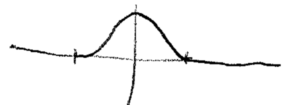
August 23, 1987

I want to understand Friedrichs's mollifier method starting on the circle. Fix a $\varphi \in C_0^\infty(-\delta, \delta)$ with shape  such that $\int \varphi(x)\,dx = 1$. Set

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right).$$

Given a distribution u we can then form

$$(\varphi_\varepsilon * u)(x) = \int \varphi_\varepsilon(x-y)\,u(y)\,dy$$

which is a smooth function. As $\varepsilon \downarrow 0$, φ_ε approaches $\delta(x)$ and $\varphi_\varepsilon * u$ approaches u. To be more specific, let's work on the F.T. side

$$\hat\varphi_\varepsilon(\xi) = \int e^{-ix\xi}\,\frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right)dx$$

$$= \int e^{-i\varepsilon x\xi}\,\varphi(x)\,dx = \hat\varphi(\varepsilon\xi)$$

and

$$\widehat{\varphi_\varepsilon * u}(\xi) = \hat\varphi(\varepsilon\xi)\,\hat u(\xi) \longrightarrow \hat u(\xi).$$

~~In the above we have been working on the line $\mathbb{R}$. Now suppose we move to the circle $\mathbb{R}/2\pi\mathbb{Z}$, whence $\xi \in \mathbb{Z}$.~~

~~First notice that~~

Notice that because $\varphi(x)\,dx$ is a prob. measure we have

$$|\hat\varphi(\xi)| \le 1$$

and hence by dominated convergence, one has

$$\varphi_\varepsilon * u \longrightarrow u$$

in L^2 when $u \in L^2$, and similarly for the Sobolev spaces H_s.

In the above we have been working on the line. Now let's move to the circle $\mathbb{R}/2\pi\mathbb{Z}$, whence we essentially make $\varphi(x)$ periodic with period 2π, and restrict ξ to be in $\mathbb{Z}$. If

$$u(x) = \sum_{n\in\mathbb{Z}} e^{inx}\,\hat{u}_n$$

then
$$(\varphi_\varepsilon * u)(x) = \sum_{n\in\mathbb{Z}} e^{inx}\,\hat{\varphi}(\varepsilon n)\,\hat{u}_n$$

Notice that the effect of $\varphi_\varepsilon *$ is to dampen the high frequencies, since $\hat{\varphi}(\xi)$ decays rapidly at ∞ i.e. is $O\left(\frac{1}{|\xi|^N}\right)$ for all N. Perhaps another "cutoff" process, e.g. Abel's might be simpler in the sequel.

Next let's turn to the main result which says that if P is a first order differential operator and if $u \in L^2$ is such that $Pu \in L^2$, then $P(\varphi_\varepsilon * u) \longrightarrow Pu$ in L^2. ~~Let's look first at $P = p\partial_x$, where $p(x) \in C^\infty(S^1)$ and where in fact $p(x) = e^{i\xi x}$.~~ Since $\varphi_\varepsilon * Pu \longrightarrow Pu$ in L^2 it's enough to show

$$P(\varphi_\varepsilon * u) - \varphi_\varepsilon * (Pu) \longrightarrow 0.$$

If $P = p\partial_x + q$, this means it's enough to show for any $u \in L^2$ that

$$\varphi_\varepsilon * (pu') - p\underbrace{(\varphi_\varepsilon * u)'}_{\varphi_\varepsilon * u'}, \qquad \varphi_\varepsilon * (qu) - q(\varphi_\varepsilon * u)$$

go to zero in L^2.

To simplify suppose $g(x) = e^{ikx}$. Then

$$\varphi_\varepsilon * (gu) = \varphi_\varepsilon * \sum e^{i(k+n)x}\hat{u}_n$$

$$= \sum_n e^{i(k+n)x}\,\hat{\varphi}(\varepsilon k+\varepsilon n)\,\hat{u}_n$$

$$g(\varphi_\varepsilon * u) = \sum_n e^{i(k+n)x}\,\hat{\varphi}(\varepsilon n)\,\hat{u}_n$$

So

$$\varphi_\varepsilon * (gu) - g(\varphi_\varepsilon * u) = \sum_n e^{i(k+n)x}\left[\hat{\varphi}(\varepsilon k+\varepsilon n) - \hat{\varphi}(\varepsilon n)\right]\hat{u}_n$$

We want to show this goes to zero as $\varepsilon \downarrow 0$ for any $u \in L^2$, i.e. that $[\varphi_\varepsilon *, g]$ converges strongly to zero. We can ask for the stronger result that this operator converges to zero in norm. The strong convergence is ~~obvious~~ clear since $\hat{\varphi}(\varepsilon k+\varepsilon n) - \hat{\varphi}(\varepsilon n)$ is bounded by 2 and for each n it goes to zero.

Next consider $p\partial_x$ where $p(x) = e^{ikx}$.

Then

$$\overbrace{[\varphi_\varepsilon *, p\partial_x]}^{[\varphi_\varepsilon *,\, p]\partial_x}u = i\sum_n e^{i(k+n)x}\left[\hat{\varphi}(\varepsilon k+\varepsilon n) - \hat{\varphi}(\varepsilon n)\right]n\,\hat{u}_n.$$

Now our problem is to understand how the coefficients

$$\left[\hat{\varphi}(\varepsilon k+\varepsilon n) - \hat{\varphi}(\varepsilon n)\right]n$$

behave as $\varepsilon \to 0$.

Use MVT:

$$\hat{\varphi}(\varepsilon k+\varepsilon n) - \hat{\varphi}(\varepsilon n) = \hat{\varphi}'(\xi_1)\,\varepsilon k$$

for some $\xi_1 \in [\varepsilon k + \varepsilon n, \varepsilon n]$

Then

$$n\left[\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right] = \hat\varphi'(\xi_1)\, \overbrace{\underbrace{n\varepsilon}_{} k}^{\xi_1 + n\varepsilon - \xi_1}$$

$$\left| n\left(\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right)\right| \leq \left|\hat\varphi'(\xi_1)\,\xi_1\right| |k| + \left|\hat\varphi'(\xi_1)\right| |\varepsilon k| |k|$$

so $$\boxed{\left| n\left(\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right)\right| \leq \left(\sup_\xi \left|\hat\varphi'(\xi)\,\xi\right|\right)|k| + \left(\sup_\xi \left|\hat\varphi'(\xi)\right|\right)\varepsilon k^2}$$

showing that the sequence

$$n \longmapsto n\left[\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right]$$

is uniformly bounded as $\varepsilon \downarrow 0$. On the other hand for fixed n the limit is zero.

Thus we conclude that the operator on L^2

$$[\varphi_\varepsilon *, e^{ikx}\partial_x]u = i \sum_n e^{i(q+n)x}\, n\left[\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right]\hat u_n$$

is bounded with uniform bounded as $\varepsilon \downarrow 0$, and that as $\varepsilon \downarrow 0$ it converges strongly to zero. This is what we want.

As for norm convergence, note that if we let $\varepsilon \downarrow 0$ and $n \to \infty$ with $n\varepsilon = \xi$, then

$$n\left(\hat\varphi(\varepsilon k + \varepsilon n) - \hat\varphi(\varepsilon n)\right) = \frac{\hat\varphi(\varepsilon k + \xi) - \hat\varphi(\xi)}{\varepsilon}\,\xi$$

$$\longrightarrow \hat\varphi'(\xi)\, k\, \xi$$

showing that <u>strong convergence</u> is best possible.

Let's try to summarize what has been learned. We claim to have shown that

$$[\varphi_\varepsilon *,\ p\partial_x] = [\varphi_\varepsilon *,\ p]\,\partial_x$$

is a bounded operator on L^2 whose norm is bounded as $\varepsilon \downarrow 0$. Furthermore this operator converges strongly to zero as $\varepsilon \downarrow 0$.

First let's check that the second assertion follows from the first. Let $u \in L^2$. The first assertion says $\underbrace{[\varphi_\varepsilon *,\ p\partial_x]}_{T_\varepsilon} u$ is bounded in L^2 as $\varepsilon \downarrow 0$. We wish to show it goes to zero, and it suffices by general theory to show it converges to zero weakly. (In effect the unit ball in a reflexive Banach space is weakly compact, so one can extract from any subsequence of $T_\varepsilon u$ a weakly cgt. subsequence. So if $T_\varepsilon u \to 0$ weakly it's impossible for $\|T_\varepsilon u\| \geq \delta$ for inf. many ε.)

Now again because $\|T_\varepsilon u\|$ is bdd, if we want to check that $(v, T_\varepsilon u)$ goes to zero for some $v \in L^2$, we can approx v by a smooth v_1, and thus it suffices to show $(v, T_\varepsilon u) \to 0$ for v smooth.

But then

$$(v, [\varphi_\varepsilon *,\ p\partial_x]\,u) = (\varphi_\varepsilon * v,\ p\partial_x u) - ((p\partial_x)^* v,\ \varphi_\varepsilon * u)$$

$$= \Big(\underbrace{(p\partial_x)^*(\varphi_\varepsilon * v)}, u\Big) - \Big(\underbrace{\varphi_\varepsilon * (p\partial_x)^* v}, u\Big) = 0.$$

converges in C^∞ to $(p\partial_x)^* v$

(Rmk: Instead of ref. Ban. space theory we can just argue by looking at each Fourier coefficient.)

Now let's return to boundedness of $[\varphi_\varepsilon *, p\partial_x]$. First we can reduce to the case where $p = e^{ikx}$ since if

$$p(x) = \sum e^{ikx} \hat{p}_k$$

then

$$\| [\varphi_\varepsilon *, p\partial_x] \| \leq \sum_k \underbrace{\| [\varphi_\varepsilon *, e^{ikx}] \|} \cdot |\hat{p}_k| .$$

So we only need a bound on which is polynomial in k.

If $p = e^{ikx}$ we are dealing with

$$\underbrace{e^{ikx}}_{\text{unitary}} \underbrace{\left\{ e^{-ikx} (\varphi_\varepsilon *) e^{ikx} - (\varphi_\varepsilon *) \right\} \partial_x}_{\text{convolution operator.}}$$

In general the norm of convolution $K*$ is bounded by the L^1 norm of K. General reason:

$$\left\| \int K(y) \; T(y) \, dy \right\| \leq \int |K(y)| \, \|T(y)\| \, dy$$

for integration in a Banach space. Specific reason is that on the Fourier transform level $K*$ becomes multiplication by $\hat{K}(\xi)$ and

$$|\hat{K}(\xi)| = \left| \int e^{-ix\xi} K(x) \, dx \right| \leq \int |K(x)| \, dx$$

Now

$$\left(e^{-ikx} (\varphi_\varepsilon *) e^{ikx} - \varphi_\varepsilon * \right) u'(x)$$

$$= \int \varphi_\varepsilon(y) \left[e^{-iky} - 1 \right] u'(x-y) \, dy$$

$$= \int (+\partial_y)\,\varphi_\varepsilon(y)\left(e^{-ik(y)} - 1\right) u(x-y)\,dy$$

Thus we have convolution by the function

$$K(y) = \partial_y\left\{\varphi_\varepsilon(y)\left(-1+e^{-iky}\right)\right\}$$

$$= \partial_y(\varphi_\varepsilon(y))\cdot\left(-1+e^{-iky}\right) + \varphi_\varepsilon(y)(-ik)$$

Now we know that $\varphi_\varepsilon(y) \geqslant 0$ and its integral is 1 so that

$$\| \varphi_\varepsilon(-ik)\|_1 = |k|.$$

Next

$$\partial_y\varphi_\varepsilon(y)\,(-1+e^{-iky}) = \partial_y\left(\frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)\right)\cdot\left(\frac{-e^{\frac{i}{2}ky}+e^{-i\frac{k}{2}y}}{+2i}\right)\left(+2i\,e^{-\frac{i}{2}ky}\right)$$

$$= \underbrace{\partial_y\left(\frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)\right)}_{\substack{\leqslant \text{ for } y>0 \\ \geqslant \text{ for } y<0}}\underbrace{\left(-2\sin\left(\frac{k}{2}y\right)\right)}_{\substack{\leq 0 \text{ for } 0\leq y\leqslant\varepsilon \\ \geqslant 0 \text{ for } -\varepsilon\leq y\leq 0}}\left(+i\,e^{-\frac{i}{2}ky}\right)$$

NEED $k\varepsilon \ll 2\pi$

$$\therefore \int \left|\partial_y\varphi_\varepsilon(y)(-1+e^{-iky})\right| dy = \int \partial_y\left(\frac{1}{\varepsilon}\varphi\left(\frac{y}{\varepsilon}\right)\right)\left(-2\sin\frac{k}{2}y\right)dy$$

$$= \int \varphi\left(\frac{y}{\varepsilon}\right) k\cos\frac{k}{2}y\,\frac{dy}{\varepsilon}$$

$$= \int \varphi(y)\,k\cos\frac{k\varepsilon}{2}y\,dy \longrightarrow k$$

as $\varepsilon \to 0$.

This assumes $k>0$; $k<0$ similar

Thus it seems that

$$\| [\varphi_\varepsilon *, e^{ikx}]\partial_x \| \le \quad 2|k|$$

where φ is assumed to have shape 
and $|k|\varepsilon \le 2\pi$. This is an improvement over the estimate 4 pages back.

<u>August 25, 1987</u>

It might be useful to concentrate on the problem of coupling ∂_x to an even K-class. This might force us to avoid arguments relying on the special features of graphs.

Consider the simplest case of a map from S^1 to the Grassmannian $\mathbb{P}^1(\mathbb{C})$. Recall that if X is a skew-symmetric matrix function on S^1 odd relative to ε, then the coupled operator is

$$\varepsilon \partial_x + X.$$

~~Hence if~~ $\cancel{X = \begin{pmatrix} 0 & -\bar{f} \\ f & 0 \end{pmatrix}}$ ~~we have the operator~~

$$\cancel{\begin{pmatrix} \partial_x & -\bar{f} \\ f & -\partial_x \end{pmatrix} = \varepsilon \partial_x + \gamma^1(\mathrm{Re} f) + \gamma^2(\mathrm{Im} f)}$$

so if $X = i\begin{pmatrix} 0 & a-ib \\ a+ib & 0 \end{pmatrix}$, then we have

$$\varepsilon \partial_x + X = i \begin{pmatrix} \frac{1}{i}\partial_x & a-ib \\ a+ib & \frac{1}{i}\partial_x \end{pmatrix} = i\left[\varepsilon \tfrac{1}{i}\partial_x + \gamma^1 a + \gamma^2 b\right]$$

Thus on the symbol level we have the map

$$(*) \qquad S^1 \times \mathbb{P}^1 \longrightarrow SU(2)$$

which extends the map

$$\mathbb{R} \times \mathbb{C} \longrightarrow su(2)$$

$$\xi, z \longmapsto i\begin{pmatrix} \xi & \bar{z} \\ z & -\xi \end{pmatrix},$$

and we apply it where z is the function $a+ib$ on S!

The map ⊛ is continuous. In effect $SU(2) = S^3 = su(2) \cup \{\infty\}$, and as

$$\begin{pmatrix} \xi & \bar{z} \\ z & -\xi \end{pmatrix}^2 = \Big(\varepsilon\xi + \underbrace{\gamma^1 \operatorname{Re} z + \gamma^2 \operatorname{Im} z}_{\gamma(z)}\Big)^2 = |\xi|^2 + |z|^2$$

the two eigenvalues of $\varepsilon\xi + \gamma(z)$, namely $\pm\sqrt{|\xi|^2+|z|^2}$ go to ∞ as either ξ or z goes to ∞.

I now want to analyze the singularities of ⊛. These occur over the point $-1 \in SU(2)$, and we have a good coordinate system around this point by using the map

$$g = \frac{1+X}{1-X} \longmapsto X^{-1}$$

Thus I want to look at the matrix

$$(\varepsilon\xi + \gamma(z))^{-1} = \frac{\varepsilon\xi + \gamma(z)}{\xi^2 + |z|^2}$$

where either ξ or z is near ∞.

If $z \neq \infty$, and ξ is near ∞, then

$$\frac{\varepsilon\xi + \gamma(z)}{\xi^2 + |z|^2} = \frac{\varepsilon + \gamma(z)\xi^{-1}}{1 + |z|^2\xi^{-2}}\,\xi^{-1}$$

is obviously smooth in ξ, z.

If $\xi \neq \infty$ and z is near ∞, then

$$\frac{\varepsilon\xi + \gamma(z)}{\xi^2 + |z|^2} = \begin{pmatrix} \frac{\xi}{\xi^2+|z|^2} & \frac{\bar{z}}{\xi^2+|z|^2} \\ \frac{z}{\xi^2+|z|^2} & \frac{-\xi}{\xi^2+|z|^2} \end{pmatrix} =$$

$$\frac{1}{1+\xi^2|z^{-1}|^2}\begin{pmatrix} \xi\,|z^{-1}|^2 & z^{-1} \\ \bar{z}^{-1} & -\xi|z^{-1}|^2 \end{pmatrix}$$

which is smooth in ξ and z^{-1}.

If ξ, z are both near ∞, then we get

$$\frac{1}{\xi^{-2}+|z^{-1}|^2}\begin{pmatrix} |z^{-1}|^2 & z^{-1}\xi^{-1} \\ \bar{z}^{-1}\xi^{-1} & -|z^{-1}|^2 \end{pmatrix}\xi^{-1}$$

which homogeneous of degree 1 in the variables ξ, $\mathrm{Re}(z^{-1})$, $\mathrm{Im}(z^{-1})$.

I claim I can smooth the singularity at $(\xi,z)=(\infty,\infty)$, by replacing the map by

$$(\varepsilon\xi+\gamma(z))/r(z)$$

where r is a smooth function on $\mathbb{P}^1$ which is $\neq 0$ for $z\neq\infty$, and which has a zero of infinite order at $z=\infty$. In effect we have using the above coord. system

$$\underbrace{\frac{1}{\xi^{-2}+|z^{-1}|^2}\left(\varepsilon|z^{-1}|^2+\gamma(\bar{z}^{-1})\xi^{-1}\right)\xi^{-1}\cdot|z^{-1}|^{2N}}_{\substack{\text{homog of degree } 1+2N \\ \therefore \text{ in class } C^{2N}}}\cdot\underbrace{\frac{r(z)}{|z^{-1}|^{2N}}}_{\text{smooth}}$$

This shows the map is C^{2N} for all N.

Let's check that a homogeneous function $f(x)$ on $\mathbb{R}^n$ of degree $d+\varepsilon$ $\underset{\varepsilon>0}{\text{is}}$ in class C^d. I assume

f is smooth away from 0. This is clear for $d=0$. In effect by compactness of the sphere we get a bound

$$|f(x)| \leq C|x|^{d+\varepsilon}$$

which proves continuity at $x=0$. Next use induction: If $d \geq 1$, then $\partial_\mu f$ is homogeneous of degree $d-1+\varepsilon$; note that $(\partial_\mu f)(0)=0$ by the above bound. So $\partial_\mu f \in C^{d-1}$ for all μ by induction hypothesis.

Program: We want to go from a map $S^1 \longrightarrow \mathbb{P}^1(\mathbb{C})$ to a unitary operator U congruent to -1 modulo compacts on $L^2(S^1)^2$. We will try to construct the unitary U as the Cayley transform of a skew-adjoint operator Lr^{-1}, where L is first order. Thus

$$U = \frac{1+Lr^{-1}}{1-Lr^{-1}} = (1+Lr^{-1})\{(r-L)r^{-1}\}^{-1}$$

$$= (r+L)(r-L)^{-1}$$

We will want $\qquad \|(r+L)u\|^2 = \|(r-L)u\|^2$

i.e. $\qquad (ru, Lu) + (Lu, ru) = 0 \qquad$ or

$$r^*L + L^*r = 0$$

which is our old condition assuming $r^*=r$.

(Note:

$$Lr^{-1} \text{ skew adj} \iff \underbrace{r^*(Lr^{-1})r = r^*L}_{\text{skew adj}}$$

assuming r invertible.)

Our problem is to find r and the coefficients p, q of L in terms of the map $S^1 \longrightarrow \mathbb{P}^1$. We hope to give formulas for p, q, r by applying functions defined on $\mathbb{T}$ to the unitary $g : S^1 \longrightarrow SU(2)$ reversed by ε corresponding to $S^1 \longrightarrow \mathbb{P}^1$.

Let's work out the formulas relating g and $z \in \mathbb{P}^1$. These formulas hold more generally when g corresponds to the graph of T.

$$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \qquad g = \frac{1+X}{1-X}$$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1+T^*T} & 0 \\ 0 & \sqrt{1+TT^*} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{1+T^*T}} & -T^* \frac{1}{\sqrt{1+TT^*}} \\ T \frac{1}{\sqrt{1+T^*T}} & \frac{1}{\sqrt{1+TT^*}} \end{pmatrix}$$

$g^{-1/2}$ is obtained by the sign change $T \mapsto -T$, so

$$\text{"}p\text{"} = \frac{g^{1/2} + g^{-1/2}}{2} = \begin{pmatrix} \frac{1}{\sqrt{1+T^*T}} & 0 \\ 0 & \frac{1}{\sqrt{1+TT^*}} \end{pmatrix}$$

$$\text{"}iq\text{"} = \frac{g^{1/2} - g^{-1/2}}{2} = \begin{pmatrix} 0 & -T^* \frac{1}{\sqrt{1+TT^*}} \\ T \frac{1}{\sqrt{1+T^*T}} & 0 \end{pmatrix}$$

These are the simplest choices for p, q.

Note that on $\mathbb{P}^1$ the function $\frac{g^{1/2}+g^{-1/2}}{2}$ is well-defined & continuous, but not smooth, since if $z^{-1} = u+iv$

$$\frac{1}{\sqrt{1+|z|^2}} = \frac{1}{\sqrt{1+|z^{-1}|^2}}\ \frac{1}{|z|} = \frac{1}{\sqrt{1+u^2+v^2}}\ \overbrace{\sqrt{u^2+v^2}}^{|z^{-1}|}$$

On the other hand its square is smooth:

$$\left(\frac{g^{1/2}+g^{-1/2}}{2}\right)^2 = \frac{g+2+g^{-1}}{4}$$

Note that the function on the unit circle

$$\frac{e^{i\theta}+2+e^{-i\theta}}{4} = \frac{1+\cos\theta}{2}$$

has a 2nd order zero at $\theta = \pi$; this perhaps makes it an even nicer choice for p.

On $\mathbb{P}^1$ the function $\frac{g^{1/2}-g^{-1/2}}{2}$ is not defined as $z=\infty$, but is essentially homogeneous of degree zero in the coordinate z^{-1}:

$$z\frac{1}{\sqrt{1+|z|^2}} = \underbrace{\frac{z}{|z|}}_{\text{phase of } z \,=\, \frac{1}{\text{phase } z^{-1}}} \overbrace{\frac{1}{\sqrt{1+|z^{-1}|^2}}}^{\text{smooth}}$$

<u>August 26, 1987</u>

The problem is now to start with a g and to find p, q, r such that

$$(r+L)(r-L)^{-1}$$

is a unitary operator. Here L is to be a first order differential operator roughly of the form $hp\partial_x + iq$, and p appears as the coefficient of $h\partial_x$, while iq is what survives as we let $h \to 0$. The above unitary is supposed to be congruent to -1 modulo compacts – this is what the term $p\partial_x$ is supposed to do. However it seems reasonable to suppose we obtain a unitary by letting h go to zero. It just won't be restricted.

So the idea will be start with g and choose q, r so that

$$(r+iq)(r-iq)^{-1}$$

is nicely defined by roughly the same process we propose to use for L. This means that we need to specify the operators $r \pm iq$ on a certain class of functions.

It seems that our program has a big defect, namely the fact that $L = hp\partial_x + iq$ is not defined everywhere. However the problem might not be serious. We have to define the C.T. of Lr^{-1} defined on $\mathrm{Im}\, r$ and q is defined on $\mathrm{Im}\, r$. And $\mathrm{Im}\, r = \mathrm{Im}\, r^{1/2}$.

Let's return to a simple example where we start with $g : S^1 \to \mathbb{P}^1$, better g is a 2×2 unitary matrix inverted by ε with $\det = 1$. Recall formulas

$$g^{1/2} = \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \frac{1}{\sqrt{1+|z|^2}}$$

$$\frac{g^{1/2}+g^{-1/2}}{2} = \frac{1}{\sqrt{1+|z|^2}} \quad , \quad \frac{g^{1/2}-g^{1/2}}{2} = \begin{pmatrix} 0 & \frac{-\bar{z}}{\sqrt{1+z^2}} \\ \frac{z}{\sqrt{1+|z|^2}} & 0 \end{pmatrix}$$

In this case we want r to be a function of $|z|$ vanishing in a nbd of $z = \infty$. So we have the closed set $Z \subset S^1$ where $g = -1$ contained in the open set where $|z| > \frac{1}{\varepsilon}$. If we choose ε generic there will be finitely many points where $|z| = \frac{1}{\varepsilon}$ and we divide up the circle into intervals which alternate between $|z| > \frac{1}{\varepsilon}$ and $|z| < \frac{1}{\varepsilon}$. Assume $r = 0$ for $|z| \geqslant \frac{1}{\varepsilon}$.

Look at each interval where $|z| \leqslant \frac{1}{\varepsilon}$ and note that there is no difficulty defining $g^{1/2}$ on a nbd of such an interval. So it appears there's no obstruction to carrying out the program for such an r.

August 28, 1987

Program: Suppose we are given $g: S^1 \to U(V)$ inverted by ε. Fix a function r on the unit circle which is smooth, invariant under conjugation and which is zero near -1 then rises to be 1 near $+1$. Let q be the function $q(e^{i\theta}) = \sin\left(\frac{\theta}{2}\right)$, $\theta \in (-\pi, \pi)$; it is defined except at -1. We use the spectral theorem to apply r to g to get an operator $r(g)$ on $L^2(S^1, V)$ which is self-adjoint between $0, 1$. $q(g)$ is defined where $g \neq -1$; hence $q(g)$ is defined on the image of $r(g)$. Note $\varepsilon q(g) \varepsilon = q(g^{-1}) = -q(g)$.

As $r = r(g)$ is a bounded self-adjoint operator its closed image $rL^2(S^1, V)$ is the orthogonal complement of its kernel. I want to find a first order differential operator L essentially of the form $\varepsilon h p \partial_x + q$ which will be defined, actually an operator on a dense subspace of $\mathrm{Im}(r)$, and further I would like Lr^{-1} to be skew-adjoint so that it defines a unitary operator on $\mathrm{Im}(r)$ by means of the Cayley transform.

Review of formulas:

$$g = \frac{1+X}{1-X} = \Bigg(\underbrace{\frac{1+X}{\sqrt{1-X^2}}}_{g^{1/2}}\Bigg)^2$$

$$\frac{g^{1/2} + g^{-1/2}}{2} = \frac{1}{\sqrt{1-X^2}} \qquad \frac{g^{1/2} - g^{-1/2}}{2} = \frac{X}{\sqrt{1-X^2}}$$

$$\frac{g + 2 + g^{-1}}{2} = \frac{1}{1-X^2}$$

$$\frac{g+1}{2} = \frac{1}{1-X} \qquad \frac{g^{-1}+1}{2} = \frac{1}{1+X}$$

One way to define a unitary g would be to define $A = \frac{g+1}{2}$. Then $g = -1+2A$ is unitary iff

$$A + A^* = 2A^*A = 2AA^*$$

(e.g. if $A = \frac{1}{1-X}$, $\frac{1}{1-X} + \frac{1}{1+X} = \frac{2}{1-X^2}$).

In the case of interest where $X = Lr^{-1}$ on $\operatorname{Im} r$ we have

$$A = \frac{1}{1-Lr^{-1}} = r\frac{1}{r-L}$$

Also $Lr^{-1} = -(Lr^{-1})^* = -r^{-1}L^*$, so

$$A = \frac{1}{1+r^{-1}L^*} = \frac{1}{r+L^*}r$$

Similarly

$$A^* = \frac{1}{1+Lr^{-1}} = r\frac{1}{r+L} = \frac{1}{r-L^*}r$$

Now the above is all formal, but it should be possible to define L in terms of g. Furthermore A can be defined on all of $L^2(S^1,V)$ by letting it be zero on $\operatorname{Ker}(r)$. I'd like A to be a ΨDO, whose complete symbol can be nicely described.

Let's discuss the choices for L. We want rL to be skew-adjoint; this is equivalent to $r^{-1}(rL)r^{-1} = Lr^{-1}$ being skew-adjoint.

Now a first order operator

$$a\partial_x + b$$

is skew-adjoint when

$$a\partial_x + b = \partial_x a^* - b^*$$

i.e. $a = a^*$ and

$$\frac{b+b^*}{2} = \frac{a'}{2}$$

Thus a skew-adjoint operator has the form

$$a\partial_x + \frac{a'}{2} + \underset{\text{0th order}}{\text{skew-adjoint}}, \quad \text{where } a = a^*$$

For example if $a = \alpha^2$, with $\alpha = \alpha^*$, then we have two self-adjoint operators

$$a\partial_x + \frac{a'}{2} = \alpha^2\partial_x + \tfrac{1}{2}(\alpha'\alpha + \alpha\alpha')$$

$$\alpha\partial_x\alpha = \alpha^2\partial_x + \alpha\alpha'$$

and these differ by the skew-adjoint operator $\frac{1}{2}(\alpha'\alpha - \alpha\alpha')$.

Now we will be considering $a\partial_x + b$ where $a = rp$, and p is probably a function ≥ 0 applied to q. Thus $rp = pr$ is self-adjoint, and there are two natural candidates

$$rp\partial_x + \tfrac{1}{2}(rp)'$$

$$(rp)^{1/2}\partial_x(pr)^{1/2}$$

When we combine with q to get L we get

$$h\varepsilon p\partial_x + \frac{h}{2}\varepsilon r^{-1}(rp)' + iq \quad \text{or}$$

$$h\varepsilon\left(\frac{p}{r}\right)^{1/2}\partial_x(pr)^{1/2} + iq$$

As the difference is an $O(h)$ change to the 0th order term, there shouldn't be any difference. Except that we ~~[illegible]~~ might want L to be free from r^{-1} and then the two behave differently

$$r^{-1}(rp' + r'p) = p' + r^{-1}r'p \qquad \text{smooth}$$

$$(pr)^{1/2}\, r^{-1}\, \partial_x\left[(pr)^{1/2}\right] \qquad \text{smooth.}$$

Notice that in either case there is something to prove if p, r are defined from functions on the circle.

Next consider the symbol for $A = r\left(\frac{1}{\lambda - L}\right)$

August 29, 1987

Let's now try to construct the complete symbol of $rL^{-1} = r(p\partial_x + q)^{-1}$ as a ψDO in the matrix case. Here p, q, r are to be functions applied to a loop $g : S^1 \longrightarrow U(V)$.

Recall the identity

$$\frac{1}{A} B = B\frac{1}{A} - [A,B]\frac{1}{A^2} + [A,[A,B]]\frac{1}{A^3} - \cdots$$

$$= \sum_{n\geq 0} (-1)^n B^{(n)} \frac{1}{A^{n+1}} \qquad B^{(n)} = [A, B^{(n-1)}]$$

Write $L = p\partial_x + q = p\varphi\partial_x\varphi^{-1}$ where φ is an invertible matrix such that

$$p\varphi' + q\varphi = 0.$$

Then

$$rL^{-1} = r\varphi\, \partial_x^{-1} \overbrace{(p\varphi)^{-1}}^{B} \qquad A = \partial_x$$

$$= r\varphi \sum_{n\geq 0} (-1)^n \left(\partial_x^n (p\varphi)^{-1}\right) \partial_x^{-n-1}$$

$$= \sum_{n\geq 0} (-1)^n \left(r\,(\underbrace{\varphi\partial_x\varphi^{-1}}_{\partial_x + p^{-1}q})^n (p^{-1})\right) \partial_x^{-n-1}$$

Thus we want to insure that

$$r\,(\partial_x + p^{-1}q)^n (p^{-1})$$

is smooth for all n.

August 30, 1987

Recall

$$r(p\partial_x+q)^{-1} \doteq \sum_{n\geq 0}(-1)^n\{r(\partial_x+p^{-1}q)^n(p^{-1})\}\,\partial_x^{-n-1}$$

As a check

$$(p\partial_x+q)\cancel{r^{-1}}\sum(-1)^n\{\cancel{r}(\partial_x+p^{-1}q)^n(p^{-1})\}\,\partial_x^{-n-1}$$

$$= p\sum_{n\geq 0}(-1)^n\{(\partial_x+p^{-1}q)^{n+1}(p^{-1})\}\partial_x^{-n-1} + (-1)^n\{(\partial_x+p^{-1}q)^n(p^{-1})\}\partial_x\partial_x^{-n-1}$$

$$= pp^{-1} = 1.$$

Thus we want to choose p, q, r so that

$$r(p^{-1}L)^n(p^{-1}) \quad \text{is smooth for } n\geq 0.$$

where $L = p\partial_x + q$.

$n=0$ says rp^{-1} smooth

$n=1$ says $r(\partial_x+p^{-1}q)(p^{-1}) = \underbrace{r\,\partial_x(p^{-1})}_{-rp^{-1}p'p^{-1}} + rp^{-1}qp^{-1}$ is smooth

Before going on let's recall our adjointness condition. We want to have another first order operator L' so that the rows of $\begin{pmatrix} r & L' \\ L & r \end{pmatrix}$ are $\perp$. Thus $rL' + L^*r = 0$, or $r^{-1}L^*r = -L'$ is smooth, or $rLr^{-1} = -(L')^*$ is smooth. Natural condition:

$$rLr^{-1} \quad \text{is smooth}$$

Now we can write

$$r(p^{-1}L)^n(p^{-1}) = (rp^{-1}Lpr^{-1})(rp^{-2}Lp^2r^{-1})\cdots(rp^{-n}Lp^nr^{-1})(rp^{-n-1})$$

7 hence it seems reasonable to ask that the operators

$$(rp^{-n})L(rp^{-n})^{-1}$$

be smooth for all n.

September 1, 1987.

Let's consider the question of whether we can find real functions p, r on $\mathbb{T}$ depending only on $\mathrm{Re}(\zeta)$, vanishing at $\zeta = -1$, such that when extended to unitary matrices satisfy

$$r^{-1}dr\,p \quad \text{is smooth}$$

We use the map $T \times U/T \longrightarrow U = U_n$ which assigns to a diagonal matrix $\zeta = (\zeta_i)$ and a flag g (standard flag in $\mathbb{C}^n$) the unitary matrix $g\zeta g^{-1}$. Then $r(g\zeta g^{-1}) = g\,r(\zeta)\,g^{-1}$, where $r(\zeta)$ is the diagonal matrix with entries $r(\zeta)_i = r(\zeta_i)$. Also

$$dr(g\zeta g^{-1}) = g\,dr(\zeta)g^{-1} + dg\,r(\zeta)g^{-1} - g\,r(\zeta)g^{-1}dg\,g^{-1}$$

so $r(g\zeta g^{-1})^{-1}dr(g\zeta g^{-1})\,p(g\zeta g^{-1}) =$ sum of

$$g\,r(\zeta)^{-1}dr(\zeta)p(\zeta)g^{-1} = g\big(r(\zeta)^{-1}r'(\zeta)p(\zeta)\,d\zeta\big)g^{-1}$$

which is smooth assuming $rr'p$ is a smooth fn. on $\mathbb{T}$, and

$$g\Big\{\big(r(\zeta)^{-1}g^{-1}dg\,r(\zeta) - g^{-1}dg\big)p(\zeta)\Big\}g^{-1}$$

Now $g^{-1}dg$ can be an arbitrary skew-adjoint matrix, and the (i,j) entry of the matrix in braces is

$$r(\zeta_i)^{-1}(g^{-1}dg)_{ij}\,r(\zeta_j)p(\zeta_j) - (g^{-1}dg)_{ij}\,p(\zeta_j)$$

and this is definitely not smooth as it blows up when $\zeta_i \to -1$.

September 2, 1987

Here's a special case to understand. Let $g: S^1 \longrightarrow U(V)$ only have the eigenvalues ± 1, i.e. $g^2 = 1$. Then $\tilde{V}$ decomposes into $E \oplus E^{\perp}$ where $g = 1$ on E and -1 on $E^{\perp}$. It's clear that any of our functions $r(g)$, $p(g)$ depend only on the values of $r(\zeta)$, $p(\zeta)$ when $\zeta = \pm 1$. Thus $r = p =$ projection onto E, at least up to constant factors.

Let's recall our program of defining a skew-adjoint operator, roughly of the form Lr^{-1} defined on Im r. In the present example g is defined only on E, and it is zero. I think the operator we want is

$$hp\,\partial_x\, p = hp\,\partial_x + hpp'$$

where p is the projection onto E. Thus we are taking the induced Dirac operator on the subbundle E.

It's difficult to think of another operator we could define on sections of E. Somehow we have to obtain this operator from a general formula.

It seems now that the case to handle is where we have a loop $g: S^1 \longrightarrow U(2)$, where one eigenvalue stays $\neq -1$ and the other passes through -1. This is a mixture;

rather it is a direct sum, of two scalar situations which maybe we can handle. Notice we have a splitting $\tilde{V} = E_1 \oplus E_2$, where each E_i is a line bundle, and where the splitting is g-invariant. Since E_1, E_2 are trivial, there is a gauge transformation on $\tilde{V}$ which transforms g to diagonal form. This means that we can suppose g diagonal, but there will be a connection term. Thus the $h\partial_x$ will become $h\nabla_x$ where ∇ is a unitary connection on $\tilde{V}$.

Here seems to be a new proof of the extension by -1 theorem for superconnections. Consider the resolvent of the superconnection operator $X\sigma + D$, namely

$$\frac{1}{\lambda - X\sigma - D} \qquad X = \frac{g-1}{g+1}$$

This can be written in two ways

$$(g+1)\frac{1}{\lambda(g+1) - (g-1)\sigma - D(g+1)}$$

$$= \frac{1}{\lambda(g+1) - (g-1)\sigma - (g+1)D}(g+1)$$

The inverses actually exist, provided $\lambda \notin i\mathbb{R}$. This is because $\sigma = \pm 1$ on even, resp. odd forms, because D belongs to a nilpotent ideal (raises form degree by 1), and because $\lambda(g+1) \pm (g-1)$ is invertible for g unitary and $\lambda \notin i\mathbb{R}$.

But now use the old argument $E = E' \oplus E''$

$$g \doteq \begin{pmatrix} g' & 0 \\ 0 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} \overbrace{i^*Di}^{D'} & i^*Dj \\ j^*Di & j^*Dj \end{pmatrix}$$

$$(g+1)\left(\lambda(g+1) - (g-1)\sigma - D(g+1)\right)^{-1} =$$

$$\begin{pmatrix} g'+1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda(g'+1) - (g'-1)\sigma - D'(g'+1) & 0 \\ -(j^*Di)(g'+1) & 2\sigma \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} g'+1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [\lambda(g'+1) - (g'-1)\sigma - D'(g'+1)]^{-1} & 0 \\ * & (2\sigma)^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} (g'+1)[\lambda(g'+1) - (g'-1)\sigma - D'(g'+1)]^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

So we see that the actual superconnection resolvent behaves nicely under extension by -1. Add this assertion to itself ~~with~~ sign change $\lambda \to -\lambda$ to get the desired result for the curvature:

$$+\frac{1}{\lambda - X\sigma - D} - \frac{1}{(-\lambda) - X\sigma - D} = 2\lambda \frac{1}{\lambda^2 - (X\sigma + D)^2}$$

September 3, 1987

Yesterday I discovered that I could simplify the proof of the extension by -1 theorem by working with the resolvent of the superconnection operator $D + X\sigma$. One has when g is the Cayley transform of X the formula

$$\frac{1}{\lambda - X\sigma - D} = (g+1)\frac{1}{\lambda(g+1) - (g-1)\sigma - D(g+1)}$$

$$= \frac{1}{\lambda(g+1) - (g-1)\sigma - (g+1)D}(g+1)$$

and the expressions on the right make sense for g arbitrary. Here g, D, σ are viewed as operators on $\Omega(M,E)$, and the inverses for $\mathrm{Re}(\lambda) \neq 0$ are given by geometric series, which are finite as D raises the degree as differential form.

Our problem is to "quantize" this. The first idea was to replace $D = dx^\mu D_\mu$ with $h\not{D} = h\gamma^\mu D_\mu$, where σ, γ^μ all anti-commute. For example if $\dim M = 1$, then we take

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

and get

$$h\not{D} + X\sigma = \begin{pmatrix} 0 & h\partial_x - \frac{1}{i}\frac{g-1}{g+1} \\ h\partial_x + \frac{1}{i}\frac{g-1}{g+1} & 0 \end{pmatrix}$$

This is a skew-adjoint differential operator whose Cayley transform is the unitary inverted by ε corresponding to the graph of $h\partial_x + \frac{1}{i}\frac{g-1}{g+1}$.

Let $\tilde{X} = h\emptyset + X\sigma$,

and $\tilde{g} = \frac{1+\tilde{X}}{1-\tilde{X}}$. $\tilde{g}$ is supposed to be congruent to -1 modulo Hilbert-Schmidt operators. We have

$$\frac{1}{\lambda - \tilde{X}} = \frac{1}{\lambda - \frac{\tilde{g}-1}{\tilde{g}+1}} = \frac{\tilde{g}+1}{\lambda(\tilde{g}+1) - (\tilde{g}-1)}$$

In particular $\quad \frac{1}{1-\tilde{X}} = \frac{\tilde{g}+1}{2}$.

Thus the resolvent $\frac{1}{\lambda - \tilde{X}}$ is supposed to be Hilbert-Schmidt. One way to obtain this is to make the resolvent a ψDO of order -1. It would be very nice if we could show that $\frac{1}{\lambda - \tilde{X}}$ is a ψDO of order -1, because then the Cayley transform $\tilde{g}$ would depend smoothly on g.

So one goal would be to produce the complete symbol of $\frac{1}{\lambda - \tilde{X}}$ as a ψDO of order -1, and also as a smooth function of g.

Let's review how to invert a first order DE $a\partial_x + b$ within the algebra of "classical" ψDO's. We can write

$$a\partial_x + b = a\varphi\, \partial_x\, \varphi^{-1}$$

where φ is a fundamental matrix. Then

$$\begin{aligned}
(a\partial_x + b)^{-1} &= \varphi\, \partial_x^{-1}\, \varphi^{-1} a^{-1} \\
&= \varphi \sum_{n \geq 0} (-1)^n\, \partial_x^n(\varphi^{-1} a^{-1})\, \partial_x^{-n-1} \\
&= \sum_{n \geq 0} (-1)^n\, (\varphi \partial_x \varphi^{-1})^n (a^{-1})\, \partial_x^{-n-1}
\end{aligned}$$

$$= \sum_{n\geq 0} (-1)^n \left((\partial_x + a^{-1}b)^n (a^{-1}) \right) \partial_x^{-n-1}$$

Actually it might be more convenient to start with $\partial_x a + b$, whence

$$(\partial_x a + b)^{-1} = \left(\varphi\, \partial_x \varphi^{-1} a \right)^{-1} \qquad (\partial_x + ba^{-1})\varphi = 0$$

$$= a^{-1}\varphi\, \partial_x^{-1} \varphi^{-1} = \sum_{n\geq 0} (-1)^n\, a^{-1}\varphi\, \partial_x^n(\varphi^{-1})\, \partial_x^{-n-1}$$

$$= \sum_{n\geq 0} (-1)^n \left\{ a^{-1} (\partial_x + ba^{-1})^n (1) \right\} \partial_x^{-n-1}$$

Thus we have to know that all the matrix functions

$$a^{-1}(\partial_x + ba^{-1})^n (1) \qquad n \geq 0$$

are smooth.

The goal now will be to decide once and for all whether I can succeed with a smooth map to ψDO's of order -1, or whether a more subtle class of operators has to be used.

First look at

$$-\lambda + h\not{\partial} + X\sigma = \partial_x \overbrace{h\gamma^1}^{a} + \overbrace{X\sigma - \lambda}^{b}$$

better

$$(-\lambda + h\not{\partial} + X\sigma)(h\gamma^1)^{-1} = \partial_x + \overbrace{(X\sigma - \lambda)(h\gamma^1)^{-1}}^{b}$$

where $a = 1$. This won't work unless b is smooth, which rules out X becoming singular.

The following might be useful

$$(\partial_x + c)\sum_{n\geq 0}(-1)^n\{(\partial_x+c)^n(f)\}\,\partial_x^{-n-1}$$

$$= \sum_{n\geq 0}(-1)^n\left\{(\partial_x+c)^{n+1}(f)\,\partial_x^{-n-1} + (\partial_x+c)^n(f)\,\partial_x^{-n}\right\}$$

$$= f$$

Thus

$$\boxed{(\partial_x+c)^{-1}f = \sum_{n\geq 0}(-1)^n\{(\partial_x+c)^n(f)\}\,\partial_x^{-n-1}}$$

The idea: We are trying to define an operator on the image of r. This operator is supposed to be the sum of a differentiation operator and a "potential" q. Try to identify the differentiation operator with a "connection induced on $\operatorname{Im} r$." For example if r is a projector then we want the induced connection on the subbundle $\operatorname{Im} r$ which is $r.d.r$.

September 4, 1987

Let's review the scalar case. We suppose $g = -1$ outside an open interval. On this interval we have a skew-adjoint operator which we write in the form

$$\begin{pmatrix} 0 & a^{1/2}\partial_x a^{1/2} - b \\ a^{1/2}\partial_x a^{1/2} + b & 0 \end{pmatrix}$$

The solutions of the two homogeneous DE's are found as follows:

$$\left(a^{1/2}\partial_x a^{1/2} + b\right)\varphi = a^{1/2}\left(\partial_x + \frac{b}{a}\right)a^{1/2}\varphi = 0$$

so
$$\varphi = a^{-1/2} e^{-\int \frac{b}{a}}$$

similarly
$$\psi = a^{-1/2} e^{+\int \frac{b}{a}}$$

In order for the above operator to be essentially skew-adjoint on the interval, it's necessary + sufficient that at each endpt one of φ, ψ is not in L^2. Call this the L^2 condition.

There's also the ψDO condition

$$\left(a^{1/2}\partial_x a^{1/2} + b\right)^{-1} = a^{-1/2}\left(\partial_x + \frac{b}{a}\right)^{-1} a^{-1/2}$$

$$= \sum_{n \geq 0} (-1)^n \left\{ a^{-1/2}\left(\partial_x + \frac{b}{a}\right)^n \left(a^{-1/2}\right)\right\} \partial_x^{-n-1}$$

Thus we want $a^{-1/2}\left(\partial_x + \frac{b}{a}\right)^n\left(a^{-1/2}\right)$ to be smooth for all $n \geq 0$. For $n = 0$, this says a^{-1} is smooth.

Let $f = \frac{b}{a}$. This has to blowup at the endpoints sufficiently fast that not both φ, ψ are in L^2. Thus a^{-1} has to vanish sufficiently fast to kill the singularitees arising from f, yet not so fast as to make both φ, ψ in L^2.

Old notation

$$a^{1/2}\partial_x a^{1/2} + b = \left(\frac{p}{r}\right)^{1/2}\partial_x\left(\frac{p}{r}\right)^{1/2} + \frac{q}{r}$$

so that $a = \frac{p}{r}$ $b = \frac{q}{r}$, so $\frac{b}{a} = \frac{q}{p}$. Thus we assume $\frac{b}{a}$ has poles at the ends of order >1. Typically $q = \pm 1$ at the endpoints so $\frac{b}{a} \sim \frac{1}{p}$ and p has a 2nd order zero, or maybe third order.

What you should remember is that $f = \frac{b}{a}$ is to blow up algebraically and that a^{-1} is to vanish to infinite order but not too fast.

<u>Remarks</u>: The problem is to go from g to $a^{1/2}\partial_x a^{1/2} + b$ a first order DE with singularities, whose inverse is a ψDO. <u>First</u> let's stop using the p, q, r notation as this isn't intrinsic and ~~it distorts~~ the ~~intuition~~.

<u>2nd</u>. It should be enough to give formulas for $g = \frac{1+X}{1-X}$. This is OK when we deal with g reversed by ε and the ~~operator~~

$$\varepsilon\left(a^{1/2}\partial_x a^{1/2}\right) + ib$$

since any loop $g \longrightarrow$ Grass can be deformed off the complex hypersurface where g has the eigenvalue -1. (It's first necessary to get in the right component of Grass.)

<u>3rd</u> When it comes to the asymptotic analysis, we need $[b,a] = O(h)$. One has

$$\left(h\gamma^1 a^{1/2}\partial_x a^{1/2} + \gamma^2 b\right)^2 = h^2\left(a^{1/2}\partial_x a^{1/2}\right)^2 + b^2 + h\gamma^1\gamma^2\left[a^{1/2}\partial_x a^{1/2}, b\right]$$

$$\left[a^{1/2}\partial_x a^{1/2}, b\right] \equiv \left[a\partial_x, b\right] \equiv [a,b]\partial_x$$

modulo 0th order operators, so one needs an extra h factor to get this in Getzler's algebra. (Note that $\gamma^2 = \sigma$ is not scaled.)

In practical terms, since we will choose $a^{-1/2}$ to be a smooth function applied to g, this means that the leading term in $b = b_0 + b_1 h + \cdots$ must be also a function applied to g.

<u>4th.</u> We have with $\rho = a^{-1/2}$

$$\left(a^{1/2}\partial_x a^{1/2} + b\right)^{-1} = \rho\left(\partial_x + f\right)^{-1}\rho \qquad f = \rho b \rho$$

$$\doteq \sum_{n\geq 0}(-1)^n\left\{\rho\left(\partial_x + f\right)^n(\rho)\right\}\ \partial_x^{-n-1}$$

Suppose g is a direct sum of scalar operators i.e. there is a g-invariant decomposition $\tilde{V} = \bigoplus E_j$ with g scalar on each E_j. Then we can take $\partial_x + f$ of the form $\partial_x + A + f_1$, where $\partial_x + A$

is a connection preserving the decomposition and where f_1 is a suitable function applied to g. This reduces everything to the scalar case.

September 5, 1987

We have a path $g = g(x)$ in the unitary group and wish to assign to it a ΨDO symbol

$$\sum_{n \geq 0} (-1)^n \{\rho(\partial_x + f)^n(\rho)\}\, \partial_x^{-n-1}$$

where ρ and f are to be constructed from g. I assume that ρ is obtained by applying a function on S^1 to g. f is similar except there has to be a correction term in the nonabelian case. We will take $f = \frac{1}{i}\frac{g-1}{g+1}$ + correction term until it is found out this doesn't work.

The problem is to show $\rho(\partial_x + f)^n(\rho)$ is smooth, and I propose to do this by inserting:

$$\rho_i \rho_i^{-1}(\partial_x + f)\rho_{i+1}\rho_{i+1}^{-1}$$

I am therefore interested in pairs ρ_1, ρ such that

1) These are functions on S^1 applied to g
2) They vanish only at $\zeta = -1$.
3) $\rho_1^{-1}(\partial_x + f)\rho$ is smooth

Let's suppose $g = g(x)$ can be lifted to

$T \times U/T$, i.e. we have

$$g = u \xi u^{-1}$$

where $\xi = \xi(x)$ is a path in the subgroup of diagonal matrices, and $u = u(x)$ is a path in the unitary group. Then

$$\rho(g) = u \rho(\xi) u^{-1}$$

$$d\rho(g) = u \rho'(\xi) d\xi\, u^{-1} + [(du)u^{-1}, \rho(g)]$$

$$= u \{ \rho'(\xi) d\xi + [u^{-1}du, \rho(\xi)] \} u^{-1}$$

In a generic situation this means $d\rho(g)$ determines the off-diagonal part of $u^{-1}du$. (In fact it would be better to use

$$dg = u \{ d\xi + [u^{-1}du, \xi] \} u^{-1}.)$$

So

$$\rho_1(g)^{-1} \partial_x \rho(g) = u \{ \rho_1(\xi)^{-1} \rho'(\xi) +$$

$$\rho_1(\xi)^{-1} [u^{-1}\partial_x u, \rho(\xi)] \} u^{-1}$$

Since $\rho_1(\xi)^{-1}[u^{-1}\partial_x u, \rho(\xi)]$ is a non-diagonal matrix, the correction term to be added to $\partial_x + \frac{1}{i}\frac{g-1}{g+1}$ must kill any singularity obtained from this non-diagonal part. One possible thing is to kill the term $u^{-1}\partial_x u$, and this is probably the same thing as using the connection preserving the eigenspace splitting.

However it might be possible to use part of this term $u^{-1}\partial_x u$. Somehow one only cares about the link given by $u^{-1}\partial_x u$ with the $g=-1$ eigenspace with its orthogonal complement.

Now this seems to be a key idea. At any point x we have the splitting into the $g(x)=-1$ eigenspace and its orthogonal complement. Near x we can extend this splitting canonically and we can consider the variation of this splitting at x, which is a map from the $g(x)\neq -1$ eigenspace to the $g(x)=-1$ eigenspace. It is this map which causes the trouble and which has to be eliminated by a suitable connection term.

So the sort of thing to look for is a skew-hermitian matrix 1-form A on the unitary group such that at any unitary matrix g, the block of A going from the $g\neq -1$ eigenspace to the $g=-1$ eigenspace coincides with the same block of the variation of g. Locally such a thing might exist, and then one can patch using a partition of unity.

It's not so obvious locally, since moving can increase the $g\neq -1$ eigenspace, so instead you must examine relative to the stratification; Take $n=2$, and see what is needed to go, after doing for the hypersurface with one eigenvalue $=-1$, to handle $g=-1$.

Let A be a hermitian matrix; then it determines a splitting

$$V = \operatorname{Ker} A \oplus \operatorname{Im} A$$

If we vary A slightly we can extend this splitting as follows. Fix $\varepsilon > 0$ such that all non zero eigenvalues of A have abs. value $> \varepsilon$. Then

$$P = \frac{1}{2\pi i} \oint_{|\lambda| = \varepsilon} \frac{1}{\lambda - A} \, d\lambda$$

is projection on the sum of the eigenspaces belonging to eigenvalues of abs. val. $< \varepsilon$. This formula defines a projection as long as the spectrum of A stays off the contour.

We have corresponding to an infinitesimal variation δA

$$\delta P = \frac{1}{2\pi i} \oint_{|\lambda| = \varepsilon} \frac{1}{\lambda - A} \, \delta A \, \frac{1}{\lambda - A} \, d\lambda$$

We break up δA into four blocks corresponding to the decomposition $V = \operatorname{Im} P + \operatorname{Im}(1-P)$. The $(1-P), (1-P)$ block is zero as the integrand is analytic inside the contour. Similarly the P, P block is zero because we can push the contour to infinity. This checks with δP having only off diagonal blocks for projectors. Next we evaluate use eigenvectors and find using Cauchy's thm. that for $|\lambda_i| < \varepsilon$ and $|\lambda_j| > \varepsilon$

$$(\delta P)_{ij} = \frac{1}{2\pi i}\int \frac{1}{\lambda-\lambda_i}(\delta A)_{ij}\frac{1}{\lambda-\lambda_j}\,d\lambda$$

$$= \frac{(\delta A)_{ij}}{\lambda_i-\lambda_j} \qquad \text{for } |\lambda_i| < \varepsilon < |\lambda_j|$$

$$= \frac{(\delta A)_{ij}}{\lambda_j-\lambda_i} \qquad \text{for } |\lambda_j| < \varepsilon < |\lambda_i|$$

So in particular if P is projector on the kernel of A we have

$$\delta P = P\,\delta A\left(\frac{-1}{A}\right)(1-P) + (1-P)\left(\frac{-1}{A}\right)\delta A\,P$$

Next suppose that $P = uP_0u^{-1}$ with u unitary varying. Then

$$\delta P = \delta u\,P_0\,u^{-1} - u\,P_0\,u^{-1}\delta u\,u^{-1}$$

$$= [\delta u\cdot u^{-1}, P]$$

so that if $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\delta u\cdot u^{-1} = \begin{pmatrix} X & -T^* \\ T & Y \end{pmatrix}$

then

$$\delta P = \begin{pmatrix} X & 0 \\ T & 0 \end{pmatrix} - \begin{pmatrix} X & -T^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$$

This identifies the off-diagonal blocks of $\delta u\cdot u^{-1}$ with δP up to sign.

The next project should be to consider the space of hermitian matrices. At each A we have a linear form on the tangent space having values in $\mathrm{Hom}(\mathrm{Im}A, \mathrm{Ker}A)$, namely

$$P\, \delta A \left(\frac{1}{A}\right)(1-P)$$

where $P =$ projection on $\mathrm{Ker}\,A$. What I would like to know is whether there is a herm. matrix valued 1-form ω on the space of hermitian matrices which extends this family.

September 6, 1987

Review the problem. Suppose g is a unitary matrix function over M. We would like to find a connection $d+B$ such that if ρ_1, ρ are suitable functions on S^1 vanishing only at $\zeta = -1$, then $\rho_1(g)^{-1}(d+B)\rho(g)$ is smooth.

Assuming we can smoothly diagonalize g, i.e. $g = u\zeta u^{-1}$ with ζ diagonal, u unitary functions we have

$$\rho_1(g)^{-1} d\rho(g) = u\rho_1(\zeta)u^{-1}\, d(u\rho(\zeta)u^{-1})$$

$$= u\{\rho_1(\zeta)^{-1}\rho'(\zeta)\,d\zeta + \rho_1(\zeta)^{-1}u^{-1}du\,\rho(\zeta)$$

$$-\rho_1(\zeta)^{-1}\rho(\zeta)\,u^{-1}du\,\}u^{-1}$$

The bad term is

$$u\rho_1(\zeta)\,u^{-1}du\,\rho(\zeta)u^{-1} = \rho_1(g)^{-1}(du\,u^{-1})\rho(g)$$

This will not be defined $(u^{-1}du)_{ij} \neq 0$ with $\zeta_i = -1$ and $\zeta_j \neq -1$, i.e. if $du\,u^{-1}$ induces a non-zero map from the $g \neq -1$ generalized eigenspace to the $g = -1$ eigenspace. In order to interpret this induced map, let P be the projection onto the $g=-1$ eigenspace at the point of interest. We have $P = uP_0u^{-1}$ where P_0 is diagonal with 1's in the rows where $\xi_i = -1$. As we vary around the point P projects onto the (gen) eigenspace with g is close to -1. We have

$$dP = du\,P_0u^{-1} - uP_0u^{-1}du\,u^{-1}$$

$$= [du\cdot u^{-1}, P]$$

Thus if $du\cdot u^{-1} = \begin{pmatrix} X & -T^* \\ T & Y \end{pmatrix}$, then

$$dP = [du\cdot u^{-1}, P] = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}.$$

We want the connection form B to be such that $P(B + du\cdot u^{-1})(1-P) = 0$, so if B is skew adjoint we want B at the point of interest to be

$$\begin{pmatrix} 0 & T^* \\ -T & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dP = F\,dP$$

where $F = P - (1-P)$ is the involution, plus some diagonal block form.

Thus the condition on the connection $d+B$ is that at each point it preserve the decomposition into $g=-1$ and $g\neq -1$ eigenspaces. For example if the number of -1 eigenvalues doesn't change, ~~the~~ simplest connection is

$$P\,d\,P + (1-P)\,d\,(1-P) = d + F\,dP$$

and we can add to this matrix 1-form like $PA_1P + (1-P)A_2(1-P)$.

Next, again assuming smooth diagonalization, let's construct such connections. Thus we suppose g factored into $u\,\xi\,u^{-1}$, and we wish to find a matrix 1-form B such that it's skew-adjoint and at each point

$$P(du\,u^{-1} + B)(1-P) = 0$$

where P is the orthogonal projection on the kernel, i.e. the $g=-1$ eigenspace. Conjugating by u we want at each point

$$P_0(u^{-1}du + u^{-1}Bu)(1-P_0) = 0$$

where P_0 projects onto the kernel of $\xi+1$. Now, as we pointed out before one can always take $B = -du\,u^{-1}$, however it would be nice if B were introduced only to take care of the eigenvalues close to -1.

One possibility is to choose a real function on S^1 k which is 1 at $\xi=-1$, and zero for most of the circle. Then we can take $-u^{-1}Bu$ to be

$$k(u^{-1}du)(1-k) + (1-k)(u^{-1}du)k$$

where k is applied to ξ. This works OK at points where $k(\xi)$ is a projection. This gives same result effectively as

$$k(u^{-1}du) + (u^{-1}du)k$$

The sort of connection I want doesn't seem to exist. It is too much to ask that we can control things as an eigenvalue becomes $\neq 0$.

Suppose we formulate the problem on the space of hermitian matrices A. Recall from yesterday

$$P_A = \frac{1}{2\pi i} \int_{|\lambda|=\varepsilon} \frac{1}{\lambda - A}\, d\lambda$$

$$\delta P_A = \frac{1}{2\pi i} \int \frac{1}{\lambda - A}\, \delta A\, \frac{1}{\lambda - A}$$

$$= P_A\, \delta A \left(\frac{-1}{A}\right)(1-P_A) \quad + \quad \text{c.c.}$$

We seek a smooth matrix 1-form $\omega(A;\delta A)$ such that for all A

$$P_A\, \omega(A;\delta A)(1-P_A) = P_A\, \delta A \left(\frac{-1}{A}\right)(1-P_A)$$

Consider this equation under scaling $A \mapsto tA$. The right side doesn't change, and the the left becomes

$$t\; P_A\, \omega(tA;\delta A)(1-P_A)$$

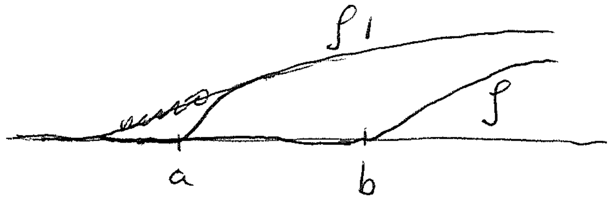
Let $t \to 0$ gives a contradiction, except in the 1-dimensional case where one of P_A and $(1-P_A)$ is always zero.

September 7, 1987

I propose to finish up something from yesterday. Recall the problem is to find a 1-form B such that

$$\rho_1(g)^{-1}(d+B)\,\rho(g)$$

is smooth. Here ρ, ρ_1 vanish at $\xi=-1$ and we suppose ρ_1 vanishes to lower order in the sense that $\rho_1^{-1}\rho$ is smooth. I've established more or less that it's impossible to find B when ρ_1, ρ only vanish at $\xi=-1$. However suppose that $\rho_1 \neq 0$ on $\mathrm{Supp}(\rho)$, and to simplify suppose that both $\rho_1, \rho \geq 0$, that $\rho_1(\xi)>0$ for $\mathrm{Re}(\xi)>a$, that $\rho(\xi)=0$ for $\mathrm{Re}(\xi)\leq b$ where $a<b$.



Now I would like to show that a B can be found such that $\rho_1^{-1}(d+B)\rho$ is smooth.

First of all I might as well replace g by the hermitian matrix $A=\frac{g+g^{-1}}{2}$. I will suppose as before the existence of a smooth diagonalization

$$A=u\lambda u^{-1} \qquad \lambda \text{ diagonal.}$$

The problem term in $\rho_1^{-1}d\rho$ is

$$u\{\rho_1(\lambda)^{-1}u^{-1}du\,\rho(\lambda)\}u^{-1} = \rho_1(A)^{-1}du\,u^{-1}\rho(A)$$

in particular, the block of $du\,u^{-1}$ which goes from $\mathrm{Im}\,\rho(A)$ to $\mathrm{Ker}\,\rho_1(A)$. We want to eliminate the term with $\rho_1(A)^{-1} B \rho(A)$. Let choose a function $k(x) = 1$ for $x \leq a$ and $= 0$ for $x \geq b$, and set

$$-B = k(A)\, du u^{-1} \big(1 - k(A)\big) + \big(1 - k(A)\big)\, du u^{-1} k(A)$$

Applying B to $\mathrm{Im}\, \rho(A)$, the second term vanishes since $k\rho = 0$, and then projecting into $\mathrm{Ker}\, \rho_1(A)$, on which $k(A) = 1$, we get exactly $-(du u^{-1})$ induced from $\mathrm{Im}\, \rho(A)$ to $\mathrm{Ker}\, \rho_1(A)$. Thus $\rho_1^{-1}(d+B)\rho$ is smooth as claimed.

However its not clear that B is defined intrinsically independent of the assumption that A can be diagonalized. A purpose of the factor $1-k$ in the above expression is to try to get the variation in the splitting between where $k=1$ and $k=0$. This really makes sense only when $k(A)$ is a projection, i.e. when the spectrum of A has a gap in (a,b).

<u>Better approach</u>. Fix an interval (a,b). I claim one can find $(d+B)$ so that if s is a section supported where $A > b$, then $(d+B)s$ is supported where $A > a$. This is a local problem since connections can be pieced together by a partition of 1. Locally we can always fix an x not in the spectrum of A and use the direct sum of the Grassmannian connections on the subbundles with $A < x$ and $A > x$.

This is a curious construction. For each λ consider the open set U_λ, where $A - \lambda$ is invertible.

Over U_λ one has a splitting of $\tilde{V}$ and an induced connection preserving the splitting. On overlaps $U_{\lambda_1,\dots,\lambda_k} = U_{\lambda_1} \cap \dots \cap U_{\lambda_k}$ one has also induced connections preserving the k-fold splitting.

<u>September 8, 1987</u>

It seems worthwhile to look for alternative approaches to constructing the pairing between ∂_x and g. Recall that my first approach was to consider the graph of

$$\partial_x + \frac{1}{i}\frac{g-1}{g+1},$$

but this graph, as an element of the Hilbert-Schmidt restricted Grassmannian, is not a smooth function of g. Since for invertible T

$$\mathrm{Im}\begin{pmatrix}1\\T\end{pmatrix} = \mathrm{Im}\begin{pmatrix}T^{-1}\\1\end{pmatrix}$$

we are considering the behavior of the Hilbert-Schmidt operator $\left(\partial_x + \frac{1}{i}\frac{g-1}{g+1}\right)^{-1}$ as a function of g. Suppose g constant near -1, so $\frac{1}{i}\frac{g-1}{g+1} = \frac{1}{a}$ where a is near zero.

$$\frac{1}{\partial_x + \frac{1}{a}} = \frac{a}{1+a\partial_x}$$

$$\mathcal{L}^2\text{-norm of } \frac{1}{1+a\partial_x} \doteq \left(\frac{C}{a}\int \frac{a\,d\xi}{|1+ai\xi|^2}\right)^{1/2} = \frac{C}{\sqrt{a}}$$

Thus $\left\|\frac{1}{\partial_x+\frac{1}{a}}\right\|_2 = O(\sqrt{a})$ showing it isn't differentiable in a at $a=0$.

To remedy this one can consider instead

the operator $\dfrac{r(a)}{1+a\partial_x}$ where $r(a)$ vanishes to higher order at $a=0$. Note

$$\partial_a^n \frac{r(a)}{1+a\partial_x} = \sum_{i=0}^{n} \binom{n}{i} r^{(n-i)}(a) \underbrace{(\partial_a)^i \frac{1}{1+a\partial_x}}$$

$$= (-1)^i \, i! \, \frac{1}{(1+a\partial_x)^{i+1}} \partial_x^i$$

$\mathcal{L}^2$ norm of $\dfrac{1}{(1+a\partial_x)^{j+1}} \partial_x^j \sim \left\{ \dfrac{C}{a^{2j+1}} \displaystyle\int \frac{a\,d\xi \quad |i\xi|^{2j} a^{2j}}{|1+ai\xi|^{2(j+1)}} \right\}^{1/2}$

$$= O\left(\frac{1}{a^{j+1/2}}\right)$$

Thus $\partial_a^n \dfrac{r(a)}{1+a\partial_x}$ will be continuous at $a=0$ provided that $r^{(n-j)}(a)\, a^{-j-1/2}$ vanishes, say $r(a) = O(a^{n+1})$.

ΨDO approach

$$\frac{1}{\partial_x + a^{-1}} = \frac{1}{1+a^{-1}\partial_x^{-1}} \partial_x^{-1}$$

$$= \partial_x^{-1} - a^{-1}\partial_x^{-2} + a^{-2}\partial_x^{-3} - \cdots$$

leads to the same singularities.

One thing we could try to do would be to construct a solution to our problem in the algebra of ΨDO's symbols. Thus look for

$$\frac{1}{1-X} = A = a\partial_x^{-1} + b\partial_x^{-2} + \cdots$$

satisfying the unitarity condition

$$A + A^* = 2AA^* = 2A^*A$$

Actually it's enough to give a skew-adjoint symbol

$$\frac{1}{X} = a\partial_x^{-1} + b\partial_x^{-2} + \cdots\cdots$$

since then, we have with $Y = \frac{1}{X}$

$$R_\lambda = \frac{1}{\lambda - X} = -\frac{Y}{1-\lambda Y} = -Y - \lambda Y^2 - \lambda^2 Y^3 - \cdots$$

and this satisfies $\quad R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$

as well as $\quad R_\lambda^* = \dfrac{Y}{1+\bar\lambda Y} = -R_{-\bar\lambda}$.

It's clear that each coefficient of $\frac{1}{X}$ is independent subject to the appropriate skew-adjointness conditions. Hence one has to find some process for cranking out the coefficients starting from the loop g.

September 9, 1987

Consider the "cup product" map

$$\begin{array}{ccc} (\mathbb{R}\cup\{\infty\}) \times (\mathbb{R}\cup\{\infty\}) & \longrightarrow & \mathbb{C}\cup\{\infty\} \\ \| & & \| \\ S^1 \times S^1 & & S^2 \end{array}$$

given by $(x,y) \longmapsto z = x+iy$. We have seen that this is smooth except at the point (∞,∞), where in terms of the local coordinates $\frac{1}{z}, \frac{1}{x}, \frac{1}{y}$ we have

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x^{-1}y^{-1}}{y^{-1}+ix^{-1}}$$

This is homogeneous of degree 1 in x^{-1}, y^{-1}; hence it is continuous but not differentiable at the origin.

The standard way to smooth the singularity of a homogeneous function is to multiply by a smooth function vanishing to infinite order at the origin. Review this. Let $f(x)$ be smooth + vanish to infinite order at 0, let $h(x)$ be homogeneous of degree d and smooth except at the origin. Then certainly $f(x)h(x)$ is continuous at $x=0$ with value zero, and when we differentiate using Leibniz

$$\partial_x^\alpha (fh) = \sum \binom{\alpha}{\beta} \partial_x^\beta f \underbrace{\partial_x^{\alpha-\beta} h}_{\text{homog. of degree } d-|\alpha-\beta|}$$

we see that any derivative is continuous with value 0 at $x=0$.

Thus the weakest possible smoothing of the above would be

$$f(x^{-1},y^{-1})\frac{1}{z} = \frac{x^{-1}y^{-1}}{y^{-1}+ix^{-1}} f(x^{-1},y^{-1})$$

where $f \equiv 1$ except for a small nbd of (∞,∞), and where f vanishes to infinite order at ∞,∞.

For example $f(x^{-1}, y^{-1}) = e^{-\frac{1}{x^2+y^2}} = e^{-\frac{x^2y^2}{x^2+y^2}}$. This example can be written in a nicer form by noting that

$$\frac{x^2y^2}{x^2+y^2} = \left|\frac{xy}{x+iy}\right|^2 = \frac{1}{|x^{-1}+iy^{-1}|^2} = \frac{x^2y^2}{|z|^2}$$

This doesn't seem very interesting.

It still seems that the simplest way to smooth the singularity is asymmetric, e.g. using

$$\frac{f(x)}{z} = \frac{f(x)}{x+iy}$$

where f vanishes to infinite order at $x=\infty$. This for example $f(x) = e^{-x^2}$.

Let ask when $(x, \xi) \longmapsto a(x)\, i\xi + b(x)$ is smooth from $S^1 \times S^1 \longrightarrow S^2$. Here a, b are smooth functions of x and as

$$|ai\xi + b|^2 = a^2\xi^2 + b^2$$

we want $a(x) \neq 0$, and $|b(x)| \longrightarrow \infty$ as $x \longrightarrow \infty$.

Now this map is smooth except where $x=\infty$ or $\xi=\infty$.

Case $\xi \neq \infty$. We have

$$\frac{1}{ai\xi + b} = \frac{1}{b}\,\frac{1}{1+\frac{ai\xi}{b}} = \sum_{n\geq 0} \frac{a^n}{b^{n+1}}(-i\xi)^n$$

Thus near $x=\infty$ we want $\frac{a^n}{b^{n+1}}$ to be smooth for all $n \geq 0$. Thus above is a form expansion giving the partial derivatives of $\frac{1}{z}$ with respect to ξ at $\xi=0$

Case: ξ near ∞. We have

$$\frac{1}{ai\xi+b} = \frac{1}{ai\xi\left(1+\frac{b}{ai\xi}\right)} = \sum_{n\geq 0}\frac{b^n}{a^{n+1}}\left(\frac{1}{-i\xi}\right)^n$$

This is a formal expansion at $\xi=\infty$ giving all the partial derivatives wrt ξ^{-1}. We want $\frac{b^n}{a^{n+1}}$ to be smooth for all $n\geq 0$.

We next put these conditions together

$$\frac{b^n}{a^{n+1}} = \frac{1}{a}\left(\frac{b}{a}\right)^n = \frac{1}{a}\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n-1}}{b^{-n}}$$

for all $-n\leq 0$, or $-n-1<0$. Thus the condition, we find is that

$$\frac{1}{a}\left(\frac{b}{a}\right)^n \text{ is smooth at } x=\infty \qquad \text{for all } n\in\mathbb{Z}.$$

Note that we could also write

$$\frac{1}{a}\left(\frac{b}{a}\right)^n = \frac{1}{b}\left(\frac{b}{a}\right)^{n+1}$$

so $\frac{1}{b}$ is smooth at ∞.

So we conclude that the singularity (or zero) of $\frac{b}{a}$ at ∞ is weaker than the order of vanishing of $\frac{1}{a}$. So the simplest example becomes $\frac{b}{a}=x$, $a(x)$ vanishes to infinite order at ∞), e.g. $a(x)=e^{-x^2}$.

September 10, 1987

It's probably worthwhile to review the simplest index 1 case. Thus I work on an interval, say $(-1,1)$, with a loop g starting at -1 at $x=-1$ and then moving counterclockwise to -1 at $x=+1$. We want g to cross 1 transversally at $x=0$.

The goal is to produce an operator whose graph is to be a point in the restricted Grassmannian. ~~Suppose the oper~~ We would like the operator to be a differential operator $a\partial_x + b$ if possible.

From studying the cup product map we are led to expect $a^{-1}(x)$ to be a smooth function on $\mathbb{R}$ which is >0 on $(-1,1)$ and zero outside. $\frac{b}{a}$ is to be smooth on $(-1,1)$ and to approach $-\infty$ as $x \downarrow -1$ and $+\infty$ as $x \uparrow 1$, yet slow enough that $\frac{1}{a}\left(\frac{b}{a}\right)^n$ is smooth on $\mathbb{R}$ when extended by zero outside $(-1,1)$, for all $n \in \mathbb{Z}$.

Next we have to make some sort of conventions concerning how to quantize $a i\xi$, and guess the simplest thing is Weyl quantization which leads to

$$a^{1/2}\partial_x a^{1/2} = a\partial_x + \tfrac{1}{2}a'$$

Then the homog. solutions for $a^{1/2}\partial_x a^{1/2} + b$ and its adjoint are

$$a^{-1/2}\, e^{\pm\int^x -\frac{b}{a}}$$

The Weyl limit point analysis tells us that one of this should fail to be in L^2 near each endpoint. This means the singularity of $\frac{b}{a}$ at the ends has to be enough to overcome the decay of $a^{-1/2}$. Another thing which is similar is for the ambiguity in $\frac{b}{a}$ due to the different quantizations, namely $\frac{a'}{a}$, to be insignificant at the endpoints. This leads to the condition that the order of vanishing of $\frac{a}{b}$ at the ends kills the singularity of $\frac{a'}{a}$; ~~[illegible]~~ my old "$p\frac{r'}{r}$ is smooth" conditions.

September 12, 1987

Let's try again to see if we can get the non abelian case to work. Thus I want to associate to a loop g a ψDO of order -1 looking like

$$\circledast \qquad r(h\partial_x + f)^{-1}$$

where $r = r(g)$ is a smooth function on $\mathbb{T}$ vanishing at $\zeta = -1$ to infinite order applied to g. Here $f = f_0 + hf_1 + \cdots$, where f_0 is a function on $\mathbb{T}$ with pole at $\zeta = -1$ applied to g.

This time I want to see if I can make all the coefficients of the symbol of $\circledast$ smooth to order 1 in h. We have

$$r(h\partial_x + f)^{-1} = \sum_{n\geq 0} (-1)^n \left\{ r(h\partial_x + f)^n(1) \right\} \partial_x^{-n-1}$$

$$r(h\partial_x + f_0 + hf_1 + \cdots)^n(1) = rf_0^n + h\sum_{i=0}^{n-1} rf_0^{n-1-i}(\partial_x + f_1)(f_0^i) + O(h^2)$$

Thus our problem is to see if f_1 can be found such that multiplying by r kills the singularities of

$$\sum_{i=0}^{n-1} f_0^{n-1-i}(\partial_x + f_1) f_0^i$$

for all $n \geq 1$.

Take $n = 1$ and we find rf_1 is to be smooth.

For $n=2$ we have

$$f_0\,(\partial_x + f_1)\,1 \;+\; \cancel{f_0}\,(\partial_x + f_1)\,f_0$$

$$= \;\partial_x f_0 + f_0 f_1 + f_1 f_0$$

is to be smooth when multiplied by r. If r commutes with f_0, then as we know already that $r f_1$ is smooth, [redacted] we have to be fairly careful. I'm assuming the singularities of f_1 are algebraic, so that at first sight $r(f_0 f_1 + f_1 f_0)$ is smooth, leading to problems as $r\,\partial_x f_0$ isn't. However r doesn't have to commute with f_1.

September 14, 1987

I return to trying to arrange that

$$r \sum_{i=0}^{n-1} f_0^{n-1-i} (\partial_x + f_1)(f_0^i)$$

be smooth for all $n \geq 1$. One has

$$(\partial_x + f_1)(f_0^i) = \underbrace{[\partial_x + f_1, f_0^i]}_{\sum_{j=1}^{i} f_0^{j-i}[\partial_x + f_1, f_0] f_0^{j-1}} + f_0^i f_1$$

so it seems we have to arrange that

$$r f_0^n f_1 \quad \text{smooth} \quad \forall\, n \geq 0$$

$$r f_0^m [\partial_x + f_1, f_0] f_0^j \quad \text{smooth} \quad \forall n, j \geq 0$$

(I might try putting $r^{1/2}$ on both sides of $(\partial_x + f)^{-1}$, and so arrange to have damping fns. on both sides of $[\partial_x + f_1, f_0]$, however it seems that we then get problems when we do the $O(h^2)$ terms. It's better therefore to insist that r comes from the left and has to kill the singularities.)

Thus it seems that I want to be able to move the damping function r through $[\partial_x + f_1, f_0]$. One way, perhaps too strong, to guarantee this is to ask this bracket to commute with g.

$$\partial_x f(g) = \partial_x (u f(\gamma) u^{-1})$$
$$= u f'(\gamma) \partial_x \gamma \; u^{-1} + [(\partial_x u) u^{-1}, f(g)]$$

For example take $f = 1$, whence

$$\partial_x g = u (\partial_x \gamma) u^{-1} + [(\partial_x u) u^{-1}, g].$$

This is the decomposition of the tangent vector to the curve $g = g(x)$ into vectors perpendicular and parallel to the conjugacy class. So it is an intrinsic decomposition, although $(\partial_x u) u^{-1}$ is not intrinsic.

If we could arrange $-f_1$ to agree with $(\partial_x u) u^{-1}$ modulo the centralizer of g, then we would have $[\partial_x + f_1, f_0] = u f_0(\gamma) \partial_x \gamma' u^{-1}$ commuting with r.

I want something weaker than this, namely I have only to worry about the $g = -1$ eigenspace.

September 15, 1987

We are trying to arrange that

$$\rho f_0^n f_1 \quad \text{smooth} \quad n \geq 0$$

$$\rho f_0^m [\partial_x + f_1, f_0] f_0^n \quad \text{smooth} \quad m, n \geq 0$$

I have decided to try taking $f_0 = \frac{g-1}{g+1} = 1 - \frac{2}{g+1}$ whence we want

$$\rho \frac{1}{(g+1)^n} f_1 \quad \text{smooth} \quad n \geq 0$$

$$\rho \frac{1}{(g+1)^{m+1}} [\partial_x + f_1, g] \frac{1}{(g+1)^{n+1}} \quad \text{smooth} \quad m, n \geq 0$$

It seems that the totality of these conditions will also work if $f_0(\zeta)$ has a higher order pole at $g = -1$.

We can also shift to hermitian matrices:

$$A^{-1} = \frac{1}{i}\left(\frac{g-1}{g+1}\right)$$

so that $A = 0 \iff g = -1$. Thus we want to find f_1 such that

$$\rho A^{-n} f_1 \quad \text{smooth} \quad \forall n \geq 0$$

$$\rho A^{-m-1} [\partial_x + f_1, A] A^{-n-1} \quad \text{smooth} \quad \forall m, n \geq 0.$$

The real problem is to arrange f_1 so that in $\rho [\partial_x + f_1, A] \frac{1}{A^{n+1}}$ the vanishing of ρ kills the singularites on the right.

Thus the problem becomes to find f_1 so that $\rho [\partial_x + f_1, A]$ can be divided by powers of A on the right.

Now we have to look at $\partial_x A$ which is the tangent vector to our curve in the hermitian matrices. Recall that at each point A the tangent space splits orthogonally into the tangent space to the conjugacy class of A, which consists of $[X, A]$ with $X \in \mathfrak{u}_n$, and its complement which is the centralizer of A, i.e. $\{B \text{ herm.} \mid [B,A]=0\}$.

We now have to find out what f_1 has to do. First note that as rf_1 has to be smooth, it follows that

$$r[\partial_x + f_1, A] = r\,\partial_x A + (rf_1)A - A(rf_1)$$

is smooth.

Lets fix a point A_0, let P_0 be the orth. projector on $\operatorname{Ker} A_0$, and then extend P_0 to a family of projectors near A_0 by

$$P = \frac{1}{2\pi i}\int_{|\lambda|=\varepsilon} \frac{1}{\lambda - A}\, d\lambda$$

where ε is small enough to exclude the non-zero eigenvalues of A_0. Certainly we must have

$$(*) \qquad r[\partial_x + f_1, A]P = 0 \qquad \text{when } A = A_0$$

Now as

$$r[\partial_x + f_1, A] = r[\partial_x + f_1, A]P + r[\partial_x + f_1, A](1-P)$$

and $\frac{1-P}{A}$ is smooth for A near A_0, $(*)$ can be strengthened to requiring that

$r[\partial_x + f_1, A]P$ divisible by A on the right near A_0.

Further review. Suppose we can smoothly diagonalize A:

$$A = u \lambda u^{-1}$$

with u unitary and λ diagonal. Then

$$\partial_x A = u\, \partial_x \lambda\, u^{-1} + [(\partial_x u)u^{-1}, A]$$

and if E projects onto the null space for λ at $A = A_0$, then

$$P = u E u^{-1}$$

$$\partial_x P = \partial_x u\, E\, u^{-1} - u E u^{-1} \partial_x u\, u^{-1}$$

$$= [(\partial_x u)u^{-1}, P].$$

[redacted] Let's try to construct f_1 by using the stratification defined by the dimension of $\operatorname{Ker} A$. Where A is invertible, there is no condition on f_1, except that it be smooth since we are supposing that $r f_1$ is smooth and $r(A)$ is invertible where A is.

Next suppose we are near a point A_0 where $\dim(\operatorname{Ker} A_0) = 1$. This is a nice hypersurface in the space of hermitian matrices. Along this hypersurface we have the projector P onto the kernel, and we can extend to a nbd of the hypersurface. This nbd can be taken to consist of A's such that one eigenvalue is smaller in size than all the others.

Now what conditions are required on f_1 in order that

$$r\,[d + f_1, A]$$

be divisible on the right by A^{+n}. Here f_1 is supposed to be a 1-form. Previously I supposed it to be smooth and got into difficulties, but I want to see if I can get it to work if I allow f_1 to have singularities killed by r.

As we have said before we must have that

$$(1-P)\,[d+f_1, A]\,P$$

be divisible by A^n on the right. In particular it must vanish along the hypersurface. Thus

$$(1-P)\Big((dA)P + \underbrace{[f_1, A]P}_{f_1 \cancel{AP} - A f_1 P}\Big) = 0$$

so

$$(1-P)A f_1 P = (1-P)(dA)P$$

or

$$\boxed{\begin{gathered}(1-P)\, f_1\, P = \frac{1}{A}(1-P)\, dA\, P \\ \text{along the hypersurface}\end{gathered}}$$

September 16, 1987

Let's consider the open set of hermitian matrices A such that $\dim(\mathrm{Ker}\, A) \leq 1$, better let's take the smaller open set containing those A such that there is one eigenvalue with absolute value smaller than the rest. Over this open set we have a decomposition

$$\vec{V} = E' \oplus E''$$

where E' is a line bundle, the eigenbundle for the smallest eigenvalue. Relative to this decomp. one has

$$A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} \qquad d = \begin{pmatrix} i^* di & i^* dj \\ j^* di & j^* dj \end{pmatrix} = \begin{pmatrix} D' & C \\ B & D'' \end{pmatrix}$$

$$[d, A] = \begin{pmatrix} D'A' - A'D & CA'' - A'C \\ BA' - A''B & D''A'' - A''D'' \end{pmatrix}$$

We wish to find $f_1 = \begin{pmatrix} K & M \\ L & N \end{pmatrix}$ such that $r\{[d + f_1, A]\}$ is right-divisible by A^{n-1}, and also such that rf_1 is smooth.

$$[d + f_1, A] = \begin{pmatrix} [D' + K, A'] & (C+M)A'' - A'(C+M) \\ (B+L)A' - A''(B+L) & [D'' + N, A''] \end{pmatrix}$$

$$r(A)[d + f_1, A]A^{-n} = \begin{pmatrix} r(A')[D+K, A'](A')^{-n} & r(A')\{(C+M)A'' - A'(C+M)\}(A'')^{-n} \\ r(A'')\{(B+L)A' - A''(B+L)\}(A')^{-n} & r(A'')[D''+N, A''](A'')^{-n} \end{pmatrix}$$

Now A'' is always invertible, hence so is $r(A'')$ assuming r vanishes only at 0. Thus rf_1 smooth means $r(A')K$, $r(A')M$ are smooth and L, N are smooth. Thus the 2nd column of $r[d+f_1, A]A^{-n}$ is smooth.

The upper left entry is a 1×1 matrix, so it's smoothness is clear. Thus we have only to worry about the lower left entry, i.e. whether

$$\{(B+L)A' - A''(B+L)\}(A')^{-n}$$

is smooth for all n. But this is clear provided $B+L$ is right divisible by $(A')^n$ for all n.

Thus we are free to choose

$$L = -B + T\, s(A')$$

where s vanishes to infinite order at 0.

Let us now consider the space of 2×2 hermitian matrices. We want to see if it is possible to find f_1 so that

$$r(A)\ [d+f_1, A] = T\, s(A)$$

with $s(x)$ vanishing to high order at 0. I think that the problem will occur at the point $A=0$.

Let's pull back to $U(2)\times\mathbb{R}^2$, that is work with $A = u\lambda u^{-1}$, with $u\in U(2)$ and λ diagonal. Then

$$[d, A] = u\{d\lambda + [u^{-1}du, \lambda]\}u^{-1}$$

Let's write

$$u^{-1}du = \begin{pmatrix} * & C \\ B & * \end{pmatrix}$$

$$f_1 = u \begin{pmatrix} K & M \\ L & N \end{pmatrix} u^{-1}$$

Then

$$r(A)\left[d+f_1, A\right] = u\left\{\begin{pmatrix} r(\lambda_1) & 0 \\ 0 & r(\lambda_2)\end{pmatrix}\begin{pmatrix} d\lambda_1 & (C+M)\lambda_2-\lambda_1(C+M) \\ (B+L)\lambda_1-\lambda_2(B+L) & d\lambda_2 \end{pmatrix}\right\}u^{-1}$$

$$= u\begin{pmatrix} r(\lambda_1)\,d\lambda_1 & r(\lambda_1)(C+M)(\lambda_2-\lambda_1) \\ r(\lambda_2)(B+L)(\lambda_1-\lambda_2) & r(\lambda_2)\,d\lambda_2 \end{pmatrix}u^{-1}$$

is supposed to be in the form

$$T\,s(A) = u\begin{pmatrix} T_{11}\,s(\lambda_1) & T_{12}\,s(\lambda_2) \\ T_{21}\,s(\lambda_1) & T_{22}\,s(\lambda_2) \end{pmatrix}u^{-1}$$

Thus we want

$$r(\lambda_2)(B+L)(\lambda_1-\lambda_2) = T_{21}\,s(\lambda_1)$$

which means $s(\lambda_1)$ divides $B+L$. Thus

$$L = -B + s(\lambda_1)\,V$$

and similarly

$$M = -C + s(\lambda_2)\,W$$

Here B, C are the off-diagonal entries of $u^{-1}du$; they depend on the eigenspace variation and not on the eigenvalues.

Now our problem is to see if by a suitable choice of V, W, K, N we can obtain an f_1 which descends to $i\,\underline{u}(2)$.

I propose to try to do things first over the open set where the eigenvalues are distinct. The decomposition

$$[d, A] = u\,d\lambda\,u^{-1} + [du\,u^{-1}, A]$$

is intrinsic, i.e. depends on A; it is the splitting of tangent vectors into components $\perp$ and $\parallel$ to the conjugacy class. Thus over the open set the B, C parts are intrinsic.

A good way to think perhaps is using the eigenspace decomposition

$$\tilde{V} = E' \oplus E''$$

where the eigenvalues λ_1, λ_2 of A on E', E'' are ordered so that $\lambda_1 < \lambda_2$. We can consider the connection $D' \oplus D'' = i^* d\, i + j^* d j$, and this differs from d by the part of f, due to $-B, -C$.

It will be necessary to understand this connection a lot better. It differs from the flat connection d by a matrix 1-form which has singularities along the subspace of scalar matrices. This subspace has real codimension 3.

In order to see how it behaves it's probably OKAY to consider hermitian matrices of trace zero. If we remove the origin we have a map to the 2-sphere obtained by taking the positive eigenline. This is just $A \to \frac{A}{|A|}$

September 17, 1987

The question is whether on the space of hermitian matrices we can find f_1 such that rf_1 is smooth and

$$r[d+f_1, A] = Ts$$

where T is smooth and $s = s(A)$ is a smooth function applied to A of high order at 0.

Take 2×2 matrices with $\det A = 0$ first. On this hypersurface outside $A=0$, we have that $\operatorname{Ker} A$ is a line. Let P be projection on that line. We want

$$r\,dA + rf_1 A - A rf_1 = T\cdot s(A)$$

so

$$[r\,dA - A rf_1)P = 0$$

Presumably rf_1 has it image contained in $\operatorname{Im} r = \operatorname{Im} A$. Thus we have

$$rf_1 P = \frac{1}{A} r(dA)P = \frac{r(A)}{A}(dA)P$$

This gives a candidate for rf_1 namely

$$rf_1 \overset{?}{=} \frac{r(A)}{A}(dA)$$

Then

$$r\,dA + \left(\frac{r}{A}(dA)A\right) - A\left(\frac{r}{A}dA\right) = \frac{r}{A}(dA)A$$

is divisible on the right by a single power of A. I have to decide whether this is a natural limitation, or whether we can do better by changing the candidate on $\operatorname{Im}(1-P) = \operatorname{Im} A$.

Let us describe the $A \neq 0$ with $\det A = 0$ as $A = \lambda Q$, where $Q = 1 - P$, and $\lambda \neq 0$. Then $r(A) = r(\lambda) Q$ as $r(0) = 0$

$$dA = (d\lambda) Q + \lambda\, dQ$$

$$[f, A] = \lambda (fQ - Qf)$$

so

$$r(A)\big(dA + [f,A]\big) = r(\lambda) Q \{ (d\lambda) Q + \lambda\, dQ + \lambda (fQ - Qf) \}$$

$$= r(\lambda)\, d\lambda\, Q + r(\lambda)\lambda \{ \underbrace{Q\, dQ - Qf(1-Q)}_{\underbrace{Q}_{1-P}(dQ - f)\underbrace{(1-Q)}_{P}} \}$$

Now we want this to be

$$T\, s(A) = T\, s(\lambda) Q$$

This implies in particular that f has to be chosen so the second term is zero. If we can arrange this, then T has to satisfy

$$T s(A) Q = r(\lambda)\, d\lambda\, Q$$

and we can take $T = A(dA)A = (\lambda Q)(d\lambda\, Q + \lambda\, dQ)\lambda Q$ $= \lambda^2 d\lambda\, Q$ and $s(\lambda) = \dfrac{r(\lambda)}{\lambda^2}$

So now take $rf = \dfrac{r(A)}{A} dA$ and we should win.

September 18, 1987

Let's review. We are working over the space of 2×2 hermitian matrices of rank ≤ 1. We look for a 1-form rf such that

$$r\,dA + rfA - Arf$$

is of the form $T\,s(A)$ with $s(x)$ vanishing to large order at $x=0$.

First candidate is $rf = \frac{r(A)}{A}dA$, for then

$$r\,dA + \frac{r}{A}dA\,A - A\frac{r}{A}dA = \frac{r(A)}{A}(dA)A$$

Now writing $A = \lambda Q$ with Q a rank 1 projection, we find

$$\frac{r(A)}{A}(dA)A = \frac{r(\lambda)}{\lambda}Q\left(d\lambda\, Q + \lambda\, dQ\right)\lambda Q$$

$$= \frac{r(\lambda)}{\lambda}d\lambda\,\lambda Q + r(\lambda)\lambda \underbrace{Q(dQ)Q}_{0}$$

$$= r(\lambda)\,d\lambda\, Q$$

coincides with

$$A(dA)\frac{r(A)}{A} = \lambda Q\left(d\lambda\, Q + \cancel{\lambda\, dQ}\right)\frac{r(\lambda)}{\lambda}Q$$

$$= Q\,d\lambda\, r(\lambda)$$

Thus we can take $T(A) = A(dA)$ and $s(x) = \frac{r(x)}{x}$.

Next let us consider 2×2 matrices in general and write them

$$A = \lambda P + \mu Q \qquad\qquad Q = (1-P).$$

Then $dA = (d\lambda)P + (d\mu)Q + (\lambda-\mu)\,dP$. Let's use block form relative to the eigendecomposition

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad dP = \begin{pmatrix} 0 & \overbrace{P\,dP\,(1-P)}^{C} \\ \underbrace{(1-P)(dP)P}_{B} & 0 \end{pmatrix}$$

so

$$dA = \begin{pmatrix} d\lambda & (\lambda-\mu)C \\ (\lambda-\mu)B & d\mu \end{pmatrix}.$$

Let $f = \begin{pmatrix} * & M \\ L & * \end{pmatrix}$. Then

$$r\big(dA + [f,A]\big)s^{-1} = \begin{pmatrix} \dfrac{r(\lambda)\,d\lambda}{s(\lambda)} & \dfrac{r(\lambda)(\lambda-\mu)(C+M)}{s(\mu)} \\ \underbrace{\dfrac{r(\mu)(\lambda-\mu)(B+L)}{s(\lambda)}}_{T_{21}} & \dfrac{r(\mu)\,d\mu}{s(\mu)} \end{pmatrix}$$

This has to be smooth which means that

$$L = -B + \frac{s(\lambda)}{r(\mu)}\,\frac{1}{\lambda-\mu}\,\overset{\text{smooth}}{T_{21}}$$

~~Now we have the candidate for f~~

$$\cancel{f = \frac{1}{A}\,dA - \begin{pmatrix} * & \frac{\lambda-\mu}{\lambda}C \\ \frac{\lambda-\mu}{\mu}B & * \end{pmatrix}}$$

We want rf to be some function of A and dA, such as the $\frac{r(A)}{A}\,dA$ we tried before. This implies that $r(\mu)L$ has the form

$$r(\mu)L = h(\mu)(\lambda-\mu)B\,k(\lambda)$$

since the only kind of thing I can see is of the form $h(A)\cdot dA\cdot k(A)$.

For example if I take $\frac{r(A)}{A}dA$, then I can realize

$$r(\mu)L = \frac{r(\mu)}{\mu}(\lambda-\mu)B$$

or
$$L = \frac{\lambda-\mu}{\mu}B = -B + \frac{\lambda}{\mu}B$$

but this is quite far from

$$L = -B + \frac{s(\lambda)}{r(\mu)(\lambda-\mu)}T_{21}$$

Wait: We ought to allow for things of the form
$$\sum_j h_j(\mu)(\lambda-\mu)B\,k_j(\lambda)$$

Here seems to be the solution of the problem. We can take $s=r$.

$$r(\mu)L = -r(\mu)B + \frac{s(\lambda)}{\lambda-\mu}T_{21}$$

Take $T_{21} = \rho B$, where ρ is smooth. Then

$$r(\mu)L = \left(-r(\mu) + \frac{s(\lambda)}{\lambda-\mu}\rho\right)B$$

and we know we can realize something of the form

$$\varphi(\lambda,\mu)(\lambda-\mu)B$$

by combining things of the form $h(A)\,dA\,k(A)$.

We want

$$-r(\mu)+\frac{s(\lambda)}{\lambda-\mu}\rho(\lambda,\mu)$$

to be smooth with ρ smooth. Multiplying by $\lambda-\mu$ and setting $\lambda=\mu$, we see that $\rho(\lambda,\mu)$ is divisible by $\lambda-\mu$, say $\rho=(\lambda-\mu)\sigma$; then

$$-r(\mu)+s(\lambda)\sigma(\lambda,\mu)$$

should vanish when $\lambda=\mu$, which means that s divides r.

Take $s=r=x^2$. Then

$$r(\mu)L = (\lambda^2-\mu^2)B=(\lambda+\mu)(\lambda-\mu)B$$

should be obtained from $A\,dA+(dA)A$. Thus let's try

$$rf = A^2f = A\,dA+(dA)A$$

$$A^2\,dA+[A^2f,A] = A^2dA+[A\,dA+(dA)A,A]$$

$$= \cancel{A^2dA}-\cancel{A^2dA}-\cancel{A(dA)A}+\cancel{A\,dA\,A}+(dA)A^2$$

$$= dA\,A^2.$$

So in general it is clear what is going on namely we are writing $r(A)\,dA-dA\,r(A)$ in the form $[h,A]$, and letting $f=r^{-1}h$

September 19, 1987

Let's return to the connection idea and see if some understanding can be achieved by using our 2×2 calculations. The idea is to see if we can arrange r, s, f so that $s^{-1}(d+f)r$ is smooth. Here $r = r(A)$ where $r(x)$ is a function vanishing at $x=0$; s is similar. The smoothness can be interpreted as saying that the connection "$d+f$" carries rC^∞ into sC^∞.

$$A = \lambda P + \mu Q = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad Q = 1-P$$

$$dA = \begin{pmatrix} d\lambda & (\mu-\lambda)C \\ (\lambda-\mu)B & d\mu \end{pmatrix} \quad \text{where} \quad dP = \begin{pmatrix} 0 & -C \\ B & 0 \end{pmatrix}$$

$$dr(A) = d\big(r(\lambda)P + r(\mu)Q\big) = r'(\lambda)d\lambda\, P + r'(\mu)d\mu\, Q + \big(r(\lambda)-r(\mu)\big)dP$$

$$= \begin{pmatrix} r'(\lambda)d\lambda & (r(\mu)-r(\lambda))C \\ (r(\lambda)-r(\mu))B & r'(\mu)d\mu \end{pmatrix}$$

Let $f = \begin{pmatrix} K & M \\ L & N \end{pmatrix}$. $\quad s^{-1}fr = \begin{pmatrix} s(\lambda)^{-1}Kr(\lambda) & s(\lambda)^{-1}Mr(\mu) \\ s(\mu)^{-1}Lr(\lambda) & s(\mu)^{-1}Nr(\mu) \end{pmatrix}$

Then

$$s^{-1}(dr+fr) = \begin{pmatrix} \dfrac{r'(\lambda)d\lambda + Kr(\lambda)}{s(\lambda)} & \dfrac{(r(\mu)-r(\lambda))C + Mr(\mu)}{s(\lambda)} \\[2ex] \dfrac{(r(\lambda)-r(\mu))B + Lr(\lambda)}{s(\mu)} & \dfrac{r'(\mu)d\mu + Nr(\mu)}{s(\mu)} \end{pmatrix}$$

Now we want this to be smooth, and of course f should be a function of A and dA. This means, according to yesterday's work, that L has the form $\rho(\lambda,\mu)(\lambda-\mu)B$. Thus we have to arrange

$$\frac{r(\lambda)-r(\mu)+(\lambda-\mu)\rho(\lambda,\mu)}{s(\lambda)}$$

<u>September 26, 1987</u>

Let's recall what happens when g has a constant number of eigenvalues $=-1$. Then we have a splitting $\tilde{V} = E' \oplus E''$ where

$$g = \begin{pmatrix} g' & 0 \\ 0 & -1 \end{pmatrix} \qquad \partial_\mu = \begin{pmatrix} i^*\partial_\mu i & i^*\partial_\mu j \\ j^*\partial_\mu i & j^*\partial_\mu j \end{pmatrix}$$

$$g' = \left(\frac{1+X'}{1-X'}\right)$$

and

$$\frac{1}{\lambda - \gamma^\mu\partial_\mu - \sigma\frac{g-1}{g+1}} = (g+1)\frac{1}{(\lambda-\gamma^\mu\partial_\mu)(g+1) - \sigma(g-1)}$$

$$= \begin{pmatrix} g'+1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (\lambda - \gamma^\mu D'_\mu)(g'+1) - \sigma(g'-1) & 0 \\ -\gamma^\mu j^*\partial_\mu i\,(g'+1) & +2\sigma \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} (g'+1)\dfrac{1}{(\lambda-\gamma^\mu D'_\mu)(g'+1) - \sigma(g'-1)} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{\lambda - \not{D}' - \sigma X'} & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus in the case where g had a constant number of eigenvalues $= -1$, the unbounded operator $\gamma^\mu\partial_\mu + \sigma\left(\frac{g-1}{g+1}\right)$ gives a good answer.

This suggests that we should examine the natural stratification of $U(V)$ by the number of eigenvalues $= -1$. We can handle $g: M \longrightarrow U(V)$ lying in a single stratum, so maybe what we need to do in the general case can be discovered by looking at the links between the different strata.

Let's review the nature of the stratification. It's related to the Morse flow which sends the eigenvalues towards $+1$. The critical points are the involutions. Set F_k = involutions with k eigenvalues $= +1$, so $F_k \cong Gr_k(V)$. Through F_k pass the ascending and descending submanifolds. The ascending Y_k^+ consists of points flowing into F_k so

$$Y_k^+ = \{g \mid \text{exactly } k \text{ eigenvalues} \neq -1\}$$

and similarly

$$Y_k^- = \{g \mid \text{exactly } k \text{ eigenvalues} \neq +1\}.$$

We are interested in the number of eigenvalues $= -1$. Thus the Y_k^+ are the strata. Y_n^+ = image of Cayley transform $Y_0^+ = \{-1\}$.

The idea motivating me now is that I have good control over the situation when the values of g lie in a single stratum, and the necessary changes when g crosses strata are determined by the way the strata are linked together. Thus I should be able to use the standard connection d over most of the open stratum Y_n^+, and there should be changes as we move to a lower stratum.

Let's try to list some useful ideas. We keep track of the eigenvalues of g as we move over M. Where the eigenvalues are well away from -1, say $\mathrm{Re}(g+1) > \varepsilon$, we use the operator $\not\partial + \sigma\left(\frac{g-1}{g+1}\right)$.

September 27, 1987

Let $a \in (-1,1)$, and let us consider the open set U_a of M where $a \notin \mathrm{Spec}(\mathrm{Re}(g))$. Over this open set we can consider the subbundle where $\mathrm{Re}(g) > a$ with the induced connection and the skew-adjoint endom. given by g. In other words we have thrown all the eigenvalues ξ with $\mathrm{Re}(\xi) < a$ to -1.

Now M is covered by the U_a for different a. Over each U_a we have a different Dirac operator. The goal is somehow to assemble them into a global operator, i.e. something like a partition of unity construction. To do this we have to have a means of comparison between the Diracs on U_a and U_b, where say $a < b$. Now, on $U_a \cap U_b$ the bundle E_b is the part of E_a where $\mathrm{Re}(g) > b$, so the operator D_b etc over $U_a \cap U_b$ is the "smaller" Dirac operator obtained by shoving the eigenvalues ξ of g with $\mathrm{Re}\,\xi \in (a,b)$ off to -1. The essential analytical difficulty is that this deformation is not smooth when one works in the appropriate Hilbert-Schmidt class.

September 28, 1987

List some ingredients for the theory sought.

First of all given E, D, X we can form $\not{D} + \sigma X$ acting on $L^2(M, S\otimes E)$, and the C.T. of $\not{D} + \sigma X$ is a point in the restricted Grass. (or unitary group), where restricted is relative to a Schatten ideal depending on dim M.

Secondly given (E, D, g) where $g' \neq -1$ on E' and $g = -1$ on a subbundle E'', then we can form $\not{D}' + \sigma X'$ acting on $L^2(M, S\otimes E') \subset L^2(M, S\otimes E)$. This gives another point of the restricted Grassmannian.

So in this way to any (E, D, g) over M, where the multiplicity of -1 as an eigenvalue of g is locally constant, we can assign a point in the restricted Grassmannian of $L^2(M, S\otimes E)$.

It's a fact that even on this class of g the map to the restricted Grass. is not smooth. We saw this using constant coefficient examples. Specifically we consider over the torus M the Dirac operator $\not{\partial} + \sigma \frac{g-1}{g+1}$ with g constant. Take $M = S^1$ whence

$$\not{\partial} + \sigma \frac{g-1}{g+1} = \begin{pmatrix} 0 & \partial_x - A \\ \partial_x + A & 0 \end{pmatrix} \qquad g = \frac{1+iA}{1-iA}$$

If $A = a \in \mathbb{R}$, then we are concerned with the graph of $\partial_x + a$, which is essentially the same as the graph of $\frac{1}{\partial_x + a}$ which is in $\mathcal{L}^2$. As $a \to \infty$ we use $\frac{1}{a}$ as coordinate, whence we have the map

$$\frac{1}{a} \longmapsto \frac{1/a}{1+\frac{1}{a}\partial_x}$$

and the point is that this is not C^1 as the Hilbert-Schmidt norm of $(1+\frac{1}{a}\partial_x)^{-1}$ is $O(\sqrt{a})$.

We have a scheme for handling this which amounts to replacing the map

$$S^1 \times S^1 \longrightarrow S^2$$

$$(\xi, a) \longmapsto i\xi + a$$

which is continuous by the smooth map

$$(\xi, a) \longmapsto \frac{i\xi + a}{r(a)}$$

where r vanishes to infinite order at $a = \infty$.

I should check that this yields a smooth map on the matrix level, i.e. that difficulties with the variation of the eigenspaces do not occur.

More generally consider (E, D, X) where $X = \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}$ relative to $E = E' \oplus E''$, and where X'' is invertible. Consider (E, D, X_t) where $X_t = \begin{pmatrix} X' & 0 \\ 0 & tX'' \end{pmatrix}$ and $t \uparrow \infty$. The problem is whether the proposed modification of $\not{D} + \sigma X_t$ is smooth in $\frac{1}{t}$

Let's consider the cup product map

$$S^1 \times \mathcal{U}(V) \longrightarrow Gr(\mathbb{C}^2 \otimes V)$$

$$(t, X) \longmapsto \text{graph of } t + X$$

$$= \text{C.T. of } \gamma^1 X + \gamma^2 \tfrac{1}{i} t$$

and see if we can show that the modification

$$(t, X) \longmapsto \text{graph of } (t+X)/r(X)$$

is smooth. Here $r(X)$ is something like e^{X^2}.

First let's show the original map is well-defined and continuous. We have

$$\text{graph}\,(t+X) = \text{Im}\begin{pmatrix} 1 \\ t + \frac{g-1}{g+1} \end{pmatrix} = \text{Im}\begin{pmatrix} g+1 \\ t(g+1)+(g-1) \end{pmatrix}$$

Recall that $t(g+1)+(g-1)$ is invertible for $Re(t) \neq 0$, as

$$\left| t(g+1)+(g-1) \right| = \left| (t+1)g + (t-1) \right|$$

$$\geq \left| |t+1| - |t-1| \right| = \frac{|t+1|^2 - |t-1|^2}{|t+1| + |t-1|} \geq \frac{4\,Re(t)}{2(1+|t|)}$$

Thus it's clear that we have a smooth map from $R \times U(V)$ to $Gr(\mathbb{C}^2 \otimes V)$. (Note: to handle $t=0$ it would have been nice to note simply that $\begin{pmatrix} g+1 \\ t(g+1)-(g-1) \end{pmatrix}$ is injective for any $(g,t) \in (\text{End}\,V) \times \mathbb{C}$.

What happens if t is near ∞. We have

$$\text{graph}(t+X) = \text{Im}\begin{pmatrix} \frac{1}{t}(g+1) \\ (g+1) + \frac{1}{t}(g-1) \end{pmatrix}$$

I want to see that this is uniformly close to $\text{Im}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if $|t|$ is large. Write it as

$$\operatorname{Im}\begin{pmatrix} \frac{1}{t} \dfrac{g+1}{(g+1)+\frac{1}{t}(g-1)} \\ 1 \end{pmatrix}$$

Now we know that

$$\frac{1}{|(g+1)+\frac{1}{t}(g-1)|} \leq \frac{1+|\frac{1}{t}|}{2|\mathrm{Re}(\frac{1}{t})|}$$

$$\frac{1}{|t(g+1)+(g-1)|} \leq \frac{1+|t|}{2|\mathrm{Re}(t)|}$$

but this doesn't help. Another approach is to ask for

$$\sup_{a\in\mathbb{R}} \left|\frac{1}{t+ia}\right| = \frac{1}{|t|}$$

Better:

$$\operatorname{graph}(t+X) = \operatorname{Im}\begin{pmatrix} \frac{1}{t+X} \\ 1 \end{pmatrix}$$

and the eigenvalues of $\frac{1}{t+X}$ are $\frac{1}{t+ia}$ with $a\in\mathbb{R}$. Thus the norm is $\leq \frac{1}{|t|}$. Alternatively use

$$\begin{pmatrix} 0 & -t+X \\ t+X & 0 \end{pmatrix} \quad \text{whose ~~square~~ is} \quad t^2-X^2 \geq t^2.$$

I can summarize the above by saying that I've shown the map $S^1\times U(V) \to Gr(\mathbb{C}^2\otimes V)$ is continuous.

Next we propose to examine the smoothness of the modified map

$$\operatorname{graph}\left(\frac{t+X}{r(X)}\right) = \operatorname{Im}\begin{pmatrix} \frac{r(X)}{t+X} \\ 1 \end{pmatrix}$$

For $t \neq \infty$, this can be written

$$\operatorname{Im}\begin{pmatrix} r(g+1) \\ t(g+1)+(g-1) \end{pmatrix}$$

which depends smoothly on t and g since this map is always injective. In effect the map $t(g+1)-(g-1)$ is injective for $\operatorname{Re}(t) \neq 0$. If $t=0$, then $r\cdot(g+1)$ is injective on $\operatorname{Ker}(g-1)$ so it's OK.

Consider now t near ∞. Recall

$$\frac{1}{1-z} = \sum_0^{N-1} z^n + \frac{z^N}{1-z}$$

hence

$$\frac{r(X)}{t+X} = \frac{r(X)}{t}\,\frac{1}{1+\frac{X}{t}}$$

$$= \sum_0^{N-1} \frac{r(X)(-X)^n}{t^{n+1}} + \frac{r(X)}{t}\,\frac{\left(\frac{-X}{t}\right)^N}{1+\frac{X}{t}}$$

$$= \sum_0^{N-1} \left(r(X)(-X)^n\right)\frac{1}{t^{n+1}} + \frac{r(X)(-X)^N}{t^N}\,\frac{1}{t+X}$$

Now we know that $r(X)(-X)^N$ extends to a smooth function of g by the smooth functional calculus, and we know that $\frac{1}{t+X}$ extends to a continuous function of g.

Actually the easiest way to establish that $\frac{r(X)}{t+X}$ is a smooth fn. of $\frac{1}{t}$ and g is to use the smooth fnl.

calculus. This reduces one to the case where $g \in U(1)$, where we have checked it.

September 29, 1987

Goal: Review the cup product map

$$S^1 \times S^1 \longrightarrow S^2 \qquad x, y \longmapsto x + iy$$

and its smoothings, and see if ~~they~~ lead to a smooth map

$$U(V) \times U(W) \longrightarrow Gr(\mathbb{C}^2 \otimes (V \otimes W))$$

The last step should be a consequence of the smooth functional calculus.

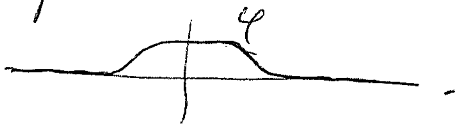
Recall that if $x, y \neq 0$, then

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x^{-1}y^{-1}}{y^{-1} + i x^{-1}}$$

is homogeneous of degree 1 in x^{-1}, y^{-1}, so it's continuous but not smooth. We can smooth it by multiplying by a function φ vanishing to infinite order at $x^{-1} = y^{-1} = 0$. For example $\varphi = e^{-x^2}$ or $\varphi = e^{-y^2}$.

Let's then consider the map

$$\frac{1}{z} = \frac{\varphi(x)}{x + iy}$$

where $\varphi(x)$ vanishes to infinite order at ∞. This means that φ belongs to the Schwartz space, if I remember correctly. I also want $\varphi(0) = 1$ so as to get the correct behavior for (x,y) near $(0,0)$. For example we can take $\varphi(x) = e^{-x^2}$ or even  so that $\varphi \in C_0^\infty(\mathbb{R})$.

For each $y \neq 0$ the function of x

$$\frac{\varphi(x)}{x+iy}$$

is in the Schwartz class $\mathcal{S}$. We claim this is smooth near $y=\infty$. We use the series

$$\frac{\varphi(x)}{x+iy} = \frac{\varphi(x)}{iy}\,\frac{1}{1+\frac{x}{iy}} = \sum_{n \geq 0} \left(\varphi(x)(-x)^n\right)\frac{1}{(iy)^{n+1}}$$

~~Note When $\varphi \in C_0^\infty(\mathbb{R})$ the series converges for~~

This is a power series in $\frac{1}{y}$ which for y large converges in the C^∞ topology on C^∞ functions of x. In effect

$$D^k(x^n\varphi) = \sum_{i+j=k} \frac{k!}{i!\,j!}\underbrace{(D^i x^n)}_{n(n-1)\cdots(n-i+1)\,x^{n-i}}(D^j\varphi)$$

will be bounded over any compact set K by a linear combination of terms of the form $n^a|K|^b$ where $|K| = \sup\{|x|, x \in K\}$. Thus the series converges for $|y| > |K|$.

So when $\varphi(x)$ has compact support we see that $\frac{\varphi(x)}{x+iy}$ is an analytic function of $\frac{1}{y}$ with values in $C_0^\infty(\operatorname{supp}\varphi)$. This proves the claim for $\varphi \in C_0^\infty$, Actually it's interesting to consider the function $y \longmapsto \dfrac{1}{x+iy}$ on the Fourier transform side:

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\xi x}\,\frac{1}{x+iy}\,dx = \begin{cases} e^{-y\xi}\,\theta(\xi) & y>0 \\ +\,e^{-y\xi}\,\theta(-\xi) & y<0 \end{cases}$$

Thus as $y \to \infty$ this goes rapidly to zero - it's the smooth fn. $e^{-\frac{1}{x}}\theta(x)$ behavior. And as y passes zero the F.T. jumps by a constant fn, corresponding to the fact that $\dfrac{1}{x+iy}$ jumps by a multiple of $\delta(x)$.

Let's now embark on the program of linking two strata in a general situation. I want to consider over M (a torus say), a triple (E, D, X), and to suppose that the spectrum of X has a gap. Specifically let's assume that X has no eigenvalue of abs. value 1 at any point of M. Then we can split $E = E' \oplus E''$ $X = \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}$ where the spectrum of X' (resp. X'') is inside (resp. outside) the unit circle. We want to compare the Dirac operators

$$\not{D} + \sigma X \qquad \text{and} \qquad \not{D}' + \sigma X'$$

in some way. More precisely we want to deform $\not{D} + \sigma X$ to $\not{D}' + \sigma X'$ by sending X'' off to infinity. For example we might let

$$X_t = \begin{pmatrix} X' & 0 \\ 0 & tX'' \end{pmatrix}$$

and then study $\not{D} + \sigma X_t$ as $t \to \infty$.

Recall that

$$g_t = \frac{1+X_t}{1-X_t} \longrightarrow \begin{pmatrix} g' & 0 \\ 0 & -1 \end{pmatrix}$$

and also that the resolvent of $\not{D} + \sigma X_t$ approaches the resolvent of $\not{D}' + \sigma X'$ extended by zero. It's necessary to go over this last point.

We have

$$\frac{1}{\lambda - \not{D} - \sigma X} = \begin{pmatrix} \lambda - \not{D}' - \sigma X' & -i^* \not{D} j \\ -j^* \not{D} i & \lambda - \not{D}'' - \sigma X'' \end{pmatrix}^{-1}$$

and the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} (a-bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

[Digression: This identity can be understood by diagrams. Thus think of have a 2 stage system with free Hamiltonian $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ and interaction $\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$. The 1,1 block of the propagator $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ is a sum of terms belonging to the diagrams

$$1 \xleftarrow{a^{-1}} 1$$

$$\begin{tikzcd} 1 & 1 \arrow[l, "a^{-1}"'] & 1 \arrow[d, "c"] & 1 \arrow[l, "a^{-1}"'] \\ & 2 \arrow[u, "b"] & 2 \arrow[l, "d^{-1}"'] & \end{tikzcd}$$

$$\begin{tikzcd} 1 & 1 \arrow[l, dash] & 1 \arrow[d, dash] & 1 \arrow[l, dash] & 1 \arrow[d] & 1 \arrow[l] \\ & 2 \arrow[u, dash] & 2 \arrow[l, dash] & 2 \arrow[u] & 2 \arrow[l, dash] & \end{tikzcd}$$

which leads to the series

$$a^{-1} + a^{-1}bd^{-1}ca^{-1} + a^{-1}bd^{-1}ca^{-1}bd^{-1}ca^{-1} + \cdots\cdots$$

$$= a^{-1}\left(1 - bd^{-1}ca^{-1}\right)^{-1} = \left(a - bd^{-1}c\right)^{-1} \quad].$$

What's happening in our situation is that X'' is replaced by tX''. So

$$d = \lambda - \not{D}'' - t\sigma X''$$

where X'' is invertible and $t \to \infty$. Thus d^{-1} should be approaching zero, whereas $j^*\not{D}i$ and i^*Dj are fixed bounded operators. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \longrightarrow \begin{pmatrix} a^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

at least as bounded operators.

<u>Question</u>: Take M to be a torus and let $\tilde{V} = E' \oplus E''$ be a splitting of the trivial bundle $\tilde{V}$ over M. I can construct the resolvent of $\not{D}$ on $L^2(M, S \otimes \tilde{V})$ easily, and then by perturbation series I can construct $\not{D} + \sigma X$. Can one take X to be 0 on E'

and t on E'' and obtain the resolvent of $D\!\!\!/\,'$ by letting $t \to \infty$?

In any case there is the problem of the rate of convergence in the Schatten class to be understood.

The goal is a prescription for an "operator" associated to g. In the constant coefficient case we take something like

$$\circledast \qquad \frac{r(g)}{\partial\!\!\!/ + \sigma \frac{g-1}{g+1}}$$

where $r(\zeta) = 1$ for ζ near $+1$ and $r(\zeta) = 0$ near $\zeta = -1$. Why **do** we take this? Because it corresponds to the smoothing

$$\frac{\varphi(x)}{i\xi + x} \quad \text{of} \quad \frac{1}{i\xi + x}$$

I think it's important to have the asymptotic expansion

$$\frac{\varphi(x)}{i\xi + x} = \sum_{n \geq 0} \left[\varphi(x)(-x)^n\right] \frac{1}{(i\xi)^{n+1}}$$

near $\xi = \infty$. First of all this is related to the smoothness and secondly it is related to the resolvent of our operator being a ψDO.

When g is not constant we might be after a non-commutative version of $\circledast$. Our next project might be to see if it is possible to handle the situation where g = C.T. of $X_t = \begin{pmatrix} X' & 0 \\ 0 & tX'' \end{pmatrix}$, in which case we want $(\not{\partial} + \sigma X_t)^{-1}_{\text{(modified)}}$ to converge to $(\not{\partial}' + \sigma X')^{-1}$ in space of ΨDO symbols.

September 30, 1987

Consider $\quad X = \begin{pmatrix} X' & 0 \\ 0 & X'' \end{pmatrix}$ with X'' invertible and

$$\not{D} + \sigma X = \begin{pmatrix} \not{D}' + \sigma X' & \iota^* \not{D} j \\ j^* \not{D} i & \not{D}'' + \sigma X'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and form the inverse in the space of ψDO symbols. We can ask whether upon replacing X'' by tX'' and letting $t \to \infty$, does $(\not{D} + \sigma X)^{-1}$ approach $\begin{pmatrix} (\not{D}' + \sigma X')^{-1} & 0 \\ 0 & 0 \end{pmatrix}$. Using

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

we have to worry about

$$d^{-1} = (\not{D}'' + t\sigma X'')^{-1}$$

as $t \to \infty$. But this doesn't approach zero in the space of ψDO symbols. For example

$$\frac{1}{\partial_x + t} = \sum_{n \geq 0} (-t)^n \frac{1}{\partial_x^{n+1}}$$

doesn't go to zero as $t \to \infty$.

So we next want to see if something like

$$\frac{r(X)}{\not{D} + \sigma X}$$

does have the appropriate continuity property. The above is ambiguous because of the non-commutativity, but one possible way to make it precise while preserving the skew-adjointness is

$$r^{1/2} \frac{1}{\not{D} + \sigma X} r^{1/2}$$

Here r is a function on $i\mathbb{R}$ applied to skew-hermitian matrices; $r(iy)$ is 1 for $|y| \leq 1$ and 0 for out, ~~and~~ ≥ 0, so

$$r(X) = \begin{pmatrix} 1 & 0 \\ 0 & r(X'') \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix}$$

where $s = r(X'')^{1/2}$. Then

$$r^{1/2}(\not{D} + \sigma X)^{-1} r^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix}\begin{pmatrix} (\quad)^{-1} & \\ & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -sd^{-1}c & 1 \end{pmatrix}\begin{pmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & sd^{-1}s \end{pmatrix}\begin{pmatrix} 1 & -bd^{-1}s \\ 0 & 1 \end{pmatrix}$$

Now we try to replace X'' by tX'' and let $t \to \infty$. It's reasonable to assume inductively that $sd^{-1}s = r(X'')^{1/2} \dfrac{1}{\not{D}'' + \sigma X''} r(X'')^{1/2}$ goes to zero as a ΨDO symbol, say by induction. It's less reasonable to expect $sd^{-1}c$ and $bd^{-1}s$ to go to zero, but this would maybe be OK in the case that X'' is a 1×1 matrix. However the upper left block

$$\begin{pmatrix} 1 & 0 \end{pmatrix} r^{1/2}(\not{D} + \sigma X)^{-1/2} r^{1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (a - bd^{-1}c)^{-1}$$

is completely unaffected by s, and there is no hope of convergence.

<u>Conclusion</u>: The difficulty with eigenspace variation is already apparent in this simple example. Notice that a, b, c are fixed as X'' is changed.

Suppose we try to deform the metric. Suppose we change the metric on E to $\|Bs\|^2$ where $B > 0$. Then we have associated to the change $\|s\|^2 \longrightarrow \|Bs\|^2$ in metric the change in connection $D \longrightarrow B^{-1}DB$. In effect

$$\left(B(B^{-1}DB)s_1, Bs_2\right) + \left(Bs_1, B(B^{-1}DB)s_2\right)$$

$$= \left(D(Bs_1), Bs_2\right) + \left(Bs_1, D(Bs_2)\right) = 0$$

so $B^{-1}DB$ preserves the new metric.

Take $E = E' \oplus E''$, $B = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$. Then

$$B^{-1}DB = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} D' & i^*Dj \\ j^*Di & D'' \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} D' & (i^*Dj)\alpha \\ \alpha^{-1}j^*Di & D'' \end{pmatrix}$$

Unfortunately this appears unlikely to help the problems with $(a - bd^{-1}c)^{-1}$ failing to have a limit as a ψDO.

Philosophy: The critical case seems to be when we have $E = E' \oplus E''$ and $X = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}$ where t is constant. Thus we have constant eigenvalues over M but non-trivial eigenspace variation. We are trying to set up a bounded ψDO starting with $\dfrac{1}{\not{D}}$ at $t = 0$ and ending with $\dfrac{1}{\not{D}'}$ extended by zero at $t = \infty$.

We ~~[illegible]~~ want to assign a ψDO to $E = E' \oplus E''$, $X = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$. It seems that something like the diagrams described briefly above might be useful. Thus one will want "free" propagators a^{-1}, d^{-1} and transitions b, c.

October 1, 1987

So far we have run into difficulties in trying to couple ∂_x to an arbitrary loop g. Let us consider the known method of doing the pairing of g with the fundamental class of the circle. The latter is represented by the Hilbert transform operator F. One has a natural map from loops g to the restricted Grassmannian obtained by sending g to gFg^{-1}.

Suppose we take the basic grading ε on $L^2(S^1, V)$ to be $\varepsilon = -F$, so that gFg^{-1} lies in the restricted Grassmannian of involutions congruent to $-\varepsilon$. The corresp. unitary inverted by ε is

$$\tilde{g} = (gFg^{-1})(-\varepsilon) = -gFg^{-1}F$$

Then

$$\tilde{g}+1 = -gFg^{-1}F + gg^{-1}FF$$

$$= -g[F, g^{-1}]F$$

It might be simpler to work with $\varphi = g^{-1}$. Then the relevant compact operator is

$$\tilde{g}+1 = -\varphi^{-1}[F,\varphi]F$$

and this is a <u>smoothing operator</u>, since $[F,\varphi]$ is a smoothing op. In effect the kernel of $[F,\varphi]$ is essentially

$$\frac{1}{x-y}\varphi(y) - \varphi(x)\frac{1}{x-y} = -\frac{\varphi(x)-\varphi(y)}{x-y}$$

So the [redacted] first observation seems to be that we end up with a nice smooth map from loops to the "smooth" restricted Grassmannian.

What I'd like to do next is to try to link up various constructions in this Toeplitz setting. First of all given the loop φ there is the associated Toeplitz operator $P\varphi P$ on H_+, which is a Fredholm operator, and a contraction.

At some time in the past (visit to Chicago) I [redacted] had some ideas [redacted] linking essentially unitary contractions and points in the restricted Grassmannian. Let's review this.

Let a be an essentially unitary contraction, whence $A = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ is an essentially involutive self-adjoint contraction. It then has a modified Cayley transform

$$\tilde{g} = \left(\sqrt{1-A^2} + iA\right)^2$$

instead of the familiar $g = \left(\frac{1+X}{\sqrt{1-X^2}}\right)^2$. Thus the link between A and X is

$$X = i\frac{A}{\sqrt{1-A^2}} \qquad iA = \frac{X}{\sqrt{1-X^2}}$$

~~[illegible] to consider the Toeplitz operator~~

So

$$X = i\begin{pmatrix} 0 & a^*/\sqrt{1-aa^*} \\ a/\sqrt{1-a^*a} & 0 \end{pmatrix}$$

and hence up to rotation by i or so, we have that $\tilde{g}$ corresponds to the graph of $a/\sqrt{1-a^*a}$

Now suppose that $a = P\varphi P$ is a Toeplitz operator. Thus relative to $L^2 = H_+ \oplus H_-$ we have

$$\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Because φ is unitary

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \quad a^*a + c^*c = 1. \quad \Longrightarrow \quad (c^*c)^{1/2} = \sqrt{1-a^*a}$$

In the generic case we expect there to be an isomorphism $H_+ \simeq H_-$ relative to which φ is the standard dilation of a to a unitary:

$$\varphi \sim \begin{pmatrix} a & \sqrt{1-aa^*} \\ \sqrt{1-a^*a} & -a^* \end{pmatrix}$$

Thus the graph of $a/\sqrt{1-a^*a}$ corresponds to the graph of ac^{-1} which is essentially the image of $\binom{a}{c}$, i.e. φH_+.
The corresponding involution is $\varphi F \varphi^{-1}$

<u>October 2, 1987</u>

Consider $g: S^1 \longrightarrow U(V)$ and [redacted] an associated ψDO of order zero whose complete symbol is g for $\xi > 0$ and 1 for $\xi < 0$. Recall that it makes sense [redacted] because the Hilbert transform lies in the center of the algebra of ψDO's symbols, i.e. $\psi DO^0 /$ smooth kernel ops. [redacted] Examples of a ψDO with this complete symbol are $P_+ g + P_-$, $g P_+ + P_-$, $P_+ g P_+ + P_-$.

[redacted] Let's now try to understand the K-theoretic significance of this construction. The idea is that the symbol of the operator determines a K-class on the cotangent bundle $T^*(S^1) = S^1 \times \mathbb{R}$. The index information [redacted] depends on this K-class.

We have given an odd K-class on $S^*(S^1) = S^1 \times \{-1, 1\}$, and there is a corresponding even class on $T^*(S^1)$ because of the exact sequence

$$K^1(SE) \longrightarrow K^0(DE, SE) \longrightarrow K^0(DE) \longrightarrow K^0(SE) \longrightarrow$$

It might be clearer in the general case, where one has bundles E, F over M and an isomorphism $\sigma : \pi^* E \xrightarrow{\sim} \pi^* F$ over $S^*(M)$.

[redacted] I'm struck by the similarity between the symbol map above and the Bott map

$$S^1 \times U(V) \longrightarrow Gr(\mathbb{C}^2 \otimes V)$$

which is based upon the graph of tg for half of the circle and the graph of t on the other half. I might hope to link the explicit Bott map of this sort to the analysis.

To be more explicit, the explicit map depends on a partition type function . Once this is chosen, one has a precise map. Similarly once one chooses an F one gets a precise operator. It might be the case that the explicit Bott map is related to the actual operator.

The first difficulty involves linking the operator, which I would like to view as a point in a restricted Grassmannian, to the Bott map. For example how is one supposed to relate a map

$$\begin{array}{ccccc} S^1 & \times & S^1 & \longrightarrow & Gr(\mathbb{C}^2 \otimes V) \\ \| & & \| & & \\ \{\xi\} \cup \infty & & \{x \in \mathbb{R}/\mathbb{Z}\} & & \end{array}$$

to an operator on $L^2(S^1, V)$, or maybe a subspace thereof.

At this point it is clear that there should be much gained by understanding the different versions of the canonical map

$$S \times U \longrightarrow BU.$$

We have two maps

$$S^1 \times U(V) \longrightarrow Gr(\mathbb{C}^2 \otimes V);$$

the first is the Bott map which on $[0, \infty]$ uses the graph of tg and on $[-\infty, 0]$

uses the graph of t. The second is the map given ~~as~~ the Cayley transform used by

$$(\xi, X) \longmapsto \text{graph}(\xi + X).$$

These maps are continuous and they should be homotopic.

Our goal now should be ~~too~~ understand these maps as well as possible. In particular it would be nice to deform one to the other.

First recall that a map to a Grassmannian is the same as a vector bundle with an embedding into a trivial vector bundle. The vector bundle over $S^1 \times U(V)$ should be the canonical vector bundle with partial connection in the S^1 direction. It is obtained by ~~starting~~ with the trivial bundle $\tilde{V}$ over $I \times U(V)$, and using the canonical autom of $\tilde{V}$ over $U(V)$ to glue the ends of the interval.

Let E be this canonical vector bundle. A section of E is an $f(x,g) : \mathbb{R} \times U(V) \longrightarrow V$ such that $f(x+1, g) = g^{-1} f(x,g)$. There's an obvious connection ∂_x in the circle direction. It's possible to trivialize the bundle if we leave out a point of the circle.

I guess the next problem is to begin with the map $(\xi, X) \longmapsto \text{graph}(\xi + X)$ and to identify the inverse image of the canonical subbundle.

Clearly

$$\mathrm{graph}\left(\xi + \frac{g-1}{g+1}\right) = \mathrm{Im}\begin{pmatrix} g+1 \\ \xi(g+1)+(g-1) \end{pmatrix}$$

and we know $\xi(g+1)+(g-1)$ is injective for $\xi \neq 0$ (in fact $\mathrm{Re}(\xi) \neq 0$), so this shows smoothness of $(\xi, g) \longmapsto \mathrm{graph}\left(\xi + \frac{g-1}{g+1}\right)$ for $\xi \neq \infty$.

But I want to compute the monodromy relative to the Grassmannian connection for this loop as ξ goes from $-\infty$ to $+\infty$. Use eigenspaces of g to reduce to the case $\dim V = 1$. In this case we have a loop in $\mathbb{P}^1$ and the monodromy of the subbundle should be related to the area of the loop.

To compute, recall the monodromy is $e^{\int A}$ where A is the connection form, and by Stokes

$$\int A = \iint dA$$

where the curvature dA is a constant times the volume form. We have

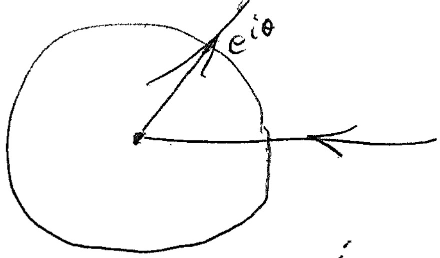
$$\iint_{S^2} \frac{i}{2\pi} dA = \pm 1 \qquad \iint_{S^2} \mathrm{vol} = 4\pi$$

$$\therefore \quad dA = \pm \frac{i}{2} \mathrm{vol}$$

As a check note that if we take a great circle, i.e. $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$, we know the monodromy is -1, and

$$\exp\left\{ \pm \iint_{\text{hemisphere}} \frac{i}{2} \mathrm{vol} \right\} = e^{\pm \frac{i}{2} 2\pi} = e^{\pm i\pi} = -1.$$

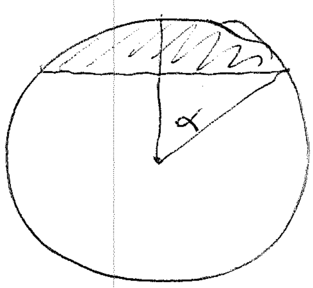
As another check consider the Bott map where we go from 0 to ∞ along $\arg = \theta$ and return by $\arg = 0$.



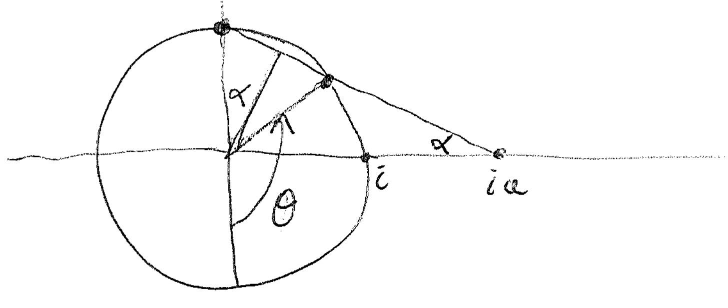
Then monodromy $= e^{\frac{i}{2}\,\text{area}} = e^{\frac{i}{2}\left(\frac{\theta}{2\pi}\right)4\pi} = e^{i\theta}$ which checks.

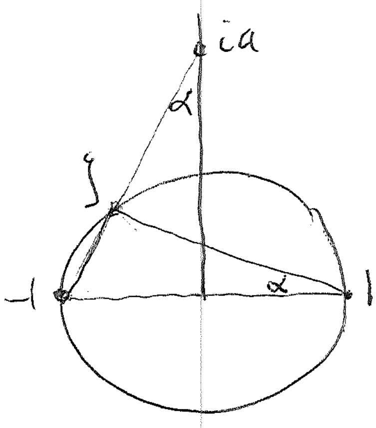
So it appears that in the case of $\xi \mapsto$ graph $\left(\xi + \frac{g-1}{g+1}\right)$, the monodromy for the loop $-\infty \leq \xi \leq \infty$ will be quadratic in $\frac{g-1}{g+1}$. More precisely we need the area of the circle $\xi + ia$.

First the area of a circle of angle α



is $2\pi(1-\cos\alpha)$. Second we use stereographic projection $\zeta = e^{i\theta}$





$$\frac{|\zeta - 1|}{|\zeta + 1|} = \cot\alpha = a \qquad \therefore \quad \frac{\zeta - 1}{\zeta + 1} = ia$$

$$2\alpha = \pi - \theta \qquad \alpha = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\cos\alpha = \sin\left(\frac{\theta}{2}\right).$$

$\therefore$ area of spherical circle is $2\pi\left(1 - \sin\frac{\theta}{2}\right)$

This agrees with

$$\cos\alpha = \frac{a}{\sqrt{1+a^2}}, \qquad \xi = \frac{1+ia}{1-ia} = \left(\frac{1+ia}{\sqrt{1+a^2}}\right)^2$$

$$\Rightarrow \quad \frac{a}{\sqrt{1+a^2}} = \operatorname{Im}(\xi^{1/2}) = \frac{\xi^{1/2}-\xi^{-1/2}}{2i} = \sin\frac{\theta}{2}.$$

Thus the monodromy is

$$e^{i\pi(1-\cos\alpha)} = e^{i\pi\left(1-\sin\frac{\theta}{2}\right)}.$$

This isn't simple enough to be useful.

October 3, 1987

A basic problem seems to be that the existing operator theory produces Fredholm operators to represent K-classes, whereas I would like points in a restricted Grassmannian.

For example one starts with an invertible symbol g on S^* then produces a Fredholm operator T_g. It would be really nice if one could directly relate K-class representatives on T^* to operators so that the following diagram commutes

$$\begin{tikzcd}[column sep=huge, row sep=huge]
g \text{ on } S^* \arrow[r, mapsto] \arrow[d, mapsto, "\substack{\text{boundary}\\ \text{map in}\\ \text{Gysin}\\ \text{sequence}}"'] & T_g \text{ Fred op} \arrow[d, mapsto, "\substack{\text{A.S.}\\ \text{corresp.}}"] \\
\tilde{g} \text{ on } T^* \arrow[r, mapsto, "?"] & \text{pt in restricted Grass.}
\end{tikzcd}$$

Consider the symbol

$$\begin{cases} g(x) & \xi > 0 \\ -1 & \xi < 0 \end{cases}$$

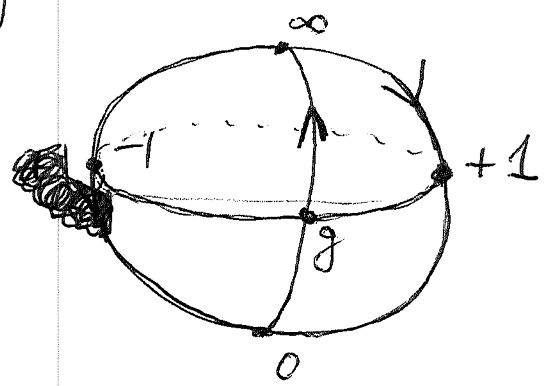
which we can realize via the ψDO $gP_+ - P_-$.
Suppose we want a map

$$(\mathbb{R} \cup \infty) \times (\mathbb{R}/\mathbb{Z}) \longrightarrow Gr(\mathbb{C}^2 \otimes V)$$

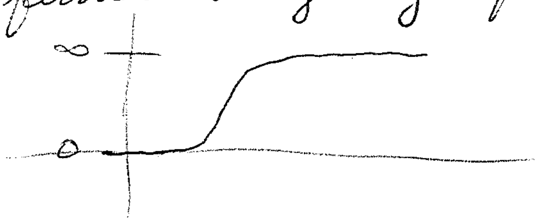
which ought to be related to it. (Here I am thinking of the Bott map $S^1 \times U(V) \to Gr(\mathbb{C}^2 \otimes V)$ as related to the above symbol in some way.)
What we can do is to use

$$\text{graph} \begin{cases} g\xi & \xi > 0 \\ -\xi & \xi < 0 \end{cases}$$

This gives a continuous map with corners

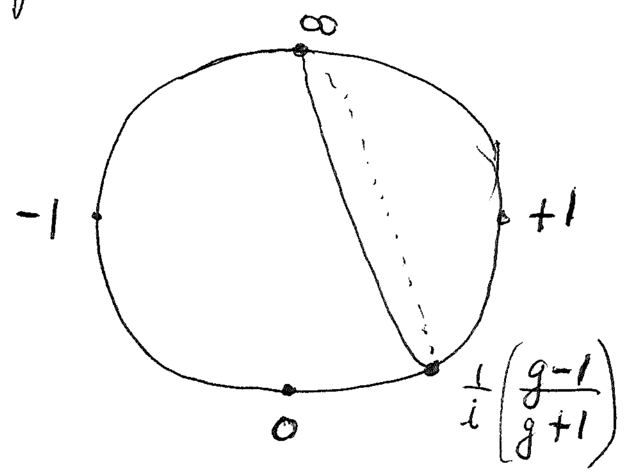


We can smooth the corners by replacing ξ by a function going from 0 to ∞

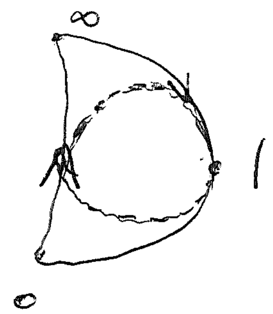


much faster.

On the other hand we have the map $\xi + \dfrac{g-1}{g+1}$ which gives the following picture



There seems to be an obvious homotopy between the two, namely, to fit inside



Now all of this is not going to be useful unless we can find a way to interpret these maps as operators. [redacted] Thus I would really like a way to interpret any of these maps $(\mathbb{R} \cup \infty) \times (\mathbb{R}/\mathbb{Z}) \longrightarrow Gr(\mathbb{C}^2 \otimes V)$ as a point in a restricted Grassmannian. But how does this work?

October 5, 1987

The question is whether a map
$$T^*(S^1) \longrightarrow Gr(\mathbb{C}^2 \otimes V)$$
can be turned into an operator. It seems that Connes tangent groupoid theory says this is possible.

Recall that the convolution algebra on the tangent groupoid of M comes with canonical maps.

$$\begin{pmatrix}\text{fns} \\ \text{on } T^*\end{pmatrix} \xleftarrow{\ h=0\ } \begin{pmatrix}\text{conv. alg.} \\ \text{of tgt} \\ \text{groupoid}\end{pmatrix} \xrightarrow{\ (h\neq 0)\ } \begin{pmatrix}\text{alg of} \\ \text{smooth} \\ \text{kernels}\end{pmatrix}$$

I believe Connes can show the first map induces an isomorphism on K-theory, whence one gets an index map
$$K(T^*) \longrightarrow K\begin{pmatrix}\text{smooth} \\ \text{kernels}\end{pmatrix} = \mathbb{Z}.$$

Thus it should be possible to take a K-class on T^* and lift it to the tangent groupoid. Perhaps one can even do this on the level of unitaries reversed by ε. I recall that Cuntz has a general way of describing the K-theory of a C^*-algebra using pairs of idempotents. In fact he uses

free product in C^*algs. without 1.
$$q\mathbb{C} = \operatorname{Ker}\{\mathbb{C} * \mathbb{C} \longrightarrow \mathbb{C}\}$$

Then $\mathbb{C} * \mathbb{C}$ is the free $\mathbb{C}^*$-alg. generated by two idempotents $=$ C^* alg of dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$. I think that
$$KA = [q\mathbb{C}, A \otimes \mathcal{K}]$$