January 29, 1987

Let $X \in \mathrm{End}_{sk}(V)$, $Y \in \mathrm{End}_{sk}(W)$ and consider

$$\gamma^1 \otimes X \otimes 1 + \gamma^2 \otimes 1 \otimes Y \quad \in \mathrm{End}(\mathbb{C}^2 \otimes V \otimes W).$$

This is skew hermitian. Also

$$(\gamma^1 \otimes X \otimes 1 + \gamma^2 \otimes 1 \otimes Y)^2 = 1 \otimes X^2 \otimes 1 + 1 \otimes 1 \otimes Y^2.$$

and it anti-commutes with $\varepsilon \otimes 1 \otimes 1$. A natural question is whether the map $X, Y \to \gamma^1 \otimes X \otimes 1 + \gamma^2 \otimes 1 \otimes Y$ (denote this simply $\gamma^1 X + \gamma^2 Y$) extends via the C.T. to a map

$$U(V) \times U(W) \longrightarrow Gr(\mathbb{C}^2 \otimes V \otimes W)$$

Suppose $V, W = \mathbb{C}$. Then $X = ix$, $Y = iy$ where $x, y \in \mathbb{R}$ and

$$\gamma^1 X + \gamma^2 Y = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} ix + \begin{pmatrix} & -i \\ i & \end{pmatrix} iy$$

$$= i \begin{pmatrix} 0 & x - iy \\ x + iy & 0 \end{pmatrix}$$

and so we have essentially the map

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & = & \mathbb{C} \\ \cap & & \cap \\ P_1(\mathbb{R}) \times P_1(\mathbb{R}) & & \mathbb{P}_1(\mathbb{C}). \end{array} \qquad x, y \longmapsto x + iy$$

Does this extend to a smooth map $S^1 \times S^1 \to S^2$?

We have that $x + iy$ is smooth in x, y for

$|x|, |y| < \infty$. Next if $\left|\frac{1}{x}\right| < \infty$, $|y| < \infty$ we have

$$\frac{1}{x+iy} = \frac{\frac{1}{x}}{1 + i\frac{y}{x}} \quad \text{is smooth}$$

And if $|x| < \infty$, $\left|\frac{1}{y}\right| < \infty$ we see

$$\frac{1}{x+iy} = \frac{\frac{1}{y}}{\frac{x}{y} + i} \quad \text{is smooth}$$

Finally if $\left|\frac{1}{x}\right| < \infty$ and $\left|\frac{1}{y}\right| < \infty$, then

$$\frac{1}{x+iy} = \frac{\frac{1}{x}\frac{1}{y}}{\frac{1}{y} + i\frac{1}{x}} .$$

Question: Is $\dfrac{xy}{x+iy}$ smooth? NO, for [redacted] it [redacted] is homogeneous of degree 1, so if it were smooth, even differentiable at zero, it would have to be linear:

$$\begin{array}{c} \dfrac{f(tx, ty)}{t} \longrightarrow (\partial_x f)(0) \cdot x + (\partial_y f)(0) y \\ \shortparallel \\ f(x, y) \end{array}$$

Question: How can we desingularize this map? The idea is to adopt the correspondence idea. A singular map has a singular graph, but we can always replace this graph by [redacted] a desingularization + obtain a correspondence.

If we blow-up the origin in the x, y plane then we can lift $\dfrac{xy}{x+iy}$ back and see if it extends smooth. Coords for the blowup are $(y, \frac{x}{y})$ for $y \neq 0$ and $(x, \frac{y}{x})$ for $x \neq 0$. (Better: a point of the blowup is a line thru the origin + a point on that line. If the line projects onto the x axis, then the line is $y = \lambda x$ for some $\lambda \in \mathbb{R}$ and a point on this line is given by its x coordinate. Thus $(x, \lambda) \longmapsto (x, \lambda x)$ gives the blowup over the open set of lines $\neq$ y axis.

The function $\frac{xy}{x+iy}$ on this open set is

$$(x, \lambda) \longmapsto \frac{x \lambda x}{x + i \lambda x} = x \frac{\lambda}{1 + i\lambda}$$

which is certainly smooth for $(x,\lambda) \in \mathbb{R}^2$.

Similar over the open set of lines $\neq$ x axis the blowup is parametrized by $(y, \mu) \in \mathbb{R}^2$ where the line is $x = \mu y$ and the point on this line is $(\mu y, y)$. The map to $\mathbb{R}^2$ is $(y, \mu) \longmapsto (\mu y, y)$ and so the function on this open set is

$$\frac{\mu y^2}{\mu y + i y} = \frac{\mu}{\mu + i} y$$

which is smooth.

Now our natural step is to return to the higher dimensional situation. Here we must worry about some eigenvalues approaching ∞ and some staying finite.

Let us first consider the case where one of the spaces [redacted] is one-dimensional, so that we are trying to get a map

$$U(1) \times U(V) \longrightarrow Gr(V \oplus V)$$

by [redacted] extending the map

$$(*) \qquad \mathbb{R} \times \mathrm{End}_{sk}(V) \longrightarrow Gr(V \oplus V)$$
$$(t \;,\; X) \longmapsto \text{graph of } t + X$$

Recall that we studied this map a bit in November 86, see page 265. A formula (p. 269) for the map [redacted] in terms of $g = \frac{1+X}{1-X}$ is

$$(**) \qquad (t,g) \longmapsto \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mathrm{Im}\begin{pmatrix} g+1 \\ g-1 \end{pmatrix}.$$

It would be better to say this gives the extension of $(*)$ above to a map

$$(***) \qquad \mathbb{R} \times U(V) \longrightarrow Gr(V \oplus V)$$

I showed how to deform the Bott map into $(**)$ essentially by writing the generator of $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ as a limit of semi-simple elements.

Now the question is whether [redacted], or really how to go about extending $(***)$ to $t = \infty$. This involves some sort of desingularizing. Note that

$$\text{graph}(t+X) = \mathrm{Im}\begin{pmatrix} g+1 \\ t(g+1) + (g-1) \end{pmatrix} = \mathrm{Im}\underbrace{\begin{pmatrix} (g+1)t^{-1} \\ (g+1) + (g-1)t^{-1} \end{pmatrix}}$$

[redacted]

$$\xrightarrow[\text{as } t \to \infty]{} \begin{pmatrix} 0 \\ g+1 \end{pmatrix}$$

can have rank $< \dim V$.

January 31, 1987

Program: In general terms I am trying to acquire a detailed understanding of K-theory, by which I mean the theory involving Fredholm operators which has been developed by Kasparov. I would like a version in which differential forms play an important role.

From my work on superconnections I was led to think of a K-cohomology class on M as being represented by a map to a suitably restricted unitary group or Grassmannian. Two days ago I discovered that the natural product of such representatives does not preserve smoothness. I am puzzled with what this means.

In any case I have to ultimately understand the Gysin homomorphism in K-theory. This is the integration over the fibre map. The simplest case is integration over the circle or line.

So ultimately I have to take an odd K-class on S^1 and integrate it over the circle.

So suppose the K-class represented by a map $g: S^1 \longrightarrow U(V)$. Set $M = S^1$. According to KK theory the fundamental class of the circle is represented certain operators on Hilbert spaces which are modules over functions on S^1. In the C^*-viewpoint these are ψDO's of order 0. We know that a metric on S^1 gives rise to a well-defined ungraded involution in $\Psi^0/\Psi^{-1} =$ functions on the cosphere.

The following works very smoothly. Take $H = L^2(S^1)$, let F be the Hilbert involution. The by letting a loop g in $U(V)$ act as a unitary on $H \otimes V$ we can move $F \otimes 1$ around. Thus essentially using F as a basepoint of the Grassmannian we can move it around by g. More generally if instead of F you were to take a self-adjoint contraction A which agrees with F modulo compacts, then A can be dilated to an involution and one can proceed as above.

What this means is that somehow without changing g we can proceed provided we take the limit where the Dirac operator has been rendered an involution. Notice that any of these constructions with the loop g do not single out $g = -1$ as being a special point.

It seems that I want to emphasize the fact that the flow g_t on the unitary group is very important. And similarly it seems that I want to use the scaling on the Dirac operator side: $h\gamma^\mu \partial_\mu$. So you should go back to the operator $\gamma^\mu \partial_\mu + \sigma X$ and put in the two possible scales.

Notice that $\begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$ is the modification of the de Rham operator $d + d^*$ discussed by Hörmander years ago and Witten in connection with Morse theory. Hörmander's inequalities should therefore handle much of the analysis I need.

February 1, 1987

Integration by parts: Consider $\begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$ with f real. Then

$$\|(\partial_x+f)u\|^2 = \langle u, (-\partial_x+f)(\partial_x+f)u\rangle$$
$$= \langle u, (-\partial_x^2 - f' + f^2)u\rangle$$
$$= \|\partial_x u\|^2 + \langle u, (-f'+f^2)u\rangle$$

Replacing f by tf alters $-f'+f^2$ to $-tf'+t^2f^2$ which becomes more positive as $t\to\infty$. Recall Hörmander's device of replacing f by $\chi(f)$ so as to make this term $\geq$ constant.

What is the point of these inequalities? What have they to do with the problem of showing that $L = \begin{pmatrix} 0 & \partial_x - f \\ \partial_x + f & 0 \end{pmatrix}$ is essentially skew-adjoint? For example take $f = x$ on $\mathbb{R}$. This example brings up an interesting point about the Schatten class. Thus not only do I want to know that L is skew-adjoint, I also want to know that its Cayley transform $\frac{1+L}{1-L}$ is congruent to -1 modulo some Schatten ideal.

Now what is involved in establishing the properties of L?

February 3, 1987

Goddard - Olive: Kac-Moody and Virasoro algebras in relation to quantum physics.

I would like to put this paper into some sort of logical order. I would like suitable principles to organize the theory. One place to start is with symmetry. One has a symmetry type, i.e. a group and one wants to study a theory with this symmetry. Examples: gauge ~~[illegible]~~ invariance, Lorentz-invariance, conformal invariance. The method is to look at the unitary representations of the group. The different irreducible representations form ~~[illegible]~~ building blocks (atoms) for the theories.

Not all unitary representations occur - there is ~~[illegible]~~ a positivity condition. ~~[illegible]~~ Also, as with rotational symmetry, one is interested in projective representations, i.e. representations of a central extension of the symmetry group.

Ch. I of this paper treats the following.

1.1. Background - the commutation relations of the T_n^a and L_n are given.

1.2. The loop group + diff gp are described

1.3. Central extensions of $\hat{g}$ and Virasoro are classified

1.4. Highest weight repn. For $\hat{g}$ they are determined by the vacuum rep of g and the central scalar k. For $Vect(S^1)$ they are determined by the eigenvalue h of L_0 on the vacuum and the central scalar c. Description of the possible (c, h).

I guess the important point so far is that we have specified the symmetry groups and we want to look at "theories" with these kinds of symmetry subject to the positivity condition.

Question: Why do they consider real fermions? It must have something to do with the Clifford algebra depending on the underlying real vector space.

They consider a "single real fermion field" ψ. This has two components ψ_-, ψ_+ which are real, and which have the equation of motion

$$(\partial_t + \partial_x)\psi^- = 0 \qquad (\partial_t - \partial_x)\psi^+ = 0$$

and which when quantized will becomes hermitian operators such that

$$\{\psi^+(x), \psi^+(y)\} = \{\psi^-(x), \psi^-(y)\} = \delta(x-y)$$

$$\{\psi^+(x), \psi^-(y)\} = 0.$$

Let's adopt the viewpoint that physicists want in 2 diml space time to see particles moving in both directions. Even a boson theory where the action involves $\partial_t^2 - \partial_x^2$ is to have both types of particles.

If this is so, then the above is just the direct sum of the fields $\psi^\pm$. From my viewpoint one looks at a single component, say ψ^-. The commutation relations give the Clifford algebra based on the real Hilbert space of real L^2 functions.

I wanted to consider the Fock space of $L^2(S^1, \mathbb{C})$. This gives an irreducible representation of the Clifford algebra based on the underlying real Hilbert space of $L^2(S^1, \mathbb{C})$. So it's twice as big as $L^2(S^1, \mathbb{R})$.

February 5, 1987

$L^2(S^1)$ has o.basis $\langle x|k\rangle = \frac{1}{\sqrt{L}} e^{ikx}$; let $\mathcal{F}$ be the fermion Fock space and let

$$\psi(x) = \sum' \langle x|k\rangle \psi_k = \frac{1}{\sqrt{L}} \sum e^{ikx} \psi_k$$

$$\psi^*(y) = \frac{1}{\sqrt{L}} \sum' e^{-ily} \psi_l^* .$$

Where are these operators defined? At best they are operator-valued distributions, ~~the Hilbert~~ ~~However it seems possible to consider~~ [redacted] i.e. distributions on S^1 with values in operators on the Hilbert space $\mathcal{F}$.

However we can consider their matrix elements, e.g. $\langle e_S, \psi(x) e_{S'}\rangle = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \langle e_S, \psi_k e_{S'}\rangle$

This is a finite sum, since

$$\langle e_S, \psi_k e_{S'}\rangle \neq 0 \implies k \in S' \text{ and } k \notin S$$

$$\Updownarrow$$

$$k \in S' - S \quad \text{which is finite.}$$

So [redacted] as [redacted] maps $\underset{\bigoplus \mathbb{C} e_S}{\overset{\vee}{\mathcal{F}}} \longrightarrow \underset{\prod \mathbb{C} e_S}{\hat{\mathcal{F}}}$, $\psi(x), \psi^*(y)$ are entire functions.

Next let us consider

$$\psi^*(x)\psi(y) = \frac{1}{L} \sum_{k,l} e^{-ikx + iky} \psi_l^* \psi_k$$

from the same viewpoint. Look at

$$\langle e_S | \psi_l^* \psi_k | e_{S'}\rangle$$

This is zero for $k \in (-S')$, or [redacted] $l \in (-S)$. And if $l \neq k$, then it is $-\langle e_S | \psi_k \psi_l^* | e_{S'}\rangle$ which is zero for $l \in S'$ or $k \in S$.

Thus when $l \neq k$ the region where it can be $\neq 0$ is

$$(k,l) \in (S' \cap (-S)) \times (S \cap (-S'))$$

which is finite.

Next let's look at the diagonal part where $k = l$.

$$\langle e_S | \psi_k^* \psi_k | e_{S'} \rangle \neq 0 \implies k \in S' \cap S$$

Thus $\langle e_S | \psi^*(x) \psi(y) | e_{S'} \rangle$ will be a finite series

$$\frac{1}{L} \sum_{k \neq l} e^{-ilx} e^{iky} \langle e_S | \psi_l^* \psi_k | e_{S'} \rangle$$

plus the diagonal term

$$* \qquad \frac{1}{L} \sum_{k} e^{-ik(x-y)} \langle e_S | \psi_k^* \psi_k | e_{S'} \rangle$$

which will be convergent when the exponential $e^{-ik(x-y)}$ decays as $k \to -\infty$. This means when $\operatorname{Im}(x-y) > 0$.

Now
$$\psi_k^* \psi_k e_S = \begin{cases} e_S & k \in S \\ 0 & k \notin S \end{cases}$$

The diagonal term $*$ is zero unless $S = S'$ in which case it is a geometric series.

So what I would like to say is that $\psi^*(x)\psi(y)$ makes sense as an operator $\check{\mathcal{F}} \to \hat{\mathcal{F}}$ for $\operatorname{Im}(x) > \operatorname{Im}(y)$ and that it has a certain singularity. To specify the singularity we introduce normal ordering with respect to a "ground" state e_{S_0} say the basis vector satisfying

$$\psi_k e_{S_0} = 0 \qquad k > 0$$
$$\psi_k^* e_{S_0} = 0 \qquad k \leq 0$$

In other words $S_0 = \{0, -1, -2, -3, \dots\}$. A Normal ordered ~~means that any~~ monomial in the $\psi_k \psi_k^*$ has all the destruction operators to the right of the creation operators.

destruction		creation	
ψ_k	$k > 0$	ψ_k^*	$k > 0$
ψ_k^*	$k \leq 0$	ψ_k	$k \leq 0$.

There is a normal ordering operation which takes any monomial and rearranges the factors putting in the appropriate sign so that it is normal ordered. For example

$$n(\psi_l^* \psi_k) = \begin{cases} \psi_l^* \psi_k & \text{if } k > 0 \text{ or if } l > 0 \\ -\psi_l \psi_k^* & \text{if } k, l \leq 0 \end{cases}$$

Note that this operation only changes the diagonal bilinears and

$$n(\psi_k^* \psi_k) = \begin{cases} \psi_k^* \psi_k & k > 0 \\ -\psi_k \psi_k^* = \psi_k^* \psi_k - 1 & k \leq 0 \end{cases}$$

$$\therefore \quad \psi_k^* \psi_k - n(\psi_k^* \psi_k) = \begin{cases} 0 & k > 0 \\ 1 & k \leq 0 \end{cases}$$

$$\psi^*(x)\psi(y) = \mathcal{N}(\psi^*(x)\psi(y)) + \underbrace{\frac{1}{L}\sum_{k\leq 0} e^{-ik(x-y)}}$$

$$= \frac{1}{L}\,\frac{1}{1-e^{+i(x-y)}}$$

Next I should check that given any ~~normal ordered monomial in the ψ_k, ψ_k^* has~~ S, S' there are only finitely many normally ordered monomials in the ψ_k, ψ_k^* which have a non-zero matrix coefficient of type S, S'. It's enough to check this in the standard exterior algebra case using the isom.

$$\mathcal{F} = \Lambda H^+ \otimes \Lambda(\overline{H^-})$$

Then it's clear because S, S' become finite and a monomial $\mu = a_I^* a_J$ will have nonzero matrix coefficient

$$\langle e_S, a_I^* a_J e_{S'} \rangle = \langle a_I e_S, a_J e_{S'} \rangle$$

only when $I \subset S$, $J \subset S'$, so there are only finitely many possibilities.

Now I want to shift to z notation. The general principle seems to be that $\psi^*(z)\psi(\zeta)$ should be defined for $|z| > |\zeta|$. This corresponds to usual ordering in time.

So we want $z = e^{-ix}$ and

$$\psi^*(z) = \sum_k z^k \psi_k^* \qquad \psi(\zeta) = \sum_\ell \zeta^{-\ell}\psi_\ell$$

Check the argument

$$\psi^*(z)\psi(\zeta) = \sum_{k\neq\ell} z^k \zeta^{-\ell}\psi_k^*\psi_\ell$$

$$+ \sum_{k=\ell} \left(\frac{\zeta}{z}\right)^{-k} \psi_k^*\psi_k \qquad \psi_k^*\psi_k = \begin{cases} \mathcal{N}(\psi_k^*\psi_k) & k>0 \\ \mathcal{N}(\psi_k^*\psi_k)+1 & k\leq 0 \end{cases}$$

$$= \mathcal{N}\big(\psi^*(z)\psi(\zeta)\big) + \underbrace{\sum_{k\leq 0} (\zeta/z)^{-k}}_{\dfrac{1}{1-\zeta/z}}$$

Thus we have the operator product expansion

$$\psi^*(z)\psi(\zeta) = \frac{z}{z-\zeta} + \underbrace{\mathcal{N}\big(\psi^*(z)\psi(\zeta)\big)}_{\text{analytic}}$$

Notice that the left side is defined initially only for $|z|>|\zeta|$, but then this formula shows it is defined by analytic continuation for all $z\neq\zeta$ on the cylinder $\mathbb{C}^*$.

Next let's go on to

$$J(z) = \mathcal{N}\big(\psi^*(z)\psi(z)\big)$$

$$= \sum_{k,\ell} z^{k-\ell} :\psi_k^*\psi_\ell: = \sum_q z^q \underbrace{\Big(\sum_\ell :\psi_{\ell+q}^*\psi_\ell:\Big)}_{J_q}$$

Let's check that J_q operates on $\check{\mathcal{F}}$ (direct sum Fock space). For $q \neq 0$

$$\psi^*_{\ell+q}\psi_\ell \, e_S \neq 0 \quad \Rightarrow \quad \begin{array}{l} \ell \in S \\ \ell+q \in \mathbb{Z}-S \end{array}$$

$$\Rightarrow \ell \in S \cap (-q)+(\mathbb{Z}-S)$$

Thus $J_q e_S$ is a finite sum. Similarly for $q=0$, because $J_0 e_S = \text{charge}(S) \cdot e_S$.

Note that $J_q \, e_{S_0} = 0$ for $q<0$.

In fact we have

$$J_q \, e_{S_n} = 0 \qquad \text{where} \qquad S_n = \{n, n-1, \dots\}$$

for $q<0$.

Further facts are the commutation relations

$$[J_p, J_q] = -p\,\delta_{p,q}$$

I should go over the proof, and see if I can understand it in terms of the operator product expansion for $J(z)J(\zeta)$.

$p-q \neq 0$.

$$J_p J_q \, e_S = \sum_\ell \psi^*_{\ell+p}\psi_\ell \sum_k \psi^*_{k+q}\psi_k \, e_S$$

$$[J_p, J_q]\, e_S = \sum_{|\ell| \leq N} \sum_{|k| \leq N} \underbrace{[\psi^*_{\ell+p}\psi_\ell, \psi^*_{k+q}\psi_k]}\, e_S$$

$$[\psi^*_{\ell+p}\psi_\ell, \psi^*_{k+q}]\psi_k + \psi^*_{k+q}[\psi^*_{\ell+p}\psi_\ell, \psi_k]$$

$$\psi^*_{\ell+p}\,\delta_{\ell,k+q}\,\psi_k - \psi^*_{k+q}\,\delta_{\ell+p,k}\,\psi_\ell$$

$\therefore$ $[J_p, J_q] e_S$

$$= \Big\{ \sum_{\substack{|k|,|l| \le N \\ l = k+q}} \psi^*_{\substack{l+p \\ k+q}} \psi_k - \sum_{\substack{|k|,|l| \le N \\ l+p = k}} \psi^*_{\substack{k+q \\ l+p}} \psi_l \Big\} e_S$$

where we know this doesn't change if N is large enough. If $p \neq -q$, we can let $N \to \infty$ in both series. Actually we write the above as

$$= \Big\{ \sum_{|k|,|k+q| \le N} \psi^*_{k+q+p} \psi_k - \sum_{|k|,|k+p| \le N} \psi^*_{k+p+q} \psi_k \Big\} e_S$$

If $p+q \neq 0$, then we can let $N \to \infty$ in each sum and we get $(J_{q+p} - J_{p+q}) e_S = 0$. But if $p+q = 0$, then we get (say $p > 0$

$$\Big(\sum_{-N+p \le k \le N} - \sum_{-N \le k \le N-p} \Big) \psi^*_k \psi_k \; e_S = -p e_S$$

For J_0 one has $J_0 e_S = \text{charge}(S) e_S$ and J_p preserves this so J_0 commutes with all the J_p.

Shift: Operator which shifts S up one. so that $\tau e_S = e_{\tau(S)}$ $\quad \tau(S) = 1 + S \subset \mathbb{Z}$.

What effect does τ have on the generators ψ_k, ψ_k^*?

$$\tau |k\rangle = |k+1\rangle \qquad \tau \psi^*_k \tau^{-1} = \psi^*_{k+1}$$

$$\tau \psi_k \tau^{-1} = \psi_{k+1}$$

$$\therefore \quad \tau J_q \tau^{-1} = J_q \quad q \neq 0 \qquad \tau J_0 \tau^{-1} = J_0 - 1$$

February 7, 1987

Consider the transmission line equations

(1) $$\begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \partial_t \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

To begin suppose $C(x), L(x)$ are smooth >0. The signal speed in general is $\frac{1}{\sqrt{LC}}$ and we can suppose by reparametrizing that it is 1: $C = L^{-1}$.

Recall that the equation can be written in "Dirac" form

(2) $$\partial_t \begin{pmatrix} L^{-1/2} V \\ L^{1/2} I \end{pmatrix} = \begin{pmatrix} 0 & L^{1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x L^{1/2} & 0 \end{pmatrix} \begin{pmatrix} L^{-1/2} V \\ L^{1/2} I \end{pmatrix}$$

The energy associated to (1) is

$$\frac{1}{2}\int (CV^2 + LI^2)\,dx = \frac{1}{2}\int \left((L^{-1/2}V)^2 + (L^{1/2}I)^2 \right) dx$$

We can also write (1) in "string" form, where "string" is used in the sense of the Dym-McKean book

$$\begin{pmatrix} L^{-1}C & 0 \\ 0 & 1 \end{pmatrix} \partial_t \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & \partial_y \\ \partial_y & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

where $\partial_y = L^{-1}\partial_x$ or $dy = L\,dx$. Then

$$\boxed{\begin{aligned} &\text{string length} = \int dy = \int L\,dx = \text{total inductance} \\ &\text{string mass} = \int L^{-1}C\,dy = \int C\,dx = \text{total capacitance} \end{aligned}}$$

Let us now work over an interval $[0, R)$

with the boundary condition at 0 considered by DM, namely

$$I(0) = 0.$$

If I understand their work correctly, then the operator

$$\begin{pmatrix} 0 & L^{+1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x L^{1/2} & 0 \end{pmatrix}$$

with this boundary condition is essentially skew-adjoint iff

$$\int_0^R C\,dx + \int_0^R L\,dx = \infty$$

Otherwise it is necessary to specify a boundary condition at the end R. It appears that this boundary condition is just to put a resistance k, with $0 \leq k \leq \infty$ at the end of the line. (This may not be a completely correct translation of the boundary condition at the right endpoint used by DM. But we aren't interested in this case.)

Another result is that the Green's function or resolvent for the skew-adjoint operator is Hilbert-Schmidt iff either of

$$\int_0^R y\,dm = \int_0^R \left(\int_0^x L\right) C\,dx$$

$$\int_0^R m\,dy = \int_0^R \left(\int_0^x C\right) L\,dx$$

is finite. Note that the sum is

$$\int_0^R \left(\int_0^x L\right) d\left(\int_0^x C\right) + \left(d\int_0^x L\right)\cdot \int_0^x C$$
$$= \left(\int_0^R L\right)\left(\int_0^R C\right) = \infty .$$

Example: Let's consider the operator $\begin{pmatrix} 0 & \partial_x - x \\ \partial_x + x & 0 \end{pmatrix}$ which we know is related to the harmonic oscillator. Work on the interval $(-\infty, \infty)$; we can handle $(0,\infty)$ by reflection.

$$L^{-1/2}\,\partial_x(L^{1/2}) = x \qquad L = e^{x^2}, \; C = e^{-x^2}$$

Thus the string length $\int e^{x^2}dx = \infty$, and the string mass $\int e^{-x^2}dx$ is finite. The operator is essentially skew adjoint consistent with $\int L + \int C = \infty$.

Next look at the integrals

$$\int \left(\int_0^x e^{-x^2}\right) e^{x^2}dx \quad , \quad \int \left(\left(\int_0^x e^{x^2}\right) e^{-x^2} dx\right)$$

The former is obviously infinite since $\int_0^x e^{-x^2} \to \text{const} \neq 0$ as $x \to \infty$. As for the latter we have for $x \geq \delta > 0$

$$\int_\delta^x e^{y^2}dy = \int_\delta^x \left(e^{y^2}2y\right)\frac{1}{2y}\,dy$$
$$= \left[e^{y^2}\frac{1}{2y}\right]_\delta^x - \int_\delta^x e^{y^2}\left(\frac{1}{2}\,\frac{-1}{y^2}\right)dy$$
$$\geq e^{x^2}\frac{1}{2x} - \text{const}.$$

Thus the second integral is divergent, and probably logarithmically divergent. This agrees with the fact the eigenvalues are $\pm i\sqrt{n}$, $n \geq 0$ and 0. The resolvent just fails to be Hilbert-Schmidt

<u>Problem</u>: Find proofs of these assertions about essential skew-adjointness and the resolvent being Hilbert-Schmidt.

The following problem might facilitate the understanding of singular Dirac operators. Let us consider two measures on the line which I will denote dm and dl with

$$dm = C\,dx \quad , \quad dl = L\,dx$$

in the smooth cases. I want to consider the transmission line equation (1) in the form

$$(3) \qquad \begin{pmatrix} (C\,dx) & 0 \\ 0 & (L\,dx) \end{pmatrix} \begin{pmatrix} \dot{V} \\ \dot{I} \end{pmatrix} = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix}$$

and try to make sense out of it for a general pair of measures dl, dm.

We can consider the energy and use it to define a Hilbert space with norm

$$\left\| \begin{pmatrix} V \\ I \end{pmatrix} \right\|^2 = \int V^2 dm + I^2 dl .$$

The Hilbert space is thus $L^2(\mathbb{R}, dm) \oplus L^2(\mathbb{R}, dl)$; these are real L^2 spaces.

The question is whether in the general case where dl, dm are singular, does (3) give rise to a one-parameter orthogonal group on the Hilbert space.

Now if this is true, then this one parameter group has an infinitesimal generator which is a skew-adjoint operator X on the Hilbert space. Formally

$$\partial_t \psi = X\psi \qquad \psi = \begin{pmatrix} V \\ I \end{pmatrix}$$

so

$$X = \begin{pmatrix} 0 & \frac{d}{Cdx} \\ \frac{d}{Ldx} & 0 \end{pmatrix}.$$

Thus we ask how to interpret

$$\frac{d}{Cdx} : L^2(\mathbb{R}, Idx) \longrightarrow L^2(\mathbb{R}, Cdx).$$

where Cdx, Ldx become dm, dl. $\blacksquare$

Suppose we write the eigenvalue equation in general

$$(4) \qquad \begin{aligned} dI &= \lambda V\,(dm) \\ dV &= \lambda I \cdot (dl) \end{aligned}$$

I guess it's clear from what we have read in DM + de Branges's books that there is a definite way to interpret these equations and to grind out solutions by Volterra iteration. In the case where dm, dl are discrete measures which are disjoint, then V has to be extended from a function on the support of dm to a step function.

Note that (4) imply V is constant outside $\mathrm{Supp}(dl)$ and I is constant outside $\mathrm{Supp}(dm)$.

Conclusion: We have just found again that the singular string or transmission line is not the singular Dirac I am interested in. The singularities I want to put in will make it impossible to propagate signals through them.

I think I have to aim directly for the resolvent which should make sense even though the operator X won't. A persistent problem is what is the appropriate space for the resolvent to act.

We will now ~~prove~~ discuss the assertions about essential skew-adjointness and the resolvent being Hilbert-Schmidt for the operator

$$X = \begin{pmatrix} 0 & L^{+1/2}\partial_x L^{-1/2} \\ L^{-1/2}\partial_x L^{1/2} & 0 \end{pmatrix}$$

on an interval $[0, R)$. The point will be that we can easily describe the inverse of this operator, that is, the resolvent at $\lambda = 0$.

The two linearly independent solutions of $X\psi = 0$ are

$$\begin{pmatrix} 0 \\ L^{+1/2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} L^{-1/2} \\ 0 \end{pmatrix}$$

We have the limit circle behavior at the right endpoint R when both of these are in L^2, i.e. when

$$\int_0^R L^{-1}\,dx \quad \text{and} \quad \int_0^R L\,dx < \infty.$$

Thus we have essential skew-adjointness by Weyl's analysis when one of these integrals is infinite as claimed. ~~I think Weyl's analysis shows that there is unique linear combination of~~ $\cancel{\begin{pmatrix} 0 \\ L^{1/2} \end{pmatrix}}$ ~~and~~ $\cancel{\begin{pmatrix} L^{1/2} \\ 0 \end{pmatrix}}$ ~~up to a scalar factor which is in~~ $\cancel{L^2}$. ~~To simplify let~~

I want to now place myself in a situation where I can determine the Green's fn. for X. Suppose then $p = \dfrac{a}{R-x}$ near R so that

$$L^{1/2} = c\, e^{-a \log(R-x)} = c/(R-x)^a$$

and $\displaystyle\int_0^R L\,dx = \infty$ for $a > \frac{1}{2}$, whereas $\displaystyle\int_0^R L^{-1}\,dx < \infty$. Thus there is an L^2 solution of $X\psi = 0$, namely $\begin{pmatrix} L^{-1/2} \\ 0 \end{pmatrix}$.

I combine this the solution $\begin{pmatrix} 0 \\ L^{1/2} \end{pmatrix}$ having the boundary condition $\mathbb{C}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ at $x = 0$. Now I can calculate the Green's function

$$G(x,x') = \begin{pmatrix} 0 & 0 \\ -L^{1/2}(x)\,L^{-1/2}(x') & 0 \end{pmatrix} \qquad x < x'$$

$$= \begin{pmatrix} 0 & L^{-1/2}(x)\,L^{1/2}(x') \\ 0 & 0 \end{pmatrix} \qquad x > x'$$

This obviously is killed by X for $x \neq x'$ and it jumps by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as x passes x'. so $X\, G(x,x') = \delta(x-x')$. Compute the Hilbert-Schmidt norm of G.

$$-\mathrm{tr}(G^2) = -\iint \mathrm{tr}\big(G(x,y)\, G(y,x)\big)\, dx\, dy .$$

Observe G is skew-symmetric so this is

$$2 \iint_{x<y} \mathrm{tr}\begin{pmatrix} 0 & 0 \\ -L^{1/2}(x) L^{-1/2}(y) & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -L^{1/2}(x) L^{-1/2}(y) & 0 \end{pmatrix}^t dx\, dy$$

$$= 2 \iint_{x<y} L(x)\, L^{-1}(y)\, dx\, dy$$

$$= 2 \int_0^R \left(\int_0^y L(x)\, dx \right) L^{-1}(y)\, dy$$

Thus G is Hilbert-Schmidt iff

$$\int \left(\int^x L \right) L^{-1}\, dx < \infty , \qquad \text{as on page 403.}$$

The above treats the case where $\lambda = 0$ is not in the spectrum. The general case is handled by treating the resolvent $(\lambda - X)^{-1}$ with $\lambda > 0$. This time one doesn't have a formula for the solutions of $X\psi = \lambda\psi$ (needed to construct the Green's function) in terms of L, and one must use estimates.

February 8, 1987

Remark that the correspondence between transmission lines and Diracs is not 1-1. The correspondence sends a transmission line equation

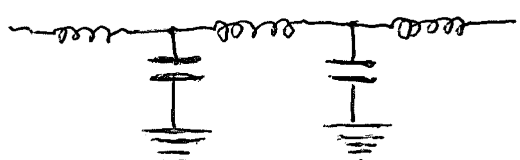
$$\partial_t \begin{pmatrix} C & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} \qquad CL = 1$$

to the hyperbolic Dirac equation

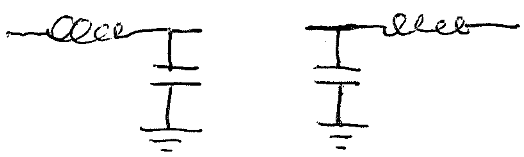
$$\partial_t \psi = \begin{pmatrix} 0 & L^{1/2} \partial_x L^{-1/2} \\ L^{-1/2} \partial_x L^{1/2} & 0 \end{pmatrix} \psi .$$

Hence two L's differing by a positive constant scalar factor give rise to the same Dirac.

The problem now I want to look at is to reformulate the transmission line equations in terms of scattering data. I think this will enable me to handle singularities. For example consider a discrete line



If one of the inductances is ∞ then it's as if there were a break in the line



So waves get reflected perfectly. It's impossible to describe this situation by a propagation equation, but the electrical system has modes of vibration, standing waves, etc.

Let's consider the Dirac equation

$$\lambda\psi = \begin{pmatrix} 0 & \partial_x - p \\ \partial_x + p & 0 \end{pmatrix}\psi = \left(\gamma^1\partial_x + \gamma^2 \tfrac{1}{i} p\right)\psi$$

and rewrite it in the form

$$\gamma^1 \partial_x \psi = (\lambda + \gamma^2 i p)\psi$$

$$\partial_x \psi = (\lambda\gamma^1 + \varepsilon p)\psi$$

By conjugation we can move $\gamma^1, \varepsilon \mapsto \varepsilon, \gamma^1$
This leads to the equation

$$* \qquad \boxed{\partial_x \tilde{\psi} = (\lambda\varepsilon + \gamma^1 p)\tilde{\psi} = \begin{pmatrix} \lambda & p \\ p & -\lambda \end{pmatrix}\tilde{\psi}}$$

Let's work out the conjugation. Put $\tilde{\psi} = u\psi$ whence $\psi = u^{-1}\tilde{\psi}$ and we want u chosen so that

$$u\gamma^1 u^{-1} = \varepsilon \qquad u\varepsilon u^{-1} = \gamma^1$$

The first equation says that the matrix u^{-1} is a matrix of eigenvectors for γ^1, and the second will follow if u^2 is scalar. So try

$$u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & +1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\varepsilon + \gamma^1)$$

Then $\quad u^2 = 1 \quad$ and $\quad u\gamma^1 = \frac{1}{\sqrt{2}}(\varepsilon\gamma^1 + 1)$

$\varepsilon u = \frac{1}{\sqrt{2}}(1 + \varepsilon\gamma^1)$, so it works.
Thus

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \implies \tilde{\psi} = \frac{1}{\sqrt{2}}\begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 - \psi_2 \end{pmatrix}$$

Let's now consider the solutions of

$$\partial_x \tilde{\psi} = (\varepsilon\lambda + \gamma' p)\tilde{\psi}$$

and in particular the propagator $\Phi_\lambda(x, x')$ from values of solutions at position x' to values at x. For $\lambda \in i\mathbb{R}$ the matrix $\varepsilon\lambda + \gamma' p$ is in the Lie algebra of $SU(1,1)$, and for $\lambda \in \mathbb{R}$ it is in the Lie algebra of $SL(2,\mathbb{R})$; for general λ it is in $sl(2,\mathbb{C})$.

What I want to do next is to convert $\Phi_\lambda(x,x')$ to a scattering matrix. What I would like is to define a system of compatible scattering matrices $S_\lambda(x,x')$ which would be more general than the Dirac equation.

Let's discuss the scattering. I want the scattering matrix associated to the propagation over the interval $[b,c]$. This means I want solutions of the equation where p is set $=0$ outside this interval. To keep things simple suppose that the interval $[b,c]$ is very small about $x=0$; say we have a singularity delta function for p.

Typically we work with λ such that $Re(\lambda) > 0$. Then incoming solutions decay as we move away from the obstacle and outgoing solutions grow exponentially. Thus we have

$$\psi_{left,in} = \underset{x \ll 0}{\begin{pmatrix} e^{\lambda x} \\ 0 \end{pmatrix}} \longleftrightarrow \underset{x \gg 0}{\begin{pmatrix} A(\lambda)e^{\lambda x} \\ B(\lambda)e^{-\lambda x} \end{pmatrix}} = A(\lambda)\underset{\psi_{right,out}}{\begin{pmatrix} e^{\lambda x} \\ 0 \end{pmatrix}} + B(\lambda)\underset{\psi_{right,in}}{\begin{pmatrix} 0 \\ e^{-\lambda x} \end{pmatrix}}$$

$$\psi_{r,out} \underset{x \ll 0}{=} \begin{pmatrix} 0 \\ e^{-\lambda x} \end{pmatrix} \longleftrightarrow \tilde{B}(\lambda)\begin{pmatrix} e^{\lambda x} \\ 0 \end{pmatrix} + \tilde{A}(\lambda)\begin{pmatrix} 0 \\ e^{-\lambda x} \end{pmatrix}$$

So we have

$$\psi_{l,in} = A\,\psi_{r,out} + B\,\psi_{r,in}$$

$$\psi_{l,out} = \tilde{B}\,\psi_{r,out} + \tilde{A}\,\psi_{r,in}$$

Thus

$$\left(\psi_{l,in}\ \ \psi_{l,out}\right) = \left(\psi_{r,out}\ \ \psi_{r,in}\right)\begin{pmatrix} A & \tilde{B} \\ B & \tilde{A} \end{pmatrix}$$

so that

$$\Phi_\lambda(0^+, 0^-) = \begin{pmatrix} A & \tilde{B} \\ B & \tilde{A} \end{pmatrix} \qquad A\tilde{A} - B\tilde{B} = 1.$$

Now when it comes to scattering we express the out states in term of the in states.

$$\psi_{r,out} = -\frac{B}{A}\psi_{r,in} + \frac{1}{A}\psi_{l,in}$$

$$\psi_{l,out} = \underbrace{\left(-\frac{\tilde{B}B}{A} + \tilde{A}\right)}_{\frac{1}{A}}\psi_{r,in} + \frac{\tilde{B}}{A}\psi_{l,in}$$

Thus the scattering matrix is

$$\left(\psi_{r,out}\ \ \psi_{l,out}\right) = \left(\psi_{r,in}\ \ \psi_{l,in}\right)\begin{pmatrix} -\frac{B}{A} & \frac{1}{A} \\ \frac{1}{A} & \frac{\tilde{B}}{A} \end{pmatrix}$$

The scattering matrix is symmetric and analytic for $Re(\lambda) > 0$. To see this we only have to see that A doesn't vanish. But if $A(\lambda) = 0$, then $\psi_{\ell,in} = B\psi_{r,in}$ decays contradicting skew-adjointness.

Next for $\lambda \in i\mathbb{R}$ one has $\tilde{B} = \bar{B}$ and $\tilde{A} = \bar{A}$. This shows the scattering matrix is symmetric unitary for $\lambda \in i\mathbb{R}$.

Now that we have the formulas for the scattering matrix $S(x,x')$ in terms of the propagator $\Phi(x,x')$, the natural step is to understand composition, and to find a differential equation satisfied by S.

I feel as if I don't yet have a good approach to the question. I feel that there should be a link with the Green's function $G_\lambda(x,x')$.

Instead of using formulas let's try to be a little bit more intrinsic. What is bothering me about the scattering matrix $S_\lambda(x,x')$ is the specific basis of solutions for the $p=0$ equation. Due to ε?

Let V_x be the space of boundary values at the point x. The propagator $\Phi_\lambda(x,x')$ is an isomorphism $V_{x'} \xrightarrow{\sim} V_x$ which is symplectic. The fact that for $\lambda \in i\mathbb{R}$, $\Phi_\lambda(x,x') \in SU(1,1)$ means that in this case the propagator preserves real symplectic structures.

So I have for each x, x' a Lagrangian subspace of $V_x \times V_{x'}$, which is a graph, depending nicely on λ.

February 9, 1987

Let $\underset{\overset{\parallel}{W}}{\Phi : \mathbb{C}^2} \longrightarrow \underset{\overset{\parallel}{V}}{\mathbb{C}^2}$ satisfy

$$\langle \Phi w | \varepsilon | \Phi w \rangle = \langle w | \varepsilon | w \rangle ,$$

i.e. $\Phi \in U(1,1)$ when $W = V$. The graph of Φ is an isotropic subspace for the hermitian form

$$\langle v | \varepsilon | v \rangle - \langle w | \varepsilon | w \rangle = \|v_+\|^2 - \|v_-\|^2 - \|w_+\|^2 + \|w_-\|^2$$

So we can view the graph of Φ as the graph of a unitary transformation $V_+ \oplus W_- \longrightarrow V_- \oplus W_+$. This gives an embedding $U(1,1) \subset U(2)$ which is the scattering matrix construction.

Suppose $\Phi = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$\text{graph } \Phi \text{ in } \begin{matrix} W \\ \oplus \\ V \end{matrix} = \operatorname{Im} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & c \\ b & d \end{pmatrix}$$

$$\text{graph } \Phi \text{ in } \begin{matrix} W_+ \\ \oplus \\ V_- \\ \oplus \\ W_- \\ \oplus \\ V_+ \end{matrix} = \operatorname{Im} \begin{pmatrix} 1 & 0 \\ b & d \\ 0 & 1 \\ a & c \end{pmatrix}$$

so the unitary matrix $S : W_+ \oplus V_- \longrightarrow W_- \oplus V_+$ is

$$\begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & d \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{b}{d} & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} -\frac{b}{d} & \frac{1}{d} \\ a - \frac{cb}{d} & \frac{c}{d} \end{pmatrix}$$

I will suppose $\det(\Phi) = ad - bc = 1$, i.e. $\Phi \in SU(1,1)$, whence

$$S = \begin{pmatrix} -\frac{b}{d} & \frac{1}{d} \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

Let us put

$$T = \frac{1}{d} \qquad R = \frac{c}{d} \qquad \tilde{R} = +\frac{b}{d}$$

so that

$$S = \begin{pmatrix} -\tilde{R} & T \\ T & R \end{pmatrix} = \begin{pmatrix} -\frac{b}{d} & \frac{1}{d} \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} T + \frac{\tilde{R}R}{T} & \frac{R}{T} \\ \frac{\tilde{R}}{T} & \frac{1}{T} \end{pmatrix}$$

($\tilde{R}$ will not be $\overline{R}$ in the unitary case.)

Assume now that Φ satisfies

$$\Phi' = \begin{pmatrix} \lambda & p \\ p & -\lambda \end{pmatrix} \Phi \qquad \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix} = \begin{pmatrix} \lambda & p \\ p & -\lambda \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and let's calculate how S varies.

$$T' = \left(\frac{1}{d}\right)' = -\frac{1}{d^2} d' = -T^2(pc - \lambda d)$$

$$= -T^2\left(p\frac{R}{T} - \lambda \frac{1}{T}\right) = T(\lambda - pR)$$

$$R' = \left(\frac{c}{d}\right)' = T'c + Tc' \qquad \lambda c + pd = \lambda \frac{R}{T} + p\frac{1}{T}$$

$$= \cancel{T}(\lambda - pR)\frac{R}{\cancel{T}} + \lambda R + p = 2\lambda R - pR^2 + p$$

$$\begin{cases} R' = 2\lambda R + p(1 - R^2) \\ T' = T(\lambda - pR) \end{cases}$$

February 10, 1987

$V = L^2(S^1)$ has orth basis z^n

$\mathcal{F}$ = fermion Fock space with operators ψ_k, ψ_k^* with orth. basis $e_S = e_{i_1} \wedge e_{i_2} \wedge \cdots$ where $S = \{i_1, i_2, \cdots$ $i_1 > i_2 > i_3 > \cdots$ and $i_{k+1} = (i_k) - 1$ for k suff large

$$\check{\mathcal{F}} = \bigoplus_S \mathbb{C}e_S \subset \hat{\mathcal{F}} = \prod_S \mathbb{C}e_S$$

Set
$$\psi^*(z) = \sum z^k \psi_k^* \qquad \psi(z) = \sum z^{-k}\psi_k$$

I want to be able to work with products of field operators such as

$$\psi^*(z_1)\,\psi(z_2)\,\psi(z_3).$$

This gets involved because $\psi^*(z), \psi(z)$ go from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$, so it is not immediately clear how such operators are to be composed. It turns out one can compose them in a time-ordered way, that is, when $|z_1| > |z_2| > |z_3|$ etc.

Let's go over this.

$$\psi^*(z)\,\psi(\zeta) = \sum z^l \zeta^{-k}\, \psi_l^* \psi_k$$

Now introduce the normal ordering relative to the ground state $e_{\{0,-1,-2,\cdots\}}$.

$$:\psi_l^*\psi_k: \;=\; \begin{cases} \psi_l^*\psi_k & \text{except when } l = k \le 0 \\ -\psi_k\psi_k^* = \psi_k^*\psi_k - 1 & \text{\_\_\_\_\_\_\_\_} . \end{cases}$$

Then we have that for any S, S'

$$\langle e_S \,|\, :\psi_l^*\psi_k: \,|\, e_{S'}\rangle$$

is nonzero for only finitely many l, k. Thus

$$\psi^*(z)\psi(\zeta) = \underbrace{\sum_{l,k} z^l \zeta^{-k} :\psi_l^* \psi_k:}_{:\psi^*(z)\psi(\zeta):} + \underbrace{\sum_{k\le 0} z^k \zeta^{-k}}_{\frac{1}{1-(\zeta/z)}}$$

The operator $:\psi^*(z)\psi(\zeta):$ makes sense as a map from $\check{\mathcal{F}}$ to $\hat{\mathcal{F}}$ for all $z,\zeta \in \mathbb{C}^\times$, and the last [redacted] series converges for $|z|>|\zeta|$. So it's clear the product $\psi^*(z)\psi(\zeta)$ makes sense for $|z|>|\zeta|$, and can be extended by analytic continuation for $z \neq \zeta$.

Now to continue further I need to have an understanding of the normal ordering process. Note that the elements ψ_k^*, ψ_k generate a Clifford algebra, denote it $C(E)$. [redacted] A Clifford algebra has a [redacted] canonical filtration which is increasing, and the associated graded algebra is the exterior algebra

$$\mathrm{gr}\, C(E) = \Lambda(E)$$

Normal ordering has something to do with splitting the filtration, that is, with defining an additive isomorphism

$$[redacted] \quad \Lambda(E) \simeq C(E)$$

Let's recall that we can construct a map from $C(E)$ to $\Lambda(E)$ as follows. Choose a linear map $B: E \longrightarrow E^*$, and associate to $\xi \in E$ the operator

$$(*) \qquad e(\xi) + i(B\xi) \quad \text{on } \Lambda E.$$

This extends to an action on $C(E)$ on ΛE provided

$$\left(e(\xi) + i(B\xi)\right)^2 = \langle B\xi, \xi\rangle$$

coincides with the quadratic form Q on E defining $C(E)$.

This means that $Q = \frac{1}{2}(B + B^t)$. [redacted] Now that we have $C(E)$ acting on $\Lambda(E)$ we can define a map $C(E) \longrightarrow \Lambda(E)$ by $\alpha \mapsto \alpha \cdot 1$. In view of $\circledast$ we see this map is compatible with filtrations, and on passage to the graded spaces it becomes the identity map on $\Lambda(E)$.

Now suppose $E = W \oplus W^*$ with the quadratic function $Q(w + \lambda) = \langle \lambda, w \rangle$. Identify E^* with $W^* \oplus W$. Then the map $Q : E \to E^*$ is given by the matrix $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ as

$$\begin{pmatrix} w \\ \lambda \end{pmatrix}^t \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} w \\ \lambda \end{pmatrix} = \frac{1}{2}\Big(\lambda(w) + w(\lambda)\Big) = \langle \lambda, w \rangle$$

so we can take $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Let's take $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ so that $B\begin{pmatrix} w \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$

Think of $C(E)$ as $\mathrm{End}(\Lambda W)$ and use the standard normal ordering isomorphism

$$C(E) = \mathrm{End}(\Lambda W) = \Lambda W \otimes \Lambda W^*$$

Then [redacted] left multiplication by $e(w)$ acts as $e(w) \hat{\otimes} 1$ on $\Lambda W \otimes \Lambda W^*$. Also left mult. by $i(\lambda)$ acts as $i(\lambda) \hat{\otimes} 1 + 1 \hat{\otimes} e(\lambda)$. Thus we have that left mult. by $e(w) + i(\lambda)$ on $C(E)$ acts as $e(\xi) + i(B\xi)$ on $\Lambda(E)$ where

$$B(w + \lambda) = \lambda.$$

Summarizing I learn that a normal ordering process on $C(E)$ is [redacted] associated to a bilinear form B on E whose symmetrization gives the quadratic form defining $C(E)$.

We now want examples of normal ordering processes. We have already seen examples arising from a vacuum state, but there should also be examples due to thermal states. (This I feel is the case because Fetter-Walecka discuss the Wick theorem in the thermal setting.)

Let's go over the thermal setup for a Fermi gas with non-interacting particles, say in one dimension. Field operators

$$\psi^*(x) = \frac{1}{\sqrt{L}} \sum e^{-ikx} \psi_k^*$$
$$\psi(x) = \frac{1}{\sqrt{L}} \sum e^{ikx} \psi_k$$

We need imaginary time evolution to go with the thermal average

$$\langle A \rangle = \frac{tr\,(e^{-\beta H} A)}{tr\,(e^{-\beta H})}$$

Thus we set

$$\psi^*(x,t) = e^{tH} \psi^*(x) e^{-tH}$$
$$= \frac{1}{\sqrt{L}} \sum_k e^{-ikx} \underbrace{e^{tH} \psi_k^* e^{-tH}}_{\psi_k^*(t)}$$

$$\partial_t \psi_k^*(t) = e^{tH} \underbrace{[H, \psi_k^*]}_{[\varepsilon_k \psi_k^* \psi_k,\, \psi_k^*] \,=\, \varepsilon_k \psi_k^*} e^{-tH}$$

$$= \varepsilon_k \psi_k^*(t) \implies \psi_k^*(t) = e^{\varepsilon_k t} \psi_k^*$$

$$\therefore \boxed{\begin{aligned} \psi^*(x,t) &= \frac{1}{\sqrt{L}} \sum e^{-ikx} e^{\varepsilon_k t} \psi_k^* \\ \psi(x,t) &= \frac{1}{\sqrt{L}} \sum e^{ikt} e^{-\varepsilon_k t} \psi_k \end{aligned}}$$

In the case where $H = \partial_x$, $\varepsilon_k = k$
and we have ($L = 2\pi$ and absorb L into $\frac{dx}{2\pi}$)

$$\psi^*(x,t) = \sum \left(e^{-ix+t}\right)^k \psi_k^*$$

So if we set $z = e^{-ix+t}$, then

$$\begin{cases} \psi^*(z) = \sum z^k \psi_k^* \\ \psi(z) = \sum z^{-\ell} \psi_\ell \end{cases}$$

Moreover, increasing time corresponds to increasing $|z|$.

I now want to discuss the thermal Green's function and to try to understand the associated normal ordering process (if there is one).

$$G(x,t\,;\,x',t') = \langle T[\psi^*(x,t)\,\psi(x',t')] \rangle = \sum_k e^{-ik(x-x')} G_k(t,t')$$

$$G_k(t,t') = \langle T[\psi_k^*(t)\psi_k(t')] \rangle. \quad \text{Say } t > t'$$

$$= \langle \psi_k^* \psi_k \rangle e^{k(t-t')} = \frac{e^{-\beta k}}{1+e^{-\beta k}} e^{k(t-t')}$$

If $t < t'$, then

$$G_k(t,t') = -\langle \psi_k(t)\psi_k^*(t') \rangle$$

$$= -\langle \psi_k \psi_k^* \rangle e^{k(t-t')} = -\frac{1}{1+e^{-\beta k}} e^{k(t-t')}$$

So

$$G_k(t,t') = \begin{cases} -\dfrac{1}{1+e^{-\beta k}} e^{k(t-t')} & t < t' \\[2ex] \dfrac{e^{-\beta k}}{1+e^{-\beta k}} e^{k(t-t')} & t > t' \end{cases}$$

Check
$$G_k(t'_+,t') - G_k(t'_-,t') = 1$$
$$G_k(\beta,t') = -G_k(0,t') \qquad \text{anti-periodic b.c. in } t.$$

So now that we know the Green's function, we can look at whether there is an associated normal ordering. It seems the question depends only on the Clifford algebra with generators ψ_k, ψ_k^* for a single k.

So let's consider the Clifford algebra with generators a^*, a satisfying $a^{*2} = a^2 = 0, aa^* + a^*a = 1$. The possible normal orderings are found by letting the Clifford algebra act on the exterior alg with generators $\mu, \bar{\mu}$ corresp. to a^*, a:

$$a^* = e(\mu) + c\, i(\bar{\mu})$$

no $i(\mu)$ term if $(a^*)^2 = 0$.

$$a = e(\bar{\mu}) + c'\, i(\mu)$$

$$\{a^*, a\} = c + c' = 1$$

The isomorphism between the Cliff alg C and the exterior algebra Λ is given by

$$1 \longrightarrow 1$$
$$a^* \longrightarrow \mu$$
$$a \longrightarrow \bar{\mu}$$

$$a^*a \longrightarrow \big(e(\mu) + c\, i(\bar{\mu})\big)\underbrace{\big(e(\bar{\mu}) + c'\, i(\mu)\big)1}_{\bar{\mu}}$$

$$= \mu\bar{\mu} + c$$

Thus $:a^*a: = a^*a - c$. Now a natural condition on an expectation $\langle\ \rangle$ linked to this normal ordering is that $\langle :a^*a: \rangle = 0$ or

$$\langle a^*a \rangle = c$$

In the thermal state $\langle a^*a \rangle = \dfrac{e^{-\beta\varepsilon}}{1+e^{-\beta\varepsilon}}$, so we get as ε varies all energies between 0 and 1.

Let's see if we can generalize this. This will get very messy if we work with a Clifford algebra $C(E)$ where E is a real vector space with inner product. But let's discuss the general case first without calculating. [redacted] Let's recall how the Gaussian states look on this Clifford algebra. The irreducible Gaussian states correspond to complex structures on E, which gives the symmetric space $O(2n)/U(n)$. A complex structure is an orthogonal transformation J such that $J^2 = -1$. One could say skew-symmetric as well as orthogonal. The general Gaussian state on $C(E)$ is given by a skew-symmetric transf. K such that $K^2 \geq -1$.

To simplify the discussion let us now fix a complex structure i on E and consider only states compatible with the corresponding circle action. This means J [redacted], K which commute with i whence $J = iF$, $F^2 = 1$ or $K = iA$ where $-1 \leq A \leq 1$.

Because of this complex structure we can describe $C(E)$ as generated by a_k^*, a_k satisfying the usual relations. Let $N(E)$ have the corresponding generators $\mu_k, \bar{\mu}_k$. Now for the circle action generated by i a_k^*, μ_k have weight 1 and a_k and $\bar{\mu}_k$ have weight -1. Thus a normal ordering scheme compatible with the circle action has the form:

$$a_k^* = e(\mu_k) + c_{kl}\, i(\bar{\mu}_l)$$
$$a_m = e(\bar{\mu}_m) + c'_{mn}\, i(\mu_n)$$

where
$$\{a_k^*, a_m\} = c'_{mk} + c_{km} = \delta_{km}$$

i.e. $\mathbf{c} + (\mathbf{c}')^t = \mathbf{1}$.

Now count dimensions. The real dimension of the possible A is n^2, $n =$ no of a_k^* and the complex dimension of the possible $\{c_{kl}\}$ is n^2. So as in the $n=1$ case there is a reality condition.

$$a_k^* a_m \longmapsto \left(e(\mu_k) + c_{kl}\ i(\bar{\mu}_l)\right) \bar{\mu}_m$$
$$= \mu_k \bar{\mu}_m + c_{km}$$

so that
$$a_k^* a_m = :a_k^* a_m: + c_{km}$$
$$\langle a_k^* a_m \rangle = c_{km}$$

Thus
$$\overline{c_{km}} = \langle (a_k^* a_m)^* \rangle = \langle a_m^* a_k \rangle = c_{mk}$$

showing $\mathbf{c}$ is hermitian. I think the rest is clear, probably c is the same as the A.

At this point I more or less understand normal ordering processes on C(E) and their relation to Gaussian states. Now I have to understand Wick's theorem so I can calculate.

February 11, 1987

More on normal ordering and Wick's theorem.

Given a bilinear form $R \in E^* \otimes E^*$ on a (f.d.) vector space E we can use it to define a twisted multiplication on either $S(E)$ or ΛE. Let's treat the boson case $S(E)$, and let $\mu : S(E) \otimes S(E) \to S(E)$ denote the usual multiplication. Then we interpret R as a differential operator on $S(E) \otimes S(E)$ and define the twisted multiplication to be

$$(1) \qquad f * g = \mu\left(e^R (f \otimes g)\right)$$

This multiplication is associative and the algebra obtained is the Weyl algebra associated to the skew form on E obtained by skew-symmetrizing R.

In effect if $\xi, \eta \in E$, then

$$\xi * \eta = \xi\eta + R(\xi,\eta)$$

and so

$$\xi * \eta - \eta * \xi = R(\xi,\eta) - R(\eta,\xi)$$

One can verify associativity by showing

$$(f*g)*h = \mu_{123}\left(e^{R_{12}+R_{13}+R_{23}} (f \otimes g \otimes h)\right)$$
$$= f*(g*h)$$

(but this isn't very convincing). Recall the twisted convolution algebra you encountered with Getzler's ideas. You have translation operators $T(x)$ $x \in E$ satisfying

$$T(x)\,T(y) = c(x,y)\,T(x+y)$$

where c is a cocycle: $\quad c(x,y)\,c(x+y,z)^{-1}\,c(x,y+z)\,c(y,z)^{-1} = 1$

for example $c(x,y) = e^{R(x,y)}$ where R is bilinear. Then by considering the operator associated to $f(x)$ on E given by

$$\int f(x)\, T(x)\, dx$$

we get the twisted convolution product

$$(2) \qquad (f * g)(z) = \int_{x+y=z} f(x)\, g(y)\, c(x,y)$$

Now think of $S(E)$ as the distributions on E supported at the origin. It's fairly clear that (1) results from (2). As (2) is associative as c is a cocycle, it follows that (1) is associative.

Now the formula (1) is essentially Wick's theorem. Thus suppose $f = \xi_1 \cdots \xi_k$ $\quad g = \eta_1 \cdots \eta_\ell$ where the $\xi_i, \eta_j \in E$. Let $R = \sum_a \lambda_a \otimes \mu_a$ where the $\lambda_a, \mu_a \in E^*$. Then

$$R(f \otimes g) = \sum_a \partial_{\lambda_a}(\xi_1 \cdots \xi_k) \otimes \partial_{\mu_a}(\eta_1 \cdots \eta_\ell)$$

$$= \sum_{\substack{1 \le i \le k \\ 1 \le j \le \ell}} \sum_a \partial_{\lambda_a}(\xi_i)\, \partial_{\mu_a}(\eta_j)\, (\xi_1 \cdot \hat{\xi}_i \cdot \xi_k) \otimes (\eta_1 \cdot \hat{\eta}_j \cdots \eta_\ell)$$

$$= \sum_{i,j} R(\xi_i, \eta_j)\, (\xi_1 \cdots \hat{\xi}_i \cdots \xi_k) \otimes (\eta_1 \cdot \hat{\eta}_j \cdots \eta_\ell)$$

$$R^q(f \otimes g) = \sum_{\substack{i_1,\dots,i_q \\ j_1,\dots,j_q}} \prod_{b=1}^{q} R(\xi_{i_b}, \eta_{j_b})\, (\hat{\xi}_{i_1} \cdots \hat{\xi}_{i_q}) \otimes (\hat{\eta}_{j_1} \cdots \hat{\eta}_{j_q})$$

where $i_1, \dots, i_q$ are distinct in $\{1, k\}$
$\qquad j_1, \dots, j_q$ ——— ——— $\{1, \ell\}$

Thus $\frac{1}{q!} R^q(f \otimes g)$ is the same sum [redacted] taken over $i_1 < \cdots < i_q$ in $\{1, \ldots, k\}$ and distinct $j_1, \ldots, j_q$ in $\{1, \ldots, q\}$.

The rest is clear.

Presumably the same sort of thing holds in the fermion case provided one follows the super sign conventions.

Now let's return to the free complex fermion field over S^1. Here the Clifford algebra is generated by the ψ_k, ψ_k^*. We have the normal ordering prescription

$$\psi_k^* \psi_\ell = :\psi_k^* \psi_\ell: + \begin{cases} 1 & \text{if } k = \ell \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

[redacted]

$$\psi_\ell \psi_k^* = :\psi_\ell \psi_k^*: + \begin{cases} 1 & \text{if } k = \ell > 0 \\ 0 & \text{otherwise.} \end{cases}$$

[redacted]

$$\psi_k^* \psi_\ell^* = :\psi_k^* \psi_\ell^*: \quad , \quad \psi_k \psi_\ell = :\psi_k \psi_\ell:$$

Thus we have

$$R(\psi_k^*, \psi_\ell) = \begin{cases} 1 & \text{if } k = \ell \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(\psi_\ell, \psi_k^*) = \begin{cases} 1 & \text{if } k = \ell > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$R(\psi_k, \psi_\ell) = R(\psi_k^*, \psi_\ell^*) = 0$$

I want to write this in terms of

$$\psi^*(z) = \sum z^k \psi_k^* \quad , \quad \psi(z) = \sum z^{-k} \psi_k$$

Then

$$R\big(\psi^*(z),\psi(\zeta)\big) = \sum_{k,l} z^k \zeta^{-l} R(\psi^*_k,\psi_l)$$

$$= \sum_{k=l\le 0} z^k \zeta^{-l} = \sum_{k\le 0} (z/\zeta)^k = \frac{1}{1-\zeta/z}$$

$$R\big(\psi(\zeta),\psi^*(z)\big) = \sum_{k=l>0} (z/\zeta)^k = \frac{z/\zeta}{1-z/\zeta}$$

Thus we have the operator product expansions that we obtained before

$$\boxed{\begin{aligned} \psi^*(z)\psi(\zeta) &= :\psi^*(z)\psi(\zeta): + \frac{z}{z-\zeta} \qquad |z|>|\zeta| \\ \psi(\zeta)\psi^*(z) &= \underbrace{:\psi(\zeta)\psi^*(z):}_{\textstyle = -:\psi^*(z)\psi(\zeta):} + \frac{z}{\zeta-z} \qquad |\zeta|>|z| \end{aligned}}$$

Now let's apply Wick's thm. to derive some other operator products

$$\boxed{\begin{aligned} \big(:\psi^*(z)\psi(z):\big)\psi^*(\zeta) &= :\psi^*(z)\psi(z)\psi^*(\zeta): + \psi^*(z)\frac{\zeta}{z-\zeta} \\ \psi^*(\zeta)\big(:\psi^*(z)\psi(z):\big) &= :\psi^*(\zeta)\psi^*(z)\psi(z): + \psi^*(z)\frac{-\zeta}{\zeta-z} \end{aligned}}$$

Let's use these to derive commutation relations. Fix ζ and use the upper formula on $|z|=a$ where $a>|\zeta|$.

$$\underbrace{\frac{1}{2\pi i}\int_{|z|=a} \frac{dz}{z^{q+1}} :\psi^*(z)\psi(z): \psi^*(\zeta)}_{J_q\psi^*(\zeta)} = \frac{1}{2\pi i}\int_{|z|=a} \frac{dz}{z^{q+1}} :\psi^*(z)\psi(z)\psi^*(\zeta):$$

$$+ \frac{1}{2\pi i}\int_{|z|=a} \frac{dz}{z^{q+1}} \psi^*(z)\frac{\zeta}{z-\zeta}$$

Next use the lower formula on $|z|=b$, where $|\zeta|>b$.

$$\psi^*(\zeta)\, J_q = \int_{|z|=b} \frac{dz}{2\pi i\, z^{q+1}}\, \psi^*(\zeta) : \psi^*(z)\psi(z)$$

$$= \int_{|z|=b} \frac{dz}{2\pi i\, z^{q+1}} :\psi^*(z)\psi(z)\psi^*(\zeta): \; + \int_{|z|=b} \frac{dz}{2\pi i\, z^{q+1}}\, \psi^*(z) \frac{\zeta}{z-\zeta}$$

Now subtract. Cauchy's formula implies the integrals with $:\psi^*(z)\psi(z)\psi^*(\zeta):$ cancel so we find by residues

$$[J_q, \psi^*(\zeta)] = \int_{\substack{\text{small} \\ \text{circle around} \\ \zeta}} \frac{dz}{2\pi i\, z^{q+1}}\, \psi^*(z) \frac{\zeta}{z-\zeta} = \psi^*(\zeta)\, \zeta^{-q}$$

Check: $$[J_q, \psi^*(\zeta)] = \Big[\sum_\ell \psi^*_{\ell+q}\psi_\ell\, ,\ \sum_k \zeta^k \psi^*_k\Big]$$
$$= \sum \zeta^k \psi^*_{k+q} = \zeta^{-q}\psi^*(\zeta).$$

Similarly I could have done the contour integral for $|\zeta| = a > |z|$ and $|\zeta| = b < |z|$, and I would have found

$$[\psi^*_k, J(z)] = \operatorname*{Res}_{\zeta=z}\left\{\zeta^{-k-1}\psi^*(z)\frac{-\zeta}{\zeta-z}\right\} = -\psi^*(z)z^{-k}$$

$$[J_q, \psi^*_k] = \psi^*_{q+k} \quad ✓.$$

Next we go onto the operator products for the J's. Using Wicks formula we have

$$\cancel{:\psi^*(z)\psi(z)::\psi^*(\zeta)\psi(\zeta): = :\psi^*(z)\psi(z)\psi^*(\zeta)\psi(\zeta): + \langle\psi^*(z)\psi(\zeta)\rangle :\psi(z)\psi^*(\zeta):}$$

$$J(z)J(\zeta) = \left(:\psi^*(z)\psi(z):\right)\left(:\psi^*(\zeta)\psi(\zeta):\right)$$

$$= \ :\psi^*(z)\psi(z)\psi^*(\zeta)\psi(\zeta):$$

$$+ \underbrace{\langle \psi^*(z)\psi(\zeta)\rangle}_{\frac{z}{z-\zeta}} :\psi(z)\psi^*(\zeta): \ + \underbrace{\langle \psi(z)\psi^*(\zeta)\rangle}_{\frac{\zeta}{z-\zeta}} :\psi^*(z)\psi(\zeta):$$

$$+ \langle \psi^*(z)\psi(\zeta)\rangle\langle \psi(z)\psi^*(\zeta)\rangle$$

$$\boxed{\begin{aligned} J(z)J(\zeta) = \ & :\psi^*(z)\psi(z)\psi^*(\zeta)\psi(\zeta): \\ & + \frac{z}{z-\zeta} :\psi(z)\psi^*(\zeta): \ + \frac{\zeta}{z-\zeta} :\psi^*(z)\psi(\zeta): \\ & + \frac{z\zeta}{(z-\zeta)^2} \end{aligned}}$$

Then doing the contour integrals over $|z| = a > |\zeta|$ and $|z| = b < |\zeta|$ and subtracting gives

$$[J_k, J(\zeta)] = \operatorname*{Res}_{z=\zeta}\left(\frac{z\zeta}{(z-\zeta)^2}\frac{1}{z^{k+1}}\right)$$

as the two level one contraction terms have opposite sign residues at $z=\zeta$. Thus

$$[J_k, J(\zeta)] = \operatorname*{Res}_{z=\zeta} \frac{\zeta(\zeta+z-\zeta)^{-k}}{(z-\zeta)^2} = (-k)\zeta^{-k-1}\zeta = -k\zeta^{-k}$$

or

$$[J_k, J_\ell] = -k\,\delta_{k+\ell}$$

Formulas for reference

$$\boxed{\begin{aligned} \langle \psi^*(z)\psi(\zeta)\rangle &= \frac{z}{z-\zeta} \\ \langle \psi(\zeta)\psi^*(z)\rangle &= \frac{z}{\zeta-z} \end{aligned}}$$

$$\boxed{\begin{aligned} J(z)\psi(\zeta) &= :\psi^*(z)\psi(z)\psi(\zeta): + \frac{z}{\zeta-z}\psi(z) \\ \psi(\zeta)J(z) &= : \text{———} : + \frac{z}{\zeta-z}\psi(z) \end{aligned}}$$

$$\begin{aligned} [J_q, \psi_\ell] &= -\psi_{\ell-q} \\ [J_q, \psi^*_k] &= \psi_{q+k} \end{aligned}$$

Action of Virasoro algebra. Let's consider a vector field $X = a(x)\partial_x$ on the circle (or line) and let make it act on $L^2(S^1)$ by a skew-adjoint operator. Thus we seek an operator $D_X = a\partial_x + b$ on L^2 such that $(D_X f, g) + (f, D_X g) = 0$.

$$((a\partial_x + b)f)g + f(a\partial_x + b)g$$

$$= \partial_x(afg) - a'fg + 2bfg$$

so we want $b = \frac{1}{2}a'$.

We are just defining the natural action on $\frac{1}{2}$ densities. Thus we have $\mathcal{L}_X dx = di_X dx = da = a'\,dx$, and $(\mathcal{L}_X (dx)^{1/2})\,dx^{1/2} + (dx^{1/2})\mathcal{L}_X (dx)^{1/2} = a'dx$ whence $\mathcal{L}_X (dx)^{1/2} = \frac{1}{2}a'(dx)^{1/2}$ and then

$$\mathcal{L}_X\left(f(dx)^{1/2}\right) = \left(a\partial_x f\right)(dx)^{1/2} + f\tfrac{1}{2}a'(dx^{1/2})$$

$$= (a\partial_x + \tfrac{1}{2}a')f \cdot (dx)^{1/2}$$

Note also that we have a Lie algebra

$$\left[a\partial_x + \tfrac{1}{2}a',\ b\partial_x + \tfrac{1}{2}b'\right] = (ab' - ba')\partial_x + \tfrac{1}{2}(ab'' - a''b).$$

The complexified Lie algebra consists of complex functions with the bracket $[a, b] = ab' - ba'$.

Associated to the operator $a\partial_x + \frac{1}{2}a'$ on $V = L^2(S^1)$ we have the operator on Fock space defined by

$$\sum \psi_\ell^* \langle \ell |\, a\partial_x + \tfrac{1}{2}a' \,| k \rangle \psi_k$$

normally ordered. Take $a = e^{iqx}$. Then

$$e^{iqx}(\partial_x + \tfrac{1}{2}iq)\, e^{ikx} = e^{i(q+k)x}\, i(k + \tfrac{1}{2}q)$$

so we get the operator

$$i \sum_k (k+\tfrac{1}{2}q)\, \psi^*_{k+q} \psi_k$$

Notice that for $q=0$ we have essentially the Hamiltonian except for the i factor. So let's set

$$\boxed{L_q = \sum_k (k+\tfrac{1}{2}q)\psi^*_{k+q}\psi_k}$$

(normal ordered). Then L_q corresponds to the operator $e^{iqx}(\frac{1}{i}\partial_x + \frac{1}{2}q)$. What's the commutation relation?

$$\left[e^{ipx}(\tfrac{1}{i}\partial_x + \tfrac{1}{2}p), e^{iqx}(\tfrac{1}{i}\partial_x + \tfrac{1}{2}q)\right]$$

$$= (q-p)\, e^{i(p+q)x}\left(\tfrac{1}{i}\partial_x + \tfrac{p+q}{2}\right)$$

Thus $[L_p, L_q] = (q-p)L_{p+q} + \text{const}$ which means that my L_p differs in sign from the one uses by physicists.

February 12, 1987

Vertex operators. Let's review how the formula is obtained. $\psi^*(z) = \sum z^k \psi_k^*$, $\psi(z) = \sum z^{-k}\psi_k$

$$J_q = \sum_k \psi_{q+k}^* \psi_k \qquad [J_{-p}, J_q] = p\,\delta_{pq}$$

$$[J_p, \psi_k^*] = \psi_{p+k}^* \implies [J_p, \psi^*(z)] = z^{-p}\psi^*(z)$$

$$[J_p, \psi_\ell] = -\psi_{\ell-p} \implies [J_p, \psi(z)] = -z^{-p}\psi(z)$$

τ shift: $\tau e_s = e_{1+s}$.

$$\tau \psi_k^* \tau^{-1} = \psi_{k+1}^* \qquad \tau\psi^*(z)\tau^{-1} = z^{-1}\psi^*(z)$$

$$\tau \psi_k \tau^{-1} = \psi_{k+1} \qquad \tau\psi(z)\tau^{-1} = z\psi(z)$$

$$\tau J_p \tau^{-1} = J_p \qquad p \neq 0$$

$$\tau J_0 \tau^{-1} = J_0 - 1 \qquad [J_0, \tau] = \tau$$

So now recall that we have boson creation + ann. operators given by $a_p^* = \frac{1}{\sqrt{p}} J_p \quad p > 0$.

$$[a_p, \psi^*(z)] = \frac{1}{\sqrt{p}}[J_{-p}, \psi^*(z)] = \frac{1}{\sqrt{p}} z^p \psi^*(z)$$

$$\implies \psi^*(z) = c\, \exp\left\{ \frac{z^p}{\sqrt{p}} a_p^* \right\} = c \cdot e^{\frac{z^p}{p} J_p}$$

where c commutes with a_p. Similarly

$$[a_p^*, \psi^*(z)] = \frac{1}{\sqrt{p}}[J_p, \psi^*(z)] = \frac{1}{\sqrt{p}} z^{-p}\psi^*(z)$$

$$\implies \psi^*(z) = c\, \exp\left(-\frac{1}{\sqrt{p}} z^{-p} a_p\right) = c\, e^{-\frac{z^{-p}}{p} J_{-p}}$$

where c commutes with a_p^*.

$$\tau\psi^*(z)\tau^{-1} = z^{-1}\psi^*(z) \implies \psi^*(z) = c\, z^{J_0}$$

$$[J_0, \psi^*(z)] = \psi^*(z) \implies \psi^*(z) = c\,\tau$$

So we can manufacture an operator which has to be $\psi^*(z)$ up to a constant factor. Better, we have

$$\boxed{\psi^*(z) = c\, \tau\, z^{J_0}\, e^{\sum_{p>0} \frac{z^p}{p} J_p}\, e^{-\sum_{p>0} \frac{z^{-p}}{p} J_{-p}}}$$

where c commutes with the boson operators τ, J_n.

Next let's work on $\psi(z)$.

$$[a_p, \psi(z)] = \frac{1}{\sqrt{p}}[J_{-p}, \psi(z)] = \frac{1}{\sqrt{p}}(-z^p)\psi(z)$$

$$\Rightarrow \psi(z) = c \cdot \exp\left(-\frac{z^p}{p} J_p\right)$$

$$[a_p^*, \psi(z)] = \frac{1}{\sqrt{p}}[J_p, \psi(z)] = \frac{1}{\sqrt{p}}(-z^{-p})\psi(z)$$

$$\Rightarrow \psi(z) = c \cdot \exp\left(\frac{z^{-p}}{p} J_{-p}\right)$$

$$\tau\, \psi(z)\, \tau^{-1} = z\,\psi(z) \qquad \Rightarrow \qquad \psi(z) = c\, z^{-J_0}$$

$$[J_0, \psi(z)] = -\psi(z) \qquad \Rightarrow \qquad \psi(z) = c\,\tau^{-1}$$

$$\boxed{\psi(z) = c'\, \tau^{-1} z^{-J_0}\, e^{-\sum \frac{z^p}{p} J_p} \cdot e^{\sum \frac{z^{-p}}{p} J_{-p}}}$$

We can determine c, c' by looking at the effect on the vacuum state.

$$\underbrace{\langle \tau|0\rangle}_{\psi_1^*|0\rangle} |\, \psi^*(z)|0\rangle = \langle 0|\psi_1 \psi^*(z)|0\rangle$$

$$= \sum z^k \langle 0|\psi_1 \psi_k^*|0\rangle = z$$

$$\langle 0|\tau^{-1}\, c\,\tau\, z^{J_0} e^{a^*\text{'s}} e^{-a\text{'s}}|0\rangle = c\langle 0|e^{a^*\text{'s}} e^{a\text{'s}}|0\rangle = c$$

~~Similarly~~ $\cancel{\langle \tau^{-1}|0\rangle \,|\, \psi}$

Similarly

$$\langle \tau^{-1}|0\rangle \,\big|\, \psi(z)|0\rangle$$

$$= \langle \psi_0|0\rangle \,\big|\, \psi(z)|0\rangle = \langle 0|\psi_0^* \sum z^{-k}\psi_k|0\rangle = 1$$

And it also equals

$$\langle 0|\tau c' \tau^{-1} z^{-J_0} e^{(a^* \xi)} e^{(a's)}|0\rangle = c'.$$

Let's now use B-notation where B stands for blip, and put

$$\boxed{\begin{aligned} B^+(z) &= z^{J_0}\tau\, e^{\sum \frac{z^p}{p} J_p}\, e^{-\sum \frac{z^{-p}}{p} J_{-p}} \\ B^-(\xi) &= \tau^{-1}\xi^{-J_0} e^{-\sum \frac{\xi^p}{p} J_p}\, e^{\sum \frac{\xi^{-p}}{p} J_{-p}} \end{aligned}}$$

Let's now compute the operator products

$$B^+(z)\,B^-(\xi) = (z/\xi)^{J_0}\, e^{\left[-\sum \frac{z^{-p}}{p} J_{-p},\, -\sum \frac{\xi^p}{p} J_p\right]}\, e^{\sum \frac{z^p - \xi^p}{p} J_p}$$

$$\times\, e^{-\sum \frac{z^{-p} - \xi^{-p}}{p} J_{-p}}$$

Now $\exp\left[-\sum \frac{z^{-p}}{p} J_{-p},\, -\sum \frac{\xi^p}{p} J_p\right] = \exp\left(\sum \frac{(\xi/z)^p}{p}\right) = \frac{1}{1-\xi/z} = \frac{z}{z-\xi}$

Similarly in the other direction we will get

$$B^-(\xi)\,B^+(z) = \underbrace{\tau^{-1}(z/\xi)^{J_0}\tau}_{(z/\xi)^{J_0}\,(z/\xi)}\; \underbrace{\exp\left(\left[\sum \frac{\xi^{-p}}{p} J_{-p},\, \sum \frac{z^p}{p} J_p\right]\right)}_{\frac{1}{1-z/\xi}}\; \exp(a^*)\,\exp(a)$$

$$\frac{z}{\xi - z}$$

So we get the following operator products

$$B^+(z)\,B^-(\zeta) = \frac{z}{z-\zeta}\left[(z/\zeta)^{J_0}\, e^{\sum' \frac{z^p-\zeta^p}{p} J_p}\, e^{-\sum' \frac{z^{-p}-\zeta^{-p}}{p} J_{-p}}\right] \quad |z|>|\zeta|$$

$$B^-(\zeta)\,B^+(z) = \frac{z}{\zeta - z}\left[\text{———————————}\right] \quad |\zeta|>|z|$$

The term in brackets is analytic in z, ζ and has the value 1 when $z=\zeta$.

Note that when we compute $B^+(z)B^+(\zeta)$ then the commutator term is

$$\exp\left[-\sum' \frac{z^{-p}}{p} J_{-p}\, ,\, \sum \frac{\zeta^p}{p} J_p\right] = \exp\left(-\sum \frac{(\zeta/z)^p}{p}\right) = 1-(\zeta/z)$$

which is regular at $z=\zeta$. Thus out come the required commutation relations from the contour integral theory.