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February 18, 1984

I want to check the index theorem over a torus for a Dirac operator of the form

$$-i\not{D} = -i\gamma^\mu D_\mu + \varepsilon L \qquad (L \text{ self adj.})$$

acting on $S \otimes E$, where $\varepsilon = \varepsilon_S$ gives the grading on the spinors S.

We use the heat kernel method:

$$\text{Index} = tr_s\left(e^{+\not{D}^2}\right)$$

and the fact the index is stable under perturbation. This means if we introduce a parameter h:

$$-i\not{D} = \frac{h}{i}\gamma^\mu D_\mu + \varepsilon L$$

we can evaluate the index in the "classical" limit $h \longrightarrow 0$.

$$\not{D}^2 = \left(h\gamma^\mu D_\mu + i\varepsilon L\right)^2$$

$$= h^2 D_\mu^2 - L^2 + h\gamma^\mu i\varepsilon[D_\mu, L] + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

Now we use the perturbation expansion

$$e^{\not{D}^2} = e^{h^2 D_\mu^2 - L^2} + \int_0^1 dt_1\, e^{(1-t_1)(h^2 D_\mu^2 - L^2)}\left\{h\gamma^\mu i\varepsilon[D_\mu, L] + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\right\} e^{t_1(h^2 D_\mu^2 - L)}$$

$$+ \cdots\cdots$$

Think of there being 2 perturbations $h\gamma^\mu i\varepsilon[D_\mu, L]$, $\frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$. Then by basic diagram calculus there is one term in the perturbation expansion for each word in the free monoid with 2 generators. Call the generators a, b then the words are

$$\underset{\text{empty word}}{1}, \quad a, \quad b, \quad a^2, \quad ab, \quad ba, \quad b^2$$

Corresponding to the word ab is the term

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \quad e^{(1-t_1)\Delta} A \, e^{(t_1-t_2)\Delta} B \, e^{t_2\Delta}$$

where $\Delta = h^2 D_\mu^2 - L^2$ is the socalled "free" propagator, $A = h\gamma^\mu i\varepsilon [D_\mu, L]$ is the first perturbation, and $B = \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$ is the second.

Now let us assign each term a degree which gives the degree of the power of h. Thus A has degree 1 and B has degree 2. This is also the number of γ factors. In this Euclidean situation the operator $\Delta = h^2 D_\mu^2 - L^2$ on $S \otimes E$ is $1 \otimes$ an operator on E, if we think of S as being a finite diml space of dim 2^n.

Ultimately I would like to learn Getzler's filtration techniques in this situation. He works in the algebra of ψDO's on the bundle of spinors.

February 19, 1984

Let's see if we can find formulas for the wave packet transform in Euclidean space, say for $\mathbb{R}$ to begin with. A typical wave packet should have the form

$$\text{Gaussian} \times \text{exponential} = e^{-\frac{1}{2a}(x-q)^2 + i\frac{p}{h}x}$$

This state has average position q, and I would like it also to have average momentum p. This is true no matter what a is; this is clear for $p=0$, and one reduces to this case.

Take $p=q=0$ and compute the second moments.

$$\langle q^2 \rangle = \int dx\, e^{-\frac{x^2}{a}} x^2 \Big/ \int dx\, e^{-\frac{x^2}{a}} = \frac{a}{2}$$

$$\langle p^2 \rangle = \int dx \left| -\frac{h}{i}\frac{1}{a} x\, e^{-\frac{x^2}{2a}} \right|^2 \Big/ \int dx\, e^{-\frac{x^2}{a}}$$

$$= \frac{h^2}{a^2}\frac{a}{2} = \frac{h^2}{2a}$$

These second moments are equal for $a = h$.

I conclude I don't understand the uncertainty principle, however, let's take $a = h$ and try to find a transform based on these wave packets.

The above function is an eigenfunction for the operator $x + h\frac{d}{dx} = (q + ip)$, hence we have got coherent states for oscillator: $H = \frac{1}{2}(p^2+q^2)$. We know in principle how to use these coherent states to transform to the holomorphic representation which should give a nice wave packet transform. What one wants, according to Connes is a family e_λ of elements of $L^2(\mathbb{R})$ parametrized by elements of the cotangent bundle such that

$$\int |e_\lambda\rangle\langle e_\lambda| = id .$$

This allows us to quantize, i.e. associate operators to functions on T^*.

In the holomorphic representation we have exactly this sort of thing happening. Recall

$$\|f\|^2 = \int \frac{d^2z}{\pi} e^{-|z|^2} |f(z)|^2 \qquad a = \frac{d}{dz} \qquad a^* = z$$

$$\frac{(a^*)^n|0\rangle}{\sqrt{n!}} = \frac{z^n}{\sqrt{n!}} \qquad \text{is an orthonormal basis.}$$

so

$$f = \sum \frac{1}{n!} |z^n\rangle\langle z^n| f$$

or

$$f(\lambda) = \sum \frac{1}{n!} \lambda^n \int \frac{d^2z}{\pi} e^{-|z|^2} \bar{z}^n f(z)$$

$$= \int \frac{d^2z}{\pi} e^{-|z|^2} e^{\lambda\bar{z}} f(z) = \langle e_\lambda | f\rangle$$

where $e_\lambda = e^{\bar{\lambda} z}$. Thus we have the reproducing formula

$$id = \int \frac{d^2z}{\pi} e^{-|z|^2} \; |e_z\rangle\langle e_z|$$

Next we work out the transform to the holom. representation. We want the annilation + creation operators

$$a = c(q+ip) , \quad a^* = \bar{c}(q-ip)$$

$$1 = [a,a^*] = |c|^2 \, 2h , \quad \text{so take } c = \frac{1}{\sqrt{2h}}$$

$$e_\lambda = e^{\bar\lambda z}$$ is an eigenfunction for a:

$$a\, e_\lambda = \bar\lambda\, e_\lambda$$

So what corresponds to e_λ is killed by

$$\frac{1}{\sqrt{2h}}\left(x + h\frac{d}{dx}\right) - \bar\lambda \qquad \text{or by} \qquad \frac{d}{dx} + \frac{x}{h} - \frac{\sqrt{2h}\,\bar\lambda}{h}$$

i.e.

$$e_\lambda = \text{const}\; e^{-\frac{x^2}{2h} + \frac{\sqrt{2h}}{h}\bar\lambda}$$

Now we want $\langle e_\lambda | e_\mu \rangle = e_\mu(\lambda) = e^{\lambda\bar\mu}$, and

$$\int dx\; e^{-\frac{x^2}{2h} + \frac{\sqrt{2h}}{h}\lambda - \frac{x^2}{2h} + \frac{\sqrt{2h}}{h}\bar\mu} = \int dx\; e^{-\frac{x^2}{h} + \frac{\sqrt{2h}(\lambda+\bar\mu)}{h}}$$

$$= \sqrt{\pi h}\; e^{\frac{1}{h}\left(\frac{\sqrt{h}}{\sqrt{2}}(\lambda+\bar\mu)\right)^2} = \sqrt{\pi h}\; e^{\frac{1}{2}(\lambda^2 + 2\lambda\bar\mu + \bar\mu^2)}$$

so the good choice seems to be

$$e_\lambda = (\pi h)^{-1/4}\, e^{-\frac{1}{2}\bar\lambda^2}\, e^{-\frac{x^2}{2h} + \frac{\sqrt{2h}\,\bar\lambda}{h}x}$$

Unfortunately this is not centered in the right place

Let's leave explicit formulas for a wave packet transform and review some general ideas developed yesterday.

I started with trying to prove rigorously the index thm. for a Dirac operator $-i\gamma^\mu D_\mu + \varepsilon L$ over a flat Riemannian manifold. The method was to introduce a Planck's constant h and to consider $-i\not\partial = -ih\gamma^\mu D_\mu + L$ and its square $+(-h^2 D_\mu^2 + L^2 + \cdots)$.

One then evaluates the index $tr_s(e^{\not{D}^2})$ in the limit as $h \to 0$.

This fits the following bunch of ideas from physics. A typical Hamiltonian $H = \frac{p^2}{2} + V(q)$ is interpreted as an operator with $q = x$, $p = \frac{h}{i}\partial_x$, whence $H = -\frac{h^2}{2}\Delta + V(x)$. The quantum partition function $Tr(e^{-\beta H})$ can be evaluated asymptotically in the classical limit: $h \to 0$. The leading term of the expansion is the integral of $e^{-\beta H}$ over phase space T^* with the Liouville volume $\left(\frac{dp\,dq}{2\pi h}\right)^n$. In 1-dimension we get

$$\int \frac{dp\,dq}{2\pi h} e^{-\beta\frac{p^2}{2} - \beta V(q)} = \frac{1}{\sqrt{2\pi\beta}\,h}\int dq\, e^{-\beta V(q)}$$

When we take H to be the square of the Dirac operator, and take the super trace, the factor h^n in the denominator ($n = \dim.$) cancels a similar factor in the numerator.

Now what is important for me is that, because of the limiting process I take where L stays fixed, I really am interested in the classical limit as opposed to t or $\beta \to 0$ limits.

Another idea which came up yesterday is the wave packet transform. Originally I thought of a transform depending on h, so that for example the packet corresponding to a point q, p of phase space is something like

$$e^{\frac{(x-q)^2}{2h} + i\frac{p}{h}x}$$

This suggested to me [redacted] various ideas of Bott. The wave packet transform might be a diagonal approximation.

Let me now try to describe Bott's idea which came from his work on the Lefschetz Fixpt theorem. The index or Lefschetz number is the trace of an endomorphism of an infinite-dimensional, but perfect complex.

Formally the trace of an operator is the "intersection number" of its kernel with the diagonal. This generally doesn't make sense, however, for an elliptic complex, the operators give one a deformation of the diagonal δ function into smooth kernel operators. This is the idea of a parametrix for a complex.

We should separate two ideas. One involves deforming the identity to a smooth kernel operator via e^{-tA} with $A \geq 0$, and then using this deformation to define

$$\text{"tr"}(B) = \underset{t\to 0}{\text{"lim"}} \ \mathrm{tr}(e^{-tA}B)$$

where $\underset{t\to 0}{\text{"lim"}}$ means take a coefficients in the asymptotic expansion. The other idea involves a homotopy operator P on a complex such that $DP + PD = I - K$ with K smoothing. (One is a geometric homotopy and the other is algebraic.)

These two ideas can be related. Take $\Delta = D^*D + DD^*$. This is homotopic to zero, hence so will be $f(\Delta)$ if

$f(0) = 0$. For example

$$\int_0^1 dt \, \frac{d}{dt}(-e^{-tx}) = \int_0^1 e^{-tx} x \, dt$$

$$1 - e^{-x} = \int_0^1 e^{-tx} x \, dt$$

$$\Longrightarrow \qquad 1 - e^{-\Delta} = \int_0^1 e^{-t\Delta} \Delta \, dt = \int_0^1 e^{-t\Delta} (D^*D + DD^*) dt$$

$$= \left(\int_0^1 e^{-t\Delta} D^* dt \right) D + D \left(\int_0^1 e^{-t\Delta} D^* dt \right)$$

and so we see that

$$P_a = \int_0^a e^{-t\Delta} D^* \, dt$$

will be a parametrix for any $a > 0$. Then we could average such things for different a if one wanted to.

For example integrate the following

$$1 - e^{-a\Delta} = [D, P_a]$$

against $e^{-a\lambda}$ and you see

$$\int_0^\infty (1 - e^{-a\Delta}) e^{-a\lambda} \, da = \frac{1}{\lambda} - \frac{1}{\lambda + \Delta}$$

is homotopic to zero. This is just the same as ~~starting~~ with $f(x) = \frac{1}{\lambda} - \frac{1}{\lambda + x}$. More useful might be functions like $\frac{1}{\lambda^n} - \frac{1}{(\lambda + x)^n}$ where n is large enough so that $\frac{1}{(\lambda + \Delta)^n}$ has trace class.

Let's continue with Bott's ideas. He gave in his LFF course an example of a parametrix for the DR complex.

I examined this a bit and concluded it was too naive to work in general. One shouldn't be able to construct a parametrix for the Laplacean without using Fourier analysis in some way.

Conclusions:

In order to really understand a proof of the index thm. one must have at the same time a proof of the finite dimensionality of the kernel + cokernel. So one is obliged to rely on a machine like ΨDO's to construct a parametrix.

Because of the formal similarity between heat kernels and the Chern character, the heat kernel approach [redacted] is preferable.

February 20, 1984

The problem I have is to rigorously prove the classical limit formula

$$\mathrm{Tr}\left(e^{-\beta H}\right) \sim \int \left(\frac{dq\,dp}{2\pi h}\right)^n e^{-\beta H} \qquad \text{as } h \to 0.$$

where $H = \frac{p^2}{2} + V(q)$ is quantized as the operator $-\frac{h^2}{2}\nabla^2 + V(x)$.

(I want to work with heat kernels $e^{-\beta H}$ because of the connection with the Chern character and with the $h \to 0$ limit because I start with Dirac ops $\not{D} = \gamma \cdot D + \varepsilon L$.)

The first difficulty consists in the existence of the operator $e^{-\beta H}$. Presumably if you had a formula for $\mathrm{Tr}\left(e^{-\beta H}\right)$ then you could let $h \to 0$ in the formula. This is what happens in a not-so-rigorous way in the path integral expression

$$\cancel{\mathrm{Tr}\left(e^{-\beta H}\right) = \int_{x(0)=x(\beta)} [\mathcal{D}x(\tau)]\, e^{-\frac{1}{h}\int_0^\beta \left[\frac{1}{2}\dot{x}^2 + V(x)\right] d\tau}}$$

for $\mathrm{Tr}\left(e^{-\beta H}\right)$. Let's derive this expression.

$$\langle x | e^{-\beta \frac{\hat{p}^2}{2}} | x' \rangle = \int \frac{dp}{2\pi h} \langle x | p \rangle e^{-\beta \frac{p^2}{2}} \langle p | x' \rangle$$

$$= \int \frac{dp}{2\pi h}\, e^{-\beta \frac{p^2}{2} + i\frac{p}{h}(x-x')} = \frac{1}{\sqrt{2\pi\beta}\, h}\, e^{-\frac{1}{2\beta h^2}(x-x')^2}$$

(in this $\beta = d\tau$)

So we get

$$\mathrm{Tr}\left(e^{-\beta H}\right) = \binom{\text{norm}}{\text{const}} \int_{x(0)=x(\beta)} \mathcal{D}x(\tau)\; e^{-\int_0^\beta \left[\frac{1}{2h^2}\dot{x}^2 + V(x)\right] d\tau}$$

and we see that ~~at~~ $h \to 0$ the dominant contributions should come from curves with $\dot{x} = 0$.

My problem remains to construct the operator $e^{-\beta H}$ in some form, & then to analyze the $h \to 0$ limit. It is clear that I probably want to use some version of the subdivision idea that Rosenberg showed me last year, namely that if you take a path of operators $\gamma(t)$ starting at the identity $\gamma(0) = I$, then

$$\lim_{N\to\infty} \gamma\left(\frac{t}{N}\right)^N = e^{\gamma'(0)t}$$

This sort of idea requires some justification, but it captures the point that $e^{-\beta H}$ has to be calculated in the algebra of operators not the [redacted] algebra of symbols. I should be able to do the calculation in the ~~[illegible]~~ algebra of operators depending on the parameter h, that is, Connes groupoid algebra.

Let review the program. I have decided to try to prove the index theorem for Dirac operators by the heat kernel approach, especially using the idea that the quantum ~~partition~~ function $tr(e^{-H})$ has as classical limit the classical ~~partition~~ function which is an integral over the cotangent bundle. One thing that occurred to me is that the heat kernel is given by a formula

$$\lim_{N\to\infty} \gamma\left(\frac{t}{N}\right)^N = e^{-tH}$$

where $\gamma(t)$ is a path of operators with derivative H. This formula fits very nicely with second order Laplaceans operators, hence seems natural for dealing with Dirac operators.

Recall the derivation of the differential equation satisfied by a limit $\lim_{N\to\infty} (L_{t/N})^N = K_t$

$$\int dx\, f(x)\left[K_{t+\Delta t}(x,y)\right] = \int dx\, f(x) \int dz\, L_{\Delta t}(x,z)\, K_t(z,y)$$

$$= \int dz \left[\underbrace{\int dx\, f(x)\, L_{\Delta t}(x,z)}\right] K_t(z,y)$$

$$\int dx \left(f(z) + f'(z)(x-z) + \frac{f''(z)}{2}(x-z)^2 + \dots \right) L_{\Delta t}(x,z)$$

Now one assumes that for Δt small, $L_{\Delta t}(x,z)$ is close to the δ-function at z with moments

$$\int (x-z)\, L_{\Delta t}(x,z) \sim \Delta t\, M(z)$$

$$\int (x-z)^2 L_{\Delta t}(x,z) \sim \Delta t\, N(t)$$

and higher moments ~ 0 as $\Delta t \to 0$.

Then we get

$$\frac{\partial}{\partial t}\int dx\, f(x)\, K_t(x,y) = \int dz \left[f'(z)M(z) + \tfrac{1}{2}f''(z)N(z)\right] K_t(z,y)$$

whence

$$\frac{\partial}{\partial t} K_t(x,y) = \left(\frac{-\partial}{\partial x} M + \frac{1}{2}\frac{\partial^2}{\partial x^2} N\right) K_t(x,y)$$

Thus the first + second moments of $L_t(x,z)$ for small t should be proportional to t.

Two basic problems or difficulties are.

1) How to bring in the cotangent bundle?

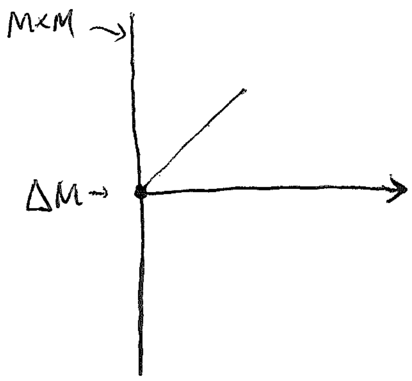
2) How are the $\hat{A}$-genus or Td genus of T^* to come in to the picture?

The first problem is more basic, and already occurs for a simple scalar Hamiltonian. The quantum partition function as $h \to 0$ is equivalent to the classical partition function which is an integral over phase space with respect to $\left(\frac{dq\,dp}{2\pi h}\right)^n$. So we have to understand how a trace becomes such an integral. Supposedly one can do this via the path integral as Alvarez-Gaumé does it in the more complicated situation where fermions are present.

February 21, 1984

Today I want to try to decipher Connes algebra of operators depending on h. Let's recall first his groupoid. The objects of the groupoid are pairs (h, x) where $h \geq 0$ and $x \in M$. The set of morphisms with source (h, y) will be $\{(h, x, y) \mid x \in M\}$ if $h > 0$ (the target of (h, x, y) is (h, x)) and if $h = 0$ the set of morphisms with source $(0, y)$ is $\{(0, v, y) \mid v \in T_M(y)\}$ (the target of $(0, v, y)$ being $(0, y)$). It is clear how the objects form a manifold with bdry isomorphic to $\mathbb{R}_{\geq 0} \times M$. The morphisms form a manifold with bdry isomorphic to the result of blowing up $\mathbb{R}_{\geq 0} \times M \times M$ along $0 \times \Delta M$ in some sense.

Picture:



A point of the blowup consists of a half line normal to $0 \times \Delta M$ together with a point on that line. So the fibre over $0 \times \Delta M$ consists of the normal half lines to $0 \times \Delta M$. Those which project non-trivially into the h axis can be identified with a tangent vector. Those lines which are vertical are thrown away.

This groupoid gives rise to a convolution algebra which consists of kernels $K(h, x, y)$ which are some sort of functions on the set of morphisms. For each $h \neq 0$ I get the algebra of (say smooth) kernels on M. Really, if $h \neq 0$ is fixed, then I get a homom. of this algebra into the algebra of operators on M. If

$h=0$ we [redacted] get a homomorphism into the convolution algebra of the tangent bundle, and this by Fourier is isomorphic to the functions on T^* under multiplication. So far, I have ignored the problem of decay at ∞.

Next let's take [redacted] an operator depending on h such as $H = -\frac{h^2}{2}\Delta + V$ and form $e^{-\beta H}$ which is a smooth kernel operator on M for each $h>0$. A natural question is whether this family of $e^{-\beta H}$ operators for $h>0$ extends to $h=0$. Also what happens to the trace $tr(e^{-\beta H})$ as $h \to 0$. (Is there a natural trace on the Connes algebra of some sort?)

What I want to do is to find enough concrete things to calculate so that I can get a feeling for Connes algebra, and whether it really can be used to understand the classical limit.

Let's do some [redacted] calculations in Euclidean space. Take $H = \frac{p^2}{2}$. Then the kernel of $e^{-\beta H}$ is

$$\langle x | e^{-\beta \frac{p^2}{2}} | y \rangle = \int \frac{dp}{2\pi h} e^{-\beta \frac{p^2}{2} + i \frac{p}{h}(x-y)}$$

$$= \frac{1}{\sqrt{2\pi\beta}\, h} e^{-\frac{(x-y)^2}{2\beta h^2}}$$

Does this extend to a kernel in Connes's algebra? At this point I really do have to worry about the volumes we define convolution by.

February 22, 1984

Program for the index theorem for Dirac operators: Start with a Dirac operator $\gamma \cdot D + \varepsilon L$ which by its nature is a supersymmetry operator mixing boson and fermion degree of freedom. Its square is the Hamiltonian H and the superpartition function $\mathrm{tr}_s(e^{-\beta H})$ can be identified with the index by spectral theory. Now take the classical limit as Planck's constant $h \to 0$. The quantum partition fn. should be asymptotic to the classical partition function.

Now a classical partition function with bosons is simply an integral over phase space with respect to the Liouville measure. What should the classical partition function be [redacted] for fermions? It should be something like differential forms. More precisely the fermion operators belong to a kind of Clifford algebra on which the super trace is the natural trace. As $h \to 0$ the Clifford algebra deforms into the exterior algebra and it should be possible to see what happens to the trace.

Examples of fermion partition functions and the classical limit: We have already seen an example in trying to prove the index formula over a torus (p.502), namely

$$\mathrm{tr}_s\left(e^{\not{D}^2}\right) = \mathrm{tr}_s\left(e^{h^2 D_\mu^2 - L^2 + h\gamma^\mu i\varepsilon[D_\mu, L] + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}}\right)$$

Lets ignore the first part $h^2 D_\mu^2$ whose effect ultimately is only to contribute an h^n in the denominator. So I take as my example

$$\operatorname{tr}_s \left(e^{-L^2 + h\gamma^\mu i\varepsilon[D_\mu, L] + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} \right)$$

where the tr_s is taken only over the spinor indices. Here L, $[D_\mu, L]$, $F_{\mu\nu}$ are vector bundle homomorphisms. So we get the following setup. The γ^μ are operators on the spinors S, then $(-L^2)$, $F_{\mu\nu}$ are even endos and $i[D_\mu, L]$ are odd endos of a super vector space E. Put $A = (-L^2)$, $B_\mu = i[D_\mu, L]$. Then we have

$$(*) \qquad \operatorname{tr}_s \left(e^{A + h\gamma^\mu \varepsilon B_\mu + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}} \right)$$

where the exponent is an operator on $S \otimes E$. Notice that $\gamma^\mu \varepsilon B_\mu$ is actually $\gamma^\mu \hat{\otimes} B_\mu$; in other words we are working in the algebra

$$C \,\hat{\otimes}\, (\operatorname{End} E)$$

and we just take the trace over C.

Another way to obtain the same sort of example is to start with a Clifford algebra $C_{k+l} = C_k \hat{\otimes} C_l$ and a quadratic Hamiltonian. Let γ^μ denote the generators of C_k and γ^j denote the generators of C_l. Then a quadratic element of C_{k+l} is of the form

$$\underbrace{\tfrac{1}{2}\gamma^i\gamma^j A_{ij}} + \gamma^\mu \underbrace{\gamma^j B_{\mu j}} + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

$$= \quad A \quad + \gamma^\mu B_\mu + \tfrac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

so if we scale the γ^μ : $\gamma^\mu \longrightarrow h\gamma^\mu$ but not the others then we obtain the same partition function $(*)$ as above.

Now that I know what fermion partition functions are, I can ask about the classical limit. This means we want the leading term. We use that fact the supertrace on C kills all monomials $\gamma^{i_1} \ldots \gamma^{i_k}$ except the highest one. So if we have $C = C_n$ one sees that the smallest term is of order h^n and the coeff can be obtained by replacing the γ's by generators of the exterior algebra and taking the projection onto the highest degree line Λ^{max}.

The Clifford algebra C_n is an example of a filtered algebra with increasing filtration. Recall that if $A = \bigcup_{n \geq 0} F_n A$ is a filtered algebra, I found it convenient in my paper on higher algebraic K-theory to introduce the graded algebra

$$\tilde{A} = \bigoplus_{n \geq 0} F_n A \subset \bigoplus_{n \geq 0} h^n A = A[h]$$

In the case where

$$\operatorname{gr} A = \bigoplus_{n \geq 0} F_n A / F_{n-1} A = \tilde{A} / h \tilde{A}$$

is regular noetherian (not nec. commutative) I was able to use the localization thm. for the following

$$\begin{tikzcd}
\mathrm{Modfgr}(\tilde{A})_{h\text{ tors}} \arrow[r, hook] \arrow[d, dotted, dash, "K\text{ equiv}"] & \mathrm{Modfgr}(\tilde{A}) \arrow[r] \arrow[d, dotted, dash, "K\text{ equiv}"] & \mathrm{Modfgr}(\underbrace{\tilde{A}[h^{-1}]}_{A[h,h^{-1}]}) \arrow[d, phantom, "\wr"] \\
\mathrm{Modfgr}(\operatorname{gr} A) \arrow[d, dotted, dash, "K\text{ equiv}"] & \mathrm{Modfgr}(A_0) & \mathrm{Modf}\ A \\
\mathrm{Modfgr}(A_0) & &
\end{tikzcd}$$

It's not clear that to think in algebra K-terms about C_n and $\Lambda \mathbb{C}^n$ is a good idea. In fact we have seen that good K-theory only arises in the Kasparov setup.

However let's look carefully at $\tilde{A}$ and its [redacted] analogue for the Weyl algebra which seems to be a minimal algebraic gadget for Connes asymptotic operators.

$$\tilde{A} = k \oplus h\overbrace{(k \oplus V)}^{F_1 C} \oplus h^2\overbrace{(k \oplus V \oplus \Lambda^2 V)}^{F_2 C} \oplus \cdots$$

is generated over $k[h]$ by the $h\gamma^\mu$. So it is a Clifford algebra over the ring $k[h]$ generated by anti-commuting elts $h\gamma^\mu$ of square h^2. Specializing at any invertible value of h we get

$$\tilde{A}[h^{-1}] = C \otimes k[h, h^{-1}]$$

and on the other hand

$$\tilde{A}/h\tilde{A} \cong \Lambda V$$

Finally the [redacted] super-trace. What is $[\tilde{A}, \tilde{A}]$? Do for C_1, whence

$$[h^i \gamma^1, h^j \gamma^1] = h^{i+j}[\gamma^1, \gamma^1] = 2\, h^{i+j}$$

and so in this case we get $[\tilde{A}, \tilde{A}] = h^2 k[h]$. In general it seems that $h^2 (h\gamma^{i_1}) \cdots (h\gamma^{i_p})$, $p < n$ is a supercommutator. This agrees with the idea that reducing mod h^2 makes the algebra $\tilde{A}/h^2\tilde{A}$ into an exterior alg. over ~~the~~ algebra of dual numbers $k[h]/(h^2)$.

In any case the interesting super-trace projects [redacted] modulo $k[h] F_{<n} \tilde{A}$ onto $k[h] h^n \gamma^1 \cdots \gamma^n$.

[redacted]

Actually something slightly more natural might be the following. Instead of $\tilde{A}$ which is generated over $k[h]$ with the generators $h\gamma\partial$ we could take the algebra over $k[h^2]$ with the generators $h\gamma\partial$. Notice that this is the universal enveloping algebra of the <u>superalgebra spanned by</u> $(h\gamma\partial)$ and h^2.

Next project will to go onto the Weyl algebra. The Weyl algebra is generated by elements x^μ, ∂_μ such that $[x^\mu, x^\nu] = [\partial_\mu, \partial_\nu] = 0$, $[\partial_\mu, x^\nu] = \delta_\mu^\nu$. Stick to one dimension. Then we have two generators $a^* = x$, $a = \partial$ such that $[a, a^*] = 1$. I now consider the graded algebra $\tilde{W} = \bigoplus h^n F_n W$ which is generated over $k[h]$ with generators ha, ha^* satisfying $[ha, ha^*] = h^2$. Again it looks more natural to consider the algebras over $k[h]$ with generators p, q satisfying $[p, q] = h$. This is then the universal enveloping algebra of the Heisenberg algebra.

February 24, 1984

Lecture on Clifford algebras.

$$C(V,Q) = T(V)/(v^2 - Q(v)) \qquad \mathbb{Z}_2\text{-graded alg.}$$

Q is a quadratic function on V:

$$Q(cv) = c^2 Q(v)$$
$$B(v,w) = Q(v+w) - Q(v) - Q(w) \quad \text{is bilinear.}$$

Then
$$Q(\textstyle\sum c_j v_j) = Q(c_1 v_1) + B(c_1 v_1, \sum_{j>1} c_j v_j) + Q(\sum c_j v_j)$$

and by induction
$$Q(\textstyle\sum c_j v_j) = \sum_j c_j^2 Q(v_j) + \sum_{i<j} c_i c_j B(v_i, v_j)$$

Ex: $Q=0 \implies C = C(V,Q) = \Lambda V$.

$C(V,Q)$ has generator v_j a basis for V with the relations $v_i^2 = Q(v_i)$, $v_i v_j + v_j v_i = B(v_i, v_j)$. It follows that C is spanned by the monomials

$$v_{i_1} \cdots v_{i_p} \qquad i_1 < \cdots < i_p$$

and hence $\dim C \leq 2^n$, $n = \dim V$ supposed finite.

Action of C on ΛV. Recall ΛV has operators

exterior mult. $e_v : \Lambda V \to \Lambda V$, $e_v \omega = v \cdot \omega$, degree 1

interior mult $\iota_\lambda : \Lambda V \to \Lambda V$, degree -1 derivation

such that $\iota_\lambda(v) = \lambda(v)$. Identities:

$$e_v^2 = 0, \quad \iota_\lambda^2 = 0$$
$$\iota_\lambda e_v + e_v \iota_\lambda = \lambda(v) \qquad \text{expresses derivation property.}$$

This defines an action of the Clifford algebra $C(V^*\oplus V)$ on ΛV, where $V^*\oplus V$ is equipped with the hyperbolic quadratic form $\lambda + v \mapsto \lambda(v)$. In effect the map $\lambda + v \longrightarrow \iota_\lambda + e_v$ from $V^*\oplus V$ to $\mathrm{End}(\Lambda V)$ satisfies $(\iota_\lambda + e_v)^2 = \cancel{\iota_\lambda^2} + [\iota_\lambda, e_v] + \cancel{e_v^2} = \lambda(v)$.

But now I want an action of $C(V,Q)$. What I can do is to use the bilinear function B to define a map $V \xrightarrow{\lambda} V^*$, $v \mapsto B(v,?)$ and then associate to v the element $\frac{1}{2}\lambda_v + v$ in $V^*\oplus V$. This gives me an embedding of (V,Q) inside $V^*\oplus V$ with the hyperbolic form. Of course I have used that $\frac{1}{2}\in k$. More generally all we need is a bilinear form, or equivalently a map $\lambda: V \longrightarrow V^*$ such that $v \mapsto \lambda_v + v$ induces Q from the ~~hyperbolic~~ form, i.e.

$$\lambda_v(v) = Q(v)$$

Such a bilinear form always exists, e.g. use the formula for $Q(\sum c_j v_j)$ on the preceding page and take λ to be given by the matrix

$$\begin{pmatrix} Q(v_1) & B(v_1,v_2) & \cdots & B(v_1,v_n) \\ & Q(v_2) & \cdots & \vdots \\ & & \ddots & \vdots \\ 0 & & & Q(v_n) \end{pmatrix}$$

So we see that we have a map $(V,Q) \longrightarrow (V^*\oplus V, h)$ hence a map of algebras

$$C(V,Q) \longrightarrow C(V^*\oplus V)$$

and so an action of $C(V,Q)$ on ΛV where v acts as the operator $\iota_{\lambda_v} + e_v$ on ΛV.

Now consider the map $C = C(V,Q) \longrightarrow \Lambda V$, $\alpha \mapsto \alpha\cdot 1$ and let F_pC be the subspace of C spanned by products of at most p elements of V. This map carries $F_p C$ into $F_p \Lambda = \bigoplus_{d \le p} \Lambda^d V$. From the formula

$$v_1 \cdots v_k \longrightarrow (i_{\lambda_{v_1}} + e_{v_1}) \cdots (i_{\lambda_k} + e_{v_k}) \, 1$$

and the fact that the i_λ have degree -1 we see that

$$F_p C / F_{p-1} C \longrightarrow \Lambda^p V$$

is an isom. We know the former is spanned by monomials $v_{i_1} v_{i_2} \cdots v_{i_p}$ $\quad i_1 < \cdots < i_p$ which get mapped to a basis of the latter. Put another way, we have algebra maps

$$\Lambda V \twoheadrightarrow \mathrm{gr}\, C \longrightarrow \Lambda V$$

whose composition is the identity.

Now this is the first result about Clifford algs, namely that $C(V, Q)$ is additively isomorphic to ΛV, or more precisely that

$$\mathrm{gr}\, C \simeq \Lambda V$$

in the above way. (This result should be maybe amplified to an assertion that <u>the Clifford algebra is a deformation of the exterior algebra</u>. In a precise sense if we adjoin a variable h to k, then we can form the Clifford alg. over $k[h]$ generated by V with the relations $v^2 = hQ(v)$. Then we get an algebra over $k[h]$, which is free as a $k[h]$-module which specializes to $C(V, Q)$ at $h = 1$ and ΛV at $h = 0$.)

The second result is that for W finite-diml one has an isomorphism of super algebras

$$\star \qquad C(W^* \oplus W) \simeq \mathrm{End}(\Lambda W).$$

I don't have a direct proof of this, direct in the sense of being $GL(W)$ invariant. The proof given in class is to reduce by direct sums to the case where W [redacted] is 1-dimensional, whence

$$\Lambda W = k \oplus W \cong k \oplus k$$

if we use a basis $\overset{w}{\wedge}$ for W. Corresponding to this basis w is the dual basis [redacted] w^* of W^* and the operators are

$$a^* = e_w = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a = i_{w^*} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

from which we see that the map (A) is an isomorphism in this case. The next point is to use direct sums

$$C(V \oplus V', Q + Q') = C(V, Q) \hat{\otimes} C(V, Q')$$

$$\begin{aligned} \mathrm{End}\left(\Lambda(W \oplus W')\right) &= \mathrm{End}\left(\Lambda W \otimes \Lambda W'\right) \\ &= \mathrm{End}(\Lambda W) \hat{\otimes} \mathrm{End}(\Lambda W') \end{aligned}$$

and check compatibility.

[redacted] The point is that the **Clifford** algebra functor maps non-degenerate quadratic forms into central simple (?) superalgebras, and so gives a map from the Witt group of k to the graded Brauer group.

[redacted] Further questions. Over an alg. closed field of char $\neq 2$ any non-degenerate q. form on an even diml space is hyperbolic so C gives an elt. of the usual Brauer group. Relation of graded + usual Brauer groups.

Let us now review the status of the index thm. problem as of yesterday. I more or less came to an understanding of Connes algebra of operators on functions. Thus the next step is to bring in the fermions and Getzler's idea.

Always think in terms of operators first, when possible, and then pass to the algebras. The operators I am interested in all operate on $L^2(M, S\otimes E)$ where S is the bundle of spinors. For example

$$i\not{D} = h\gamma^\mu D_\mu + i\varepsilon L$$

$$-H = (i\not{D})^2 = h^2 D_\mu^2 - L^2 + h\gamma^\mu i\varepsilon [D_\mu, L] + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

We want the heat kernel $e^{-\beta H}$ to be in the Connes algebra, and then to evaluate $tr_s(e^{-\beta H})$ in the classical limit $h \to 0$.

What should the Connes algebra consist of? It should contain the scalar operators $f(q,p)$ where p is quantized as $\frac{h}{i}\partial_x$, and q as x, and it should contain the operators $h\gamma^\mu$. Notice the Dirac operator itself is not in the algebra as $h\gamma^\mu \partial_\mu$ doesn't have enough h factors. However the Hamiltonian H is OKAY.

Let's first look at the "polynomials" or differential operators in the algebra. This is like the universal env. alg. of the Heisenberg algebra. In the present case we have the ~~Lie~~ superalgebra spanned by $q^\mu, p^\mu, h, h\gamma^\mu$ with the relations

$$[p^\mu, q^\nu] = \frac{h}{i}\delta^{\mu\nu} \qquad [h\gamma^\mu, h\gamma^\nu] = h^2\delta^{\mu\nu}$$

and the rest of the brackets being zero. $\dim = 3n+1$

Next I want the supertrace on the corresponding algebra of smoothing operators. The first case is constant coefficient ops, i.e. those generated by p^μ, $h\gamma^\mu$. A typical operator is

$$\sum_I f_I(p)\, h^{|I|}\, \gamma_I \qquad \gamma_I = \gamma^{\mu_1}\cdots\gamma^{\mu_p} \quad \mu_1 < \cdots < \mu_p$$

and its supertrace is

$$\int d^n q \int \frac{d^n p}{(2\pi h)^n}\, f_{\{1,\dots,n\}}(p)\, h^n (2i)^{n/2}$$

which is independent of h.

In general the algebra should consist of

$$\sum_I f_I(h,q,p)\, h^{|I|}\gamma_I$$

where $f_I(h,q,p)$ is represented in the standard ψDO way, and then the super trace is

$$\int \frac{d^n q\, d^n p}{(2\pi h)^n}\, f_{\{1,\dots,n\}}(h,q,p)\, h^n (2i)^{n/2}$$

which has a nice limit as $h \longrightarrow 0$.

The above is somewhat disappointing. One has half as many fermions as bosons. The boson picture is pretty satisfying in that one ended up in the classical limit with functions on T^* and with the trace given by integration wrt the natural measure. But in the above situation the classical limit would be an algebra ~~of something~~ like forms of type $(0,*)$ on T^*. It's not all differential forms, which one could integrate naturally over T^* if they decayed fast enough.

Another thing that looks strange is the way the $h\gamma^\mu$ square to h^2 whereas p,q bracket to h/i.

In thinking about this ~~situation~~ I was reminded about Atiyah's assertion that the index theorem for an elliptic operator P over M is the same as the index thm. for the $\bar\partial$-operator (or Dirac operator) on T^* twisted by the symbol of P. One way to see this is to use the cohomological formula for the index - the index is the integral over T^*_M of the character of the symbol times $Td(T^*_M)$.

K-theoretically one has the following picture. The manifold M has a fundamental class in K-theory which is a K-homology class of T^*_M. (In fact $K^*(M) = K_*(T^*_M)$ by "Poincaré duality" and the fundamental class is the element of $K_*(T^*_M)$ corresponding to $1 \in K^0(M)$.)

I can describe this fundamental class in $K_0(T^*_M)$ in two ways:

(i) As the Brown, Douglas, Fillmore K_1-element given by the extension of functions on $S(T^*_M)$ by the ψDO's of order 0.

(ii) As the $\bar\partial$-operator on T^*_M.

So I have made an assertion that the K-elts (i) and (ii) are equivalent. If so, then capping with an element of $K^0(T_M^*)$ will give the same index. In the first case you get the ψDO on M with the given symbol, in the second the Dirac operator on T_M^* twisted by the symbol.

So now it seems that the interesting question is the equivalence of (i), (ii) especially [redacted] whether there exists a canonical analytical equivalence.

February 25, 1984

We learned yesterday that there are three ways to think of the fundamental class in K-theory of the manifold M. The first is the extension of functions on the cosphere bundle

$$0 \longrightarrow \Psi^{-1}(M) \longrightarrow \Psi^{0}(M) \longrightarrow C(S^*M) \longrightarrow 0$$

given by ΨDO's of order zero. By the BDF theory this gives an element of $K_1(S^*M) = K_0(T^*_M)$.

The second way is the Dirac operator on T^*_M. T^*_M is a symplectic manifold, and the maximal compact subgroup of the symplectic group is the unitary group, hence T^*_M has an almost complex structure unique up to homotopy. More directly the tangent space to T^*_M at (x,ξ) is an extension of the tangent space along the fibre, which can be identified with T^*_x by translation, by the space T_x from the basis.

$$0 \longrightarrow (T^*_M)_x \longrightarrow (T_{T^*_M})_{(x,\xi)} \longrightarrow (T_M)_x \longrightarrow 0$$

This gives a natural hyperbolic quadratic form on $(T_{T^*_M})_{(x,\xi)}$. If we choose a Riemannian metric on M, then we obtain a splitting of the above sequence from the Levi-Civita connection and also an isomorphism of T^* with T. Hence $T_{T^*M} \cong (p^*T_M)^{\oplus 2}$ which can be identified with the complexification of p^*T_M.

The third way is the operator $d+\delta$ on $\Omega^*(M)$ regarded as an operator commuting modulo lower order operators with multiplication by $\mathrm{Cliff}(T_M)$. By

Kasparov theory one gets an element of $KK(\mathrm{Cliff}(T_M), \mathbb{C})$ which by the Thom isomorphism can be identified with a K-homology class of T^*_M.

Summary: We have three descriptions of the fundamental class in K-theory of a smooth manifold. We want to understand the relation. The Dirac operator on T^*_M determines an element of

$$KK_0(C_c(T^*_M), \mathbb{C})$$

where $C_c(T^*_M)$ denotes functions vanishing at ∞, i.e. functions on the Thom space $T^*_M \cup \{\infty\} = \mathbb{D}T^*/\mathbb{S}T^*$. Thus we have a K-homology class of $\mathbb{D}T^*/\mathbb{S}T^*$. Let's now use the exact sequence

$$\begin{array}{ccccccc} K_0(\mathbb{D}T^*_M) & \longrightarrow & K_0(\mathbb{D}T^*_M, \mathbb{S}T^*_M) & \xrightarrow{\delta} & K_{-1}(\mathbb{S}T^*_M) & \longrightarrow & K_{-1}(\mathbb{D}T^*_M) \\ \| & & & & & & \| \\ K_0(M) & & & & & & K_{-1}(M) \end{array}$$

Presumably δ carries the fundamental class in $\tilde{K}_0(\mathbb{D}T^*_M/\mathbb{S}T^*_M)$ into the K_{-1} class corresponding under the BDF theory to the ψDO extension of $C(\mathbb{S}T^*)$.

Thus we conclude that the Dirac operator on T^*_M is more basic than the ψDO extension. From another viewpt, think of this fundamental class as giving an index map $K^0(T^*_M) \longrightarrow \mathbb{Z}$. We have dual to the above sequence an exact sequence

$$\begin{array}{ccccccc} K_{-1}(\mathbb{D}T^*) & \longrightarrow & K_{-1}(\mathbb{S}T^*) & \xrightarrow{\delta} & \tilde{K}_0(\mathbb{D}T^*/\mathbb{S}T^*) & \longrightarrow & K_0(\mathbb{D}T^*) \\ & & & & & & \| \\ & & & & & & K_0(M) \end{array}$$

which shows that the symbols coming from bundle autos over ST^* (these are the things one pairs with the ψDO extension) are symbols of elliptic operators on the same bundle (i.e. $D: E \to F \Rightarrow [E]-[F]=0$ in $K^0(M)$)

Review: I have three descriptions of the fundamental class in K-theory of a smooth manifold: the ψDO extension, the Dirac operator on T^*, the operator $d+\delta$ on the Cliff(T)-module ΛT^*. The first seems to be less basic hence I reject it in favor of the Dirac operator on T^*.

Now the Dirac operator on T^* can be twisted by an element of $K^0(T^*_M)$, say, represented by a graded bundle with odd endomorphism. However the simplest elements of $K^0(T^*_M)$ are given by super modules over Cliff(T). In fact the Thom isomorphism theorem says the K-cohomology of Cliff(T) and T^*_M are isomorphic. To establish this isomorphism I need some sort of operator on the fibres of T^* over M with a module structure over Cliff(T).

Sometime you have to go over the operator version of Bott periodicity and the Thom isomorphism.

In any case I know that Dirac operators on M are described by Cliff(T)-modules with superconnection. To such a thing, I can associate a bundle with superconnection over T^* and then get a Dirac operator over T^*. Now it should be clear that the two Dirac operators have the same index. The reason roughly is that the smaller one has been tensored with the standard $\frac{d}{dp} + p$ Koszul complex associated to the linear structure on each fibre of T, and this Koszul complex has index 1.

Now if this is all so, then it's hard to see the advantage of using the Dirac on T^* to get at the index thm. for a Dirac operator on M.

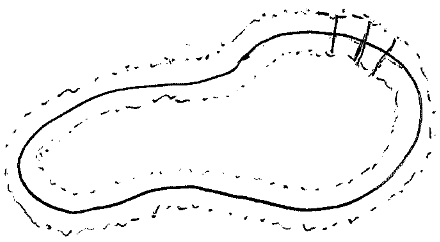
My idea was ~~to~~ explore the possibility that a Dirac on T^* is easier to handle [redacted] than [redacted] one on M. The reason was that Toeplitz operators + wave packet transforms might help. However the essential difficulty in the problem is [redacted] the geometry of M which enters thru the $\hat{A}$ genus, or Todd genus on T^*. It is not immediately clear that the Todd genus on T^* is simpler than the $\hat{A}$ genus on M. One can, of course, work with a simple model of the spinors on T^*, given essentially [redacted] by ΛT^* pulled up to T^* from M.

So [redacted] what [redacted] I conclude is although some computations might be simpler up on T^*, the end effect is the same, namely you are computing the index density of a Dirac operator on a curved manifold. The critical question is how the curvature of M enters.

Now then we come back to Getzler's

<u>February 26, 1984</u>

Program: To derive the index theorem for Dirac operators by using the isometric embedding theorem. Thus think of M as embedded in Euclidean space $\mathbb{R}^d$ with induced metric. Let N be the normal bundle and let U be the tubular nbd of M corresponding to the unit disk bundle in N. Here we rescale the [redacted] size of M so that distance ≤ 1 from M gives the tubular nbd. U.



So let's start with a Dirac operator $\not{D}$ on E over M. Thus E is a $C(T)$-module. We propose to get at the index of $\not{D}$ by the Atiyah-Singer method of tensoring with an operator in the normal direction with index 1. This will give an operator on U, or $\mathbb{R}^d$, with the same index. Then we have to know what to do over $\mathbb{R}^d$.

Now consider what to do [redacted] within the family of Dirac operators. We have started with a $C(T_M)$-module E. To get the operator in the normal direction we will probably, following Hörmander, use the Gaussian DR complex in the normal direction.

This is confusing. Instead think of the case where M is Spin^c and we have the Dirac operator on M so that $E =$ the spinor bundle S on M. Then on each normal space we want the Gaussian DR complex, which is the exterior algebra complex belonging to the operators $\frac{d}{dy^\mu} + y^\mu$ where the y^μ are orthonormal coordinates. I have seen that this is a Dirac operator with coefficients in the bundle + odd autom. given by the spinors ~~with~~ odd endo. $\sum_\mu \gamma^\mu y^\mu$.

So the operator in the normal direction we are using is on $S_N \otimes S_N = \Lambda N$ and is a Dirac operator with coefficients in spinors with Clifford multiplication.

Notice that in order to construct this we use N as a bundle with metric but not the fact that this is the normal bundle of an embedding in Euclidean space. One of the points to be understood is the difference in metric between N and the tubular nbd. U. So far we have constructed a Dirac operator on N for its metric, and the index density for this can not be evaluated immediately using the formulas on Euclidean space. Presumably N and U agree so well along M that the argument can be carried out.

This last point is what Getzler must prove in his latest proof, where he compares two Laplaceans at the same point, and argues they have to have the same index density.

It's still not clear that we get a proof using an embedding.

Let's go over the ~~situation~~. We start with a Riemannian Spin^c manifold M and then consider all Dirac operators on M. These are given by graded vector bundles with superconnection. Each Dirac operator determines an index density on M which we can think of as an n-form as M is oriented, [redacted] where $n = \dim M$. The theorem to be proved is that [redacted] for $\not{D} = \gamma \cdot D + \varepsilon L$ on $S \otimes E$ the index density is

$$\left[ch(D, L)\, \hat{A}(M) \right]_{(n)}$$

Now we consider our embedding $M \subset \mathbb{R}^d$ and let N be the normal bundle and U a tubular nbd so that we have the exponential map

[redacted]

$$\begin{tikzcd} N \arrow[r] \arrow[d] & \mathbb{R}^d \\ & U \arrow[u, phantom, "\cup"] \\ DN \arrow[r, "\sim"] & U \end{tikzcd}$$

Now N and U have different Riemannian structures, but these structures should coincide extremely well along M.

Because M is $Spin^c$ and Euclidean space is also it follows that N has a $Spin^c$ structure. Hence there is a K-Thom class on N which is represented by the irreducible $Cliff(N)$-module S_N given by the $Spin^c$ structure. The bundle S_N over M is pulled back to N and equipped with the evident degree one map provided by $Cliff(N)$-multiplication.

I am going to need the character of this Thom class. It is a specific [redacted] form on N which decays in a Gaussian fashion as we tend to ∞. By scaling on N we can make this form peak very sharply along M. Now I can integrate this form over the fibre and I get a form on M which should [redacted] represent the class $\hat{A}(M)$.

Now it's essential to my method that I [redacted] understand the behavior of my differential forms relative to the Thom isomorphism. There is also a technical problem in that I would expect the form [redacted] which is the inverse $\hat{A}$ genus of the bundle N to occur rather than $\hat{A}$ applied to T_M. Now $T_M \oplus N$ is trivial, but why should $\hat{A}(T_M)\cdot\hat{A}(N) = 1$?

This follows from the fact that for the Grassmannian connections one must have $\varphi(\mathrm{Sub})$ [redacted] $\varphi(\mathrm{Quot}) = 1$ for any exponential class, since the cohomology of the Grassmannian is exactly represented by invariant forms. (see p. 581, Jan. 17, 1983.)

[redacted] Let's try to understand the normal bundle embedding and how the $\hat{A}$ class arises.

So we are given a real, (even-dim) vector bundle E with connection and metric. We suppose E given a Spin^c structure, i.e. an irreducible $\mathrm{Cliff}(E)$ module S. For example when E is equipped with a complex structure we can take S to be $\Lambda_{\mathbb{C}} E$.

(Digression: As $\det : U(m) \longrightarrow S^1$ is an isom. on π_1, it follows $U(m)$ has a unique double covering $\widetilde{U(m)}$ obtained by extracting $\sqrt{\det}$. This is the pull-back of the double covering [redacted] $\mathrm{Spin}(2m)$ of $SO(2m)$ under the inclusion $U(m) \subset SO(2m)$. The spinor representation of

$\mathcal{U}(m)$ is $(\Lambda^{max} V)^{-1/2} \otimes (\Lambda V)$.

$Spin^c(2m)$ arises as follows. An irreducible $C(E)$-module, where E is an oriented $2m$-diml bundle can be multiplied by a line bundle. Hence such an irreducible $C(E)$-module is a principal bundle for the group

$$Spin^c(2m) = Spin(2m) \times_{\mathbb{Z}_2} S^1$$

Thus one has an exact sequence

$$1 \longrightarrow S^1 \longrightarrow Spin^c(2m) \longrightarrow SO(2m) \longrightarrow 1$$

so a fibration

$$\underset{K(\mathbb{Z},2)}{\overset{\|}{BS^1}} \longrightarrow BSpin^c(2m) \longrightarrow BSO(2m) \longrightarrow K(\mathbb{Z},3)$$

where the last map is described by the generator of

$$\Big(H^3(BSO(2m),\mathbb{Z}) = \mathrm{Ext}^1(H_2(BSO(2m)),\mathbb{Z}) = \mathrm{Ext}^1(\mathbb{Z}_2,\mathbb{Z}) = \mathbb{Z}_2 . \Big)$$

So lets return to our $Spin^c(2m)$-bundle belonging to the real bundle E, then we construct the Thom class for K of E by lifting the irreducible $C(E)$ bundle S up to E and then using Clifford multiplication. Thus we have at a point $\xi \in E_x$ an endo of S_x which is the fibre of $\pi^*(S)$ at ξ, where $\pi : E \to M$ is the projection.

When E is complex and $S = \Lambda E$, then the Clifford multiplication by $\xi \in E_x$ is $i_\xi + e_\xi$ on ΛE_x.

Now let E be equipped with a connection. It obviously doesnt gives a connection in S because the line bundle ambiguity can vary. At this point we have to explain what to do to get a connection on S

February 27, 1984

Thom isomorphism in K-theory for complex vector bundles E. Suppose E equipped with hermitian inner product, and form the Clifford algebra bundle C for the underlying real bundle. Then ΛE and ΛE^* are both irreducible Clifford modules. Hence one is a line bundle tensored with the other. In fact

$$(\ast) \qquad \Lambda E \otimes \lambda(E^*) \xrightarrow{\sim} \Lambda E^*$$

$$\xi_1 \cdots \xi_p \otimes \omega \longmapsto \iota_{\xi_1} \cdots \iota_{\xi_p} \omega$$

Then
$$e_\xi(\xi_1 \cdots \xi_p \otimes \omega) \longmapsto \iota_\xi(\iota_{\xi_1} \cdots \iota_{\xi_p} \omega)$$

$$i_{\xi^*}(\xi_1 \cdots \xi_p \otimes \omega) = \sum (-1)^{j-1} \langle \xi | \xi_j \rangle \, \xi_1 \cdots \hat{\xi}_j \cdots \xi_p \otimes \omega$$

$$\longmapsto \sum (-1)^{j-1} \langle \xi | \xi_j \rangle \, \iota_{\xi_1} \cdots \widehat{\iota_{\xi_j}} \cdots i_{\xi_p} \omega$$

$$e_{\xi^*}(\iota_{\xi_1} \cdots \iota_{\xi_p} \omega) = \langle \xi | \xi_1 \rangle \iota_{\xi_2} \cdots \iota_{\xi_p} \omega - \iota_{\xi_1} e_{\xi^*} \iota_{\xi_2} \cdots i_{\xi_p} \omega$$

etc.

so we see the isomorphism $(\ast)$ transforms e_ξ on the left (resp. ι_{ξ^*}) into i_ξ (resp e_{ξ^*}) on the right, and so is compatible with $e_\xi + i_{\xi^*}$ on the left and $\iota_\xi + e_{\xi^*}$ on the right.

But observe that $(\ast)$ does not preserve the $\mathbb{Z}_2$ grading when the complex dimension of E is odd. This means that [redacted] the two K-classes in $K(E)$ associated to these Clifford modules will satisfy
$$(-1)^{\dim E} [\lambda(E)^*] \cdot U_{(\Lambda E)} = U_{(\Lambda E^*)}$$

From algebraic geometry the natural thing to use is the class of ΛE^*. This is because if $\pi : E \to M$

the projection, then the complex of vector bundles

$$\cdots \xrightarrow{d} \pi^*\Lambda^1 E^* \xrightarrow{d} \pi^*\Lambda^0 E^*,$$

with d_ξ at a point $\xi \in E$ given by ι_ξ on $(\pi^*\Lambda E^*)_\xi = \Lambda E^*_x$, is a resolution of the sheaf of functions on M extended by 0 to be a sheaf on E.

Thus we have described the Thom class in $K(E)$. The Thom class is given by an irreducible $C(E)$-module and a choice of that module is the same as the choice of Spin^c structure on E.

The next step is to compute the Chern character of this Thom class.

Repeat: Let E be a complex vector bundle over M with inner product, and $C = \mathrm{Cliff}(E)$ the Clifford algebra bundle of the underlying real orthogonal bundle. A **Spin^c [redacted] -structure** on E is [redacted] the same as an irreducible C-module, i.e. a vector bundle over M with C-action such that each fibre is an irreducible module over [redacted] C at that point. For example ΛE or ΛE^*.

To make the relation with the Kasparov theory we want **right** graded Clifford modules. In effect the Thom isomorphism theorem for a general real vector bundle E says that the K-theory of $\mathrm{Cliff}(E)$ graded modules (with Kasparov F) is equivalent to the K-theory of the bundle E. Hence there are basic Kasparov-type elements in

$$KK(\mathrm{Cliff}\,E, C_c(E)) \qquad KK(C_c(E), \mathrm{Cliff}\,E)$$

setting up the isomorphism. Here $C_c(E)$ is the continuous functions on E vanishing at ∞.

Now in the Kasparov theory $KK(A,B)$ is defined using graded $\mathcal{H}$ a bimodules, left A-modules, right B-modules together with the F which allows one to handle the infinite dimensions. Thus $? \otimes_A \mathcal{H}$ gives the functor from right A-modules to right B-modules.

When E is equipped with a Spin^c structure this means we have an equivalence between K-theory of M and that of $\text{Cliff}(E)$. In this, the Kasparov stuff becomes the usual Morita equivalence so the spinor bundle should be

$$ {}_{C(M)}S_{\text{Cliff}} $$

and its dual bundle is then a left Cliff-module and we have

$$ S \otimes_{\text{Cliff}} S^* = \mathbb{1} \qquad\qquad S^* \otimes S = \text{Cliff}. $$

So it appears that I have to get used to the idea of $\text{Cliff}(E)$ right multiplying on S. Actually there is really no reason why I can't get used to the idea of interior mult. on the right i.e.

$$ (\xi_1 \cdots \xi_k) \lrcorner\, \xi^* = \xi_1 \cdots \hat{\xi}_k\, \xi^*(\xi_k) - \xi_1 \cdots \hat{\xi}_{k-1} \xi_k\, \xi^* \xi_{k-1} + \cdots $$
$$ = (-1)^{k-1} \iota_{\xi^*}(\xi_1 \cdots \xi_k) $$

so it is the familiar way of converting left to right for supercommutative rings.

Let's now return to the problem of computing the Chern character of the Thom class.

Consider a Dirac operator $D\!\!\!/$ on an even-dimensional manifold M, $\dim M = n = 2m$. Then it determines a metric on T and a Cliff(T)-module structure on the underlying bundle E. I have been thinking in terms of $D\!\!\!/$ as being of degree one relative to a given grading on E, so that in fact E is a graded Cliff(T)-module. But I recall reading that a grading is defined by Clifford multiplication with the orientation element.

Let's explore this in more detail. Let T be a real vector space of dim $2m$ with Euclidean structure, thus $T \simeq \mathbb{R}^n$, $n = 2m$. We know that Clifford mult. gives an isomorphism

$$C \xrightarrow{\sim} \Lambda T$$

which is natural, so that in fact there is a canonical line in C isomorphic to $\Lambda^n T$ which is complementary to $F_{n-1}C$. (This holds quite generally over any field with any quadratic form not necessarily non-degenerate.) In practice the way you find this line is by anti-symmetrization, at least in char. 0.

If $v_1, \dots, v_n$ are mutually orthogonal elements and a basis for T, then the Clifford product satisfies

$$v_{\sigma 1} v_{\sigma 2} \cdots v_{\sigma n} = \operatorname{sgn}(\sigma)\, v_1 v_2 \cdots v_n$$

so that $v_1 \cdots v_n \in C$ is the unique element of the canonical line mapping to $\Lambda^n T$ which goes to $v_1 \cdots v_n \in \Lambda^n T$. Thus we see that an orientation of T determines a canonical element of C, namely $v_1 \cdots v_n$ where $v_1, \ldots, v_n$ is an orthonormal basis with the correct orientation.

So now let's calculate the effect on a Clifford module. Take $T = \mathbb{R}^n$, $n = 2m$, with the usual orientation and let $\gamma^1, \ldots, \gamma^n \in C_n$ be the standard generators. Then

$$(\gamma^1 \cdots \gamma^n)^2 = (-1)^{\frac{n(n-1)}{2}} = (-1)^m$$

and for the module of spinors we normalize things so that

$$\gamma^1 \cdots \gamma^n = i^m \varepsilon .$$

For example for C_2 we want $\gamma^1, \gamma^2, \varepsilon$ to be the three Pauli matrices, hence $\gamma^1 \gamma^2 \varepsilon = i$ or $\gamma^1 \gamma^2 = i\varepsilon$. (Notice: For $n=4$, $\gamma^1 \cdots \gamma^4 = -\varepsilon$, which differs from physics notation).

We conclude that when we take a graded Clifford module of the form $S \otimes E$, then unless E happens to be entirely of degree 0, the grading $\varepsilon = \varepsilon_S \cdot \varepsilon_E$ will be different for $i^{-m} \gamma^1 \cdots \gamma^n = \varepsilon_S$. What this means is that we must take a Dirac operator to be odd degree operator on a $\mathbb{Z}_2$-graded bundle. The index should then be defined as a density.

For example the signature operator is the operator $d + \delta$ on forms but with a grading different from the usual degree mod 2 of a form.

Let's calculate the orientation element $\gamma^1 \cdots \gamma^n$

acting on ΛT by left Clifford multiplication. Now we have the isomorphism

$$\bigstar \qquad C(T) \xrightarrow{\sim} \Lambda T \qquad\qquad \alpha \longmapsto \alpha * 1$$

of Clifford modules. One of the things that follows from the above stuff about the canonical line at the top of a Clifford algebra is that to any subspace of T belongs a canonical line in the Clifford algebra and it compatible with the above isomorphism. To be more specific if $v_1, \dots, v_k$ are orthogonal in T, then by induction

$$(i_{v_1} + e_{v_1}) \cdots (i_{v_k} + e_{v_k}) 1 = v_1 \cdots v_k .$$

In effect multiplying by $i_{v_0} + e_{v_0}$, where v_0 is orth. to $v_1, \dots, v_k$ gives

$$(i_{v_0} + e_{v_0})(v_1 \cdots\cdots v_k) = \underbrace{i_{v_0}(v_1 \cdots v_k)}_{0} + v_0 v_1 \cdots v_k$$

~~So therefore we can identify oriented subspaces~~

So to compute the effect of $\gamma^1 \cdots \gamma^n$ on ΛT we use the above isomorphism $\bigstar$, and we use the fact that the basis element $e_I = e_{i_1} \cdots e_{i_p}$ of $\Lambda^p T$ lifts to the product $\gamma^{i_1} \cdots \gamma^{i_p}$. Let's choose $j_1 \cdots j_q$ in the appropriate order such that

$$e_{j_1} \cdots e_{j_q} e_{i_1} \cdots e_{i_p} = e_1 \cdots e_n$$

$$\text{or} \quad \gamma^{j_1} \cdots \gamma^{j_q} \gamma^{i_1} \cdots \gamma^{i_p} = \gamma^1 \cdots \gamma^n$$

and then clearly

$$(\gamma^1 \cdots \gamma^n) \cdot (\gamma^{i_1} \cdots \gamma^{i_p}) = \gamma^{j_1} \cdots \gamma^{j_q} (-1)^{\frac{p(p-1)}{2}}$$

Thus given a subspace W with orientation let $\gamma_W = w_1 \cdots w_p$ where $w_1, \ldots, w_p$ is an orthonormal basis with the given orientation. Then we can write

$$\gamma^1 \cdots \gamma^n = \gamma_{W^\perp} \gamma_W$$

where $W^\perp$ is the orthogonal space with that orientation such that the natural orientation on $W^\perp \oplus W$ agrees with the given one on T. Then

$$(\gamma^1 \cdots \gamma^n)\gamma_W = \gamma_{W^\perp}\gamma_W^2 = (-1)^{\frac{p(p-1)}{2}} \gamma_{W^\perp}.$$

Check

$$\begin{aligned}(\gamma^1 \cdots \gamma^n)^2 \gamma_W &= (-1)^{\frac{p(p-1)}{2}} (\gamma^1 \cdots \gamma^n)\gamma_{W^\perp} \\ &= (-1)^{\frac{p(p-1)}{2}} (-1)^{\frac{q(q-1)}{2}} (-1)^{pq} \gamma_W\end{aligned}$$

where $q = n-p$. (Here $\gamma_{(W^\perp)^\perp} = (-1)^{pq}\gamma_W$). As a check note that

$$\frac{p(p-1)}{2} + \frac{q(q-1)}{2} + pq = \frac{(p+q)^2 - (p+q)}{2} = \frac{n(n-1)}{2}.$$

So therefore we see that Clifford multiplication by the orientation element $\gamma^1 \cdots \gamma^n$ is closed related to the Hodge duality operator. Sometime later I should learn about the signature Dirac operator — it would seem from the above that the signature operator is the Dirac operator will coefficients in the spinors where one forgets the $\mathbb{Z}_2$-grading on the spinors. Check the L-genus

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Let's begin with a general discussion about the index theorem for Dirac operators. Like the RR theorem, we want to generalize this theorem to an assertion about maps, so that the index thm. results by taking the map to a point.

To keep things simple I suppose my manifolds are even dimensional and are equipped with Riemannian metrics and Spin structures. Then a Dirac operator can be identified with a super vector bundle with superconnection. The classes of these gadgets is the K-theory of the manifold $K(M)$. On the other hand we have the DR cohomology $H^*(M)$ and we have a basic isomorphism given by the Chern character

$$K(M)\otimes\mathbb{C} \xrightarrow{\sim} H^{ev}(M)$$

Then generalization of the index theorem to maps $f: M \to M'$ would consist in defining Gysin maps $f_!: K(M) \longrightarrow K(M')$, $f_*: H(M) \longrightarrow H(M')$ and proving the Grothendieck type formula

$$ch(f_! x) = f_*(ch(x)\,Td(f))$$

Of course $f_!$ should be defined analytically so that the map to a point gives the index.

Now if it is possible to carry out this program, then the proof of the theorem reduces to the special cases where f is a submersion and f is an embedding. So we can look at these cases first to see what happens.

I should emphasize that I am ultimately interested [redacted] not in the classes in K and H^*, but rather with actual operators on one hand and differential forms on the other.

When we have a submersion $f: M \longrightarrow M'$, hence a fibre bundle, assuming these manifolds are compact, then I see more or less how to do things [redacted] on the cochain level. A Dirac operator on M can be viewed as a Dirac operator on M' with coefficients in a Dirac operator along the fibres. (This would be true if the metric on M in the normal direction to the fibres is the same as the induced metric from M'. [redacted] Let's ignore this difficulty, hoping that it can be absorbed by the normal rescaling process.) Thus we have an obvious $f_!$ on the level of Dirac operators, up to the difficulty that the super-bundle of coefficients becomes infinite dimensional.

Next for a submersion, since our manifolds are oriented, we also have an explicit f_* on the form level

$$f_* : \Omega(M) \longrightarrow \Omega(M')$$

given by integration over the fibre. [redacted] So we are in a position to expect the index formula to hold on the level of differential forms for a submersion.

It occurs to me that perhaps when we really do carefully treat the fibre bundle situation, that [redacted] various "gravity" type differential forms will become important.

Next let's consider an embedding $M \overset{i}{\hookrightarrow} M'$. Here it is clear that we are going to have some sort of problem on the form level. Forms on M become currents on M' which then have to be smoothed normally so as to give forms on M'. So we have no i_* on forms.

To keep things simple let's suppose that we have the embedding of the zero section in a vector bundle N over M. N is equipped with metric and connection and hence is a Riemannian manifold. We take the Dirac operator on M, say it is <u>the</u> Dirac operator belonging to the spin structure, and then tensor it with the Gaussian DR complex along the fibres to get a Dirac operator on N. It is the Dirac operator associated to the superbundles on N given by the spinor bundle for $Cliff(N)$.

I think what I get in this way is an explicit way to go from bundles with <u>super</u> connection on M to bundles with super connection on N such that the final Dirac operator is the initial Dirac op. tensored with the Gaussian DR complex along the fibres. This is now an explicit $i_!$ which is nothing other than the Thom isomorphism in K-theory.

However if there is an interpretation of

$$ch\, i_!(x) = i_*(Td(i)\cdot ch\, x)$$

on the form level it should be possible to define i_* on forms (at least $i_* \cdot Td(i)$). In any case it would be nice to have a Gaussian representative for the Thom class $i_* 1$.

Repeat: The question is whether, corresponding to the $i_!$ we have on bundles, there is Gysin homomorphism i_* on forms. We know that

$$ch(i_! x) = ch(i_! 1 \cdot \pi^* x)$$
$$= ch(i_! 1) \cdot \pi^*(ch\, x)$$

so if we can solve the equation

$$ch(i_! 1) = \underbrace{i_* 1}_{\substack{\text{a form of}\\ \text{the fixed}\\ \text{degree } n\\ = \dim (N)}} \cdot \underbrace{\pi^*(Td\, i)}_{\substack{\text{a form coming}\\ \text{from the base.}}}$$

Then we can define

$$i_*(x) = i_* 1 \cdot \pi^* x$$

whence

$$\begin{aligned} ch(i_! x) &= ch(i_! 1 \cdot \pi^* x) \\ &= ch(i_! 1) \cdot \pi^*(ch\, x) \qquad \text{uses mult. of } ch \text{ for } \otimes \text{ of superconnections.} \\ &= i_* 1 \cdot \pi^*(Td(i))\, \pi^*(ch\, x) \\ &= i_* 1 \cdot \pi^*(Td(i) \cdot ch\, x) \\ &= i_*(Td(i) \cdot ch\, x) \end{aligned}$$

Notice that as $Td\, i$ is supposed to have leading term 1, $i_* 1$ has to be the leading term of $ch(i_! 1)$. Hence it is really a question of whether the higher components are $i_* 1$, multiplied by a form from the base, the latter being unique as it can be obtained by integrating over the fibre.

The first step must be to calculate $ch(i_! 1)$.

What is $i_! 1$? Recall that N is Spin^c bundle, say, of even dim over M. So there is a principal bundle behind it, P, for Spin$^c(n)$, $n = 2m$. We then get a super bundle S of spinors over M associated to the spin repn. Δ of Spin$^c(n)$. S is a Cliff(N)-module. $i_! 1$ is $\pi^*(S)$ equipped with Cliff mult.

Now we need a connection in P which will then give us connections in N, Cliff(N), S. We pull back the connection D on S to $\pi^* S$. Then we have on the superbundle $\pi^* S$ over N, a connection D and an odd degree endomorphism L, hence a super connection. So we can compute the Chern character $tr_s(e^{(D+L)^2})$ which gives the form we want for $ch(i_! 1)$. This is a differential form on N peaking in [redacted] Gaussian fashion along the zero-section.

As usual we do the calculation locally, i.e. up in the principal bundle. This means that we lift up to P, whence

$$P \times_M N = P \times \mathbb{R}^n$$

$$P \times_M S = P \times \Delta .$$

[redacted] This part namely the bundle $\pi^* S / N$ with L is obtained by descending $\mathbb{R}^n \times \Delta / \mathbb{R}^n$ with $i \gamma^\mu x^\mu$.

But we have also the connection on the bundle $\pi^* S$ over N, which has been lifted to $P \times \mathbb{R}^n \times \Delta$ over $P \times \mathbb{R}^n$. This is the pull-back of the connection in $P \times \Delta / P$, which is $D = d + \rho(\theta)$. Here [redacted] $\theta \in \Omega^1(P) \otimes \mathfrak{g}$, $\mathfrak{g} = \mathrm{Lie}(\mathrm{Spin}^c$[redacted]$(n))$ is the connection form in P, and $\rho : G \longrightarrow \mathrm{Aut}(\Delta)$ is the spin representation.

In order to carry out the calculation I should perhaps be thinking in terms of equivariant forms. We have $G = \mathrm{Spin}^c(n)$ acting on $\mathbb{R}^n$ and we have the equivariant bundle $\mathbb{R}^n \times \Delta / \mathbb{R}^n$ with the invariant connection d. Also the odd equivariant map L on $\mathbb{R}^n \times \Delta$ over $\mathbb{R}^n$. We therefore get an equivariant super bundle with invariant superconnection. This gets modified by the connection form θ: $DL = d + \rho(\theta) + L$, so that it will descend.