilon)^2}{4}$$

Thus

$$\tau(F\mu^{2n}) = \tau\left(F\left(\frac{F+\varepsilon}{2}\right)^{2n}\right)$$

$$\tau(\varepsilon\mu^{2n}) = \tau\left(\varepsilon\left(\frac{F+\varepsilon}{2}\right)^{2n}\right)$$

so

$$\text{Index} = \tau\left((e-e')\mu^{2n}\right) = \tau\left(\frac{F+\varepsilon}{2}\mu^{2n}\right) = \tau\left(\left(\frac{F+\varepsilon}{2}\right)^{2n+1}\right)$$

$$= \tau\left((e-e')^{2n+1}\right)$$

It looks a ~~little~~ strange to have τ a trace on $\mathcal{A}^n$ and to use such a high power of $F+\varepsilon$. Let's check that the index is represented by a lower power when the trace is defined. Let's assume that the trace vanishes on $[a, a^{2n-1}]$. Then we can write

$$\tau\left(\varepsilon\left(\frac{F+\varepsilon}{2}\right)^{2n}\right) = \tau\left(\varepsilon\left(\frac{F+\varepsilon}{2}\right)\left(\frac{F+\varepsilon}{2}\right)^{2n-1}\right)$$

$$= \tau\left(\frac{F+\varepsilon}{2}F\left(\frac{F+\varepsilon}{2}\right)^{2n-1}\right) = \tau\left(F\left(\frac{F+\varepsilon}{2}\right)^{2n}\right)$$

and so we conclude

$$\tau\left(\varepsilon\left(\frac{F+\varepsilon}{2}\right)^{2n}\right) = \tau\left(F\left(\frac{F+\varepsilon}{2}\right)^{2n}\right) = \tau\left(\left(\frac{F+\varepsilon}{2}\right)^{2n+1}\right) = \text{Index}.$$

Also

$$\tau\left(\varepsilon\left(\frac{F+\varepsilon}{2}\right)\left(\frac{F+\varepsilon}{2}\right)^{2n-1}\right) = \frac{1}{2}\tau\left(\left(\frac{F+\varepsilon}{2}\right)^{2n-1}\right) + \frac{1}{2}\tau\Big(\varepsilon\underbrace{F\left(\frac{F+\varepsilon}{2}\right)^{2n-1}}_{\left(\frac{F+\varepsilon}{2}\right)^{2n-1}\varepsilon}\Big)$$

$$= \tau\left(\frac{F+\varepsilon}{2}\right)^{2n-1}$$

At this point I have calculated in $a^n/[a^n, a^n]$ the "trace" of ~~[illegible]~~ an element of $K_0(a)$. The formula shows that it lies in $a^{2n+1} + [a^n, a^n]/[a^n, a^n]$. Let us consider the tower

$$\begin{tikzcd}
K_0(a^n) \arrow[r] \arrow[d] & K_0(a^{n-1}) \arrow[r] \arrow[d] & \cdots \arrow[r, dashed] & K_0(a) \arrow[d] \arrow[dll, dashed] \arrow[dlll, dashed] \\
a^n/[a^n, a^n] \arrow[r] & a^{n-1}/[a^{n-1}, a^{n-1}] \arrow[r] & \cdots \arrow[r] & a/[a,a]
\end{tikzcd}$$

and the fact that $K_0(a)$ lifts. Thus we have in general a map

$$K_0(\mathcal{Q}) \longrightarrow \varprojlim_n \left(\mathcal{a}^n / [\mathcal{a}^n, \mathcal{a}^n] \right)$$

A natural question is what this looks like for $q\mathbb{C}$. We need to calculate commutators in $\mathcal{a}$.

We review the structure of $A = \mathbb{C}[g, g^{-1}] \tilde{\otimes} \mathbb{C}[\varepsilon]$ where $\varepsilon g \varepsilon^{-1} = g^{-1}$. Then $\mathcal{a}^n = \mathbb{C}[g,g^{-1}]\left(\frac{g+1}{2}\right)^n \tilde{\otimes} \mathbb{C}[\varepsilon]$. Let's single out the element

$$\left(\frac{g+1}{2}\right)\left(\frac{g^{-1}+1}{2}\right) = \frac{2+g+g^{-1}}{4} = \left(\frac{F+\varepsilon}{2}\right)^2$$

which is in the center of A. In fact it generates the center of A. (The center of A consists of $\sum a_n g^n$ with $a_n = a_{-n}$. Thus the center of A is a polynomial ring generated by $\frac{g+g^{-1}}{2}$. A is of rank 4 over its center.)

Let's put $z = \left(\frac{F+\varepsilon}{2}\right)^2$. What I want to do is say that $\mathcal{a}^n$ is generated over $k[z]$ by certain elements, and hence $[\mathcal{a}^n, \mathcal{a}^n]$ is generated over $k[z]$ by the brackets of the generators. Since A is of rank 4 over $k[z]$, so will be $\mathcal{a}^n = A\left(\frac{g+1}{2}\right)^n$

Suppose n even. Then $\mathcal{a}^n = z^m A$ where $m = \frac{n}{2}$.

so

$$\mathcal{a}^{2n} / [\mathcal{a}^n, \mathcal{a}^n] = z^{2m}A / z^{2m}[A,A]$$

$$\simeq A/[A,A]$$

But A is the group algebra of the infinite dihedral group, so $A/[A,A]$ is the vector space generated by the conjugacy classes. The conjugacy classes are $\{g^n, g^{-n}\}$ for $n \geq 0$, and all other elements $\{g^n \varepsilon, n \in \mathbb{Z}\}$ form a single conjugacy class. But recall that the index class

$$\text{index} = \text{image of } \varepsilon\left(\frac{F+\varepsilon}{2}\right)^{2n} \text{ in } a^{2n}/[a^n, a^n]$$

and if $n = 2m$, this is

$$\text{image of } \varepsilon z^{2m} \text{ in } z^{2m}A/[z^mA, z^mA].$$

Thus all the interest seems to be in this single conjugacy class.

Note that one has the grading of A given by $\mathbb{C}[g,g^{-1}] \oplus \mathbb{C}[g,g^{-1}]\varepsilon$, and that the corresponding splitting of $A/[A,A]$ separates the involution conjugacy class from the others. The odd part is always $\mathbb{C}$, the even part is $\mathbb{C}[g,g^{-1}]_{\mathbb{Z}/2}$. Next observe that the inverse limit of

$$\longrightarrow \mathbb{C}[g,g^{-1}] \xrightarrow{z} \mathbb{C}[g,g^{-1}] \longrightarrow$$

is zero, and this will also be the case if we take coinvariants under the $\mathbb{Z}/2$-action. The conclusion is that

$$\varprojlim_n \left(a^n/[a^n,a^n]\right) = \varprojlim_n \left(a^{2n}/[a^n,a^n]\right)$$

$$\xleftarrow{\sim} \mathbb{C}$$

with $1 \in \mathbb{C}$ going the class of εz^n in $a^{2n}/[a^n, a^n]$.

At this point I have probably found

out everything that can be expected concerning [redacted] the effect of traces on $\mathfrak{a}^n$ on $K_0(\mathfrak{a})$. Except perhaps one should recall the link between q and non-commutative differential forms.

Idea: Up to [redacted] now we have explored $K_0(\mathfrak{a})$ for $\mathfrak{a}$ non-unital by using [redacted] traces on $\mathfrak{a}^n$. Now the other thing one could do is to use excision, i.e. $K_0(\mathfrak{a})$ is independent of the ring $\mathfrak{a}$ sits as an ideal in. This is the multiplier algebra approach.

It seems that a more promising approach is the following. Given A nonunital [redacted] one forms Cuntz's construction:

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r] & A*A \arrow[r] & A \arrow[l, dashed, bend right] \arrow[r] & 0 \end{tikzcd}$$

so then on K-groups one has

$$\begin{tikzcd} & & K_0(A) \arrow[d, "\iota_*-\bar\iota_*"] \arrow[dl, dashed] & & \\ 0 \arrow[r] & K_0(qA) \arrow[r] & K_0(A*A) \arrow[r] & K_0(A) \arrow[r] & 0 \end{tikzcd}$$

and one can consider the effect of traces on the powers $(qA)^n$. Somehow I have to [redacted] find the link between Connes "cycles" and such higher traces. It has something to do with

$$gr(A*A) \simeq \Omega(A)$$

October 28, 1987

Suppose we take the Fredholm module situation, where we have $\mathcal{A}$ acting on a graded Hilbert space $H^+ \oplus H^-$ preserving the grading, and an odd $F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$ of square 1 such that for any $a \in \mathcal{A}$

$$[F, a] = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} - \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}\begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & p^{-1}\bar{a} - ap^{-1} \\ pa - \bar{a}p & 0 \end{pmatrix}$$

lies in a Schatten ideal. Then we have two homomorphism

$$\mathcal{A} \underset{a \mapsto p^{-1}\bar{a}p}{\overset{a \mapsto a}{\rightrightarrows}} \mathcal{B}(H^+)$$

which are congruent module $\mathcal{L}^p$. Thus we have

$$\begin{tikzcd}
q\mathcal{A} \arrow[r] \arrow[d, dashed] & \mathcal{A} * \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\
\mathcal{L}^p \arrow[r] & \mathcal{B}(H^+) \arrow[r] & \mathcal{B}(H^+)/\mathcal{L}^p
\end{tikzcd}$$

and thus we are in the situation where we have a trace on a power of $q\mathcal{A}$, which can be used to calculate the index associated to an element of $K_0(\mathcal{A})$.

October 29, 1987

Let $\mathcal{A}$ be the algebra of $f(h,x,p)$, either in the circle case, or the line case. On $\mathcal{A}$ we have a trace

$$\tau : f \longmapsto \int \frac{dx\,dp}{2\pi h} f(h,x,p)$$

which has values in $\frac{1}{h}C^\infty(h)$. Let F be an involutive matrix of the form $F = \varepsilon + \alpha$ where ε is constant, and α has entries in $\mathcal{A}$. Thus F represents an element of $K_0(\mathcal{A})$ and we can apply the above trace map to it to get a function of h with possibly a pole at $h=0$:

$$\tau([F]-[\varepsilon])(h) = \int \frac{dx\,dp}{2\pi h} \operatorname{tr}\big(F(h,x,p) - \varepsilon\big)$$

Now in fact we know by analysis, i.e. representations ρ_h for $h \neq 0$, that this index function of h is in fact independent of h.

Problem: Give a formal proof that the index is independent of h, in particular that there is no pole at $h=0$. Formal means the following. One can expand $F(h,x,p) = F_0 + hF_1 + \cdots$ as a formal power series in h, and then it should be an algebraic matter to see that the resulting series in h is constant.

There is some sort of analogy here with residues, perhaps.

Let's pursue the formal aspects of the problem. I know that the answer depends on $F(0,x,p) = F_0$, and I think that one doesn't have to work with an $F(h,x,p)$ smooth in h, but rather, one can work with formal power series in h.

Thus it should be possible somehow to start with F_0 over $\mathcal{A}_0 = \mathcal{A}/h\mathcal{A}$, lift F_0 to F over $\mathcal{A}/h^N\mathcal{A}$ using the nilpotence of the ideal $h\mathcal{A}/h^N\mathcal{A}$, then look at $\tau([F]-[\mathcal{E}])(h)$ in $\mathbb{C}[h]h^{-1}/\mathbb{C}[h]h^{N-1}$. Hopefully by a variation of the homotopy argument we will be able to see τ is constant in h.

October 30, 1987

Formal problem: a_0 = Schwartz functions on $T^*(M)$, $\hat{a}$ algebra over $\mathbb{C}[[h]]$ which is the deformation obtained from the tangent groupoid. On $\hat{a}$ we should have a trace with values in $h^{-n}\mathbb{C}[[h]]$. Now take an involution F_0 over a_0. One knows it can be lifted over $\hat{a}$ to define an element of $K_0\hat{a}$. This can then be paired with the trace. The problem is to compute this "index". The answer is the character of $[F_0]$ times Todd of $T^*(M)$ integrated over $T^*(M)$.

Since this is the answer, it is sort of clear that even for M a torus we are going to get involved with with the lower dim'l components of the character of $[F_0]$.

Ideas: Rescaling $(h,p) \longmapsto (th, tp)$ should give an action of $\mathbb{G}_m$ on on $\hat{a}$ leaving the trace invariant. It should be possible to show by a variant of homotopy-invariance that the index is constant. Perhaps also using translations in x, p (in the torus case), one can use the proof of invariance of the index to link up the de Rham complex of T^*.

October 31, 1987

There appears to be a formal theorem (perhaps valid for the tangent groupoid of a general manifold - I think this is what Connes described in a version of Ch. I, where he used formal ψDO's with an h.) To fix the ideas let's consider a torus $\mathbb{R}^n/\Gamma$. Then we have the algebra $\mathcal{A}_0$ of Schwartz fns. on $T^*(M) = \mathbb{R}^n/\Gamma \times \mathbb{R}^n$ and ~~the~~ deformation of it denoted $\hat{\mathcal{A}}$ consisting of formal series in h

$$f(h,x,p) = f_0(x,p) + h f_1(x,p) + \cdots$$

whose coefficients lie in $\mathcal{A}_0$. The multiplication is the twisted one $e^{-i\gamma x} f(p) e^{i\gamma x} = f(p+h\gamma)$. On $\mathcal{A}$ we have a trace with values in $h^{-n}\mathbb{C}[[h]]$:

$$\tau(f)(h) = \int \left(\frac{dx\,dp}{2\pi h}\right)^n f(h,x,p)$$

Given an element of $K_0(\mathcal{A}_0)$ represented by an involution F_0 congruent modulo $\mathcal{A}_0$ to a standard involution ε, one know it is possible to lift F_0 to an involution F over $\mathcal{A}$. The claim is that $\tau(F-\varepsilon)$ is ~~independent of~~ constant in h and depends only on F_0. And ultimately one wants a formula for $\tau(F-\varepsilon)$ in terms of the character of $[F_0]$.

The reason $\tau(F-\varepsilon)$ depends only on the choice of F_0 is because $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{A}_0)$. (It's enough to use $\mathcal{A}/h^N\mathcal{A}$ instead of $\mathcal{A}$.)

To see that $\tau(F-\varepsilon)$ is ~~indepen~~ constant we consider rescaling $\sigma_t f(h,x,p) = f(th,x,tp)$.

This gives an action of $\mathbb{R}^\times_{>0}$ on $\mathcal{A}$ such that

$$\tau(\alpha_t f)(h) = \int \left(\frac{dx\,dp}{2\pi h}\right)^n f(th, x, tp)$$

$$= \int \left(\frac{dx\,dp}{2\pi th}\right)^n f(th, x, p) = \tau(f)(th)$$

Check multiplication:

$$\alpha_t\left(e^{-i\gamma x} f(h,x,p) e^{i\gamma x}\right) = \alpha_t\{f(h,x,p+h\gamma)\}$$

$$= f(th, x, tp + th\gamma) = e^{-i\gamma x} \underbrace{f(th, x, tp)}_{\alpha_t f(h,x,p)} e^{i\gamma x}$$

Set $\mathring{F} = \partial_t \alpha_t(F)\big|_{t=1} = (h\partial_h + p\partial_p)f$. Then from $F^2 = 1$ we have $\alpha_t(F)^2 = 1$, so we have $\mathring{F}F + F\mathring{F} = 0$. Thus

$$\tau(\mathring{F}) = \tau(F^2\mathring{F}) = \tau(F\mathring{F}F) = -\tau(F^2\mathring{F}) = -\tau(\mathring{F})$$

$$\Longrightarrow \tau(\mathring{F}) = 0.$$

Now by differentiating

$$\tau(\alpha_t F - \varepsilon)(h) = \tau(F-\varepsilon)(th)$$

and setting $t=1$, we find

$$\underset{\substack{\shortparallel \\ 0}}{\tau(\mathring{F})(h)} = h\partial_h \tau(F-\varepsilon)(h)$$

which shows $\tau(F-\varepsilon)(h)$ is constant in h. $\therefore$

<u>Prop</u>. $\tau(F-\varepsilon)(h)$ is constant in h and it depends only on $[F_0] \in K_0(\mathcal{A}_0)$

So next we consider the problem of finding a formula for $\tau(F-\varepsilon)$ in terms of F_0.

November 1, 1987

Discussion of the problem. We've seen that the index of an involution F over $\mathcal{A}$ in the case $n=1$ is the integral of the character form $\mathrm{tr}\left(F_0(dF_0)^2\right)$ up to a constant. The problem is to generalize this to higher n.

A first idea is to do non-commutative differential calculus, that is, to do in a non-commutative setting the formalism which leads to the result that $\int \mathrm{tr}\, F_0(dF_0)^2$ depends only on the K-class of F_0. Thus we can embed the fixed-h algebra $\mathcal{A}_h$ into a de Rham complex, that is, a differential graded algebra:

$$(*) \qquad \mathcal{A}_h \otimes \Lambda \mathbb{R}^{2n}$$

using the fact that the derivations $\partial_{x_i}, \partial_{p_i}$ on $\mathcal{A}_h$ commute. This diff graded algebra is a deformation of the de Rham complex of $\mathcal{A}_0$. Integration of $2n$-forms gives a graded closed trace on $(*)$. It should follow from the scaling argument that given any F over $\mathcal{A}$, the number

$$\int \mathrm{tr}\left(F_h(dF_h)^{2n}\right)$$

is independent of h, so can be evaluated at $h=0$.

But now comes the real problem of relating

The index of F, i.e. the trace of $F-\varepsilon$ acting via ρ_h on $L^2(S^1)$ to the non-comm. characteristic number $\int \operatorname{tr} F_h (dF_h)^{2n}$.

This should involve cyclic cohomology, namely, the trace on $\mathcal{A}_h$ defined by ρ_h is a cyclic 0 cocycle, which via Connes S-operator gives rise to a cyclic $2n$-cocycle. The cyclic cohomology class of this $2n$-cocycle should be the same as the one defined by the non-comm. de Rham complex.

It seems now that the thing to understand is why a graded trace on a differential graded algebra should induce a linear functional on K_0 of the algebra.

For tomorrow's lecture we should prove

$$K_0(A) \xrightarrow{\sim} K_0(A/I) \qquad \text{if } I \text{ nilpotent.}$$

We can suppose A has a unit, and we can work with involutions. First we show that any involution over A/I lifts to A. Given f over A/I with $f^2 = 1$, lift f to u over A. Then $u^2 = 1-\alpha$ with α a matrix with entries from I. Assuming I nilpotent we can form the element

$$(1-\alpha)^{-1/2} = \sum_{k \geq 0} \frac{1 \cdot 3 \cdots (2k-1)}{2^k k!} \alpha^k$$

which commutes with u. Then $F = u(1-\alpha)^{-1/2}$ is an involution over A lifting f.

Next we show two involutions F, ε over A, which are congruent modulo I, are conjugate by an invertible matrix which is $\equiv 1 \bmod I$. This follows from

$$F \, \frac{F\varepsilon + 1}{2} = \frac{F\varepsilon + 1}{2} \, \varepsilon$$

and the fact that $\frac{F\varepsilon+1}{2} \equiv \frac{\varepsilon^2+1}{2} = 1 \mod I$, so $\frac{F\varepsilon+1}{2}$ is invertible. (Instead of $\frac{F\varepsilon+1}{2} = \frac{g+1}{2}$ I could use $g^{1/2}$ since $g^{-1/2}\frac{g+1}{2} = \frac{g^{1/2}+g^{-1/2}}{2}$ commutes with ε. Here $g^{1/2}$ can be defined using the exponential and logarithm series.)

The same conclusion holds when I is topologically nilpotent and sufficiently complete, so that the series above converge.

We have to <u>correct an error</u> about the conjugacy classes in the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$ with generators F, ε or $\mathbb{Z} \rtimes (\mathbb{Z}/2)$ with generators $g = F\varepsilon$, ε. All the elements $g^n\varepsilon$ are involutions, but there are two conjugacy classes, since

$$g(g^n\varepsilon)g^{-1} = g^{n+2}\varepsilon$$

Thus F and ε are in different conjugacy classes, which can also be seen by the map $\mathbb{Z}/2 * \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Consider then $A = \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2]$

$= \mathbb{C}[u,u^{-1}] \oplus \mathbb{C}[u,u^{-1}]\varepsilon$

where we write u instead of g.

Let's consider A to be $\mathbb{Z}/2$-graded, and consider the induced grading on $A/[A,A]$, which is a vector space having a basis in one-one correspondence with the conjugacy classes of the inf. dihedral group. Look at the odd part

$$[u^m, u^n\varepsilon] = u^n(u^m\varepsilon - \varepsilon u^m) = u^n(u^m - u^{-m})\varepsilon$$

Thus the odd part $[A,A]^-$ is an ideal in $\mathbb{C}[u,u^{-1}]$ times ε. The ideal is obviously generated by $u-u^{-1}$. So

$$[A,A]^- = (u-u^{-1})\mathbb{C}[u,u^{-1}]\varepsilon$$

and

$$A^-/[A,A]^- = \Big(\underbrace{\mathbb{C}[u,u^{-1}]/(u-u^{-1})}\Big)\cdot\varepsilon$$

$$\xrightarrow{\sim} \mathbb{C}\times\mathbb{C} \quad \text{corresp to } u\mapsto \pm 1.$$

is 2 dimensional.

Also

$$[u^m\varepsilon, u^n\varepsilon] = u^{m-n} - u^{n-m}$$

so

$$A^+/[A^+,A^+] \simeq \mathbb{C}[u,u^{-1}]_{\mathbb{Z}/2}$$

Next we consider $\mathfrak{a} = g\mathbb{C}$:

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \xrightarrow{\text{fold}} \mathbb{C}[\mathbb{Z}/2] \longrightarrow 0$$

Thus $\mathfrak{a}$ is generated by $(u-1)$. We wish to find

$$\varprojlim_n \mathfrak{a}^n/[\mathfrak{a}^n,\mathfrak{a}^n]$$

since $K_0(\mathfrak{a})$ maps naturally to this inverse limit. As before this is the same as

$$\varprojlim_n \mathfrak{a}^{2n}/[\mathfrak{a}^n,\mathfrak{a}^n]$$

and we can suppose $n = 2m$ is even whence
$$a^{2m} = z^m A$$
where $z = (u-1)(u^{-1}-1) = 2-u-u^{-1}$ generates the center. We have

$$\begin{array}{ccccc} a^{4m} & = & z^{2m}A & \subset & A \\ \cup & & \cup & & \cup \\ [a^{2m}, a^{2m}] & = & z^{2m}[A,A] & \subset & [A,A] \end{array}$$

Recall that the even part gives zero in the inverse limit. This is because we can identify

$$\left(a^{2m}/[a^{2m},a^{2m}]\right)^+ = \left(z^{2m}\mathbb{C}[u,u^{-1}]\right)_{\mathbb{Z}/2}$$
$$\uparrow \wr$$
$$\left(z^{2m}\mathbb{C}[u,u^{-1}]\right)^{\mathbb{Z}/2}$$

and coinvariants are the same as invariants and $\bigcap_m z^{2m}\mathbb{C}[u,u^{-1}] = 0$.

As for the odd part note that $z = 2-u-u^{-1}$ goes to zero under $u \mapsto 1$, and is non-zero as $u \mapsto -1$. So in forming the inverse limit we can replace a^{4m} by $z^{2m}a$ whence

$$\left(z^{2m}a/[a^{2m},a^{2m}]\right)^- = \left(z^{2m}\, a/[A,A]\right)^-$$

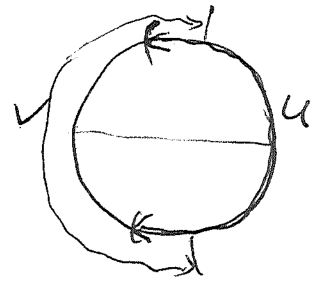
is one dimensional. So we can still conclude that

$$\varprojlim_n a^n/[a^n,a^n] \xrightarrow{\sim} \mathbb{C}$$

with the same <u>generators</u> as before.

It seems I can now calculate K_0 for $A = \mathbb{C}[\mathbb{Z}/2 * \mathbb{Z}/2]$ and for the C^* group algebra and the smooth group algebra. Let's take the latter cases first. The C^* group algebra of $\mathbb{Z}$ is $C(\mathbb{T})$, so the C^*-version of A is the cross product $C(\mathbb{T}) \times (\mathbb{Z}/2)$ with $\mathbb{Z}/2$ acting on $\mathbb{T}$ by conjugation. Finite proj modules over $\mathbb{C}(\mathbb{T}) \times (\mathbb{Z}/2)$ are the same thing as equivariant bundles on $\mathbb{T}$ for the $(\mathbb{Z}/2)$-action, so $K_0(C^* \text{ version of } A) = K^0_{\mathbb{Z}/2}(\mathbb{T})$.

Now let's use the open covering of $\mathbb{T}$



and we have MV + homotopy

$$\begin{array}{ccccccccc}
K^1_{\mathbb{Z}/2}(\{\pm i\}) & \longrightarrow & K^0_{\mathbb{Z}/2}(\mathbb{T}) & \longrightarrow & K^0_{\mathbb{Z}/2}(pt) & \oplus & K^0_{\mathbb{Z}/2}(pt) & \longrightarrow & K^0_{\mathbb{Z}/2}(\{\pm i\}) \\
\| & & & & \| & & \| & & \| \\
K^1(pt) = 0 & & & & R(\mathbb{Z}/2) & & R(\mathbb{Z}/2) & & K^0(pt) = \mathbb{Z}
\end{array}$$

(Put another way, an equivariant bundle on $\mathbb{T}$ is given by two representations of $\mathbb{Z}/2$ and an isomorphism of their underlying vector spaces.)
The two maps $R(\mathbb{Z}/2) \longrightarrow \mathbb{Z}$ are the augmentation so one sees that $K^0_{\mathbb{Z}/2}(\mathbb{T}) \simeq \mathbb{Z}^3$.

The same argument should be valid in the smooth case.

Next consider the algebraic situation, where A is the cross product $\mathbb{C}[u, u^{-1}] \times \mathbb{Z}/2$. Modules over this are equivariant $\mathbb{C}[u, u^{-1}]$ modules. Because the order of the group is invertible, finite proj A-modules should be equivariant $\mathbb{C}[u, u^{-1}]$-modules which are finite proj. over $\mathbb{C}[u,u^{-1}]$.

Now this A is regular and we should have a localization sequence for localizing with respect to $v = u - u^{-1}$

$$v\text{-torsion modules} \longrightarrow \mathrm{Modf}(A) \longrightarrow \mathrm{Modf}(A[v^{-1}])$$

This should give a long exact sequence:

$$\to K_1\left(A[v^{-1}]\right) \xrightarrow{\partial} K_0\left(\mathbb{C}[\mathbb{Z}/2]\times\mathbb{C}[\mathbb{Z}/2]\right) \xrightarrow{\circledast} K_0(A) \longrightarrow K_0\left(A[v^{-1}]\right) \to 0$$

Now $A[v^{-1}]$ is the cross product of $\mathbb{C}[u,u^{-1}][v]$ and $\mathbb{Z}/2$, and since $\mathbb{Z}/2$ acts freely on $\mathbb{C}-\{0,1,-1\}$ with the action $\zeta \longmapsto \zeta^{-1}$, it should follow by Galois descent that the modules over $A[v^{-1}]$ as the same as the modules over the invariants, which is $\mathbb{C}[x]\,[(x^2-1)^{-1}]$, where $x = \dfrac{u+u^{-1}}{2}$. Note that

$$\frac{u+u^{-1}}{2} = x \iff u^2 - 2xu + 1 = 0 \iff u = x \pm \sqrt{x^2-1}$$

so $(\mathbb{C}-\{0,1,-1\})/\mathbb{Z}/2 = \mathbb{C} - \{\pm 1\}$. So we should know that

$$\begin{cases} K_0(A[v^{-1}]) = \mathbb{Z} \\ K_1(A[v^{-1}]) = K_1(\mathbb{C}) \oplus \mathbb{Z} \oplus \mathbb{Z} \end{cases}$$

It remains to do the calculation of ∂. It seems reasonable to expect that the two units $u \pm 1$ in $A[v^{-1}]$, ought to give interesting factors in each $K_0(\mathbb{C}[\mathbb{Z}/2])$, which would then show that $K_0(A) \simeq \mathbb{Z}^3$. In any case one ought to be able to compute the map $\circledast$ easily using resolution.

Let's try it as follows: Over A we

have $\mathbb{C}$ considered as a left-module in 4 ways where $u = \pm 1$, $\varepsilon = \pm 1$. To compute $\circledast$: $K_0(\mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2]) \longrightarrow K_0(A)$, we must take each of these and resolve them by finite proj. resolutions over A. For example we have

$$0 \longrightarrow A \xrightarrow{\cdot(u-1)} A \longrightarrow A/(u-1)A \longrightarrow 0$$
$$A/(u-1)A = \mathbb{C}\binom{u=1}{\varepsilon=1} \oplus \mathbb{C}\binom{u=1}{\varepsilon=-1}$$

$$0 \longrightarrow A \xrightarrow{\cdot(u+1)} A \longrightarrow A/(u+1)A \longrightarrow 0$$
$$A/(u+1)A = \mathbb{C}\binom{u=-1}{\varepsilon=1} \oplus \mathbb{C}\binom{u=1}{\varepsilon=-1}$$

which shows that $\circledast$ half of the four $\mathbb{Z}$'s in $K_0(\mathbb{C}[\mathbb{Z}/2 \times \mathbb{Z}/2])$. It follows that $K_0(A)$ has 3 generators, hence $K_0(A) \cong \mathbb{Z}^3$ since [redacted] it generates the C^* [redacted] K_0.

Let us consider the f. proj module $A\left(\frac{1+\varepsilon}{2}\right)$ which maps onto $\mathbb{C}\binom{u=1}{\varepsilon=1} = A/(u-1, \varepsilon-1)$ and try to determine the kernel K:

$$0 \longrightarrow K \longrightarrow A\left(\frac{1+\varepsilon}{2}\right) \longrightarrow \mathbb{C} \longrightarrow 0$$

$A\left(\frac{1+\varepsilon}{2}\right)$ consists of $\left(f(u)+g(u)\varepsilon\right)\left(\frac{1+\varepsilon}{2}\right) = \left(f(u)+g(u)\right)\left(\frac{1+\varepsilon}{2}\right)$

$\therefore$ $A\left(\frac{1+\varepsilon}{2}\right)$ consists of $\left\{f(u)\left(\frac{1+\varepsilon}{2}\right)\right\} \subset A$. Such an element goes to zero in $\mathbb{C}\binom{u=1}{\varepsilon=1}$, when $f(1) = 0$ whence $f(u) \in (u-1)\mathbb{C}[u, u^{-1}]$. Thus

$$K = \left\{ f(u)(u-1)\left(\frac{1+\varepsilon}{2}\right)\right\} \subset A.$$

is generated by $(u-1)\left(\frac{1+\varepsilon}{2}\right)$. Let's determine the relations:

$$\left(f(u)+g(u)\varepsilon\right)(u-1)\left(\frac{1+\varepsilon}{2}\right) = f(u)(u-1)+g(u)(u^{-1}-1)]\left(\frac{1+\varepsilon}{2}\right)$$

$$= \left(f(u)-g(u)u^{-1}\right)(u-1)\left(\frac{1+\varepsilon}{2}\right)$$

This is zero $\iff f(u)u = g(u)$.

$$\iff f(u)+g(u)\varepsilon = f(u)(1+u\varepsilon)$$

This implies $K = A\cdot\frac{1+u\varepsilon}{2}$ and $\frac{1+u\varepsilon}{2} = \frac{1+F}{2}$

Thus we have an exact sequence

$$0 \longrightarrow A\left(\frac{1+F}{2}\right) \longrightarrow A\left(\frac{1+\varepsilon}{2}\right) \longrightarrow \mathbb{C}\begin{pmatrix} u=1 \\ \varepsilon=1 \end{pmatrix} \longrightarrow 0$$

Similarly we should have

$$0 \longrightarrow A\left(\frac{1-F}{2}\right) \longrightarrow A\left(\frac{1-\varepsilon}{2}\right) \longrightarrow \mathbb{C}\begin{pmatrix} u=1 \\ \varepsilon=-1 \end{pmatrix} \longrightarrow 0$$

and so forth.

November 2, 1987

Here seems to be the central problem: There are many ways to detect elements of $K_0\mathcal{A}$ by infinitesimal methods:

1) Given a trace τ on $\mathcal{A}^n$ it induces from K_0 by:

$$\begin{tikzcd} K_0\mathcal{A}^n \arrow[r, two heads] \arrow[d, "\tau"] & K_0\mathcal{A} \arrow[dl, dotted] \\ \mathbb{C} & \end{tikzcd}$$

2) Given a deformation $\mathcal{A} = \mathcal{B}/I$ with $I^n = 0$ and a trace τ on $\mathcal{B}$, it defines a maps by

$$\begin{tikzcd} K_0(\mathcal{B}) \arrow[r, "\sim"] \arrow[d, "\tau"] & K_0(\mathcal{A}) \\ \mathbb{C} & \end{tikzcd}$$

3) Given a differential graded algebra $\Omega^\cdot$ starting with $\mathcal{A}$ and a graded trace on $\Omega^\cdot$, it defines a functional on $K_0\mathcal{A}$ by the connection, curvature, etc. scheme.

The problem is how to compare these methods. An important example should be the one I found for the index theorem on the circle: Here the ring is $\mathcal{A}_0 = S(T^*(S^1))$ and the deformation is $\mathcal{B} = \mathcal{A}/h^N\mathcal{A}$, where $\mathcal{A}$ is the algebra of $f(h,x,p)$ as before. The trace starts with $\int \frac{dx\,dp}{2\pi h}$ which has values in $h^{-1}\mathbb{C}[[h]]/h^{N-1}\mathbb{C}[[h]]$, and τ picks out the constant term.

Thus we have the two approaches, one based on the de Rham complex

$$\mathcal{a}_0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \cdots \qquad \Omega^i = \mathcal{a}_0 \otimes \Lambda^i(\mathbb{R}^2)$$

and the other on the deformation $\mathcal{a}/h^N\mathcal{a} \to \mathcal{a}_0$. We could try to fit these together using the diffl algebra

$$\mathcal{a} \xrightarrow{d} \Lambda^1\mathbb{R}^2 \otimes \mathcal{a} \xrightarrow{d} \Lambda^2\mathbb{R}^2 \otimes \mathcal{a} \xrightarrow{d} \cdots$$

Then the problem is to link up the two traces.

KK-theory:

<u>Def</u>: A, B C^*-algebras (possibly $\mathbb{Z}/2$-graded) a <u>Kasparov module for (A,B)</u> is a triple (E, ϕ, F) where E is a graded Hilbert B-module (countably generated), $\phi: A \longrightarrow \mathbb{B}(E)$ is a graded $*$ hom., and $F \in \mathbb{B}(E)$ is an operator of odd degree such that

$$\left.\begin{matrix} [F, \phi(a)] \\ (F^2-1)\phi(a) \\ (F-F^*)\phi(a) \end{matrix}\right\} \in \mathbb{K}(E) \qquad \forall a \in A.$$

I would like to think of a Kasparov (A,B)-module as defining a map from the "K-theory" of A to the "K-theory" of B. Hopefully the theory shows that ~~[illegible]~~ $KK(A,B) \stackrel{\text{defn}}{=}$ the equivalence classes of Kasparov (A,B)-modules is indeed the set of natural transformations from the K-theory of A to the K-theory of B.

Examples: 1) A homom. $A \to B$, more generally a homomorphism $A \to B \otimes \mathbb{K}$, determines a natural transf from K-th of A to K-th of B.

2) A split exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & D \arrow[r] & A \arrow[r] \arrow[l, bend right, dashed, "s"'] & 0 \end{tikzcd}$$

by the K-theory exact sequence, determines a map from K-th (A) to K-th(B). There's a corresp. Kasparov (D, B)-module.

3) Consider the canonical extension

$$0 \longrightarrow \underset{\mathbb{K}(\mathbb{H}_B)}{\overset{\shortparallel}{B \otimes \mathbb{K}}} \longrightarrow \underset{\mathbb{B}(\mathbb{H}_B)}{\overset{\shortparallel}{M^s(B)}} \longrightarrow Q^s(B) \longrightarrow 0$$

here $\mathbb{H}_B$ = the Hilbert module $\ell^2 \otimes B$. Then there's an isom

$$K_1(Q^s(B)) \xrightarrow{\sim} K_0(B)$$

so in some sense an equivalence of the K-theory of $Q^s(B)$ with that of $B \hat{\otimes} C_1$. Thus a map $A \longrightarrow Q^s(B)$ induces a map from K-th of A to K-th of $B \hat{\otimes} C_1$.

<u>Problem</u>: In the algebraic context with $A \to A/I$, I nilpotent, one has a map $K_0(A/I) \xrightarrow{\sim} K_0(A)$. Is there a kind of Kasparov construction, i.e. a Kasparov $(A/I, A)$-modules, even though there needn't be a lifting of A/I back to A?

Example. $\mathcal{a} = \{f(h,x,p)\}$ as usual on the circle. Then we have an algebra extension

$$0 \longrightarrow h\mathcal{a}/h^2\mathcal{a} \longrightarrow \mathcal{a}/h^2\mathcal{a} \longrightarrow \mathcal{a}_0 \longrightarrow 0$$

and we have a trace on $\mathcal{a}/h^2\mathcal{a}$ given by

$$f(h,x,p) = f_0(x,p) + h f_1(x,p) \overset{\tau}{\longmapsto} \int \frac{dx\,dp}{2\pi} f_1(x,p)$$

Notice that $[\mathcal{a}/h^2\mathcal{a}, \mathcal{a}/h^2\mathcal{a}] = h\{\mathcal{a}_0,\mathcal{a}_0\} \subset h\mathcal{a}/h^2\mathcal{a}$, so traces on $\mathcal{a}/h^2\mathcal{a}$ are linear functionals vanishing on $h\{\mathcal{a},\mathcal{a}\}$. Here $\{f,g\} = \partial_p f\,\partial_x g - \partial_x f\,\partial_p g$, so that $[f,g] = \frac{h}{i}\{f,g\}$.

[redacted] What are the linear functionals on $\mathcal{a}_0/\{\mathcal{a}_0,\mathcal{a}_0\}$? Since $\{f,g\}\,dp\,dx = df\,dg$ we are asking for **linear** functionals [redacted] on 2-forms vanishing on closed ones, and the unique possibility up to a scalar is $\int f\,dp\,dx$.

We have a canonical ~~[redacted]~~ functional on $K_0(\mathcal{a}_0)$ given by

$$\begin{array}{ccc} K_0(\mathcal{a}/h^2\mathcal{a}) & \xrightarrow{\ \sim\ } & K_0(\mathcal{a}_0) \\ \downarrow{\scriptstyle\tau} & & \\ \mathbb{C} & & \end{array}$$

which we want to understand. Try DR cx

[redacted]

$$(\mathcal{a}/h^2) \xrightarrow{d} \Lambda^1\mathbb{R}^2 \otimes (\mathcal{a}/h^2) \xrightarrow{d} \Lambda^2\mathbb{R}^2 \otimes (\mathcal{a}/h^2) \to 0$$

which will give *cyclic* cocycles.