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August 3, 1985 (after returning to Bures from the week in the US)

It seems desirable to work out the theory of 1-diml Gaussian integrals, both bosonic and fermionic. Such integrals are supposed to give the quantum mechanics belonging to quadratic Hamiltonians. I will take this as point of departure.

Thus consider the ~~[illegible]~~ unique irreducible repn. of the CCR's: $q_1, \dots, q_n, p_1, \dots, p_n$ on $\mathcal{H}$ ∋

$$[q_i, q_j] = [p_i, p_j] = 0 \qquad [p_i, q_j] = \frac{1}{i}\delta_{ij}$$

There is a canonical action of a double covering of the symplectic group $Sp(n, \mathbb{R})$ called the metaplectic group $Mp(n, \mathbb{R})$ on $\mathcal{H}$. The Lie algebras of $Sp(n, \mathbb{R})$ (also $Mp(n, \mathbb{R})$) can be identified with the space of quadratic Hamiltonians $H(q, p)$ (spanned by squares of linear functions $a.q + b.p$, $a, b \in \mathbb{R}$) Such a quad. Hamilton H gives rise to a 1-param. unitary group e^{-itH}.

But also in QM one wants H positive and to take t to be imaginary: $t = -i\tau$, which leads to operators $e^{-\tau H}$ where $\mathrm{Re}\,\tau > 0$, and these operators leads to Gaussian type path integrals. So it seems that the general case involves a continuous product of $e^{-dt H_t}$ where H_t is a positive quadratic Ham. More generally we need only its real part to be positive.

we have [illegible]

of complex symmetric $n \times n$ matrices with positive definite imaginary part. Such a path with appropriate boundary conditions should lead to a Gaussian type path integral.

Let's now take a constant (purely imaginary) path so that we are dealing with $e^{-\tau H}$ with H pos. definite. Then by changing variables we have (say $n=1$)

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) = \omega\left(a^* a + \tfrac{1}{2}\right)$$

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \qquad a^* = \frac{1}{\sqrt{2\omega}}(\omega q - ip).$$

Now I want to understand the corresponding path integral.

According to physics we integrate over paths $(q(t), p(t))$ the exponential of

$$\int (ip\dot{q} - H)\, dt$$

Now

$$q = \frac{a + a^*}{\sqrt{2\omega}} \qquad p = \sqrt{\frac{\omega}{2}}\,(ia^* - ia)$$

and

$$ip\dot{q} = i\sqrt{\frac{\omega}{2}}\,(ia^* - ia)\,\frac{1}{\sqrt{2\omega}}(\dot{a} + \dot{a}^*)$$

$$= \tfrac{1}{2}\left(-a^*\dot{a} - a^*\dot{a}^* + a\dot{a} + a\dot{a}^*\right)$$

$$= -a^*\dot{a} + \frac{d}{dt}(\cdots)$$

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) = \omega a^* a \qquad \text{(classically)}$$

so we get the integral over complex paths $a(t)$ of the exponential of $\quad -a^*\left(\frac{d}{dt} + \omega\right) a$

So physics give a formula such as

$$\mathrm{tr}\left(e^{-\beta H}\right) = \int \mathcal{D}a^* \mathcal{D}a \; e^{-\int_0^\beta a^*(\partial_t+\omega)a\,dt}$$

where the integral is taken over periodic paths: $a(0)=a(\beta)$. Formally the integral is

$$\frac{1}{\det(\partial_t+\omega)}$$

and

$$\det(\partial_t+\omega) = \prod_{k\in\frac{2\pi}{\beta}\mathbb{Z}} (ik+\omega)$$

This det vanishes when $i\frac{2\pi}{\beta}n+\omega=0$ or $\beta\omega \in 2\pi i\mathbb{Z}$, so

$$\det(\partial_t+\omega) = 1-e^{-\beta\omega}$$

up to a non-vanishing factor. Also

$$\mathrm{tr}\left(e^{-\beta H}\right) = \sum_{n\geq 0} e^{-\beta(n+\frac{1}{2})\omega} = \frac{e^{-\frac{1}{2}\beta\omega}}{1-e^{-\beta\omega}}.$$

Let's try to describe what the mathematical problem is. We have the complex vector space of periodic functions $a(t)$ with values in $\mathbb{C}$ and the operator $\partial_t+\omega$ which has positive definite hermitian part.

August 4, 1985

Yesterday I looked at the path integral involved with the simple harmonic oscillator and got

$$\int \mathcal{D}\phi\,\mathcal{D}\bar{\phi}\; e^{-\int \bar{\phi}(\partial_t+\omega)\phi\,dt}$$

where the integral is taken over complex valued fields $\phi(t)$, say periodic: $\phi(t+\beta)=\phi(t)$. The important observation is that the exponent is not real-valued. In fact if we use Fourier series

$$\phi(t) = \sum_{k\in\frac{2\pi}{\beta}\mathbb{Z}} e^{ikt}\phi_k$$

Then

$$\int_0^\beta \bar{\phi}(\partial_t+\omega)\phi\,dt = \beta\sum_k (ik+\omega)|\phi_k|^2.$$

Notice the eigenvalues $\beta(ik+\omega)= i2\pi n+\beta\omega \qquad n\in\mathbb{Z}$ are not real, but have positive real part.

This path integral will split formally as a product over k. So we should look at the integral

$$\boxed{\begin{aligned}&\int_{\mathbb{C}} e^{-\alpha|z|^2+\bar{J}z+J\bar{z}}\;\frac{d^2z}{\pi}\\ &= \frac{1}{\alpha}\,e^{+\frac{|J|^2}{\alpha}} \qquad \mathrm{Re}(\alpha)>0\end{aligned}}$$

The formula is easily checked, and may be generalized to the case where $\bar{J}$ is a variable independent of J.

The obvious generalization of this formula is

$$\int \mathcal{D}\phi \mathcal{D}\bar{\phi} \; e^{-(\bar{\phi}, (\partial_t + \omega)\phi) + (\bar{J}, \phi) + (\bar{\phi}, J)}$$

$$= \frac{1}{\det(\partial_t + \omega)} \; e^{(\bar{J}, (\partial_t + \omega)^{-1} J)}$$

I now want to understand whether there exists countably-additive complex-valued measure with the required properties. This measure should perhaps live on the space of distributions $\phi(t)$.

Using Fourier series we want to make sense of a product measure

$$\prod_k e^{-\alpha_k |z_k|^2} \alpha_k \frac{d^2 z_k}{\pi}$$

on a suitable sequence space. In our case $\alpha_k = \omega + ik$ where we can suppose $\omega = 1$. So we want the product measure

$$\prod_k \left\{ (1+ik) \, e^{-ik|z_k|^2} \right\} \cdot \prod_k e^{-|z_k|^2} \frac{d^2 z_k}{\pi}$$

It appears there are problems with defining this as a countably-additive measure. On a sequence space where the Gaussian measure makes sense, the series $\sum k|z_k|^2$ would not converge a.e.

Let's review Gaussian prob measures on a (real) Hilbert space. By principal axes one can suppose the Hilbert space is ℓ^2_r and the measure is

$$d\mu = \prod_{n=1}^{\infty} e^{-\frac{1}{2}a_n x_n^2} \sqrt{\frac{a_n}{2\pi}}\, dx_n$$

A necessary condition that this be countably additive is that $e^{-\frac{1}{2}\tau\|x\|^2}$ have finite >0 integral, that is

$$\prod_1^{\infty} \left(\frac{a_n}{\tau + a_n}\right)^{1/2} = \left\{ \prod_{n=1}^{\infty} \left(1 + \frac{\tau}{a_n}\right)\right\}^{-1/2}$$

converge, equivalently

$$\sum \frac{1}{a_n} < \infty .$$

Now we can consider multiplying $d\mu$ by

$$e^{-\sum_1^{\infty} \frac{1}{2} i b_n x_n^2}$$

The series $\sum b_n x_n^2$ converges in L^1 provided

$$\int \sum |b_n| x_n^2 \, d\mu = \sum \frac{|b_n|}{a_n} < \infty .$$

Thus we can produce Gaussian measures on ℓ^2 of the form

$$\prod_1^{\infty} e^{-\frac{1}{2}(a_n + i b_n) x_n^2} \sqrt{\frac{a_n}{2\pi}}\, dx_n$$

provided $\sum \frac{|b_n|}{a_n} < \infty$. This condition implies the convergence of $\prod \left(\frac{a_n + i b_n}{a_n}\right)$ which shows that we get Gaussian measures on ℓ^2

of the form

$$\prod_{1}^{\infty} e^{-\frac{1}{2}\alpha_n x_n^2} \sqrt{\frac{\alpha_n}{2\pi}}\, dx_n$$

where $\alpha_n = a_n + i b_n$, $\sum \frac{1}{a_n} < \infty$, $\sum \frac{|b_n|}{a_n} < \infty$.

Unfortunately this doesn't come close to handling the case I'm interested in, when $\frac{|b_n|}{a_n} \to \infty$

Thus we ought to look at the usual path integral treatment of the simple harmonic oscillator. Here one starts with the exponent

$$\int \left(i p \dot{q} - \tfrac{1}{2}(p^2 + \omega^2 q^2) \right) dt$$

and one does the integral over p obtaining the exponent

$$-\frac{1}{2}\int \left(\dot{q}^2 + \omega^2 q^2\right) dt = -\frac{1}{2}\int q\,(-\partial_t^2 + \omega^2)\, q \, dt.$$

Now this is supposed to lead to a well-defined Gaussian probability measure on the space of real L^2 functions of t. One has pushed forward something under the map $a \mapsto q = \frac{a + a^*}{\sqrt{2\omega}}$

August 6, 1985

Project: Investigate the Hilbert space attached to an infinite-dimensional Gaussian measure.

Start with $\mathbb{R}$, [redacted] dimension one. Let $\mathcal{H}$ be the Fock space with $a, a^*, |0\rangle$; let $\omega > 0$.

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \qquad \left.\begin{array}{l} q = x \\ p = \frac{1}{i}\partial_x \end{array}\right\} \text{ on } L^2(\mathbb{R})$$

$$= \frac{1}{\sqrt{2\omega}}(\partial_x + \omega x) \qquad \text{kills} \qquad e^{-\frac{1}{2}\omega x^2}$$

so we get isom

$$\mathcal{H} \xrightarrow{\sim} L^2(\mathbb{R})$$

$$|0\rangle \longleftrightarrow e^{-\frac{1}{2}\omega x^2}\left(\frac{\omega}{\pi}\right)^{1/4}$$

$$\frac{1}{\sqrt{2\omega}}(a^* + a) \longleftrightarrow x$$

Note that the probability measure on $\mathbb{R}$ corresp. to $|0\rangle$ is

$$e^{-\omega x^2}\sqrt{\frac{\omega}{\pi}}\,dx$$

Recall

$$e^{\gamma a^* - \bar{\gamma} a} = e^{-\frac{1}{2}|\gamma|^2} e^{\gamma a^*} e^{-\bar{\gamma} a}$$

then

$$\langle 0|e^{i\xi x}|0\rangle = \langle 0|e^{i\frac{\xi}{\sqrt{2\omega}}(a^* + a)}|0\rangle$$

$$= e^{-\frac{1}{2}\left|\frac{i\xi}{\sqrt{2\omega}}\right|^2} = e^{-\frac{\xi^2}{4\omega}}$$

The simplest choice for ω is $\omega = \frac{1}{2}$ whence $x = a + a^*$

Now I want to take [redacted] finitely many dimensions. Let's keep the ω in, but choose it to be diagonal:

$$q_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a_k^*)$$

Then

$$|0\rangle = e^{-\frac{1}{2}\sum_k \omega_k x_k^2} \prod_k \left(\frac{\omega_k}{\pi}\right)^{1/4}$$

and the prob. measure is

$$(\star) \qquad e^{-\sum \omega_k x_k^2} \prod_k \left(\frac{\omega_k}{\pi}\right)^{1/2} \cdot dx_k$$

[redacted] Functions of the x_k become operators. Of particular interest are Gaussian functions.

Let's pass now to infinite dimensions, and try to correlate functions of the x_k with operators. (In other words I don't really want to use any measure theory. More precisely we [redacted] know that $(\star)$ [redacted] defines a probability measure on various subspaces of the space of all sequences (x_k). [redacted] The operators will correspond to a.e. defined functions modulo null-equivalence for any of these subspaces. Somehow this whole measure theory is completely ugly - highly non canonical. In the end one gets the existence of operators from the theory. I'd like to bypass the measure theory if possible)

Let's consider a linear function $\sum \xi_k x_k$ or better, its exponential $e^{i\xi\cdot x}$.

We know this leads to an operator on $\mathcal{H}_F$ when

$$\sum \frac{\xi_k^2}{\omega_k} < \infty$$

August 7, 1985

Let's go over the Shale-Stinespring business. Let $\mathcal{H}_F$ be boson Fock space with $a_k^*, a_k, |0\rangle$ as usual. I can view elements of $\mathcal{H}_F$ as certain power series in variables z_k. Now suppose I have a transformation

$$\begin{aligned} a_k &\longrightarrow \alpha_k a_k + \beta_k a_k^* \\ a_k^* &\longrightarrow \bar\beta_k a_k + \bar\alpha_k a_k \end{aligned} \qquad |\alpha_k|^2 - |\beta_k|^2 = 1.$$

which is symplectic and compatible with $*$.

It would be better to first look at the simple case with only a pair a^*, a. I want a unitary operator T on $\mathcal{H}_F$ such that

$$TaT^{-1} = \alpha a + \beta a^*$$

Then $$Ta^*T^{-1} = \bar\beta a + \bar\alpha a^*$$

and $$|\alpha|^2 - |\beta|^2 = 1.$$

Look at $T|0\rangle = f(z)$.

$$(\alpha a + \beta a^*) T|0\rangle = TaT^{-1}T|0\rangle = Ta|0\rangle = 0$$

$$\left(\alpha \frac{d}{dz} + \beta z\right) f(z) = 0 \qquad \Longrightarrow \qquad f(z) = e^{-\frac{\beta}{\alpha}\frac{z^2}{2}}\cdot\text{const.}$$

Note that $\left|\frac{\beta}{\alpha}\right| < 1$.

Let's compute

$$\left\| e^{\tau \frac{z^2}{2}} \right\|^2 = \sum_n \frac{1}{(n!)^2} \left\| \left(\frac{\tau}{2}\right)^n z^{2n} \right\|^2$$

$$= \sum \underbrace{\frac{(2n)!}{(n!)^2\, 2^{2n}}}\, (|\tau|^2)^n$$

$$\frac{1\cdot 3 \cdots (2n-1)}{n!\; 2^n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}(-1)^n$$

$$\| e^{\tau \frac{z^2}{2}} \|^2 = (1-|\tau|^2)^{-1/2}$$

So if we want to find a T unitary on $\mathcal{H}_F$ in the infinite dimensional case with

$$T a_k T^{-1} = \alpha_k a_k + \beta_k a_k^*$$

then we must have

$$T|0\rangle = e^{\sum (-\frac{\beta_k}{\alpha_k}) \frac{z_k^2}{2}} \cdot \text{const.}$$

and for this to be in the Hilbert space means

$$\prod (1-|\tau_k|^2)^{-1/2} < \infty \qquad \tau_k = -\frac{\beta_k}{\alpha_k}$$

which gives us the condition

$$\sum \left| \frac{\beta_k}{\alpha_k} \right|^2 < \infty$$

for the unitary implementability of the symplectic transformation.

Let's consider $\mathcal{H}_F$ with $a_k^*, a_k, |0\rangle$ as usual and try to identify $\mathcal{H}_F$ with L^2 of the Gaussian cylinder measure

$$\prod_k e^{-\frac{1}{2}x_k^2} \frac{dx_k}{\sqrt{2\pi}}$$

on $\ell^2_{\mathbb{R}}$. This means that we want to identify multiplication by x_k with the operator

$$q_k = a_k^* + a_k \quad \text{on } \mathcal{H}_F.$$

Recall that an orthonormal basis for $\mathcal{H}_F$ is given by

$$1, \quad x_k, \quad \begin{cases} x_k x_\ell & k<\ell \\ \frac{1}{\sqrt{2}}(x_k^2-1) \end{cases}$$

A typical 2-particle element of $\mathcal{H}_F$ is thus of the form

$$\sum_{k<l} c_{kl}\, x_k x_l \;+\; \sum_k{}' \tfrac{1}{2} c_{kk}(x_k^2-1)$$

and it L^2 norm squared is

$$\sum_{k<l} |c_{kl}|^2 \;+\; \sum_k \tfrac{1}{4}|c_{kk}|^2\, 2$$

$$= \sum_{k,l}{}' \tfrac{1}{2}|c_{kl}|^2$$

provided we set $c_{kl} = c_{lk}$. The conclusion is that the 2-particle subspace of $\mathcal{H}_F$ is isomorphic to the space of quadratic functions

$$Q = \tfrac{1}{2}\sum{}' c_{kl}\, x_k x_l \qquad \text{on} \qquad \ell^2_{\mathbb{R}}$$

with the norm squared

$$\|Q\|^2 = \tfrac{1}{2}\sum{}' |c_{kl}|^2 \qquad \text{finite.}$$

Suppose the c_{kl} real. Then Q is a real symmetric Hilbert-Schmidt matrix. Now what is important is the difference between Hilbert-Schmidt and trace class. If Q is of trace class, then it defines a canonical element in the Fock space, namely

$$\sum \tfrac{1}{2} c_{kl}\, x_k x_l = :\sum{}' \tfrac{1}{2} c_{kl}\, x_k x_l: \;+\; \tfrac{1}{2}\sum_k{}' c_{kk}$$

For example suppose we have a Hilbert Schmidt map

$$T : \ell^2_{\mathbb{R}} \longrightarrow W$$

i.e. $\operatorname{tr}(T^*T) < \infty$. Then the quadratic function on $\ell^2_{\mathbb{R}}$ given by

$$Q(x) = \|Tx\|^2_W = (x, T^*Tx)$$

is the quadratic form belonging to the trace class symm. operator T^*T, so this function Q defines an element of the Fock space. Similarly polynomial functions on W become elements of $\mathcal{H}_F$.

However there are many elements of $\mathcal{H}_F$ that do not come from obviously defined functions on W, and ultimately are realized only by a.e. defined functions on W. My viewpoint is that it is perhaps preferable to understand how such elements are constructed, and to do the construction back in $\mathcal{H}_F$ if possible, instead of relying on measure theory on W.

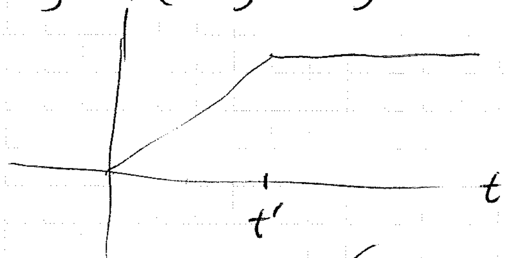
<u>8 August 1985</u>

Let's consider some path integrals which are Gaussian. First consider Brownian motion on the line. This means we consider paths $x(t)$ defined for $t \geq 0$ with $x(0) = 0$ with covariance

$$\int_0^\infty \frac{1}{2} \dot{x}^2 \, dt .$$

The variance $\langle x_t x_{t'} \rangle$ is ~~the~~ Green's fn. for $-\partial_t^2$ on $[0, \infty)$ with the b.c. $x(0) = 0$, $\dot{x}(\infty) = 0$, i.e.

$$\langle x_t x_{t'} \rangle = \min(t, t')$$



Thus the (1-particle) Hilbert space is the space H_1. (To make coordinates let us work on an interval $[0, \beta]$ and use the orthonormal basis

$$e_k(t) = \left(\frac{2}{\beta}\right)^{1/2} \sin kt \qquad k\beta \in \left\{ \left(\frac{\pi}{2} + n\pi \right) \,\middle|\, n \geq 0 \right\}.$$

for $L^2(0, \beta)$. This might not be necessary.)

Now the real Hilbert space over which we have the Gaussian cylinder measure is the H_1-space i.e. real functions $x(t)$ on $(0, \infty)$ with $\dot{x} \in L^2$ and $x(0) = 0$. The map $x \longmapsto \dot{x}$ identifies this space with $L^2(0, \infty)$.

Hence we see that the linear ~~function~~ elements in $\mathcal{H}_F$ are continuous linear fnls. on this H_1-space. Such a fnl is of the form

$$x \longmapsto \int f \dot{x} \, dt$$

for $f \in L^2(0,\infty)$. This is the embedding due to Paley + Wiener: [redacted]

$$L^2(0,\infty) \hookrightarrow L^2(\text{Wiener space})$$
$$f \longmapsto \int f\,dx$$

In particular evaluation at time t is in this space ($f = \chi_{[0,t]}$). On the other hand $x \mapsto \dot{x}_t$ is not in the Fock space.

Now the next thing to look at are quadratic functions. There are three things [redacted] $\int \dot{x}_t^2$ $\quad \int x_t \dot{x}_t$ $\quad \int x_t^2$

to consider first. [redacted] Let's cacalate using the orth basis [redacted]. $\frac{1}{k} e_k$ for H_1. Thus a typical element of H_1 is

$$x_t = \sum c_k \frac{1}{k} e_k(t) \quad \text{with norm} \quad \sum c_k^2 = \int \dot{x}_t^2$$

Then $\int x_t^2 = \sum' c_k^2 \frac{1}{k^2}$ which is of trace class [redacted] as $\sum \frac{1}{k^2} < \infty$.

But $$\int x_t \dot{x}_t \, dt = \sum_k{}' \frac{c_k^2}{k}$$

and this is only H.S.

What this means is that the quadratic function

$$x \longmapsto \int_0^\beta x_t \dot{x}_t \, dt$$

is not canonically an element [redacted] in Fock space.

A natural question is what exactly is the function

$$: \int_0^\beta x_t \mathring{x}_t \, dt :$$

It should of course be related to $\frac{1}{2} x_\beta^2$, but it is not this, because we know x_β is a Gaussian r.v. with [redacted] variance β. Thus we expect

$$: \int_0^\beta x_t \mathring{x}_t \, dt : \;=\; \frac{1}{2}\left(x_\beta^2 - \beta\right)$$

August 9, 1985

Goal: To formulate nicely the link between functional integrals and QM. The formulation should somehow encompass what I learned from Stroock's book about Markov chains, and hopefully it might link up with what I saw in Fried's papers about zeta functions and Markov partitions.

Review what I learned from Stroock's book.

Cramer thm: μ probability measure on V

$$Z(J) = \int_V e^{Jx}\,\mu(dx) \qquad J \in V^*$$

μ_n = image of $\mu^{\otimes n}$ under $V^n \longrightarrow V$, $(x_1,\dots,x_n) \longmapsto \frac{1}{n}\sum x_j$

$$Z(J,\mu_n) = Z(J/n)^n = \Big(1 + \underbrace{Z'(0)}_{\langle x\rangle}\tfrac{J}{n} + \dots\Big)^n$$

$$\longrightarrow e^{J\langle x\rangle} \qquad \text{as } n \longrightarrow \infty$$

WLLN: $\mu_n \longrightarrow \delta_{\langle x\rangle}$

Cramèr thm says:

$$\mu_n(x) \sim e^{-nW(x)}$$

$$W(x) = \sup_J\big(Jx - \log Z(J)\big)$$

Sanov thm. is special case where $V = \mathcal{M}(M)$ and μ starts on M. Then μ_n = image of $\mu^{\otimes n}$ under

$$M^n \longrightarrow \mathcal{M}(M)$$
$$(x_1,\dots,x_n) \longmapsto \tfrac{1}{n}(\delta_{x_1} + \dots + \delta_{x_n})$$

Here J becomes $f \in C_b(M)$ and

$$Z(f) = \int_M e^f\,d\mu$$

$$Z(f, \mu_n) = \int_{M^n} e^{\frac{1}{n}\sum f(x_j)} \mu(dx_1)\cdots\mu(dx_n)$$

$$= Z\left(\tfrac{1}{n}f\right)^n \xrightarrow[n\to\infty]{} e^{\int f\,d\mu}$$

$$\mu_n \longrightarrow \delta_\mu$$

Sanov thm. says

$$\mu_n(\alpha) \sim e^{-nW(\alpha)}$$

$$W(\alpha) = \sup_f \int f\,d\alpha - \log Z(f) \qquad \text{etc.}$$

Next discuss Markov chain variant. Suppose M finite to simplify, [redacted] let $p(x,y)$, $(x,y)\in M$ be a positive matrix (strictly >0 to simplify). Let $\mu_{(n)}$ be the measure on M^n

$$\mu_{(n)}(x_1,\dots,x_n) = [redacted]\, p(x_1,x_2)\cdots p(x_{n-1},x_n)\, [redacted] \,/\,\text{norm}$$

and given $f\in C(M)$, set

$$Z(f,\mu_n) = \int_{M^n} e^{\frac{1}{n}\sum f(x_j)} \mu_{(n)}(x_1,\dots,x_n)$$

$$= [redacted]\, \langle 1| e^{\frac{1}{n}f} \underbrace{p\, e^{\frac{1}{n}f}\cdots\cdots p\, e^{\frac{1}{n}f}}_{n \text{ times.}} |1\rangle \Big/ \langle 1|p^n|1\rangle$$

[redacted] Let $\langle 0|$, $|0\rangle$ be the right + left positive eigenvectors for p normalized so that $\langle 0|0\rangle = 1$. Then

$$Z(f,\mu_n) \longrightarrow e^{\langle 0|f|0\rangle}$$

(This is a calc. using perturbation theory of first order)

[redacted]

It would have been simpler to take p stochastic

$\sum_y p(x,y) = 1$, and let $\langle 0| = \nu$ so that $\sum_x \nu(x)\, p(x,y) = \nu(y)$ and $\sum \nu(x) = 1$. Then we set

$$\mu_{(n)}(x_1, \dots, x_n) = \nu(x_1)\, p(x_1,x_2) \cdots\cdots p(x_{n-1},x_n)$$

and have

$$Z(f, \mu_n) = \langle \nu | e^{\frac{1}{n}f} (p e^{\frac{1}{n}f})^n | 1 \rangle$$

$$\longrightarrow e^{\int f d\nu}$$

reflecting the fact that $\mu_n \longrightarrow \delta_\nu$ as measures on $\mathcal{M}(M)$.

Now when we come to discuss the large deviations result we want the behavior of

$$Z(nf, \mu_n) = \langle \nu | e^f (p e^f)^n | 1 \rangle$$

$$\sim \lambda_{max}(p e^f)^n$$

and the result should be

$$\mu_n(\alpha) \sim e^{-nW(\alpha)}$$

$$W(\alpha) = \sup_f \left\{ \int f d\alpha - \log \lambda_{max}(p e^f) \right\}$$

Next we pass to the continuous case where the semi group p^n is replaced by e^{-tH} with $H \geq 0$ (so that we have a contraction). (Note that the fact p isn't symmetric necessarily suggests H needn't be self-adjoint, but we assume it is as well as having a non-degenerate ground state $|0\rangle$ with $H|0\rangle = 0$.)

Then

$$Z(f, \mu_\beta) = \langle 0 | e^{-\beta(H - \frac{1}{\beta} f)} | 0 \rangle$$

$$\longrightarrow e^{\langle 0 | f | 0 \rangle}$$

and

$$Z(\beta f, \mu_\beta) = \langle 0 | e^{-\beta(H - f)} | 0 \rangle$$

$$\sim e^{-\beta \lambda_{min}(H-f)}$$

should be the continuous formulas.

And this is reasonable because suppose $e^{-\beta H}$ were to be given by a path integral:

$$\int \mathcal{D}x \; \underbrace{e^{-\int_0^\beta [\frac{1}{2}\dot{x} + V(x)]\, dt}}_{e^{-S(x)}}$$

so that $e^{-S(x)} \mathcal{D}x$ is the analogue of $\mu_{(n)}$ on M^n.

Then the generating function for μ_β should be

$$\int \mathcal{D}x \, e^{-S(x)} \; e^{\frac{1}{\beta} \int_0^\beta f(x_t)\, dt}$$

which indeed corresponds to the action

$$+ \int_0^\beta \left(\frac{1}{2} \dot{x}^2 + V(x) - \frac{1}{\beta} f(x) \right) dt$$

and hence to the operator $e^{-\beta(H - \frac{1}{\beta} f)}$.

August 10, 1985

I still am trying to reach an understanding of the Osterwalder-Schrader reconstruction of a QFT from Schwinger functions. I am working with ordinary QM of a particle on the line. The Schwinger functions are

$$\langle 0 | T[x(t_1)\cdots x(t_n)] | 0 \rangle$$

$$= \int \mathcal{D}x \; e^{-S(x)} \; x_{t_1} [redacted] \cdots x_{t_n} [redacted] .$$

These are just the moments of the path integral prob. measure.

I have decided to assume that the path integral probability measure is Gaussian and stationary. This case occurs with the s.h.o. So I have a stationary Gaussian process x_t, and we know this is completely determined by the variance

$$G(t-t') = \langle x_t x_{t'} \rangle$$

which has the representation

$$\begin{cases} G(t) = \int e^{-ikt} \, d\mu(k) \\ d\mu(k) = d\mu(-k) \end{cases}$$

(This comes from the fact that the 1-particle space for the Gaussian measure is a real Hilb. space with unitary one-param. gp. and cyclic vector x_0.)

Let's start again. I am trying to understand better the link between the path integral and QM, and have decided to look carefully at the case where the path integral is Gaussian. In this case the path integral amounts to the probability measure describing a stationary Gaussian process x_t; and it is determined by the variance

$$G(t-t') = \langle x_t x_{t'} \rangle$$

which we know has the form

$$G(t) = \int e^{i\omega t} \, d\mu(\omega) \qquad d\mu(-\omega) = d\mu(\omega).$$

On the other hand we have

$$G(t) = \langle 0 | T[x(t), x] | 0 \rangle$$

$$= \langle 0 | x e^{-tH} x | 0 \rangle \qquad \text{if } t > 0$$

so applying the **spectral** thm. to H gives

$$G(t) = \int e^{-\lambda |t|} \, d\rho(\lambda)$$

In the example of the simple harmonic oscillator $H = \omega_0 a^* a$, we have for $\omega_0 > 0$

$$G(t) = \frac{e^{-\omega_0 |t|}}{2\omega_0} = \langle t | \frac{1}{-\partial_t^2 + \omega_0^2} | 0 \rangle$$

$$= \int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\omega^2 + \omega_0^2}$$

Thus given

$$G(t) = \int_0^\infty e^{-\lambda|t|} \, d\rho(\lambda)$$

we can write it

$$G(t) = \int_0^\infty \frac{e^{-\lambda|t|}}{2\lambda} \, 2\lambda \, d\rho(\lambda)$$

$$= \int_0^\infty \left(\int \frac{d\omega}{2\pi} \frac{e^{i\omega t}}{\lambda^2+\omega^2} \right) 2\lambda \, d\rho(\lambda)$$

$$= \int \frac{d\omega}{2\pi} e^{i\omega t} \left\{ \int_0^\infty \frac{2\lambda \, d\rho(\lambda)}{\lambda^2+\omega^2} \right\}$$

It would therefore, at least formally, appear that if one has the QM situation: $G(t) = \langle 0 | T[x(t)x] | 0 \rangle$ one gets the stationary Gaussian process. It seems one must add the assumption that $H > 0$.

It is interesting to construct these QM theories. I already have the s.h.o. which corresponds to $d\rho(\lambda)$ a δ-function at the frequency. So let's take a multiple oscillator with different frequencies ω_k:

$$H = \sum \omega_k a_k^* a_k$$

and take x to be a suitable linear combination of the $x_k = \frac{1}{\sqrt{2\omega_k}}(a_k^* + a_k)$ Then

$$x = \sum_k c_k x_k = \sum_k c_k \frac{(a_k^* + a_k)}{\sqrt{2\omega_k}}.$$

August 11, 1985

One of the things we discovered yesterday is the following. Normally the QM of a particle moving in M is described by a path integral over the paths in M, so that one expects the Hilbert space $L^2(\mathbb{R})$ to occur when using paths in $\mathbb{R}$. This is also what one expects from the matrix multiplication picture of Feynman — the reason one gets paths in M is that one is using the basis $|x\rangle$, $x \in M$, for $L^2(M)$.

However it seems that we can actually describe a much larger Hilbert space than $L^2(\mathbb{R})$ using paths in $\mathbb{R}$. We have examples illustrating how a multiple harmonic oscillator can be described by ordinary Gaussian integral on a space of paths $x(t)$ in $\mathbb{R}$. I'd like now to check this assertion carefully.

First let's analyze the simple h.o. carefully. Consider the Schwinger functions

$$\langle 0 | T[x(t_1) \cdots x(t_n)] | 0 \rangle$$

$$= \langle 0 | x e^{-(t_1-t_2)H} x e^{-(t_2-t_3)H} x \cdots e^{-(t_{n-1}-t_n)H} x | 0 \rangle$$

if $t_1 \geq t_2 \geq \cdots \geq t_n$.

These Schwinger functions can all be obtained from a generating function

$$Z(J) = \langle 0 | S_J | 0 \rangle$$

where $J = J(t)$ is an elt of $C_c^\infty(\mathbb{R})$ and

S_J is the scattering operator for the time dependent perturbation

$$H - J(t)\, x = \omega a^* a - J(t) \frac{a^* + a}{\sqrt{2\omega}}$$

Dyson's formula gives

$$S_J = T\left\{ e^{\int J(t)\, e^{tH} \frac{a^*+a}{\sqrt{2\omega}} e^{-tH}\, dt} \right\}.$$

More precisely ~~the~~ propagator (we in imag. time) is defined by

$$\partial_t U_J(t) = -(H - J(t)x)\, U_J(t)$$

$$U_J(0) = I,$$

so

$$\partial_t\, e^{tH} U_J(t) = e^{tH}\{H - H + J(t)x\}\ U_J(t)$$

$$= \left(e^{tH} J(t) x e^{-tH}\right) e^{tH} U_J(t)$$

and

$$S_J = e^{TH}\, U_J(T, T')\, e^{-T'H} \qquad \begin{array}{l} T \gg 0 \\ T' \ll 0 \end{array}$$

But next we evaluate S_J to get (move $e^{tH} a e^{-tH} = e^{-\omega t} a$ past the $e^{t'H} a^* e^{-t'H} = e^{\omega t'} a^*$ for $t > t'$ to get)

$$\langle 0 | S_J | 0 \rangle = \exp\left\{ \iint_{t > t'} \frac{1}{2\omega} e^{-\omega t + \omega t'} J(t) J(t')\, dt\, dt' \right\}$$

$$= \exp\left\{ \frac{1}{2} \iint \frac{e^{-\omega |t-t'|}}{2\omega} J(t) J(t')\, dt\, dt' \right\}$$

What the calculation amounts to is a verification of the fact that the Schwinger functions $\langle 0 | T[x(t_1) \ldots x(t_n)] | 0 \rangle$ are the moments of the Gaussian process with variance $\langle x_t x_{t'} \rangle = \dfrac{e^{-\omega|t-t'|}}{2\omega}$

Next let us consider a multiple harm. oscillator with distinct frequencies ω_k : $H = \sum \omega_k a_k^* a_k$. Then we consider the perturbed Ham.

$$H_J = H - \sum J_k(t) \underbrace{\frac{a_k^* + a_k}{\sqrt{2\omega_k}}}_{x_k}$$

and have the generating fn.

$$\langle 0 | S_J | 0 \rangle = \exp\left\{ \frac{1}{2} \iint \sum \frac{e^{-\omega_k|t-t'|}}{2\omega_k} J_k(t) J_k(t')\, dt\, dt' \right\}$$

At this point I want to let

$$x = \sum c_k x_k$$

where each c_k is real $\neq 0$. The generating fn. for the Schwinger fns.

$$\langle 0 | T[x(t_1) \ldots x(t_n)] | 0 \rangle$$

is clearly

$$\exp\left\{ \frac{1}{2} \iint \left(\sum_k \frac{e^{-\omega_k|t-t'|}}{2\omega_k} c_k^2 \right) J(t) J(t')\, dt\, dt' \right\}.$$

What this means is that we have constructed a Q.M. setup $(\mathcal{H}, x, H, |0\rangle)$ whose Schwinger fns. are the moments of the Gaussian process with

$$\langle x_t x_{t'} \rangle = \sum_k \frac{e^{-\omega_k |t-t'|}}{2\omega_k} c_k^2$$

Notice that

$$e^{tH} x e^{-tH} = \sum c_k \frac{e^{\omega_k t} a_k^* + e^{-\omega_k t} a_k}{\sqrt{2\omega_k}}$$

and since the ω_k are distinct and the $c_k \neq 0$, it follows that the alg. generated by x, H contains all the a_k, a_k^*. Hence $\mathcal{H}$ has to be the full Hilbert space of the multiple oscillator.

<u>Remark:</u> I was reminded, in the course of the calculation of $\langle 0 | S_J | 0 \rangle$, of the fermionic calculation

$$tr_s\left(e^{a^*\omega a + \tilde{J} a + a J}\right) = tr_s\left(e^{(a^* + \tilde{J}\omega^{-1})\omega(a + \omega^{-1}J)} e^{-\tilde{J}\omega^{-1}J}\right)$$

$$= tr_s\left(e^{a^*\omega a}\right) e^{-\tilde{J}\omega^{-1}J}$$

which identifies the former non-comm. generating fn. with the Gaussian one

$$\int \mathcal{D}\psi^* \mathcal{D}\psi \; e^{\psi^*\omega\psi + \tilde{J}\psi + \psi^* J}$$

up to some constant.

August 12, 1985

When considering the s.h.o. we used the formulas $x = \frac{1}{\sqrt{2\omega}}(a^* + a)$, $p = \sqrt{\frac{\omega}{2}}(ia^* - ia)$. But this is based on the mass being 1 and $H = \frac{1}{2}(p^2 + \omega^2 q^2)$. Recall in general:

$$x = \mathrm{Re}(Ae^{-i\omega t})$$

$$\dot{x} = \omega\,\mathrm{Im}(Ae^{-i\omega t})$$

$$m\omega^2 = k \qquad \omega = \sqrt{\tfrac{k}{m}}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}(m\omega^2\,\mathrm{Im}^2 + k\,\mathrm{Re}^2)$$

$$= \frac{1}{2}m\omega^2|A|^2$$

Quantization $\Rightarrow$ $= \omega a^* a \quad \Longrightarrow \quad |A|^2 = \frac{2}{\omega m}|a|^2$

$$\Longrightarrow \quad A = \sqrt{\frac{2}{\omega m}}\, a$$

$$\therefore \quad x(t) = \frac{1}{2}\sqrt{\frac{2}{\omega m}}\left(a e^{-i\omega t} + a^* e^{i\omega t}\right)$$

$$\therefore \quad \boxed{x = \frac{1}{\sqrt{2\omega m}}(a^* + a) \qquad \text{when } m \neq 1}$$

The point is that there is nothing sacred about which multiple of $a^* + a$ we take to be x.

So now suppose we are given

$$* \qquad G(t) = \int_0^\infty e^{-\lambda|t|}\, d\rho(\lambda)$$

and that we want to construct a Gaussian field theory (0 space dimension, real scalar) with $\langle 0|xe^{-tH}x|0\rangle = G(t)$ for $t > 0$. A multiple harmonic oscillator is given by a complex Hilbert space V together with a self-adjoint $H > 0$.

Let's suppose there is a cyclic vector v for H. Then we get a measure $d\rho$ on $\mathbb{R}_{\geq 0}$ without atom at 0 and isoms.

$$\begin{array}{ccc} V & \xrightarrow{\sim} & L^2(\mathbb{R}_{\geq 0}, d\rho) \\ H & \longleftrightarrow & \text{mult by } \lambda \\ v & \longleftrightarrow & \underline{1} \end{array}$$

In particle V acquires a real reduction. So what I can do is the following:

Given $G(t) = \int_0^\infty e^{-\lambda|t|}\, d\rho(\lambda)$, set $V = L^2(\mathbb{R}_{\geq 0}, d\rho)$ form the Fock space $\hat{S}(V)$ and let

$$x = a^*(v) + a(v)$$

Then
$$\langle 0 | x e^{-tH} x | 0 \rangle = \langle v | e^{-tH} | v \rangle$$
$$= \int_0^\infty e^{-\lambda t}\, d\rho(\lambda).$$

as desired.

The question then becomes which Gaussian processes are such that $G(t)$ is the Laplace transform of a measure. Using

$$e^{-\lambda|t|} = \int \frac{d\omega}{2\pi} e^{i\omega t} \frac{2\lambda}{\omega^2+\lambda^2}$$

we get
$$G(t) = \int_0^\infty e^{-\lambda|t|} d\rho(\lambda) = \int \frac{d\omega}{2\pi} e^{i\omega t} \int_0^\infty \frac{2\lambda}{\omega^2+\lambda^2}\, d\rho(\lambda)$$

But $\varphi(\omega) = \int_0^\infty \frac{2\lambda}{\omega^2+\lambda^2}\, d\rho(\lambda)$ is always > 0

so one can't get all processes, e.g. $\varphi(\omega) = \delta(\omega-\omega_0) + \delta(\omega+\omega_0)$

At this point I want to see how the reflection-positivity works in the Gaussian case. Thus I consider a distribution $G(t)$ on $0<t<\infty$ satisfying the positivity condition

$$\underbrace{\int_0^\infty\int_0^\infty dt\,dt'\; G(t+t')\,f(t)\,f(t')}_{\|f\|^2} \geq 0$$

for all $f(t) \in C_c^\infty(\underbrace{(0,\infty)}_{\mathbb{R}_+})$ (real). By completing with respect to this inner product on $C_c^\infty(\mathbb{R}_+)$ I get a Hilbert space; denote it $\mathcal{H}$.

What I want to do it to identify $\mathcal{H}$ with $L^2(\mathbb{R}, d\rho)$ where $d\rho$ is a measure on $\mathbb{R}$ whose support is bounded below.

Before I describe the properties of this identification, I should discuss translation. Given $a \geq 0$ let $(T_a f)(t) = f(t-a)$ on $C_0^\infty(\mathbb{R}_+)$. Then

$$\begin{aligned}
\langle g | T_a f\rangle &= \iint dt\,dt'\; G(t+t')\,g(t)\,f(t'-a) \\
&= \iint dt\,dt'\; G(t+t'+a)\,g(t)\,f(t') \\
&= \langle T_a g | f\rangle \\
&= \langle T_{a/2}\, g | T_{a/2} f\rangle
\end{aligned}$$

I can now consider the closure of T_a; T_a is a densely-defined symmetric operator such that

$$\langle f | T_a f\rangle = \| T_{a/2} f\|^2 \geq 0$$

Without further hypotheses I don't think it is possible to prove that T_a is bounded.

In fact suppose we take

$$* \qquad G(t) = \int_{\mathbb{R}} e^{-\lambda|t|}\, d\rho(\lambda)$$

where $d\rho$ has support not bounded below?

Let's work on the picture we want. We want to express G in the form $*$, which implies

$$G(t) = \langle v | e^{-tH} | v \rangle$$

where H is a self-adjoint operator and v is a cyclic vector. Hence $H =$ mult. by λ on $L^2(\mathbb{R}, d\rho)$ and $v = 1$. If we have this representation, then we have a map from $C_c^\infty(\mathbb{R}_+)$ to $L^2(\mathbb{R}, d\rho)$ given by

$$f(t) \longmapsto \int_0^\infty f(t) e^{-\lambda t}\, dt = \int_0^\infty f(t) e^{-tH} v\, dt$$

and this map is an isometry, so it extends to an embedding $\mathcal{H} \hookrightarrow L^2(\mathbb{R}, d\rho)$. It's an isom. when the functions

$$\hat{f}(\lambda) = \int_0^\infty f(t)\, e^{-\lambda t}\, dt \qquad \text{for } f \in C_c^\infty(\mathbb{R}_+)$$

are dense in $L^2(\mathbb{R}, d\rho)$. The functions $\hat{f}(\lambda)$ are a subalgebra of continuous functions on $\mathbb{R}$ vanishing at $+\infty$. It's clear this subalgebra separates points, so by Stone-Weierstrass this algebra of functions is uniformly dense in the alg. of continuous functions on $[a, \infty)$ vanishing at $+\infty$ for any a.

Let's summarize what we have so far. Given $G(t)$ a distribution on $(0, \infty)$ satisfying the pos. condition we can make a Hilbert space by completing $C_c^\infty(\mathbb{R}_+)$. Indeed let's start with $C_c^\infty(\mathbb{R}_+)$ as an algebra under convolution:

$$(f*g)(t) = \int_0^t f(t-t')\,g(t')\,dt'$$

and consider the linear fnl. on it

$$\tau(f) = \int_0^\infty G(t)f(t)\,dt.$$

This linear functional is positive

$$\tau(f*f) = \int_0^\infty G(t)\left(\int_0^t f(t-t')f(t')\,dt'\right)dt$$

$$= \iint_{0\le t'\le t} G(\overset{u+t'}{t})\, f(\overset{u}{t-t'})f(t')\,dt'\,dt$$

$$= \iint_{0\le u,t'} G(u+t')\,f(u)f(t')\,du\,dt' \geq 0.$$

(Formally analogous to having a positive linear functional on the algebra of polynomials, i.e. a Hankel matrix.)

So I am looking at a continuous analogue of the moment problem.

So I have this positive linear functional on the algebra of $\{\hat{f}(\lambda) \mid f\in C_c^\infty(\mathbb{R}_+)\}$ of Laplace transforms.

August 13, 1985

Review some gas theory. Grand partition fn is $\mathcal{Z}(T,V,\mu) = \operatorname{tr}\left(e^{-\beta(H-\mu N)}\right)$. In terms of this one has

$$p = \frac{1}{\beta}\partial_V \log \mathcal{Z}$$
$$N = \frac{1}{\beta}\partial_\mu \log \mathcal{Z}$$

In the infinite volume limit

$$\partial_V \log \mathcal{Z} = \frac{\log \mathcal{Z}}{V} \quad \text{fn. of } \beta, \mu$$

and we have

$$\begin{cases} p = \frac{1}{\beta}\left(\frac{\log \mathcal{Z}}{V}\right)(\beta,\mu) \\ \rho = \frac{1}{\beta}\,\partial_\mu\left(\frac{\log \mathcal{Z}}{V}\right)(\beta,\mu) \end{cases}$$

from which we eliminate μ to get the eqn. of state relating p, ρ, β.

Next consider an Ising model. We can think of this as a lattice gas and essentially identify H the applied magnetic field with μ. Take $\beta = 1$ and the linear Ising model to fix the ideas. The energy of the configuration $s = (s_n)$ is

$$-E = H\sum s_n + J\sum s_n s_{n+1}$$

and the partition function with periodic b.c. is

$$\mathcal{Z} = \sum_s \prod_n e^{Hs_n} \cdot e^{Js_n s_{n+1}} = \operatorname{tr}\left(e^{Hs}P\right)^N$$

where

$$e^{Hs} = \begin{pmatrix} e^H & 0 \\ 0 & e^{-H} \end{pmatrix} \qquad P = \begin{pmatrix} e^J & e^{-J} \\ e^{-J} & e^J \end{pmatrix}$$

Note this is the analogue of the grand partition function for the lattice gas. The analogue of the pressure is

$$P = \lim_{N\to\infty} \frac{\log \mathcal{Z}(H,N)}{N} = \log \lambda_{max}(e^{Hs}p)$$

and the analogue of the density is

$$\rho = \frac{1}{N}\,\partial_H \log \mathcal{Z}(H,N) = \bar{s}$$

the average spin.

So far I haven't brought in the measure that one expects to see on $\{\pm 1\}^N$. There appears to be a different probability measure for each H.

Also I would like to bring in the Lee-Yang thm. which says the complex zeroes of $\mathcal{Z}(H,N)$ are located on $Re(H)=0$.

August 14, 1985:

Goal: To get a feeling for Gaussian Markov chains.

Recall that a Markov chain (= discrete-time Markov process) is given by a state space M and transition probabilities - these are probability measures

$$p(x,y)\,dy \text{ on } M \qquad \forall x \in M.$$

giving the transition probability starting from x. Then one puts a probability measure on M^n

$$p^{(n)}(x_1,\dots,x_n) = \mu(x_1)\,p(x_1,x_2)\cdots p(x_{n-1},x_n)\,dx_1\cdots dx_n$$

where $\mu(x)dx$ is an initial probability distribution.

Suppose $M = \mathbb{R}$ and that the probability measures $p^{(n)}$ are Gaussian. This implies ~~means~~ that

$$\mu(x) = e^{-\frac{1}{2}\alpha x^2}\sqrt{\frac{\alpha}{2\pi}}\,dx$$

$$(*) \qquad \mu(x)\,p(x,y) = e^{-\frac{1}{2}Q(x,y)}\,\frac{(\det Q)^{1/2}}{2\pi}\,dx\,dy$$

where $Q(x,y)$ is a positive quadratic form.

A natural condition to impose is to have a limiting distribution. Thus if

$$\mu_2(y) = \int \mu_1(x)\,p(x,y)\,dx \qquad \text{where } \mu = \mu_1$$

etc. we want μ_n to converge.

From $(*)$ we see that

$$p(x,y) = e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}\,dy \cdot \text{constant}$$

and the only way for this to satisfy

$$\int p(x,y)\,dy = 1 \qquad \text{for all } x$$

is to have

$$p(x,y) = e^{-\frac{c}{2}\left(\frac{b}{c}x+y\right)^2}\sqrt{\frac{c}{2\pi}}\,dy$$

so that $\qquad a = \dfrac{b^2}{c}.$

Let's first consider ~~the~~ general ~~situation~~:

$$p(x,y) = e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}$$

Think of this as a positive matrix and look for its largest eigenvector. Assume a Gaussian input

$$\mu(x) = e^{-\frac{1}{2}\alpha x^2}$$

and then evaluate $\displaystyle\int \mu(x)\,p(x,y)\,dx$ by the critical point

$$\delta_x \tfrac{1}{2}\left(\alpha x^2 + ax^2 + 2bxy + cy^2\right) = (\alpha+a)x + by = 0$$

$$x = \frac{-by}{\alpha+a}$$

$$\frac{1}{2}\left((\alpha+a)\left(\frac{-by}{\alpha+a}\right)^2 + 2b\left(\frac{-by}{\alpha+a}\right)y + cy^2\right)$$

$$= \frac{1}{2}\left(\frac{-b^2}{\alpha+a}+c\right)y^2.$$

<u>Formula:</u>

$$\int e^{-\frac{1}{2}\alpha x^2}\,e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}\,dx$$

$$= \sqrt{\frac{\alpha+a}{2\pi}}\;e^{-\frac{1}{2}\left(\frac{-b^2}{\alpha+a}+c\right)y^2}$$

Similarly

$$\int e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}\, e^{-\frac{1}{2}\beta y^2}\, dy$$

$$= \sqrt{\frac{\beta+c}{2\pi}}\; e^{-\frac{1}{2}\left(\frac{-b^2}{\beta+c}+a\right)x^2}$$

The condition that $\mu(x) = e^{-\frac{1}{2}\alpha x^2}$ be a right eigenvector (i.e. $\int \mu(x)\, p(x,y)\,dx = \mu(y)$) is therefore that

$$\alpha = \frac{-b^2}{\alpha+a} + c$$

The eigenvalue is $\sqrt{\frac{a+\alpha}{2\pi}}$

or that $\boxed{(a+\alpha)(c-\alpha) = b^2}$

The condition the $\nu(x) = e^{-\frac{1}{2}\beta x^2}$ be a left eigenvector for p is that

$$\beta = \frac{-b^2}{\beta+c} + a$$

or $\boxed{(a-\beta)(c+\beta) = b^2}$ The eigenvalue is $\sqrt{\frac{c+\beta}{2\pi}}$.

These are consistent as

$$a+\alpha = c+\beta \iff a-\beta = c-\alpha$$

In order that the eigenvectors exist we need a real root of the quadratic equation

$$\alpha^2 + (a-c)\alpha + b^2 - ac = 0$$

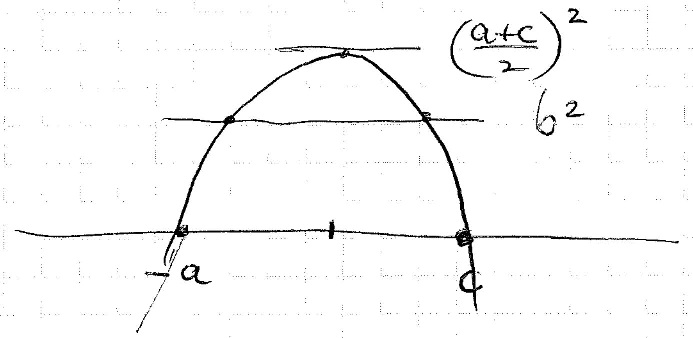
whence $(a-c)^2 \geq 4(b^2-ac)$

or $(a+c)^2 \geq 4b^2$ or $\left(\frac{a+c}{2}\right)^2 \geq b^2$

We also want $a+\alpha>0$, whence $c-\alpha>0$, and

$$-a<\alpha<c \qquad \Longrightarrow \quad -a<c \Longleftrightarrow \quad a+c>0.$$

Picture of $f(\alpha)=(a+\alpha)(c-\alpha)$



Thus if one assumes that $a+c>0$ and that $\frac{a+c}{2} > |b|$, there are two roots α.
We are interested in the largest eigenvalue $\alpha+a$ hence the larger of the two roots.

The other point is that if $a, c>0$ and $ac>b^2$ so that $ax^2+2bxy+cy^2$ is pos. definite. then as $\left(\frac{a+c}{2}\right)^2 \geqslant ac$ always, one necessarily has the conditions $\frac{a+c}{2}>|b|$ satisfied.

<u>Summary</u>: The matrix $p(x,y)=e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}$ has left + right eigenvectors which are Gaussian when $\frac{a+c}{2}\geqslant|b|$.

Actually, the following is clearer. We can consider the fractional linear transformation

$$\alpha \longmapsto \frac{-b^2}{a+\alpha}+c = \frac{c\alpha+(ac-b^2)}{\alpha+a}$$

$$=\begin{pmatrix} c & ac-b^2 \\ 1 & a \end{pmatrix}(\alpha)$$

whose fixpoints give the Gaussian right eigenvectors. The characteristic equation of this matrix is

$$\lambda^2 - (a+c)\lambda + b^2 = 0$$

and there are real fixpoints α when there are real eigenvalues, i.e.

$$\left(\frac{a+c}{2}\right)^2 \geqslant b^2$$

Moreover if we have $>$ here, then there are two real eigenvalues of the same sign (in fact >0 if $a+c>0$), and the α corresponding to the larger eigenvalue in abs. value is an attractive fixpoint for the fractional linear transformation.

<u>Recap</u>. We have investigated the Perron-Frobenius thm. for the Gaussian matrix

$$p(x,y) = e^{-\frac{1}{2}(ax^2+2bxy+cy^2)}$$

and now we should get back to a Gaussian Markov process. Here one wants $\beta = 0$ to give a fixpoint, i.e. $ac = b^2$, whence

$$\alpha = c - a$$

So we see that the transition prob.

$$p(x,y)dy = e^{-\frac{1}{2}c\left(\frac{b}{c}x+y\right)^2}\sqrt{\frac{c}{2\pi}}\,dy$$

where $|b| < c$ defines a Gaussian Markov process with limiting distribution

$$\mu(x) = e^{-\frac{1}{2}\alpha x^2}\sqrt{\frac{\alpha}{2\pi}}\,dx \qquad \alpha = c - \frac{b^2}{c}$$

Check: Let this Markov process be denoted x_n and start with $x_0 = 0$. Then x_1 is Gaussian with $\langle x_1^2 \rangle = \frac{1}{c}$ and

$$x_2 = \frac{-b}{c} x_1 + y_1$$

where y_1 is independent of x_1 and y_1 is also Gaussian of variance $\frac{1}{c}$. Thus

$$\langle x_2^2 \rangle = \frac{b^2}{c^2}\frac{1}{c} + \frac{1}{c}$$

In general

$$\langle x_n^2 \rangle = \frac{1}{c} + \frac{1}{c}\frac{b^2}{c^2} + \cdots + \frac{1}{c}\left(\frac{b^2}{c^2}\right)^n$$

$$\longrightarrow \frac{1}{c}\frac{1}{1 - \frac{b^2}{c^2}} = \frac{1}{c - \frac{b^2}{c}}$$

August 15, 1985

Yesterday we learned about Gaussian Markov chains. Such a thing is a sequence of real random variables x_n which are jointly Gaussian such that when x_n is projected onto the subspace spanned by the x_k for $k<n$ one gets hx_{n-1} for some number h. Thus we have

$$x_n = hx_{n-1} + y_n$$

where the y_n are independent identical r.v. Supposing $\langle y_n^2 \rangle = 1$, we have

$$\langle x_n^2 \rangle = h^2 \langle x_{n-1}^2 \rangle + 1$$

$$= 1 + h^2 + h^4 + \cdots + h^{2n}\langle x_0^2 \rangle$$

There are two cases. If $|h|<1$, then we see x_n has a limit. We can then construct a process for $n \in \mathbb{Z}$ by taking independent y_n and letting

$$x_n = y_n + hy_{n-1} + h^2 y_{n-2} + \cdots$$

This gives a stationary Gaussian (discrete-time) process with

$$\langle x_n x_0 \rangle = \langle y_n + hy_{n-1} + \cdots + h^n y_0 + h^{n+1}y_{-1} + \cdots \mid y_0 + hy_{-1} + \cdots \rangle$$

$$= h^n + h^{n+2} + \cdots = \frac{h^n}{1-h^2} \qquad n \geq 0$$

so

$$\boxed{\langle x_n x_0 \rangle = \frac{h^{|n|}}{1-h^2}}$$

If $|h| \geq 1$, then there is no limit of x_n as $n \to \infty$, and the process has to start somewhere, i.e. with $x_0 = 0$. For instance if $h = 1$ we have a discrete Brownian motion

$$x_n = y_n + \cdots + y_1 \qquad \langle x_n^2 \rangle = n.$$

Next lets look at the continuous case. Thus we suppose given a Gaussian process x_t such that if $t_1 < t$ and if we project x_t onto the subspace spanned by the $x_{t'}$ for $t' \leq t_1$ we get a multiple of x_{t_1}. Call the space spanned by the $x_{t'}$ for $t' \leq t$, $\mathcal{H}_t$. Then we have for $t_2 < t_1 < t$

$$P_{\mathcal{H}_{t_1}} x_t = \lambda_{t,t_1} x_{t_1}$$

$$\underbrace{P_{\mathcal{H}_{t_2}} P_{\mathcal{H}_{t_1}}}_{\parallel} x_t = \lambda_{t,t_1} P_{\mathcal{H}_{t_2}} x_{t_1} = \lambda_{t,t_1} \lambda_{t_1,t_2} x_{t_2}$$

$$P_{\mathcal{H}_{t_2}} x_t = \lambda_{t,t_2} x_{t_2}$$

so

$$\lambda_{t,t_2} = \lambda_{t,t_1} \lambda_{t_1,t_2}$$

If we require stationarity: $\lambda_{t,t'} = \lambda_{t-t'}$, then we have

$$\lambda_{t,t'} = e^{-\omega(t-t')} \qquad \begin{array}{l} t > t' \\ \omega > 0. \end{array}$$

so

$$\langle x_t x_{t'} \rangle = e^{-\omega(t-t')} \underbrace{\langle x_{t'}^2 \rangle}_{\text{const.}}$$

Here is the missing argument concerning reflection positivity (after Glimm-Jaffe)

Suppose given the real-valued distribution $G(t)$ on $\mathbb{R}$ $\ni$ $G(-t) = G(t)$ and such that the following two positivity conditions hold:

1) $G(t-t')$ is positive for t,t' in $\mathbb{R}$
2) $G(t+t')$ is positive for $t,t' > 0$.

I have two Hilbert spaces I can form. First I can take $f \in C_0^\infty(\mathbb{R})$ and complete wrt.

$$\|f\|_V^2 = \iint f(t)\,G(t-t')\,f(t')\,dt\,dt' \quad \geq 0 \text{ by } 1).$$

This gives the Hilbert space V which is the 1 particle Hilbert space for the Gaussian measure. Secondly I can form $\mathcal{H}$ which is the completion of $C_0^\infty(\mathbb{R}_{>0})$ with

$$\|f\|_{\mathcal{H}}^2 = \langle \theta f, f\rangle_V = \iint f(-t)\,G(t-t')\,f(t')\,dt'\,dt$$

$$= \iint f(t)\,G(t+t')\,f(t')\,dt'\,dt'$$

First note that by Cauchy-Schwarz

$$\|f\|_{\mathcal{H}}^2 \leq \|\theta f\|_V \cdot \|f\|_V = \|f\|_V^2$$

Secondly let $(T_a f)(t) = f(t-a)$; this maps $C_0^\infty(\mathbb{R}_{>0})$ into itself for $a > 0$, and we have

$$\|T_a f\|_{\mathcal{H}}^2 = \langle f, T_{2a} f\rangle_{\mathcal{H}}$$

So

$$\|T_a f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}^{1/2}\,\|T_{2a} f\|_{\mathcal{H}}^{1/2}$$

$$\leq \|f\|_{\mathcal{H}}^{1/2} \|f\|_{\mathcal{H}}^{1/4} \|T_{4a}\|_{\mathcal{H}}^{1/4}$$

$$\cdots\cdots\cdots \leq \|f\|_{\mathcal{H}}^{1-\frac{1}{2^n}} \|T_{2^n a} f\|_{\mathcal{H}}^{1/2^n}$$

where
$$\|T_{2^n a} f\|_{\mathcal{H}} \leq \|T_{2^n a} f\|_V = \|f\|_V .$$

Thus letting $n \longrightarrow \infty$ gives

$$\|T_a f\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}$$

showing T_a defines a contraction operator on $\mathcal{H}$.

In fact one gets a semigroup of self-adjoint contraction operators. This can be seen to be strongly continuous, hence of the form

$$T_a = e^{-aH} \qquad \text{with} \quad H \geq 0.$$

Next I want to understand the discrete time case of this reflection-positivity of Osterwalder-Schrader.

Suppose given a Hilbert space V with a unitary operator T, and a subspace W of V such that $T(W) \subset W$, and an operator $\theta : W \longrightarrow V$ such that

$$\langle w, w' \rangle_{\mathcal{H}} = \langle \theta w, w' \rangle_V$$

is a non-negative inner product on W, and such that $\theta T = T^{-1} \theta$.

Let $\mathcal{H}$ be the completion of W with respect to the inner product $\langle w, w' \rangle_{\mathcal{H}}$. We claim that T induces a self-adjoint contraction operator on $\mathcal{H}$.

Proof. First

$$\|w\|_{\mathcal{H}}^2 = \langle \Theta w, w \rangle_V \le \|\Theta w\|_V \, \|w\|_V$$

Next

$$\begin{aligned}\langle Tw, w' \rangle_{\mathcal{H}} &= \langle \Theta T w, w' \rangle_V \\ &= \langle T^{-1} \Theta w, w' \rangle_V \\ &= \langle \Theta w, T w' \rangle_V = \langle w, T w' \rangle_{\mathcal{H}}\end{aligned}$$

Next

$$\begin{aligned}\|T^n w\|_{\mathcal{H}} &= \langle T^n w, T^n w \rangle_{\mathcal{H}}^{1/2} \\ &= \langle w, T^{2n} w \rangle_{\mathcal{H}}^{1/2} \\ &\le \|w\|_{\mathcal{H}}^{1/2} \, \|T^{2n} w\|_{\mathcal{H}}^{1/2}\end{aligned}$$

so

$$\begin{aligned}\|Tw\|_{\mathcal{H}} &\le \|w\|_{\mathcal{H}}^{1/2} \, \|T^2 w\|_{\mathcal{H}}^{1/2} \\ &\le \|w\|_{\mathcal{H}}^{1/2} \, \|w\|_{\mathcal{H}}^{1/4} \, \|T^4 w\|_{\mathcal{H}}^{1/4} \\ &\le \|w\|_{\mathcal{H}}^{1 - \frac{1}{2^n}} \, \underbrace{\|T^{2^n} w\|_{\mathcal{H}}^{\frac{1}{2^n}}}\end{aligned}$$

$$\le \underbrace{\|\underbrace{\Theta T^{2^n} w}_{T^{-2^n}\Theta w}\|_V^{1/2} \, \|T^{2^n} w\|_V^{1/2}}_{\|\Theta w\|_V^{1/2} \, \|w\|_V^{1/2}}$$

Taking the limit as $n \to \infty$ yields

$$\|Tw\|_{\mathcal{H}} \le \|w\|_{\mathcal{H}}$$

showing that T extends to a contraction on $\mathcal{H}$, which is necessarily self-adjoint. Q.E.D.

Notice also that Θ has to satisfy

$$\overline{\langle \Theta w', w \rangle}_V = \langle \Theta w, w' \rangle_V$$

$$\overline{\langle \Theta w', w \rangle}_V = \langle w, \Theta w' \rangle_V$$

which will be the case if Θ is self adjoint on V. ~~[illegible]~~ Better - if Θ is a symmetric operator with domain including W.

August 16, 1985

Program: I am still far from an understanding of the relation between the QM and the path integral. It appears that there might be a general thing one can say about the transfer matrix for a 1-dimensional system. This should be independent of reflection-positivity and deals mainly with the path integral

Probably I should look at an Ising model with more than nearest-neighbor interaction.

Let's consider a linear Ising model, that is, with spins $s_n = \pm 1$ for $n \in \mathbb{Z}$. I want the energy to be

$$-E(s) = H\sum_n s_n + J\sum_n s_n s_{n+1} + K\sum_n s_n s_{n+2}$$

so that sites of distance 1 and 2 interact. As before we form the partition function $Z_N(H)$ for a circular ring of N sites and we want to know about the "pressure"

$$\lim_N \frac{1}{N} \log Z_N(H)$$

Maybe more important would be to know whether on $\Omega = (\mathbb{Z}/2)^N$ there is a limiting probability measure, as we know in the ~~case~~ case of nearest neighbor interactions.

The idea will be to work with ~~~~ pairs $\sigma_n = (s_n, s_{n+1})$ of consecutive spins instead of the individual spins. Thus a configuration is to be viewed as a sequence $\sigma = (\sigma_n)_{n \in \mathbb{Z}}$ and the energy of such a configuration will be infinite unless $pr_2(\sigma_n) = pr_1(\sigma_{n+1})$ for each n.

Now I want to see that the energy can be written in the form

$$-E(\sigma) = \sum f(\sigma_n) + \sum g(\sigma_n, \sigma_{n+1})$$

in which ~~[illegible]~~ case it follows that the partitition function will have the form

$$\sum_{(\sigma_1, \cdots, \sigma_N)} \prod_{n=1}^{N} e^{f(\sigma_n)} e^{g(\sigma_n, \sigma_{n+1})} = tr\,(e^f p)^N$$

where $p(\sigma, \sigma') = e^{g(\sigma, \sigma')}$. Clearly we want

$$f(\sigma) = H\, pr_1(\sigma) + J\, pr_1(\sigma) pr_2(\sigma)$$

and we want

$$g(\sigma, \sigma') = \begin{cases} -\infty & \text{if } pr_2(\sigma) \neq pr_1(\sigma') \\ K\, pr_1(\sigma)\, pr_2(\sigma') & \text{otherwise} \end{cases}$$

~~[illegible]~~

Let's try to generalize and consider a general interaction of finite range. This means that we have a function $K(s_1, s_2, \cdots, s_r)$ of r spins and the energy is

$$-E(s) = \sum_n K(s_n, s_{n+1}, \cdots, s_{n+r-1})$$

Let ~~[illegible]~~ $M = \{\pm 1\}^r$ and consider the 0-1 matrix $p(\sigma,\sigma')$ on M such that $p(\sigma, \sigma') = 1$ exactly when the last $r-1$ spins of σ coincide with the first $r-1$ spins of σ'. Then $\{\pm 1\}^N$ can be identified with the sequences (σ_n) in $M^{\mathbb{N}}$ such that $p(\sigma_n, \sigma_{n+1}) = 1$ for all n. The partition function is

$$Z_N = \sum_{(\sigma_1, \ldots, \sigma_N) \in M^N} \prod_n e^{K(\sigma_n)} \prod_n p(\sigma_n, \sigma_{n+1}) = \operatorname{tr} (e^K p)^N$$

Now the existence of the limiting probability measure on $\{\pm 1\}^{\mathbb{N}} \subset M^{\mathbb{N}}$ is a consequence of the non-negative matrix $e^K p$ having unique non-negative left and right eigenvectors. (Limiting measure is an imprecise concept and a better one is Gibbs state). Thus given the left and right eigenvectors I can define consistent measures on the finite products. (The uniqueness of the Gibbs state probably also follows from the Frobenius thm. and this argument explains why there are no phase transitions in such 1-diml models.)

Next, suppose we take one of these models and try to work out what the $\mathcal{H}$ picture is. Suppose I start with the linear Ising model with nearest nbr interaction. So we have a simple Markov type process; ~~governed by a doubly stoch. symmetric matrix. The~~ the partition fn. is

$$Z_N = \operatorname{tr} (e^{Hs} p)^N$$

Now I have to work with a fixed H in order to get the path integral, or prob. measure on $\{\pm 1\}^{\mathbb{N}}$. The moments of the measure are

$$\langle 0 \,|\, T[s(t_1) \cdots s(t_n)] \,|\, 0 \rangle$$

where the $t_i \in \mathbb{Z}$.

Here s denotes the embedding $\{\pm 1\} \subset \mathbb{R}$.

I have the idea that quite generally a path integral is supposed to determine an $\mathcal{H}$.

Let's start with a positive matrix $p(x,y)$ $x, y \in M$, say $M = \{\pm 1\}$ and a diagonal matrix s. Let $\mathcal{H} = \mathbb{R}^M$

<u>August 19, 1985</u>

Let's recall the s.h.o. formula

$$\operatorname{tr}\left(e^{-\omega a^*a + a^*J + \bar{J}a}\right) = \operatorname{tr}\left(e^{-\omega a^* a}\right) e^{\bar{J}\omega^{-1}J}$$

(1)

$$= \frac{\omega}{1-e^{-\omega}} \int e^{-\omega \bar{z}z + zJ + \bar{z}\bar{J}} \frac{d^2z}{\pi}$$

and the fermion analogue.

I have been looking at the path integral representation for QM. For the s.h.o. we get

(2) $$\operatorname{tr}\left(U_J(\beta,0)\right) = N \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-\int \bar{\psi}(\partial_t+\omega)\psi\,dt + \int J\psi\,dt + \int \bar{J}\bar{\psi}\,dt}$$

where $U_J(t,0)$ is the propagator for

$$\partial_t + \omega a^*a - J(t)a^* - \bar{J}(t)a$$

The idea I have is that the path integral is analogous to the Gaussian integral in (1), and so maybe there is some sort of analogue of the left hand side of (2), an analogue in the sense of having to do with a Weyl algebra.

August 20, 1985

Again let us consider the s.h.o. and its path integral representation. The path integral is some kind of Gaussian integral taken over paths $\psi(t)$ in $\mathbb{C}$. Among the functions one wants to integrate are the exponentials

$$\psi \longmapsto \exp \int \left(J(t)\psi(t) + \bar{J}(t)\bar{\psi}(t) \right) dt$$

Thus to $J(t)$ belongs a function on the path space. But also there is an operator and the path integral is supposed to give the trace or VEV of the operator.

Let me try to construct the operator working in real time. Thus I want the scattering operator S_J belonging to the perturbation $-(J(t)a^* + \bar{J}(t)a)$ of $H = \omega a^* a$.

$$S_J = T\left\{ e^{i\left(\int J(t)e^{i\omega t} a^* dt + \int \bar{J}(t) e^{-i\omega t} a\, dt\right)} \right\}$$

$$= e^{-\iint_{t>t'} \bar{J}(t) e^{-i\omega(t-t')} J(t')} \underbrace{e^{\overbrace{\left(i\int J(t)e^{i\omega t}dt\right)}^{\gamma} a^*} e^{\overbrace{\left(i\int \bar{J}(t)e^{-i\omega t}dt\right)}^{-\bar{\gamma}} a}}_{e^{\frac{1}{2}|\gamma|^2} \underbrace{e^{-\frac{1}{2}|\gamma|^2} e^{\gamma a^*} e^{-\bar{\gamma} a}}}$$

Now S_J is unitary, so it differs from T_γ by a factor of abs. value 1. Check:

$$\mathrm{Re} \iint_{t>t'} \bar{J}(t) e^{-i\omega(t-t')} J(t') = \frac{1}{2} \iint_{t>t'} \bar{J}(t) e^{-i\omega(t-t')} J(t')$$
$$+ \frac{1}{2} \iint_{\cancel{t>t'}\ t<t'} J(t') e^{-i\omega(t-t')} \bar{J}(t)$$

$$= \frac{1}{2} \iint \bar{J}(t) e^{-i\omega t} e^{+i\omega t'} J(t') = -\frac{1}{2} |\gamma|^2$$

Now

$$i \operatorname{Im} \iint_{t>t'} \bar{J}(t) e^{-i\omega(t-t')} J(t')\, dt\, dt'$$

$$= \frac{1}{2} \left[\iint_{t>t'} \bar{J}(t) e^{-i\omega(t-t')} J(t') - \iint_{t<t'} \bar{J}(t) e^{-i\omega(t-t')} J(t') \right]$$

$$= \iint \bar{J}(t)\, G(t,t')\, J(t')\, dt\, dt'$$

where
$$G(t,t') = \begin{cases} \frac{1}{2} e^{-i\omega(t-t')} & t > t' \\ -\frac{1}{2} e^{-i\omega(t-t')} & t < t' \end{cases}$$

is a Green's function for $\partial_t + i\omega$.

I don't know how to interpret this. Roughly what we have done is to map the exponential function

$$e^{i\left(\int J\psi + \bar{J}\bar{\psi}\right)}$$

into the unitary operator T_γ, $\gamma = i\int J e^{i\omega t} dt$, times the Gaussian factor

$$e^{\iint \bar{J} G J}.$$

Recall the periodicity thm. à la Atiyah-Singer. Let $\mathcal{F}_0$ (resp. $\mathcal{F}_1$) be the space of Fredholm operators (resp. s.a. Fredholm operators with essential spectrum both >0 and <0). By polar decomposition these spaces retract onto $\mathcal{F}_0^u$ = the space of T which are unitary mod $\mathcal{K}$ (resp. $\mathcal{F}_1^u$ = the space of s.a. T which are non-trivial ~~projectors~~ involutions mod $\mathcal{K}$). Periodicity consists of two h. eq.'s

$$\mathcal{F}_1 \sim \Omega\mathcal{F}_0 \qquad\qquad \mathcal{F}_0 \sim \Omega\mathcal{F}_1$$

which are explicitly realized by Bott maps down on [redacted] involutions and unitaries in the Calkin algebra.

Now when one wants to be concrete one works with Dirac operators. These are unbounded operators, however this [redacted] is not important since this disappears under polar decomposition in some sense. A basic problem is to see the link between the Dirac operator and the corresponding unitary or involution mod $\mathcal{K}$.

Another problem is to see the inverse maps $\Omega\mathcal{F}_0 \longrightarrow \mathcal{F}_1$, $\Omega\mathcal{F}_1 \longrightarrow \mathcal{F}_0$ as the Bott maps go the other way.

I would like to understand things along the following lines. Let's ~~Realize $\mathcal{F}_1$ as a~~ [redacted] take a [redacted] loop in $\mathcal{F}_1$ which we suppose to come from a family of self-adjoint odd-diml. Dirac operators, denoted D_t.

Then we can form a Dirac operator in one higher dimensions, roughly

$$\partial_t + D_t$$

In this way we get a map from loops in $\mathcal{F}_1$ to $\mathcal{F}_0$ provided we use these **models** for $\mathcal{F}_1$ and $\mathcal{F}_0$.