August 22, 1985

I now want to work on periodicity
$$\Omega \mathcal{F}_1 \sim \mathcal{F}_0 \qquad \text{or} \qquad \Omega U \sim \mathbb{Z} \times BU$$
but on the level of Dirac operators. What I mean by this is roughly as follows. Suppose we consider a family of Dirac operators on an even diml. manifold parametrized by a space Y. The index of this family gives a well-defined homotopy class $Y \longrightarrow \mathbb{Z} \times BU$. Thus Y is an approximation to $\mathbb{Z} \times BU$. By taking more complicated families we might hope to see more of the structure of $\mathbb{Z} \times BU = \mathcal{F}_0$.

What appears to be very significant is the dimension of M. The way this enters is that the index map $Y \longrightarrow \mathcal{F}_0$ assigning to y the Dirac operator corresponding to y will map tangent vectors to operators with some Schatten class property.

So it is time to get more specific. Suppose M is the circle $\mathbb{R}/\mathbb{Z}$. I fix the Riemannian structure on M. A Dirac op. is then given by a vector bundle E over M with inner product and connection preserving the inner product. We fix E and its inner product and let $\mathcal{A}$ be the space of (unitary) connections and $\mathcal{G}$ the group of gauge transformations. Let $\mathcal{H} = L^2(M, E)$. Then $\mathcal{G}$ acts on $\mathcal{H}$ by unitary transformations. To each $A \in \mathcal{A}$ we have D_A self-adjoint on $\mathcal{H}$. Thus we get a map

$$\mathcal{A} \longrightarrow \text{s.a. ops on } \mathcal{H}$$

equivariant with respect to $\mathcal{G}$.

To get even more specific, we trivialize E whence $\mathcal{G} = C^\infty(S^1, U_n)$ = smooth loop group of U_n and

$$D_A = \frac{1}{i}\left(\partial_x + A(x)\right)$$

where $\quad A \in \mathrm{Lie}(\mathcal{G}) = C^\infty(S^1, \mathrm{Lie}\, U_n)$. [redacted]

[redacted]

I've learned from Atiyah + Bott that [redacted] we should think in terms of having a family of elliptic operators on M parametrized by the homotopy orbit space $\mathcal{G}\,\|\,\mathcal{A} \sim B\mathcal{G}$. The index of this family is a map

$$\mathcal{G}\,\|\,\mathcal{A} \longrightarrow \mathcal{F}_1$$

In the case where $M = $ [redacted] S^1, $\mathcal{G} = \mathcal{L}U_n$ and we can identify $B\mathcal{G}$ as follows. $\mathcal{G} = U_n \ltimes \Omega U_n$ and ΩU_n acts freely on $\mathcal{A}$ with quotient U_n given by the monodromy of a connection at the basepoint of S^1. Thus dividing out by ΩU_n gives

$$\mathcal{G}\,\|\,\mathcal{A} \xrightarrow{\ \sim\ } U_n \,\|\, U_n$$

where U_n acts on itself by conjugation

[redacted] Now $\quad U_n \,\|\, U_n = \Delta U_n \,\|\, (U_n \times U_n / \Delta U_n)$

fits into a fibre square

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] & BU_n \arrow[d] \\
BU_n \arrow[r, "\Delta"] & BU_n \times BU_n
\end{tikzcd}$$

whence we know [redacted] that

$$U_r /\!/ U_r \simeq \mathcal{L}(BU_r)$$

(More generally for G connected we have

$$B(\mathcal{L}G) \sim \mathcal{L}(BG) \sim G /\!/ G$$ (for the conj. action).

Now evidently there is a map

$U_r /\!/ U_r \longrightarrow$ [redacted] $\mathcal{F}_1 \sim U.$

It is obtained by the [redacted] Dirac operator machine I have just described. Let's go over the steps. [redacted]

I start with $g \in U_r$ and then choose a connection A with the monodromy g. Then I form the operator $\frac{1}{i}(\partial_x + A)$ which is a self-adjoint Fredholm operator on $L^2(S^1)^{\oplus r}$. [redacted]

[redacted] Now the group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}$ compatible with the conjugation action of U_r. So we have a map

$$\mathcal{A} \longrightarrow \mathcal{F}_1$$

$$\begin{array}{ccc} \mathcal{G} /\!/ \mathcal{A} & \longrightarrow & \tilde{U} /\!/ \mathcal{F}_1 \\ \wr & & \wr \\ U_r /\!/ U_r & & \mathcal{F}_1 \end{array}$$

$\tilde{U}$ unitary gp which is contractible by Kuiper's thm.

So we see ultimately that the map needs Kuiper's thm. But then we have the following simpler way to proceed, namely, we take the embedding $U_r \subset U$ which is equivariant for the conjugation action of U_r and of $\tilde{U}$; so we get

$$U_1 \| U_r \longrightarrow \tilde{U} \| U \sim U$$

Now so far I haven't used any periodicity, except possibly $\Omega BG \sim G$, $B(\Omega G) \sim G$. So we next want to look at analogues of the periodicity map $\Omega \mathcal{F}_1 \to \mathcal{F}_0$.

Thus I want to take a loop of Dirac operators on the circle and associate something like a Fredholm operator. There are two ways I see to do this.

1) To a loop of Dirac operators on S^1 one can associate a Dirac operator over the 2-torus.

2) Since $\mathcal{A}$ is contractible and our model for the space of Dirac ops is $\mathcal{G} \| \mathcal{A} \sim B\mathcal{G}$, the loop space of $\mathcal{G} \| \mathcal{A}$ is equivalent to $\mathcal{G}$. Then we have the Toeplitz operator construction $\mathcal{G} \longrightarrow \mathcal{F}_0$.

It seems worthwhile to relate these two things. They are two ways of assigning to a gauge transformation a Fredholm operator. Better, we have two maps from $\mathcal{G}$ to (different versions of) the space of Fredholm operators.

Let's describe 1). Fix a basepoint A_0 of $\mathcal{A}$ so that we get a map $\mathcal{G} \longrightarrow \mathcal{A}$. Then we can use the linear path from A_0 to $g^* A_0$ to get a path A_t enabling us to form the 2diml

Dirac operator $\partial_t + \frac{1}{i}(\partial_x + A_t)$. Now what? We have produced a vector bundle over $S^1 \times S^1$ by a clutching construction.

Next we consider 2). Given a gauge transformations we form the Toeplitz operator ege where e is projection on a half-space. It might be more reasonable to think in terms of the map $g \longmapsto geg^{-1}$ from the loop group to an infinite-dimensional Grassmannian.

The real problem is to relate these two maps.

August 23, 1985

Today's work: I began with the problem of linking two Fredholm operators belonging to a loop of ~~the~~ Dirac operators on the circle.

Actually we have to consider a bundle E over $Y \times M$ $(Y = S^1, M = S^1)$ with a connection. Then we have the 2-diml Dirac operator. Also by means of parallel transport in the Y direction we get a gauge transformation, essentially a clutching function for E. This gauge transf. determines a Fredholm operator, by Toeplitz's construction.

Next I remembered Graeme's formulas

$$\det(\partial_t + A) = \det(M - I)$$

$$\operatorname{index}(\partial_t + A) = \operatorname{index}(M - I)$$

and that he wanted to do adiabatic approx. on the latter ~~to~~ prove spectral flow = index of $\partial_t + A$.

This suggested abstracting to the case of a loop in $\mathcal{F}_1$; call it $A = A(t)$, and relating

$$A \longmapsto \partial_t + A$$

to the periodicity map $\Omega\mathcal{F}_1 \sim \mathcal{F}_0$. This led to a digression thinking about Kasparov cup products. The two periodicity maps are given by such cup products. In particular given $A \in \Omega\mathcal{F}_1$, this should represent an element of $KK^1(\mathbb{C}, C^0(S^1))$. We are then supposed to bring in the Dirac operator on S^1; in its ψDO form it is the Hilbert operator

$$\frac{\frac{1}{i} d_t}{|d_t|}$$

So forming $\partial_t + A$ should be essentially like taking the Kasparov product.

I thought about taking the Kasparov product of two odd operators to get an even operator. In particular why is the product of the Bott classes in $K^{+1}(\mathbb{R})$ and $K^{+1}(\mathbb{R})$ the Bott class in $K^0(\mathbb{C})$.

The point is that given two graded C_1- modules their tensor product is a graded C_2-module which is equivalent to a graded vector space. Specifically a graded C_1-module is of the form $S_2 \otimes V$, and we tensor with another $S_2 \otimes W$ (here V, W are vector spaces). Then we get the graded C_2 module $(S_2 \otimes S_2) \otimes V \otimes W$ which is equivalent to $S_2 \otimes (V \otimes W)$. Given endos a, b of V, W the corresponding endo of $S_2 \otimes (V \otimes W)$ is

$$a\gamma^1 + b\gamma^2 = \begin{pmatrix} & a-ib \\ a+ib & \end{pmatrix}$$

where $a = a \otimes 1$, $b = 1 \otimes b$ on $V \otimes W$.

August 24, 1985

I want to go through the Kasparov cup product in a special case. I want to construct a map $\Omega\mathcal{F}_1 \longrightarrow \mathcal{F}_0$ which is part of the periodicity thm. $\mathcal{F}_1$ is the space of self-adjoint ops F on Hilbert space such that $F^2 - I \in \mathcal{K}$ from which we have removed those F such that $F-I$ or $F+I \in \mathcal{K}$. According to A-S the exponential map $F \longmapsto -\exp(i\pi F)$ is a homotopy equivalence of $\mathcal{F}_1$ with $\mathcal{U}$ = the group of unitaries congruent to I mod $\mathcal{K}$. $\mathcal{F}_0$ is the space of essentially unitary operators (i.e. T such that T^*T and TT^* are $\equiv I$ mod $\mathcal{K}$).

I recall roughly the definition of $KK(A,B)$. Here A, B are supposed to be C^*-algebras. An element of $KK(A,B)$ is represented by a right C^*-module M over B which is also a left $*$-A-module together with an F such that

$$a(F^2 - I) \in \mathcal{K}_B \qquad \forall a \in A$$
$$[F, a] \in \mathcal{K}_B \qquad \text{"} \quad .$$

A C^*-module over B is a module equipped with an inner product with values in B (roughly); one can then define $\mathcal{K}_B$ as the closure of linear combinations of operators of the form $|m\rangle\langle m'|$. The typical example is given by taking a bundle of Hilbert spaces $\mathcal{H}_y$ $y \in Y$ over a space Y; then $\Gamma(Y, \{\mathcal{H}_y\})$ is a C^*-module over $C(Y)$, and $\mathcal{K}_B$ is the family of operators commuting with B which are compact in each fibre + are a continuous family.

Examples: 1) $KK^0(\mathbb{C}, A) = K_0(A)$. An elt. of $KK(\mathbb{C}, A)$ is represented by a Fredholm F acting on a (Hilbert) C^*-module over A, and F commutes with A. Geometrically: $A = C(X)$ and one has a family of Fredholms parametrized by X.

2) $KK(A, \mathbb{C})$. An element of this is represented by a Hilbert space $\mathcal{H}$ with A-module structure and an $F \ni [F, a] \in \mathcal{K}$ for all a. Geometrically $A = C(X)$, $\mathcal{H} \simeq L^2(X, E)$, and F is a ψDO of order zero with $F^2 - I$ of order -1.

Now let us take up the Kasparov product in the special case of interest. I have ~~first~~ of all a loop in $\mathcal{F}_1$, denoted $A(t)$, $t \in S^1$. This represents an elt. of

$$KK^1(\mathbb{C}, C(S^1))$$

as follows. We have the trivial Hilbert space bundle over S^1 and can ~~let~~ $\mathcal{M} = \Gamma(S^1, S^1 \times \mathcal{H})$ $= C(S^1, \mathcal{H})$. This consists of all continuous maps from S^1 to Hilbert space with pointwise inner product. Clearly the family $A(t)$ gives an F on $\mathcal{M}$ commuting with $C(S^1)$.

To construct an example of a loop in $\mathcal{F}_1$ we can start with the family of s.a. operators on $L^2(S^1)$

$$\frac{1}{i}\partial_x + t \qquad 0 \leq t \leq 2\pi$$

This is a loop if one uses the gauge transformation of mult. by $e^{2\pi i x}$ to go between $t = 0$, $t = 2\pi$.

These operators are unbounded, but we can make bounded ones by dividing by $\sqrt{1-(\partial_x+it)^2}$. Or passing to ℓ^2 via the F.T. we consider the multip. operators given by the functions

$$\frac{k+t}{\sqrt{1+(k+t)^2}} \qquad 0 \le t \le 2\pi$$

together with the shift.

The second K-class is the element of

$$KK^1(C(S^1), \mathbb{C})$$

represented by a ΨDO on $L^2(S^1)$ of order zero having the symbol $\operatorname{sgn}(k)$. In other words a version of the Hilbert transform. For example multiplication by $\begin{cases} 1 & k \ge 0 \\ -1 & k < 0 \end{cases}$ in the F.T. picture or

$$\frac{\frac{1}{i}\partial_t}{\sqrt{1-\partial_t^2}}.$$

When we take the Kasparov product we first form the tensor product

$$\mathcal{M} \otimes_{C(S^1)} L^2(S^1)$$

It's clear this is supposed to be (possibly after completion) the Hilbert space

$$\mathcal{H} \otimes L^2(S^1) \simeq L^2(S^1, \mathcal{H})$$

of L^2 functions on S^1 with values in $\mathcal{H}$.

To keep [redacted] ideas fixed let's think of $\mathcal{H}$ as being $L^2(S^1)$ with [redacted] this S^1 having the coordinate x. Then the Hilbert space of our Kasparov product is $L^2(S^1 \times S^1)$, and from the viewpoint of the example discussed above of $A(t)$, we "really" have $L^2(S^1 \times S^1, E)$ for some vector bundle E on $S^1 \times S^1$. (But maybe it is simpler to take a product setup, say where $A(t)$ is constant. No: you have to take the Bott class to get something interesting.)

[redacted] I want to get the simplest example possible. There are two simple models, [redacted] which are unbounded.

1) harmonic oscillator. Here the loop is represented by the trivial Hilbert line bundle over $\mathbb{R}$ with the s.a. operator of multiplication by x. The Dirac operator on $\mathbb{R}$ (which represents the class in $KK(C_c(\mathbb{R}), \mathbb{C})$) is $\frac{1}{i}\partial_x$. Put these together to get

$$\begin{pmatrix} 0 & \frac{1}{i}\partial_x - ix \\ \frac{1}{i}\partial_x + ix & 0 \end{pmatrix}$$

(this x should be t)

which gives essentially the Fredholm operator $\partial_x + x$ on $L^2(\mathbb{R})$, and its adjoint. The operator $\partial_x + x$ has index 1.

2) Hopf line bundle over $S^1 \times S^1$. Here we combine [redacted] $\frac{1}{i}\partial_t$ with $\frac{1}{i}\partial_x + t$ to get

$$\partial_t + \frac{1}{i}\partial_x + t$$

which is a Dirac operator over $S^1 \times S^1$. Again get the index ± 1.

It is curious that these two operators are *isomorphic*.

Let's now try to understand the construction of the Kasparov tensor product. YUCK

August 25, 1985:

$$F(x,y) = \sum_n e^{2\pi i n y} f(x-n) \qquad f \in \mathcal{S}(\mathbb{R})$$

$$F(x,y+1) = F(x,y)$$

$$F(x+1,y) = e^{2\pi i y} F(x,y)$$

$$f(x) = \int_0^1 F(x,y)\,dy$$

Put

$$G(x,y) = \cancel{\sum_m e^{2\pi i m x} g(y+m)}\; e^{-2\pi i x y} F(x,y)$$

$$G(x+1,y) = G(x,y)$$

$$G(x,y+1) = e^{-2\pi i x} G(x,y)$$

so

$$G(x,y) = \sum_m e^{2\pi i m x} g(y+m)$$

$$\begin{aligned} g(y) &= \int_0^1 G(x,y)\,dx \\ &= \int_0^1 e^{-2\pi i x y} \sum_n e^{2\pi i n y} f(x-n)\,dx \\ &= \int_0^1 \sum_n e^{-2\pi i (x-n) y} f(x-n)\,dx \\ &= \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x)\,dx = \hat{f}(y) \end{aligned}$$

Thus we have the formula

$$\boxed{\sum_n e^{2\pi i n y} f(x-n) = e^{2\pi i x y} \sum_m e^{2\pi i m x} \hat{f}(y+m)}$$

which is essentially the Poisson summation formula, e.g. if $x = y = 0$, then

$$\sum_n f(n) = \sum_m \hat{f}(m)$$

Recall that one ~~reason to~~ use of Gaussians is to compute the volume of spheres:

$$1 = \left(\int e^{-\pi x^2} dx\right)^n = \int_0^\infty e^{-\pi r^2} r^{n-1} \operatorname{vol}(S^{n-1}) dr$$

$$= \operatorname{vol}(S^{n-1}) \int_0^\infty e^{-\pi r^2} r^n \frac{dr}{r} = \operatorname{vol}(S^{n-1}) \frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}$$

$$\therefore \boxed{\operatorname{vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}}$$

However we recall that $\pi^{-s/2}\Gamma(s/2)$ is what one multiplies $\zeta(s)$ by to get something symmetric under $s \mapsto 1-s$. So

$$\frac{\zeta(s)}{\operatorname{vol}(S^{s-1})}$$

is the [redacted] zeta of $\mathbb{Z} \cup \{\infty\}$

Question: Are the other factors of ζ related to p-adic volumes?

In $\mathbb{Q}_p$ the sphere $\{x \mid |x| = 1\}$ is $\mathbb{Z}_p - p\mathbb{Z}_p$ which is the union of $(p-1)$ cosets of $p\mathbb{Z}_p$. Thus if $\mathbb{Z}_p$ has volume 1, the volume of $\{x \mid |x| = 1\} = 1 - \frac{1}{p}$

Let's equip $\mathbb{Q}_p^n$ with the norm

$$\|\vec{x}\| = \max |x_j|$$

whence $\{\vec{x} \mid \|\vec{x}\| \le 1\} = \mathbb{Z}_p^n$ and $\{x \mid \|x\| < 1\} = (p\mathbb{Z}_p)^n$ and the "sphere" is

$$S^{n-1} = \{\vec{x} \mid \|\vec{x}\| = 1\} = \mathbb{Z}_p^n - (p\mathbb{Z}_p)^n$$

This has volume $1 - \frac{1}{p^n}$.

Thus we see the factors

$$\zeta(s) = \prod \frac{1}{\left(1 - \frac{1}{p^s}\right)}$$

in the Euler product expansion do seem to be related to volume of spheres p-adically.

We learn that s in $\zeta(s)$ is closely connected to <u>dimension</u>.

One big mystery in the ζ function business is why one should want to integrate the Θ-function

$$\Theta(t) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$$

over $0 < t < \infty$. From the elliptic curve viewpoint we are integrating over the part of the moduli space where the period $\tau = it$ is purely imaginary. From the harmonic oscillator viewpoint we are integrating $e^{-\pi t x^2}$ for $t > 0$, which is like looking at the ground states for different frequencies.

However, note

$$(*) \qquad \int_0^\infty e^{-\pi t x^2} t^{\frac{s}{2}} \frac{dt}{t} = \frac{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}{|x|^s}$$

and as the F.T. of $e^{-\pi t x^2}$ is

$$\int e^{-2\pi i x y - \pi t x^2} dx = \int e^{-\pi t (x + \frac{i}{t} y)^2 - \frac{\pi}{t} y^2} dx$$

$$= \frac{e^{-\frac{\pi}{t} y^2}}{\sqrt{t}}$$

We see that the F.T. of $(*)$ is

$$\int_0^\infty e^{-\frac{\pi}{t} y^2} t^{\frac{s-1}{2}} \frac{dt}{t} = \int_0^\infty e^{-\pi t y^2} t^{\frac{1-s}{2}} \frac{dt}{t}$$

$$= \frac{\pi^{-\frac{(1-s)}{2}} \Gamma\left(\frac{1-s}{2}\right)}{|y|^{1-s}}$$

These formulas are true without problems if $0 < \mathrm{Re}(s) < 1$ otherwise one has to somehow resort to distribution theory.

August 26, 1985

[redacted] If we let μ be the Haar measure on $\mathbb{Q}_p^\times$ normalized so that $\mu(\mathbb{Z}_p^\times) = 1$, then

$$\int_{\mathbb{Q}_p^\times} \chi_{\mathbb{Z}_p}(x)\, |x|^s \, d\mu = \sum_{n \geq 0} (p^{-n})^s \underbrace{\mu(p^n \mathbb{Z}_p^\times)}_{1}$$

$$= \frac{1}{1-p^{-s}}$$

in exact analogy with

$$\int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{x} = \pi^{-s/2} \Gamma(s/2)$$

Idea: We know that

$$\theta\left(\tfrac{1}{t}\right) = \sum_{n \in \mathbb{Z}} e^{-\frac{\pi}{t} n^2}$$

is supposed to be analogous to $q^{h^0(L)}$; [redacted] because the functional equation

$$\theta(t) = \frac{1}{\sqrt{t}}\, \theta\left(\tfrac{1}{t}\right) \quad \text{or} \quad \frac{\theta\left(\frac{1}{t}\right)}{\theta(t)} = \sqrt{t}$$

is the analogue of R.R.

$$q^{h^0(L) - h^1(L)} = q^{1-g+\deg L)}$$

Thus a natural thing to ask [redacted] would be whether $\log \theta\left(\frac{1}{t}\right)$ can be interpreted as some kind of dimension. Note that

$$\theta\left(\tfrac{1}{t}\right) > 1 \quad \text{so that} \quad \log \theta\left(\tfrac{1}{t}\right) > 0$$

Now in order to interpret $\log \Theta(1/t)$ as a dimension we need a setting where dimensions are real numbers. This suggests type II von Neumann algebras.

Let's review a little of Araki-Woods. Let's take the boson gas which single particles are described by $L^2(V)$, where $V = \mathbb{R}^n/L\mathbb{Z}^n$. This has the orthonormal basis $|k\rangle = \frac{1}{\sqrt{V}} e^{ik\cdot x}$, $k \in \frac{2\pi}{L}\mathbb{Z}^n$, and we suppose the energy diagonal in this basis with eigenvalues ε_k. Form the boson Fock space and consider the state describing the gas at inverse temperature β. On Fock space we have the unitary operators

$$T_f = e^{a^*(f) - a(f)} = e^{-\frac{1}{2}\|f\|^2} e^{a^*(f)} e^{-a(f)}$$

$$a^*(f) = \sum c_k a_k^* \quad , \quad a(f) = \sum \bar{c}_k a_k$$

where $f \in L^2(V)$ and $c_k = \langle k|f\rangle$. The thermal state gives the generating fn.

$$\frac{\operatorname{tr}(e^{-\beta H} T_f)}{\operatorname{tr}(e^{-\beta H})} \qquad H = \sum \varepsilon_k a_k^* a_k$$

Recall the s.h.o. formula

$$\frac{\operatorname{tr}(e^{-\omega a^* a} e^{ca^* - \bar{c}a})}{\operatorname{tr}(e^{-\omega a^* a})} = e^{-\frac{1}{2}|c|^2 \frac{e^{\omega}+1}{e^{\omega}-1}}$$

$$= e^{-|c|^2 \left\{ \frac{1}{2} + \frac{1}{e^{\omega}-1} \right\}}$$

Thus we have

$$\log\left(\frac{tr(e^{-\beta H}T_f)}{tr(e^{-\beta H})}\right) = -\sum_k |c_k|^2 \left\{\frac{1}{2} + \frac{1}{e^{\beta\varepsilon_k}-1}\right\}$$

$$c_k = \langle k|f\rangle = \frac{1}{\sqrt{V}}\int_V e^{-ikx} f(x)\,dx = \frac{1}{\sqrt{V}}\hat{f}(k)$$

Thus in the $V \to \infty$ limit we find

$$\boxed{\frac{tr(e^{-\beta H}T_f)}{tr(e^{-\beta H})} = e^{-\int \frac{d^n k}{(2\pi)^n} |\hat{f}(k)|^2 \left\{\frac{1}{2} + \frac{1}{e^{\beta\varepsilon_k}-1}\right\}}}$$

August 27, 1985

I would like to make an effort to get a good picture of Gaussian integrals, both boson and fermion, in infinite dimensions. I have the feeling that in the cases where these make sense one is employing a sort of Eisenstein summation process. If this feeling could be properly expressed it might be important.

Let's consider the path integral behind the simple harmonic oscillator. I want to work with a^*, a so that q, p are on an equal footing to begin with. So $H = \omega a^* a$. The path integral is supposed to yield Schwinger's generating function

$$\langle 0 | S_J | 0 \rangle$$

where S_J is the scattering operator associated to the perturbation

$$H_J(t) = H - J(t) a^* - \bar{J}(t) a.$$

We have calculated that

$$\langle 0 | S_J | 0 \rangle = \exp \left\{ \iint_{t>t'} \bar{J}(t)\, e^{-\omega(t-t')} J(t')\, dt\, dt' \right\}$$

is the generating function of a Gaussian integral with variance

$$\langle 0 | T[a(t)\, a(t')] | 0 \rangle = \begin{cases} e^{-\omega(t-t')} & t > t' \\ 0 & t < t' . \end{cases}$$

This is the Green's fn. for $\partial_t + \omega$; we see that $\langle 0 | S_J | 0 \rangle$ is the same as the Gaussian integral

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\int \bar{\psi}(\partial_t+\omega)\psi\,dt \;+\; \int(\bar{\psi}J + \bar{J}\psi)\,dt}$$

[redacted] Some remarks: The above-path integral is supposed to be taken over complex-valued path $\psi(t)$ with $\bar{\psi}(t)$ = conjugate of ψ. The exponent is not real: $\partial_t+\omega$ has the eigenvalues $ik+\omega$, k real. The above integral is not a [redacted] countably additive integral on the Hilbert space $L^2(\mathbb{R})$, since $\frac{1}{\partial_t+\omega}$ is not of trace class.

Next we should consider the fermion analogue of the preceding. Here $H = \omega a^* a$ acting on 2-diml spin space. We [redacted] now consider a pertubation

$$H_J = H - a^* J(t) - \tilde{J}(t)\,a$$

where $J(t)$, $\tilde{J}(t)$ are Grassman quantities anti commuting with a^*, a. Then

$$S_J = T\left\{ e^{\int (a^* e^{\omega t} J(t) \,+\, e^{-\omega t}\tilde{J}(t)\,a)\,dt} \right\}$$

[redacted] $\quad [\tilde{J}a, a^*J] = \tilde{J}\,[a, a^*]\,J = \tilde{J}J$

so

$$\langle 0 | S_J | 0 \rangle = \exp\left\{ \iint_{t>t'} \tilde{J}(t)\, e^{-\omega(t-t')} J(t')\,dt'\,dt \right\}$$

So it would appear that $\langle 0|S_J|0\rangle$ coincides with the fermionic Gauss integral

$$\int \mathcal{D}\tilde{\psi}\,\mathcal{D}\psi \; e^{-\int \tilde{\psi}(\partial_t+\omega)\psi\,dt \;+\; \int(\tilde{\psi}J + \tilde{J}\psi)\,dt}$$

Let's recall a little about the theory of Gaussian integrals. First the algebraic theory. Suppose we are given a fermion Gaussian integral

$$\int d\psi \, e^{\frac{1}{2}\psi^t \omega \psi} \quad ?$$

where ω is non-degenerate and the integral is normalized so that $\int 1 = 1$.
Here the integral is a linear functional $\Lambda V \to k$ where V is spanned by the ψs. What happens is that ω determines an element $\omega^{-1} \in \Lambda^2(V^*)$ and the integral is essentially pairing with $e^{-\frac{1}{2}\omega^{-1}} \in \Lambda(V^*)$.

So algebraically a normalized fermion Gaussian integral is given by $\omega^{-1} \in \Lambda^2(V^*)$, i.e. a skew form on V. This form is then extended to ΛV by "Wick" contraction. Similarly a boson Gaussian integral is given by a quadratic form $\omega^{-1} \in S^2(V^*)$ and is extended to a linear fnl. on $S(V)$.

Now in the boson case there is a natural Hilbert space, a Fock space, around. Let's start with boson Fock space generated by operators a_k^*, a_k and a cyclic vector $|0\rangle$ satisfying the usual identities. The Fock space is the Hilbert space $S(V)$ where V is the Hilbert space with orthonormal basis $a_k^*|0\rangle$. We have the space of s.a. ops $a^*(v) + a(v)$, $v \in V$ which is naturally a real vector space with a skew form given by commutator. Actually V is a complex Hilbert space and the symplectic form is the imaginary part of $\langle v|w\rangle$. So a real reduction of V gives a commuting family of operators, and $|0\rangle$ is cyclic, hence we end up with a probability measure space.

Suppose we try the same thing fermionically. We consider fermion Fock space ΛV and the operators $a^*(v)+a(v)$. These generate a Clifford algebra. These form a real vector space, in fact Hilbert space with norm

$$\left(a^*(v)+a(v)\right)^2 = \|v\|^2 .$$

It would be nice to find a commutative (anti-) subspace, but there are no isotropic subspaces for this quadratic form except 0. [redacted] Because of this we can't hope to find a fermion analogue of the countably-additive Gaussian measure.

August 28, 1985

$\mathcal{A}_\theta$ = smooth rotation algebra. It consists of series $\sum a_{mn} U^m V^n$ where $UVU^{-1}V^{-1} = e^{-2\pi i\theta}$ and a_{mn} is rapidly decreasing. It is convenient to think of $\mathcal{A}_\theta$ as an algebra of operators on $\mathcal{S}(\mathbb{R})$ where

$$U = e^{2\pi i x}$$

$$V = e^{\theta\partial_x}$$

Then a typical element of $\mathcal{A}_\theta$ is of the form

$$\sum f_n(x)\, e^{n\theta\partial_x}$$

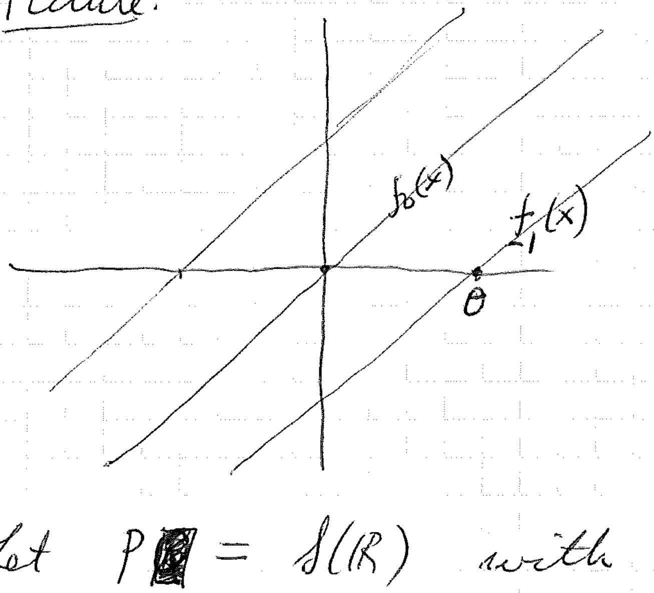
where $f_n(x+1) = f_n(x)$ are smooth and decreases rapidly to zero as $|n| \to \infty$. Such an operator has the Schwartz kernel

$$\sum f_n(x)\langle x|e^{n\theta\partial_x}|y\rangle = \sum f_n(x)\,\delta(x+n\theta-y)$$

since

$$e^{n\theta\partial_x} f(x) = f(x+n\theta) = \int \delta(x+n\theta-y) f(y)\,dy.$$

Picture:



Let $P = \mathcal{S}(\mathbb{R})$ with left $\mathcal{A}_\theta$ and right $\mathcal{A}_{\theta^{-1}}$ module structure

$$U = e^{2\pi i x} \qquad\qquad U' = e^{2\pi i \theta^{-1} x}$$

$$V = e^{\theta\partial_x} \qquad\qquad V' = e^{\partial_x}$$

and let $Q = \mathcal{S}(\mathbb{R})$ with left $\mathcal{A}_{(-\theta^{-1})}$ and right $\mathcal{A}_\theta$ structure given by

$$U' = e^{2\pi i\theta^{-1}x} \qquad U = e^{2\pi i x}$$

$$V' = e^{-\partial_x} \qquad V = e^{-\theta\partial_x}$$

Then we define

$$P \otimes_{\mathcal{A}_{(-\theta^{-1})}} Q \longrightarrow \mathcal{A}_\theta$$

$$f \otimes g \longmapsto \sum_{m,n} f(x+m)\, g(x+m+n\theta)\, e^{n\theta\partial_x}$$

Claim well-defined:
$$(fV', g) \longmapsto \sum f(x+m+1)\, g(x+m+n\theta) e^{n\theta\partial_x}$$
$$(f, V'g) \longmapsto \sum f(x+m)\, g(x+m+n\theta-1) e^{n\theta\partial_x}$$

$$fU', g \qquad f(x+m) e^{2\pi i\theta^{-1}(x+m)}\, g(x+m+n\theta)\, e^{n\theta\partial_x}$$

$$f, U'g \qquad f(x+m)\, e^{2\pi i\theta^{-1}(x+m+n\theta)}\, g(x+m+n\theta) e^{n\theta\partial_x}$$

Compatible with left $\mathcal{A}_\theta$-mult.

$$e^{+\theta\partial_x} \sum f(x+m)\, g(x+m+n\theta)\, e^{n\theta\partial_x} = \sum f(x+m+\theta)\, g(x+m+(n+1)\theta)\, e^{(n+1)\theta\partial_x}$$
$$= \sum (e^{+\theta\partial_x} f)(\quad)\, g(\quad) \text{''}.$$

and right $\mathcal{A}_\theta$ - mult

$$\left(\sum f(x+m)\, g(x+m+n\theta)\, e^{n\theta\partial_x}\right) e^{+\theta\partial_x} = \sum f(x+m) \underbrace{g(x+\theta+m+n\theta)}_{(gV)(x+m+n\theta)} e^{n\theta\partial_x}$$

etc.

If we identify $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$ with $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ then the map

$$P \otimes Q \longrightarrow \mathcal{A}_\theta$$

$$\cancel{K(x,y) \longmapsto \sum_{m,n} K(x+m, x+m+n\theta)\,\delta(x+n\theta-y)}$$

sends $K(x,y)$ to its restriction to the lines $x-y \in \theta\mathbb{Z}$ and then it sums over the diagonal action of $\mathbb{Z}$. Also can do this in the reverse order

$$K(x,y) \longmapsto \sum_m K(x+m, y+m)$$

$$\longmapsto \sum_{m,n} K(x+m, x+n\theta+m)\,\delta(x+n\theta-y)$$

From this I recall expressing $1 \in \mathcal{A}_\theta$ as a finite sum of kernels $f_i \otimes g_i$, which implies P is finite projective over $\mathcal{A}_\theta$.

Now think of $\mathcal{S}(\mathbb{R})$ as a left $\mathcal{A}_\theta$ as the analogue of the sections of the degree 1 line bundle over $S^1\times S^1$. Recall the isomorphism

$$\mathcal{S}(\mathbb{R}) \longmapsto \Gamma(S^1\times S^1, L)$$

$$f(x) \longmapsto F(x,y) = \sum e^{2\pi i m y} f(x-m)$$

$$\begin{cases} F(x,y+1) = F(x,y) \\ F(x+1,y) = e^{2\pi i y} F(x,y) \end{cases}$$

$$\partial_x f(x) \longmapsto \partial_x F(x,y)$$

$$x\, f(x) \longmapsto \left(x - \frac{1}{2\pi i}\partial_y\right) F(x,y)$$

$$(\partial_x + 2\pi x) f(x) \longmapsto (\partial_x + i\partial_y + 2\pi x) F(x,y)$$

One sees the $\bar\partial$ operator on $\Gamma(S^1\times S^1, L)$ corresponds to $\partial_x + 2\pi x$ acting on $\mathcal{S}(\mathbb{R})$. Note that $\partial_x + 2\pi x$ doesn't commute with ~~multiplication~~ the operators $e^{2\pi i x}, e^{\theta\partial_x}$, but does modulo lower order (constants in this case.)

It would appear one has a Kasparov, (really Atiyah) operator. We have a Hilbert space which is a module over $\mathcal{A}_\theta$ and a Fredholm operator $\partial_x + 2\pi x$ which commutes with elements of $\mathcal{A}_\theta$ up to lower order.

By KK theory one gets a map $K_0(\mathcal{A}_\theta) \longrightarrow \mathbb{Z}$ which probably agrees with the torus case.

Actually we can consider $\partial_x + 2\pi x$ as defining a K-homology class for the Weyl algebra. ∎

August 29, 1985

Let's consider the free fermion gas belonging to the 1-particle space $L^2(\mathbb{R}/L\mathbb{Z})$ with $H = \frac{1}{i}\partial_x$. We have the orthonormal basis $\langle x|k\rangle = \frac{1}{\sqrt{L}}e^{ikx}$, $k \in \frac{2\pi}{L}\mathbb{Z}$ with energies $H|k\rangle = \varepsilon_k|k\rangle$ where $\varepsilon_k = k$. Let μ be a chemical potential.

First I have to review how to handle μ and for this purpose let's simplify and suppose $\varepsilon_k = k^2$, so that there are only finitely many states [redacted] with energy below a given bound. The Hilbert space of the gas is $\Lambda L^2(\mathbb{R}/L\mathbb{Z})$ and the Hamiltonian is $\tilde{H} = \sum(\varepsilon_k - \mu)a_k^* a_k$. The partition function is

$$\operatorname{tr}(e^{-\beta\tilde{H}}) = \prod_k \left(1 + e^{-\beta(\varepsilon_k - \mu)}\right)$$

If we let $\beta \to \infty$, i.e. $T \to 0$, then we see that

$$\operatorname{tr}(e^{-\beta\tilde{H}}) \sim e^{-\beta\sum_{\varepsilon_k<\mu}(\varepsilon_k-\mu)}$$

so the ground state [redacted] is the line in $\Lambda L^2(\mathbb{R}/L\mathbb{Z})$ belonging to the subspace spanned [redacted] $|k\rangle$ with $\varepsilon_k < \mu$. This is the Fermi sea; μ is adjusted so as to obtain the correct N. The ground energy is

$$\sum_{\varepsilon_k<\mu} \varepsilon_k - \mu = \sum_{\varepsilon_k<\mu} \varepsilon_k - N\mu$$

This becomes negatively infinite as $V \to \infty$, but the density

$$\int_{\varepsilon_k<\mu} \frac{dk}{2\pi}\varepsilon_k - \rho\mu$$

makes sense.

So lets now consider the case $\varepsilon_k = k$. Then we form the Fock space $\Lambda(\mathcal{H}^-)^* \otimes \Lambda \mathcal{H}^+$ where $\mathcal{H}^\pm \subset L^2(\mathbb{R}/L\mathbb{Z})$ are the subspaces when $k > \mu$ and $k < \mu$ resp. The partition function is

$$\prod_{k>\mu} \left(1 + e^{-\beta(k-\mu)}\right) \prod_{k<\mu} \left(1 + e^{-\beta(\mu-k)}\right)$$

It appears that we have arranged the ground energy to be zero. Thus we have taken the formal Hamiltonian

$$\tilde{H} = H - \mu N = \sum (\varepsilon_k - \mu) a_k^* a_k$$

and renormalized it by normal ordering

$$: \sum (\varepsilon_k - \mu) a_k^* a_k : \;=\; \sum_{\varepsilon_k > \mu}{}' (\varepsilon_k - \mu) a_k^* a_k$$

$$+ \sum_{\varepsilon_k < \mu}{}' (\mu - \varepsilon_k) a_k a_k^*$$

Thus we change $\tilde{H}$ by the "constant"

$$\sum_{\varepsilon_k - \mu < 0}{}' (\varepsilon_k - \mu)$$

$\blacksquare$

Another way to obtain a partition function would be to define $H = :\sum' \varepsilon_k a_k^* a_k:$ and $N = :\sum a_k^* a_k:$ relative to a fixed splitting $\mathcal{H}^+ \oplus \mathcal{H}^-$, and then form

$$tr\left(e^{-\beta(H-\mu N)}\right) = \prod_{k<0} \left(1 + e^{+\beta(\varepsilon_k - \mu)}\right) \prod_{k>0} \left(1 + e^{-\beta(\varepsilon_k - \mu)}\right)$$

Now that I have some idea about the partition function I can let $V \longrightarrow \infty$. We start with the grand partition fn.

$$\mathcal{Z} = \operatorname{tr} e^{-\beta(H-\mu N)} = \prod_{k\leq 0}\left(1+e^{\beta(k-\mu)}\right)\prod_{k>0}\left(1+e^{-\beta(k-\mu)}\right)$$

Then

$$p = \lim \frac{\log \mathcal{Z}}{\beta V} = \int_{-\infty}^{0}\frac{dk}{2\pi\beta}\log\left(1+z^{-1}e^{\beta k}\right) + \int_{0}^{\infty}\frac{dk}{2\pi\beta}\log\left(1+ze^{-\beta k}\right)$$

$$\rho = z\frac{\partial}{\partial z}\frac{\log\mathcal{Z}}{V} = \int_{-\infty}^{0}\frac{dk}{2\pi}\,\frac{-z^{-1}e^{\beta k}}{1+z^{-1}e^{\beta k}} + \int_{0}^{\infty}\frac{dk}{2\pi}\,\frac{ze^{-\beta k}}{1+ze^{-\beta k}}$$

$$\rho = \int_{0}^{\infty}\frac{dx}{2\pi\beta}\underbrace{\frac{-z^{-1}e^{-x}}{1+z^{-1}e^{-x}}}_{\partial_x \log(1+z^{-1}e^{-x})} + \int_{0}^{\infty}\frac{dx}{2\pi\beta}\underbrace{\frac{ze^{-x}}{1+ze^{-x}}}_{-\partial_x \log(1+ze^{-x})}$$

$$= \frac{1}{2\pi\beta}\left(-\log(1+z^{-1}) + \log(1+z)\right) = \frac{1}{2\pi\beta}\log\left(\frac{1+z}{1+z^{-1}}\cdot\frac{z}{z}\right)$$

$$\boxed{\rho = \frac{1}{2\pi\beta}\log z = \frac{\mu}{2\pi}}$$

Thus the density is independent of β. (Check: At $\beta = \infty$ the ground state has all $k \leq \mu$ and the number exceeds the $k \leq 0$ by $\mu / \frac{2\pi}{L} = \frac{\mu L}{2\pi}$ so the density is $\frac{\mu}{2\pi}$.

Now if we know that

$$\underbrace{z\frac{\partial}{\partial z}}_{\frac{1}{\beta}\frac{\partial}{\partial\mu}}\left(\lim\frac{\log\mathcal{Z}}{V}\right) = \frac{\mu}{2\pi} \quad \text{then} \quad \boxed{\lim\frac{\log\mathcal{Z}}{V} = \frac{\mu^2\beta}{4\pi} + f(\beta)}$$

and we can evaluate $f(\beta)$ by putting $\mu=0$

$$\frac{\log \mathcal{Z}}{V} = \int_{-\infty}^{0} \frac{dk}{2\pi} \log(1+e^{\beta k}) + \int_{0}^{\infty} \frac{dk}{2\pi} \log(1+e^{-\beta k})$$

$$= \frac{1}{\pi\beta} \int_0^\infty dx \underbrace{\log(1+e^{-x})}_{e^{-x} - \frac{e^{-2x}}{2} + \frac{e^{-3x}}{3} \cdots}$$

$$= \frac{1}{\pi\beta}\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots\right) = \frac{1}{2\pi\beta} \zeta(2)$$

Thus we obtain

$$\boxed{\lim \frac{\log \mathcal{Z}}{V} = \frac{\mu^2\beta}{4\pi} + \frac{1}{2\pi\beta} \overbrace{\zeta(2)}^{\pi^2/6}}$$

Note that $\frac{1}{4}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots$

so $\underbrace{\left(1 - \frac{2}{4}\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)}_{\frac{1}{2}\zeta(2)} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

Also we can check using the Jacobi triple product identity

$$\prod_{n\geq 0}(1+z^{-1}q^n) \prod_{n\geq 1}(1+zq^n) = \frac{\sum q^{\frac{n(n+1)}{2}} z^n}{\prod_{n\geq 1}(1-q^n)}$$

Here $z = e^{\beta\mu}$, $q = e^{-\frac{2\pi\beta}{L}}$ and $k = \frac{2\pi}{L}n$

$$\boxed{\zeta(2) = \frac{\pi^2}{6}}$$

$$\frac{1}{L}\log\left(\prod(1-q^n)\right)^{-1} = \frac{1}{L}\sum_{k>0} -\log(1-e^{-\beta k})$$

$$\to \int_0^\infty \frac{dk}{2\pi}\left(-\log(1-e^{-\beta k})\right) = \int \frac{dk}{2\pi}\left(e^{-\beta k} + \frac{e^{-2\beta k}}{2} + \cdots\right) = \frac{1}{2\pi\beta}\underbrace{\left(1 + \frac{1}{2^2} + \cdots\right)}_{\zeta(2)}$$

August 30, 1985

I am looking at the fermion gas associated to $L^2(\mathbb{R}/L\mathbb{Z})$ with $H = \frac{1}{i}\partial_x$. I consider the gas at inverse temperature β with chemical potential μ.

Associated to this data, better, one interprets this data as follows: One forms the fermion Fock space of $\mathcal{H} = L^2(\mathbb{R}/L\mathbb{Z})$ relative to the splitting $\mathcal{H} = \mathcal{H}^- \oplus \mathcal{H}^+$ into subspace where $\frac{1}{i}\partial_x \leq 0$ and > 0. One then normalizes the Hamiltonian H and number of particles operator N so as to vanish on the state corresponding to $\mathcal{H}^-$. Thus

$$H = \sum_{k>0} k\, a_k^* a_k - \sum_{k\leq 0} k\, a_k a_k^*$$

$$N = \sum_{k>0} a_k^* a_k - \sum_{k\leq 0} a_k a_k^*$$

Then one can form the partition function

(1) $$\operatorname{tr}\left(e^{-\beta(H-\mu N)}\right)$$

and one has the state

(2) $$\frac{\operatorname{tr}\left(e^{-\beta(H-\mu N)}\ ?\right)}{\operatorname{tr}\left(e^{-\beta(H-\mu N)}\right)}$$

on the algebra of operators on Fock space $\mathcal{H}_F$.

(Basic question: What is the link between (1) + (2) – why do these appear and function differently?)

The next step is to take the infinite volume limit $L \to \infty$. I've done this step for the partition function, so it remains to do it for the state (2).

Following the Araki-Woods paper we [redacted] focus not on Fock space $\mathcal{H}_F$ but on the algebra defined by the canonical commutation relations. This is the algebra generated by $a^*(f)+a(f)$ $f \in \mathcal{H}$ with relations of linearity over $\mathbb{R}$ and

$$\left(a^*(f)+a(f)\right)^2 = \|f\|^2.$$

We want to calculate the state (2) on this CAR algebra and then take the limit in the formulas as $L \to \infty$. This should then give us a state on the CAR algebra belonging to $L^2(\mathbb{R})$.

Digression. Let V be the complexification of a real Hilbert space V_0, let $C(V)$ be the Clifford algebra; its a star algebra generated by hermitian elements $c(v)$ $v \in V_0$ such that $c(v)^2 = \|v\|^2$. We can define an action of $C(V)$ on ΛV by letting

$$c(v) = v\wedge + v\lrcorner$$

[redacted] We get in this way a $*$ action of $C(V)$ on ΛV, where ΛV is equipped with the usual inner product. By [redacted] letting elts of $C(V)$ act on 1 we get a map

$$C(V) \longrightarrow \Lambda V$$

which in finite dimensions is an isomorphism. Hence provided $C(V)$ and ΛV are constructed algebraically it will be an isomorphism in infinite dimensions.

Corresponding to an orthonormal basis e_k of V we get anti-commuting hermitian involutions $\gamma^k = c(e_k)$ in $C(V)$. The map sends

$$\gamma^{k_1} \cdots \gamma^{k_l} \longmapsto e_{k_1} \wedge \cdots \wedge e_{k_l}$$

if the k_i are distinct.

Let us next consider the Hilbert space version of ΛV which has the orthonormal basis e_I. The question is whether Clifford multiplecation extends. In other words if we use the isomorphism $C(V) \simeq \Lambda V$ to define Clifford mult. on the algebraic ΛV, does it extend to the Fock space ΛV?

I seem to have two answers to this question. Actually I decided to take an element ω of $\Lambda^2(V)^\wedge$ say $\frac{1}{2}\omega_{ij} e_i e_j$ where ω_{ij} is Hilbert Schmidt. Then I would like to see whether lifting back to the Clifford algebra and forming the exponential, the terms are in Fock space.

Suppose ω is real; then ω is a skew-adjoint Hilbert Schmidt operator on V_0, so it is diagonalizable and we can suppose

$$\omega = \sum \omega_k \gamma^{2k-1}\gamma^{2k} \qquad \text{where} \quad \sum \omega_k^2 < \infty.$$

But

$$e^{\omega\gamma^1\gamma^2} = e^{i\omega\varepsilon} = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}$$

$$= \cos(\omega) + \sin\omega \, i\varepsilon$$

$$= \cos\omega + \sin\omega \, \gamma^1\gamma^2 = \cos\omega(1 + \tan\omega \, \gamma^1\gamma^2)$$

Thus

$$e^{\sum \omega_k \gamma^{2k-1}\gamma^{2k}} = \prod_k (\cos\omega_k)(1+\tan\omega_k\, \gamma^{2k-1}\gamma^{2k})$$

and this will make sense in $(\Lambda V)^\wedge$ because $\cos\omega \sim 1-\frac{\omega^2}{2}$ as $\omega \to 0$ so

$$\sum \omega^2 < \infty \implies \prod \cos\omega_k \text{ converges.}$$

On the other hand we might want to think of the Clifford algebra as an algebra of operators on Fock space [redacted] with a_k^*, a_k linked to $\gamma^{2k-1}, \gamma^{2k}$ in the usual way. Then

$$\sum \omega_k \gamma^{2k-1}\gamma^{2k} = i\sum \omega_k (1-2a_k^* a_k)$$

and

$$\langle 0 | e^{\sum \omega_k \gamma^{2k-1}\gamma^{2k}} | 0\rangle = e^{i\sum \omega_k}$$

won't make sense.

It appears the problem has to be formulated more carefully.

<u>August 31, 1985</u>

A state on the Clifford algebra with the generators a_k^*, a_k is a particular kind of linear functional on the algebra, hence it is determined by its values on the basis elements $a_I^* a_J$ (or γ_I if one wants). Hence a state is a certain kind of function on the set of finite subsets I, i.e. a function on an elementary abelian 2 gp.

Let's now consider the state

$$\langle ? \rangle = \operatorname{tr}\left(e^{-\sum \omega_k a_k^* a_k} ?\right) \Big/ \operatorname{tr}\left(e^{-\sum \omega_k a_k^* a_k}\right)$$

In the case of 1 "degree of freedom" we have

$$\left\langle \begin{matrix} 1 \\ a \\ a^* \\ a^* a \end{matrix} \right\rangle = \begin{matrix} 1 \\ 0 \\ 0 \\ \dfrac{e^{-\omega}}{1-e^{-\omega}} \end{matrix}$$

It follows that the only monomials $a_I^* a_J$ having non zero expectation are finite products of $a_k^* a_k$.

We want to use this state in the case of the fermion Fock space of $L^2(\underset{\overset{\shortparallel}{\mathbb{R}/L\mathbb{Z}}}{S^1})$. In this case we have

$$a_k^* = \psi_k^* \qquad k > 0$$

$$a_k^* = \psi_k \qquad k \leq 0$$

$$\varepsilon_k = \begin{cases} k-\mu & k > 0 \\ -(k-\mu) & k \leq 0 \end{cases}$$

so that

$$H - \mu N = \sum_{k>0} (k-\mu)\psi_k^* \psi_k - \sum_{k\leq 0} (k-\mu)\psi_k \psi_k^*$$

as expected.

The problem is to let the volume L become infinite. For each L I have a state on the Clifford algebra associated to $L^2(\mathbb{R}/L\mathbb{Z})$ considered as a real Hilbert space. Somehow I want there to be a limiting state on the Clifford algebra of $L^2(\mathbb{R})$.

The only way to make sense out of this appears to be by analogy with the boson setup. In the boson case one has a Weyl algebra instead of a Clifford algebra, and [redacted] the Weyl algebra [redacted] has the "basis" $T_f = e^{a^*(f) - a(f)}$ for $f \in L^2(\mathbb{R}/L\mathbb{Z})$. The state is then determined by the function on V

$$\langle T_f \rangle = \operatorname{tr}\left(e^{-\sum \beta\omega_k a_k^* a_k}\, T_f\right) / \text{norm}.$$

Recall the s.h.o. formula

$$\frac{\operatorname{tr}\left(e^{-\omega a^* a}\, e^{c a^* - \bar{c} a}\right)}{\operatorname{tr}\left(e^{-\omega a^* a}\right)} = e^{-|c|^2\left(\frac{1}{2} + \frac{1}{e^{\omega}-1}\right)}$$

so that in the general case

$$e^{-\sum_k |f_k|^2 \left(\frac{1}{2} + \frac{1}{e^{\beta\omega_k}-1}\right)}$$

where the f_k are the components $\langle k | f \rangle = \frac{1}{\sqrt{V}} \underbrace{\int e^{-ikx} f(x)\,dx}_{\hat{f}(k)}$

Thus we get

$$e^{-\frac{1}{V}\sum_k |\hat{f}(k)|^2 \left(\frac{1}{2} + \frac{1}{e^{\beta\omega_k}-1}\right)}$$

which has a clear limit as $V \longrightarrow \infty$, where $\frac{1}{V}\sum_k \to \int \frac{dk}{2\pi}$

Now I would like to do the fermion analogue, however I don't have the family of exponentials T_f in the Clifford algebra. We can tensor the Clifford algebra with an exterior algebra on a vector space containing elements J_k, $\tilde{J}_k$ and form

$$e^{\sum_k a_k^* J_k + \tilde{J}_k a_k}$$

In the case of a single fermion we have

$$\left\langle e^{a^* J + \tilde{J} a} \right\rangle = \left\langle 1 + a^* J + \tilde{J} a + \frac{(a^* J + \tilde{J} a)^2}{2} \right\rangle$$

$$= 1 + \frac{1}{2} J\tilde{J} \langle a^* a - a a^* \rangle$$

$$= 1 + \tilde{J} J \langle a^* a - \tfrac{1}{2} \rangle$$

$$= 1 + \tilde{J} J \left\{ \frac{e^{-\omega}}{1+e^{-\omega}} - \frac{1}{2} \right\}$$

$$= e^{-\tilde{J} J \underbrace{\left\{ \frac{1}{2} - \frac{1}{e^{\omega}+1} \right\}}}$$

as $\omega > 0$ normally this no is > 0.

So in the general case we have

$$\left\langle e^{\sum_k a_k^* J_k + \tilde{J}_k a_k} \right\rangle = e^{-\sum_k \tilde{J}_k J_k \left(\frac{1}{2} - \frac{1}{e^{\beta\omega_k}+1} \right)}$$

It's clear that I can take the infinite volume limit in this expression. What is not clear is how this is to be interpreted.

Let's take advantage of the Gaussian nature of things, so that the only part which is critical is the "variance"

$$\langle a_k^* a_{k'} \rangle = \delta(k-k') \frac{1}{e^{\beta\omega_k}+1}$$

In the case at hand we have to be careful about the change as we pass from $k \le 0$ to $k > 0$, especially since the a_k^* change from ψ_k to ψ_k^*. I therefore have to write this variance carefully.

The Clifford algebra is generated by the operators

$$\psi^*(f) = \sum \psi_k^* \langle k | f \rangle$$
$$\psi(g) = \sum \langle g | k \rangle \psi_k \qquad f, g \in \underbrace{L^2(\mathbb{R}/L\mathbb{Z})}_{V}$$

It is therefore the Clifford algebra of the complex vector space $V \oplus \bar{V}$ equipped with the quadratic form

$$(f,g) \longmapsto (\psi^*(f) + \psi(g))^2 = \langle g | f \rangle$$

$V \oplus \bar{V}$ has a conjugation $\overline{(f,g)} = (\bar{g}, \bar{f})$ and the real subspace corresponding is $V \xrightarrow{\Delta} V \oplus \bar{V}$, and the corresponding hermitian operators are

$$\psi^*(f) + \psi(f) \qquad f \in V.$$

The variance for the thermal state $\langle\ \rangle$ is this state applied to quadratic elements in the Clifford algebra. It is therefore a skew symmetric form on $V \oplus \bar{V}$, or a skew symmetric form on the real subspace. The way to obtain quadratic elements of the Clifford algebra is to take the ordinary commutator of linear elements. Thus the skew-form is essentially

$$f, g \longmapsto \langle [\psi^*(f) + \psi(f), \psi^*(g) + \psi(g)] \rangle$$

Let's define the bilinear form $B(f,g) = \langle (\psi^*(f) + \psi(f))(\psi^*(g) + \psi(g)) \rangle$

From general properties of the trace we have

$$\overline{B(f,g)} = B(g,f)$$

Hence the real part of B is symmetric and the imaginary part is skew-symmetric. Alternatively the symmetrization of B is real and the skew-symmetrization is purely imaginary. As

$$B(f,f) = |f|^2$$

$$B(f,g) + B(g,f) = \langle f|g\rangle + \langle g|f\rangle .$$

we check half this statement.

Next suppose we consider the actual example at hand where $a_k^* a_k$ and $a_k a_k^*$ are the only quadratic monomials in a_k, a_ℓ^* with $\langle\ \rangle \neq 0$. It follows that $\psi_k^* \psi_k$ and $\psi_k \psi_k^*$ are the only quadratic monomials in ψ_k, ψ_k^* with $\langle\ \rangle \neq 0$. We have

$$\langle (\psi^*(f) + \psi(f)) \cdot (\psi^*(g) + \psi(g)) \rangle \qquad f_k = \langle k|f\rangle$$

$$= \langle \sum f_k \psi_k^* + \bar{f}_k \psi_k , \sum g_\ell \psi_\ell^* + \bar{g}_\ell \psi_\ell \rangle$$

$$= \sum f_k \bar{g}_k \langle \psi_k^* \psi_k \rangle + \bar{f}_k g_k \langle \psi_k \psi_k^* \rangle$$

$$= \sum \bar{f}_k g_k + \underbrace{\sum_k (f_k \bar{g}_k - \bar{f}_k g_k) \langle \psi_k^* \psi_k \rangle}$$

$$= \frac{1}{2} \sum (\bar{f}_k g_k + f_k \bar{g}_k) + \frac{1}{2} \sum_k (\bar{f}_k g_k - f_k \bar{g}_k) +$$

$$= \frac{1}{2}(\langle f|g\rangle + \langle g|f\rangle) + \sum_k (\bar{g}_k f_k - \bar{f}_k g_k) \left\{ -\frac{1}{2} + \langle \psi_k^* \psi_k \rangle \right\}$$

$$= \frac{1}{2}(\langle f|g\rangle + \langle g|f\rangle) + \langle g| -\tfrac{1}{2} + n |f\rangle - \langle f| -\tfrac{1}{2} + n |g\rangle$$

Thus the skew-symmetric part is purely imaginary, namely

$$2i \operatorname{Im} \langle g | -\tfrac{1}{2}+n | f \rangle$$

where $-\frac{1}{2}+n$ is the self-adjoint operator with the eigenvalues

$$-\frac{1}{2} + \langle \psi_k^* \psi_k \rangle = \begin{cases} -\frac{1}{2} + \dfrac{1}{e^{\beta\omega_k}+1} & k > 0 \\[2ex] -\frac{1}{2} + \dfrac{1}{1+e^{-\beta\omega_k}} & k \leq 0 \end{cases}$$

$$= -\frac{1}{2} + \frac{1}{e^{\beta(k-\mu)}+1} \qquad \text{for all } k.$$

September 1, 1985

From yesterday we have

$$\langle \psi_k^* \psi_k \rangle = n_k = \frac{1}{e^{\beta(k-\mu)}+1}$$

and this determines the thermal ~~state~~ on the Clifford algebra as it's a Gaussian state. Note that n_k goes from 1 to 0 as k goes from $-\infty$ to ∞; if β is large there is a sharp jump from 1 to 0 as k crosses μ.

Sample computation. Let $r>0$, so ρ_{-r} destroys

$$\langle \rho_{+r}\,\rho_{-r} \rangle = \Big\langle \sum_k \psi_{k+r}^* \psi_k \sum_\ell \psi_{\ell-r}^* \psi_\ell \Big\rangle$$

$$= \sum_{k,\ell} \langle \psi_k \psi_{\ell-r}^* \rangle \langle \psi_{k+r}^* \psi_\ell \rangle$$

$$= \sum_k \langle \psi_k \psi_k^* \rangle \langle \psi_{k+r}^* \psi_{k+r} \rangle$$

$$= \sum_k (1-n_k)(n_{k+r})$$

$$= \sum_k \frac{1}{1+e^{-\beta(k-\mu)}} \, \frac{1}{e^{\beta(k+r-\mu)}+1}$$

$a = \frac{2\pi}{L}$

$k = an,\ r = pa$

$e^{-\beta a} = q$

$$= \sum_n \frac{1}{1+q^n z} \, \cancel{\frac{1}{q^{-n-p}z^{-1}+1}} \, \frac{q^{n+p}z}{1+q^{n+p}z}$$

$$= \sum_n \left(\frac{1}{1+q^{n+p}z} - \frac{1}{1+q^n z} \right) \frac{1}{q^{-p}-1}$$

$$= \frac{p}{q^{-p}-1} = \frac{r}{a} \, \frac{1}{e^{\beta r}-1}$$

This checks with

$$[\rho_{-r}, \rho_r] = \frac{r}{a}$$

$\Longrightarrow$ $\sqrt{\frac{a}{r}}\,\rho_r$, $\sqrt{\frac{a}{r}}\,\rho_{-r}$ are creation + ann. ops

$$\Longrightarrow \quad \left\langle \sqrt{\tfrac{a}{r}}\,\rho_r \, \sqrt{\tfrac{a}{r}}\,\rho_{-r} \right\rangle = \frac{1}{e^{\beta r} - 1}$$

by standard h. osc. theory of Planck.

The program now is to look at things from the boson viewpoint. We know that the fermion Fock space is actually an irreducible representation of a Heisenberg group which is a central extension of the loop group of $U(1) = \mathbb{T}$.

Recall that $\mathcal{L}(\mathbb{T})$ contains $\mathbb{T} \times \mathbb{Z}$ = loops of the form ζz^n, $\zeta \in \mathbb{T}$, $n \in \mathbb{Z}$ where $z = e^{2\pi i x}$. To describe a complementary subgroup, let's first look at loops $u(x)$ of degree 0. Such a loop has a cont. logarithm: $u(x) = e^{2\pi i f(x)}$ which is unique up to $\mathbb{Z}$ (constant fns). If we associate to $u(x)$ the element $\exp^{2\pi i}\left(\int_0^1 f(x)\,dx\right)$, we get a well-defined map to $\mathbb{T}$. The kernel is the subgroup of loops of the form $e^{2\pi i f(x)}$ where $f(x)$ is real valued, periodic, and $\int_0^1 f(x)\,dx = 0$. Call this subgroup $\mathcal{L}_0(\mathbb{T})$ so that

$$\mathcal{L}(\mathbb{T}) = \mathcal{L}_0(\mathbb{T}) \times (\mathbb{T} \times \mathbb{Z})$$

Let's go back to $\mathcal{H}_F$ and use that it is a tensor product of the irreducible representations of the Heisenberg groups belonging to $\mathcal{L}_0(\mathbb{T})$ and $\mathbb{T} \times \mathbb{Z}$. The $\mathcal{L}_0(\mathbb{T})$ part is described by the operators ρ_r, $r \neq 0$, the $\mathbb{T} \times \mathbb{Z}$ part by the operators ρ_0 and σ_0

Here σ is the unitary operator on Fock space which corresponds to the shift operator on $L^2(\mathbb{R}/L\mathbb{Z})$

$$|k\rangle \longmapsto |k+a\rangle \qquad a = \frac{2\pi}{L}$$

i.e. multiplication by e^{iax}. More general let σ_r $\quad r \in \frac{2\pi}{L}\mathbb{Z}$ be the operator on $\mathcal{H}_F$ corresponding to mult. by e^{-irx}. We will suppose the multiplicative constant is pinned down by requiring

$$\sigma_r = \sigma^{\frac{r}{a}}$$

$$\sigma\,|k, k-a, k-2a, \ldots\rangle = |k+a, k, k-a, \ldots\rangle$$

Then one has

$$\sigma_r \psi_k^* \sigma_r^{-1} = \psi_{k+r}^*$$

$$\sigma_r \psi_k \sigma_r^{-1} = \psi_{k+r}$$

$$\sigma_r \rho_k = \rho_k \sigma_r \qquad k \neq 0$$

$$\sigma_r \rho_0 \sigma_r^{-1} = \rho_0 - r$$

Let's concentrate on the subspace of $\mathcal{H}_F$ with the orthonormal basis

$$|k, k-a, \ldots\rangle = \sigma_k \cdot (\text{ground state})$$

This subspace is isomorphic to $\ell^2(\frac{2\pi}{L}\mathbb{Z})$ and ρ_0 is the function multiplication by k.

What is $H - \mu N$? Clearly

$$H\,|k, k-a, \ldots\rangle = \underbrace{(k + (k-a) + \cdots + a)}_{\frac{k}{a}\,\frac{1}{2}(k+a)}\,|k, k-a, \ldots\rangle$$

$$N\,|k, k-a, \ldots\rangle = \frac{k}{a}\,|k, k-a, \ldots\rangle$$

So $H-\mu N$ on $l^2(\frac{2\pi}{L}\mathbb{Z})$ is multiplication by

$$\frac{1}{2}\frac{k(k+a)}{a} - \mu\frac{k}{a}.$$

and the partition function ~~on~~ $l^2(\frac{2\pi}{L}\mathbb{Z})$ is

$$\operatorname{tr} e^{-\beta(H-\mu N)} = \sum_k e^{-\beta\left(\frac{a}{2}\frac{k}{a}\left(\frac{k}{a}+1\right) - \mu\frac{k}{a}\right)}$$

$$= \sum_n e^{-\beta a\frac{n(n+1)}{2}} e^{\beta\mu n}$$

$$= \sum_n q^{\frac{n(n+1)}{2}} z^n \qquad \begin{array}{l} q = e^{-\beta a} \\ z = e^{\beta\mu} \end{array}$$

The thermal state is determined ~~by~~ the function on

$$\mathbb{T}\times\mathbb{Z} = [redacted] \ \mathbb{R}/L\mathbb{Z} \times \frac{2\pi}{L}\mathbb{Z}$$

$$= \left\{ e^{ikx}\sigma_r \;\middle|\; x\in\mathbb{R}/L\mathbb{Z},\ r\in\frac{2\pi}{L}\mathbb{Z} \right\}$$

given by

$$\langle e^{ix?}\sigma_r\rangle = \begin{cases} 0 & r\neq 0 \\ \dfrac{\sum_k e^{-\beta\left(\frac{a}{2}\frac{k}{a}\frac{k}{a}+1 - \mu\frac{k}{a}\right)} e^{ixk}}{\sum_k e^{-\beta\left(\frac{a}{2}\cdot\frac{k}{a}\cdot\frac{k+a}{a} - \mu\frac{k}{a}\right)}} & r=0 \end{cases}$$

$$\frac{\sum_n [redacted]\, e^{-\beta a\frac{n(n+1)}{2}} e^{(\beta\mu + iax)n}}{\sum_n e^{-\beta a\frac{n(n+1)}{2}} e^{(\beta\mu)n}}$$

Now comes the question as to what sort of limit to take?

At this point I should like to go over the Araki–Woods construction. What I would like to do is to link the non type I to the non-implementability of symplectic transformations.

Let $\mathcal{H}_F$ be the boson Fock space with $|0\rangle, a_k^*, a_k$ as usual. Thus $\mathcal{H}_F$ is the Hilbert space symmetric tensor space $S(V)$ where V has the orthonormal basis $|k\rangle$. $\mathcal{H}_F$ is a cyclic representation of the CCR and is determined by the generating fn.

$$\langle 0|T_f|0\rangle = \langle 0|\, e^{\sum f_k a_k^* - \bar{f}_k a_k}\,|0\rangle$$
$$= e^{-\frac{1}{2}\|f\|^2}$$

If we have a Hamiltonian $H = \sum \omega_k a_k^* a_k$, then we can consider the corresponding thermal state at inverse temperature β. This gives the generating function

$$\langle T_f\rangle = \frac{tr\left(e^{-\beta H} T_f\right)}{tr\left(e^{-\beta H}\right)} = e^{-\sum_k |f_k|^2\left(\frac{1}{2} + \frac{1}{e^{\beta\omega_k} - 1}\right)}.$$

By general theory, there is a cyclic representation of the CCR corresponding to this state, and we now construct it.

We consider $\mathcal{H}_F \otimes \mathcal{H}_F$ and define

$$\tilde{a}_k^* = s_k\, a_k'^* + t_k\, a_k''$$
$$\tilde{a}_k = \bar{s}_k\, a_k' + \bar{t}_k\, a_k''^*$$

which we satisfy the CCR provided $|s_k|^2 - |t_k|^2 = 1$. The generating function for the cyclic vector $|0,0\rangle = |0\rangle \otimes |0\rangle$

so
$$\left\langle e^{\sum f_k \tilde{a}_k^* - \bar{f}_k \tilde{a}_k} \right\rangle = \left\langle e^{\sum f_k s_k a_k'^* - \sum \bar{f}_k \bar{s}_k a_k'} \, e^{\sum -\bar{f}_k \bar{t}_k a_k''^* + f_k t_k a_k''} \right\rangle$$
$$= e^{-\frac{1}{2}\sum |f_k s_k|^2} \, e^{-\frac{1}{2}\sum |-\bar{f}_k \bar{t}_k|^2}$$
$$= e^{-\frac{1}{2}\sum |f_k|^2 \left(|s_k|^2 + |t_k|^2\right)}$$

In order that $e^{\sum f_k \tilde{a}_k^* - \bar{f}_k \tilde{a}_k}$ be well-defined it is n.a.s. that $\sum |f_k|^2 (|s_k|^2 + |t_k|^2) < \infty$, and we want this for all $f \in V$. Thus we need
$$|s_k|^2 + |t_k|^2 = 1 + 2|t_k|^2 \quad \text{bounded.}$$

September 2, 1985

Suppose we have $\underbrace{\text{creation}}_{}$ independent creation + annihilation operators
$$\begin{matrix} a^* & b^* \\ a & b \end{matrix}$$
and we make a symplectic transformation
$$\tilde{a}^* = s a^* + t b \qquad [\tilde{a}, \tilde{a}^*] = |s|^2 - |t|^2 = 1$$
$$\tilde{a} = \bar{s} a + \bar{t} b^*$$
$$\tilde{b}^* = u b^* + v a \qquad [\tilde{b}, \tilde{b}^*] = |u|^2 - |v|^2 = 1$$
$$\tilde{b} = \bar{u} b + \bar{v} a^*$$

$$[\tilde{b}^*, \tilde{a}^*] = vs - ut = 0 \quad \Rightarrow [\tilde{a}, \tilde{b}] = 0$$
$$[\tilde{b}^*, \tilde{a}] = [\tilde{b}, \tilde{a}^*] = 0 \quad \text{clear.}$$

$\therefore$ If $\frac{u}{s} = \theta$, then $v = \theta t$ and
$$1 = |u|^2 - |v|^2 = |\theta|^2 (|s|^2 - |t|^2) \Rightarrow |\theta| = 1.$$
This θ rotates on the $\tilde{b}^*, \tilde{b}$ side, so let's suppose $\theta = 1$, whence
$$\left(\begin{aligned} \tilde{a}^* &= s a^* + t b \\ \tilde{b}^* &= s b^* + t a \end{aligned} \right. \qquad |s|^2 - |t|^2 = 1$$

So if $f(w) = \sum c_n w^n$

$$\| e^{\lambda z w} f(w) \|^2 = \int e^{-(1-|\lambda|^2)|w|^2} |f(w)|^2 \frac{d^2 w}{\pi}$$

$$= \frac{1}{1-|\lambda|^2} \int e^{-|w|^2} \left| f\left(\frac{w}{\sqrt{1-|\lambda|^2}}\right)\right|^2 \frac{d^2 w}{\pi}$$

$$= \frac{1}{1-|\lambda|^2} \sum_n \frac{|c_n|^2}{(1-|\lambda|^2)^n} n!$$

Conclusion is that the minimum norm solution of $\tilde{a} \Psi = 0$, $\langle 0 | \Psi \rangle = 1$ is $e^{\lambda z w} = e^{\lambda a^* b^*}|0\rangle$ and that its norm squared is

$$\| e^{\lambda a^* b^*} |0\rangle \|^2 = \frac{1}{(1-|\lambda|^2)} \qquad \lambda = -\frac{\bar{t}}{\bar{s}}.$$

Here's the application I had in mind. Take ind. creation and ann. operators a_k^*, a_k acting on $\mathcal{H}_F$, form $\mathcal{H}_F \otimes \mathcal{H}_F$ and write $a_k^* \otimes 1 \mapsto a_k^*$, $1 \otimes a_k^* \mapsto b_k$. Then define operators on $\mathcal{H}_F \otimes \mathcal{H}_F$

$$\tilde{a}_k^* = s_k a_k^* + t_k b_k$$

$$\tilde{a}_k = \bar{s}_k a_k + \bar{t}_k b_k^* \qquad \text{where } |s_k|^2 - |t_k|^2 = 1.$$

Provided the sequence t_k is bounded we get a representation of the CCR in this way. I mean that [redacted] one has a unitary operator on $\mathcal{H}_F \otimes \mathcal{H}_F$

$$e^{\sum c_k \tilde{a}_k^* - \bar{c}_k \tilde{a}_k}$$

for each ℓ^2 sequence c_k:

$$c_k \tilde{a}_k^* - \bar{c}_k a_k = c_k s_k a_k^* - \bar{c}_k \bar{t}_k b_k^* - \text{c.c.}$$

$$\sum |c_k s_k|^2 + \sum |-\bar{c}_k \bar{t}_k|^2 = \sum |c_k|^2 (|s_k|^2 + |t_k|^2)$$

New ground state

$$\tilde{a}\Psi = (\bar{s}a + \bar{t}b^*)\Psi = 0 \qquad \left(\frac{d}{dz} + \frac{\bar{t}}{\bar{s}}w\right)\Psi = 0$$

$$\tilde{b}\Psi = (\bar{s}b + \bar{t}a^*)\Psi = 0 \qquad \left(\frac{d}{dw} + \frac{\bar{t}}{\bar{s}}z\right)\Psi = 0$$

$$\therefore\ \Psi = \text{const}\ e^{-\frac{\bar{t}}{\bar{s}}wz} = \text{const}\ e^{-\left(\frac{\bar{t}}{\bar{s}}\right)a^*b^*}|0\rangle$$

Now

$$e^{\lambda a^* b^*}|0\rangle = \sum \frac{\lambda^n}{n!} a^{*n} b^{*n}|0\rangle$$

$$= \sum \lambda^n \underbrace{\frac{a^{*n}}{\sqrt{n!}}\frac{b^{*n}}{\sqrt{n!}}|0\rangle}_{\text{orthonormal}}$$

$$\boxed{\lambda = -\frac{\bar{t}}{\bar{s}}}$$

so

$$\| e^{\lambda a^* b^*}|0\rangle \|^2 = \sum |\lambda^n|^2 = \sum (|\lambda|^2)^n = \frac{1}{1-|\lambda|^2}$$

For later purposes what I need to know is the minimum norm of a solution of

$$\tilde{a}\Psi = 0$$

subject to the condition that $\langle 0|\Psi\rangle = 1$. Solutions of $\tilde{a}\Psi = 0$ are of the form

$$e^{\lambda z w} f(w).$$

Compute

$$\int e^{-|z|^2 - |w|^2} e^{\lambda z w + \bar{\lambda}\bar{z}\bar{w}} |f(w)|^2 \frac{d^2 z}{\pi}\frac{d^2 w}{\pi}$$

$$-\bar{z}z + \lambda w z + \bar{z}\bar{\lambda}\bar{w} - |\lambda w|^2 + |\lambda w|^2$$

$$= \int e^{-|w|^2}\frac{d^2 w}{\pi}|f(w)|^2 \underbrace{\int e^{-|z-\bar{\lambda}\bar{w}|^2}\frac{d^2 z}{\pi}}_{1} e^{|\lambda w|^2}$$

$$= \int e^{-(1-|\lambda|^2)|w|^2} |f(w)|^2 \frac{d^2 w}{\pi}$$

Now one can ask whether this representation of the CCR is a direct sum of the standard irreducible representation. For this it is necessary to find vectors in $\mathcal{H}_F \otimes \mathcal{H}_F$ killed by all $\tilde{a}_k$. Things killed by all the $\tilde{a}_k$ have the form

$$e^{\sum \lambda_k z_k w_k} \underbrace{f(w)}_{\sum c_\alpha w^\alpha}$$

and this has norm squared

$$\prod_k \frac{1}{1-|\lambda_k|^2} \sum_\alpha \frac{|c_\alpha|^2 \alpha!}{(1-|\lambda|^2)^\alpha}$$

(More precisely the only formal power series in z_k, w_k killed by all $\tilde{a}_k$ are in the above form.)

One concludes that unless

$$(*) \qquad \sum |\lambda_k|^2 < \infty$$

there are no vectors in $\mathcal{H}_F \otimes \mathcal{H}_F$ killed by all $\tilde{a}_k$.

According to Araki + Woods the von-Neumann operator algebra R on $\mathcal{H}_F \otimes \mathcal{H}_F$ generated by the $\tilde{a}_k^*, a_k$ is a factor. The reason is that R' contains the operators

$$\tilde{b}_k^* = s_k b_k^* + t_k a_k$$
$$\tilde{b}_k = \bar{s}_k b_k + \bar{t}_k a_k^*$$

and that $R \cup R'$ contains all the a_k^*, a_k, b_k^*, b_k hence $R \cap R' \subset (R \cup R')' = \mathbb{C}$. So when $(*)$ fails we get a non type I factor, which is either II_∞ or III as II_1 is not compatible with the CCR.