January 10, 1984

It seems that I have a new proof that the Chern characteristic class is independent of the choice of connection.

Let \( E^+ = E^- \) have two connections \( D^\pm \).

Let

\[
D = \begin{pmatrix} 0^+ & 0^- \\ 0^+ & 0^- \end{pmatrix} = d^+ \begin{pmatrix} A^+ & 0 \\ 0 & A^- \end{pmatrix} \quad \text{and} \quad E^+ \ominus E^- = \mathfrak{m}(V \otimes V)
\]

where \( B = A^+ - A^- \)

\[
B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

\[
[d, L] = \begin{pmatrix} 0 & A^+ - A^- \\ A^- A^+ & 0 \end{pmatrix} = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

\[
B^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

We know that \( D = 0^2 + t[d, L] + t^2 L^2 \), then

\[
\frac{d}{dt} \text{tr}_S(e^{\Omega t}) = d \text{ tr}_S(e^{\Omega t} L)
\]

hence

\[
\text{tr}_S(e^{\Omega t}) = d \left\{ -\int_0^\infty dt \text{ tr}_S(e^{\Omega t} L) \right\}
\]

provided we integrate along a path along which \( \text{tr}_S(e^{\Omega t}) \) goes to zero. More precisely

\[
e^{\Omega t} = e^{0^2 + t[d, L] + t^2 L^2} \quad L^2 = I
\]

so we take the path in the complex plane to be the positive imaginary axis: \( 0 \) to \( i\infty \).

Let's recall that \( e^{\Omega t} \) is computed in the algebra \( \Omega(M) \otimes \text{End}(V) \), which acts as operators on \( \Omega(M) \otimes V \). The natural way to write such operators is as matrix forms. To incorporate the signs we should write \( \xi_0 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) for \( L \). Then
\[ \text{tr}_V(e^{\mathcal{Q} t}) = \text{tr}_V\left( \frac{\exp \left( t^2 + D^2 + tB \mathcal{E}_\Omega \right.}{\exp \left( (0,1) \mathcal{E}_\Omega (1,0) (0,-1) \right)} \right) \]

\[ = e^{t^2} \text{tr}_V \left( e^{F + tB \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \right) \]

Now we expand the exponential using the standard perturbation series

\[ e^{F + G} = \sum_{n \geq 0} \frac{1}{n!} \int dt_1 ... dt_n e^{(1-t_1)F} e^{(t_1-t_2)F} ... e^{t_nF} t^nB \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \]

Bring all the $\mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right)$ to the right and use that

\[ \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) = - \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \]

together with the fact that $\mathcal{E}_\Omega$ anti-commutes with $B$. Only an odd number of $B$'s will have a non-zero trace. Then

\[ \text{tr}_V \left( e^{F + tB \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \mathcal{E}_\Omega \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \right) \right) = \sum_{n \text{ odd}} \frac{1}{n!} \int dt_1 ... dt_n \text{tr} \left\{ e^{(1-t_1)F} e^{(t_1-t_2)F} e^{t_3F} ... e^{t_nF} \right\} \]

\[ \times e^{(t_1-t_2)(F^- F^+)} tB ... tB e^{t_n(F^- F^+)} \}

It seems that this is the same as

\[ \text{tr}_V \left( e^{F + tB \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \right) \right) = \text{tr}_V \left( e^{tB \left( F^- F^+ \right) \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \left( \begin{array}{c} 0,1 \\ 1,0 \end{array} \right) \right) \right) \]

Thus we reach the following formula for the difference of character forms computed
with respect to two connections:

\[
\text{tr}(e^{F^+}) - \text{tr}(e^{F^-}) = d \left\{ \int_0^\infty dt \, t^2 \, \text{tr} \left( e^{(F^+ t B)} (O \, I) \right) \right\}
\]

This should be contrasted with the old formula

\[
\text{tr}(e^{F^+}) - \text{tr}(e^{F^-}) = d \left\{ \int_0^1 dt \, \text{tr} \left( e^{(1-t)F^- + tF^+} (O \, I) \right) \right\}
\]

I think these formulas are different, i.e. the forms in braces are different in general. For example, to first order in \( B \) we have

\[
\text{tr} \left( e^{(F^+ t B)} (O \, I) \right) = \int_0^1 ds \, \text{tr} \left[ e^{\left( -s F^- \right)} t B e^{\left( -s F^+ \right)} \right]
\]

\[
= \int_0^1 ds \, \text{tr} \left[ e^{(1-s)F^+} t B e^{s F^-} + e^{(1-s)F^-} t B e^{s F^+} \right]
\]

\[
= \left[ \text{tr} \left( e^{(1-s)F^+} t B e^{s F^-} + e^{(1-s)F^-} t B e^{s F^+} \right) \right]
\]

Now

\[
- \int_0^\infty dt \, t e^{t^2} = -\frac{1}{2} \int_0^\infty du \, e^{-u} = \frac{1}{2} \left( -\int_0^\infty du \right) = \frac{1}{2}
\]

So, to first order in \( B \)

\[
\int_0^\infty dt \, t e^{t^2} \text{tr} \left[ e^{(F^+ t B)} (O \, I) \right] = \int_0^1 ds \, \text{tr} \left[ \frac{e^{s F^+} e^{(1-s)F^+} e^{s F^-}}{2} B \right]
\]

whereas the Chern-Simons form gives

\[
\int_0^1 dt \, \text{tr} \left( e^{(1-t)F^- + tF^+} B \right)
\]

to first order in \( B \). (Note that at a given pt. \( B \)
Example: Consider the case where \( F^+ = F^- = 0 \), for example when we have a flat connection on the trivial bundle. Then

\[
\text{tr} \ e^{(0 \ 1) \ 0 \ 1} = \sum_{\text{modd}} \frac{t^n}{\Omega_n!} \ 2 \text{tr} (B^n)
\]

\[
- \int_0^\infty \ e^{t^2 + 2n + 1} \ dt = - \int_0^\infty u^n \ e^{-u} \ du = \int_0^\infty u^n \ e^{-u} \ (-u)^n \ du = (-1)^n \ n!
\]

So

\[
- \int_0^\infty \ e^{t^2 + 2n + 1} \ dt \ \text{tr} \ e^{(0 \ 1) \ 0 \ 1} = \sum_{n=0}^{\infty} \frac{(-1)^n \ n!}{(2n+1)!} \ \text{tr} (B^{2n+1})
\]

The Chern-Simons form is

\[
\int_0^1 \ \text{tr} (e^{(t^2-t)B} B) = \sum_{n=0}^{\infty} \frac{1}{n!} \ \text{tr} (B^{2n+1}) \int_0^1 \ (t^2-t)^n \ dt \ (-1)^n \ \frac{n! \ n!}{(2n+1)!}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \ n!}{(2n+1)!} \ \text{tr} (B^{2n+1})
\]
Example: Consider our $\mathbb{R}^n$ in even the virtual bundle with compact support given by the spinors $S$. Thus we have the trivial bundle over $\mathbb{R}^n$ with fibre $S = S^+ \oplus S^-$ and the odd degree endomorphism

$L = \gamma^i x_i$ at $x = (x^i) \in \mathbb{R}^n$.

Take $D = d$. Then $D^2 = 0$, $L^2 = x^2$

$$[D, L] = \left[ \partial_{x^i}, \gamma^j x_j \right] = dx^i \gamma^j$$

so the character form is

$$
\left( e^{u(x^2) + t[D, L] + t^2 L^2} \right) = e^{ut^2 x^2} tr_s \left( e^{ut \gamma^i x_i} \right)
$$

$$
= e^{ut^2 x^2} \left( ut \right)^n + tr_s \left( \begin{array}{c}
\overbrace{dx^1 \gamma^1 dx^2 \gamma^2 \cdots dx^n \gamma^n}^{g_{\alpha}}
\end{array} \right)
$$

$$
= e^{ut^2 x^2} (ut)^n dx^1 \cdots dx^n tr_s \left( g^{\alpha} \cdots g^1 \right)
$$

Change $t \rightarrow \frac{it}{\sqrt{u}}$, the above becomes

$$
= e^{\frac{-t^2 x^2 n}{2}} \left( -u \right)^{n/2} \left( 2i \right)^{n/2} \frac{dx^1 \cdots dx^n \left( -1 \right)^{\frac{n(n-1)}{2}}}{\frac{i}{2\pi}}
$$

$$
= e^{-t^2 x^2} (ut)^n \frac{dx^1 \cdots dx^n \left( -1 \right)^{\frac{n(n-1)}{2}}}{\left( 2\pi i \right)^{n/2}}
$$

For $n=2$ we have

$L = \gamma^1 x^1 + \gamma^2 y = \begin{pmatrix} 0 & x-iy \\ x+iy & 0 \end{pmatrix}$
so we have the $K$-element corresponding to the complex \[ 1 \xrightarrow{z} 1. \]

This looks like it should have degree $-1$ on $O$ as an element of $K$. This should explain the factor \( (-1)^{\frac{h(n,0)}{2}} \).

One of the things I should go over carefully is the limit process where $t$ goes to infinity. The problem is that we can only show that a non-homogeneous expression goes to zero; so let me fix a number $u$ and consider the DR class of the form

\[ t^r e^{u(0^+ + t[0,1] + t^2[1])} \]

This is an even form, and I know its DR class is independent of $t$, so on letting $t \to 0$, I see it represents the character of $E^+ -$ the character of $E^-$. On the other hand if $L^2 > 0$, then by letting $t \to \infty$ in such a way that $ut^3 \to -\infty$, we can conclude that the form goes to zero.

This last step requires some estimates. Let's work in a normed algebra so that we have inequalities like

\[ \|A + B\| \leq \|A\| + \|B\| \]

\[ \|AB\| \leq \|A\| \cdot \|B\|. \]

Then use

\[ e^{A+B} = e^A + \int_0^t e^{(t-s)A} B e^s A \, ds + \int_0^t \int_0^{t_1} e^{(t-t_2)A} B e^{t_1} A \, dt_2 \, ds + \ldots \]
and assume known that
\[ \| e^{tA} \| \leq e^{ct} \quad \text{for } t > 0, \quad c \in \mathbb{R}. \]
and we get
\[
\begin{aligned}
\| e^{A+B} \| & \leq \| e^A \| + \int_0^t \| e^{(1-t)A} \| \| B \| \| e^{tA} \| + \ldots \\
& \leq e^c + \int_0^t e^c \| B \| + \int_0^t \int_0^t e^c \| B \|^2 + \ldots \\
& = e^c \left( 1 + \| B \| + \frac{\| B \|^2}{2} + \ldots \right) = e^c e^{\| B \|}.
\end{aligned}
\]

Now we apply this to the operator
\[ e^{(D^2 + t[L, L] + t^2L^2)} \]
where \( u t^2 \rightarrow -\infty \). Put \( A = -L^2 \). If \( L^2 > 0 \) is hermitian, then by using eigenvectors we can see that
\[ \| e^{-tL^2} \| \leq e^{-\lambda t} \quad \text{for } t > 0, \quad \lambda > 0. \]
So the above estimate will give us
\[
\begin{aligned}
\| e^{uD^2 + ut[L, L] + (ut^2)(-L^2)} \| & \leq e^{-\lambda(ut^2)} e^{\| uD^2 + ut[L, L] \|} \\
& \leq e^{-\lambda(ut^2)} + \| uD^2 \| + |t| \| u[L, L] \|
\end{aligned}
\]
and as \( t \to \infty \) the quadratic term swamps the rest.

More precise and rigorous version of \( \ast \). Use Duhamel formula
\[ e^{t(A+B)} = e^{tA} + \int_0^t ds e^{(t-s)A} B e^{s(A+B)} \]
whence we get the inequality
\[ \| e^{t(A+B)} \| \leq \| e^{tA} \| + \int_0^t \| e^{(t-s)A} \| \| B \| \| e^{s(A+B)} \| \] \[ \leq e^{ct} + \int_0^t e^{c(t-s)} \| B \| \| e^{s(A+B)} \|. \]

So if we put \( f(t) = e^{-ct} \| e^{t(A+B)} \| \) we have
\[ f(t) \leq 1 + \| B \| \int_0^t d\tau \ f(\tau) \]

Now iterate this inequality:
\[ f(t) \leq 1 + \| B \| \int_0^t d\tau_1 (1 + \| B \| \int_0^{\tau_1} d\tau_2 \ f(\tau_2)) \] \[ \leq 1 + \| B \| t + \| B \|^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ f(\tau_2) \] \[ \leq 1 + \| B \| t + \| B \|^2 \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \ f(\tau_3). \]

Thus as in Picard's theorem one finds that if \( |f(t)| \leq M \) on the interval, then in fact
\[ |f(t)| \leq e^{\| B \| t}. \]

So again one gets
\[ \| e^{t(A+B)} \| \leq e^{ct} \| B \| t \]
January 11, 1984

On p. 367 the signs aren't quite correct. Let $D$ be a connection on $E = E^0 + E^1$, a $\mathbb{Z}_2$-graded bundle. We extend $D$ to

$$\omega(M,E) = \omega(M) \otimes \Omega^0(M,E)$$

so that

$$D(\omega \otimes x) = dx \cdot x + (-1)^{\text{deg}(\omega)} \omega \cdot Dx.$$ 

However we can also regard $\omega(M,E)$ as a right $\Omega(M)$-module by defining

$$x \cdot \omega = (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} \omega \cdot x$$

Here $\text{deg}(x)$ is the total $\mathbb{Z}_2$ degree on $\Omega(M,E)$.

Then

$$D(x \cdot \omega) = (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} D(\omega \cdot x)$$

$$= (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} \left( d\omega \cdot x + (-1)^{\text{deg}(\omega)} \omega \cdot Dx \right)$$

$$= (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} \left[ (-1)^{\text{deg}(x) \cdot (\text{deg}(\omega) + 1)} \omega \cdot dx + (-1)^{\text{deg}(\omega)} \omega \cdot (dx + 1) \right]$$

$$= (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} x \cdot d\omega + Dx \cdot \omega$$

so that

$$D(x \cdot \omega) = Dx \cdot \omega + (-1)^{\text{deg}(x) \cdot \text{deg}(\omega)} x \cdot d\omega$$

Conclude: The difference of two connections is a right $\Omega(M)$-homomorphism of $\Omega(M,E)$ of degree 1. I think I can conclude that we have an algebra isomorphism

$$\Omega(M) \otimes \Omega^0(M,E) \otimes \Omega^0(M,End E) \rightarrow \text{End}_{\Omega(M)^{op}}(\Omega(M,E))$$

Check that the map is well-defined. First given
Given $\omega \in \Omega(M)$ I want to check that left multiplication by $\omega$ is a right $\Omega(M)$-module map:

$$\omega (\cdot \eta) = (-1)^{\deg \omega \deg \eta} \omega \eta \cdot \eta$$

$$= (-1)^{\deg \omega \deg \eta + (-1) \deg \omega \deg \eta} \omega \cdot \eta$$

$$= (\omega \cdot \eta) \eta$$

Finally I want to compare the super-trace on $\text{End}_{\Omega(M)}(\Omega(M,E))$ with the way I have defined it on $\Omega(M) \otimes \Omega(M,EndE)$. To take an endo of the form $e \otimes \omega \lambda$, $e \in E$, $\omega \in \Omega$, $\lambda \in E^*$.

Its super-trace as an endo is $(-1)^{\deg \omega \deg \lambda} \omega \lambda(e)$. It comes from $(-1)^{\deg \omega \deg \lambda} \omega \lambda \otimes e \cdot \lambda \in \Omega(M) \otimes \Omega(M,EndE)$, and I calculate the super-trace here as

$$(-1)^{\deg \omega \deg \lambda} \omega \cdot (\lambda(e))$$

Thus things check.
If $A$ is an element of a Banach algebra, then
\[ e^{tA} = \frac{1}{2\pi i} \oint_C e^{\lambda} (\lambda - A)^{-1} d\lambda \]
where the contour $C$ encloses $\text{Sp}(A)$. So if $\text{Re}(\lambda) < a$ for all $\lambda \in \text{Sp}(A)$ we can choose $C$ so that
\[ |e^{t\lambda}| = e^{t\text{Re}(\lambda)} \leq e^{ta} \quad t \geq 0 \]
Then
\[ \|e^{tA}\| \leq \frac{1}{2\pi} \int_C e^{ta} \|\lambda^{-1}\| |d\lambda| \leq \text{Const} e^{ta}, \quad t \geq 0. \]

Examples such as $A = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ with $x$ large and a small $> 0$, show that the constant cannot be replaced by one.

Suppose now that we have
\[ \|e^{tA}\| \leq Me^{ta} \quad t \geq 0. \]
Then consider the Dyson expansion
\[
e^{t(A+B)} = e^{tA} + \int_0^t \frac{e^{(t-s)A}B e^{sA}}{s!} ds + \int_0^t \int_0^{t_1} \frac{e^{(t-s_1)A}B e^{s_1A}}{s_1!} ds_1 ds + \ldots
\]
Then
\[ \|e^{t(A+B)}\| \leq Me^{ta} + \ldots + \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} \frac{e^{(t-s_{n-1})A}B \ldots B e^{s_{n-1}A}}{s_{n-1}!} ds_{n-1} ds_1 \ldots ds_0 + \ldots \leq Me^{ta} \left(1 + t \|B\| + \frac{t^2}{2!} \|B\|^2 + \ldots \right) = Me^{ta + t\|B\| + \frac{t^2}{2!} \|B\|^2 + \ldots} \]
Next, I want to consider the case where $A$ is an endomorphism of a vector bundle, hence is a section of the algebra $\Omega(M, \text{End} E)$. This space has a $C^k$ topology and can be completed to be a Banach space. I want to see that it actually has a $C^k$ norm which makes it into a Banach algebra. For then, if I know that the spectrum of $A$ at each point is contained in $\text{Re}(A) < 0$, I can conclude that
\[
\| e^{tA} \| \leq \text{Const} \ e^{Ct} \quad t > 0.
\]

It is clear that I can define some sort of $\kappa$-norm which will satisfy
\[
\| x y \| \leq C \| x \| \| y \| \quad x, y \in \mathfrak{A} = C(M, \text{End} E)
\]
and the point will be to modify the norm so as to obtain a normed algebra. Obvious choice is the norm of left multiplication
\[
\| x \| = \sup \left\{ \frac{| x y |}{| y |} : y \neq 0 \right\}
\]
which is just the pull-back of the norm on the algebra $L(\mathfrak{A}, \mathfrak{A})$ of bounded operators on $\mathfrak{A}$. Hence it will satisfy
\[
\| x y \| \leq \| x \| \| y \|.
\]
On the other hand from (x) above we have
\[
\| x \| < C \| x \|
\]
and using the $1 \in \mathfrak{A}$, we have
\[
| x | = | x 1 | \leq \| x \| | 1 |
\]
so that we conclude the norms $| x |$, $\| x \|$ are equivalent.
Odd case. Let $C_1$ be the first Clifford algebra. It is super: $C_1 = C_0 \oplus C_1$, where $\alpha^2 = 1$. Hence it is not supercommutative, although it is commutative.

For the odd case we want to consider super vector bundles $E = E_0 \oplus E_1$ which are $C_1$-modules. This means that multiplication by $\alpha$ is an isomorphism of $E_0$ with $E_1$. The first problem is whether $E$ should be a left $C_1$-module or a right $C_1$-module. At this point it makes no difference as in either case we have an isomorphism of $E_0$ with $E_1$.

It makes a difference when we try to describe endomorphisms. Suppose $V = V_0 \oplus V_1$ is a graded $C_1$-module. Multiplication by $\alpha$ is an isomorphism of $V_0$ with $V_1$. Let $T$ be an odd degree endo. of $V$. If $V$ is a right $C_1$-module we have $T \alpha = \alpha T$, but if $V$ is a left $C_1$-module, then $T \alpha = -\alpha T$.

In the interests of operators I should first work out the theory for right $C_1$-modules. We can write

$$V = V_0 \otimes C_1$$

and then

$$V_0 \otimes C_1 \otimes (V_0)^* \rightarrow \text{End}_{C_0 \oplus}(V_0 \otimes C_1).$$

This is really

$$V \otimes C_1 \rightarrow \text{Hom}_{C_0 \oplus}(V, C_1) \rightarrow \text{End}_{C_0 \oplus}(V)$$

so to get the super-trace we use the map

$$V \otimes V^* \rightarrow V^* \otimes V \rightarrow C_1 \rightarrow C_1 \rightarrow \lambda(0).$$
and then we have to divide out by \([C_0, C_1]\) to get something well-defined.

Notice! The category of graded \(C_1\)-modules is equivalent to the category of ungraded \(C_0\)-modules, i.e. vector spaces. However, the graded \(C_1\)-modules have a richer Hom of some sort. What does this mean?

It seems that I should concentrate on right \(C_1\)-super modules. The super algebra \(C_1\) is non-commutative, so you have to keep right and lift straight. Then a super \(C_1\)-module is of the form

\[V = V^0 + V^1 = V^0 \otimes C_1\]

and if we identify \(V^0\) and \(V^1\) via \(\alpha\), then endos with commute with \(\lambda\). So \(\text{End}_{C_0}(V)\) consists of operators

\[
\begin{pmatrix}
  u & 0 \\
  0 & u
\end{pmatrix}
\quad \text{even}
\quad \begin{pmatrix}
  0 & v \\
  v & 0
\end{pmatrix}
\quad \text{odd}
\]

with \(u, v \in \text{End}(V^0)\).

Examples of super commutators are

\[
\left[ \begin{pmatrix}
  u & 0 \\
  0 & u
\end{pmatrix}, \begin{pmatrix}
  u_1 & 0 \\
  0 & u_2
\end{pmatrix} \right] = \begin{pmatrix}
  [u_1, u_2] & 0 \\
  0 & [u_1, u_2]
\end{pmatrix}
\]

\[
\left[ \begin{pmatrix}
  0 & v \\
  v & 0
\end{pmatrix}, \begin{pmatrix}
  0 & v \\
  v & 0
\end{pmatrix} \right] = \begin{pmatrix}
  [u, v] & 0 \\
  0 & [u, v]
\end{pmatrix}
\]

\[
\left[ \begin{pmatrix}
  0 & v_1 \\
  v_1 & 0
\end{pmatrix}, \begin{pmatrix}
  0 & v_2 \\
  v_2 & 0
\end{pmatrix} \right] = \begin{pmatrix}
  v_1 v_2 + v_2 v_1 & 0 \\
  0 & v_1 v_2 + v_2 v_1
\end{pmatrix}
\]

so the only possible super trace is \(\begin{pmatrix}
  u & v \\
  v & u
\end{pmatrix} \mapsto tr v\).
Now the idea is that when we come to a super $C_1$-vector bundle $E = E^0 \oplus E'$, the super algebra being commutative, it does not matter whether we consider

$$\Omega(M, E) = \Omega(M) \otimes_{\Omega(M)} \Omega(M, E)$$

as a right or left super $\Omega(M)$ module. But we still want $C_1$ to stay on the right.

\[ \text{January 16, 1984} \]

There are points to discuss in the odd case.

Consider a graded $C_1$-bundle $E = E^0 \oplus E'$ over $M$. I think of $C_1$ as acting on the right. Then when I compute endomorphisms I consider endos commuting with $\alpha$. So then when I consider connections I won't violate the sign rule. More precisely if I start with a connection on $E$ commuting with $\alpha$, so that it is the same on $E^0$ and $E'$ under the isomorphism given by $\alpha$, then we extended to $\Omega(M, E)$ in the obvious way we have $D$ commuting with $\alpha$. Writing $D$ on the right, this becomes

$$D(\eta \alpha) = (D\eta)\alpha, \quad \eta \in \Omega(M, E)$$

and there is no violation of the sign rule.

Then if we identify $E^0, E'$, an endomorphism of $E$ as $C_1$-module is of the form

$$\begin{pmatrix} u & v \\ v & u \end{pmatrix}$$

with $u, v \in \text{End}(E^0)$. In particular we get

$$L = i \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$$
where \( v = v^* \), the same sort of formula encountered in the even case.

Thus what we have is a single vector bundle \( E^0 \) with a connection \( D^0 \) and self-adjoint endomorphism \( u \). Such data determine an odd K-class on \( M \) with supports in the singular set of \( u \). In effect we associate to \( u \) the bundle automorphism \( e^{i\pi u} \). Over the open set where \( u \) is invertible we have a definite deformation of this automorphism to the identity, namely, we push the eigenvalues of \( u \) to \( \pm 2\pi \) without crossing zero, so this defines an odd K-class on \( M \) modulo the open set where \( u \) is invertible.

Next I want to work through the superconnection business. The point will be that the calculations in the odd case are exactly the same except they take place in a smaller algebra, namely

\[
\Omega(M) \otimes \Omega^0(M, \text{End}_\omega(E)) = \Omega(M, \text{End}_\omega E)
\]

and this time the supertrace

\[
\text{tr}_S : \Omega(M, \text{End}_\omega E) \longrightarrow \Omega(M)
\]

is of odd degree.
January 17, 1984

The odd case: Given a vector bundle $F$ with metric $\mathcal{M}$, I get a super vector bundle

$$E = F \otimes C_1, \quad C_1 = \mathcal{C} \otimes \mathcal{C}$$

with right $C_1$ action. We've already understood the algebra

$$\Omega(M, \text{End } E) = \Omega(M) \otimes \Omega^0(M, \text{End } E) \otimes \Omega^0(M)$$

as an algebra of operators in $\Omega(M, E)$. Now I want to consider the subalgebra

$$A_y = \Omega(M, \text{End}_y E) = \Omega(M) \otimes \Omega^0(M, \text{End}_y E) \otimes \Omega^0(M)$$

of those operators which commute with right mult.

It's clear that

$$\text{End}_y E = \text{End } F \otimes C_1$$

so that

$$A_y = \Omega(M, \text{End}_y E) = \Omega(M, \text{End } E) \otimes C_1$$

In other words we can represent the operators in $A_y$, or elements in $A_y$ in the form

$$\alpha + \beta \mathbf{j}$$

where $\alpha, \beta \in \Omega(M, F)$ and $\mathbf{j} = 1 \otimes \mathbf{Y} \text{ in } A$. Thus $\mathbf{j}$ commutes with even $\alpha$ and anti-commutes with odd $\alpha$.

So now consider super connections on $E$ commuting with $\mathbf{Y}$. The curvature + difference of two superconnections
is in $A_g$.

The supertrace on $A$ vanishes on $A_g$; the appropriate super trace on $A_g$ is the map

$$A_g = \Omega(M, \text{End}_g E) = \Omega(M, \text{End} F) \otimes \mathcal{C}_1 \rightarrow \Omega(M) \otimes \mathcal{C}_1 / [\mathcal{C}_1]$$

$$\alpha + \beta \gamma \rightarrow \text{tr}(\beta) \gamma$$

so when we take the curvature $\tilde{\nabla}^2$ of a superconnection and form $\text{tr}_s e^{\tilde{\nabla}^2}$ using this trace, we get an odd form on $M$.

Take now $\tilde{\nabla} = D + L = d + A + B_j$ on $\Omega(M, F) \otimes \mathcal{C}_1$, $D$ is connection on $F$. The curvature is

$$\tilde{\nabla}^2 = (D^2 + B_j^2) + \frac{[D, B_j]}{2} \beta$$

where $\alpha$ is even, hence commutes with $\beta$, and $\beta$ is odd, hence anti-commutes with $\beta$. Now in

$$e^{\alpha + \beta \gamma} = e^{\alpha} + \int_0^1 dt e^{(1-t)\alpha} \beta_j e^{\alpha} + \int_0^1 dt_1 \int_0^{t_1} dt_2 e^{(1-t_1)\alpha} \beta_j e^{t_1 \alpha} \beta_j e^{t_2 \alpha} + \cdots$$

we move all the $\beta_j$'s to the right. The signs are

$$\beta_j^2 = -\beta^2$$

$$\beta_j^3 = -\beta^3 \beta$$

$$\beta_j^4 = -\beta^3 \beta^3 \beta = \beta^4$$

It's clear then that

$$\text{tr}_s(e^{\alpha + \beta \gamma}) = \frac{1}{i} (\text{odd part of } \text{tr } e^{\alpha + i \beta}) \gamma$$
\[ \text{tr}_s(e^{x+if}) = \frac{1}{2i}(\text{tr} e^{x+if} - \text{tr} e^{x-if}) \]

Let's check this out:

\[ \text{tr}_s(e^{(0+ib)j}) = \text{tr}_s(e^{tb^2 + t[D,j]j + 0^2}) \]

\[ = \frac{1}{2i} \text{tr}(e^{tb^2 + t[D,j]j + 0^2} - e^{-t^2b^2 + t[D,j]j + 0^2}) \]

To take \( L = i \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} \), or \( B = iu \) and this becomes:

\[ \frac{1}{2i} \text{tr}(e^{-t^2u^2 - t[D,u] + 0^2} - e^{-t^2u^2 + t[D,u] + 0^2}) \]

which up to a normalization constant is just the odd degree part of

\[ \text{tr}(e^{-t^2u^2 + t[D,u] + 0^2}) \]

Here \( D \) is a connection and \( u \) is a self-adjoint operator.
January 18, 1984

Conversation with Connes: 

Topics:

Trick to get from \( F^2 = I \) (mod \( X \)) to \( F^2 = I \).

Kasparov cup product, positivity condition, convexity
(with Skandalis)

Bordism (with Baum) represents \( R(G) \) not \( K(BG) \).

Description of \( BG \), \( G \) a semi-simple Lie gp., \( G/K \)
in the final object for proper \( G \)-actions, barycenters.

Pseudo differential operators + blow-up of \( \mathbb{V} \times \mathbb{V} \) along \( \Delta \)

Physics: space-time doesn't exist, there is a groupoid of charts, why do structure groups reduce to \( SO(3,1) \), \( C^* \)-alg. cosheaf over space-time

Coadjoint of a Lie group on a \( C^* \)-algebra and the non-commutative picture of principal bundles, how to see the \( C^* \)-alg in physics are real.

YM fields on \( \Delta \), like \( d(x,0) \) on \( G/K \).

Helfand - Furko + cyclic cocycles, interpretation of Godbillon - Vey.

Actually masses should be points for the groupoid, i.e. independent of the observer.
Let me see how much I can reconstruct.

Consider his trick for replacing a Fredholm operator $P$ by an invertible one, in order to obtain a cocycle. This involves the ring $A \otimes A_e$ with $e \otimes 0$. Somehow he now does this for $A = C$ and then takes cup product. The idea is to consider the operator of multiplication by a real number $A$ on $C$. Then upon cup product one has something like $P \otimes 1 + 1 \otimes A$ whose square is

$$(P \otimes 1 + 1 \otimes A)^2 = P^2 + A^2 = 0,$$

hence it is invertible. Now one can compute the cocycle using the invertible operator, since the class is independent of $A$, and is a (polynomial?) function in $A$, one gets the same class by letting $A = 0$, or $\infty$.

Kasparov cup product: When constructing the Kasparov product one must make choices. It is important that the choices lies in a contractible set, e.g. a convex set, so that the result is unique up to homotopy.

An example of a Kasparov product is furnished by tensoring an operator $P$ with a vector bundle. If the v.b. is described by an idempotent $e$, then I have seen that the interaction of $e$ and $P$ is more or less the same as a connection $D$. Therefore the Kasparov product is something like the sum $D \otimes 1 + 1 \otimes P$. 

Cannes and Skandalis characterize when an operator
like $D \otimes 1 \otimes P$ represents the Kasparov product. The conditions are that the operator have a derivation (or connection) property, and also some sort of positivity property. Note that a choice of connection + positive operator are convex choices. Their result describes the Dirac operator with coefficients in a complex of v.b.'s as the Kasparov product of the Dirac K-homology class with the K-homology class represented by the complex of vector bundles. So given a total Dirac operator in a fiber bundle $X \to M$, one checks it has the derivation property and positivity property, so is the Kasparov cup product of the elements:  

\[
\text{KK}(C(X), C(M)) \times \text{KK}(C(M), C) \to \text{KK}(C(X), C) \text{.}
\]

\[\begin{align*}
\text{family of Diracs} & \quad \text{Dirac on M} \\
\text{on the fibres} & \quad K\text{-homology class}
\end{align*}\]

Pseudo differential operators & $V \times V$ blown up along $\Delta V$. We form a topological groupoid. The objects are pairs $(t, x)$ with $t \in [0, 1]$, $x \in V$. For $t \neq 0$ there is a unique morphism from $(t, x)$ to $(t, y)$, and for $t = 0$ there is one morphism from $(0, x)$ to $(0, x)$ given by each tangent vector at $x$. This is a smooth groupoid and has an associated convolution algebra which can be completed to a $C^*$-algebra. We have two sub-groupoids given by $t = 0$, $t = 1$. 
The groupoid of \((0, x), x \in V\) has for convolution algebra the smooth families of convolution algebras in the tangent spaces. By Fourier transform this algebra is isomorphic to functions on \(T^*_V\) vanishing at \(\infty\).

Let us now consider the restriction map from the \(C^*\)-algebra associated to the full groupoid to the \(t=0\) subgroupoid. The kernel consists of families with parameter \(k\), vanishing at \(t=0\), of ordinary smooth kernels on \(V \times V\). The \(K\)-theory is the same as that of \([0,1] \mod \{0,1\}\), and this is trivial.

I should have mentioned that the algebra for \(t=1\) or for any \(t>0\) is just the algebra of kernels on \(V \times V\) under convolution, hence its \(K\)-theory is that of a point. Thus we get

\[
K^*(\text{fns. on } T^*_V) \leftarrow K^*(V \times V, \Delta V, \text{alg}) \rightarrow K^*(pt)
\]

which is just the index map defined analytically.

In effect defining a PDO from a symbol involves choosing a section of the restriction of the algebra of the \((V \times V, \Delta V)\) groupoid to the \(T^*_V\) groupoid. This construction is needed when there is a \(g\) acting.

Crunes would like to lift all the \(C^*\)-alg. stuff back to (smooth?) geometry. Central to this aim is a conjecture to the effect that the \(K\)-cohomology of the \(C^*\)-algebra is the same as the \(K\)-homology of the classifying space, at least up to some orientation business. At the moment he doesn't have a perfect
A description of the classifying space $K$-homology, only his work with Baum.

The first point is that $K$-homology classes of $BG$ are compactly supported, hence are realized by compact things mapping in. A second point is that one doesn't want just free actions. For example, when $G$ is compact a map $M \to BG$ is a principal $G$-bundle over $M$, so one is getting the $K$-theory of $BG$ which is a big limit (direct for homology, inverse for coh.). For the $G$-index thm. one wants arbitrary $G$-actions on manifolds -- equivariant bundles.

A fact is that if one asks for a way to assign an element of an abelian gps to a compact $G$-manifold + bundle, compatible with $G$-equivarian maps, then one gets $R(G)$. Here compatible with $G$-equivarian maps is easier to work with, but equivariant to, bordism.

For $G$ a Lie gps. one considers proper $G$-actions with compact quotient. Up to homotopy there is a unique map to $G/K$. This explains the importance of the Dirac operator on $G/K$ as a means to go from $R(K)$ to operators (?), and lets to the conjecture

$$R(K) \Rightarrow K(C^*_\text{red}(G))$$

proved by Pennington + Waner.

Cannas says that on a Riemannian manifold, if the curvature is $< 0$, then the weighted sum

$$f(x) = \sum t_i d(x, x_i)$$

has a unique critical point, so defines barycentric cords.
All this is convexity of the function \( x \cdot \text{dil}(x,y) \) along geodesics.

Physics: The standard C*-alg approach to quantized field theory says one should have a cochain of C*-algebras over space-time, such that open sets which are space-like separated have commuting algebras. Finally, there should be some global Lorentz invariance.

Connes objects to the latter as being inconsistent with general relativity. His viewpoint is that there is no space-time in the same way that there is no quotient to a foliation in general. What one has is observers; each observer has a chart, hence his C*-algebra. Two observers that can relate to each other have overlapping charts. Thus there is a groupoid. (Notice the connection with Kaluza-Klein.)

Why is the structural group \( \text{SO}(3,1) \)? There perhaps is a ergodic type obstruction to reducing the group further, and this ergodic stuff should be essential for the existence of thermodynamics.

I asked him how one sees that the C*-alg on space-time is actually comes from a real three dimensional time-like section. He said that there is a coaction of \( \text{SO}(3) \) on the C*-algebra.
Covers proof of the index theorem.

M manifold, consider Clifford algebra of $T_M$ to define a $K$-homology class $[\mathfrak{a}]$ in this algebra. Operator $D = d + s$ on forms on $M$, tensors with complex $\mathfrak{a} \to \mathbb{C}$ where $\mathbb{C}$ acts as zero on the second. This gives $D \otimes 1$ which is invertible form cocycle $\lambda \to 0$ (or $\infty$). Evaluating cocycle in the limit involves asymptotic diff operators; local on $M$; Göttingen proof is simpler with same result. Works equivariantly important that one has only to compute cocycle in this canonical $d + s$ situation. Morita equivalence of $\mathfrak{a}$ with endpoints of a vector bundle $E$ involves character of $E$; this introduces $\mathfrak{a}$ somehow.

Jan 19, 1987

Hodge $\ast$ operator is Clifford multiplication with the orientation.

Let's go over Gromov's proof of the index theorem. One idea which seems to be very good is the following:

The basic operator will be $d + s$ on forms on $M$. But you can think of this operator in two ways. The normal way is as an operator over $\Omega^\bullet(M)$-module. The interesting way is as an operator over $\Omega^\bullet(M)$ as a module over the Clifford algebra. In the first case we know the index problem is relatively trivial, because the $\hat{A}$ denominator is missing. However this is exactly what is killed under the forgetful map.

Algebroid cohomology

$$\int_{T^2} \sum_{i=1}^{n} \frac{d f_i}{d^2} = \text{area inside } (f_0, \ldots, f_n)(2) \subset \mathbb{R}^{n+1}$$

Novikov Conjecture: (Morgan. Last remaining step in
It has to do with signature of inverse images.

Witnars says it as follows: Given a map \( f: M \to S^n \)
the signature of the inverse image of a point is not to be
homotopy invariant of \( \text{ker}(M,f) \). This is because the portygen-
classes, better the L-classes, are not homotopy invariant except
the top one. But given \( M \to K(\pi,1) \) and a cohomology
class in \( K(\pi,1) \), say represented by a submanifold, transversally
oriented, then the signature of the inverse image is supposed
by the Novikov conjecture to be a homotopy invariant.

Morgan: For example the Novikov fibering theorem says
that an \( M \) homotopy equivalent to \( N \times S^1 \) is of the form \( M = 
N' \times S^1 \) and \( N_1 \times S^l \sim N_2 \times S^l \Rightarrow N_1 \simeq N_2 \).

Hence \( M \to \text{Signature}(N) \) is a homotopy invariant. This
result proves Novikov for \( \pi \in \mathbb{Z} \), and similarly for \( \pi \) abelian.

Group cocycles give cyclic cocycles on the group algebra
you could describe these for Heisenberg groups.
Old conventions. Given $n = 2m - 1$ and an ungraded $\mathbb{C}_n$-module $\mathcal{W}$, the involutions being $\alpha^\mu$, we get a graded $\mathbb{C}_{n+1} = \mathbb{C}_{2m}$-module

$$\mathcal{V} = \mathcal{W} \oplus \mathcal{W}$$

with the following:

$$\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \alpha^\mu \\ \alpha^\mu & 0 \end{pmatrix} = \alpha^\mu \gamma \quad 1 \leq \mu < 2m$$

$$\gamma^{2m} = -\varepsilon \gamma \gamma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Thus for example if $n = 1$ and $\alpha^1 = 1$ on $\mathbb{R}$ then we get for $\varepsilon, \gamma, \gamma^2, \varepsilon = \gamma$ the three Pauli matrices.

Also we have

$$\text{tr}_\mathcal{V}(\varepsilon \gamma^1 \cdots \gamma^{n+1}) = \text{tr}_\mathcal{V}(\alpha^1 \gamma^2 \cdots \alpha^n \gamma (-\varepsilon \gamma)^{n+1}) = 2i \text{tr}_\mathcal{W}(\alpha^1 \cdots \alpha^n)$$

so that if we take the preferred ungraded $\mathbb{C}_n$-module to have dim $2^{m-1}$ with $\alpha^1 \cdots \alpha^n = i^{m-1}$, then $\mathcal{V}$ becomes the spinors. Good choice is

$$\text{tr}_\mathcal{W}(\alpha^1 \cdots \alpha^n) = (2i)^{m-1}.$$

Notice that $\text{tr}_\mathcal{W}(\alpha^{i_1} \cdots \alpha^{i_p})$ is 0 for $1 \leq p < n$.

$$\text{tr}_\mathcal{W}(\alpha^{i_1} \cdots \alpha^{i_p}) = \begin{cases} 0 & 1 \leq p < n \\ 2^{m-1} & p = n \end{cases}$$

$\alpha^1 \cdots \alpha^n$ anti-commutes
with a missing A^+; Thus it is odd relative to the
A^+ grading and has zero trace. If p is even and > 0,
then \( \text{tr}(\alpha^{i_1} \ldots \alpha^{i_p}) = - \text{tr}(\alpha^{i_{p-1}} \alpha^{i_p}) \) and all this,
hence we get 0.

Finally if I take the corresponding character
I get

\[
\text{tr}_w(e^{-t^2 x^2 + tdx^* x^*}) = e^{\frac{t^2}{2}} \prod_{\mu} (1 + tdx^* x^*_\mu)
\]

\[
= e^{\frac{t^2 x^2}{2}} [2^{m-1} + t^n \sum_{n} \text{tr}_w(d^i x^i_1 \ldots d^i x^i_n)]
\]

\[
= e^{\frac{t^2 x^2}{2}} \left[ 2^{m-1} + t^n \sum_{n} \frac{\text{tr}_w(\alpha_n \ldots \alpha_1)}{(2i)^{m-1} (-1)^{2n-2}} \right]
\]

Thus the character form is

\[
\text{odd part of } \text{tr}_w(e^{-t^2 x^2 + tdx^* x^*}) = (-2i)^{m-1} e^{-\frac{t^2 x^2}{2}} t^n dx^i \ldots dx^n
\]
Problem: Consider a Clifford algebra bundle $C(T)$ where $T$ is a vector bundle over $M$. Then super $C(T)$-modules give rise to elements of $K(T)$. The problem will be to construct their Chern characters.

Start with the simplest case where $T$ is trivial. Then I have a super vector bundle $E$ with $C_n$-action. Suppose $n$ is even. Then I know that super $C_n$-modules are the same as super $C_0$-modules. On the other hand, I know how to associate to a super $C_n$-module an element of $K(R^n)$. This should somehow be Bott periodicity. Therefore Bott periodicity seems to be a kind of Morita equivalence.
January 27, 1984

I want to consider the local index formula for the family of all Dirac operators with coefficients in the vector bundle $E$. Let $H = L^2(M, S \otimes E)$, $S = \text{gauge transf. } \phi$, $A = \text{connections}$, $A \rightarrow \Phi_A$, the equivariant map from $A$ to $(\text{End } H)'$. Then by passing to the quotient we get a Fredholm operator on the induced Hilbert bundle $A \times \mathfrak{g}H$ over $A/\mathfrak{g}$. To apply the character formula I need a connection on $A \times \mathfrak{g}H$, i.e., a connection on $A \times \mathfrak{g}H$ which descends.

Look at this problem first. Suppose we have a representation $\rho : G \rightarrow \text{Aut}(V)$ of $G$ and a principal $G$-bundle $P/M$. Then we want a connection on the bundle $P \times G V$ over $M$. Start with the invariant connection $\theta$ on $P \times V$ and let $\epsilon \in \Omega^1(P) \otimes \mathfrak{g}$ be a connection form on $P$. Then $\theta + \rho(\epsilon)$ is a connection which descends.

Start again: let $H, \Phi_A : A \rightarrow (\text{End } H)'$ be as above. Then we get a Fredholm map $F$ on the super Hilbert bundle $A \times \mathfrak{g}H$ over $A/\mathfrak{g}$. This defines an element of $K^0(A/\mathfrak{g})$. We are trying to transgress this $K$-element to something perhaps in $K^{-1}(\mathfrak{g})$, which would be roughly the difference of the two maps of $\mathfrak{g}$ to the unitary group.

Look at finite dimensions first. Given a compact group $G$ and a principal $G$-bundle $P/M$, we get the bundle $P \times G V$.
Representing an elt of $K^0(M)$. This transgresses to an elt $K^{-1}(G)$ which is the map $G \to U$ given by the representation.

In terms of characteristic classes we get the following. If $\Theta$ is a connection for $P/M$, then $d + s(\Theta)$ on $\Omega(P) \otimes V$ descends to give a connection on $P \times G V$. Drop $s$ from the notation, so that $D = d + \Theta$. Then the Chern character is the class on $M$ represented by the form
\[ \text{tr}(e^{d\Theta + \Theta^2}) \text{ on } P. \]

To compute the transgression we use the family of connections $d + t\Theta$ which gives
\[ \text{tr}(e^{d\Theta + \Theta^2}) - n = d\int_0^1 dt \text{ tr}(e^{td\Theta + t^2\Theta^2}). \]

When we restrict to a fibre, the form $\Theta$ becomes the MC form, and $d\Theta = -\Theta^2$ so that we get the form
\[ \int_0^1 dt \text{ tr}(e^{t^2\Theta^2}). \]

Representing the transgressions.

Now I want to do something similar in the infinite-dimensional setup. Again let $\Theta$ be a connection form in the principal $G$-bundle $A$ over $A/G$. Then on $A \times H$ I have the connection $d + \Theta$ which descends to gives a connection on $A \times \mathfrak{g}H$ over $A/G$. I would like to use the forms
\[ \text{tr}(e^{d\Theta + \Theta^2}). \]
but the trace doesn't make sense. $d\theta + \theta^2 = \Omega$

is a two-form with values in $(\text{End } \mathcal{H})^\circ$. It is

essentially an operator on $L^2(S \otimes E)$ coming from

a vector bundle endo. of $E$, so it's a multiplication

operator and doesn't have a trace. Somehow only

the difference of the operators on $\mathcal{H}^\circ$ and $\mathcal{H}^!$ has

a trace. (?)

The next step is then to use the Chern

character form with the odd degree endo $\varphi$

$$\text{tr}_g \left( e^{-t^2 \varphi^2} + it [D, \varphi] + D^2 \right),$$

where $D = d + \Theta$. This is a form on $A$ which

descends to $A/4$. I want to go thru the

transgression process which means that I have to

write this form as $\text{d}$ of something, and then

restrict the something to $\mathcal{H}$. So how does this go?

The only technique we have is to change

the superconnection

$$\tilde{D} = d + \Theta + it \varphi$$
Bott's theorem on Chern numbers. Let $M^{2n}$ be a $2n$-dimensional manifold in which the circle $S^1$ acts, and let $E$ be an equivariant vector bundle. Let $p$ be an invariant polynomial of degree $n$. Bott's theorem expresses the Chern numbers of $E$ as sums of contributions over the fixed points. I want to see if I can see this result geometrically.

More precisely, I would like to deform the differential form representing $p(E)$ into a neighborhood of the fixed point set.

To assume the action is free. Then $E$ descends to $E$ over the orbit space $M/S^1$ which has dimension $2n-1$, hence $p(E) = 0$ and $p(E) = 0$ for dimensional reasons. On the differential form level, let $D$ be a connection on $E$ which is invariant, and let $\Theta$ be a connection form for the $S^1$ action. Then $D + \Theta q$, where $\Phi$ is the Higgs field, descends. So $p(D + \Theta q)$ is a $2n$ form which is basic. But a $2n$ form killed by $i(x)$ has to be zero. Then

$$p(D + \Theta q) - p(D) = d \int_{S^1} p'(D + \Theta q, \Theta q)$$

allows one to express $p(D)$ as a coboundary.
Consider a circle action on $M$ and an equivariant bundle $E$. The localization theorem says that in some sense, equivariant cohomology classes on $M$ when localized are described by their restriction to the fixpoint set. I would like to see this done geometrically, for example, I would like to have differential forms representing equivariant classes, such as the equivariant characteristic classes of $E$, given by forms concentrated near the fixpoints.

Now one has to incorporate the localization process into the picture. I can't expect to represent a class by a form near the fixpts unless it somehow vanishes when the fixpoint set is removed.

Witten seems to do the localization process in an analytical way which goes as follows. He considers the forms on $M$ with the differential operator $d + t \xi$, where $t$ is a parameter. Notice that this preserves only the $\mathbb{Z}_2$-grading on forms. He has some argument to get to the case of invariant forms in which case, because

$$(d + t \xi)^2 = t (d \xi + \xi d) = t 2 \xi$$

one gets $(d + t \xi)^2 = 0$ on $\Omega(M)^S$.

In order to keep the connection with equivariant form, I should use $d - t \xi$, so let's change the notation.

So now we have

$$\Omega^{\text{ev}}(M)^S \leftrightarrow \Omega^{\text{odd}}(M)^S$$
and we can form the Laplacean and do Hodge theory to realize the cohomology $H_\tau$ with respect to $\text{d} - \text{tr}_\chi$ via harmonic forms.

Atiyah and Bott have pointed out that $H_\tau$ for $\tau \neq 0$ is essentially the localized equivariant cohomology. The equivariant cohomology $H_\tau(M)$ is a graded module over $H_\tau(pt) = k[u]/u^2$ where $\text{deg} u = 2$.

When localized, $H_\tau(M)[u^{-1}]$ is a module over $k[u,u^{-1}]$ which one can think of as a vector bundle over the $\bar{\mathbb{C}}$-plane minus the origin. So I can specialize at a value $t$ of $u$ and get a super vector space.

They show this is the same as Witten's $H_\tau$ as follows. Introduce the complex of equivariant forms $k[u] \otimes \Omega(M)^S$ with diff $d - \text{tr}_\chi$.

Localization is exact so $H_\tau(M)[u^{-1}]$ is the homology of $k[u,u^{-1}] \otimes \Omega(M)^S$ with diff $d - \text{tr}_\chi$.

These homology groups are free over $k[u,u^{-1}]$, so specialization is also exact, i.e.

$$H[\frac{k[t]}{k[u]} \otimes k[u,u^{-1}] \otimes \Omega(M)^S] = k[t] \otimes k[u,u^{-1}] H_\tau(M)[u^{-1}]$$

$$H(\Omega(M)^S \text{ with } d - \text{tr}_\chi) = H_\tau.$$  

So what we see is that Witten's cohomology $H_\tau$ can be identified with the localized equivariant cohomology.

I gather that Witten shows as $t \to \infty$ that
harmonic forms for $d + t i x$ actually concentrate at the fixpoints. In this way he is able to show that the $\mathbb{R}$ cohomology coincides with that of the fixpoint set.

But I can ask a related question: What happens to characteristic forms for an equivariant vector bundle as $t \to \infty$. The equivariant curvature of $E$ relative to an invariant connection $D$ is $D^2 + \alpha q$, where $q = -[\iota_x D]$ is the Higgs field. So we can look at the form

$$\text{tr } e^{D^2 + t q}$$

as $t \to \infty$.

Let's be specific and take an equivariant line bundle. We have $d q = \iota_x D^2$, and we are looking at the form

$$e^{t q + D^2}$$

In practice, we write $i D^2 = \omega$ and $i q = H$, so that $\omega$ and $H$ are real satisfying $\iota_x \omega = d H$. This means that $H$ is the Hamiltonian generating $X$ when $\omega$ is symplectic. So we have the form

$$e^{-i t H - i \omega}$$

which doesn't concentrate at the stationary points as $t \to \infty$.

On the other hand when $M$ has dim $2n$

$$\left(\frac{i}{2\pi}\right)^n \int e^{-i t H - i \omega} = \int_M e^{-i t H} \frac{\omega^n}{(2\pi)^n}$$
is the quantity which has the exact stationary phase approximation, so something is happening.

Atiyah mentioned the following. Frankel's basic observation is that one has inequalities:

Morse: \[ H(M) \leq H(M^{\text{crit}}) \]
equivariant: \[ H(M^S) \leq H(M) \]

so in the case of an $S$-action on a Kahler manifold (with at least one fixed so that the Hamiltonian exists), the critical points for $H$ are the same as the fixed points for the circle action; hence the Morse theory is perfect.

(Why doesn't this work for circle actions on symplectic manifolds?)

We have seen that Witten's complex

\[ \Omega^{2\mathbb{Z}}(M)^S \leftrightarrow \Omega^{2\mathbb{Z}}(M)^S \]

with differential $d_{-t\chi}$ represents essentially the equivariant cohomology. In particular any class for this complex should be represented by a form supported near the fixpoint set. In fact he must have a way to identify harmonic forms on the fixed set with zero modes for $d_{-t\chi}$ for large $t$.

The simplest case would be when the fixpoint set is empty. Then he must see that the Laplacian of this operator is strictly positive and so there is no
If there are no fixed points, then \( X \) is never zero, so we can find an invariant 1-form \( \Theta \in \Omega^1(M)^S \) such that \( X \Theta = 1 \). But then

\[
[d - u_x, \Theta] = d\Theta - u
\]

so that multiplying by \( d\Theta \) and by \( u \) in \( k[u] \otimes \Omega^1(M)^S \) are homotopic. On the other hand \( d\Theta \) is nilpotent since \( d\Theta \in \Omega^2(M)^S \), which is finite dimensional. Thus \( u \) is nilpotent on \( H_0(M) \), etc.

The same argument works if \( u \) is a parameter \( \lambda \) and we consider \( d - \lambda x \Theta \) in \( \Omega^1(M)^S \).
January 30, 1987

Conversations with Michael

Frankel observation holds for a circle action as a symplectic manifold. Also the spectral sequence for equivariant cohomology is degenerate, i.e., additively
\[ H^*_s(pt) \otimes H(M) = H^*_s(M) \]

and hence there is no \( H^*_s(pt) \) torsion in \( H^*_s(M) \).

Yang-Mills = norm squared of the moment map. It leads to equivariantly perfect Morse function. Example

Then \( x^2 \) has maximum at \( \pi \) poles and minimum on equator.

\[
\begin{align*}
\min & \quad \max \\
1 + \frac{2t^2}{1-t^2} &= \frac{1+t^2}{1-t^2} = P.S. \quad H^*_s(S^2)
\end{align*}
\]

This is quite different from the Morse function \( x \)

\[
\begin{align*}
\min & \quad \max \\
\frac{1}{1-t^2} + \frac{t^2}{1-t^2} &= \frac{1+t^2}{1-t^2}
\end{align*}
\]

In the Kähler case the Morse cells are complex and the circle action extends to a \( \mathbb{C}^* \)-action preserving the cells. In Kirwan's work the cells are affine bundles over the critical submanifolds, affine relative to nilpotent groups. This explains why the counting formulas over finite fields works. (There are two logical problems relating Morse results with counting formulas. The Morse decompositions give long exact sequences, hence are not a priori additive. The finite field methods are additive but

\[
\begin{align*}
1 + \frac{2t^2}{1-t^2} &= \frac{1+t^2}{1-t^2} = P.S. \quad H^*_s(S^2)
\end{align*}
\]
a priori, the stratum attached to a critical set there might be a complicated relation between the numbers of retinal points. Both difficulties don't occur in Kirwan's setup so that is why results agree.)

Review of Donaldson's proof that stable bundles have unique YM-connections. Picture from Mumford-Kirwan-theory: YM = square of moment map which is defined by K action (K = max. comp. of G); moduli space is X/G or YM minima. The unique K-orbit which minimizes YM in a G-orbit is also the part of the G-orbit in the line bundle (or vector space if you think of G as acting linearly) closest to the zero section.

Donaldson therefore has to minimize the distance to the origin on the sphere of metrics on a fixed holomorphic bundle. He deforms the Kähler metric until it becomes a S-function concentrated on a hyperplane section. We can control this change, and then using that a stable bundle on the surface restricts to a stable bundle on the hyperplane section, he sees the limit is bounded below.

\[ \text{Novikov conjecture. Let } \pi \text{ be a f.p. group. Given a manifold } M \text{ with fundamental group } \pi \text{ and a coh. class in } B\pi, \text{ one takes the submanifold dual to this coh. class and computes the index. Is this operation } \{ M \} \to \mathbb{Z} \text{ an invariant? An equivalent description is to push forward the } K\text{-homology class in } M \text{ given by the signature operator into } B\pi \text{ and ask whether this is a homotopy invariant? Pairing with a representation of } \pi \text{ gives a homotopy invariant, since it gives the signature of } M. \]
with coefficients in the representation. More generally, this conclusion holds for "representation" generalized to "element of $K(C^*_r(\pi))$", where $C^*_r(\pi)$ is the reduced $C^*$-algebra of $\pi$. So we want to know the relation between $K_*(B\pi)$ and $K^*(C^*_r(\pi))$.

I probably have $K$ homology and cohomology for $C^*_r(\pi)$ confused. Connes' conjecture is that
\[ \mu : K_*(B\pi) \to K^*(C^*_r(\pi)) \]

is an isomorphism and a representation of $\pi$ should pair with a finite projective module to give an integer. Thus "representations" are in $K$-homology $KK(C^*_r(\pi), C)$, and so we see that if $\mu$ is injective, then the Novikov conjecture follows (more or less)."

John Roe: Let $A$ be an algebra contained in $B$ such that $D \in B$ is invertible mod $A$, $A$ being assumed to be an ideal in $A$. Then from the exact sequence
\[ 0 \to A \to B \to A/B \to 0 \]
is a connecting homomorphism
\[ K_1(A/B) \to K_0(A). \]
Roe describes this connecting homomorphism as follows. Given $D$ and a $P \in B$ with $DP = PD = I$ mod $A$, one obtains the image of $D$ under $\delta$.
as the difference of two $2 \times 2$ matrices over $A$
which are projectors
\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix} - \begin{pmatrix}
\rho & \rho \phi \\
\rho \phi & \rho
\end{pmatrix}.
\]


Scott: He explained to me what a Dixmier - Heckman type proof of the index theorem would amount to: Namely an actual evaluation at any time of the traces of the Dirac heat kernel. This is instead of an asymptotic evaluation.

On spinors over a Riemann spin manifold with Killing vector field $X$, we should have
\[
[X, \phi] = L_X
\]
where $X$ is identified with an element of the Clifford algebra.

Arajah summarizes talk by Berry: Phase +
Degeneracy. Deform a Hamiltonian with many parameters, following the eigenvectors one gets a line bundle, a connection, hence going around a loop results in a phase shift. Generic case encounters degeneracy in codim 3.

Classical example of a bead running around a closed curve. When rotated adiabatically there is a phase shift $\frac{1}{4} \left( 1 - \frac{\pi^2}{4A} \right)$ which is always $\leq 0$ and $= 0$ for the circle by the isoperimetric inequalities.
Localization thm. in equivariant cohomology implies that in the localized cohomology we have

\[ \text{i}_* \frac{\text{i}^* (\alpha)}{c(\nu)} = \alpha \]

The problem I would like to understand is how to do this explicitly in Witten’s cohomology. For example, can I define \( \text{i}_* \)?

Cyclic cohomology = Witten cohomology of the loop space.
Conversation with Connes: Kasparov KK, seminar on Hörmander's several complex variables in order to get the Grothendieck result. Morita equivalence formulas in Ch. II. Calculation of cocycle for $C^*\Gamma$ and the $\bar{\partial}$ operator. Bivariant cyclic homology, rotation alg. on $S^1$ and index thm. for Mathieu. Example of Hilbert Schmidt operators, cyclic homology of $C^0(V)$ = proper homology, circle bundle.

Gleason: Today wants to relate cyclic homology to some kind invented by Deligne + Beilinson.

Problem with cyclic theory when the ring has no unit. Consider the algebra of Hilbert Schmidt operators. One has cyclic cocycle $\frac{1}{n!} \sum \text{tr}(x_0 x_1 x_2) = \tau(x_0, x_1, x_2)$ which is the Hochschild coboundary $b\eta$ where $\eta(x_0 x_1) = \text{tr}(x_0 x_1)$. But then the cocycle $\tau$ should be in the image of $S$. For some reason the candidate should be $\text{tr}(x)$ which isn't defined for Hilbert Schmidt operators.

Morita equivalence: see Chap. II. Note that the element he adjoins to make a complex, i.e. to kill the curvature, is something like the $dl$ or $d\epsilon$. This builds in the $S$-operator. He said something about passing from $A$ to $\text{End}(E) = B$ and then from $B$ back to $A$.
If an abelian group $G$ acts on $A$, then $A \times G$ has a natural $\hat{G}$ action. Moreover $(A \times G) \times \hat{G}$ is Morita equiv. to $A$ and this equiv. is compatible with the $G$-action.

Given a circle bundle, the crossed product of the functions on $P$ with $S$ admits a $\mathbb{Z}$-action (which is inner: as the action on the base is trivial?) somehow the effect on cyclic cohomology has to do with multiplication by the Euler class of the circle bundle.

Other topics: computing the cocycles for the Hilbert transform on $R$ and $\mathbb{D}$.

Idea: Can his new theory be used to construct characteristic classes for flat bundles in “real” K-theory? Atiyah-Segal’s question for non-unitary representations.
February 2, 1984.

Lunch with Katy, Yackle, Deligne.

Differential Galois theory. Start with a linear DE over $\mathbb{C}$. This is a holomorphic vector bundle $E$ with flat connection. We are primarily interested in the case where there is a non-regular singular point at $\infty$.

There is an intelligent way to take the Zariski closure of the monodromy based on Tannakian ideas. The monodromy is trivial for $\mathbb{C}$, but one can look flat sections of $E \otimes \mathbb{A}$ which are algebraic. Put another way, if one started with an algebraic variety over a field of char zero, and a flat bundle, then by embedding the ground field in $\mathbb{C}$ and taking Zariski closure of the monodromy, one gets various algebraic groups depending on the embedding. A good algebraic definition based on alg. flat sections of $E \otimes \mathbb{A}$ solves this difficulty.

One uses the Chevalley theorem, that an algebraic subgroup $G$ of $GL_n$ stabilizes, if one in some sense.

**Example:** Take the fn. $e^z$ which belongs to the operator $\frac{d}{dz} - 1$ on $E = 1$. Tensoring $N$-times leads to $e^{Nz}$. None of these are algebraic for $N \neq 0$, hence the differential Galois group is $GL_1 = \mathbb{G}_{m}$.

**Example:** Airy: $\frac{d^2}{dz^2} - z$. Here the differential Galois group is all of $SL_2(\mathbb{C})$.

Effect of wild ramification on $H^*$. Take an affine non-singular curve and an $\ell$-adic sheaf $F$. Then $H^0$ and $H^2$ are zero, hence the Euler char. $x$ is $-\dim H^1_\mathbb{C}$.

Naively one expects $x(F) = x(Q_{\mathbb{C}}) \cdot \text{rank } F$, but there is
If one wants something like the \( \Gamma \) function factor in the usual Fourier transform between \( x^s \) and \( x^{1-s} \), then one has to put some kind of rational structure into these spaces.

Deligne: \( \pi_1(\mathbb{A}/\mathbb{Q}) \) acts on \( \pi_1(P^1_{\mathbb{Q}} - \{0, 1, \infty\}) \), \( \pi_0(P^1_{\mathbb{Q}} - \{0, 1, \infty\}) \) is a free \( \mathbb{Q} \)-group on 2-generators, isomorphic to free Lie on \( H' = \mathbb{Z}/(\mathbb{Q})^{\oplus \mathbb{Q}} \). The isomorphism is non-canalical and upon writing out exact sequences using the descending central series, one finds classes in

\[
H'(\mathbb{Q}/\mathbb{Q}, \mathbb{Z}/(\mathbb{Q})) \sim K_{1-2n}(\mathbb{Q})
\]

One can view \( \pi_1(P^1_{\mathbb{Q}} - \{0, 1, \infty\}) \) as representing an elt. in a non-abelian coh. of \( G \) \( H'(\mathbb{Q}/\mathbb{Q}, G) \) where \( G \) is a group of auto of the free lie algebra. The idea is to reduce \( G \) as small as possible, in which case it should have generators corresponding to the infinite generators of \( K_{1}(\mathbb{Q}) \).

Over \( \mathbb{C} \) one can do the mixed Hodge analogues.

On \( X = P^1_{\mathbb{Q}} - \{0, 1, \infty\} \) there is an \( H_1 \) and an \( H_2 \), both \( 2\mathrm{dim} \), and the pairing gives a 1-form on \( X \) with values in the \( H_1 \). Then construct a differential equation on \( X \) whose monodromy is the free lie algebra gen. by \( H_1 \). The point is that \( FL \) \( H_1 \) has a mixed Hodge structure, the integrality coming from topology - Chen + Sullivan + Morgan; really should say rational lattice. On the other hand upon integrating the DE one gets a transcendental function (essentially the polylogarithm). Then one gets another rational structure from the natural diff. form basis times \( 2\pi i \) factors. Bannakrishna computes the ratio and gets \( S(\text{integer}) \mod \mathbb{Q}^\star \).
an error term due to wild ramification. The formula involves the multiplicity of the Swan map, and this is zero if the ramification is tame.

Analogy between wild ramification in char. $p$ and irregular singular points over $C$. For example at an irregular singular point one can define an integer (called the Swan number) which measures the irregularity. Two approaches:

1) Look for largest lattice $\mathbb{Z}^n$ stable under $z_n \frac{2 \partial}{\partial \bar{z}}$ ($n \in \mathbb{Q}$ by adjoining $\mathbb{Z}^{2n}$), get a Newton polygon and compare slopes. 2) Take $\mathbb{D}$-module viewpoint and look at the characteristic variety (which sits in the bundle $T^*(\log z)$ dual to the operator $z \frac{d}{dz}$). Then the Swan is the multiplicity of the vertical component over $z = 0$.

Fourier transform for $\mathbb{D}$-modules: In the case of $C$ a $\mathbb{D}$-module is simply a f.g. module over the Weyl group $W(x, \frac{dx}{x})$. Its F.T. is obtained by $\frac{d}{dx} \mapsto -x, x \mapsto \frac{d}{dx}$.

A general definition is possible using the appropriate derived category language.

Over a finite field a sheaf gives a function on the set of rational points, and then the function belonging to the F.T. is the F.T. of the function.

Example: Take the sheaf $F$ on the line extension of zero from the sheaf over $\mathbb{G}_m$ associated to a complex character of $\mathbb{G}_m$ and the principal $\mathbb{G}_m$-bundle $0 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0$. Maybe this should be: leaking.

Then the F.T. of the sheaf $F$ is essentially itself, twisted by a line bundle over the finite field. The corresponding formula involves the Gaussian sum, which occurs because of this line bundle.
February 3, 1984

Connes: before lunch: if \( \Gamma \) is torsion-free, conjecture
\[ H^* (\Gamma) \cong H^* (\Sigma \Gamma) \text{, important to have statements using} \]
the group ring as an algebra without the augmentation given by the trivial repns because \( C_0(\Gamma) \) doesn't map to \( C \), example of free group.

If \( G \) acts on \( A \), then \( A \times G \) admits a dual action of \( G \). Dual action can be described in three equivalent ways:

1) using positive functionals
2) using representations: A dual action of \( G \) on \( B \) allows one to associate to each \( \theta \) representation of \( G \) a \( B \)-bimodule, so \( R(G) \rightarrow KK(B,B) \) is an alg. map.
3) Using Hoff algebra type maps: \( B \rightarrow B \otimes C(G) \).

For a circle bundle \( P \rightarrow M \), let \( A = C(P) \).
Then \( C(P) \times G \cong C(M) \). We can now define an element of \( KK(M,M) \) as follows

\[ K \rightarrow KC(M) \xrightarrow{\text{inclusion of the center}} KC(P) \times G \xrightarrow{\text{action of the generator of } G = \mathbb{Z}} KC(M) \]

On \( K \)-theory and also on cohomology this is multiplication by \((1, \text{ euler class})\).

Problem of the conjecture \( K_* (\mathbb{B} \Gamma) = K_* (\mathbb{C} \Gamma) \) when \( \Gamma \) not torsion-free. Here one replaces \( K_* (\mathbb{B} \Gamma) \) by a geometric group consisting of classes of proper \( \Gamma \)-actions on \((M,E)\), instead of free actions. Gives correct answer for \( G \) finite.

Noiriall conjecture is improved for elements of \( H^* (\mathbb{B} \Gamma) \) which are represented by the following "geometric cycles for \( \Gamma \)":

\( (V, E^\pm, \sigma, \omega) \), where \( V \) is a \( \Gamma \)-manifold
$E^\pm$ are $\Gamma$-bundles over $V$, $\sigma$ = degree 1 map on $E = E^+ \oplus E^-$, which is an isomorphism outside a compact subset, and such that $\sigma - \sigma^3 \to 0$ at $\infty$, $\omega$ is an odd form on $V^\Gamma$-invariant.

Here is how such a geometric cycle for $\Gamma$ determines an element of $H^*(B\Gamma)$. Form over $P_\Gamma \times \Gamma V$ the $K$-element represented by $P_\Gamma \times P E^\pm$ and the map $\sigma$. It will have support proper over $B\Gamma$, hence so will its character. Then as $\omega$ is an odd form $\Gamma$-inv, it gives an odd form on $P_\Gamma \times \Gamma V$, so it makes sense to integrate over the fibre:

$$\pi_\ast (\text{ch}(E^\pm, \sigma) \omega) \in H^*(B\Gamma).$$

Groups cocycles and cyclic cocycles. $G$ Lie gp, van Est:

$$H^n_{\text{ant}}(G, \mathbb{R}) = H^n(\Omega(G/K)^G)$$

Let $c(g_1, \ldots, g^n)$ be a differentiable group cocycle. Then it defines a cyclic cocycle on $C_c^\infty(G)$ by

$$\tau(f_0, f_1, \ldots, f^n) = \int f_0(g_0) \cdots f^n(g^n) c(g_1, g_2, \ldots, g^n) \, dg_1 \cdots dg^n$$

Haar measure.

Somedewhere there is a pairing $\langle \cdot, \cdot \rangle$ with representations of $G$. For example if one started with $1 \in H^0(\Omega(G/K)^G)$, then $\tau(f_0) = f_0(e)$ is somehow related to the Plancherel formula.

Connes says my character formula should be useful in the transversal elliptic operators for the action of a compact group. The symbol is defined on a singular
subset of $T^*$. Explicit formulas are lacking when the fibres jump, even for $S^1$.

Back to $G$ Lie gp. If $G = SL_2(\mathbb{C})$, then the dual of $G$ is the space of principal series:

union of lines except for some stuff with the Weyl gp. Then one has the $\zeta$-volume class in $H^3(G/\mathbb{R}^G)$. One has from $K_x(C_\pi(G)) = K_x(G)$, and the fact $\mathcal{G}$ is a union of lines, various elements in $H_1(\mathcal{G})$. The pairing somehow involves $\mathcal{S}$. 
Connes: Lunch & at tea

He explained to Deligne + Grojnow how Kasparov improved
Douglas etc. to include the even case. The rough
idea: start with a BDF extension, say a map $A \to C^*$-algebra.
Then under certain conditions (A a nuclear $C^*$-algebra)
one can lift this map to a positive map $A \to B(H)$.
Now use Stinespring to construct a larger Hilbert space $H'$
such that $\tilde{\gamma}$ is the compression of a representation of $A$
in $H'$. I think this means that $\gamma = \text{even}$ where

$v: A \to B(H')$ is a homomorphism of $\gamma$ algebras and $v$
is the projection of $H'$ onto $H$. Then $v$ commutes
with $A$ modulo compact, so we now have an odd
Kasparov element.

Note that the effect of the above construction is
to produce the additive inverse of the original BDF
extension.

A $C^*$-algebra $A$ is nuclear if its tensor product with
another $C^*$-algebra $B$ has a unique norm, e.g. $C^*(\Gamma)$ where
$\Gamma$ is amenable, $\Gamma$. Amenable means that given any finite
subset $S$ and $\epsilon > 0$ there is a larger finite set $X$ such
that the size of $(X - X^a) \cup (X^a - X)$ divided by
the size of $X$ is $< \epsilon$, for each $s \in S$. Solvable groups
are amenable, but free groups are not. The reduced
and maximal group rings are equal $\iff$ group is
amenable. Connes claim $C^*(\Gamma)$ is the correct thing since
you have control over the coefficients of an

I asked whether you lose representations not contained
in the regular repn. He replied given one such $H_{\Pi}$
you can consider $H_{\Pi} \otimes l^2(\Gamma)$ as a bimodule over $C^*(\Gamma)$. 
Further example: Condition $T$ says trivial repn. is isolated in the regular repn. Introduce

$$R(\Gamma) = \text{von Neumann alg} = \text{commutant of right action of } \Gamma \text{ on } L^2(\Gamma).$$

This is of type $\text{II}_1$, (unless $\Gamma$ has an abelian subgroup of finite index). Claim that $R(\Gamma)$ satisfies $T \iff \Gamma$ satisfies $T$. Here we consider $L^2(\Gamma)$ as a bimodule over $R(\Gamma)$ as the analogue of the trivial representation. By looking at coefficients one can describe what it means for bimodules to be closed. Application: Take $\Gamma = SL_3(\mathbb{Z})$ which satisfies $T$, then $R(\Gamma)$ is of type $\text{II}_1$ but is not isom. to $R(\Gamma) \otimes M_2$.

Wave packet transform, string G"arding inequality, positive lifting, blowup of $V \times V$ along $\Delta V$, Maslov formula + Lagrangian submanifolds as $h \to 0$.

Wave packet transform. To each element of $T^*$ one constructs a wave packet. Then one uses these wave packets to decompose the identity operator into rank one operators

$$\text{Id} = \int_{T^*} \delta_1 \times s |_{S \in T^*} \quad d^2\nu = \text{symplectic volume}$$

Enough to do approximately with a partition of 1, then using $\sqrt{\cdot}$ you can make it exact.

String G"arding inequality proved by Friedrichs. Given $A$ with positive symbol, one introduces

$$B = \int_{T^*} \delta_{t > a(s)} |_{S \in T^*}$$
so that $B > 0$ and $A - B$ is lower order. In fact we get an estimate
\[
(f, Bf) = \int d^2x \ l(a(\xi)) \left\langle K \phi \right| f \right\rangle^2
\]
from which one can compare $(f, Af)$ and derive the inequality.

The basic decomposition $(x)$ provides a quantization process. In fact it gives a positive lifting
\[
\begin{array}{c}
\text{functions on } T^* \\
\longrightarrow \\
\text{operators on } L^2
\end{array}
\]
which is a concrete realization of the Stinespring + Kasparov business (Kengo says to look at Kasparov’s version in the Romanian Journal of Operator Theory.)

Asymptotic differential operators: We start with the groupoid obtained from $V \times V \times [0, 1]$ consisting of $x, y, \varepsilon$ by adjoining $T^* \times \{0\}$ as the limit of $x \cdot \varepsilon$ as $\varepsilon \to 0$. The point is now to study asymptotic behavior of any operator as $\varepsilon \to 0$. Somehow what happens is the Maslov theory in which what occurs in the classical limit is a Lagrangian submanifold of $T^*$, phase problems along this submanifold, and some recursive calculation of coefficients in the asymptotic expansion.

Geometric proof of the index thm: Connes proposes to replace all $C^*$ algebras by geometrically defined groups. There is a functor $\mu$ from the geometrical to the analytic (or $C^*$) $K$-groups, so when one has a geometric isomorphism one gets also an analytic isom.
Discussion of the Novikov conjecture. Think of defining for a manifold $\hat{\Pi}$ with fundamental group $\Pi$ an equivariant signature. This lies in the Witt group $W(\mathbb{C} \hat{\Pi})$, and is obviously a homotopy invariant of the operator. On the other hand, the equivariant signature operator of the manifold pushes forward under the classifying map $f: M \to B\hat{\Pi}$ to give an element of $K_0(BG)$ which rationally is $H^*_*(BG)$.

Somehow the problem comes down to mapping the Witt ring to $K$-theory, i.e., taking a quadratic form and splitting it into positive and negative bundles. This has to be done via Cauchy's theorem so can't be done over $\mathbb{C} \hat{\Pi}$ but rather something analytical like $C_k(\hat{\Pi})$. Thus one gets a map

$$W(\mathbb{C} \hat{\Pi}) \to K_0(C_k(\hat{\Pi}))$$

such that the two signatures have the same image:

$$W(\mathbb{C} \Pi) \to K_0(C_k(\Pi)) = KK(\mathbb{C}, C_k(\Pi)) \to K_0(B\Pi)$$

Novikov follows from injectivity of $\mu$.

In turn we can try to construct enough elements of $KK(C_k(\Pi), \mathbb{C})$ (generalized representations of $\Pi$) to detect non-zero elements of $K_0(B\Pi)$. Thus it is really a question of generating $H^*(B\Pi)$ by "reps. of $\Pi".

Connes claims that the Novikov conjecture is true for the following cohomology classes of $\Pi$. Take a
family of representations of $\mathbb{T}$ parameterized by a space $X$; the character of the family is a cohomology class on $X \times B\mathbb{T}$, and capping with a homology class on $X$ gives a cohomology class of $\mathbb{T}$.

(This seems to be a less general form of his "geometric cycles for $\mathbb{T}$" where $\mathbb{T}$ is allowed to act on $X$ and one takes a $\mathbb{T}$-invariant chain on $X$, see p. 444.)


Let $M$ be a compact oriented manifold of dim = 0 mod 4. Given a vector bundle $E$ over $M$ I can twist the signature operator on forms and get an index. This is given by

$$\text{index} = \int_M \text{ch}(E) L(M)$$

where $L(M)$ is the $L$-genus. These indices are supposedly not usually homotopy-invariant in the sense that given a heg $f: M' \to M$, then the index of $M', f^*E$ is not the same as that for $M, E$.

Actually I first should have taken a cohomology class $\alpha$ on $M$ and looked at the number

$$\int_{M'} f^* \alpha \cdot L(M') = \int_M \alpha \cdot f_* L(M')$$

for $f: M' \to M$ a heg. Rationally, $\alpha$ is realized by an embedded submanifold with trivial normal bundle, or more accurately a framed map $N \to M$, and then the number is the signature of $f^{-1}N$. This is in fact the way Thom defines rational Pontryagin classes.

The Novikov conjecture asserts the homotopy invariance of the above number ($\star$) when $\alpha$ comes from $H^*(\mathbb{B}\pi)$ $\pi$ = fundamental group of $M$. Equivalently that $\mathbb{Z}$
\[ \chi_L(M) \in H^*(\mathbb{B}_\pi) \] and \( \phi : M \to \mathbb{B}_\pi \) the canonical map is a homotopy-invariant.

Now if \( E \) is a flat unitary bundle on \( M \), then I think the cohomology \( H(M, E) \) in the middle dimension is a finite-dimensional vector space with hermitian inner product, so there is a signature. This signature is given by the index of the signature operator, but perhaps coincide with \( \text{sign}(M) \cdot \text{rank} E \) as \( \chi(E) = \text{rank} E \) for flat bundles.

Question: What can we say when we have a family of flat unitary bundles?

So we suppose given a family of representations of \( \pi \) parametrized by \( \Lambda \). Then we get a bundle \( \tilde{E} \) over \( Y \times M \) with partial flat connection in the \( M \)-direction. Then I can do two things:

1) twist the signature operator on \( M \) with \( \tilde{E} \) to get a family of operators parametrized by \( Y \). The index is an element of \( K(Y) \) which we get from the index theorem for families:

\[ \chi(\text{index}) = \int_M \chi(E) \xi(M) \in H^*(Y) \]

If \( \chi(E) \in H^*(Y \times M) = H^*(Y) \otimes H^*(M) \), then there are \( x \in H^*(M) \) which are obtained from \( \chi(E) \) and cycles on \( Y \). So if \( \chi(\text{index}) \) is a homotopy-invariant of \( M \) then we see \( x \) for these \( E \) is homotopy-invarient. This follows from the second way of getting at \( \chi(\text{index}) \).

2) We look at the \( K \)-element on \( Y \) obtained from the family of cochains \( \alpha^M \) with coefficients in \( E_\pi \) with its hermitian structure. This is the topological signature for the family. (Lüst's the work)
February 4, 1984

I want to go over Correa's method of extending a cyclic cocycle on $A$ to one on $\text{End}_A(E)$, where $E$ is a finite projective module over $A$. I have the idea that this can be understood by analogy with defining invariant forms on the gauge group associated to an arbitrary, not necessarily trivial, vector bundle.

Review the construction: Let $E$ be a vector bundle over $M$, and $G$ be the group of automorphisms, say unitary, relative to some inner product. Let $\mathfrak{g} = \text{Lie}(G)$. In order to produce Lie algebra forms, let $\mathfrak{g}$ be the space of invariant differential forms on $G$, we consider a flat connection on the trivial $G$-bundle over $Y$. This is the same as giving $\theta \in \Omega^1(Y, \Omega^0(M, \text{End}(E)))$ skew-adjoint such that $d^{\prime} + \theta^2 = 0$. Then $d^{\prime} + \theta$ can be viewed as a flat partial connection on the bundle $\text{pr}_2^*(E)$ over $Y \times M$ in the $Y$-direction. Now choose a connection on $E$, call it $D$, and let $D''$ be the obvious vertical connection on $\text{pr}_2^*(E)$ which is the pull-back of $D$. Then we get two connections $d^{\prime} + D''$, $\theta + D''$ on $\text{pr}_2^*(E)$ over $Y \times M$ which are flat in the $Y$-direction.

A better notation seems as follows. Write $d = d^{\prime} + d^\prime$ over $Y \times M$, where $d^\prime = dy$, $d = dM$. Then the pull-back of $D$ on $E$ to $\text{pr}_2^*(E)$ is $\tilde{D} = d + D$ and we consider the 1-parameter family of connections $\tilde{D} + t\theta$.

The curvature is
\[ \tilde{D}^2 + t\tilde{D}\theta + t^2\theta^2 = D^2 + tD\theta + (t^2 - t)\theta^2. \]
Hence the basic formula is
\[ \text{tr}(e^{D^2 + t\omega}) - \text{tr}(e^{D^2}) = \int_0^1 dt \text{ tr}(e^{D^2 + t\omega + (\omega - t)\omega^2}) \]

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Viewpoint: I am reviewing the way to construct invariant forms on \( G \). I recall having too ways to do this, one being slightly different from the other, which leads to left-invariant forms, the other being the one I used in the letter tolinger which produced right invariant forms.

Now ultimately the idea is that we produce two connections on \( \mathfrak{pr}^\sharp(E) \) over \( G \times M \) which are partially flat in the \( G \)-direction. Then the Bott theorem gives vanishing for characteristic classes, and so we get difference elements. This means that the characteristic class gives a form on \( G \times M \) of dimension \( 2n-1 \) whose boundary has filtration \( > n \). Hence integrating over a \( p \)-cycle in \( M \) with \( p < n \) will give a closed form over \( G \).

To start with a \( p \)-cycle \( \gamma \) in \( M \). Then pick an \( n \) and we get a closed invariant form on \( G \) of degree \( 2n-1-p \). It should be a cyclic \( 2n-1-p-1 \) cocycle. (It depends on \( 2n-1-p \) elements of \( \mathfrak{g} \) and \( \omega \) is a \( 2n-1-p-1 \) cochain on the ring \( \Omega^0(M, End E) \).) The obvious choice seems to be to take \( n = p+1 \), for then \( n > p \) and \( 2n-2-p = p \), so that the \( p \)-cycle \( \gamma \) gives rise to a \( p \)-cocycle

\[ \frac{1}{p!} \int_0^1 dt \int_0^\gamma \text{ tr}(e^{D^2 + t\omega + (\omega - t)\omega^2})^p \omega \]
Now \( \Theta^2 \in C^2(\mathcal{F}_0, \Omega^0(M, \text{End } E)) \)
\( D\Theta \in C^1(\mathcal{F}_0, \Omega^1(M, \text{End } E)) \)
\( D^2 \in C^0(\mathcal{F}_0, \Omega^2(M, \text{End } E)) \)

and in
\[
\int_0^1 \int_\mathcal{F}_p t \operatorname{tr} \left( D^2 + t D\Theta + (t^2 - t) \Theta^2 \right) \Theta \frac{1}{p!} \]

we want to collect terms of degree \( p \) in \( \mathcal{F}_p \) so as to get a non-trivial integral over \( \mathcal{F}_p \). These will be of the following sort

\[ p \text{ D}\Theta \text{'s} \quad \int_0^1 \! dt \operatorname{tr} \left( D\Theta \right)^p t^p = \frac{1}{(p+1)!} \operatorname{tr} \Theta (D\Theta)^p \]

\[ p-2 \text{ D}\Theta \text{'s} \; \text{; this will have symmetrized products of} \]
\( D^2 \), \( (D\Theta)^{p-2} \), \( \Theta^2 \)

\[ p-4 \text{ D}\Theta \text{'s} \; \text{; this will involve} \]
\( (D^2)^2 \), \( (D\Theta)^{p-4} \), \( \Theta^2 \) etc.

Notice that there are \( p \Theta \)-terms as there should be.

Now I can’t seem to see a simple formula for this thing so I should look at Connes approach.