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(Problem: Generalize Kronecker foliation theory to complex torus)

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# September 23, 1985



Let us consider a simple harmonic oscillator with time dependent forcing.

$$H = \overbrace{\omega a^* a}^{H_0} + a^* J(t) + \tilde{J}(t)\, a$$

And let us calculate the (imaginary) time-evolution

$$(\partial_t + H)\, U(t,0) = 0$$
$$U(0,0) = I.$$

Then

$$e^{\beta H_0} U(\beta,0) = T\left\{ e^{-\int_0^\beta dt[a^* e^{\omega t} J(t) + \tilde{J}(t) e^{-\omega t} a]} \right\}$$

$$= e^{\int_0^\beta dt \int_0^t dt' \; \tilde{J}(t) e^{-\omega(t-t')} J(t')} \; e^{-a^*\left(\int_0^\beta dt\, e^{\omega t} J(t)\right)}$$
$$\times e^{-\left(\int_0^\beta \tilde{J}(t) e^{-\omega t} dt\right) a}$$

Suppose I now compute $\operatorname{tr} U(\beta,0)$.

Formula:

$$\operatorname{tr}\left(e^{-\omega a^* a}\, e^{a^* v}\, e^{u a}\right) = \frac{1}{1-e^{-\omega}}\, e^{u \frac{1}{e^{\omega}-1} v}$$

$$\|$$

Proof:

$$\operatorname{tr}\left(e^{a^* e^{-\omega} v}\, \underbrace{e^{-\omega a^* a}}_{:e^{-(1-e^{-\omega}) a^* a}:}\, e^{u a}\right) = \operatorname{tr}\left(: e^{-(1-e^{-\omega}) a^* a + a^* e^{-\omega} v + u a} :\right)$$

complete the square here

Using this formula gives

$$\operatorname{tr}\left(\mathcal{U}(\beta,0)\right) = \frac{1}{1-e^{\beta\omega}} \exp\left\{ \int_0^\beta dt \int_0^t dt' \, \tilde{J}(t)\, e^{-\omega(t-t')} J(t') \right.$$
$$\left. + \left( \int_0^\beta dt \, \tilde{J}(t)\, e^{-\omega t} \right) \frac{1}{e^{\beta\omega}-1} \left( \int_0^\beta dt' \, e^{\omega t'} J(t') \right) \right\}$$

Now let's recall the

<u>Formula</u>: The Green's function for $\partial_t + \omega$ on $[0,\beta]$ with periodic b.c. is

$$G(t,t') = \begin{cases} e^{-\omega(t-t')} \dfrac{1}{e^{\beta\omega}-1} & t < t' \\[2ex] e^{-\omega(t-t')} \left( \dfrac{1}{e^{\beta\omega}-1} + 1 \right) & t > t' \end{cases}$$

Thus we find

$$\operatorname{tr} \mathcal{U}(\beta,0) = \underbrace{\frac{1}{1-e^{-\beta\omega}}}_{\operatorname{tr}(e^{-\beta H_0})} \exp\left\{ \iint_0^\beta \tilde{J}(t)\, G(t,t')\, J(t')\, dt\, dt' \right\}$$

It is also possible to derive this variationally à la Schwinger. ∎

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Next we consider the fermion Fock space of $L^2(S^1)$ where $S^1 = \mathbb{R}/L\mathbb{Z}$ and the current operators

$$\rho(f) = \sum \hat{f}_q \rho_q \qquad f(x) = \sum \hat{f}_q e^{iqx}$$
$$\hat{f}_q = \frac{1}{L} \int_0^L f(x)\, e^{-iqx}\, dx$$

Suppose given $f(t,x)$ on $S^1 \times S^1$, then we consider the time-dependent perturbation

$$H = H_0 + \rho(f(t,\cdot))$$

where

$$H_0 = :\sum k \psi_k^* \psi_k:$$
$$= \frac{2\pi}{L}\left(\sum_{q>0} \rho_q \rho_{-q} + \frac{1}{2}\rho_0(\rho_0+1)\right)$$

$$\rho(f(t,\cdot)) = \sum_q \hat{f}_q(t)\,\rho_q$$

The $S$ operator for the perturbation in the time interval $[0,\beta]$ is

$$S = e^{\beta H_0}\, U(\beta,0) = T\left\{ e^{-\int_0^\beta dt \sum_q \hat{f}_q(t)\, e^{qt} \rho_q} \right\}$$

$$= e^{-\left(\int_0^\beta dt\, f_0(t)\right)\rho_0} \; e^{\sum_{q>0} \iint_{t>t'} \hat{f}_{-q}(t)\, e^{-q(t-t')} \hat{f}_q(t') \frac{qL}{2\pi}\, dt\, dt'}$$

$$\times\; e^{-\sum_{q>0}\left(\int_0^\beta dt\, \hat{f}_q(t) e^{qt}\right)\rho_q} \; e^{-\sum_{q>0}\left(\int_0^\beta dt\, \hat{f}_{-q}(t) e^{-qt}\right)\rho_{-q}}$$

Suppose that $f(t,x) = \left(\partial_t + \frac{1}{i}\partial_x\right) g(x,t)$, where $g$ is **doubly-periodic**. Then putting

$$g(t,x) = \sum_q \hat{g}_q(t)\, e^{iqx}$$

we have

$$f(t,x) = \sum_q (\partial_t + q)\,\hat{g}_q(t)\, e^{iqx}$$

$$\Longrightarrow \quad \hat{f}_q(t) = (\partial_t + q)\,\hat{g}_q(t) \quad \text{and}$$

$$\int_0^\beta \hat{f}_q(t)\, e^{qt}\, dt = \int_0^\beta \left[(\partial_t + q)\,\hat{g}_q(t)\right] e^{qt}\, dt = \int_0^\beta \partial_t\left(\hat{g}_q(t)\, e^{qt}\right) dt = 0$$

September 25, 1985

Consider the torus $T = \mathbb{R}/\beta\mathbb{Z} \times \mathbb{R}/L\mathbb{Z}$ with coords $t, x$ and the complex structure given by the operator

$$\partial_t + \frac{1}{i}\partial_x$$

To fix the ideas suppose we consider the trivial line bundle over the torus. Then any smooth function $f(t,x)$ on the torus defines a connection with the assoc. $\bar\partial$-operator

$$\partial_t + \frac{1}{i}\partial_x + f(t,x)$$

Modulo gauge transformations only the constant term in the Fourier series of $f$ matters, and this constant term only modulo the lattice

$$\Gamma' = \left\{ \frac{2\pi}{\beta} i m + \frac{2\pi}{L} n \;\middle|\; m, n \in \mathbb{Z} \right\}$$

What I am looking at is the family of flat line bundles over the torus and I know there is a universal family parametrized by the dual torus $T'$.

I can look at the family of $\bar\partial$ operators. The index of this family is the K-class on $T'$ which is of rank 0 and degree 1. It is the K-class represented by the origin of $T'$, i.e., $\iota_! 1$ where $\iota : 0 \hookrightarrow T'$.

One of the problems I am having is to think of this family as defining a map from $T'$ to the space $\mathcal{F}_0$ of operators which are essentially unitary: $u^*u - I$, $uu^* - I$ compact.

The problem is that the [redacted] family of $\bar\partial$ operators works on the Hilbert bundle over $T'$ whose fibre at $x$ is the space of $L^2$ sections of $L_x$ where $L$ is the Poincare line bundle over $T' \times T$. $L$ is not trivial, so that in order to trivialize the Hilbert bundle one needs to use Kuiper's thm.

To be more explicit, we can start with the vector space of constant $f$, [redacted] i.e. $\mathbb{C}$ and the trivial Hilbert bundle over it, and the family of $\bar\partial$ operators in the fibres of this Hilbert bundle. Then we have the action of $\Gamma'$ on this setup and we take the quotient.

September 26, 1985

Let us consider the family of flat line bundles on the circle. Let the circle be $\mathbb{R}/\mathbb{Z}$ with coordinate $x$; for each $y \in \mathbb{R}/\mathbb{Z}$ we have a character $m \longrightarrow e^{2\pi i m y}$ of $\mathbb{Z}$, and sections of the corresponding flat line bundle $L_y$ over $\mathbb{R}/\mathbb{Z}$ can be viewed as functions $f(x)$ on $\mathbb{R}$ satisfying

$$f(x+1) = e^{2\pi i y} f(x).$$

The Hilbert space $\mathcal{H}_y = L^2(\mathbb{R}/\mathbb{Z}, L_y)$ has the orth. basis $\langle x|k\rangle = e^{ikx}$ where $k \in 2\pi(y+\mathbb{Z})$.

Now we have the Hilbert space bundle $\{\mathcal{H}_y\}$ over the "dual" circle and the skew-adjoint operator $\partial_x$ in each fibre. We can form the Fock space bundle. Now there is no problem doing this. Recall that Fock space has a natural basis consisting of subsets of a given orthonormal basis which agree modulo finite sets with a basis for the negative subspace.

To be precise, we consider subsets $S \subset 2\pi(y+\mathbb{Z})$ which agree modulo a finite set with $2\pi(y - \mathbb{N})$. Thus we have a natural Hilbert space bundle.

What I have done is to take the covering

$\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$, $y \longmapsto y+\mathbb{Z}$ and replace it by the covering space with fibres the different subsets $S$ in $y+\mathbb{Z}$ as above, and then take the corresp. Hilbert space bundle.

Next I want to examine the operators on this Fock space bundle. This means that I want to make current operators $\rho_q$, and a Hamiltonian $H_0$. We already know there is no problem with $\rho_q$ for $q \neq 0$. The series

$$\rho_q = \sum_k \psi^*_{k+q} \psi_k$$

makes sense, i.e. converges for $q \neq 0$. However $\rho_0$ and $H_0$ have to be normal ordered

$$\rho_0 = :\sum \psi_k^* \psi_k: \qquad H_0 = :\sum k \psi_k^* \psi_k:$$

and the normal **ordering** depends on a choice of polarization.

Normally one uses the negative energy subspace which is spanned by the $|k\rangle$ with $k \leq 0$. Unfortunately this jumps as $y$ goes thru 0. Then $\rho_0$ jumps by 1. It is clear that $\rho_0$ doesn't make sense as a continuous operator on the Fock space bundle. <u>NO</u>.

Now I should really understand the sort of structure at hand. Recall that Fock space is an irreducible representation of a Heisenberg group associated to $\mathcal{L}\,U(1)$. Let's ignore the part generated by the $\rho_q$ for $q \neq 0$. Consider the subspace of Fock space annihilated by the $\rho_q$ for $q < 0$. This is a Hilbert space bundle over the dual circle whose fibre at $y$ is $\ell^2(y + \mathbb{Z})$.

We can define $\rho_0$ on this Hilbert bundle by letting $\rho_0$ on $\ell^2(y+\mathbb{Z})$ be given by $\rho_0$ at the point $y+n$ is just $y+n$. Thus we see how to define $\rho_0$ nicely on the Fock space bundle. The shift $\sigma$ remains the same.

Next let's try to define a continuous version of $H_0$ on each fibre. Let $h(y)$ be the 'energy' of the state

$$|2\pi y,\ 2\pi(y-1),\ 2\pi(y-2),\ \cdots\ \rangle$$

Then $h(y)$ is to be continuous in $y$ and is to satisfy

$$h(y+1) - h(y) = 2\pi y$$

The simplest solution seems to be

$$h(y) = \pi(y^2 - y) + \text{const}$$

maybe $\quad h(y) = \pi\left(y - \tfrac{1}{2}\right)^2 \quad$ so that the energy is always $\geq 0$.

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My game is ultimately to explain the Morita equivalence between the two axes of the torus.

Let's look at this in the case of a foliation.

September 27, 1985

I want next to work out the formulas for $\operatorname{tr} U(\beta,0)$ where $U(\beta,0)$ is the propagator for $(\partial_t + H(t))\psi = 0$, where

$$H(t) = H_0 + \rho(f(t,\cdot))$$
$$= H_0 + \sum_q \hat{f}_q(t)\rho_q$$

First recall that for a simple harmonic oscillator and

$$H(t) = \omega a^* a + a^* J(t) + \tilde{J}(t) a$$

we found

$$\operatorname{tr} U(\beta,0) = \operatorname{tr}(e^{-\beta H_0})\, e^{\iint \tilde{J}(t) G(t,t') J(t')\, dt\, dt'}$$

where $G$ is the Greens function for $\partial_t + \omega$ with periodic b.c. on $[0,\beta]$

$$G(t,t') = \begin{cases} e^{-\omega(t-t')}\dfrac{1}{e^{\beta\omega}-1} & t<t' \\[2ex] e^{-\omega(t-t')}\dfrac{1}{1-e^{-\beta\omega}} & t>t' \end{cases}$$

Now go on to the case

$$H(t) = H_0 + \sum_q \hat{f}_q(t)\rho_q$$

and recall that we have $[\rho_{-q},\rho_q] = \frac{qL}{2\pi}$ and so we have an oscillator for each $q>0$ with

$$a_q^* = \left(\frac{qL}{2\pi}\right)^{-1/2}\rho_q \qquad a_q = \left(\frac{qL}{2\pi}\right)^{-1/2}\rho_{-q}$$

so that $$\hat{f}_q(t)\,\rho_q = \left(\frac{qL}{2\pi}\right)^{1/2} \hat{f}_q(t)\, a_q^* \qquad q>0$$

Thus we have

$$\operatorname{tr} U(\beta,0) = \operatorname{tr}\left(e^{-\beta H_0}\right)\cdot\left(\text{term having to do with } \rho_0\right) \times$$
$$e^{\sum_{q>0} \iint \hat{f}_{-q}(t)\, G_q(t,t')\, \hat{f}_q(t')\, \frac{qL}{2\pi}\, dt\, dt'}$$

I will ignore the $\rho_0$ term for the moment and instead concentrate on the Gaussian factor. Recall

$$G_q(t,t') = \frac{1}{\beta} \sum_{\omega \in \frac{2\pi}{\beta}\mathbb{Z}} \frac{e^{i\omega(t-t')}}{i\omega + q}$$

and $$f(t,x) = \sum_q \hat{f}_q(t)\, e^{iqx} = \sum_{\omega,q} \hat{f}(\omega,q)\, e^{i(\omega t + qx)}$$

$$\hat{f}(\omega,q) = \frac{1}{\beta}\int_0^\beta \hat{f}_q(t)\, e^{-i\omega t}\, dt$$

So $$\sum_{q>0} \iint \hat{f}_{-q}(t) \sum_\omega \frac{1}{\beta} \frac{e^{i\omega t}\, e^{-i\omega t'}}{i\omega + q}\, \hat{f}_q(t')\, \frac{qL}{2\pi}\, dt\, dt'$$

$$= \sum_{\substack{q>0\\ \omega}} \frac{\beta L}{2\pi}\, \hat{f}(-\omega,-q)\, \frac{q}{i\omega + q}\, \hat{f}(\omega,q)$$

$$= \frac{1}{2} \sum_{q,\omega} \frac{\beta L}{2\pi}\, \hat{f}(-\omega,-q)\, \frac{q}{i\omega+q}\, \hat{f}(\omega,q)$$

$$= \frac{1}{4\pi} \int dt\,dx \int dt'\,dx' \quad f(t\,x) \sum_{q,\omega} \frac{1}{\beta L}\, e^{i\omega(t-t') + iq(x-x')}\, \frac{q}{i\omega+q}\, f(t'x')$$

Thus the Gaussian factor is

$$G(tx,t'x') = \frac{1}{4\pi}\sum_{q\omega}\frac{1}{\beta L}\, e^{i\omega\Delta t + iq\Delta x}\,\frac{q}{i\omega+q}$$

which represents the operator

$$\frac{1}{4\pi}\left(\partial_t + \frac{1}{i}\partial_x\right)^{-1}\frac{1}{i}\partial_x$$

Let's check this by letting $\beta, L \longrightarrow \infty$. Suppose we first do the integral over $q>0$.

$$\frac{1}{4\pi}\iint_{q>0}\frac{d\omega}{2\pi}\,\frac{dq}{2\pi}\,\frac{q}{i\omega+q}\, e^{i\omega\Delta t + iq\Delta x}$$

$i\omega + q = 0$
$\omega = iq$ UHP

$$= \frac{1}{4\pi}\int_{q>0}\frac{dq}{2\pi}\, q\, e^{-q\Delta t + iq\Delta x}\,\theta(\Delta t)$$

$$= \frac{1}{2(2\pi)^2}\,\frac{1}{(\Delta t - i\Delta x)^2}\cdot\theta(\Delta t)$$

Doing $q<0$ gives the same with $\Delta t, \Delta x$ changed in sign. Thus

$$G(tx,t'x') = \frac{1}{2(2\pi)^2}\,\frac{1}{(\Delta t - i\Delta x)^2}$$

Check as follows $\quad z = x+it$

$$\partial_{\bar z} = \tfrac{1}{2}(\partial_x + i\partial_t)$$

$$\frac{2}{\partial_x + i\partial_t} = \frac{1}{\pi(x+it)} \qquad \frac{1}{\partial_t + \frac{1}{i}\partial_x} = \frac{1}{2\pi(t-ix)}$$

$$\frac{1}{i}\partial_x \frac{1}{\partial_t + \frac{1}{i}\partial_x} = \frac{-1}{2\pi\,(t-ix)^2}\;\frac{1}{i}(-i) = \frac{1}{2\pi\,(t-ix)^2}$$

$$\frac{1}{4\pi}\left(\partial_t + \frac{1}{i}\partial_x\right)^{-1}\partial_x = \frac{1}{2\,(2\pi)^2}\,\frac{1}{(t-ix)^2}$$

At this point we have found a formula for $\operatorname{tr} U(\beta,0)$ as a function of the connection $f$. We see it is a Gaussian function of $f$, and we have identified the ~~exponent~~ variance.

September 28, 1985

Review of the definition of determinants for $\partial_t + \frac{1}{i}\partial_x + f(t,x)$. I want to consider this operator as acting on functions $\psi(t,x)$ such that $\psi(t,x+L) = \psi(t,x)$ and $\psi(t+\beta,x) = -\psi(t,x)$. The reason for anti periodicity in the $t$-direction is because

$$\operatorname{tr}(\Lambda M) = \det(1+M)$$

and if $M$ is the monodromy for $\partial_t + A$, then

$$\det(\partial_t + A) \doteq \det(1-M) \qquad \text{periodic b.c.}$$

$$\doteq \det(1+M) \qquad \text{anti-periodic b.c.}$$

Here $f$ is periodic in both $t, x$. I will suppose $f$ has zero constant term in its F.S., i.e. $\iint f(t,x)\,dt\,dx = 0$. Then we know there is a gauge equivalence

$$\varphi\left(\partial_t + \frac{1}{i}\partial_x\right)\varphi^{-1} = \partial_t + \frac{1}{i}\partial_x + f$$

The Green's function for $\partial_t + \frac{1}{i}\partial_x$ with the above b.c. is

$$G(\Delta t, \Delta x) = \sum_{\substack{\xi \in \frac{2\pi}{L}\mathbb{Z} \\ \omega \in \frac{2\pi}{\beta}(\frac{1}{2}+\mathbb{Z})}} \frac{1}{\beta L} \frac{e^{i(\omega\Delta t + \xi\Delta x)}}{i\omega + \xi}$$

where $\Delta t = t-t'$, $\Delta x = x-x'$. This is holomorphic in $z = x+it$ (say $z'=0$) except for $z=0$ where it should have the singularity

$$\int \frac{d\omega}{2\pi} \int \frac{dq}{2\pi} \quad \frac{e^{i(\omega t + qx)}}{i\omega + q} \quad = \quad \frac{1}{2\pi(t - ix)} \quad = \frac{i}{2\pi z}$$

(evaluation by splitting into $q > 0, q < 0$ and doing $\omega$ integral first.)

Note that $-G(\Delta t, \Delta x) = G(-\Delta t, -\Delta x)$, hence

$$G(\Delta t, \Delta x) \sim \frac{i}{2\pi} \left\{ \frac{1}{z} + O(z) \right\} \qquad \text{as} \quad z \to 0.$$

The Green's function for $\partial_t - i\partial_x + f$ is

$$G_f(z,z') = \varphi(z)\, G(\Delta z)\, \varphi(z')^{-1}$$

$$= \frac{i}{2\pi} \left\{ \frac{1}{z - z'} + O(z) \right\} \left\{ 1 + (z-z')(\varphi^{-1}\partial_z \varphi)(z') + (\bar{z}-\bar{z}')(\varphi^{-1}\partial_{\bar{z}}\varphi)(z') \right.$$

From this we see that the natural finite part of $G_f(z,z')$ on $z = z'$ is.

$$FP\, G_f(z,z) = \frac{i}{2\pi} \partial_z \log \varphi(z).$$ So from the formal definition of determinant

$$\delta \log \det\left(\partial_t - i\partial_x + f\right) = \iint \frac{i}{2\pi} \partial_z \log\varphi \;\; \delta f$$

where $\quad (\partial_t - i\partial_x) \log\varphi = -f.\quad$ Note that

$$\partial_z = \tfrac{1}{2}\left(\partial_x - i\partial_t\right) = \frac{i}{2}\left(\tfrac{1}{i}\partial_x - \partial_t\right)$$

so

$$\delta \log \det\left(\partial_t - i\partial_x + f\right) = \iint \frac{i}{2\pi}\, \frac{i}{2} \left(\tfrac{1}{i}\partial_x - \partial_t\right)\left(\partial_t - i\partial_x\right)^{-1}(-f) \cdot \delta f$$

$$= \iint \frac{1}{4\pi} \left(\tfrac{1}{i}\partial_x - \partial_t\right)\left(\partial_t - i\partial_x\right)^{-1} f \cdot \delta f$$

and so integrating gives

$$\log\det\left(\partial_t - i\partial_x + f\right) = \iint f \, \underbrace{\frac{1}{8\pi}\left(\frac{1}{i}\partial_x - \partial_t\right)\left(\partial_t - i\partial_x\right)^{-1}}_{K} f$$

because

$$K(\Delta t, \Delta x) = \sum_{(\xi,\omega)\neq 0} \frac{1}{\beta L}\,\frac{1}{8\pi}\,\frac{-i\omega+\xi}{i\omega+\xi}\, e^{i(\omega\Delta t + \xi\Delta x)}$$

is clearly symmetric: $K(-\Delta t, -\Delta x) = K(\Delta t, \Delta x)$.

Notice that

$$\frac{1}{8\pi}\left(\frac{1}{i}\partial_x - \partial_t\right)\left(\partial_t - i\partial_x\right)^{-1} + \frac{1}{8\pi} = \frac{1}{4\pi}\left(\frac{1}{i}\partial_x\right)\left(\partial_t - i\partial_x\right)^{-1}$$

so that [redacted] $\det(\partial_t - i\partial_x + f)$ agrees with $\mathrm{tr}\; U(\beta, 0)$ except for the factor $\exp\left(\frac{1}{8\pi}\iint f^2\right)$.

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September 30, 1985

Problem: To write $\operatorname{tr} U(\beta, 0)$ as a bosonic functional integral over the torus.

We first remark that we want an integral over $\mathcal{L}(\mathcal{L}(U(1)))$ i.e. maps from $S^1 \times S^1$ to $U(1)$, and that $\mathcal{L}(U(1))$ is not configuration space but rather like phase space, since it is on $\mathcal{L}(U(1))$ that one has the skew pairing given by dlog. To fix the ideas I will ignore the $S^1 \times \mathbb{Z}$ part of $\mathcal{L}(U(1))$; thus I am dealing with the real vector space of real valued fns. $f(x)$ on $S^1$ s.t. $\int f(x)\,dx = 0$ with the skew form

$$[\rho(f), \rho(g)] = \frac{1}{2\pi i} \int f g' \, dx$$

or

$$[\rho(x), \rho(y)] = \frac{1}{2\pi i} \delta'(x-y)$$

Recall

$$f(x) = \sum_q \hat{f}_q e^{iqx} \qquad \hat{f}_q = \frac{1}{L} \int f(x) e^{-iqx} dx$$

The Hamiltonian is formed as follows:

$$[\rho_{-q}, \rho_q] = \frac{qL}{2\pi}$$

$$a_q = \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_{-q} \qquad a_q^* = \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_q \quad q>0$$

$$H = \sum_{q>0} q\, a_q^* a_q = \sum_{q>0} q \left(\frac{qL}{2\pi}\right)^{-1} \rho_q \rho_{-q} = \frac{2\pi}{L} \sum_{q>0} \rho_q \rho_{-q}$$

$$\int \rho(x)^2 dx = \int \left(\sum \tfrac{1}{L} e^{-iqx} \rho_q\right)^2 dx = \frac{1}{L} \sum_q \rho_{-q} \rho_q$$

$$\therefore \quad \boxed{H = \pi \int \rho(x)^2 dx}$$

at least in a formal sense (classically before quantization)

Check the equations of motion:

$$\dot\rho(x) = i[H, \rho(x)]$$

$$= i\pi \int [\rho(y)^2, \rho(x)]\, dy = \frac{i\pi 2}{2\pi i} \int \rho(y)\, \delta'(y-x)\, dy$$

$$= - \int \rho'(y)\, \delta(y-x)\, dy = -\rho'(x)$$

or

$$\boxed{\dot\rho(x) = -\rho'(x)}$$

$$\dot a^* = i[\omega a^* a, a^*] = i\omega a$$

Check: $\rho(tx) = \sum \frac{1}{L} e^{-iqx} e^{+iqt} \rho_q$ is a function of $x-t$ so is killed by $\partial_t + \partial_x$.

We see therefore that we are dealing with a real vector space with symplectic form and quadratic Hamiltonian.

Recall the path integral repn. for the harmonic oscillator

$$\iint \mathcal{D}q(t)\, \mathcal{D}p(t)\, \underbrace{e^{i\int p\dot q\, dt - \int H dt}}_{e^{-S}} .$$

~~The~~ where

$$S = \int \left(-ip\dot q + \tfrac{1}{2}(p^2 + \omega^2 q^2)\right) dt$$

$$a = \frac{\omega q + ip}{\sqrt{2\omega}} \qquad a^* = \frac{\omega q - ip}{\sqrt{2\omega}}$$

$$a^* \dot a \equiv \frac{1}{2\omega}\left(\omega^2 q \dot q + \omega q i \dot p - i p \omega \dot q + p \dot p\right)$$

$$\equiv -ip\dot q$$

( $\equiv$ means mod $\frac{d}{dt}$ (?) )

Thus

$$S = \int (a^* \dot a + \omega a^* a)\, dt = \int a^* (\partial_t + \omega)\, a\, dt$$

Redo this with $a_q^* = \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_q$, $a_q = \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_{-q}$

and we have $S = \int \tilde{L}\, dt$ where

$$\tilde{L} = \sum_{q>0} \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_q \left(\partial_t + q\right) \left(\frac{qL}{2\pi}\right)^{-1/2} \rho_{-q}$$

$$= \frac{2\pi}{L} \sum_{q>0} \frac{1}{q} \rho_q \cdot \partial_t \rho_{-q} + \rho_q \rho_{-q}$$

$$= \frac{\pi}{L} \sum_{q \neq 0} \left( \frac{1}{q} \rho_q \cdot \partial_t \rho_{-q} + \rho_q \rho_{-q} \right)$$

$$= \frac{\pi}{L} \sum_{q \neq 0} \left[ \rho_{-q} \cdot \left(-\frac{1}{q}\right) \partial_t \rho_q + \rho_{-q} \rho_q \right]$$

as
$$\rho(x) = \frac{1}{L} \sum e^{-iqx} \rho_q$$
$$\left(\frac{1}{i}\partial_x\right)^{-1} \rho(x) = \frac{1}{L} \sum e^{-iqx} (-q)^{-1} \rho_q$$

and so we have

$$\boxed{\tilde{L} = \pi \int \left\{ \rho(x)\, \partial_t \left(\frac{1}{i}\partial_x\right)^{-1} \rho(x) + \rho(x)^2 \right\} dx}$$

(this is for the imaginary time case so that the ~~stationary~~ points are

$$\left( \partial_t \left(\frac{1}{i}\partial_x\right)^{-1} + 1 \right) \rho(x) = 0$$

$$\left( \partial_t + \frac{1}{i}\partial_x \right) \rho(tx) = 0 \qquad \checkmark.)$$

Review vertex operators. These are [redacted] the operators $\psi(x)$ $\psi^*(x)$ written in [redacted] terms of the boson operators. Note

$$[\rho_q, \psi_k^*] = [\sum_e \psi_{e+q}^* \psi_e, \psi_k^*] = \sum_e \psi_{e+q}^* \delta_{e,k}$$
$$= \psi_{k+q}^*$$

hence
$$[\rho_q, \psi^*(x)] = [\rho_q, \sum \tfrac{1}{\sqrt{L}} e^{-ikx} \psi_k^*]$$
$$= \sum \tfrac{1}{\sqrt{L}} e^{-ikx} \psi_{k+q}^* = e^{iqx} \psi^*(x)$$

which also agrees with the idea that $\rho_q$ is the derivation of the exterior alg extending mult. by $e^{iqx}$ on $L^2(S^1)$.

Now
$$[a, e^{\lambda a^* - \mu a}] = \lambda e^{\lambda a^* - \mu a}$$
$$[a^*, e^{\lambda a^* - \mu a}] = \mu e^{\lambda a^* - \mu a}$$

which suggests we try
$$\psi^*(x) = e^{\sum_q c_q \rho_q}.$$

Then
$$[\rho_q, e^{\sum_k c_k \rho_k}] = c_{-q} [\rho_q, \rho_{-q}]\, e^{\sum_k c_k \rho_k} = c_{-q} \frac{-qL}{2\pi}\, e^{\sum_k c_k \rho_k}$$
$$\parallel$$
$$[\rho_q, \psi^*(x)] = e^{iqx} \psi^*(x)$$
$$\Longrightarrow \quad c_{+q} = \frac{2\pi}{L} \frac{e^{-iqx}}{+q}$$

What is the series
$$f(y) = \sum{}' \frac{2\pi}{L} \frac{e^{-iqx}}{q} e^{iqy} \qquad ?$$

$$f'(y) = \sum_q{}' \frac{2\pi i}{L} e^{iq(y-x)}$$

$$= 2\pi i \left( \delta(y-x) - \frac{1}{L} \right)$$

Thus $\quad f(y) = 2\pi i \left( \theta(y-x) - \frac{y}{L} + \text{const} \right)$

It remains to find the $\sigma, \rho_0$ part of the vertex operator. Note that

$$(*) \qquad \begin{aligned} [\rho_0, \psi^*(x)] &= \psi^*(x) \\ \sigma\, \psi^*(x)\, \sigma^{-1} &= e^{\frac{2\pi i}{L} x}\, \psi^*(x) \end{aligned}$$

since $\sigma$ is the autom. of the "exterior alg" extending the operator of mult by $e^{\frac{2\pi i}{L}x}$ on $L^2$. We also have

$$\sigma \rho_0 \sigma^{-1} = \rho_0 - 1 \qquad \text{or} \qquad [\rho_0, \sigma] = \sigma$$

$$\sigma e^{\lambda \rho_0} \sigma^{-1} = e^{-\lambda} e^{\lambda \rho_0}$$

Thus another operator with the same commutation relations as $\psi^*(x)$ rel. to $\rho_0, \sigma$ is

$$\sigma e^{-\frac{2\pi i x}{L}\rho_0}$$

The candidate for the vertex operator is

$$\psi^*(x) = \sigma e^{-\frac{2\pi i x}{L}\rho_0} e^{\frac{2\pi}{L}\sum_q{}' \frac{e^{-iqx}}{q} \rho_q}$$

and this [redacted] should be correct up to a scalar. [redacted]

October 2, 1985

I want to work out the theory of vertex operators. These are the operators $\psi(x)$ $\psi^*(x)$ expressed in terms of the operators $\sigma, \rho_q$. Recall the commutation relations

$$[\rho_q, \psi_k^*] = [\sum_\ell{}' \psi_{\ell+q}^* \psi_\ell, \psi_k^*] = \psi_{k+q}^*$$

$$[\rho_q, \psi^*(x)] = \sum_k \frac{1}{\sqrt{L}} e^{-ikx} \psi_{k+q}^* = e^{+iqx}\psi^*(x)$$

$$\sigma \psi^*(x) \sigma^{-1} = e^{\frac{2\pi}{L} ix} \psi^*(x)$$

$$\sigma \rho_0 \sigma^{-1} = \rho_0 - 1 \qquad \sigma\rho_0 = \rho_0\sigma - \sigma, \quad [\rho_0, \sigma] = \sigma$$
$$[\rho_0, \sigma^{-1}] = -\sigma^{-1}$$

$$\sigma e^{a\rho_0}\sigma^{-1} = e^{-a} e^{a\rho_0}$$

$$\boxed{\psi^*(x) = \text{const.}\ \sigma e^{-\frac{2\pi i}{L} x \rho_0} e^{\frac{2\pi}{L}\sum_{q>0} \frac{e^{-iqx}}{q}\rho_q} e^{-\frac{2\pi}{L}\sum_{q>0}\frac{e^{iqx}}{q}\rho_{-q}}}$$

$$[\rho_q, \psi_k] = [\sum_\ell{}' \psi_{\ell+q}^*\psi_\ell, \psi_k] = -\sum_\ell{}' \delta_{\ell+q,k}\psi_\ell$$
$$= -\psi_{k-q}$$

$$[\rho_q, \psi(y)] = -\sum \frac{1}{\sqrt{L}} e^{iky}\psi_{k-q} = -e^{iqy}\psi(y)$$

$$\sigma\psi(y)\sigma^{-1} = e^{-\frac{2\pi i}{L} y}\psi(y)$$

$$\boxed{\psi(y) = \overline{\text{const}}\ e^{\frac{2\pi i}{L} y \rho_0} \sigma^{-1} e^{-\frac{2\pi}{L}\sum_{q>0}\frac{e^{-iqy}}{q}\rho_q} e^{\frac{2\pi}{L}\sum_{q>0}\frac{e^{+iqy}}{q}\rho_{-q}}}$$

I have written things so that $\psi^*(x)$ and $\psi(x)$ are adjoints.

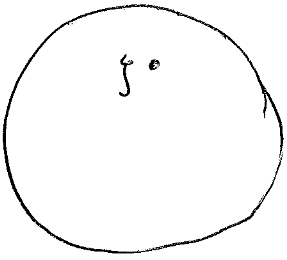
The problem now is to see that

$$\psi^*(x)\psi(y) + \psi(y)\psi^*(x) = \delta(x-y)$$

as well as the other commutation relations. I would also like to know the exact value of the constant.

The next step will be to link these formulas to the exponential operators associated to what Graeme calls blips. These are maps $S^1 \longrightarrow U(1)$ which wrap around the circle once ~~as~~ one passes thru a point on $S^1$, the analogue of a Heaviside function: $e^{2\pi i \theta(x-y)}$.

Think in complex variable terms:



$\dfrac{z-\zeta}{1-\bar{\zeta} z}$ maps $|z| \to 1$ to $U(1)$

$\parallel$

$\dfrac{1}{z}\,\dfrac{z-\zeta}{\bar{z}-\bar{\zeta}}$

If $\zeta$ is nearly on the unit circle the phase shifts rapidly by $2\pi$ as one passes past $\zeta$.

Now write in terms of $z = e^{ix}$, $\zeta = re^{iy}$ where $r = 1^-$. Normalize to have value $1$ at $x=0$

$$\frac{e^{ix}-re^{iy}}{1-re^{i(x-y)}} \cdot \underbrace{\frac{1-re^{-iy}}{1-re^{iy}}}$$

approaches as $r \nearrow 1$ $\quad e^{-iy}\dfrac{e^{+iy}-1}{1-e^{iy}} = -e^{-iy}$ (provided $y \neq 0$)

Thus we look at the loop

$$(-1)e^{i(x-y)}\,\frac{1-re^{-i(x-y)}}{1-re^{i(x-y)}}$$

To this loop belongs the operator

$$(*) \qquad \sigma\,(-1)^{\rho_0}\, e^{-iy\rho_0}\; e^{\sum \frac{r^n e^{-iny}}{n}\rho_n}\; e^{-\sum \frac{r^n e^{iny}}{n}\rho_{-n}}$$

which is unitary when multiplied by the constant

$$e^{-\frac{1}{2}\sum_{n=1}^{\infty}\frac{r^{2n}}{n}} = \sqrt{1-r^2}$$

In any case $(*)$ is a well-defined bounded operator on the Fock space whose limit as $r \nearrow 1$ is our candidate for $\psi^*(y)$ times $(-1)^{\rho_0}$.

---

Let's consider a different approach. Let's recall Deligne's (or maybe Bloch's) construction of a regular map

$$K_2(X) \longrightarrow H^1(X, \mathbb{C}^*)$$

for a Riemann surface. Actually what one constructs is a map

$$K_2(F) \longrightarrow \varinjlim_{S} H^1(X-S, \mathbb{C}^*)$$

more precisely, given two merom. fns. $f, g$ one constructs a flat line bundle on $X-S$ where $S =$ zeros + poles of $f, g$.

Now this pairing is a skew symmetric bilinear form and one might hope to produce a representation of the corresponding central extension. Unfortunately the central extension is not a Heisenberg group, so it has lots of irred. repns. parametrized by the different characters of

$$\varinjlim_{S} H^1(X-S, \mathbb{C}^*)$$

i.e. by elements of

$$\varprojlim_{S} H_1(X-S, \mathbb{Z})$$

A curve in $X$ will determine an elt of $H_1(X-S, \mathbb{Z})$ provided $S$ is disjoint from the curve. So if we ignore poles + zeroes on the curve we see that we should be thinking in terms of having an irreducible repn. belonging to each curve, and some sort of operator on this repn. attached to each merom. function on the torus which is regular + invertible on the curve.

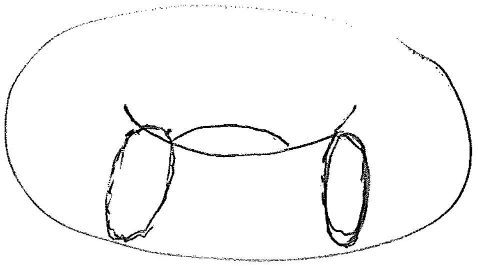
So I must look at each curve and form the Fock space with its Heisenberg group. The next point is that the rest of the Riemann surface is going to give a state on this Heisenberg group, analogous to a thermal state.

It would appear that when we take a curve which separates the surface, then we also get a state, at least a vector and dual vector on the Fock space given by the subspace of boundary values of holom. functions on either side of the curve.

Next point: The Heisenberg group is disconnected

October 3, 1985

We consider the torus $\mathbb{R}/\beta\mathbb{Z} \times \mathbb{R}/L\mathbb{Z}$ with coordinates $t, x$ and with complex structure $\ni$ $z = x + it$ is holomorphic. We are going to be considering two different times, hence we have two parallel circles on the torus.



Each circle has a Fock space attached. Think of this Fock space as the unique irreducible representation of the CCR belonging to $\mathcal{L}(U(1))$ with its dilog pairing.

The operator $e^{-tH_0}$ goes from the Fock space at time $0$ to time $t$, and $e^{-(\beta-t)H_0}$ goes from time $t$ to time $\beta$. What I want to say somehow is that the holomorphic structure gives natural operators

$$\mathcal{H}_0 \longrightarrow \mathcal{H}_t \longrightarrow \mathcal{H}_\beta = \mathcal{H}_0$$

Recall that $tr(XY) = tr(YX)$. I want to use this to link operators on $\mathcal{H}_0$ and $\mathcal{H}_t$, and by operators I mean loop operators.

The thermal state $tr(e^{-\beta H_0}?)/tr(e^{-\beta H_0})$ on the loop group is a natural gadget.

By GNS this state determines a cyclic module over the Weyl algebra of the loop group. There should be a corresponding module for any Morita equivalent algebras.

So one thing we could look at would be to take different times and look at the two Weyl algebras. Now we have decided that the thermal state on the Weyl algebra is the analogue of the irrational rotation in the foliation case. Hopefully one can make sense out of this for the s.h.o.

October 4, 1985

Discuss thermal states $\mathrm{tr}(e^{-\beta H_0}\cdot)$ for the simple harmonic oscillator. I want to identify the GNS representation belonging to this state.

General observation. Let $\mathcal{A}$ be a $*$-algebra acting on a Hilbert space $\mathcal{H}$, and let $\rho$ be a density matrix, i.e. a pos. s.a. op. of trace class. This gives a state on $\mathcal{A}$ namely

$$A \longmapsto \mathrm{tr}(\rho A)$$

We can identify the GNS representation as follows: Consider $\mathcal{H}\otimes\mathcal{H}'$ = Hilbert-Schmidt operators on $\mathcal{H}$. This is a Hilbert space with $\|X\|^2 = \mathrm{tr}(X^*X)$ and $\mathcal{A}$ acts on it by left multiplication. As

$$\langle X|AY\rangle = \mathrm{tr}\, X^*AY = \mathrm{tr}\,(A^*X)^*Y = \langle A^*X|Y\rangle$$

it is a $*$ rep. The operator $\rho^{1/2}$ is H.S. and

$$\langle \rho^{1/2}|A\rho^{1/2}\rangle = \mathrm{tr}(\rho^{1/2}A\rho^{1/2}) = \mathrm{tr}(\rho A)$$

It follows that the cyclic $\mathcal{A}$-subspace generated by $\rho^{1/2}$ is the GNS representation.

Note if $\rho = \sum \rho_i |i\rangle\langle i|$ where $\rho_i > 0$ and the $|i\rangle$ are an orthonormal basis for $\mathcal{H}$, and if $\mathcal{A}$ is dense enough in $\mathcal{B}(\mathcal{H})$, then the subspace $\overline{\mathcal{A}\rho^{1/2}}$ should be all Hilbert-Schmidt operators.

Next we want to apply these observations to $\rho = e^{-\beta\omega a^*a}$. The ~~corresponding~~ vector in $\mathcal{H}\otimes\mathcal{H}$ corresponding to $\rho^{1/2}$ is

$$\sum_{n\geq 0} e^{-\frac{1}{2}\omega n} \frac{(a^*)^n|0\rangle}{\sqrt{n!}} \otimes \frac{(a^*)^n|0\rangle}{\sqrt{n!}}$$

$$= e^{\left(e^{-\frac{1}{2}\omega} a^*\otimes a^*\right)} |0\rangle\otimes|0\rangle.$$

Next I recall another way to obtain the GNS representation belonging to this state. Again we consider the Hilbert space $\mathcal{H}\otimes\mathcal{H}$; let's put $a_1^* = a^*\otimes 1$, $a_2^* = 1\otimes a^*$, etc. and set

$$\tilde{a}_1^* = s a_1^* + t a_2 \qquad |s|^2 - |t|^2 = 1.$$

$$\tilde{a}_1 = \bar{s} a_1 + \bar{t} a_2^*$$

$$\tilde{a}_2^* = s a_2^* + t a_1$$

$$\tilde{a}_2 = \bar{s} a_2 + \bar{t} a_1^*$$

We consider the repn. of the Weyl algebra (gen. by $a^*, a$) where $a^*, a$ acts as $\tilde{a}_1^*, \tilde{a}_1$ on $\mathcal{H}\otimes\mathcal{H}$. The vector $|0\rangle\otimes|0\rangle = |0,0\rangle$ leads to the gen. fnl.

$$\langle 0,0| e^{c\tilde{a}_1^* - \bar{c}\tilde{a}_1} |0,0\rangle = \langle 0| e^{csa_1 - \bar{c}\bar{s}a_1}|0\rangle \cdot \langle 0| e^{-\bar{c}\bar{t}a_2^* + cta_2}|0\rangle$$

$$= e^{-\frac{1}{2}|cs|^2} e^{-\frac{1}{2}|-\bar{c}\bar{t}|^2} = e^{-\frac{1}{2}(|s|^2+|t|^2)|c|^2}$$

We have calculated for the thermal state

$$\frac{\operatorname{tr}\left(e^{-\omega a^* a} e^{ca^* - \bar{c}a}\right)}{\operatorname{tr}\left(e^{-\omega a^* a}\right)} = \exp\left\{-\bar{c}\left(\frac{1}{2} + \frac{1}{e^\omega - 1}\right)c\right\}$$

So it is clear we want

$$\frac{|s|^2+|t|^2}{2} = \frac{1}{2} + |t|^2 = \frac{1}{2} + \frac{1}{e^\omega - 1}$$

or $$|t|^2 = \frac{1}{e^{\omega}-1}, \qquad |s|^2 = \frac{1}{1-e^{-\omega}}$$

Next I would like to correlate these two approaches. Consider the skew-hermitian 'quadratic' operator $-a^*\otimes a^* + a\otimes a = -a_1^*a_2^* + a_1a_2$. It lies in the Lie algebra of the metaplectic group and hence generates a symplectic transformation on the span of the $a_i^*, a_i$.

$$\left[-a_1^*a_2^* + a_1a_2,\ \begin{matrix} a_1^* \\ a_1 \\ a_2^* \\ a_2 \end{matrix}\right] = \begin{matrix} a_2 \\ a_2^* \\ a_1 \\ a_1^* \end{matrix}$$

matrix:

$$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

When exponentiated we get

$$e^{\theta(1\,1\,1\,1)} = \begin{pmatrix} \cosh\theta & & & \sinh\theta \\ & \cosh\theta & \sinh\theta & \\ & \sinh\theta & \cosh\theta & \\ \sinh\theta & & & \cosh\theta \end{pmatrix}$$

and so

$$\begin{aligned} Ad\left(e^{\theta(-a_1^*a_2^*+a_1a_2)}\right) a_1^* &= \cosh\cdot a_1^* + \sinh\cdot a_2 \\ a_1 &= \cosh\cdot a_1 + \sinh\cdot a_2^* \\ a_2^* &= \sinh\cdot a_1 + \cosh\cdot a_2^* \\ a_2 &= \sinh\ a_1^* + \cosh\ a_2 \end{aligned}$$

$U a_1^* U^{-1}, |0\rangle \Leftrightarrow a_1^*, U^{-1}|0\rangle$

I now need

$$e^{-\theta(-a_1^* a_2^* + a_1 a_2)} \, |0,0\rangle$$

Clearly this is annihilated by

$$\cosh\theta \cdot a_1 - \sinh\theta \; a_2^*$$
$$\cosh\theta \cdot a_2 - \sinh\theta \; a_1^*$$

and hence it is proportional to

$$e^{+\frac{\sinh\theta}{\cosh\theta} a_1^* a_2^*} \, |0,0\rangle$$

Thus we want $\theta$ to be such that

$$e^{-\frac{1}{2}\omega} = \frac{\sinh\theta}{\cosh\theta} \qquad e^{-\omega} = \frac{\sinh^2}{\cosh^2} = \frac{\cosh^2 - 1}{\cosh^2}$$

or $$\cosh^2\theta = \frac{1}{1 - e^{-\omega}} \qquad \sinh^2\theta = \frac{1}{e^{\omega} - 1}$$

so everything checks.

October 11, 1985

Some ideas + analogies:

Consider a complex torus and a embedded oriented circle which is a generator of $\pi_1$ say for example $\mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$, $\operatorname{Im}\tau > 0$ and the circle $\mathbb{R}/\mathbb{Z}$. Then we can form the fermion Fock space $\mathcal{H}$ of $L^2$ of the circle, and the complex structure determines a contraction operator $T$ on this Fock space. I want to somehow use $(\mathcal{H}, T)$ as the analogue of $L^2(S^1)$ with the operator of rotation through $\theta$ which occurs in the theory of the Kronecker foliation.

Consider a parallel circle, e.g. $\mathbb{R}+it/\mathbb{Z} \subset \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$. This has a Fock space with contraction $(\mathcal{H}', T')$. In the general setting there is no canon. isomorphism between $\mathcal{H}$ and $\mathcal{H}'$. Rather using the complex structure to propagate one obtains contraction operators

$$\mathcal{H} \xrightarrow{T_1} \mathcal{H}' \xrightarrow{T_2} \mathcal{H}$$

such that $\quad T_2 T_1 = T \quad , \quad T_1 T_2 = T'$.

This suggests that $(\mathcal{H}, T)$ and $(\mathcal{H}', T')$ are somehow "shift-equivalent" as in the theory of dynamical systems. Perhaps I can apply ideas about Markov chains, martingales, etc. The rough idea is that there should be a big Hilbert space with a unitary operator, maybe a big probability space for a Markov process, which lies behind $(\mathcal{H}, T)$ and $(\mathcal{H}', T')$.

Furthermore, I recall the work of Burkholder and Gundy linking martingales and Hardy spaces of analytic functions. There is a strong parallel between martingales + harmonic functions which becomes more precise when one studies Brownian motion on a surface. Thus I could really hope that some of this theory might be relevant to the problems I am working on.

---

Recall the Gaussian Markov processes. In the case of a discrete Markov process $x_n$ real-valued, if it's Gaussian and stationary then the transition probability

$$p(x,y)\,dy$$

is Gaussian, say $e^{-\frac{a}{2}x^2+bxy-\frac{c}{2}y^2}\,dy \cdot \text{const}$. In order that this be a prob. measure for each $x$, the exponential is a perfect square.

October 24, 1985.

Problem: Let's consider the path integral for quantum mechanics of a particle where the paths are in phase space:

$$1) \qquad \int \mathcal{D}p\,\mathcal{D}q \; e^{\int(i p\dot{q} - H\,dt)} \; (?) \qquad \text{(imag. time)}$$

If one does the $p$-integral, supposing $H = \frac{p^2}{2} + V(q)$ one gets a path integral over paths in configuration space

$$2) \qquad \int \mathcal{D}q \; e^{-\int\left(\frac{1}{2}\dot{q}^2 + V(q)\right)dt} \; (?)$$

One knows this latter integral can be made precise via Wiener measure. The problem is whether the former integral can be made precise.

Let's go back to the s.h.o. Recall that if we put

$$\psi = \frac{1}{\sqrt{2\omega}}(\omega q + ip)$$

$$\bar{\psi} = \frac{1}{\sqrt{2\omega}}(\omega q - ip)$$

Then

$$\bar{\psi}\,\partial_t\psi \cong -ip\dot{q} \qquad \text{mod } \tfrac{d}{dt}\text{ something}$$

This is true for any $\omega > 0$, and if we take it to be the frequency of the oscillator, we have

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) = \omega\bar{\psi}\psi$$

so the path integral 1) becomes

$$3) \qquad \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-S} \; (?) \qquad , \qquad S = \int \bar{\psi}(\partial_t + \omega)\psi\,dt$$

where $\psi$ runs over paths $\psi(t)$ in $\mathbb{C}$.

Change $\omega$ to $\omega_0$, and suppose we want $\psi$ to be periodic on $[0,\beta]$. Using F.S.:

$$\psi(t) = \frac{1}{\sqrt{\beta}} \sum_{\omega} e^{i\omega t}\, \psi_\omega \qquad \omega \in \frac{2\pi}{\beta}\mathbb{Z}$$

we have

$$S = \int_0^\beta \bar{\psi}\,(\partial_t + \omega_0)\,\psi\, dt$$

$$= \sum_{\omega} (i\omega + \omega_0)\,|\psi_\omega|^2$$

Thus the path integral 3) is roughly a product over $\omega$ of Gaussian integrals over $\mathbb{C}$ of the form

$$\int d^2z\, e^{-(i\omega+\omega_0)|z|^2}\, (?) \Big/ \text{norm.}$$

We run into the following problem. We have a Gaussian integral where the covariance is not real, although its real part is negative definite. We know how to integrate polynomial functions, i.e. polys in $\psi_\omega, \bar{\psi}_\omega$ (the variables). But we don't know what else can be integrated.

This is a familiar problem. The integral is defined on a class of "elementary" functions and we want to extend it. Recall in the moment problem we knows how to integrate polynomials in one variable and the problem is to extend the integral to other functions.

Note that for $a > 0$ one has

$$\int \frac{d^2z}{\pi}\, e^{-a|z|^2} = \frac{1}{a}$$

and hence this holds for $\mathrm{Re}(a) > 0$. Also

$$\int \frac{d^2z}{\pi}\, e^{-a|z|^2 + J|z|^2} = \frac{1}{a-J} = \sum_{k\geq 0} \frac{J^k}{a^{k+1}}$$

$$\Longrightarrow \quad \int \frac{d^2z}{\pi}\, e^{-a|z|^2}\, |z|^{2k} = \frac{k!}{a^{k+1}}$$

which also follows by polar coordinates + $\Gamma$ fn.

~~Problems~~

Here's a typical problem. Let's suppose we try to integrate $e^{-b\int |\psi|^2 dt} = e^{-b\sum_\omega |\psi_\omega|^2}$, The integral formally is

$$\prod_\omega \frac{i\omega + \omega_0}{i\omega + \omega_0 + b} = \prod_\omega \left(1 + \frac{b}{i\omega + \omega_0}\right)^{-1}$$

which is divergent. T

October 29, 1985

Let's continue examining the path integral

$$(*) \qquad \int \mathcal{D}\bar\psi\, \mathcal{D}\psi \; e^{-\int \bar\psi(\partial_t+\omega_0)\psi\, dt} \quad (?)$$

to see if we can find a firm foundation for it.

To begin with, we know this integral is a linear functional on polynomials; to be precise, let $V$ be the space of real linear functions

$$\psi \longmapsto \int (\tilde{J}\psi + \bar\psi J)\, dt$$

on the space of $\mathbb{C}$-valued paths $\psi$. Then $(*)$ is a linear fcl. on $S(V)$; here $V$ is a complex vector space and $S$ is its symmetric alg. over $\mathbb{C}$. $(*)$ is the "Wick" extension of the quadratic form on $V$ given by the Green's function.

We have therefore a complex vector space with a quadratic form. ~~A natural question~~ In order to fit this into the Gaussian integral framework we might look for a real spanning subspace on which the quadratic function is positive. One also gets this idea from observing that the maximal compact of $O(n,\mathbb{C})$ is $O(n)$.

~~Thus we are led back to finding a real spanning subspace of the space of paths~~

Finding a real reduction of $V$ should be ess. equivalent to finding a real reduction of the space of pairs $(\psi(t), \bar\psi(t))$ with the quadratic function $\int \tilde{\psi}(\partial_t+\omega_0)\psi \, dt$.

It seems to be possible to do this sort of thing in a translation invariant way by looking at each frequency $\omega$. If we expand in Fourier series or integral, then we get the quadratic fn

$$\sum_{\omega} \tilde{\psi}_{-\omega}(i\omega+\omega_0)\psi_{\omega}$$

Thus ~~for~~ each $\omega$ we have a 2 diml complex vector space with coordinates $\tilde{\psi}_{-\omega}, \psi_{\omega}$ the quadratic function $Q = (\tilde{\psi}_{-\omega}\psi_{\omega})(i\omega+\omega_0)$. The idea is now to select a real 2diml subspace over which $Q$ is positive, and then use the resulting honest Gaussian integral.

It might help to think of having the quadratic function $Q(z) = az^2$ on $\mathbb{C}$. The line integrals

$$\int e^{-az^2} z^n \, dz$$

taken over a line going between the sectors where $e^{-az^2}$ decays are independent of the line.

October 30, 1985

*Prop.* Let $V = W \oplus W^*$, $Q((w,\lambda)) = \lambda(w)$ be a hyperbolic quadratic space $/\mathbb{C}$. Then any hermitian inner product on $W$ determines a conjugation on $V$ such that $Q$ is positive on the real subspace fixed by the conjugation.

Proof: An inner product $\langle \mid \rangle$ on $W$ determines a conjugate linear isomorphism $\varphi: W \to W^*$, $\varphi(w) = \langle w | \ \rangle$. Define $\overline{(w,\lambda)} = (\varphi^{-1}(\lambda), \varphi(w))$ on $V$; this is a conjugation which has fixed subspace $\{(w, \varphi(w)) \mid w \in W\}$. And

$Q(w, \varphi(w)) = \varphi(w)(w) = \langle w | w\rangle > 0$ for $w \neq 0$.

*Remark:* $S^1$ acts on $(V, Q)$ by $\zeta * (w,\lambda) = (\zeta w, \zeta^{-1}\lambda)$. Given a conjugation $\overline{\phantom{x}}$ commuting with this action, one has

$$\zeta * \overline{(w,0)} = \overline{(\zeta w, 0)} = \bar{\zeta}\,\overline{(w,0)} \implies \overline{W} \subset W^*$$

and similarly $\overline{W^*} \subset W$. This means $\overline{\phantom{x}}$ is described by a conjugate linear isom. $\varphi: W \to W^*$ and its inverse as above. If $Q(\bar{v}) = \overline{Q(v)}$, then $Q(w, \varphi(w)) = \varphi(w)(w)$ will be real and so $\langle w | w'\rangle = \varphi(w)(w')$ is hermitian. Thus the prop describes all conjugations on $(V, Q)$ commuting with the $S^1$ action which have $Q$ positive on the fixed subspace.

Example: Let $V = \mathbb{C}^2$ with $Q((z_1, z_2)) = z_1 z_2$. Then the different inner products on $W$ are $\|z\|^2 = t|z|^2$ for $t > 0$. Thus we get real spanning subspaces

$$\{(z, t\bar{z}) \mid z \in \mathbb{C}\}$$

Example: $V = \mathbb{C}$, $Q(z) = az^2$. There is only one real line in $V$ on which $Q$ is positive, namely, $\tau\mathbb{R}$ where $a\tau^2 = 1$.

These examples illustrate:

$$O(1,\mathbb{C})/O(1) = \{\pm 1\}/\{\pm 1\} = 1$$

$$O(2,\mathbb{C})/O(2) = \mathbb{Z}/2 \ltimes \mathbb{C}^\times / \mathbb{Z}/2 \ltimes S^1 = \mathbb{R}^\times_{>0}.$$

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Now given $(V, Q)$, if we pick a real form of $V$ on which $Q > 0$, then the real form determines a Gaussian integral, hence a Hilbert space which is a completion of $S(V)$. The question arises as to the relation between the Hilbert spaces associated to different real forms. Put another way suppose we have two Euclidean space

# November 13, 1985



$$\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)} = s(s+1)\cdots(s+n-1)\frac{1}{\Gamma(s+n)}$$

$$= s(1+s)\left(1+\frac{s}{2}\right)\cdots\left(1+\frac{s}{n-1}\right)\frac{(n-1)!}{\Gamma(s+n)}$$

Now

$$\frac{\Gamma(s+n)}{(n-1)!} = \frac{\Gamma(s+n)}{\Gamma(n)} = \frac{\int_0^\infty e^{-t}t^{s+n}\frac{dt}{t}}{\int_0^\infty e^{-t}t^{n}\frac{dt}{t}}$$

and the measure $e^{-t}t^n\frac{dt}{t}$ peaks at the $t \ni$

$$\frac{d}{dt}\log\left(e^{-t}t^n\right) = \frac{d}{dt}\left(-t+n\log t\right) = -1+\frac{n}{t} = 0$$

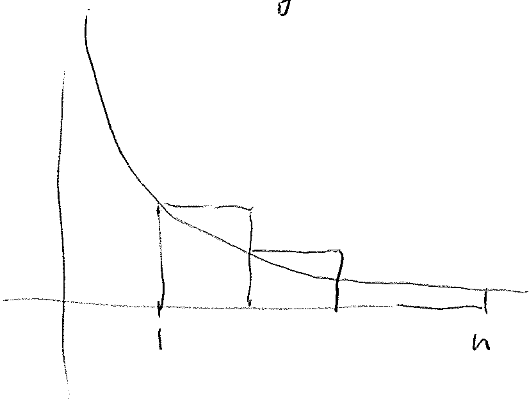
i.e. $t=n$. Thus

$$\frac{\Gamma(s+n)}{\Gamma(n)} \sim n^s \qquad \text{as } n \to \infty.$$

so

$$\frac{1}{\Gamma(s)} = s\prod_{j=0}^{n-1}\left(1+\frac{s}{j}\right)e^{-\frac{s}{j}} \quad e^{s\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)}\overbrace{\frac{\Gamma(n)\,n^s}{\Gamma(s+n)}}^{1}e^{-s\log n}$$

Now



$$1+\frac{1}{2}+\cdots+\frac{1}{n-1} > \int_1^n \frac{1}{t}\,dt = \log n$$

and

$$\gamma = \lim_{n\to\infty} 1+\frac{1}{2}+\cdots+\frac{1}{n-1} - \log n$$

is Euler's constant.

$$\therefore \frac{1}{\Gamma(s)} = e^{\gamma s}\, s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

Next log differentiate:

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right)$$

whence

$$-\Gamma'(1) = \gamma .$$

As a check consider the sign: $\gamma(\tfrac{3}{2}) = \tfrac{1}{2}\gamma(\tfrac{1}{2})$ so that $\gamma(\tfrac{1}{2}) > \gamma(\tfrac{3}{2})$. Better $\Gamma(1) = \Gamma(2)$.

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<u>Program:</u> I would like to find a continuous analogue of the Weil theory of zeta for curves over a finite field. Hopefully this could be applied to the Riemann zeta.

I find striking in the theory of correspondences on a curve the interplay between the intersection theory on the surface $X \times X$ and the algebra of correspondences. The former is a commutative type of algebra whereas the latter is non-commutative. Somehow I feel that there might be a link between this interplay and the interplay between path integrals and quantum mechanics. And it would be especially nice if the unitary nature of time evolution is a consequence of the Lorentzian structure of space-time, in analogy with the fact that positivity of the trace on the alg. of correspondences follows from the Lorentzian character of the intersection form.

I would like to replace Frobenius endomorphism $F$ by a 1-parameter semi-group $e^{tH}$. Then $FF^* = F^*F = q$ should become $[H, H^*] = 0$ and $Re(H) = \frac{1}{2} I$.

The first problem is to find a continuous analogue of the identity

$$(*) \qquad \frac{1}{\det(1-A)} = e^{\sum_1^\infty \frac{1}{k} tr(A^k)}$$

In particular

$$\frac{1}{1-qz} = e^{\sum_{k=1}^\infty \frac{1}{k} q^k z^k}.$$

Now this factor is the first singularity encountered in the curve zeta: $z = q^{-s}$ and $s = 1 \Rightarrow z = \frac{1}{q}$. The corresponding factor in $\zeta(s)$ is $\frac{1}{s-1}$. Hence we want to find $f(t)$ so that

$$\log \frac{1}{s-1} = \int_0^\infty e^{-st} f(t)\, dt$$

Then

$$\frac{1}{s-1} = \int_0^\infty e^{-st}\, t f(t)\, dt \quad \Rightarrow \quad t f(t) = e^{+t}$$

and so $f(t) = \frac{e^t}{t}$. Simpler is

$$\log \frac{1}{s} = \int_0^\infty e^{-st} \frac{1}{t}\, dt$$

Now the Laplace transform of $\frac{1}{t}$ is not defined, one has to define $\frac{1}{t}\theta(t)$ as a distribution, for example

$$\Phi(f) = \int_0^\infty \left[ f(t) - f(0) e^{-t} \right] \frac{dt}{t}$$

4 whence $\Phi$ is a distribution such that $t\Phi = 0$

and $$\Phi(e^{-st}) = \int_0^\infty [e^{-st} - e^{-t}] \frac{dt}{t} = \log \frac{1}{s}$$

It's clear that

$$\log \frac{1}{s-1} = \int_0^\infty e^{-st} \frac{e^t}{t} dt$$

is a continuous analogue of

$$\log \frac{1}{1-qz} = \sum_{m=1}^\infty \frac{1}{m} q^m z^m$$

e.g. if we use the Riemann sum approximation:

$$\int_0^\infty e^{-st} \frac{e^t}{t} dt \sim \sum_{m=1}^\infty e^{-sm\varepsilon} \frac{e^{m\varepsilon}}{m\cancel{\varepsilon}} \cancel{\varepsilon}$$

Thus $q = e^\varepsilon \to 1$ as $\varepsilon \to 0$, and

$$1 - qz = 1 - q^{1-s} = 1 - (e^\varepsilon)^{1-s} = 1 - (1 + \varepsilon(1-s) + O(\varepsilon^2))$$
$$= \varepsilon(s-1) + O(\varepsilon^2)$$

Next we take

$$\frac{1}{\det(1 - q^{-s}F)} = \exp\left\{\sum_{m=1}^\infty \frac{1}{m} q^{-sm} \operatorname{tr}(F^m)\right\}$$

and let $q = e^\varepsilon$, $F = e^{\varepsilon H}$. If we now let $\varepsilon \to 0$ we get

$$\boxed{\frac{1}{\det(s-H)} = e^{\int_0^\infty e^{-st} \operatorname{tr}(e^{tH}) \frac{dt}{t}}}$$

which is very familiar.

5. So now we have found our continuous analogue of ⊛ p. 3 $\frac{1}{\det(1-A)} = \cdots$

[redacted] This formula is ambiguous, because one has to extend the function $\mathrm{tr}(e^{tH})\frac{1}{t}$ to a distribution on $\mathbb{R}$ supported in $[0,\infty)$.

Next let's look at the zeta fn. for $\mathbb{Z} \cup \infty$ namely
$$\pi^{-s/2}\,\Gamma(s/2)\,\zeta(s) = \hat{\zeta}(s)$$
Notice that the asymptotic behavior of
$$\int_0^\infty e^{-st}\,\mathrm{tr}(e^{tH})\,\frac{dt}{t}$$
as $\mathrm{Re}(s) \to +\infty$ is determined by the asymptotics of $\mathrm{tr}(e^{tH})$ at $t \downarrow 0$. As $\zeta \to 1$ fast as $\mathrm{Re}(s) \to \infty$, we look at $\Gamma(s/2)$, or just $\Gamma(s)$.

Thus we want [redacted] $\log \Gamma(s)$ to be represented as a Laplace transform.
$$\frac{1}{\Gamma(s)} = e^{\gamma s}\, s \prod \left(1+\frac{s}{n}\right)e^{-s/n}$$

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \gamma + \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{s+n} - \frac{1}{n}\right)$$

This last is a regularization of the formal sum $\sum_{n=0}^{\infty}\frac{1}{s+n}$ which is the Laplace transform of
$$\sum_{n=0}^{\infty} e^{-nt} = \frac{1}{1-e^{-t}}$$
This is only formal, since $q$ has a simple pole at

[redacted] $t=0$ and hence its L.T. is not defined. However it is defined modulo an additive constant. So we can write

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \mathcal{L}\left\{\frac{1}{1-e^{-t}}\right\} + \text{const}$$

and hence integrating we get

$$\log\Gamma(s) = \mathcal{L}\left\{\frac{1}{(1-e^{-t})t}\right\} + \underset{as+b}{\text{linear fn}}$$

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<u>November 14, 1985</u>

Recall Bernoulli polys.

$$P_k(N) = \sum_{n=0}^{N-1} n^k$$

$$\sum_{k\geq 0} \frac{t^k}{k!} P_k(N) = \sum_{n=0}^{N-1} \sum_{k\geq 0} \frac{n^k t^k}{k!} = \sum_{n=0}^{N-1} e^{nt} = \frac{e^{Nt}-1}{e^t-1}$$

$$= \left(\sum_{l\geq 1} \frac{t^l}{l!} N^l\right)\left(\sum_{m\geq -1}{}' b_m t^m\right)$$

$$\therefore \quad P_k(N) = k! \sum_{l\geq 1} \frac{N^l}{l!} b_{k-l}$$

e.g. $$P_1(N) = \left[N b_0 + \frac{N^2}{2!} b_{-1}\right] = \frac{N^2}{2} - \frac{N}{2} = \tfrac{1}{2}N(N-1)$$

since $$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + b_3 t^3 + b_5 t^5 + \ldots\ldots$$

(Note $$\frac{1}{e^t-1} + 1 = \frac{e^t}{e^t-1} = \frac{1}{1-e^{-t}} \implies \underbrace{\frac{1}{e^t-1} + \frac{1}{2} = \frac{1}{1-e^{-t}} - \frac{1}{2}}_{\text{is odd in } t}$$

Next let's return to the formula

$$\log \Gamma(s) = \underbrace{\mathcal{L}\left\{\frac{1}{1-e^{-t}}\,\frac{1}{t}\right\}}_{\int_0^\infty e^{-st}\left(\frac{1}{1-e^{-t}}\right)\frac{dt}{t}} + as+b$$

One way to regularize this integral is to remove terms of the series

$$\frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + b_1 t^1 + b_3 t^3 + \ldots$$

Thus formally

$$(*) \qquad \log \Gamma(s) = \mathcal{L}\left\{\frac{1}{t^2} + \frac{1}{2t}\right\} + \int_0^\infty e^{-st}\left(\frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2}\right)\frac{dt}{t}$$

where $\mathcal{L}\left(\frac{1}{t}\right) \equiv -\log s$ mod constants

$\mathcal{L}\left(\frac{1}{t^2}\right) \equiv s\log s$ mod $as+b$

(Check:

$$\Gamma(s) = \int_0^\infty e^{-t}\,t^s\,\frac{dt}{t} = s^s\int_0^\infty e^{-st}\,t^s\,\frac{dt}{t} \qquad Re(s)\geq 0$$

$$= s^s\int_{-\infty}^{\infty} e^{-s(e^x-x)}\,dx$$

$$= s^s e^{-s}\underbrace{\int_{-\infty}^{\infty} e^{-s(e^x-1-x)}\,dx}_{\frac{\sqrt{2\pi}}{\sqrt{s}}\cdot\left(1+O\left(\frac{1}{s}\right)\right)} \qquad \text{as } s\to\infty \text{ in } Re(s)\geq 0$$

$$\therefore \log\Gamma(s) = s\log s - s - \tfrac{1}{2}\log s + \log\sqrt{2\pi} + O\left(\tfrac{1}{s}\right)$$

Thus we can see the rest of the asymptotic expansion of $\log \Gamma(s)$ from the formal expression

(*) :

$$\int_0^\infty e^{-st} \underbrace{\left( \frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right)}_{\sum_{k \geq 1} b_k t^k} \frac{dt}{t} \sim \sum b_k \frac{\Gamma(k)}{s^k}$$

$$= \frac{1}{12}\frac{1}{s} + b_3 \frac{\Gamma(3)}{s^3} + b_5 \frac{\Gamma(5)}{s^5} + \cdots$$

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Nov. 17, 1985

Thm. In $\mathrm{Re}(s) > 0$ one has

$$\log \Gamma(s) = s \log s - s - \tfrac{1}{2} \log s + \tfrac{1}{2} \log(2\pi)$$
$$+ \int_0^\infty e^{-st} \left\{ \frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right\} \frac{dt}{t}$$

$$\int_0^\infty e^{-st} \left\{ \frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right\} \frac{dt}{t} \sim \frac{1}{12s} + b_3 \frac{\Gamma(3)}{s^3} + b_5 \frac{\Gamma(5)}{s^5} + \cdots$$

as $|s| \to \infty$.

Proof: Define

$$F(s) = \int_0^\infty e^{-st} \left\{ \frac{1}{1-e^{-t}} - \frac{1}{t} - \frac{1}{2} \right) \frac{dt}{t} + s \log s - s - \tfrac{1}{2} \log s + \tfrac{1}{2} \log(2\pi)$$

This is analytic in $\mathrm{Re}(s) > 0$. We show it has the same second derivative as $\log \Gamma(s)$. Then $\log \Gamma(s) = F(s) + as + b$ and by Stirling's formula for the asymptotics as $s \to \infty$ we see $a = b = 0$.

$$F''(s) = \int_0^\infty e^{-st} \left\{ \frac{1}{1-e^{-t}} - \cancel{\frac{1}{t}} - \cancel{\frac{1}{2}} \right\} t\, dt + \cancel{\frac{1}{s}} - \cancel{\frac{1}{2s^2}}$$

$$= \sum_{n=0}^\infty \frac{1}{(s+n)^2} = \frac{d^2}{ds^2} \log \Gamma(s)$$

Note that this also yields

$$-\frac{\Gamma'(s)}{\Gamma(s)} = \int_0^\infty e^{-st}\left\{\frac{1}{1-e^{-t}} - \frac{1}{t} - \cancel{\frac{1}{2}}\right\} dt - \log s + \cancel{\frac{1}{2s}}$$

hence

$$\boxed{\gamma = -\frac{\Gamma'(1)}{\Gamma(1)} = \int_0^\infty e^{-t}\left\{\frac{1}{1-e^{-t}} - \frac{1}{t}\right\} dt}$$

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Interesting Phenomenon: Notice that

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \left(\theta\left(\tfrac{1}{t^2}\right) - 1\right) t^{-s}\,\frac{dt}{t}$$

$$\Gamma(s) = \int_0^\infty e^{-\frac{1}{t}}\, t^{-s}\,\frac{dt}{t}$$

viewed as [redacted] continuous [redacted] Laurent series do involve all the powers $t^{-s}$ for $0 < t < \infty$, whereas their logarithms involve [redacted] only $t^{-s}$ for $t \geq 1$ e.g.

$$\log \Gamma(s) = \mathcal{L}\left(\frac{1}{1-e^{-t}}\,\frac{1}{t}\right)$$

$$= \int_0^\infty \underbrace{e^{-st}}_{(\underbrace{e^{+t}}_{\geq 1})^{-s}} \cdots\cdots\cdots$$

This is quite strange since one would expect that if $\log \Gamma(s)$ is a continuous power series, then $\Gamma(s)$ should be also a continuous power series.