lots of interesting stuff    Feb 6 - Apr 2, 1985

Cramers thm.

Hölder ≤

Doobs ≤

AS Periodicity proof via Kuiper's thm
+ quasi fibrations

Convex Analysis + ( Fenchel Transformation
Legendre "

Martingales a.e. conv., inequalities

Hardly any — this may be
continue from end of '84

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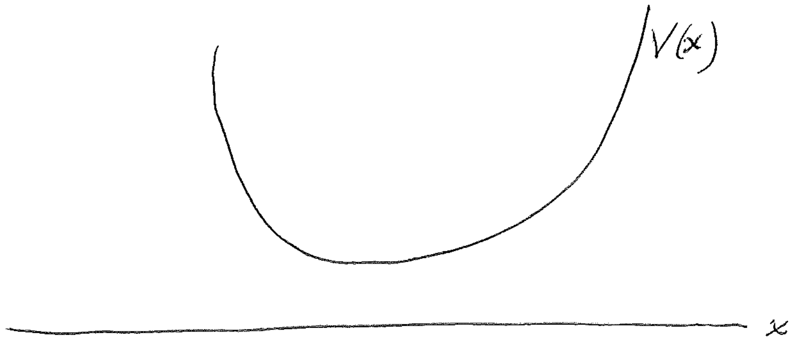
February 6, 1985

I want to understand Cramer's thm. and large deviations. The first thing is to get the physics straight and then go on to the mathematics.

I return to the effective action idea. Let me consider a particle on the line subject to the potential $V(x)$. Assume $V$ convex + smooth.



Apply force $J$ to the particle. This means the new potential $V(x) - Jx$ and the particle is found at the minimum $\bar{x}$, where

$$(*) \qquad V'(\bar{x}) = J$$

This defines $\bar{x}$ as a function of $J$.

Suppose I want $J$ as a function of $\bar{x}$. I make the Legendre transform

$$W(J) = J\bar{x} - V(\bar{x})$$

where $\bar{x}$ is regarded as a fn of $J$ by . Then if I know $W(J)$, then I get $\bar{x}$ by

$$\frac{d}{dJ}W(J) = \bar{x} + J\frac{d\bar{x}}{dJ} - \underbrace{\frac{d}{dJ}V(\bar{x})}_{V'(\bar{x})\frac{d\bar{x}}{dJ}}$$

$$= \bar{x}$$

Now suppose the particle is connected to a heat bath, whence its position is now

$$\bar{x} = \frac{\int e^{-\beta(V(x)-Jx)}\, x \, dx}{\int e^{-\beta(V(x)-Jx)}\, dx}$$

Assume $\beta = 1$, put $\mu(dx) = \frac{e^{-\beta V(x)} dx}{\int e^{-V(x)} dx}$. Then this is

$$* \quad \bar{x} = \frac{\partial}{\partial J} \log Z(J) \quad , \qquad Z(J) = \int e^{Jx} \mu(dx)$$

~~This time we let $W(J)$ be the effective action namely such that~~

~~$\bar{x} = \frac{\partial}{\partial J} W(J)$.~~

~~And the formula for $W$ is~~

~~$W(J) = J\bar{x} - \log Z(J)$~~

~~where $J$ is regarded as a function of $\bar{x}$ via $*$.~~

~~Check: $\frac{\partial}{\partial J} W(J) = \bar{x} + J \frac{\partial \bar{x}}{\partial J} - \frac{\partial}{\partial J}$~~

Let $U(\bar{x})$ be the effective action which means we want

$$J = U'(\bar{x})$$

when $\bar{x}$, $J$ are related by $*$. Then

$$U(\bar{x}) = J\bar{x} - \log Z(J)$$

Check: $$\frac{\partial}{\partial \bar{x}} U(\bar{x}) = J + \cancel{\frac{\partial J}{\partial \bar{x}} \bar{x}} - \cancel{\frac{\partial}{\partial J} \log Z(J)} \frac{\partial J}{\partial \bar{x}}$$

What is the way to think? Think of the measure as being very concentrated. No this will just give back the classical theory. Instead we want to take an ensemble of this system and averge. Then we expect deviations from the mean to be very small.

So let's go on to the statement of Cramer's thm.

$$Z(J) = \int e^{Jx} \mu(dx)$$

One assumes this is finite for all $J \in \mathbb{R}$ and defines

$$I(x) = \sup_J \{ Jx - \log Z(J) \}.$$

This is the effective potential in the cases of interest to me. Conclusion is

F closed $$\overline{\lim} \ \frac{1}{n} \log \mu_n(F) \leq - \inf_{x \in F} I(x)$$

G open $$\underline{\lim} \ \frac{1}{n} \log \mu_n(G) \geq - \inf_{x \in G} I(x)$$

These [redacted] say[redacted] roughly that

$$\mu_n(F) \leq e^{-n \inf_{x \in F} I(x) + \varepsilon n}$$

$$\mu_n(G) \geq e^{-n \inf_{x \in G} I(x) - \varepsilon n}$$

These can't be correct, but basically the leading term of the asymptotic expansion is OK.

Here is the heuristic way to look at things. We know

$$\int e^{Jx} \mu_n(dx) = \int e^{J \frac{1}{n}(x_1 + \cdots + x_n)} \mu(dx_1) \cdots \mu(dx_n)$$

$$= Z\left(\frac{J}{n}\right)^n = e^{n \log Z(J/n)}$$

so by some sort of Fourier inversion

$$\mu_n(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-xJ}\, e^{n\log Z(J/n)}\, dJ$$

$$\sim \frac{1}{2\pi i}\int e^{-n(xJ - \log Z(J))}\, n\,dJ$$

and now use steepest descent to get

$$\mu_n(x) \sim e^{-nI(x)} \times\left(\frac{n}{\sqrt{n}\,\det \cdots}\right)$$

since $I(x)$ is the minimum value of $xJ - \log Z(J)$.

This is about as far as one can go with the intuitive arguments. So where do I start. Suppose we were to start with a nice measure, i.e. smooth with compact support or a sum of $\delta$-measures. Then $Z(J)$ is an entire function so we ought to have good control over

$$\int e^{Jx}\mu_n(dx) = Z(J/n)^n$$

Now $\quad Z(J) = \int e^{Jx}\mu(dx) = 1 + m_1 J + m_2\frac{J^2}{2!} + \cdots$

so that $\quad Z(J/n)^n = \left(1 + m_1\frac{J}{n} + m_2\frac{J^2}{2!\,n^2} + \cdots\right)^n$

$$\longrightarrow e^{m_1 J}$$

if $J$ remains bounded. ~~[illegible]~~ This confirms the fact that $\mu_n \longrightarrow \delta(x - m_1)\,dx$.

February 8, 1985

$$\int e^{\psi^t \omega \tilde{\psi}} = \det(\omega)$$

provided $\int$ picks out the coefficient of $\psi^1 \cdots \psi^n \tilde{\psi}^n \cdots \tilde{\psi}^1$

Next we want to evaluate

$$\int e^{\psi^t \omega \tilde{\psi} + \psi^t J + \tilde{J}^t \tilde{\psi}}$$

in two ways. Complete square in the exponent

$$(\psi^t + \tilde{J}^t \omega^{-1}) \omega (\tilde{\psi} + \omega^{-1} J) - \tilde{J}^t \omega^{-1} J$$

and one gets

$$\det(\omega)\, e^{-\tilde{J}^t \omega^{-1} J}.$$

Next use

$$e^{\psi^t \omega \tilde{\psi}} = \sum_{|I| = |K|} \det(\omega_{I,K})\, \psi^I \tilde{\psi}^{K^*}$$

where $K^*$ denotes $K$ in reverse order. Also

$$e^{\psi^t J} = \sum (\psi^t J)^I = \sum (-1)^{\frac{1}{2}p(p-1)} \psi^I J^I \qquad p = |I|$$

$$e^{\tilde{J}^t \tilde{\psi}} = \sum_K (-1)^{\frac{1}{2}q(q-1)} J^{K^*} \tilde{\psi}^{K^*} \qquad q = |K|.$$

Now multiply and integrate.

Actually it might be easier if you wrote

$$e^{\psi^t J} = \sum \psi^I J^{I^*} \qquad e^{\tilde{J}^t \tilde{\psi}} = \sum \tilde{J}^K \psi^{K^*}$$

Then the integral is made up of the terms

$$\sum_{|I|=|K|} \det(\omega_{I',K'}) \int \psi^{I'} \tilde\psi^{K'*} \psi^{I} J^{I*} \tilde J^{K} \tilde\psi^{K*}$$

So all I have to do is figure out the sign of the latter. This integrand is

$$\psi^{I'} \tilde\psi^{K'*} \overbrace{\psi^{I} \tilde\psi^{K*}}^{\text{even}} \overbrace{J^{I*} \tilde J^{K}}^{\text{even}}$$

~~$\psi^{I'} \psi^{I} \tilde\psi^{K'*} \tilde\psi^{K*}$~~

$$= \psi^{I'} \psi^{I} \tilde\psi^{K*} \tilde\psi^{K'*} \quad J^{I*} \tilde J^{K}$$

Now $\tilde\psi^{K*} \tilde\psi^{K'*} = \tilde\psi_{k_p} \cdots \tilde\psi_{k_1} \tilde\psi_{k'_q} \cdots \tilde\psi_{k'_1}$ $\qquad p+q=m$

$$= (-1)^{\frac{m(m-1)}{2}} \tilde\psi_{k'_1} \cdots \tilde\psi_{k'_q} \tilde\psi_{k_1} \cdots \tilde\psi_{k_p}$$

$$= (-1)^{\frac{m(m-1)}{2}} \varepsilon(K',K)\, \tilde\psi_1 \cdots \tilde\psi_m$$

$$= \varepsilon(K',K)\, \tilde\psi_m \cdots \tilde\psi_1$$

Also $J^{I*} \tilde J^{K} = J^{I} \tilde J^{K*}$ since $|I| = |K|$.

Thus we get

$$\sum_{|I|=|K|} \varepsilon(I',I)\, \varepsilon(K',K) \det(\omega_{I',K'}) \quad J^{I} \tilde J^{K*}$$

So we get the formula

$$\boxed{\begin{aligned} \int e^{\psi^t\omega\tilde\psi + \tilde J^t\tilde\psi + \psi^t J} &= \det(\omega)\, e^{-\tilde J^t\omega^{-1}J} \\ &= \sum_{|I|=|K|} \varepsilon(I',I)\,\varepsilon(K',K)\,\det(\omega_{I',K'})\, J^I \tilde J^{K*}\end{aligned}}$$

We ought to be able to check this by using

$$-\tilde J^t\omega^{-1}J = J^t(+\omega^{-1})^t\tilde J$$

so this gives the formula

$$\det(\omega)\,\underbrace{\det\big((\omega^{-1})^t_{I,K}\big)}_{\det\left((\omega^{-1})_{K,I}\right)} = \varepsilon(I',I)\,\varepsilon(K',K)\,\det(\omega_{I',K'})$$

This is the generalized Cramer's rule.

Now how can I really see this is true without computation? There should be someway with the fermion integrals.

$$\frac{\int e^{\psi^t\omega\tilde\psi+\psi^t\eta\tilde\psi}}{\int e^{\psi^t\omega\tilde\psi}} = \frac{1}{\det(\omega)}\sum\int \underbrace{e^{\psi^t\omega\tilde\psi}\det(\eta_{I,J})\,\psi^I\tilde\psi^{J*}}$$

$$= \sum\frac{\det(\eta_{I,J})}{\det(\omega)}\,\varepsilon(I,I')\,\varepsilon(J,J')\,\det(\omega_{I',J'})$$

$$\frac{\det(\omega+\eta)}{\det(\omega)} = \det(1+\omega^{-1}\eta) = \sum_{I,J}\det(\omega^{-1})_{J,I}\,\det(\eta)_{I,J}$$

$$\therefore\quad \boxed{\det(\omega)\,\det(\omega^{-1})_{J,I} = \varepsilon(I,I')\,\varepsilon(J,J')\,\det(\omega_{I',J'})}$$

## February 25, 1985

Cramèr's theorem. Let $\mu$ be a probability measure on $\mathbb{R}$ such that

$$(1) \qquad Z(J) = \int e^{Jx}\,\mu(dx) < \infty$$

for all $J \in \mathbb{R}$. Put

$$(2) \qquad W(x) = \sup_J \{ Jx - \log Z(J) \}.$$

This is the so called Fenchel transform of $\log Z(J)$ and it is essentially the Legendre transform in the present case. We first discuss this.

The first remark is that if we split the integration into $x \geq 0$ and $x < 0$, then we get two Laplace transforms which are knows to be analytic in half-planes $Re(J) <$ const or $>$ const. Thus $Z(J)$ is an entire function of $J$ under the assumption (1).

Next one knows that

$$\partial_J^2 \log Z(J) = \langle (x - \langle x \rangle_J)^2 \rangle_J$$

where $\langle\ \rangle_J$ denotes the average wrt $\dfrac{e^{Jx}\mu}{Z(J)}$.

This shows that

$$J \longmapsto \bar{x}_J = \langle x \rangle_J = \partial_J \log Z(J)$$

will be smooth [redacted] with derivative $> 0$, provided (as we assume) the measure $\mu$ has support with at least 2 points.

Then $J \longmapsto \bar{x}_J$ is a diffeomorphism of $\mathbb{R}$ with an open interval $(a, b)$ of $\mathbb{R}$.

We now identify $(a, b)$ with the interior of the convex hull of the support of $\mu$. Since $\bar{x}_J < b$ we have for $J \geqslant 0$

$$\log Z(J) = \int_0^J \bar{x}_J \, dJ \leqslant bJ$$

$$Z(J) \leqslant e^{bJ} \qquad J \geqslant 0$$

Similarly

$$Z(J) \leqslant e^{-aJ} \qquad J \leqslant 0.$$

Then for $y > b$

$$Z(J) \geqslant \int_y^\infty e^{Jx} \mu(dx) \geqslant e^{Jy} \int_y^\infty \mu(dx)$$

$$\int_y^\infty \mu(dx) \leqslant Z(J) e^{-Jy} \leqslant e^{-J(y-b)} \xrightarrow[J \to +\infty]{} 0$$

So $\int_y^\infty \mu(dx) = 0$ for all $y > b$, showing $\mu$ is supported in $x \leqslant b$. Similarly it is supported in $x \geqslant a$. Thus $\mu$ is supported in $[a, b]$, and if it were supported in a smaller interval, then $\bar{x}_J$ could not approach both $a, b$.

Let's now identify $W(x)$ as defined in (2). First of all $W(x) \geqslant 0$ (take $J = 0$). If $x > b$, then

$$Jx - \log Z(J) \geqslant Jx - Jb = J(x - b) \longrightarrow +\infty$$

as $J \to +\infty$, so $W(x) = +\infty$. Similarly $W(x) = +\infty$ if $x < a$.

Next let $x \in (a, b)$. Then we know there is a unique $J_x$ such that $\bar{x}_{J_x} = x$. The function

$$J \longmapsto J x - \log Z(J)$$

has second derivative $> 0$ and the first derivative

$$x - \bar{x}_J$$

vanishes at $J = J_x$. By convexity this has to be the unique minimum, i.e.

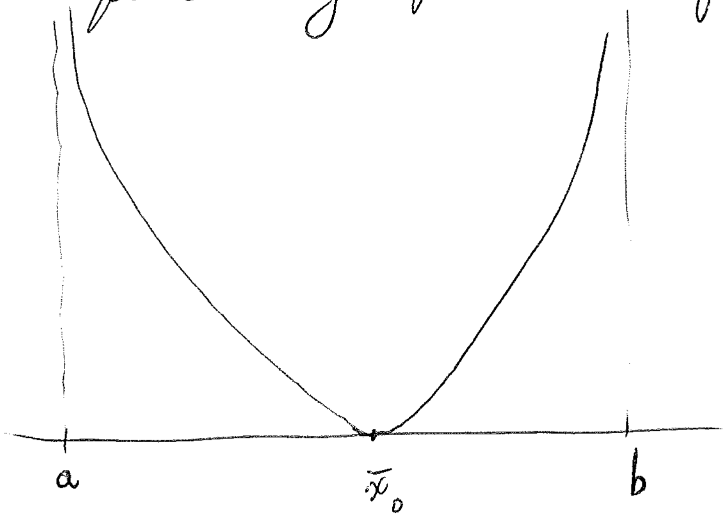
$$W(x) = J_x x - \log Z(J_x)$$

and so $W(x)$ is just the Legendre transform of $\log Z(J)$. Note that

$$\partial_x W(x) = J_x + \cancel{\partial_x J_x\, x} - \cancel{x_{\bar{J}_x} \cdot \partial_x J_x}.$$

In particular $W(x) \longrightarrow +\infty$ as $x \uparrow b$ ~~or~~ $x \downarrow a$.

Finally as $W$ is an upper envelope of a family of lines, it is convex and this implies that $W(x) = \infty$ if $x = a$ or $b$. So we have the following picture of $W(x)$



$W(x)$ is the <u>effective potential</u> if $\mu$ is the Boltzmann measure $\dfrac{e^{-\beta U(x)}dx}{\int e^{-\beta U(x)}dx}$

Next we discuss the Cramér theorem. This is a refinement of the (weak) law of large numbers which I now review.

Think of $\mu$ as the distribution of a r.v. $X$ and then take $n$ identically distributed independent r.v. $X_1, \dots, X_n$ and let $\mu_n$ be the distribution of $\frac{1}{n}\sum X_j$. Thus the Laplace transform for $\mu_n$ is

$$Z_n(J) = \int_{\mathbb{R}} e^{Jx} \mu_n(dx) = \int_{\mathbb{R}^n} e^{\frac{1}{n}J(x_1+\dots+x_n)} \mu(dx_1)\dots\mu(dx_n)$$

$$= Z(J/n)^n$$

Let's use Laplace inversion formally

$$\frac{\mu_n(dx)}{dx}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{-Jx} Z(J/n)^n \, dJ$$

$$= \frac{n}{2\pi i}\int_{-i\infty}^{i\infty} e^{-n(Jx - \log Z(J))} \, dJ$$

Now $x$ being given we know that

$$Jx - \log Z(J)$$

has a critical point at $J = J_x$. So if we use the steepest descent method and assume there are not other contributing critical points, we get

$$\frac{\mu_n(dx)}{dx}(x) \sim e^{-nW(x)} (x \dots )$$

as leading term.

Cramér's thm. is a precise version of this leading term ~~estimate~~

$$\mu_n \text{ at } x \sim e^{-nW(x)}$$

which says

$$\underline{\lim}\ \frac{1}{n} \log \mu_n(G) \geq -\inf\{W(x),\ x \in G\}$$

for $G$ open and

$$\overline{\lim}\ \frac{1}{n} \log \mu_n(F) \leq -\inf\{W(x),\ x \in F\}$$

for $F$ closed.

Actually it seems one can prove for $G$ open that

$$\circledast \qquad \lim_{n\to\infty} \frac{1}{n} \log \mu_n(G) = -\inf\{W(x), x \in G\}$$

But I see now I have forgotten to discuss the (weak) law of large numbers. This says that $\mu_n \longrightarrow$ the $\delta$ measure at the mean $\bar{x} = \bar{x}_0$ in the sense that if we take any open interval containing then the measure outside goes to zero.

The first thing we derive is an estimate for the measure outside. For any n and y

$$Z(J/n)^n \geq \int_{[y,\infty)} e^{Jx} \mu_n(dx) \geq e^{Jy} \int_{[y,\infty)} \mu_n(dx)$$

for $J \geq 0$. So.

$$\mu_n[y,\infty) \leq Z(J/n)^n e^{-Jy} = e^{-n\left(\frac{J}{n}y - \log Z(J/n)\right)}$$

so
$$\mu_n[y,\infty) \le e^{-n(Jy - \log Z(J))}$$
for all $J \geq 0$. If $y > \bar{x}$, then it is clear from the picture of $W(x)$ that the sup of $Jy - \log Z(J)$ for $J \geq 0$ is $W(y)$.

More precisely if $y \geq b$, then $Jy - \log Z(J) \to +\infty$ as $J \to +\infty$, and if $\bar{x} < y < b$, then the sup is achieved at $J = J_y$ which is $> 0$. Thus we get

(3) $$\mu_n[y,\infty) \le e^{-nW(y)} \qquad y > \bar{x}$$

and similarly

(4) $$\mu_n(-\infty, y] \le e^{-nW(y)} \qquad y < \bar{x}$$

Putting these together we find that if $\bar{x} \in (y, y')$ then

(5) $$\mu_n(y,y') \geq 1 - e^{-nW(y)} - e^{-nW(y')}$$

and thus we have the (weak) law of large nos.
$$\mu_n \longrightarrow \delta_{\bar{x}}$$

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~~So now take an arbitrary open set $G$.~~

~~If $\bar{x} \in G$, then from (5) we have~~

~~$\mu_n(G) \geq \mu_n(y,y') \geq 1 - e^{-nW(y)} - e^{-nW(y')}$~~

~~$\mu_n(G) \uparrow 1$ as $n \to \infty$~~

~~by picking $\bar{x} \in (y,y') \subset G$, verifying ⊛. Next~~

Now we want to prove Cramer's thm. Let's begin with the upper bound result. This means we look at as closed set $F$ which we can suppose such that [illegible] $\inf\{W(x) \mid x \in F\} < \infty$. If $\bar{x} \in F$, this inf is 0 and

$$(6) \qquad \overline{\lim}\ \frac{1}{n}\mu_n(F) \leq -\inf\{W(x) \mid x \in F\}$$

is trivial. So suppose $\bar{x} \notin F$. Write $F = F^+ \cup F^-$ where $F^{\pm}$ is the subset above (resp. below) $\bar{x}$. Assume both $\neq \phi$, let $y' =$ least elt of $F^+$, $y =$ greatest elt of $F^-$, so that

$$\inf\{W(x) \mid x \in F\} = \inf\{W(y), W(y')\}$$

Next we have

$$F \subset (-\infty, y] \cup [y', \infty)$$

so for any $n$

$$\mu_n(F) \leq e^{-nW(y)} + e^{-nW(y')}$$

whence

$$\mu_n(F)^{1/n} \leq \Big( \qquad\qquad\qquad\qquad \Big)^{1/n} \longrightarrow e^{\min\{W(y), W(y'}}$$

and we obtain (6).

Now look at the lower bound. ~~Let G be~~ ~~[illegible] open set such that the inf~~ If $G$ is an open set we want to prove

$$(7) \qquad \underline{\lim}\ \frac{1}{n}\log \mu_n(G) \geq -\inf\{W(x) \mid x \in G\}.$$

~~[illegible]~~ It's enough to show

$$\underline{\lim}\ \frac{1}{n}\log \mu_n(G) \geq -W(x)$$

if $x \in G$. We can suppose $W(x) < \infty$. ~~Two cases $x \geq \bar{x}$ or $x < \bar{x}$. Take the former. Then~~ ~~[illegible]~~ ~~$x_J \in G$ for $J$ near $J_x$. Let's choose a $J > J_x$ such that $x_J \in G$, and concentrate on the open interval~~

~~$(\bar{x}, x') \in G$~~

~~where $x' > x_J$~~

Let us consider an open interval $(y, y')$ ~~[illegible]~~ above $\bar{x}$ such that $W(y) < \infty$. Choose $J$ such that $\bar{x}_J \in (y, y')$, and note that $J > J_y$. Now we consider the measure

$$\mu_J = \frac{e^{Jx}\mu}{Z(J)}$$

which has its mean at $\bar{x}_J$. We want to apply the law of large numbers to this.

$$\int_{(y,y')} \mu_{J,n}(dx) = \int_{\frac{1}{n}\sum x_j \in (y,y')} e^{J\sum x_j} \mu(dx_1)\cdots\mu(dx_n) / Z(J)^n$$

$$= \int_{(y,y')} \frac{e^{nJx}}{Z(J)^n} \mu_n(dx) \leq \frac{e^{nJy'}}{Z(J)^n} \int_{(y,y')} \mu_n(dx)$$

~~w.g. $\mu_n(y,y')$~~ The law of large numbers says

$$\int_{(y,y')} \mu_{J,n}(dx) \longrightarrow 1$$

with exponential error.

$$\log \int_{(y,y')} \mu_n(dx) + n\left(Jy' - \log Z(J)\right) \geqslant \log \int_{(y,y')} \mu_{J,n}(dx)$$

$$\frac{1}{n} \log \mu_n(y,y') + Jy' - \log Z(J) \geqslant \frac{1}{n} \log \mu_{J,n}(dx) \quad \uparrow$$

$$\Rightarrow \quad \underline{\lim} \frac{1}{n} \log \mu_n(y,y') \geqslant -\left(Jy' - \log Z(J)\right)$$

for all $J$ such that $\bar{x}_J \in (y,y')$

Now consider an arbitrary open set $G$. We want to prove

$$\underline{\lim} \frac{1}{n} \mu_n(G) \geqslant - \inf_{x \in G} W(x)$$

and it is enough to take an $x \in G$ such that $W(x) < \infty$ and prove

$$\underline{\lim} \frac{1}{n} \mu_n(G) \geqslant - W(x).$$

There are three cases $x > \bar{x}$, $x = \bar{x}$, $x < \bar{x}$. If $x = \bar{x}$, then it's true by the law of large nos. For the other two it suffices to treat $x > \bar{x}$. Take and Let $(x, y') \subset G$. Then from the above

$$\underline{\lim} \frac{1}{n} \log \mu_n(G) \geqslant \underline{\lim} \frac{1}{n} \log \mu_n(x, y')$$
$$\geqslant -\left(Jx - \log Z(J)\right) - J(y' - x)$$

for all $J$ with $x_J \in (x, y')$. Let $J \downarrow J_x$ and we get

$$\underline{\lim} \frac{1}{n} \log \mu_n(G) \geqslant -W(x) - J(y' - x)$$

and then let $y' \downarrow x$ and we get the desired result.

This concludes the proof of Cramérs thm.

February 28, 1985

Discuss conditional expectation. [illegible] Start with a measure space $(X, \mathcal{F}, \mu)$; here $\mathcal{F}$ is a $\sigma$-field of subsets of $X$. One supposes given a sub $\sigma$-field $\mathcal{F}_0 \subset \mathcal{F}$ and defines the conditional expectation of a measurable function $f$ on $X$ relative to $\mathcal{F}_0$. I want to think geometrically, so I will suppose given a map

$$\pi : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{F}_Y)$$

such that $\pi^*(\mathcal{F}_Y) = \mathcal{F}_0$. To simplify, suppose $\pi$ onto, and then identify $\mathcal{F}_Y$ and $\mathcal{F}_0$ via $\pi^*$.

Now measures push forward, so that we have a measure $\pi_* \mu$ on $(Y, \mathcal{F}_0)$ determined by

$$(1) \qquad \int_Y g \; \pi_* \mu = \int_X \pi^*(g) \; \mu$$

$$(2) \qquad \int_A \pi_* \mu = \int_{\pi^{-1} A} \mu \qquad \text{all } A \in \mathcal{F}_0$$

(Actually (2) should be taken as definition, and then (1) holds for all $g \geq 0$ measurable wrt $\mathcal{F}_0$)

If $f$ is measurable on $X$, then its conditional expectation $\pi_*(f)$ is defined so that

$$(3) \qquad \int_A \pi_*(f) \; \pi_*(\mu) = \int_{\pi^{-1}(A)} f \mu$$

i.e.

$$④ \qquad \int_Y g\ \pi_*(f)\ \pi_*(\mu) = \int_X \pi^*(g) f \mu$$

Why (and when) is $\pi_*(f)$ defined? Assume $f$ integrable, then

$$\nu(A) = \int_{\pi^{-1}A} f \mu$$

is a signed measure absolutely continuous with respect to $\pi_*\mu$. $\left( \int_A \pi_*\mu = \int_{\pi^{-1}A} \mu = 0 \Rightarrow \int_{\pi^{-1}A} f\mu = 0 \right)$. Hence by the Radon-Nikodym thm. $\exists\ \pi_*(f) \in L^1(Y, \pi_*\mu) \ni$

$$\nu(A) = \int_A \pi_*(f)\ \pi_*(\mu)$$

which yields ③.

Thus we have a geometric interpretation of conditional expectation, namely, as $\pi_*(f)$.

Now I want to work in the Hilbert space picture, since I know 1) R-N thm. is proved by Hilbert space methods, 2) things simplify in the $L^2$-picture.

The first remark is that one has an isometric embedding

$$L^2(Y, \pi_*\mu) \hookrightarrow L^2(X, \mu)$$

because

$$\int_Y |g|^2\ \pi_*\mu = \int_X \pi^*|g|^2 \cdot \mu = \int_X |\pi^* g|^2 \mu$$

Hence $L^2(Y, \pi_*\mu)$ [redacted] is a closed subspace of $L^2(X,\mu)$, and there is an orthogonal projection operator $E : L^2(X,\mu) \longrightarrow L^2(Y, \pi_* \mu)$ characterized by

$$\begin{array}{ccc} \langle Ef, g \rangle_Y & = & \langle f, \pi^* g \rangle_X \\ & & \| \\ \int_Y g\, \pi_*(f)\, \pi_*\mu & = & \int_X \text{[redacted]}\, \pi^*(g) f \,\mu \end{array} \qquad \text{by (4)}$$

Therefore we conclude that conditional expectation corresponds to orthogonal projection provided we work with $L^2$ fns.

(Recall that for prob. measures $L^1 \supset L^2 \supset L^\infty$, so that $L^2$ is more restrictive than integrable.)

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Discuss martingales.

Suppose given an increasing sequence [redacted] $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ of $\sigma$-fields and a sequence of r.v. $X_n$ such that $X_n$ is $\mathcal{F}_n$-measurable (notation of Durrett: $X_n \in \mathcal{F}_n$). Then $(X_n, \mathcal{F}_n)$ is a martingale if the conditional expectation of $X_n$ relative to $\mathcal{F}_{n-1}$ is $X_{n-1}$ for all $n$. One can assume $\mathcal{F}_n = \sigma$-field generated by $X_0, \ldots, X_n$. One has to assume $X_n$ integral so things are defined

<u>$L^p$-Martingale convergence thm.</u> says that if $\sup\{E(X_n^p)\} < \infty$, then $X_n$ converges in $L^p$ for $p > 1$.

This result is obvious for $p=2$: an $L^2$ martingale is in particular a sequence $X_n$ in $L^2$ such that the increments are mutually orthogonal. So

$$X_n = \sum_{p=1}^{n} (X_p - X_{p-1}) \quad \text{is an orthogonal sum}$$

$$\|X_n\|^2 = \sum_{1}^{n} \|X_p - X_{p-1}\|^2 \qquad X_0 = 0$$

and if this is bounded, then the series $\sum_{n\geq 1}(X_n - X_{n-1})$ converges.

Notice the difference between an $L^2$ martingale and a sequence $X_n$ in $L^2$ with orthogonal increments. The $\sigma$-field $\mathcal{F}_n$ gen. by $X_1, \dots, X_n$ will determine a closed subspace of $L^2$ containing $X_1, \dots, X_n$ and functions of $X_1, \dots, X_n$. Then $X_{n+k} - X_n$ must be orthogonal to this whole subspace.

## March 6, 1985

[redacted] Yesterday's lecture contained the map from cyclic cohomology to Lie algebra cohomology of $gl_n(A)$.

$\tilde{g} = gl_n(A)$ $\qquad A = C^\infty(M)$

$\tilde{g} = \text{Lie}(\mathcal{G}) \otimes_{\mathbb{R}} \mathbb{C}$ $\qquad \mathcal{G} = C^\infty(M, G)$, $\quad G = U(n)$

$C^*(\tilde{g})$ = differential graded alg of cochains on $\tilde{g}$ which are continuous for the $C^\infty$ topology

= diffl. graded alg of left-invariant forms on $\mathcal{G}$.

Denote the differential by $\delta$. It's given by an [redacted] ugly formula, but I want to reduce all calculations to the MC form.

How to construct elements of $C^p(\tilde{g})$. Start with a multi-linear functional $\varphi(f^1, .., f^p)$ on $A^p$ which is continuous. (Such a $\varphi$ is the same thing as a distribution on $M^p$ by theory of top. tensor products.)

Extend $\varphi$ to $\tilde{g}$:

$$\text{tr}\, \varphi(X^1, .., X^p) \overset{\text{defn}}{=} \sum_{i_1, .., i_p} \varphi(X^1_{i_1 i_2}, X^2_{i_2 i_3}, .., X^n_{i_p, i_1})$$

and then skew-symmetrize to obtain

$$[\text{redacted}]\ \tilde{\varphi}(X^1, .., X^n) = \sum_{\sigma \in \Sigma_p} \text{sgn}(\sigma)\, \text{tr}\, \varphi(X^{\sigma 1}, .., X^{\sigma p})$$

Then $\tilde{\varphi}$ is obviously an element of $C^p(\tilde{g})$.

Because I don't want to compute with $\delta$ on the level of multi linear functions I now rewrite $\tilde{\varphi}$ in terms of the MC form. This is the element

$$\theta \in C^1(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}) = C^1(\tilde{\mathfrak{g}}, A) \otimes M_n$$

which is given by the identity map of $\tilde{\mathfrak{g}}$. It satisfies

$$\left|\begin{array}{ll} \iota_X \theta = X & \qquad X \in \tilde{\mathfrak{g}} \\ \delta\theta = -\theta^2 & \end{array}\right.$$

( Now I want to explain the formula

$$\tilde{\varphi} = tr\ \varphi(\theta, \cdots, \theta)$$

This doesn't seem to work because the right side doesn't have an [redacted] a priori meaning. Except

$$\theta_{ij} \in C^1(\tilde{\mathfrak{g}},\ \text{[redacted]}\ A)$$

so $\qquad tr\ \varphi(\theta, \cdots, \theta) = \sum_{i_1, \cdots, i_p} \varphi(\theta_{i_1 i_2}, \cdots, \theta_{i_p i_1})$

so it is enough to explain $\varphi(\theta_{i_1 i_2}, \cdots, \theta_{i_p i_1})$. Now

$$\varphi : A^{\otimes p} \longrightarrow \mathbb{C}$$

and $\qquad \theta_{i_1 i_2} \otimes \cdots \otimes \theta_{i_p i_1} \in C^p(\tilde{\mathfrak{g}}, A^{\otimes p}) \qquad$ etc.

Clearly from the definition of [redacted] product

$\iota_{X_p} \iota_{X_{p-1}} \cdots \iota_{X_1} (\theta_{i_1 i_2} \otimes \cdot$ [redacted] $\cdots \otimes \theta_{i_p i_1})$

$= \sum_\sigma$ [redacted] $sgn(\sigma)\ X^{\sigma 1}_{i_1 i_2} \otimes \cdots \otimes X^{\sigma p}_{i_p i_1}$.

Thus I see that in order to explain the expression $\operatorname{tr}\varphi(\theta,\dots,\theta)$ I have to define cup product on $C^*(\tilde{g}, ?)$. Once I do this, then I can compute that

$$\boxed{\iota_{X_p}\cdots\iota_{X_1}\operatorname{tr}\varphi(\theta,\dots,\theta) = \sum_\sigma \operatorname{sgn}(\sigma)\operatorname{tr}\varphi(X^{\sigma 1},\dots,X^{\sigma p})}$$

The next point is to see that $\operatorname{tr}\varphi(\theta,\dots,\theta)$ depends on the cyclic skew-symmetrization of $\varphi$.

$$\operatorname{tr}\varphi(\theta,\dots,\theta) = \sum_{i_1,\dots,i_p}\varphi(\theta_{i_1 i_2},\theta_{i_2 i_3},\dots,\theta_{i_p i_1})$$

$$= \sum_{i_1\cdots i_p}\varphi(\theta_{i_p i_1},\theta_{i_1 i_2},\dots,\theta_{i_{p-1} i_p})$$

Set $\quad \varphi'(a^1,\dots,a^p) = (-1)^{p-1}\varphi(a^p,a^1,\dots,a^{p-1})$.

$$\operatorname{tr}\varphi'(\theta,\dots,\theta) =$$

$$= \varphi'\sum\theta_{i_1 i_2}\otimes\cdots\otimes\theta_{i_p i_1}$$

You will have to use the skew commutativity of the cup product:

$$\alpha\in C^p(\tilde{g},M) \qquad \beta\in C^q(\tilde{g},N)$$

$$\alpha\otimes\beta \in C^{p+q}(\tilde{g}, M\otimes N)$$

$$\beta\in\alpha \in C^{p+q}(\tilde{g}, N\otimes M)$$

If $\tau : M\otimes N \xrightarrow{\sim} N\otimes M$ is the interchange, then

$$\tau(\alpha\otimes\beta) = (-1)^{pq}\,\beta\otimes\alpha.$$

Using this one has: $\lambda(a^1\otimes..\otimes a^p) = (-1)^{p-1}\, a^p\otimes a^1\otimes..\otimes a^{p-1}$

$$\varphi'\sum\theta_{i_1i_2}\otimes..\otimes\theta_{i_pi_1} = \varphi\lambda\sum\theta_{i_1i_2}\otimes..\otimes\theta_{i_{p-1}i_p}$$

$$= \varphi\sum\theta_{i_2i_3}\otimes..\otimes\theta_{i_{p-1}i_p}\otimes\theta_{i_1i_2}$$

$$= \operatorname{tr}\varphi(\theta,..,\theta)$$

So the essential point is that

$$\sum\theta_{i_1i_2}\otimes..\otimes\theta_{i_pi_1} \in C^p(\tilde{g}, A^{\otimes p})$$

is preserved by $\lambda$ on $A^{\otimes p}$. Thus

$$* \quad \boxed{\operatorname{tr}(\varphi\lambda)(\theta,..,\theta) = \operatorname{tr}\varphi(\theta,..,\theta)}$$

Example: $A = \mathbb{C}$:

$$\varphi(a^1,..,a^p) = a^1..a^p$$

leads to $\operatorname{tr}\varphi(\theta,..,\theta) = \operatorname{tr}(\theta^p)$. As $\varphi\lambda = -\varphi$ for $p$ even, this form ↑ will be zero for $p$ even.

$$\boxed{\iota_{X_p}\cdots\iota_{X_1}\operatorname{tr}(\theta^p) = \sum_\sigma \operatorname{sign}(\sigma)\operatorname{tr}(X^{\sigma 1}..X^{\sigma p})}$$

Next we consider the boundary operator in Hochschild cohomology. Let $\varphi: A^p \to \mathbb{C}$

$$b\varphi(a^1,..,a^{p+1}) = \sum_{j=1}^{p}(-1)^{j-1}\varphi(..,a^ja^{j+1},..) + (-1)^p\varphi(a^{p+1}a^1, a^2,..,a$$

Assume $\varphi\lambda = \varphi$, i.e. $\varphi$ is cyclically skew-symmetric.

Then

$$\varphi(a^1,..,a^ja^{j+1},..,a^{p+1}) = (-1)^{(j-1)(p-1)}\varphi(a^ja^{j+1},..,a^{p+1},a_1,..,a_{j-1})$$

and so

$$b\varphi(a^1,\dots,a^{p+1}) = \sum_{j=1}^{p\text{[redacted]}} (-1)^{(j-1)p}\,\varphi(a^j a^{j+1},\dots,a^{p+1},a^{1}\text{[redacted]},\dots,a^{j-1}\text{[redacted]})$$

$$+ (-1)^{p^2}\varphi(a^{p+1}a^1,a^2,\dots,a^p)$$

[redacted]

$$= \sum_{j=0}^{p} (\psi t^j)(a^1,\dots,a^{p+1})$$

where

$$t(a^1,\dots,a^{p+1}) = (-1)^p(a^2,\dots,a^{p+1},a^1)$$

$$\psi(a^1,\dots,a^{p+1}) = \varphi(a^1a^2,a^3,\dots,a^p)$$

Thus [redacted] using formula $*$ we get

$$\mathrm{tr}\,(b\varphi)(\overbrace{\theta,\dots,\theta}^{p+1}) = (p+1)\,\mathrm{tr}\,\psi(\overbrace{\theta,\dots,\theta}^{p+1})$$

or

① $$\boxed{\mathrm{tr}\,(b\varphi)(\overbrace{\theta,\dots,\theta}^{p+1}) = (p+1)\,\mathrm{tr}\,\varphi(\theta^2,\overbrace{\theta,\dots,\theta}^{p-1})}$$

assuming $\varphi$ is [redacted] a cyclic cochain.

Next we have

$$\delta\ \mathrm{tr}\,\varphi(\overbrace{\theta,\dots,\theta}^{p}) = -\,\mathrm{tr}\left\{\varphi(\theta^2,\theta,\dots\theta) - \varphi(\theta,\theta^2,\theta,\dots,\theta) + \dots + (-1)^{p-1}\varphi(\theta,\dots,\theta,\theta^2)\right\}$$

and because $\varphi$ is cyclic

$$\mathrm{tr}\,\varphi(\overbrace{\theta,\dots,\theta}^{j-1},\theta^2,\overbrace{\theta,\dots,\theta}^{p-j}) = (-1)^p(-1)^{p-1}\,\mathrm{tr}\,\varphi(\overbrace{\theta,\dots,\theta}^{j},\theta^2,\overbrace{\theta,\dots,\theta}^{p-j-1})$$

Conclude

② $$\boxed{\delta\ \mathrm{tr}\,\varphi(\theta,\dots,\theta) = -p\,\mathrm{tr}\,\varphi(\theta^2,\overbrace{\theta,\dots,\theta}^{p-1})}$$

Introduce $\mathcal{C}^*(A)$ the complex of cyclic cochains. First define the Hochschild complex $C^*(A, A^*)$ where $C^p(A, A^*)$ = the space of continuous multilinear map $\varphi : A^{p+1} \longrightarrow \mathbb{C}$. $\mathcal{C}^p(A) \subset C^p(A, A^*)$ is the subspace of cyclic cochains, i.e. $\varphi(a^0, \dots, a^p)$ which are cyclically skew-symmetric. On $C^*(A, A^*)$ we have

$$(b\varphi)(a^0, \dots, a^{p+1}) = \sum_{j=0}^{p} (-1)^j \varphi(\dots a^j a^{j+1} \dots) + (-1)^{p+1} \varphi(a^{p+1}a^0, a^1, \dots$$

On the preceding page we saw that if $\varphi$ is cyclic then

$$b\varphi = \sum_{j=0}^{p+1} \psi t^j$$

$t(a^0, \dots, a^{p+1}) = (-1)^{p+1}(a^1, \dots, a^{p+1}, a^0)$

$\psi(a^0, \dots, a^{p+1}) = \varphi(a^0 a^1, a^2, \dots, a^{p+1})$

$$= N\psi$$

and hence it follows that $\mathcal{C}^*(A)$ is a subcomplex of $C^*(A, A^*)$.

We now have a map of degree 1

$$\mathcal{C}^{p-1}(A) \longrightarrow C^p(\tilde{\mathfrak{g}})$$

$$\varphi \longmapsto \frac{1}{p}\,\mathrm{tr}\,\tilde{\varphi}(\overbrace{\theta, \dots, \theta}^{p}) = \tilde{\varphi}$$

and ① ② above imply

$$\delta \tilde{\varphi} = -\widetilde{b\varphi}$$

showing that $\varphi \longmapsto \tilde{\varphi}$ is a degree 1 map of complexes. Induces canonical map

$$HC^{p-1}(A) \Longrightarrow H^p(\tilde{\mathfrak{g}})$$

Examples: $A = C^\infty(M)$. Let $\gamma$ be a closed $p$-dim'l current $=$ ($\mathbb{C}$-linear fnl $\Omega^p \to \mathbb{C}$ vanishing on $d\Omega^{p-1}$). Then

$$\varphi(a^0,\dots,a^p) = \int_\gamma a^0\, da^1 \dots da^p$$

is a cyclic $p$-cocycle. Cyclic.

$$\begin{array}{rcl}
(-1)^{p-1}\varphi(a^1,\dots,a^p,a^0) & = & (-1)^{p-1}\displaystyle\int_\gamma a^1\, da^2 \dots da^p\, da^0 \\
+ & & + \\
\varphi(a^0,\dots,a^p) & = & +\displaystyle\int_\gamma (da^1\, da^2 \dots da^p)\, a^0 \\
\hline
 & = & \displaystyle\int_\gamma d(a^1\, da^2 \dots da^p\, a^0) = 0
\end{array}$$

$$\Longrightarrow \quad \varphi(a^1,\dots,a^p,a^0) = (-1)^p\,\varphi(a^0,\dots,a^p)$$

$$\begin{array}{rcl}
\varphi(a^0a^1,\dots,a^{p+1}) & = & \displaystyle\int_\gamma a^0a^1\, da^2 \dots da^{p+1} \\
-\varphi(a^0, a^1a^2, a^3,\dots) & = & -\displaystyle\int_\gamma a^0\, d(a^1a^2)\, da^3 \dots \\
+\varphi(a^0, a^1, a^2a^3,\dots) & = & \displaystyle\int_\gamma a^0\, da^1\, d(a^2a^3)\, da^4 \dots \\
 & & \\
+(-1)^{p-1}\varphi(a^0,\dots,a^pa^{p+1}) & = & (-1)^{p-1}\displaystyle\int_\gamma a^0\, da^1 \dots\dots d(a^p \cancel{d} a^{p+1}) \\
+(-1)^p\varphi(a^{p+1}a^0,\dots\dots) & = & (-1)^p\displaystyle\int_\gamma a^0\, da^1 \dots\dots da^p\, a^{p+1} \\
\hline
b\varphi(a^0,\dots,a^{p+1}) & = & 0.
\end{array}$$

This gives rise to the Lie alg cocycle

$$\frac{1}{p}\int_\gamma \operatorname{tr} \theta(d\theta)^p$$

March 8, 1985

Some analysis: Recall the Hölder inequalities:

1) $$\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p} \qquad p \geq 1$$

2) $$\int fg \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} \qquad \frac{1}{p} + \frac{1}{q} = 1$$

where we are integrating with respect to a (positive) measure.

Try to prove these. The first says that

$$\|f\|_p = \left(\int |f|^p\right)^{1/p}$$

is a norm (as it [redacted] clearly satisfies $\|tf\|_p = |t| \cdot \|f\|_p$). So what we have to do is show the unit ball is convex. (Why:

$$\frac{f+g}{\|f\|+\|g\|} = \frac{\|f\|}{\|f\|+\|g\|} \cdot \frac{f}{\|f\|} + \frac{\|g\|}{\|f\|+\|g\|} \cdot \frac{g}{\|g\|}$$

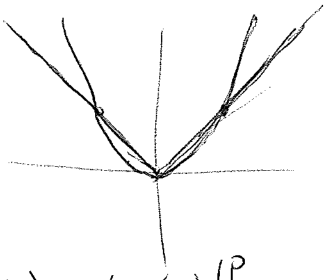
so if we know $\{f \,|\, \|f\| \leq 1\}$ is convex, then the linear combination on the right is in this ball, so

$$\|f+g\| \leq \|f\| + \|g\|.)$$

Next note that for any $x$

$$t \longmapsto |(1-t)f(x) + tg(x)|^p$$

is convex for $p \geq 1$. The function $x \longmapsto |x|^p$ is convex for $p \geq 1$:



Thus $F(t) = \int |(1-t)f(x) + tg(x)|^p$ is also convex.

$$F(t) \leq (1-t)\int |f|^p + t\int |g|^p \qquad 0 \leq t \leq 1$$

It follows that $\|f\|, \|g\| \leq 1 \Longrightarrow \|(1-t)f + tg\| \leq 1$. QED.

2): We want to start from the fact that an integral of exponential functions

$$\int e^{Jx}\, d\mu(x)$$

is logarithmically convex (assuming it is defined).

To prove 2) we can suppose f, g are $> 0$.

Then

$$\log\left(\int fg\right) = \log\left(\int e^{\frac{1}{p}\log f^p + \frac{1}{q}\log g^q}\right)$$

$$\leq \frac{1}{p}\log\int e^{\log f^p} + \frac{1}{q}\log\int e^{\log g^q}$$

which proves 2). Here we have used the function

$$F(t) = \log\left(\underbrace{\int e^{(1-t)\log f^p + t\log g^q}}\right)$$

which is convex:

$$\int e^{+t(\log g^q - \log f^p)}\left(e^{\log f^p}\, d\mu\right)$$

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Suppose we now have a probability measure.

Then $\int |f| \leq \left(\int |f|^p\right)^{1/p}\left(\int |1|^q\right)^{1/q} \Longrightarrow \|f\|_1 \leq \|f\|_p$.

And more generally if $p < p'$, then applying the above to $|f|^p$,

$$\int |f|^p \leq \left(\int \left(|f|^p\right)^{\frac{p}{p'}}\right)^{p'/p} \Longrightarrow \|f\|_p \leq \|f\|_{p'}.$$

Thus we have for a prob. measure

3) $p \leq p' \implies \|f\|_p \leq \|f\|_{p'}$

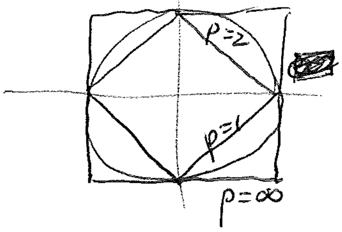
(Opposite inequalities hold for $\ell_p$. The idea is that $\|x\|_p \leq 1 \Rightarrow \sum |x_n|^p \leq 1$, hence each $|x_n| \leq 1$ and so for $p \leq p'$

$$\sum |x_n|^{p'} \leq \sum |x_n|^p \leq 1.$$

Thus the unit ball for $\ell_p$ is contained in the unit ball for $\ell_{p'}$, and so

$$\|x\|_p \geq \|x\|_{p'} . \qquad \text{for } p < p'.$$

Picture:



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Doob's inequality. Let $X_n$ $n \geq 0$ be a <u>martingale</u>. What this means is that if $\Omega_n = \mathbb{R}^{n+1}$ with the probability measure giving the distribution of $X_0, \dots, X_n$, then for the projective system

$$\cdots\cdots \longrightarrow \Omega_2 \longrightarrow \Omega_1 \longrightarrow \Omega_0$$

one has that $X_n$ on $\Omega_n$ pushes forward to $X_{n-1}$. Also we assume that each $X_n$ is integrable.

Doob's inequality says that for any $a \in \mathbb{R}$

$$a\, P\{\sup_{m \leq n} X_m \geq a\} \leq \int_{\{\sup_{m \leq n} X_m \geq a\}} X_n$$

To prove this set $\bar{X}_n = \sup_{m \leq n} X_m$ and decompose the set $K = \{\bar{X}_n \geqslant a\}$ according to where the sequence $X_0, X_1, \ldots, X_n$ first becomes $\geqslant a$. Let

$$K_m = \{\omega \mid X_m \geqslant a,\ X_0, \ldots, X_{m-1} < a\}$$

Then $K_m$ comes from $\Omega_m$, hence by definition of martingale

$$(*) \qquad \int_{K_m} X_n = \int_{K_m} X_m$$

But this is $\geqslant a P(K_m)$, so adding up for different $m$ we get the desired inequality.

Notice that instead of $(*)$ we need only

$$\int_{K_m} X_n \geq \int_{K_m} X_m$$

i.e. that the pushforward of $X_n$ from $\Omega_n$ to $\Omega_m$ is $\geqslant X_m$. Such a thing is called a <u>submartingale</u>.

It seems that if $X$ is a martingale, then $\varphi(X)$ is a submartingale for $\varphi$ convex (except for the problem of integrability). e.g. $|X|$ is submartingale.

It's clear from the above proof of Doob's inequality that the essential point is $(*)$ for different $m$. This leads us to introduce the notion of stopping time which is a random variable $N$ having value in $\mathbb{N}$ such that $\{N \leq m\}$ comes from $\Omega_m$. In the above example $N(\omega) = \inf\{n \mid X_n^{(\omega)} \geq a\}$.

Given a stopping time $N \le n$, then (*) says

$$\int X_n \ge \int X_N$$

for a submartingale.

Convergence in $L^p$: Let $X_n$ be a submartingale with $X_n \ge 0$ and set

$$\bar{X}_n = \max_{m \le n} X_m$$

Then

$$\int \bar{X}_n^{\,p} = -\int_0^\infty \lambda^p \, d P\{\bar{X}_n \ge \lambda\}$$

$$= \int_0^\infty p \lambda^{p-1} P\{\bar{X}_n \ge \lambda\} \, d\lambda$$

$$\le \int_0^\infty p \lambda^{p-1} \left( \frac{1}{\lambda} \int_{\bar{X}_n \ge \lambda} X_n \right) d\lambda$$

$$= \int X_n(x) \int_0^{\bar{X}_n(x)} p \lambda^{p-2} \, d\lambda = \frac{p}{p-1} \int X_n \bar{X}_n^{\,p-1}$$

$$\le \frac{p}{p-1} \left( \int X_n^p \right)^{1/p} \left( \int \bar{X}_n^{\,(p-1)q} \right)^{1/q} \qquad (p-1)\frac{1}{1-\frac{1}{p}} = p$$

so dividing we get

$$\left( \int \bar{X}_n^{\,p} \right)^{1-\frac{1}{q}} \le \frac{p}{p-1} \left( \int X_n^p \right)^{1/p}$$

or

$$\| \bar{X}_n \|_p \le \frac{p}{p-1} \| X_n \|_p$$

(Some limiting process is needed to do the division?

This inequality shows that if $\|X_n\|_p$ is bdd, then $\overline{X}_n$ (which is increasing in $n$) [illegible] has $\sup \overline{X}_n \in L^p$. Then one uses dominated convergence to get $X_n \to X$ in $L^p$. (One has already proved $X_n$ converges a.e.)

March 9, 1985

Go over Thursday's lecture

Theorem of invariant for $gl_n$. Let $V = \mathbb{C}^n$ and $g = gl_n = End(V)$. Then $g$ acts on tensors

$$V^{\otimes p} \otimes (V^*)^{\otimes q}$$

and we are interested in the invariant subspace.

If $p \neq q$, then $id_V \in g$ acts on this space of tensors as $p-q$, and so there are no non-trivial invariants.

If $p=q=1$, then

$$V \otimes V^* = g$$

is the adjoint representation, and the invariant subspace consists of the multiples of $id_V$. We want to generalize this.

~~We make the identifications~~

~~$V^{\otimes p} \otimes (V^*)^{\otimes p} = End(V^{\otimes p}) = (End V)^{\otimes p} = g^{\otimes p}$.~~

~~Thus the space of tensor can be identified with the ring of all operators on $V^{\otimes p}$ and also with the p-fold tensor product of the ring of operators on $V$. We want to calculate~~

We identify $V^{\otimes p} \otimes (V^*)^{\otimes p}$ with the ring $R$ of all operators on $V^{\otimes p}$ in the obvious way. We have an action of $\Sigma_p$ on $V^{\otimes p}$ which commutes with the $g$-action. Let

$$A = Im \{ k[\Sigma_p] \longrightarrow R \}$$
$$B = Im \{ U(g) \longrightarrow R \}$$

be the subrings generated by these two actions.

The space of $\mathfrak{g}$-invariants in $R$ is the commutant $B'$.

Thm. $B' = A$, that is, every invariant in $V^{\otimes p}\otimes(V^*)^{\otimes p}$ is a linear combination of the invariants given by the elements of $\Sigma_p$.

Proof. First step

$$A' = R^{\Sigma_p} = (\mathfrak{g}^{\otimes p})^{\Sigma_p}$$

is spanned by the operators $e^X\otimes\cdots\otimes e^X$ p-times for $X\in\mathfrak{g}$. (Dually any polynomial function $f$ on $\mathfrak{g}$ homogeneous of degree $p$ which vanishes on $e^X$ for $X\in\mathfrak{g}$ is zero.) Now if $\rho(X)$ is the action of $X\in\mathfrak{g}$ on $V^{\otimes p}$, then

$$e^X\otimes\cdots\otimes e^X = \exp\rho(X)$$

belongs to $B$. Thus we have $A' = B$.

2nd step is to use the von Neumann double commutant theorem which tells us that $A = A''$, provided $V^{\otimes p}$ is semi-simple as an $A$-module. This latter follows from the complete reducibility of representations of finite groups in characteristic zero.

Thus $B' = A'' = A$ proving the theorem.

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Notation: $\quad Id_V = e_i\otimes e_i^* \in V\otimes V^*$

where $e_i$ is a basis for $V$ and $e_i^*$ is the dual basis for $V^*$. [illegible] $V^{\otimes p}$ has the bases $e_{i_1}\otimes\cdots\otimes e_{i_p}$ and if $\sigma\in\Sigma_p$, then the effect of $\sigma$ on $V^{\otimes p}$ is

$$\sigma(e_{i_1}\otimes\cdots\otimes e_{i_p}) = e_{i_{\sigma^{-1}1}}\otimes\cdots\otimes e_{i_{\sigma^{-1}p}}$$

Thus the map giving the invariants is

$$k[\Sigma_p] \longrightarrow V^{\otimes p}\otimes(V^*)^{\otimes p} \simeq (V\otimes V^*)^{\otimes p}$$

$$\sigma \longmapsto e_{i_{\sigma^{-1}1}}\otimes\cdots\otimes e_{i_{\sigma^{-1}p}}\otimes e^*_{i_1}\otimes\cdots\otimes e^*_{i_p}$$

$$\longleftrightarrow (e_{i_{\sigma^{-1}1}}\otimes e^*_{i_1})\otimes\cdots\otimes(e_{i_{\sigma^{-1}p}}\otimes e^*_{i_p})$$

Now I want the invariant linear functionals and this means I use the duality somehow. I want $[(\mathfrak{g}^{\otimes p})_{\mathfrak{g}}]^* = (\mathfrak{g}^{\otimes p})^{*\mathfrak{g}} \simeq (\mathfrak{g}^{\otimes p})^{\mathfrak{g}} \twoheadleftarrow k[\Sigma_p]$. Thus each $\sigma$ determines an invariant linear functional by taking trace.

$$\mathrm{tr}_\sigma : X^1\otimes\cdots\otimes X^p \longmapsto \mathrm{tr}_{V^{\otimes p}}(\sigma\cdot(X^1\otimes\cdots\otimes X^p))$$

$$= X^1_{i_1 i_{\sigma^{-1}1}} X^2_{i_2 i_{\sigma^{-1}2}}\cdots\cdots X^p_{i_p i_{\sigma^{-1}p}}$$

Thus we have the invariant multi-linear fcns.

$$\mathrm{tr}_\sigma(X^1,\dots,X^p) = \mathrm{tr}_{V^{\otimes p}}(\sigma\cdot(X^1\otimes\cdots\otimes X^p))$$

on $\mathfrak{g}^p$, where $\sigma\in\Sigma_p$. Next note that

$$\tau(X^1\otimes\cdots\otimes X^p)\tau^{-1} = X^{\tau^{-1}1}\otimes\cdots\otimes X^{\tau^{-1}p}$$

Proof: $\tau(X^1\otimes\cdots\otimes X^p)\tau^{-1}(v_1\otimes\cdots\otimes v_p) = \tau(X^1 v_{\tau 1}\otimes\cdots\otimes X^p v_{\tau p})$

$$= X^{\tau^{-1}1} v_{\tau\tau^{-1}1} \otimes \cdots \otimes X^{\tau^{-1}p} v_{\tau\tau^{-1}p}$$

$$= \left(X^{\tau^{-1}1} \otimes \cdots \otimes X^{\tau^{-1}p}\right)\left(v_1 \otimes \cdots \otimes v_p\right).$$

It follows that

$$tr_\sigma\left(X^{\tau^{-1}1}, \ldots, X^{\tau^{-1}p}\right) = tr_{V^{\otimes p}}\left(\sigma\tau(X^1 \otimes \cdots \otimes X^p)\tau^{-1}\right)$$

$$= tr_{\tau^{-1}\sigma\tau}(X^1, \ldots, X^p)$$

or better

$$\boxed{tr_{\tau\sigma\tau^{-1}}\left(X^{\tau^{-1}1}, \ldots, X^{\tau^{-1}p}\right) = tr_\sigma(X^1, \ldots, X^p)}$$

---

Final remark is that if $n \geq p$, then one can see that

$$k[\Sigma_p] \hookrightarrow \mathrm{End}(V^{\otimes p})$$

In effect, let $v_1, \ldots, v_p$ be independent elements of $V$. Then the elements

$$\sigma(v_1 \otimes \cdots \otimes v_p) = v_{\sigma^{-1}1} \otimes \cdots \otimes v_{\sigma^{-1}p}$$

are independent, etc. Thus for $n \geq p$ one has

$$k[\Sigma_p] \xrightarrow{\sim} (\mathfrak{g}^{\otimes p})^{\mathfrak{g}}$$

and so by duality any invariant linear fnl on $\mathfrak{g}^{\otimes p}$ is **uniquely** a linear combination of the $tr_\sigma$.

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Next we consider $\varphi \in C^p(\tilde{\mathfrak{g}})^{\mathfrak{g}}$, $\tilde{\mathfrak{g}} = \mathfrak{gl}_n A = \mathfrak{g} \otimes A$ i.e. $\varphi$ is a ~~[illegible]~~ skew-symmetric linear functional on $\tilde{\mathfrak{g}}^p$ which is invariant for the adjoint action of $\mathfrak{g} \subset \tilde{\mathfrak{g}}$. Consider

$$\varphi(a^1X^1, a^2X^2, \ldots, a^pX^p)$$

where $a^1, \ldots, a^p \in A$ and $X^1, \ldots, X^p \in \mathfrak{g}$. Keeping the $a^i$ fixed we obtain a $\mathfrak{g}$-invariant multi-linear fn. on $\mathfrak{g}^p$. Supposing $n \geq p$ it can be written uniquely

$$\varphi(a^1X^1, \ldots, a^pX^p) = \sum_\sigma \varphi_\sigma(a^1, \ldots, a^p) \cdot tr_\sigma(X^1, \ldots, X^p)$$

~~where the $\varphi_\sigma$ are continuous multilinear functions on $A^p$~~ ~~Because $\varphi$ is skew symmetric~~. Clearly the $\varphi_\sigma$ are multilinear fns. on $A^p$. One has

1) $\varphi$ continuous $\Longleftrightarrow$ the $\varphi_\sigma$ all continuous

2) $\varphi$ skew-symm. $\Longleftrightarrow$ $\varphi_{\tau\sigma\tau^{-1}}(a^{\tau^{-1}1}, \ldots, a^{\tau^{-1}p}) = sgn(\tau)\, \varphi_\sigma(a^1, \ldots, a^p)$

Proof of the second:

$$\varphi(a^1X^1, \ldots, a^pX^p) = sgn(\tau)\, \varphi(a^{\tau^{-1}1}X^{\tau^{-1}1}, \ldots, a^{\tau^{-1}p}X^{\tau^{-1}p})$$

$$= sgn(\tau) \sum_\sigma \varphi_{\tau\sigma\tau^{-1}}(a^{\tau^{-1}1}, \ldots, a^{\tau^{-1}p})\, \underbrace{tr_{\tau\sigma\tau^{-1}}(X^{\tau^{-1}1}, X^{\tau^{-1}p})}_{tr_\sigma(X^1, \ldots, X^p)}$$

and it's clear.

2) enables us to decompose:

$$C^p(\tilde{\mathfrak{g}})^{\mathfrak{g}} = \bigoplus_{\substack{\pi \text{ partitions} \\ \text{of } p}} C^p(\tilde{\mathfrak{g}})^{\mathfrak{g}}_\pi$$

and to identify $C^p(\tilde{\mathfrak{g}})^{\mathfrak{g}}_p$ with $C^{p-1}(A)$.

March 10, 1985

Clean up the periodicity mechanism and how it appears on the differential form level. I worked this out last summer (p. 95-98). The following ideas are involved:

Graphs + Cayley transform
Clifford algebras should give a uniform procedure
Reducing to $F^2 = I$, $F = F^*$.

These should be organized properly. In addition I should be able to fit the infinite diml case in also.

So let's begin with the basic maps of Bott. The first embeds $U(n)$ into a space of paths in the Grassmannian $U(2n)/U(n) \times U(n)$. I see this map as associating to a unitary matrix $g$ and real number $x \geq 0$ the graph of $xg$.

$$(x, g) \longmapsto \Gamma_{xg} = \operatorname{Im}\left\{\begin{pmatrix} 1 \\ xg \end{pmatrix} : \mathbb{C}^n \to \mathbb{C}^{2n}\right\}$$

Note that this map is defined for $g \in GL(n, \mathbb{C})$. Note that as $g$ is invertible, the path starts at $\mathbb{C}^n \oplus 0 \subset \mathbb{C}^{2n}$ and ends at $0 \oplus \mathbb{C}^n \subset \mathbb{C}^{2n}$, so one obtains a map of the suspension of $U(n)$ to the Grassmannian.

We have this map

$$\mathbb{R}_{\geq 0} \times U(n) \longrightarrow U(2n)/U(n) \times U(n)$$

and we can pull back the canonical character forms on the Grassmannian. I propose to do this by looking at the pull-back of the subbundle with ~~the~~ Grassmannian connection.

We can lift the map to the principal frame bundle

$$(x, g) \longmapsto \frac{1}{\sqrt{1+x^2}}\begin{pmatrix} 1 \\ xg \end{pmatrix} : \mathbb{C}^n \longrightarrow \mathbb{C}^{2n}$$

The connection form is then

$$\theta = \frac{1}{\sqrt{1+x^2}}\begin{pmatrix} 1 \\ xg \end{pmatrix}^* d\,\frac{1}{\sqrt{1+x^2}}\begin{pmatrix} 1 \\ xg \end{pmatrix}.$$

(In general given the subbundle of a trivial bundle spanned by an orthonormal frame $(s_1, \dots, s_n)$, we calculate the connection form as follows: A section of the subbundle is $s\varphi$ where $\varphi$ is a column vector of fns. Then the Grass. connection is

$$D(s\varphi) = \underbrace{ss^*}_{\text{orth. projection}} d(s\varphi) = s[d\varphi + s^* ds \cdot \varphi]$$

hence the connection form is $\theta = s^* ds$.)

We calculate

$$\theta = \frac{1}{1+x^2}\, x g^{-1}(dx\, g + x dg) \quad + d \log \frac{1}{\sqrt{1+x^2}}$$

$$= \frac{x^2}{1+x^2} g^{-1} dg \quad + \cancel{\frac{x\,dx}{1+x^2}} - \cancel{\frac{1}{2}\frac{2x}{1+x^2}dx}$$

Thus we are getting the 1-parameter family of connection form

$$\theta_t = t g^{-1} dg \qquad \text{where } t = \frac{x^2}{1+x^2}$$

So we can conclude that the character form $\frac{1}{k!}\operatorname{tr}(D^2)^k$, $D^2 = d(de)^2$ on the Grassmannian will go in the [redacted] form on $U(n)$ which are obtains via the Chern-Simons transgression process, i.e

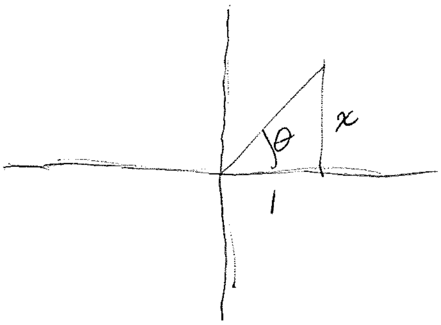
$$\frac{(-1)^{k-1}(k-1)!}{(2k-1)!} \, \mathrm{tr}\,(g^{-1}dg)^{2k-1} .$$

(A possible computation free proof would say that as the character forms on the Grass. are invariant and the map

$$I \times U(n) \longrightarrow U(2n)/U(n)\times U(n)$$

is canonical, the pull-back form on $I \times U(n)$ is biinvariant; as transgression yields a primitive form it has to be a multiple of $\mathrm{tr}\,(g^{-1}dg)^{2k+1}$.)

(What is intriguing about the above is the way the time parameter on the connection side appears out of the slope parameter $x$ in the graph constructi



$$\tan\theta = x \qquad \frac{x^2}{1+x^2} = \frac{\tan^2\theta}{\sec^2\theta} = \sin^2\theta \;\Bigg)$$

Let us now turn to the other map in the periodicity, the Bott map. This associates to a subspace of $\mathbb{C}^n$ a path from $I$ to $-I$ in $U(n)$. Let the subspace be [redacted] identified with a self-adj. involution $F$; $F = 1$ on the subspace and $-1$ on its orthog. complement. Then the path is

$$g = \frac{F+ix}{F-ix} \qquad 0 \le x < \infty .$$

I want to pull-back the odd form

$$\frac{(-1)^k k!}{(2k+1)!} \operatorname{tr} (g^{-1}dg)^{2k+1} .$$

Set $A = F + ix$, $B = F - ix$ so

$$g^{-1}dg = (AB^{-1})^{-1} d(AB^{-1}) = BA^{-1}(dA.B^{-1} - AB^{-1}.dB.B^{-1})$$

$$= B(A^{-1}dA - B^{-1}dB)B^{-1}$$

$$A^{-1}dA - B^{-1}dB = \frac{1}{F+ix}(dF + i\,dx) - \frac{1}{F-ix}(dF - i\,dx)$$

$$= \frac{-2ix}{x^2+1}dF + \frac{2F\,i\,dx}{x^2+1}$$

$$= \frac{2i}{x^2+1}(x\,dF + F\,dx)$$

Now $x\,dF$, $F\,dx$ commute (strictly) as $F$, $dF$ anti-commu

Thus

$$\frac{(-1)^k k!}{(2k+1)!} \int_0^\infty dx.\, \iota(\partial_x) \operatorname{tr}(g^{-1}dg)^{2k+1}$$

$$= \frac{(-1)^k k!}{(2k+1)!} \int_0^\infty dx \frac{(2i)^{2k+1} x^{2k}}{(1+x^2)^{2k+1}} (2k+1) \operatorname{tr} F(dF)^{2k}$$

$$= 2i \frac{k!\, 2^{2k}}{(2k)!} \left( \int_0^\infty dx \frac{x^{2k}}{(1+x^2)^{2k+1}} \right) \operatorname{tr}(F(dF)^{2k})$$

And then one can compute the integral by the substitution $t = \frac{x^2}{1+x^2}$ ending up with $\frac{1}{2}\beta(k+\frac{1}{2}, k+\frac{1}{2})$

$$2i\pi \cdot \frac{1}{2^{2k+1}} \operatorname{tr}(F(dF)^{2k})$$

What I would like to do at this point is to combine these two calculations by introducing the Clifford algebra.

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March 11, 1985

I want to present the periodicity maps in a uniform way using the Clifford algebras $C_k$. The rough idea is that [redacted] [redacted] the K-theory of $C_k$-modules is related to the $k$-th loop space of the K-theory of $\mathbb{C} = C_0$-modules.

I guess one begins with the ABS paper which shows how to go from graded $C_k$-modules to virtual vector bundles over $\mathbb{R}^k$ [redacted] with compact support. [redacted] Trivial K-classes are produced by graded $C_k$-modules which come from graded $C_{k+1}$-module by forgetting one of the generators.

Thus a trivialization of a graded $C_k$-module $V$ is provided by an odd degree s.a. involution $F$ which anti-commutes with $\gamma^1, \ldots, \gamma^k$. The space of such $F$ is [redacted] the analogue of the unitary group for the $k$-th K-theory.

For example if $k=0$, then an odd (s.a.) involution $F$ on a $\mathbb{Z}_2$-graded vector space $V = V^+ \oplus V^-$ is the same thing as a unitary isom. $P : V^+ \xrightarrow{\sim} V^-$. We have

$$F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix} \qquad \text{and} \quad F = F^* \implies P^{-1} = P^* \text{ so}$$

$P$ is unitary.

If $k=1$, then any graded $C_1$-module $V$ is of the form $V \cong S_2 \otimes W$ with $W$ an ungraded vector space, namely $W = V^+$. It might be better to say that

$$V = W \oplus W \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

An odd involution $F$ of $V$ anti-commuting with $\gamma^1$ is of the form

$$F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix} \qquad p^* = p^{-1} = -p$$

i.e. of the form

$$F = \begin{pmatrix} 0 & -iJ \\ iJ & 0 \end{pmatrix} \qquad J^2 = 1 \; , \; J = J^* .$$

Hence $F$ corresponds to a grading of $W$, i.e. an element in a <u>certain Grassmannian</u>.

Summarizing: If I consider on a graded $C_k$-module the space of odd involutions anti-commuting with $\gamma^1, .., \gamma^k$, then this space has the homotopy type of $BU$ for $k$ odd and $U$ for $k$ even. It might be simpler to consider ungraded $C_k$-modules to get the parity to work out correctly.

So what we will do is to carry out the periodicity. We start with $\gamma^1, .., \gamma^k$ fixed and consider an $F$ anti-commuting with $\gamma^1, .., \gamma^k$. Then we can form the path of involutions anti-commuting with $\gamma^1, .., \gamma^{k-1}$.

$$\frac{x\gamma^k + F}{\sqrt{1+x^2}} = (\sin\theta)\gamma^k + (\cos\theta)F$$

for $-\infty < x < \infty$, i.e. $-\pi/2 \le \theta \le \pi/2$. The problem is that the endpoint at $x=0$ is $F$.

Let's take some examples. Take $k=1$. Think of $\gamma^1$ as defining a grading of $V$:

$$\gamma^1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \text{on} \quad V = W^+ \oplus W^-$$

and then

$$F = \left(\begin{array}{c|c} & p^* \\ \hline p & \end{array}\right) \quad \text{where} \quad p : W^+ \xrightarrow{\sim} W^- \text{ is}$$

unitary. Then

$$\frac{x\gamma^1 + F}{\sqrt{1+x^2}} = \begin{pmatrix} x & p^* \\ p & -x \end{pmatrix} \frac{1}{\sqrt{1+x^2}}$$

is the involution belonging to the subspace

$$\mathrm{Im}\left\{ \begin{pmatrix} x \\ p \end{pmatrix} : W^+ \hookrightarrow W^+ \oplus W^- \right\}$$

this is wrong see below

as it is supposed to be.

Next take $k=2$. Then as a $C_2$-module

$$V = S_2 \otimes W = W \oplus W$$

with [redacted]

$$\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $F$ has the form

$$F = \begin{pmatrix} 0 & -iJ \\ iJ & 0 \end{pmatrix}$$

where $J$ is an involution on $W$. So

$$\frac{x\gamma^2 + F}{\sqrt{1+x^2}} = \begin{pmatrix} 0 & x - iJ \\ x + iJ & 0 \end{pmatrix} \frac{1}{\sqrt{1+x^2}}$$

[redacted] is the involution corresponding to the unitary matrix

$$g = \frac{x + iJ}{\sqrt{1+x^2}}$$

Actually this parameter $x$ is not the one we used before. Let's first correct the error with $k = 1$. We have the involution

$$\frac{1}{\sqrt{1+x^2}} \begin{pmatrix} x & g^{-1} \\ g & -x \end{pmatrix}$$

The corresponding projector is

$$\frac{1}{2} \left\{ \frac{1}{\sqrt{1+x^2}} \begin{pmatrix} x & g^{-1} \\ g & -x \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} \frac{1}{2}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) & \frac{1}{2\sqrt{1+x^2}} g^{-1} \\ \frac{1}{2\sqrt{1+x^2}} g & \frac{1}{2}\left(1 - \frac{x}{\sqrt{1+x^2}}\right) \end{pmatrix}$$

which is the projector onto the subspace

$$\operatorname{Im} \begin{pmatrix} y \\ g \end{pmatrix} \qquad y = \left(\sqrt{1+x^2} - x\right)^{-1} = \frac{[redacted]}{\sqrt{1+x^2}+x}$$

Thus we want $x$ to go from $-\infty$ to $+\infty$ and $y$ will go from 0 to $\infty$.

Similarly [redacted] in the case of

$$g = \frac{x + iJ}{\sqrt{1+x^2}}$$

we have

$$g = \frac{x+iJ}{\sqrt{1+x^2}} = \frac{y+iJ}{y-iJ} \qquad \begin{array}{ll} x=-\infty & y=0 \\ x=+\infty & y=\infty \end{array}$$

provided $$\frac{x+i}{\sqrt{1+x^2}} = \frac{y+i}{y-i} = \frac{(y+i)^2}{y^2+1} = \frac{(y^2-1)+2yi}{y^2+1}$$

$$\Rightarrow \quad \frac{1}{x} = \frac{2y}{y^2-1} \quad \Rightarrow \quad y^2-2xy-1=0$$

$$\Rightarrow \quad y = x \pm \sqrt{x^2+1} \quad \Rightarrow \quad \boxed{y = x+\sqrt{x^2+1}}$$

So I seem to be getting the correct answers in a mysterious way.

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Summary: If $V$ is a $C_k$-module with inner product, let $\mathcal{I}_k(V)$ be the space of self-adj involutions $F$ on $V$ anti-commuting with $\gamma^1, \dots, \gamma^k$. The periodicity map [redacted] [redacted] is

$$[-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \times \mathcal{I}_k(V) \longrightarrow \mathcal{I}_{k-1}(V)$$

$$(\theta, F) \longmapsto (\sin\theta)\gamma^k + (\cos\theta)F.$$

It thus assigns to $F$ [redacted] a path in $\mathcal{I}_{k-1}(V)$ going from $-\gamma^k$ to $+\gamma^k$.