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November 14, 1984

Go over heat operators – your ideas on a geometric approach to heat operators. I decided that the singularity in a heat kernel $K(t,x,x')$ is essentially the same as the singularity in a kernel on the tangent groupoid $K(h,x,x')$, and that one should first treat the tangent groupoid case.

I want to work this all out over a compact manifold M which I will suppose to be a torus V/Γ to begin with. Then I can restrict to translation-invariant operators.

The first thing to get straight is the difference between functions and densities. Let's suppose our operators work on functions on M. Then the Schwartz kernel of an operator [redacted] is a section over $M \times M$ of $pr_2^* \omega$ where $\omega = \left|\Lambda^{max} T^*_M\right|$ is the bundle of densities:

$$K(x,x')\,|dx'|$$

Translation invariance means $K(x,x') = K(x-x')$. Thus a translation-invariant operator on functions on M is given by a density on M:

$$f \longmapsto (Kf)(x) = \int K(x')|dx'|\, f(x-x')$$

$$= \int |dx'|\, K(x-x') f(x')$$

Composition of operators corresponds to convolution of densities

$$(KLf)(x) = \int |dx'|\, K(x-x')\underbrace{(Lf)(x')}_{\int |dx''|\, L(x'-x'') f(x'')}$$

$$= \int |dx''| \left\{ \int |dx'| \, K(x - x') L(x' - x'') \right\} f(x'') \qquad x' \to x' + x''$$

$$= \int |dx''| \left\{ \int |dx'| \, K(x - x'' - x') L(x') \right\} f(x'')$$

So

$$|dx| K(x) * |dx| L(x) = |dx| \int |dx'| \, K(x - x') L(x')$$

and a more sensible way to do this is to consider the sum map $M \times M \xrightarrow{+} M$, and say one takes the product of the two densities and pushes forward.

Next consider a density depending on h, $|dx| K(h, x)$, defined for $h \neq 0$. We blowup $\mathbb{R} \times M$ at $(0,0)$ and assume the density $|dx| K(h,x)$ extends smoothly to a section of $pr_2^*(\omega_M)$ over $\widetilde{\mathbb{R} \times M}$, which vanishes to infinite order along $\tilde{M}$.

Note that this implies for $x \neq 0$ that $K(h,x) \to 0$ faster than any power of h.

How do we describe the blowup $\widetilde{\mathbb{R} \times M}$? We take an open nbd of $(0,0)$ which we can identify with an open subset of the tangent space at that point. In this case take the open set to be $\mathbb{R} \times U$, where U is a nbd. of $0 \in M$. Identify U with a nbd. of 0 in V. Then we define

$$\widetilde{\mathbb{R} \times V} = \{ (\ell, w) \mid \ell \text{ line in } \mathbb{R} \times V, w \in \ell \}$$

and let $\widetilde{\mathbb{R} \times U} = p_2^{-1}(\mathbb{R} \times U)$. Then

$$\widetilde{\mathbb{R} \times M} = \widetilde{\mathbb{R} \times U} \cup \{ \mathbb{R} \times M - (0,0) \}$$

Next consider $|dx| K(h, x)$ and ask what it means for this density to extend smoothly over $\widetilde{R\times M}$ vanishing to infinite order on $\tilde{M}$. Clearly we can suppose that $K(h,x)$ is supported in $R_{\geq 0}\times U$ by multiplying by $\rho(x)\in C_0^\infty(U)$, $\rho\equiv 1$ near 0.

Then we just have to look at $\widetilde{R\times U}\subset\widetilde{R\times V}$. Now $\widetilde{R\times V}-\tilde{V}$ consists of (ℓ,w) with $\ell\subset R\times V$ not in V, hence $\ell = R(1,\tilde{v})$ for a unique $v\in V$ and $w = (h,hv)$. Thus

$$\widetilde{R\times V}-\tilde{V}\simeq R\times V$$

and under this isomorphism $|dx|\,K(h,x)$ becomes

$$h^n\,|dv|\,K(h,hv).$$

What should be the assertion? I could ask for smooth sections on $\widetilde{R\times V}$ of $pr_2^*(\omega_V)$ which vanish to infinite order along $\tilde{V}$.

Can we characterize ΨDO's by requiring that the kernel $K(x,x')$ be resolved by blowing up the diagonal? Look at the translation invariant case:

$$K(x)=\int\frac{d^n\xi}{(2\pi)^n}\,e^{i\xi x}\,f(\xi)$$

Let's use blowup at 0 where the fibre over 0 is the unit sphere, i.e., a point of the blowup is a ray and a point on the ray. Thus it is $R_{\geq 0}\times S^{n-1}$, a manifold with boundary and the map is $(r,u)\longrightarrow ru=x$ from $R_{\geq 0}\times S^{n-1}$ to R^n. Requiring K to extend to a smooth function on the blowup means that there is an asymptotic expansion of some sort as $r\to 0$.

$$K(ru) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi r u} f(\xi)$$

$$= \int \frac{d^n\xi}{(2\pi r)^n}\, e^{i\xi u} f\left(\frac{\xi}{r}\right)$$

Now in the definition $f(\xi) \sim \sum f_k(\xi)$ where f_k is homogeneous of degree $-k$. Then

$$r^n K(ru) \sim \sum_k \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi u} f_k\left(\frac{\xi}{r}\right) = \sum_k r^k \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi u} f_k(\xi)$$

should be an asymptotic expansion in positive powers of r, whose coefficients are functions on S^{n-1}. This is exactly what we would expect from a smooth function on $\mathbb{R}_{>0} \times S^{n-1}$.

However the above heuristics leave out the log factors which occur e.g. for $\Delta^{-1} = c \log r$ in 2 dimensions. So how can I bring these into the geometry? There really ought to be a good way since ultimately one does have the expansion as one goes to infinity in T^*.

One should first understand the Fourier transform on homogeneous distributions on $\mathbb{R}^n$.

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Let $\mathcal{D}_d$ denote the homogeneous distributions on $\mathbb{R}^n$ which are smooth outside 0 and let $\mathcal{F}_d$ be the smooth homogeneous functions of degree d on $\mathbb{R}^n-(0)$. We have a map

$$\mathcal{D}_d \longrightarrow \mathcal{F}_d$$

which is an isomorphism when $d > -n$, since any element of $\mathcal{F}_d$ is integrable, and since the distributions supported at 0 are derivatives of δ which have degree $-n, -n-1, -n-2, \ldots$. The Fourier transform gives an isomorphism

$$\mathcal{D}_d \xrightarrow{\sim} \mathcal{D}_{-n-d}$$

since

$$f(tx) = \int \frac{d^n\xi}{(2\pi)^n} e^{i\xi tx} \hat{f}_d(\xi) = \int \frac{d^n\xi}{(2\pi)^n} t^{-n} e^{i\xi x} \hat{f}_d(t^{-1}\xi)$$
$$= t^{-n-d} f(x).$$

Picture: x side $\qquad\qquad$ ξ side

$d \geq 0$

$$\begin{tikzcd}
 & 0 \arrow[d] \\
S_d(V) \arrow[r, equal] \arrow[d, phantom, "\cap"] & \{\partial^\alpha\delta\} \arrow[d] \\
\mathcal{D}_d \arrow[d, equal] & \mathcal{D}_{-n-d} \arrow[l, "\sim"'] \arrow[d] \\
\mathcal{F}_d & \mathcal{F}_{-n-d} \arrow[d] \\
 & C_{-n-d} \arrow[d] \\
 & 0
\end{tikzcd}$$

One should think in terms of taking the inverse Fourier transform of $\hat{f}(\xi)$, where f is of degree $-n-d$. This one would expect to be a function $f(x) \in \mathcal{F}_d = \mathcal{D}_d$, but $\hat{f} \in \mathcal{F}_{-n-d}$ can't be extended necessarily to a homogeneous distribution.

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, \hat{f}(\xi) = c\int_0^\infty r^{n-1}dr \int_{S^{n-1}} du\; e^{i\,r(u\cdot x)}\, \underbrace{\hat{f}(ru)}_{r^{-d-n}\hat{f}(u)}$$

$$= c\int_0^\infty \frac{dr}{r}\, r^{-d} \left\{ \int_{S^{n-1}} du\; e^{ir(u\cdot x)}\, \hat{f}(u) \right\}$$

Stationary phase implies that the integral in braces is, as $r \to \infty$, like

$$\frac{e^{\pm i r|x|}}{r^{\frac{n-1}{2}}}$$

so that except for $n=1$, $d=0$, there is absolute convergence at the $r\to\infty$ end. For $n=1$, $d=0$ one has

$$\int_0^\infty \frac{dr}{r}\left(e^{irx}\,\hat{f}(1) + e^{-irx}\hat{f}(-1)\right)$$

which are convergent at the $r\to\infty$ end.

At the $r=0$ end there is trouble unless

$$\int_{S^{n-1}} du\; e^{ir(u\cdot x)}\, \hat{f}(u) = \mathcal{O}(r^d)$$

so for $d=0$ one has to have $\int_{S^{n-1}} du\, \hat{f}(u) = 0$, and for $d \geq 0$, d integral, one wants the moments

$$\int_{S^{n-1}} du\;\; u^\alpha \hat{f}(u) = 0 \qquad |\alpha| \leq d.$$

This probably shows $\mathcal{F}_{-n-d} / C_{-n-d}$ is of the same size as S_d.

So what remains is to describe precisely the singularity obtained by regularizing. What I mean is the following. Take $d=0$ and $\hat{f}(\xi) = \frac{1}{|\xi|^n}$ so that

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{-i\xi x}\, \frac{1}{|\xi|^n}$$

is logarithmically divergent. The standard way to regularize this is to choose $\rho(|\xi|) = 0$ near $\xi=0$ $=1$ for out and form

$$\int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x}\, \frac{\rho(|\xi|)}{|\xi|^n}$$

The choice of ρ is irrelevant to the singularity $\Delta\rho$ should be such that $\frac{\Delta\rho}{|\xi|^n}$ has smooth transform at the origin which is a condition on the growth as $|\xi| \to \infty$.

But actually what you should really be doing is to understand the process of extending $\frac{1}{|\xi|^n}$ to a distribution. So you really shouldn't have to introduce ρ at all.

Let's try to understand kernels on the tangent groupoid and why they can be composed. The key idea is to think in terms of a groupoid, so there are an object manifold and an arrow manifold. Thus we have

$$\begin{tikzcd} & \mathbb{R}\times M\times M \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{R}\times M & & \mathbb{R}\times M \end{tikzcd}$$

and we get an operator on functions on the object manifold. The basic rule is that

$$(Kf)(x) = \int K(x,x')f(x')$$

so that the operator is $(p_1)_* \cdot K \cdot p_2^*$. If we are thinking of an operator on functions, then in order to do $(p_1)_*$ we need to have a density on the fibres of p_1. So it is necessary that we have a section of the bundle of relative densities for the map p_1. In our case this means K is

$$K(h,x,x')\,dx'$$

and we want to treat the blowup as fibred over $\mathbb{R}\times M$ via p_1, which sends (h,x,x') to (h,x).

It is not clear that p_1 is a fibration. However we do know that the blowup fibres over M via the map $(h,x,x') \longrightarrow x$, and that the fibre over x is the blowup of $\mathbb{R}\times M$ along $(0, x)$. In other words the restriction of $\mathbb{R}\times M\times M \xrightarrow{pr_2} M$ to $0\times\Delta M$

is a submersion, better we have that $0 \times \Delta M$ is a section of the map pr_2 and we are blowing up the image of the section. So the ~~[illegible]~~ fibre of $\widetilde{\mathbb{R}\times M\times M}$ over x is the blowup of $\mathbb{R}\times M$ at $(0,x)$. This does not fibre over the h-line since for $h \neq 0$ the fibre is M and for $h=0$ the fibre is the **union** of two divisors $(\tilde{M}$ at $x) \cup T_x(M)$.

What I **do** learn is that I should treat x as a **parameter**, and I should try to understand why

$$\int K(h,x,x')\,dx'\, f(x')$$

is supposed to be a smooth function of h. Let's look at the Euclidean space example. Typical operator is

$$(Kf)(x) = \int \frac{d^n p}{(2\pi h)^n} e^{i\frac{p}{h}x} \hat{K}(h,x,p) \underbrace{\hat{f}(p)}_{\int d^n x' e^{-i\frac{p}{h}x'} f(x')}$$

$$= \underbrace{\int d^n x' \left\{ \int \frac{d^n p}{(2\pi h)^n} e^{i\frac{p}{h}(x-x')} \hat{K}(h,x,p)\right\}}_{d^n x'\, K(h,x,x')} f(x')$$

Notice that if we set $v = \frac{x-x'}{h}$ ~~[illegible]~~ or $x' = x - hv$, $d^n x' = h^n d^n v$, then

$$d^n x'\, K(h,x,x') = d^n v \int \frac{d^n p}{(2\pi)^n} e^{ipv} \hat{K}(h,x,p)$$

which is indeed smooth in h, x, v.

I think this is a hard way to see the

fact that the operators I am interested in are just families of Schwartz functions in v depending smoothly on (h, x).

So summarize: We have learned that because the map $\widetilde{\mathbb{R}\times M\times M} \xrightarrow{p'} \mathbb{R}\times M$ is not a submersion, we have to be careful about integration. We also have learned to use relative densities for this map.

What I have to concentrate on is a density $K(h, x')\,d^n x'$ which extends to $\widetilde{\mathbb{R}\times M}$ = blowup of $\mathbb{R}\times M$ at $(0,0)$, vanishing to infinite order on $\tilde{M}$ and to explain why

$$\int d^n x'\, K(h, x')\, f(x')$$

is smooth in h for $f(x)$ smooth on M.

$$\begin{tikzcd} & \widetilde{\mathbb{R}\times M} \arrow[dl] \arrow[r] & \mathbb{R}\times M \\ \mathbb{R} & & \end{tikzcd}$$

It would seem that I ought to be able to multiply $d^n x'\, K(h, x')$ by a smooth function on $\widetilde{\mathbb{R}\times M}$. Thus the f is irrelevant, and so the problem is just to see why a family of densities on M $d^n x'\, K(h, x')$ defined for $h \neq 0$, which extend to $\widetilde{\mathbb{R}\times M}$ vanishing to infinite order on $\tilde{M}$, will give rise on integration

$$\int_M d^n x'\, K(h, x')$$

to a smooth function of h.

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Gaussian measures: Given a real vector space V, a positive definite quadratic form Q on V determines a unique Gaussian probability measure $d\mu$ on V^* such that $\langle x \rangle =$ $Q(x) = \langle x^2 \rangle$ for all $x \in V$. Here $x \in V$ is interpreted as a function on V^*. The Hilbert space $L^2(V, d\mu)$ is a suitable completion of $S(V_{\mathbb{C}})$. Problem: Do the Gram-Schmidt process on $S(V_{\mathbb{C}}) \subset L^2(V, d\mu)$ to obtain an isomorphism of $L^2(V, d\mu)$ with the symmetric tensor Hilbert space $S(V_{\mathbb{C}})$.

Let's discuss this problem. The idea should be that Q determines a harmonic oscillator Hamiltonian on $L^2(V, dx)$, that $V \oplus V^*$ becomes isomorphic to $V_{\mathbb{C}}$, and that $L^2(V, dx) \simeq S(V_{\mathbb{C}})$ under the holomorphic representation.

The point to emphasize is that a positive definite form Q on V determines a Gaussian measure $d\mu$ on V^* such that $\langle x \rangle = 0$, $\langle x^2 \rangle = Q(x)$ for all $x, y \in V$. The space of 1-particle states of $L^2(V, d\mu)$ can be identified with $V_{\mathbb{C}}$.

Next consider a Gaussian process x_t with parameter t. In this case V consists of functions (real-valued) $f(t)$ and we have an infinite-dimensional setup. The quadratic form is then

$$\|f\|^2 = \int dt\,dt' \; G(t,t') f(t) f(t')$$

where $G(t,t') = \langle x_t x_{t'} \rangle$ is positive-definite

If this process is stationary, i.e. invariant under time translation, then G is a function of $t-t'$ and conversely. Since it is symmetric, G is a function of $|t-t'|$.

The standard Bochner thm. says that

$$G(t,t') = \int$$ [redacted] $$e^{i\omega(t-t')}\, d\mu(\omega)$$

for some measure $d\mu$, which in this case is invariant under $\omega \longrightarrow -\omega$. This results by applying the spectral thm. to the one-parameter unitary group of time translations in [redacted] the Hilbert space obtained by completing $V = \{f(t)\}$ with respect to the inner product.

Evidently Osterwalder-Schrader have found that $G(|t-t'|)$ is positive definite on $\mathbb{R}$ $\Longleftarrow$ $G(t+t')$ is positive definite on $\mathbb{R}_{>0}$.

Standard example of such a $G(t)$ is

$$G(t) = \frac{e^{-\omega_0|t|}}{2\omega_0} = \int \frac{d\omega}{2\pi}\, e^{i\omega t}\, \frac{1}{\omega^2+\omega_0^2}$$

which is a Green's function for $-\frac{d}{dt^2}+\omega_0^2$. Brownian motion doesn't fit into this stationary Gaussian process pattern, since

$$G(t,t') = \langle x_t x_{t'}\rangle = \min\{t,t'\}$$

is given only for $t, t' \geq 0$.

Let's turn to fermions. In analogy with the Gaussian measures where one starts with a quadra[tic] form Q on V, Q positive definite, this time one wants a skew-adjoint operator T which is nondegenerate. Then we have something like

$$\frac{1}{N}\int_{V^*} \mathcal{D}\psi \; e^{\frac{1}{2}\psi T^{-1}\psi} \; \psi_v \psi_{v'} = vT$$[redacted]$$v'$$

Next consider a fermion process ψ_t depending on a parameter t which is Gaussian in the sense that it is given by a skewadjoint operator T. A typical example is where

$$T^{-1} = \frac{d}{dt} + A$$

is a parallel transport operator relative to some connection [redacted] preserving an inner product on a real vector bundle over the t-line.

Notice that in [redacted] order that $\frac{d}{dt} + A$ be invertible we have to have some kind of boundary conditions.

Now a rather important problem from my viewpoint is to understand the process given by the bilinears $\psi_t^\mu \psi_t^\nu$. I want to get this problem formulated carefully. Suppose we deal with $U(1) = SO(2)$ in which case we have two fields $\bar\psi_t, \psi_t$ and the [redacted] functional integral is

$$\int \mathcal{D}\bar\psi \mathcal{D}\psi \; e^{-\int(\bar\psi\dot\psi + \bar\psi A\psi)dt} = \det\left(\frac{d}{dt} + A\right)$$

where the boundary conditions have to be specified.

The simplest boundary condition is periodicity but unfortunately for $A = 0$ this doesn't work, i.e. the operator $\frac{d}{dt}$ is still not invertible. Now I tend to think of $\bar{\psi}_t \psi_t$ as being closely related to Brownian motion x_t. Also I really ought to understand the current operators.

What I should review now is the current operators $\psi^*(x)\psi(x)$ on the Fock space attached to $L^2(S^1)$. This I did: $S^1 = \mathbb{R}/L\mathbb{Z}$, $\langle x|k\rangle = \frac{1}{\sqrt{L}} e^{ikx}$ for $k \in \frac{2\pi}{L}\mathbb{Z}$ is an orthonormal basis for $L^2(S^1, dx)$.

$$\psi(x) = \sum_k \langle x|k\rangle \psi_k = \frac{1}{\sqrt{L}} \sum_k e^{ikx} a_k$$

$$\psi^*(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} a_k^*$$

If f is a function on S^1 its extension to the Fock space is the operator

$$\rho(f) = \sum_{k',k} a_{k'}^* \langle k'|f|k\rangle a_k$$

$$= \frac{1}{L} \sum_q \underbrace{\left(\sum_k a_{k+q}^* a_k \right)}_{\rho_q} \underbrace{\int dx\, e^{-iqx} f(x)}_{\hat{f}(q)}$$

$$\rho_q = \rho(e^{iqx}). \qquad f(x) = \frac{1}{L}\sum_q e^{iqx} \hat{f}(q)$$

$$\rho(f) = \int dx \underbrace{\left(\frac{1}{L} \sum_q \rho_q e^{-iqx} \right)}_{\rho(x) = \psi^*(x)\psi(x)} f(x)$$

Basic commutation relations among the ρ_q. Actually one must define them precisely on the Fock space

$$\begin{cases} \rho_q = \lim_{N\to\infty} \sum_{|k|\le N} a_{k+q}^* a_k & q \neq 0 \\ \rho_0 = \lim_{N\to\infty} \sum_{k>0} a_k^* a_k - \sum_{k\le 0} a_k a_k^* & \end{cases}$$

so that $\rho_0 = 0$ on $\bigwedge_{k \le 0} |k\rangle$. Then

$$[\rho_k, \rho_\ell] = 0 \qquad k+\ell \neq 0$$

$$[\rho_k, \rho_{-k}] = -k \frac{L}{2\pi}$$

$$[\rho(x), \rho(y)] = \frac{1}{2\pi i} \delta'(x-y)$$

$$[\rho(f), \rho(g)] = \frac{i}{2\pi} \int dx\, f'(x) g(x)$$

But I am not interested in these commutation relations directly.

What I have just reviewed concerns the quantum mechanics of the fields $\psi(x)^*$, $\psi(x)$ in one space dimension. It is related to fermion integrals over spaces of functions of t, x, hence to things like determinants of $\partial_{\bar{z}}$.

What I need to understand is fermion integrals over spaces of functions of t.

November 18, 1984

Let's recall the idea of constructing $e^{t\phi^2+\theta\phi}$ as a product of infinitesimal pieces

$$T\left\{ e^{\int \theta_t \text{[redacted]} \phi\, dt} \right\}$$

where θ_t is a 1-parameter family of anti-commuting quantities. What this means is that we have $t \mapsto \theta_t$, $\mathbb{R} \longrightarrow V$, and

$$u(t) = T\left\{ e^{\int_0^t \theta_t \phi\, dt} \right\} \quad \in \quad \Lambda V \otimes \mathrm{End}(H)$$

is the solution of

$$\partial_t u(t) = \theta_t \phi\, u(t)$$
$$u(0) = 1.$$

It's clear from

$$e^{\theta_1\phi}\, e^{\theta_2\phi} = e^{\frac{1}{2}[\theta_1\phi,\, \theta_2\phi]}\, e^{(\theta_1+\theta_2)\phi}$$
$$= e^{-\theta_1\theta_2\phi^2}\, e^{(\theta_1+\theta_2)\phi}$$

that we have

$$T\left\{ e^{\int \theta_t \phi\, dt} \right\} = e^{\left(-\int_{t_1>t_2} \theta_{t_1}\theta_{t_2}\, dt_1 dt_2\right)\phi^2}\, e^{\left(\int \theta_t dt\right)\phi}$$

It is impossible to see the positivety requirement of t in $e^{t\phi^2+\theta\phi}$ in this way. This point is the problem with this approach to the constructeon of $e^{t\phi^2+\theta\phi}$.

Review Bismut's separation of parallel transport in $S \otimes E$ using Brownian motion.

Suppose we have trivial bundles over $\mathbb{R}$ with fibres S, E and we wish to solve

$$\frac{dU_t}{dt} = U_t\left(\sum_a L_a \otimes M_a\right)$$

$$U_0 = I$$

where $L_a(t), M_a(t)$ are endos of S, E. Introduce independent Brownian motions w^a and solve the Itô diffl equations

$$dU_t' = U_t' \sum_a L_a \, dw^a \qquad U_0' = I$$

$$dU_t'' = U_t'' \sum_a M_a \, dw^a \qquad U_0'' = I$$

Then

$$U_t = \langle U_t' \otimes U_t'' \rangle$$

Proof: $d(U' \otimes U'') = dU' \otimes U'' + U' \otimes dU'' + dU' \otimes dU''$

$$= (U' \otimes U'')\left(\sum_a L_a \, dw^a\right) \otimes I + (U' \otimes U'')\left(I \otimes \sum_b M_b \, dw^b\right)$$

$$+ (U' \otimes U'') \sum_{a,b} (L_a \otimes M_b) \underbrace{dw^a dw^b}_{dt\, \delta^{ab}}$$

by rules of the Itô calculus

Taking expectations $\quad d\langle U' \otimes U'' \rangle = \langle U' \otimes U'' \rangle \left(\sum_a L_a \otimes M_a\right).$

QED.

In the (probabilistic or physicists) functional integral for the index appears a parallel transport term with potential corresponding to the part $-\frac{R}{4}+\frac{1}{2}\gamma\gamma F$ in the formula for $\not{D}^2$. Let's take the case of a line bundle with constant curvature over a 2-torus. The fermion expression for this term is

$$\int D\psi \; e^{-\int_0^1 \left(\frac{1}{4}\psi\dot{\psi}-\frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}\right)dt} = \operatorname{tr}_s\left(e^{\frac{1}{2}F_{\mu\nu}\gamma^\mu\gamma^\nu}\right)$$

which we evaluated (take $n=2$) on p. 183 and found to be

$$e^{iF}-e^{-iF}.$$

The expression used by Bismut is apparently

$$\operatorname{tr}_s T\left\{e^{\frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}}\right\}$$

$$=\int_W \operatorname{tr}_s T\left\{e^{\int_0^1\frac{1}{2}\gamma^\mu\gamma^\nu dw_{\mu\nu}}\right\} T\left\{e^{\int_0^1\frac{1}{2}F_{\mu\nu}\,dw_{\mu\nu}}\right\}$$

Let's check. Note that $\frac{1}{2}$'s appear in both because it's a sum over $\mu<\nu$ really. In the second exponential we have just (now take $n=2$)

$$T\left\{e^{\int_0^1 F\,dw}\right\} \qquad \begin{aligned} F&=F_{12}\\ dw&=dw_{12}.\end{aligned}$$

which we have evaluated before as follows. Put

$$U_t=T\left\{e^{\int_0^t F\,dw}\right\}$$

so that

$$dU_t = U_t F\,dw.$$

This is an Ito D.E. To solve use

$$d\log U_t = \frac{1}{U_t}dU_t - \frac{1}{2U_t^2}(dU_t)^2 = F\,dw - \frac{1}{2}F^2dt$$

whence $$\log U_t = F\omega_t - \tfrac{1}{2}F^2 t$$

so $$T\left\{ e^{\int_0^t F\,d\omega} \right\} = e^{F\omega_t - \frac{F^2}{2}t}$$

Similarly $$T\left\{ e^{\int_0^t \gamma^1\gamma^2\,d\omega} \right\} = e^{(\gamma^1\gamma^2)\omega_t - \frac{t}{2}(\gamma^1\gamma^2)^2}$$
$$= e^{i\varepsilon\omega_t + \frac{t}{2}}$$ and so
$$\mathrm{tr}_s\, T\left\{ e^{\int_0^1 \gamma^1\gamma^2\,d\omega} \right\} = e^{\frac{1}{2}}\left(e^{i\omega_1} - e^{-i\omega_1}\right).$$

So now we need to evaluate
$$\int_W e^{\frac{1}{2}}\left(e^{i\omega_1} - e^{-i\omega_1}\right) e^{F\omega_1 - \frac{F^2}{2}}$$
Put $F = ib$, $b \in \mathbb{R}$
$$= \int_W \left[e^{i(1+b)\omega_1} - e^{i(-1+b)\omega_1} \right] e^{\frac{1+b^2}{2}}.$$
But ω_1 is a Gaussian variable of mean 0 and variance 1:
$$= \int dx\, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left[e^{i(1+b)x} - e^{i(-1+b)x} \right] e^{\frac{1}{2}(1+b^2)}$$
$$= \left[e^{-\frac{1}{2}(1+b)^2} - e^{-\frac{1}{2}(-1+b)^2} \right] e^{\frac{1}{2}(1+b^2)} = e^{-b} - e^{+b}$$
$$= e^{iF} - e^{-iF}$$ which checks.

Now in the general case
$$T\left\{ e^{\int_0^t \frac{1}{2}\gamma^\mu\gamma^\nu\,d\omega_{\mu\nu}} \right\}$$
will be essentially the heat kernel in the spinor group, modulo problems with its going outside $\mathrm{Spin}(n)$.

November 19, 1984.

Summary of yesterday's work:

The problem was to clarify the link (if it exists) between the fermion integral in the physicists functional integral expression for the index, [redacted] the use by Bismut of Brownian motion in Lie(Spin(n)), and Vergne's Laplacean in the group direction. The latter ~~two~~ seem to be clearly related, and I thought there might be [redacted] a connection with the fermion integral.

One idea was that the integration process

$$F \longmapsto \int \mathcal{D}\psi \; e^{-\int \psi\dot{\psi}\,dt} \; F(\psi)$$

on functions of the Grassman variables ψ_t^μ gives an integration process on functions $F(\phi)$ of the commuting variables $\phi_t^{\mu\nu} = \psi_t^\mu \psi_t^\nu$, which might turn out to be a Brownian motion process in Lie SO(n). Take $n=2$, where Lie SO(2) = Lie U(1), and then one [redacted] has the integral

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\int \bar{\psi}(\dot{\psi} - A\psi)\,dt} = \det\left(\frac{d}{dt} - A\right)$$

which is a generating function for the moments

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\int \bar{\psi}\dot{\psi}\,dt} \; \phi(t_1)\ldots\phi(t_n)$$

One runs into the following difficulties. The operator $\frac{d}{dt}$ [redacted] is not invertible, so one can't divide by $\det\left(\frac{d}{dt}\right)$ so as to get a "probability measure". Nevertheless one ought to be able to, and I did in Sept 82, construct a function

$$A \longmapsto \frac{\det\left(\frac{d}{dt} - A\right)}{\det\left(\frac{d}{dt}\right)}$$

A similar problem arises with Brownian motion where the operator which isn't invertible is $-\frac{d^2}{dt^2}$. Somehow to get a process one has to restrict to $t \geq 0$.

Another point is that one would like the ϕ_t-process to be Gaussian, which means something like

$$\det\left(\frac{d}{dt} - A\right) = c\, e^{-Q(A)}$$

where $Q(A)$ is quadratic in A. But

$$-\log \det \text{[redacted]} (1 - G_0 A) = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}(G_0 A)^k$$

so we need to have $\operatorname{tr}(G_0A)^k = 0$ for $k > 2$. Now this sort of phenomenon doesn't happen for $\frac{d}{dt} - A$ on the line, but does happen for $\frac{d}{dt} + i \text{[redacted]} \frac{d}{dx} + \alpha$ on $\mathbb{C}$.

Thus we have the following problems for the future:

1. Go over the construction of $\det\left(\frac{d}{dt} - A\right)$ [redacted] from the determinant line bundle veiwpoint, and evaluate the resulting integrals

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, e^{-\int \bar{\psi}\dot{\psi}\, dt}\, \phi(t_1) \cdots \phi(t_n) \qquad \phi(t) = \bar{\psi}(t)\psi(t)$$

2. Construct bosonic integrals

$$\int \mathcal{D}x\; e^{-\int \left(x(-\ddot{x}) \text{[redacted]} + Ax^2\right) dt} = \det\left(-\frac{d^2}{dt^2} + A\right)^{-1/2}$$

on similar principles. A degenerate covariance leads to a singular variance. Maybe one can understand the $t \geq 0$ restriction as producing a positive measure

3. [redacted] What significance is there in the loop group case [redacted]?

November 20, 1984

Roe's thesis: X = compact manifold foliated by Riemann surfaces. S = compact Riemann surface. By holomorphic function [redacted] $f : X \to S$, one means a Borel function holomorphic on each [redacted] leaf.

Λ = harmonic transverse measure. Defined as follows assuming the leaves are oriented. A smooth transverse measure is a section of $\Lambda^{max}(T_X/\mathcal{F})^*$ which is positive. In general [redacted] one looks locally at the foliated manifold, where it is fibred and one takes a measure $d\xi$ on the base times a strictly positive function h on the total space. Transverse measures can be used to integrate transverse Borel sets and p-forms where $p = \dim \mathcal{F}$. The latter defines the Ruelle-Sullivan current Λ° corresponding to the transverse measure Λ.

Λ is harmonic when disintegrated locally as a product $h\,d\xi$, the function h is harmonic. This requires a conformal structure on the leaves. Existence of harmonic transverse measures [redacted] is proved by Garnett by using the leafwise heat flow or diffusion operator $D_t = e^{-t\Delta_{\mathcal{F}}}$ on the space of Radon probability measures.

Now given $X, f,$ [redacted] Λ as above, we define the divisor [redacted] $q(f,a)$ (this is $f^{-1}(a)$ with multiplicities) in the obvious way. Where f has order n is a transverse Borel set $q(f,a)_n$ and one can then take its Λ-measure

$$\Lambda(q(f,a)) = \sum_{n \geq 1} n\,\Lambda(q(f,a)_n)$$

Next pick a smooth volume form Ω on S, let u_a be the solution of Poisson's equation with logarithmic singularity at a, $\lambda_a = \frac{1}{2\pi} * du_a$.

Roe's Equidistribution thm.

X compact foliation by R.S.
f holom. from X to compact R.S. S
Λ harmonic trans. measure

If $f^*\lambda_a + \frac{1}{2\pi} f^*u_a \cdot {*}\partial\Lambda$ is controlled (this is a finiteness condition satisfied if f is continuous; note $\partial\Lambda$ is the boundary of the Ruelle-Sullivan current), then

$$\langle \Lambda^\circ, f^*\Omega \rangle = \Lambda(g(f,a))$$

In other words the degree [redacted] of the divisor $f^{-1}(a)$ is the [redacted] same as its average values. Here[redacted] Ω is normalized so that $\int_S \Omega = 1$.

Cor. If a continuous holom. fn. f [redacted] does not cover a point of S, then it is constant along almost all leaves relative to any harmonic transverse measure.

Outline of first four chapter's of Roe's thesis; these are devoted to function theory on a manifold foliated by R.S.

I. Review of Nevanlinna theory.

A. Poisson's equation on S = compact R.S.

B. First main thm. for $f: R \longrightarrow S$. $R_0 \subset R$ rel. compact smooth with ∂, then

$$\underbrace{n(R_0, a, f)}_{\substack{\text{no of pts of } f^{-1}(a) \\ \text{with mult in } R_0}} + \int_{\partial R_0} f^*\lambda_a = \frac{V(R_0, f)}{\int_{R_0} f^*\Omega}$$

C. Application to merom. fns. on $\mathbb{C}$. $N(r,a) = \int^r n(t,a)\,\frac{dt}{t}$ counting fn.

$T(r) = \int^r V(t)\,\frac{dt}{t}$, characteristic fn. Classical 1st Main thm.

II. X compact oriented with smooth $d\mu$ preserved by an ergodic action of $\mathbb{C}$ infinitesimally free. Look at holom. fns. $f: X \to S^2$.

Lebesgue measure on $\mathbb{C}$ induces longitudinal measures on each orbit, so from $d\mu$ and these we get a canonical transverse measure Λ.

A. Ergodic theorem: for this transverse measure applied to a transverse Borel set Z: For almost all $x \in X$

$$\Lambda(\mathbf{Z}) = \lim \left(\frac{1}{\pi r^2} \operatorname{Card} B(r) \cap Z_x \right)$$

B. Thm: A merom. fn. of order < 2, or order $= 2$ and minimal type is a.e. constant.

Controlled: order 2 + mean type

Prop: f controlled $\iff \int_X f^*\Omega < \infty$

Define $\int f^*\Omega = \deg(f)$.

Define $\mathbb{C}(X) =$ controlled merom. fns. mod. a.e. equality

Thm: No entire fns. in $\mathbb{C}(X)$ except constants.

C. Construction of non constant fns. in $C(X)$ using Weierstrass fn. of a pseudo-lattice

D. Equidistribution: $\Lambda(f^{-1}(a)) = \int_X f^*\Omega$ if f is a non-constant element of $C(X)$.

III. Foliations; transverse measures, diffusion

B. transverse measures Λ, desintegration, Ruelle-Sullivan current Λ^0, boundary $b\Lambda^0$, $b\Lambda^0 = 0 \iff \Lambda$ holonomy invariant

Examples.

C. Diffusion - existence of solutions to $\partial_t + \Delta_{\mathcal{F}}$ and harmonic measures.

November 22, 1984

Consider $\Omega(\mathcal{L}M) = \prod_{g\geq 0} \Omega^g(\mathcal{L}M)$ with $d - \iota_X = d_X$

This is a differential on the invariant forms. I'd like to prove that $\Omega(\mathcal{L}M)^{S^1} \longrightarrow \Omega(M)$, the restriction to the fixpoint map induces an isom. on cohomology. The idea is to use [redacted] the forms

$$[redacted]\ \alpha = \int_0^1 \dot{x}^\mu \psi^\mu \, dt , \qquad d_X \alpha = -\int_0^1 (\dot{x}^2 + \psi\dot{\psi}) \, dt .$$

Then given any form $\xi \in \Omega(\mathcal{L}M)^{S^1}$ which is d_X-closed, it should be true that the cohomology classes of the form[redacted]

$$(*) \qquad e^{t d_X \alpha} \xi$$

doesn't depend on t. (Check the proof: On $\mathbb{R} \times \mathcal{L}M$ consider the form $(dt\, \partial_t + d_X)(t\alpha) = dt \cdot \alpha + t d_X \alpha$. It is killed by $dt\,\partial_t + d_X$, hence so is

$$e^{t d_X \alpha + dt\,\alpha} = \underbrace{e^{t d_X \alpha}}_{U_t} + dt \underbrace{e^{t d_X \alpha} \alpha}_{V_t}$$

and as usual this implies $\partial_t U_t = d_X V_t$, etc.)

Now for $t = 0$, the form $(*)$ is ξ and for $t \to +\infty$ it will peak around the zero set of the energy $= +\int_0^1 \frac{1}{2}\dot{x}^2 \, dt$. [redacted] So what we really want to see is that there is some map from forms on [redacted] M to forms on $\mathcal{L}M$. What we need is a way to see that $\lim_{t\to\infty} e^{t d_X \alpha} \xi$ is a function of $\iota^* \xi$.

Specifically one can ask for a section of ι^*

$$\Omega(M) \longrightarrow \Omega(\mathcal{L}M), \quad \xi \longmapsto \tilde{\xi}$$

i.e. a way of extending forms on M to forms on $\mathcal{L}M$ compatible with d, d_X. Then we want

$e^{td_X\alpha}($[redacted]$\;\xi - \iota^*\tilde{\xi}) \longrightarrow 0$

Something I did notice is that if we attempt to use ~~the Bott~~ process to push a d_X-closed ξ ~~into~~ a tubular nbd., then what one can do by inverting $d_X\alpha = -\text{Energy} + 2\text{-form}$ is to replace ξ by an equivalent form supported in $E < \varepsilon$. One has to invert $\frac{1}{E}$ off the fixpoint set, then multiply by a smooth bump function. This bump function must more or less ~~be~~ [redacted] a fn. of the energy.

But $E(x) < \varepsilon$ is not a tubular nbd. of the constant paths since one can get broken geodesics of arbitrarily long length and energy $< \varepsilon$. NO (One has the other inequality

$$L = \left(\int |\dot{x}|\, dt\right) \leq \left(\int \dot{x}^2\, dt\right)^{1/2} \left(\int 1\, dt\right)^{1/2} = E^{1/2}$$

Actually it seems $E \downarrow 0 \Rightarrow L \downarrow 0$.

November 26, 1984

Notes on the basic two form on $\mathcal{L}M$ defined by the Riemannian structure on M. (Lectures in graduate class)

$$\mathcal{L}M = \text{Maps}(S^1, M) \quad , \quad x : t \longmapsto x_t \in M \quad , \quad t \in S^1 = \mathbb{R}/\mathbb{Z}$$

$$T_x(\mathcal{L}M) = \Gamma(S^1, x^* T(M))$$

circle action $(s * x)_t = x_{s+t}$; X = infinitesimal generator

$$X : x \longmapsto \dot{x}_t = \frac{d}{ds}\Big|_{s=0} x_{t+s}$$

Basic 1-form

$$\alpha_x(Y) = \int_0^1 \langle \dot{x}_t, Y_t \rangle \, dt$$

Another way is to note that the metric on M induces one on $\mathcal{L}M$ by integrating

$$|Y|^2 = \int_0^1 |Y_t|^2 \, dt .$$

Then α is the 1-form determined by taking the inner product with X. Note X is a Killing vector field on $\mathcal{L}M$.

Compute $d\alpha$

$$d\alpha(Y, Z) = Y\langle X, Z \rangle - Z \langle X, Y \rangle - \langle X, [Y, Z] \rangle$$

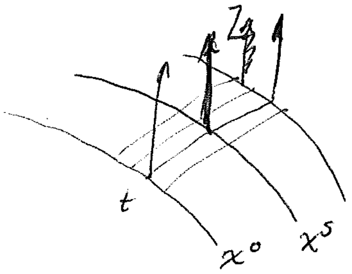
$$= \langle D_Y X, Z \rangle - \langle D_Z X, Y \rangle - \cancel{\langle X, [Y,Z] \rangle}$$
$$+ \cancel{\langle X, D_Y Z \rangle} - \cancel{\langle X, D_Z Y \rangle}$$

$$= \langle D_X Y + [Y, X], Z \rangle - \langle D_X Z + [X, Z], Y \rangle$$

$$= \langle D_X Y, Z \rangle - \langle Z, D_X Y \rangle + \underbrace{\left(\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle \right)}_{0 \text{ as } X \text{ is Killing}}$$

Here in this calculation $D_X Y$ refers to covariant diffn. on $\mathcal{L}M$.

We can compute $D_Y Z$ for two vector fields on $\mathcal{L}M$ as follows. Given $x \in \mathcal{L}M$, choose a 1-parameter family $x^s = (x^s_t)$ of loops with tangent vector Y at x at $s=0$.



For each s, Z at x^s gives a tangent field along x^s, whence $Z_{s,t} =$ Z at x^s_t is a tangent field along the 2-parameter family x^s_t. Then for t fixed we can covariantly differentiate $Z_{s,t}$ along $s \longmapsto x^s_t$, and this gives $(D_Y Z)$ at $x^0_t = x_t$.

Then for $X : x \longmapsto \dot{x}$, $D_X(Z) = \frac{D}{dt} Z$. In this case D_X is an operator on $T(\mathcal{L}M)$ which is skew-symmetric as

$$\langle \tfrac{D}{dt}Y, Z \rangle + \langle Y, \tfrac{D}{dt}Z \rangle = \int_0^1 dt\ \partial_t \langle Y_t, Z_t \rangle = 0$$

Thus

$$\boxed{d\alpha(Y,Z) = 2 \langle \frac{DY}{dt}, Z \rangle}$$

Coordinate calculation

$$\alpha = \int dt\ g_{\mu\nu}(x_t)\, \dot{x}^\mu_t\, \psi^\nu_t$$

Need Levi-Civita symbols. $X_\mu = \partial/\partial x^\mu$, $\langle X_\mu, X_\nu \rangle = g_{\mu\nu}$

$$D_\mu X_\nu = \Gamma^\lambda_{\mu,\nu} X_\lambda \iff D_\mu(f^\nu X_\nu) = (\partial_\mu f^\lambda + \Gamma^\lambda_{\mu,\nu} f^\nu)$$

D_μ charac. by

torsion-zero $\iff$ $\Gamma_{\mu,}{}^{\lambda}{}_{\nu} = \Gamma_{\nu,}{}^{\lambda}{}_{\mu}$.

preserves metric $\iff$ $\partial_\lambda g_{\mu\nu} = \langle D_\lambda X_\mu, X_\nu \rangle + \langle X_\mu, D_\lambda X_\nu \rangle$

$$= g_{\nu\rho}\Gamma_{\lambda,}{}^{\rho}{}_{\mu} + g_{\mu\rho}\underbrace{\Gamma_{\lambda,}{}^{\rho}{}_{\nu}}_{\Gamma_{\nu,}{}^{\rho}{}_{\lambda}}$$

Write this

$$\partial_1 g_{23} = g_{3\bullet}\Gamma_{1,}{}^{\bullet}{}_{2} + g_{2\bullet}\Gamma_{3,}{}^{\bullet}{}_{1}$$

$$\partial_1 g_{23} = (3,1,2) + \cancel{(2,3,1)}$$

$$\partial_2 g_{31} = \cancel{(1,2,3)} + (3,1,2)$$

$$-\partial_3 g_{12} = -\cancel{(2,3,1)} + -\cancel{(1,2,3)}$$

$$\boxed{\tfrac{1}{2}\left(\partial_1 g_{23} + \partial_2 g_{31} - \partial_3 g_{12}\right) = g_{3\bullet}\Gamma_{1,}{}^{\bullet}{}_{2}}$$

So $d\alpha = \int dt \left(dg_{\mu\nu}(x_t)\, \dot{x}_t^\mu \psi_t^\nu + g_{\mu\nu}(x_t)\, \dot{\psi}_t^\mu \psi_t^\nu \right)$

$$= \int dt \left(\partial_\lambda g_{\mu\nu}\; \dot{x}_t^\mu \psi_t^\lambda \psi_t^\nu + \text{———} \right)$$

$$\int dt \left\langle \frac{D}{dt}\psi, \psi \right\rangle = \int g_{\mu\nu}\left(\dot{\psi}^\mu + \dot{x}^1\Gamma_{1,}{}^{\mu}{}_{2}\psi^2\right)\psi^\nu$$

$$= \int \Big(g_{\mu\nu}\dot{\psi}^\mu\psi^\nu + \underbrace{\left(g_{3\bullet}\Gamma_{1,}{}^{\bullet}{}_{2}\right)}_{\frac{1}{2}\left(\cancel{\partial_1 g_{23}} + \partial_2 g_{31} - \partial_3 g_{12}\right)} \dot{x}^1\psi^2\psi^3 \Big)$$

use $\psi^2\psi^3 = -\psi^3\psi^2$

$$= \int \left(g_{\mu\nu}\dot{\psi}^\mu\psi^\nu + \partial_2 g_{13}\; \dot{x}^1\psi^2\psi^3 \right) dt \qquad = d\alpha$$

Problem: We want to make sense of the process $\int \mathcal{D}x\, \mathcal{D}\psi$ on $\Omega(\mathcal{L}M)$. The idea is to replace $\mathcal{L}M$ by a finite-dimensional approximation. So obviously we want to construct the push forward integration process on the finite dimensional approximation. Hence it might be possible to break up S^1 into N-equal steps and construct the relevant "kernel".

I have this picture of fermion integration of Gaussian functions which corresponds nicely to the idea that a path in the orthogonal group is a succession of infinitesimal Cayley transforms.

Another idea is that the classical picture (path in Lie $O(n)$) determines the fermion integrals as a line, i.e. up to a normalization constant. So my feeling is that I know everything about Gaussian fermion integrals that I need. Possibly I could adapt the approach to stochastic integrals

$$\int f(t, x_t)\, dx_t$$

which Stroock described - the original Paley-Wiener idea to make sense of this integral using special Riemann sums.

In fact I should understand completely this integration process.

November 27, 1984

Problem: Integrating differential forms on $\mathcal{L}M$.

To fix the ideas, let me consider a line bundle with connection over a torus M. I consider the integral which is to yield the index for the Dirac operator, namely,

$$tr_s\left(e^{\tau \not{D}^2}\right) = \int \mathcal{D}x\,\mathcal{D}\psi\; e^{-\frac{1}{4\tau}\int_0^1 (\dot{x}^2 + \psi\dot{\psi})\,dt} \times e^{\int_0^1 (\dot{x}^\mu A_\mu + \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu})\,dt}$$

(This seems correct: The last factor is the form constructed by Bismut on $\mathcal{L}M$ to extend e^F for the line bundle. The first exponential factor is

$$e^{\frac{1}{4\tau} d_X \alpha} \qquad \alpha = \int \dot{x}\psi\,dt$$
$$d_X\alpha = -\int (\dot{x}^2 + \psi\dot{\psi})\,dt,$$

$-d_X\alpha$ is the energy + the canonical two form on $\mathcal{L}M$.)

Thus I have a specific integral to make sense of. I believe that I have to use some kind of finite dimensional approximation at some point.

The first thing to try is to see what the probabilists do. They reduce to standard Wiener measure, so that the τ disappears ~~[illegible]~~ in front of the energy, and so that the energy factor can be combined with $\mathcal{D}x$ to get Wiener measure. This means one sets

$$x_t = x_0 + \sqrt{2\tau}\,\omega_t .$$

$M = \mathbb{R}^n/\Gamma$

Because I am dealing with a torus, x_t will be a loop when $\sqrt{2\tau}\,\omega_1$ belongs to the lattice Γ.

Hence we decompose [redacted] the path integral into Brownian 'bridge' integrals, one for each element of Γ. To simplify, set $\sqrt{2\tau} = h$. Also rescale $\psi \longmapsto h\psi$ whence we have

$$tr_s\left(e^{\frac{1}{2}h^2 \not{D}^2}\right) = \int_M d\psi_0 \int [\text{redacted}]\, Dw D\psi\, e^{-\frac{1}{2}\int(\dot{w}^2 + \psi\dot{\psi})\,dt}$$

$$\times\ e^{\int_0^1 \left[h\,\dot{w}^\mu A_\mu(x_0 + hw) - \frac{h^2}{2}\psi\psi F(x_0+hw)\right]}$$

The probabilists [redacted] have a way of handling the ψ-integral as parallel transport in $S \otimes L$.

If we take the constant coeff case, the [redacted] fermion and boson integrals are completely independent

November 28, 1984

Problem: Integration of differential forms on $\mathcal{L}M$.

The latest idea is to use a finite-diml manifold approximation to $\mathcal{L}M$ essentially based on the idea of dividing the circle into N pieces. In order to carry this out, one would need an idea of what is happening over a time interval $t' \leq t \leq t''$.

We want to do an integral such as

$$\int \mathcal{D}x\, \mathcal{D}\psi \; e^{-\frac{1}{4t}\int (\dot{x}^2 + \psi\dot{\psi})dt} \; T\left\{ e^{\int[-\dot{x}A_\mu(x) + \frac{1}{2}\psi\psi F(x)]} \right.$$

What is important to notice is that the fermion integral involves polynomials in the quantities $\psi_t^\mu \psi_t^\nu$ at different time. Similarly the boson integral involves the integral of $\dot{x}_t f(x_t)$. So we are not trying to integrate the most general form.

The first step will be to understand the conventions concerning $\dot{x}_t f(x_t)$ and $\psi_t^\mu \psi_t^\nu$. The former was partially explained by Stroock to me. What I want to compute is the details of the process

$$y_t = \int_0^t f(x_s)\, dx_s$$

where x_s is Brownian motion. However the above integral has to be defined, since x_s is not of bdd variation.

Consider $y_t = \int_0^t x_t\, dx_t$. This is to be defined as the limit of Riemann sums

$$y_t = \lim_{n\to\infty} \sum_{k=0}^{n-1} x_{\frac{k}{n}t}\left(x_{\frac{k+1}{n}t} - x_{\frac{k}{n}t}\right)$$

where, in the $f(x)\,\Delta x$ part, f is evaluated at the left point of the interval. Use

$$\Delta \frac{x^2}{2} = x\,\Delta x + \tfrac{1}{2}(\Delta x)^2.$$

Then we get

$$y_t = \lim_{n\to\infty}\left\{\sum_{k=0}^{n-1}\frac{1}{2}\left(x^2_{\frac{k+1}{n}t} - x^2_{\frac{k}{n}t}\right) - \frac{1}{2}\sum_{k=0}^{n-1}\underbrace{(\Delta x)^2}_{\Delta t}\right\}$$

$$= \frac{1}{2}x_t^2 - \frac{1}{2}t$$

it would seem. In any case one has

$$\left\langle \int_0^t f(x_s)\,dx_s \right\rangle = \int_0^t \langle f(x_s)\,dx_s\rangle = 0$$

since in the Riemann sums one has $f(x_{t_i})(x_{t_{i+1}} - x_{t_i})$ and Δx is indep. of $f(x_t)$. So $\langle y_t\rangle = 0$ which checks.

Also we can use

$$\Delta g(x) = g'(x)\,\Delta x + \tfrac{1}{2}g''(x)(\Delta x)^2 + \cdots$$

so

$$\int_0^t g'(x_t)\,dx_t = g(x_t) - g(x_0) - \frac{1}{2}\int_0^t g''(x_t)\,dt$$

which gives the same answer for $g = \frac{x^2}{2}$.

November 29, 1984

The problem is to construct an integration process for differential forms on LM. If $M = \mathbb{R}^n/\Gamma$, then the typical integral to make sense of is

$$\int \mathcal{D}x\,\mathcal{D}\psi\; e^{-\frac{1}{4t}\int_0^1(\dot{x}^2+\psi\dot{\psi})dt}\, \mathrm{tr}\, T\left\{ e^{\int(-\dot{x}^\mu A_\mu(x)+\frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)}\right.$$

From Wiener measure we learn that the natural integration process [redacted] is [redacted]

$$\int \mathcal{D}x\,\mathcal{D}\psi\; e^{-\frac{1}{4t}\int_0^1(\dot{x}^2+\psi\dot{\psi})dt} \quad (?)$$

and it is to be applied to certain forms.

To be more precise, [redacted] on the continuous functions on $[0,1]$ are various measures

$$\mathcal{D}x\; e^{-\frac{1}{4t}\int_0^1 \dot{x}^2\,dt}$$

for different t. These are probably not mutually abs. cont. Nevertheless each one gives a way of integration functions defined for continuous paths. And in fact other functions like parallel transport, which are defined a.e. for continuous paths.

Clearly a standard trick for handling the boson path integrals for different t is always to drive by the standard Wiener process, but to run the process for [redacted] the time interval t.

Let's analyze this trick in more detail. Let ω_t denote the standard Brownian motion starting at the origin at $t=0$ with $\langle \omega_t^2 \rangle = t$. This gives us a probability measure on $C(\mathbb{R}_{\geq 0})$. On the other hand I could consider the process

$$x_t = h w_t$$

which represents Brownian motion such that $\langle dx^2 \rangle = h^2 dt$. This gives us another probability measure on $\mathcal{C}(\mathbb{R}_{\geq 0})$. Now the point is that $w_{h^2 t}$ also represents the Brownian motion x_t. As

$$w_{h^2(t+dt)} - w_{h^2 t} = w_{h^2 t + h^2 dt} - w_{h^2 t}$$

is a Gaussian r.v. with variance $h^2 dt$. Thus

$$\begin{array}{ccccc} \mathcal{C}(0,1) & \xrightarrow{\sim} & \mathcal{C}(0,h^2) & \xleftarrow{\text{restriction (if } h<1)} & \mathcal{C}(0,1) \\ x_t & \longmapsto & w_{h^2 t} & \qquad w_t \longleftarrow\!\shortmid & w_t \end{array}$$

Brownian meas. of diffusion h — standard Brownian

but I don't think this shows that the process x_t is abs. cont. wrt w_t.

So I get the following impression: Just as in the Brownian motion game, I should fix the Gaussian, i.e. the energy + 2-form, but allow time to run for different amounts. Is it possible to set up an integral for certain forms on the space of paths starting at 0 at $t = 0$?

November 30, 1984

I know that for the harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + \omega^2 x^2)$ path integrals

$$\int \mathcal{D}x \; e^{-\frac{1}{2}\int(\dot{x}^2 + \omega^2 x^2)\,dt} \; x(t_1)\cdots x(t_n)$$

coincide with Green's functions

$$\langle \alpha \,|\, T[x(t_1)\cdots x(t_n)] \,|\, \alpha' \rangle \qquad x(t) = e^{-tH} x e^{tH}$$

where the initial and final states are related to the boundary conditions on the paths. I would like to find the Hilbert space picture which goes with Brownian motion.

In the case of Brownian motion we deal with paths x_t defined for $t \geq 0$ which start at 0. It appears as if there is only one endpoint condition. I want to evaluate a Green's function such as

$$G(t,t') = \frac{\langle \alpha |\, e^{-(T-t)H_0} \, x \, e^{-(t-t')H_0} \, x \, e^{-t'H_0} \alpha' \rangle}{\langle \alpha |\, e^{-TH_0} \,|\, \alpha' \rangle} \qquad 0 < t' < t < T$$

for $H_0 = \frac{p^2}{2} = -\frac{1}{2}\partial_x^2$ on $L^2(\mathbb{R})$. I think that if I take $\alpha = \alpha' = \delta(x)$, then I get the Green's fn. for the operator $-\partial_t^2$ on $[0,T]$ with the bdry conditions $x_t = 0$ at $t = 0, T$. In the limit as $T \to \infty$ I get the Green's fn. on $[0,\infty)$ with bdry conditions $x_0 = 0$, $\dot{x}_t = 0$ as $t \to \infty$. This agrees with the fact that

$$e^{-TH_0}\delta \sim \frac{1}{\sqrt{2\pi T}} \qquad \text{(const fn. of } x\text{) as } T \to +\infty.$$

I have therefore reached the operator picture belonging to Brownian motion.

Let's review. I start with construction of Wiener measure. One gives for each finite set $t_1 > t_2 > \dots > t_n > 0$ a Gaussian measure on $\mathbb{R}^n$. In other words, one will have a set of random variables x_t for $t > 0$, and one specifies the joint distributions for any finite subset. Then according to Kolmogorov, there is some unique probability measure on the product $\prod_{t>0} \mathbb{R} = \{(x_t)_{t>0}\}$. However this measure is only good for integrating measurable functions. At the moment I don't know what these are.

Now I have learned that it is useful to think in terms of Gaussian measures on a vector spaces. One starts with a pos. definite quadratic form $Q(v)$ on a real vector space V, supposed finite-dimensional to begin with. Then there is a unique Gaussian measure $d\mu$ on V^* such that, upon interpreting V as functions on V^*, one has $\langle v^2 \rangle = Q(v)$. Moreover the polynomial fns. $S(V)$ on V^* are dense in $L^2(V^*, d\mu)$.

Next one passes to the infinite diml. case. Again one is given $Q(v)$ on V from which one can construct the Hilbert space $S(V)$. The key question is to see that this Hilbert space is $L^2(V^*, d\mu)$ in the case that $V \longrightarrow \bar{V}$ is sufficiently compact. I want somehow to use standard cyclic vector theory.

Let's make precise the fermion integration process:

$$\int \mathcal{D}\psi \; e^{\frac{-1}{4t}\int_0^1 \psi\dot{\psi}\,dt}.$$

Following the practice in the boson [redacted] case, we fix the Gaussian and let the paths run over different time intervals. Let's work out the formulas carefully. Recall

$$N = \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\bar{\psi}A\psi} = \text{const } \det(A)$$

$$\Longrightarrow \frac{1}{N}\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\bar{\psi}A\psi}\,(-\bar{\psi}\,\delta A\,\psi) = \operatorname{tr}(A^{-1}\delta A)$$

$$\frac{1}{N}\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\bar{\psi}A\psi}\,\psi_i\bar{\psi}_j\,\delta A_{ji} = \sum (A^{-1})_{ij}\,\delta A_{ji}$$

$$\Longrightarrow \quad \boxed{\frac{1}{N}\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\bar{\psi}A\psi}\,\psi_i\bar{\psi}_j = (A^{-1})_{ij}}$$

(Think of ψ as a column vector, and $\bar{\psi}$ as a row vector)

Thus when we want

$$\int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \; e^{-\int \bar{\psi}\dot{\psi}\,dt}\,\psi_t\bar{\psi}_{t'} = G(t,t')$$

we get a Green's function for $\frac{d}{dt}$. In analogy with Wiener measure we probably want to work on $t \geq$ and to have $\psi_0 = 0$, whence it should be that

$$G(t,t') = H(t-t') = \begin{cases} 1 & t > t' \\ 0 & t < t'. \end{cases}$$

What is the Hilbert space picture? The Hamiltonian is 0, so we have

$$G(t,t') = \langle T[a(t)\,a^*(t')]\rangle$$

where $\langle \quad \rangle$ has to be specified.

December 1, 1984

Recall some fermion integration formulas

$$tr_s\left(e^{\frac{1}{2}\omega_{12}\gamma^1\gamma^2}\right) = e^{\frac{1}{2}i\omega_{12}} - e^{-\frac{1}{2}i\omega_{12}}$$

$$= 2i\,\frac{\omega_{12}}{2}\,\frac{\sinh(i\omega_{12}/2)}{(i\omega_{12}/2)}$$

In general

$$tr_s\left(e^{\frac{1}{4}\omega_{jk}\gamma^j\gamma^k}\right) = (2i)^{n/2}\, Pf\left(\frac{\omega}{2}\right) \det\left(\frac{\sinh(\omega/2)}{\omega/2}\right)^{1/2}$$

$$tr_s\left(e^{-a^*\omega a}\right) = \det(1-e^{-\omega}) = \det(\omega)\det\left(\frac{1-e^{-\omega}}{\omega}\right)$$

I am trying to understand properly the fermion integration process

$$\int \mathcal{D}\bar\Psi\,\mathcal{D}\psi\; e^{-\int \bar\Psi\dot\psi\,dt}\;(\cdots).$$

Is it true that

$$tr_s\left(e^{-a^*\omega a}\right) = \int_{\substack{\text{periodic}\\ \text{b.c.}}} \mathcal{D}\bar\Psi\,\mathcal{D}\psi\; e^{-\int_0^1 \bar\Psi\left(\frac{d}{dt}+\omega\right)\psi\,dt} \quad ?$$

The right side is $\det\left(\frac{d}{dt}+\omega\right)$. Eigenvalues of $\frac{d}{dt}$ on functions on S^1 are $2\pi i n$, $n\in\mathbb{Z}$, so $\det\left(\frac{d}{dt}+\omega\right)$ vanishes when $\omega\in 2\pi i\mathbb{Z}$ which is also where $1-e^{-\omega}$ vanishes.

Thus the Hamiltonian $H = a^*\omega a$ is the quantization of the Lagrangian $\bar\Psi\left(\frac{d}{dt}-\omega\right)\psi$. Check the Green's function on the infinite interval

$t>t'$ $$\langle\Phi_0|T[a(t)\,a^*(t')]|\Phi_0\rangle = \langle\Phi_0|\,a\,e^{-(t-t')H}a^*|\Phi_0\rangle$$

$$= \begin{cases} e^{-(t-t')\omega} & t > t' \\ 0 & t < t' . \end{cases}$$

Ultimately I want to interpret the case $\omega = 0$:

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \; e^{-\int \bar{\psi}\dot{\psi}\, dt} \, (\cdots)$$

as associated to $H = 0$. This means that I have to find the appropriate boundary conditions for the operator $\frac{d}{dt}$. (Recall that the Wiener process involves the boundary conditions $x_0 = 0$, $\dot{x}$ [redacted] $_\infty = 0$ for the operator $\frac{d^2}{dt^2}$.)

It occurred to [redacted] me to make a careful study of the required boundary conditions. I did this for Graeme when I first arrived in England, but now I know more. In particular I want to think of the one-dimensional fermion integral as an unfolding orthogonal transformation. (and Cayley transforms.)

So the problem is where to start. Let's keep [redacted] to the orthogonal picture where possible. When we write $\bar{\psi} A \psi \in \Lambda^2(V^* \oplus V)$ we are really dealing with the hyperbolic quadratic space $V^* \oplus V$. More generally one looks at $\frac{1}{2}$ [redacted] $a_{jk}\psi^j\psi^k$ where (a_{jk}) is skew-symmetric.

In finite dimensions, given a positive definite quadratic form Q on V, where V is a real vector space there is a unique Gaussian prob. measure on V^* such that $\langle v^2 \rangle = Q(v)$. In other words we get an integral on $S(V)$ such that $\int 1 = 1$.

Given a non-degenerate $\omega \in \Lambda^2 V$ we get an integral on ΛV by setting

$$\langle \alpha \rangle = \frac{\int e^{\omega} \alpha}{\int e^{\omega}}$$

where $\int$[redacted] projects onto $\Lambda^{\max} V$. (Question: Can one evaluate the Gaussian integral on $S(V)$ by [redacted]looking at the components of "top degree"? Is there some way to do the Gaussian integral "over the sphere"?)

Let's go back to the determinant line view of the fermion integral. I recall that given a Fredholm $T: W \longrightarrow V$, it determines a line

$$\mathcal{L}_T \subset \mathrm{Hom}(\Lambda V, \Lambda W)$$

which is the line generated by $\Lambda(T^{-1})$ in case T is invertible. ~~[illegible]~~ I want to think of T as being a Dirac, e.g. $\partial_t + A$ or $\partial_{\bar{z}} + \alpha$.

We have formally

$$\mathcal{L}_T \subset \mathrm{Hom}(\Lambda V, \Lambda W) = \Lambda W \otimes \Lambda V^*$$
$$= \Lambda(W \oplus V; V)$$

and $\mathcal{L}_T$ is the line belonging to the graph of T.

What I want to use this for is to note that ~~[illegible]~~ one has a Hilbert space version of $\Lambda(W \oplus V; V)$ called Fock space, so that one can integrate by taking the inner products in Fock space.

Next look at this in terms of what can be integrated. In general given $\Lambda(T^{-1}) : \Lambda V \to \Lambda W$ we can form its matrix elements, i.e. we get a map

$$\Lambda V \otimes (\Lambda W)^* \longrightarrow \mathbb{C}$$

But [redacted] now one [redacted] would like to enlarge the things that can be integrated. For example, one might like a Hilbert space version of $\Lambda V \otimes (\Lambda W)^*$.

It is necessary to become more specific. Let's place ourselves in a position where the operator T in question actually gives a line in Fock space.

I believe that it should be possible to see in the case of an operator $\frac{d}{dt} + A$ exactly what class of "functions" can be integrated.

December 2, 1984

Let's consider the operator $\frac{d}{dt}$ on functions on $S^1 = \mathbb{R}/\mathbb{Z}$ with anti-periodic boundary conditions, so that the operator is invertible. Eigenfunctions:

$$\frac{d}{dt} e^{ikt} = ik\, e^{ikt} \qquad k \in \underbrace{2\pi\left(\tfrac{1}{2}+\mathbb{Z}\right)}_{\Gamma}$$

Then we write functions in terms of the eigenfns. Put

$$\psi_t = \sum_{k\in\Gamma} e^{ikt}\psi_k \quad , \quad \bar\psi_t = \sum_{k\in\Gamma} e^{-ikt}\bar\psi_k$$

whence

$$\int_0^1 \bar\psi\dot\psi\, dt = \sum ik\,\bar\psi_k\psi_k .$$

and

$$\langle \cdots \rangle = \frac{1}{N}\int D\psi D\bar\psi\; e^{-\int \bar\psi\dot\psi\, dt}(\cdots) = \frac{1}{N}\int \prod_k \left(D\psi_k D\bar\psi_k\, e^{ik\psi_k\bar\psi_k}\right)(\cdots).$$

where formally $N = \prod(ik)$, but we normalize the integral so it gives $N=1$.

We apply this integral $\langle\cdots\rangle$ to elements in the exterior algebra generated by the space of fns. $\psi, \bar\psi$, i.e. the exterior algebra generated by $\psi_k, \bar\psi_k$. $\Lambda[\psi_k,\bar\psi_k]$ has the basis $\psi^I\bar\psi^J$ and

$$\langle \psi^I \bar\psi^J\rangle = \begin{cases} 0 & I \neq J \\ \prod_{k\in I}\frac{1}{ik} & \text{if } I = J. \end{cases}$$

(Here I, J are finite subsets of Γ, ψ^I = the product in order of the ψ_k, $k\in I$, and $\bar\psi^J$ = the product in reverse order of the $\bar\psi_l$, $l \in J$.)

Now I want to know what elements of the exterior algebra $\sum a_{I,J}\,\psi^I\bar\psi^J$ can be integrated, i.e. such that

$$\sum a_{I,I} \prod_{k \in I} \frac{1}{ik}$$

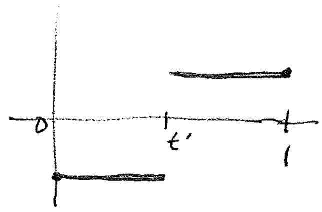
makes sense. I can becomes infinite in roughly two ways - either $|I| = p$ and I goes to infinity ??

Takes $|I| = 1$, i.e. an element in the space spanned by $\psi_t \bar{\psi}_{t'}$. For example, this element:

$$\psi_t \bar{\psi}_{t'} = \sum_k e^{ikt} \psi_k \sum_\ell e^{-i\ell t'} \bar{\psi}_\ell$$

$$\langle \psi_t \psi_{t'} \rangle = \sum_k e^{ik(t-t')} \frac{1}{ik}$$

$$= \theta(t-t') - \frac{1}{2} \quad :$$



This series in k is conditionally convergent for $t \neq t'$, but divergent for $t = t'$.

At this stage I want to describe the maximum one could naively hope for. We want to be able to integrate as much as possible. So start with a vector space V on which a skew-symmetric form ω is given. I should think in terms of V being a space of smooth functions. Then we form the algebra ΛV^* of multilinear forms on V. These are antisymmetric distributions. Now we want to integrate elements of ΛV^*.

I suppose ω is non-degenerate, whence it defines an injection $V \hookrightarrow V^*$. If V is reflexive, then the image is dense. We can transport the skew form ω on V to one on the image and then ask that it extend to a skew-form on V^*. We can also ask if ω, viewed as an element of $\Lambda^2 V^*$, comes from an element of $\Lambda^2 V$. These are probably equivalent.

(Digression: In finite dimensions suppose given $\omega \in \Lambda^2(V^*)$ with kernel K and let $W = V/K$. Then we have

$$V \twoheadrightarrow W \simeq W^* \hookrightarrow V^*$$

$$\Lambda^2 V \twoheadrightarrow \Lambda^2 W \simeq \underset{\omega/W}{\Lambda^2 W^*} \hookrightarrow \underset{\omega}{\Lambda^2 V^*}$$

Notice that if we lift ω back to $\Lambda^2 V$, then we get a skew form on V^* extending the skew form on W^*, but this skew form on V^* could be non-degenerate.)

Note that if we ~~know~~ that $\omega \in \Lambda^2 V^*$ actually comes from an element $\tilde{\omega}$ of $\Lambda^2 V$, then we get an integral defined on $\Lambda(V^*)$, namely, pairing with $e^{\tilde{\omega}}$.

So now I want to make this whole business more explicit. The idea is that given the skew form ω on V, we ought to be able to find a splitting $V = V^+ \oplus V^-$, where $V^\pm$ are isotropic subspaces. Hence ω is determined by a linear map $V^+ \longrightarrow (V^-)^*$. Now I want to write

$$\omega \in (V^+)^* \otimes (V^-)^* \subset \Lambda^2 V^*$$

which is OK for the kind of spaces I work with (nuclear). Also the "functions" to be integrated are

$$\Lambda V^* = \Lambda(V^+)^* \otimes \Lambda(V^-)^*$$

This is too confusing. Let's get concrete and consider the skew-form on anti-periodic fns. on S^1 given by the operator $\frac{d}{dt}$.

$$\omega(f,g) = \int_0^1 f g' \, dt$$

$$= \sum_k{}' f_{-k} \, ik \, g_k .$$

So we get an obvious pair of isotropic subspaces V^+, V^-.

All the action takes place between $\Lambda^2 V$ and $\Lambda^2 V^*$. I am thinking here of V as being the smooth functions and V^* as the distributions. But I have this fixed element $\omega \in \Lambda^2 V^*$ which I would like to pull-back to $\Lambda^2 V$. V^* has the basis ψ_k and

$$\omega = \sum_{k>0} \psi_{-k} \, ik \, \psi_k = \frac{1}{2} \sum_k{}' ik \, \psi_{-k} \psi_k$$

V has the basis $e^{ikt} = |k\rangle$ and the map

$$V \longrightarrow V^* \qquad v \longmapsto -\iota_v \omega$$

sends $|k\rangle$ to $ik\psi_{-k}$. Hence ω comes from the element

$$\frac{1}{2} \sum_k ik \frac{|-k\rangle}{-ik} \frac{|k\rangle}{ik} = \frac{i}{2} \sum{}' \frac{1}{k} |-k\rangle |k\rangle \quad \in (\Lambda^2 V)$$

which should be the skew-symmetric Green's function.

$$\sum{}' \frac{1}{ik} e^{ik(t-t')} = \theta(t-t') - \frac{1}{2}$$

Well, now we know this element is not a smooth kernel. The first thing we could try is to interpolate some kind of space W between V and V^*

so that $\omega \in \Lambda^2 W$. Then one could integrate elements of ΛW^*.

There are a couple of things that come to mind.

1) In finite dimensions the fermion integral on ΛV is essentially identical to tr_s on $C(V)$ defined via the spin representation. So one might to be able ~~to~~ use the Fock space representation to define the fermion integral in the S^1-case.

2) What Graeme had to say about skew forms and Pffaffians. I think he wanted to construct the spinor representation by giving the spherical function so to speak. In any case he uses formulas like

$$\frac{\int \mathcal{D}\psi \mathcal{D}\bar\psi \; e^{-\bar\psi A \psi} \; e^{-\bar\psi B \psi}}{\int \mathcal{D}\psi \mathcal{D}\bar\psi \, e^{-\bar\psi A \psi}} = \frac{\det(A+B)}{\det(A)} = \det(1+A^{-1}B)$$

Now this raises the following. In order for the determinant to be well-defined one needs that $A^{-1}B$ is of trace class. In the S^1 case $A = \frac{d}{dt}$ and B is a multiplication operator, so $A^{-1}B$ is not of trace class although it is Hilbert-Schmidt. Nevertheless there is a recipe for assigning ~~to~~ it a trace, and this step constitutes a step past the normal Hilbert space theory.

The algebra is still confused in my mind. I tend to think in terms of an operator T, or rather a linear transformation $T: V^0 \longrightarrow V^1$ and then have

$$\bar\psi T \psi \in V^1 \otimes (V^0)^*$$

I should be thinking in terms of
$$T: V^+ \longrightarrow (V^-)^*$$
and then have
$$\bar\psi T \psi \in (V^-)^* \otimes (V^+)^* \subset \Lambda^2(V^+ \oplus V^-)^*$$
So from now on think in terms of $V = V^+ \oplus V^-$ and of the skew-form on V associated to a linear map $T: V^+ \longrightarrow (V^-)^*$. In order to get a number from T we need a similar map
$$B: V^+ \longrightarrow (V^-)^*.$$
We then take the number $\operatorname{tr}(T^{-1}B)$ provided this is defined.

So basically we need
$$T^{-1} \in V^- \otimes V^+ \quad \text{to pair with} \quad B \in (V^- \otimes V^+)^*$$
Anyway what does this have to do with the main S^1-example?

Idea: In the S^1 case the kind of things to be integrated form an algebra which is a subalgebra of $\Lambda(V^*)$, V = smooth functions. I think this algebra ought to be generated by certain 1-diml and 2-diml elements.

Let make precise the setting. I want the integration process
$$\int \mathcal{D}\psi \mathcal{D}\bar\psi \, e^{-\int \bar\psi \dot\psi \, dt} (\ldots) = \int \prod_k \mathcal{D}\psi_k \mathcal{D}\bar\psi_k \, e^{-ik\bar\psi_k \psi_k} (\ldots)$$
so that V is the space of pairs of smooth anti-periodic functions, or V is rapidly decreasing linear comb. of $\psi_k, \bar\psi_k$

Write $V = V^+ \oplus V^-$ where V^+ is spanned by the ψ_k and V^- by the $\bar{\psi}_{+k}$. The basic 2-form is

$$\sum_k \left(\frac{1}{ik}\right) \psi_k \bar{\psi}_k .$$

A clearer way to describe this is to say that the basic 2-form is

$$\int f(t)\, G(t,t')\, g(t')\, dt\, dt' = \sum_k f_{-k} \frac{1}{ik} g_k .$$

When is this defined? Clearly if $f, g \in H^{-1/2}$ (i.e.

$$\|f\|^2_{-1/2} = \sum \frac{|f_k|^2}{(1+k^2)} < \infty)$$

then the series converges by Cauchy-Schwartz. Thus we should be able to integrate elements of $\Lambda(H^{+1/2})^*$.

So we start with the 2-form $\omega(f,g) = \int f' g\, dt$ defined on $H^{1/2}$ and then are able to integrate elements of $\Lambda(H^{+1/2})^* = \Lambda(H^{-1/2})$.

But also it should be possible to extend, in a more subtle way, the integral to things like

$$\int f(t)\, \psi_t \bar{\psi}_t\, dt.$$

Let's now consider trying to approach the fermion integral through the spinor super trace. Recall the formula

$$tr_s\left(e^{-\omega a^* a}\right) = \det(1 - e^{-\omega}) \sim \det(\omega)$$

$$\det(\omega) = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\ e^{-\omega \bar{\psi}\psi}$$

Thus the problem is to deform the supertrace somehow into the determinant. I want to take $\omega = \frac{d}{dt}$ and I want it to act on the functions on S^1. Then $\omega a^* a$ should be the multiplicative extension of ω to the exterior algebra

December 5, 1984

Let's consider [redacted] the Wiener process x_t on the line to fix the ideas. It is a Gaussian process defined for $t \geq 0$ with variance $\langle x_t x_{t'} \rangle = \min(t,t')$. [redacted] We know this gives a probability measure space $(W, d\mu)$, where W is a space of paths. Moreover, we have an algebraic description of $L^2(W, d\mu)$ as a Fock space based upon the Hilbert space of functions $f(t)$, $t \geq$ such that

$$\|f\|^2 = \int \overline{f(t)}\, G(t,t')\, f(t)\; dt\, dt' < \infty .$$

What I would like to do is to use the Hilbert space [redacted] $L^2(W, d\mu)$, which I can construct algebraically, to construct $W, d\mu$. The point will be to use the theorem that a $*$-repn. of a commutative C^*-alg. with a cyclic vector is the same as a measure on the maximal ideal space, or possibly the ideas in the proof of this result.

Simpler problem. Suppose we try to construct the Gaussian measure $e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ on $\mathbb{R}$ knowing the integral on polynomials. This is orthogonal polynomial theory. The space of polynomials is completed to form a Hilbert space, and [redacted] multiplication by x is a symmetric densely-defined operator. [redacted] In this case the [redacted] deficiency indices are at worst $(1,1)$, as one sees from the Jacobi matrix picture. Probably one can show directly the invertibility of $i + x$ in this case, whence one has a self-adjoint operator, and hence a measure. But in fact the operator e^{-tx^2} should be defined on the Hilbert space for $t > 0$. In fact better is the "hyperbolic" operator e^{itx}. This one-parameter unitary group should be enough to give the required measure. $\blacksquare$

Other idea is to think of a prob measure on Hilbert space supported on a Hilbert cube. Topologically the Hilbert cube is the compact space $[0,1]^{\mathbb{N}}$. So a measure on it pushes forward to a measure in ℓ^2 under the standard embedding. The key result is the Gaussian version of this idea. Evidentally it makes sense to speak of a Gaussian measure on Hilbert space whose variance matrix is Hilbert-Schmidt.

December 6, 1984

First of all, I would like to understand integration with respect to Wiener measure in terms of the Hilbert space $L^2(W)$. Let me begin with Gaussian measures in finite dimensions. Given a real vector space V and a positive definite quadratic form Q on V, there is a unique Gaussian probability measure on V^* such that $\langle v^2 \rangle = Q(v)$ for all $v \in V$. Moreover the polynomial alg. $S(V)$ is dense in $L^2(V^*)$. So what I have is a Hilbert space $\mathcal{H}$, together with a commuting family of self-adjoint operators L_v $v \in V$, and finally a cyclic vector. These satisfy certain conditions which probably can be summarized by

$$\langle e^{iL_v} \rangle = e^{-\frac{1}{2}Q(v)}.$$

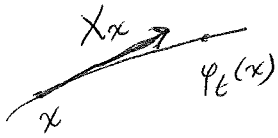
Now the sort of thing I want to get at is the idempotents in $\mathcal{H}$ which commute with the operators L_v. So what?

Let's begin again with the Wiener process. Evidently an important part of it is concerned with time evolution. This means I want to look at the increasing family of $V_t = \{\text{span of } x_s, \ s \leq t\}$

December 7, 1984

Notes on Bismut's construction.

Let φ_t be a 1-parameter group of diffeos. of N. and X the vector field on N generating this 1-parameter group



$$\frac{d}{dt}\Big|_{t=0} f(\varphi_t(x)) = (Xf)(x)$$

$$\frac{d}{dt}\Big|_{t=0} \varphi_t^* = X$$

Now use gp property $\varphi_{s+t} = \varphi_t \varphi_s$

$$\varphi_{s+t}^* = \varphi_s^* \varphi_t^*$$

$$\frac{d}{dt} \varphi_t^* = \frac{d}{ds}\Big|_{s=0} \varphi_{s+t}^* = X \varphi_t^* . \qquad \text{Similarly}$$

$$\frac{d}{dt} \varphi_t^* = \varphi_t^* X$$

Thus
$$\left| \begin{aligned} \frac{d}{dt} \varphi_t^* &= X \varphi_t^* \\ \varphi_0^* &= 1 \end{aligned} \right.$$

which justifies the notation

$$\varphi_t^* = e^{tX}$$

Now suppose we have a vector bundle E over N with connection D. Using $\|$ transport along the trajectory from x to $\varphi_t(x)$ gives us

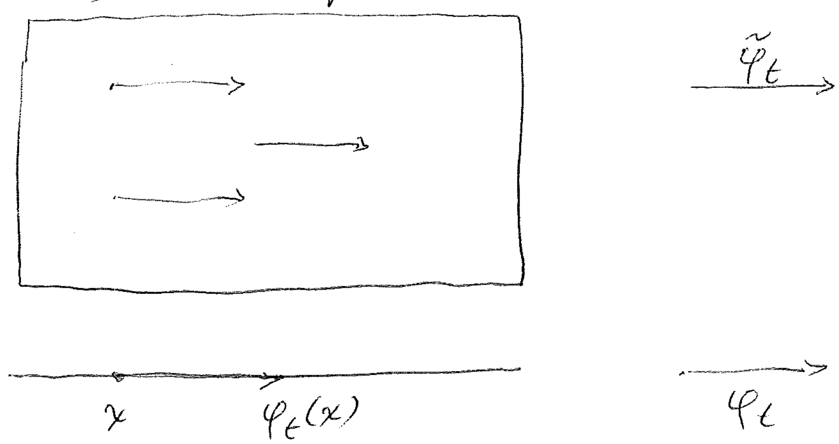
$$E_x \xrightarrow{\sim} E_{\varphi_t(x)} \qquad \forall x$$

or $\quad \tilde\varphi_t : E \xrightarrow{\sim} \varphi_t^*(E) \quad$ and hence a map $\tilde\varphi_t^*$ on $\Gamma(E)$ defined by

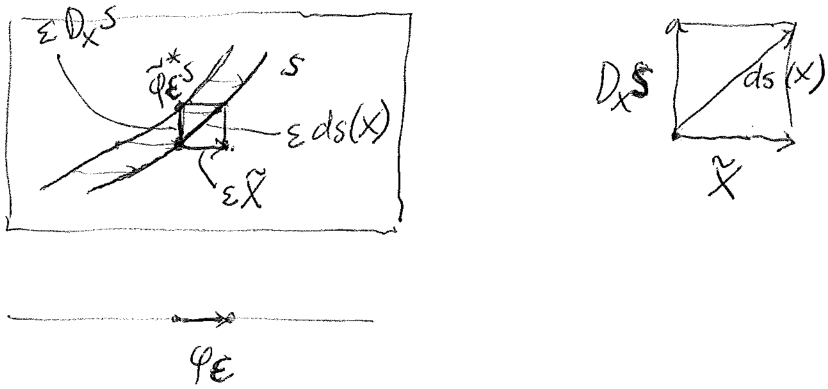
$$\Gamma(E) \longrightarrow \Gamma(\varphi_t^*(E)) \xleftarrow{\sim} \Gamma(E)$$

Geometrically we lift X to a vector field $\tilde X$ on E horizontal for the connection. Then $\tilde X$ generates a 1-parameter group $\tilde\varphi_t$ of diffeos. of E covering φ_t on X. $\tilde\varphi_t$ exists because linear ord. diff equations have global solutions.

Picture of $\tilde\varphi_t^*$. Think of a section s of E/N as a submanifold, then pull back via $\tilde\varphi_t$.



Picture of what happens to s



This picture shows

$$\frac{d}{dt}\Big|_{t=0} \tilde{\varphi}_t^*(s) = D_X s$$

for all s, hence as before

$$\frac{d}{dt}\Big|_{t=0} \tilde{\varphi}_t^* = D_X \ , \quad \frac{d}{dt}\tilde{\varphi}_t^* = D_X \quad \forall t$$

justifying the notation (and proving the existence of)

$$e^{tD_X} = \tilde{\varphi}_t^*$$

Now we have to extend to forms. First of all we have by definition of $\mathcal{L}_X$ on $\Omega(N)$:

$$\frac{d}{dt}\varphi_t^* = \mathcal{L}_X \quad \text{so} \quad \boxed{e^{t\mathcal{L}_X} = \varphi_t^*}$$

Next we have

$$(E \otimes \Lambda T^*)_x \xrightarrow{\sim} (E \otimes \Lambda T^*)_{\varphi_t(x)}$$

by combining parallel transport in E with $d\varphi_t : T_x \xrightarrow{\sim} T_{\varphi_t(x)}$. So $\tilde{\varphi}_t^*$ is defined on $\Omega(N,E)$; it is the unique extension of $\tilde{\varphi}_t^*$ on $\Omega^0(N,E)$ satisfying

$$\tilde{\varphi}_t^*(\omega\alpha) = (\varphi_t^*\omega)(\tilde{\varphi}_t^*\alpha).$$

Differentiating we see that on $\Omega(N,E)$

$$\frac{d}{dt}\Big|_{t=0}\tilde{\varphi}_t^* = D_X \quad \text{whence} \quad \tilde{\varphi}_t^* = e^{tD_X}.$$

Now we turn to Bismut's construction. We suppose given a circle action on N, $t \mapsto \varphi_t$, with generator X, and a vector bundle E over N with connection D. We consider the operator

$$D + \lambda\iota_X \quad \text{on} \quad \Omega(N, E)$$

where $\lambda \neq 0$. It's square is

$$(D + \lambda\iota_X)^2 = \lambda D_X + D^2 = \lambda(D_X + \lambda^{-1}D^2)$$

so we see

$$\bigstar \qquad [D + \lambda\iota_X, D_X + \lambda^{-1}D^2] = 0$$

set $\boxed{U_t = e^{t(D_X + \lambda^{-1}D^2)}}$ i.e. define U_t to be the solution of the IVP

$$\frac{d}{dt}U_t = (D_X + \lambda^{-1}D^2)U_t$$
$$U_0 = I$$

To see the existence & uniqueness note that

$$\frac{d}{dt}\left(e^{-tD_X}U_t\right) = e^{-tD_X}(\cancel{-D_X})U_t + e^{-tD_X}(\cancel{D_X} + \lambda^{-1}D^2)U_t$$
$$= \underbrace{e^{-tD_X}(\lambda D^2)e^{tD_X}}_{K_t}\left(e^{-tD_X}U_t\right)$$

Note that K_t is a vector bundle endom. of $\Omega(N,E)$ commuting with multiplication by $\Omega(N)$, hence the above is a ODE on each fibre of $\Lambda T^* \otimes E$ over N. ∴ We can solve uniquely and we conclude

$$e^{t(D_X+\lambda^{-1}D^2)} = e^{tD_X}\left(e^{-tD_X}U_t\right) \quad \text{where}$$

$$e^{-tD_X}U_t \in \Omega^{ev}(N, \operatorname{End} E) \quad \forall t.$$

From ☆ it follows that

$$[D+\lambda\iota_X,\ U_t] = 0.$$

Now take $t=1$, which is a period of X by assumption, so e^{D_X} is a vector bundle automorphism of E; it is called the monodromy. We see then that

$$e^{D_X+\lambda^{-1}D^2} \in \Omega^{ev}(N, \operatorname{End} E)$$

$$\operatorname{tr}\left(e^{D_X+\lambda^{-1}D^2}\right) \in \Omega^{ev}(N)$$

and

$$(d+\lambda\iota_X)\operatorname{tr}\left(e^{D_X+\lambda^{-1}D^2}\right) = \operatorname{tr}\left[D+\lambda\iota_X,\ e^{D_X+\lambda^{-1}D^2}\right] = 0$$

so

$$\boxed{\operatorname{tr}\left(e^{D_X+\lambda^{-1}D^2}\right) \text{ is equivariantly closed}}$$

Note that as $(d+\lambda\iota_X)^2 = \lambda\mathcal{L}_X$ any equivariantly closed form is automatically S^1-invariant.

Remarks. 1) The above construction is natural with respect to an equiv. map $N' \to N$. In particular taking $N' =$ fixpoint submanifold N^{S^1}, we have [redacted] $X = 0$ on N', so

$$\operatorname{tr}\left(e^{D_X + \lambda^{-1}D^2}\right)\Big|_{N'} = \operatorname{tr}\left(e^{\lambda^{-1}D^2}\right)\Big|_{N'}$$

which is essentially the Chern character of E/N' with respect to D.

2) Suppose that [redacted] E is an equivariant bundle over N for the S^1 action, and that D is an invariant connection. Because E is equivariant one has $\mathcal{L}_X$ on $\Omega(N, E)$ and

$$\mathcal{L}_X \text{ [redacted] } = D_X + J_X \qquad J_X \in \Omega^0(N, \operatorname{End} E)^X$$

where J_X is the inclination (or momentum). Then

$$e^{D_X + \lambda^{-1}D^2} = e^{\mathcal{L}_X + (\lambda^{-1}D^2 - J_X)}$$

$$= e^{\mathcal{L}_X} e^{\lambda^{-1}D^2 - J_X}$$

because $[\mathcal{L}_X, \lambda^{-1}D^2 - J_X] = 0$ as both D^2, J_X are S^1-invariant. Thus

$$\boxed{\operatorname{tr} e^{D_X + \lambda^{-1}D^2} = \operatorname{tr} e^{\lambda^{-1}(\underbrace{D^2 - \lambda J_X}))}}$$

where we ~~recall~~ now need to recall [redacted] our formulas for the equivariant curvature:

$$d - \iota_\xi \text{ diff}^l \longmapsto D - \iota_\xi \ \text{equiv. conn.} \longmapsto D^2 + J_\xi \ \text{equiv. curv.}$$

In this case $\xi = -\lambda X$. Thus we have = equiv. curvature

3) Line bundle case. Here End E is canonically trivial so $D^2 \in \Omega^2(N)$ and

$$e^{-tD_X} \lambda^{-1} D^2 e^{tD_X} = e^{-t\,\mathrm{ad}(D_X)}(\lambda^{-1}D^2)$$
$$= e^{-t\mathcal{L}_X} \lambda^{-1} D^2 = \lambda^{-1} \varphi_t^*(D^2)$$

~~Thus~~ $\cancel{e^{D_X+\lambda^{-1}D^2} = e^{D_X}}$ ~~monodromy map~~ $\cancel{N \to \mathbb{C}^*}$

Set $V_t = e^{-tD_X} U_t$. Then we have

$$\frac{d}{dt} \log V_t = \lambda^{-1} \varphi_{-t}^*(D^2)$$

$$V_t = e^{\lambda^{-1}\int_0^t \varphi_{-t}^*(D^2)\,dt}$$

so

$$e^{D_X + \lambda^{-1}D^2} = e^{D_X}\, e^{\lambda^{-1}\int_0^1 \varphi_{-t}^*(D^2)\,dt}$$

$$\boxed{e^{D_X+\lambda^{-1}D^2} = \begin{pmatrix}\text{monodromy}\\ \text{map } N \to \mathbb{C}^*\end{pmatrix} \cdot e^{\lambda^{-1}\begin{pmatrix}\text{average of } D^2\\ \text{for } S^1\text{-action}\end{pmatrix}}}$$

December 7, 1984 (cont.)

I want to get the structure of bosonic and fermionic integrals straight. Let's first look algebraically.

In the fermion case one has the integral $\langle ? \rangle = \int D\psi\, e^{\frac{1}{2}\omega\psi\psi}(?)$ which is defined on ΛV. The integral is determined from the its values on $\Lambda^2 V$ by the Wick rules. Hence we have a "fermionic Gaussian measure" on ΛV defined by a skew-symmetric form A on V. So $A \in (\Lambda^2 V)^*$ and probably the integral is pairing with $e^A \in (\Lambda V)^*$.

In the boson case one has the integral defined on $S(V)$ by its value on $S^2(V)$ and the Wick rules. I should check to see if the integral is pairing with $e^{\frac{1}{2}A}$ where $\frac{1}{2}A \in (S^2V)^*$ is the integral on $S^2(V)$.

Let's use the natural pairing of $S(V^*)$ and $S(V)$ obtained by thinking of V as differentiations on $S(V^*)$. Then given $Q \in (S^2V)^*$ we have from the Wick property

$$\langle e^{tv} \rangle = e^{\frac{1}{2}t^2 Q(v)}$$

(Recall $\frac{1}{N}\int D\varphi\, e^{-\frac{1}{2}\langle Q^{-1}\varphi,\varphi\rangle + J\varphi} = e^{\frac{1}{2}Q(J)}$.) Thus

$$\frac{\langle v^{2n}\rangle}{(2n)!} = \frac{1}{2^n n!}\, Q(v)^n$$

Now the pairing of e^{tv} and $e^{\frac{1}{2}Q}$ is

$$\left(e^{tv}\cdot e^{\frac{1}{2}Q}\right)(0) = e^{\frac{1}{2}Q(tv)} = e^{\frac{1}{2}t^2 Q(v)} = \langle e^{tv}\rangle$$

Put more simply, the integral satisfies

$$\langle e^v \rangle = e^{\frac{1}{2}Q(v)}$$

and the pairing is

$$\left(e^v \cdot e^{\frac{1}{2}Q}\right)(0) = e^{\frac{1}{2}Q(v+?)}(0) = e^{\frac{1}{2}Q(v)}$$

so the two are equal.

So far we haven't brought in any positivity. But now that we have the integral or trace on $S(V)$ we want to construct a Hilbert space representation of $S(V)$ such that $\langle f \rangle = \langle 0|f|0\rangle$. We want a $*$ representation which seems to imply a conjugation on V, hence V has a real structure. If $v = v^*$, then

$$Q(v) = \langle v^2 \rangle = \langle 0|vv|0\rangle = \| v|0\rangle\|^2 \geq 0$$

so that what we effectively have is a real Hilbert space, and Q is the metric extended linearly to be a quadratic form on the complexification.

December 8, 1984

What is the Hilbert space associated to a Gaussian measure? The simplest description is to use the holomorphic representation which exhibits $L^2(d\mu)$ as the Hilbert space symmetric algebra of the 1-particle Hilbert space. One-dimensional formula.

$$a = \underbrace{c(\omega q + ip)}_{\frac{d}{dx}+\omega x} \qquad a^* = c(\omega q - ip) \qquad [a, a^*] = c^2\, 2\omega = 1.$$

$$\therefore \quad a = \frac{1}{\sqrt{2\omega}}(\omega q + ip), \quad a^* = \frac{1}{\sqrt{2\omega}}(\omega q - ip)$$

$$a + a^* = \frac{1}{\sqrt{2\omega}}\, 2\omega q \qquad q = \frac{a+a^*}{\sqrt{2\omega}}$$

$|0\rangle$ proportional to $e^{-\frac{1}{2}\omega x^2}$. $\quad \langle 0|f|0\rangle = \int f(x) e^{-\omega x^2} dx$

so to get Gaussian measure we want $\omega = \frac{1}{2}$, whence

$$\boxed{q = a + a^*} \qquad \text{and} \qquad \langle q^2 \rangle = \langle 0 | (a+a^*)^2 | 0 \rangle$$
$$= \langle 0 | a a^* | 0 \rangle = 1$$

So the general picture will be to start with a complex Hilbert space V and form its Fock space $S(V)$ with operators a_v and $(a_v)^*$; here $a_v =$ differentiation wrt $\langle v|$ and $(a_v)^* =$ multiplication by v. Then

$$[a_v + a_v^*, a_w + a_w^*] = \langle v|w\rangle - \langle w|v\rangle$$
$$= 2i \,\mathrm{Im}\, \langle v|w\rangle$$

so that we get a commuting family of operators by singling out a real subspace of V on which the symplectic form $\mathrm{Im}\langle v|w\rangle$ is zero.

Now what does all this mean in the case of the Wiener process or perhaps another Gaussian process such as the one governed by $-\frac{d^2}{dt^2} + \omega^2$ on the line and related to the simple harmonic oscillator?

The intriguing aspect of the situation is the fact that we pass from a q.m. situation ~~to~~ a classical situation in one higher dimension, then we are forming a q.m. situation to understand the latter. Now is it possible ~~that~~ there is a self-delusion happening? Things aren't really becoming classical with the extra dimension; & although the operators become commutative, the difficulties are transferred to the kind of trace one wants to take.

Can I sort this out in the case of a Gaussian process? Suppose it arises from a q.m. situation

$$G(t,t') = \langle 0 | T[x(t)\, x(t')] | 0 \rangle.$$

Here is a Hilbert space with time-evolution, vacuum, positive energy, etc. On the other hand in order to explain the path integral I must introduce a measure on paths, which gives me another Hilbert space time-evolution, etc.

<u>Question</u>: What is the relation between these two Hilbert spaces?

December 9, 1984

Gaussian measures on Hilbert space can be understood perhaps via the Hilbert cube. Let's fix a real Hilbert space $\mathcal{H}$ with a positive-definite operator A. Assume the spectrum is discrete, whence $\mathcal{H} = \ell^2$ and $Ae_n = a_n e_n$ where the a_n are > 0. Actually we will assume that A^{-1} is Hilbert-Schmidt i.e. $\sum \frac{1}{a_n^2} < \infty$. I want to think of $\mathcal{H}$ as infinite dimensional Euclidean space with coordinates x_n, $n \geq 1$. The goal will be to produce the probability measure on $\mathcal{H}$ given in some sense by

$$\prod_{n\geq 1} e^{-\frac{1}{2}a_n x_n^2} \frac{dx_n}{\sqrt{2\pi a_n}} .$$

Take a product of intervals

$$Q = \{ (x_n) \mid |x_n| \leq b_n \}$$

where $\sum b_n^2 < \infty$. This is a Hilbert cube and is compact; it is isomorphic to the direct product of $[-1,1]$ a countable number of times. Assume the b_n are chosen so that

$$\prod_{n\geq 1} \int_{-b_n}^{b_n} e^{-\frac{1}{2}a_n x_n^2} \frac{dx_n}{\sqrt{2\pi a_n}} = \prod_{n\geq 1} \int_{-b_n\sqrt{a_n}}^{b_n\sqrt{a_n}} e^{-\frac{1}{2}y_n^2} \frac{dy_n}{\sqrt{2\pi}}$$

is positive (i.e. convergent). Assuming this can be found, then we get a nice measure on the compact space Q; there is no problem with measures on compact spaces (by Dini's thm). Also replacing Q with tQ as $t \to +\infty$, it is clear that we get a sequence of measures on $\mathcal{H}$ tending to a probability measure.

The key is to see the sequence b_n can be found. We need $b_n\sqrt{a_n} \to \infty$ fast enough so that the infinite product converges, yet we want $\sum b_n^2 < \infty$. Now if $b_n\sqrt{a_n} \to \infty$, then $b_n\sqrt{a_n} \geq 1$ for large n so $b_n \geq \frac{1}{\sqrt{a_n}}$ and $\sum b_n^2 < \infty \Longrightarrow \sum \frac{1}{a_n} < \infty$.

So what we see is that this method won't work unless $\frac{1}{a_n}$ is an ℓ^1-sequence; in fact we need something slightly better. Thus it appears we don't have the optimal viewpoint yet.

December 10, 1984

I seem to have the wrong condition for

$$d\mu = e^{-\frac{1}{2}\sum a_n x_n^2} \prod \frac{\sqrt{a_n}\,dx_n}{\sqrt{2\pi}}$$

to be a probability measure on ℓ^2. The condition has to be that $\sum \frac{1}{a_n} < \infty$. In effect

$$\int e^{-\frac{1}{2}\|x\|^2} d\mu = \lim_{N\to\infty} \int e^{-\frac{1}{2}\sum_{n\le N} x_n^2} d\mu$$

$$= \lim_{N\to\infty} \prod_{n=1}^{N} \frac{\sqrt{a_n}}{\sqrt{1+a_n}} = \lim_{N} \prod_{1}^{N} \left(1+\frac{1}{a_n}\right)^{-1/2}$$

and if $\sum \frac{1}{a_n} = \infty$, this limit will be zero. This will imply each disk $\|x\| \le k$ will have measure 0 whence the measure has to be identically zero by countable additivity.

So we assume $s = \sum \frac{1}{a_n} < \infty$. Choose a sequence $b_n \to +\infty$ of positive numbers such that $\sum \frac{b_n}{a_n} < \infty$. To do this choose n_1 so that $\sum_{n\le n_1} \frac{1}{a_n} > \frac{s}{2}$, n_2 so that $\sum_{n_1<n\le n_2} \frac{1}{a_n} > \frac{1}{2}\sum_{n_1<n} \frac{1}{a_n}$, etc. so that the series

$$\underbrace{\left(\sum_{n\le n_1} \frac{1}{a_n}\right)}_{<s} + \underbrace{\left(\sum_{n_1<n\le n_2} \frac{1}{a_n}\right)}_{<s/2} + \underbrace{\left(\sum_{n_2<n\le n_3} \frac{1}{a_n}\right)}_{<s/4} + \cdots$$

converges geometrically. Then take $b_n = k$, for $n_{k-1} < n \le n_k$. Then $\sum \frac{b_n}{a_k} < \sum_{1}^{\infty} \frac{k\,s}{2^{k-1}} < \infty$.

December 11, 1984

I consider ℓ^2 with (real) coordinates x_n, $n \geq 1$ and want to construct on ℓ^2 a prob measure given by

$$d\mu = e^{-\frac{1}{2}\sum a_n x_n^2} \prod \frac{dx_n \sqrt{a_n}}{\sqrt{2\pi}}.$$

I saw yesterday it is necessary to assume

$$\sum \frac{1}{a_n} < \infty.$$

If this is the case I can find $b_n > 0$ such that

$$\sum \frac{b_n}{a_n} < \infty, \qquad b_n \longrightarrow +\infty.$$

Consider the ellipsoid in ℓ^2

$$\sum b_n x_n^2 \leq t$$

Because $b_n \longrightarrow \infty$, this ellipsoid should be compact. To see this take a sequence x^k of points in the ellipsoid. Each x_n^k is a bdd sequence, so by a diagonal argument we can suppose $x_n^k \longrightarrow y_n$ for each n.
For any N, $\sum_{n=1}^{N} b_n (x_n^k)^2 \longrightarrow \sum_{n=1}^{N} b_n y_n^2$, so $\sum_{n=1}^{N} b_n y_n^2 \leq t$ for all N, hence y is in the ellipsoid. We want to show $x^k \longrightarrow y$ in ℓ^2, hence suppose given $\varepsilon > 0$. Choose N so that for $n > N$, $b_n > \frac{t}{\varepsilon}$. Then

$$\|y - x^k\|^2 = \sum_{n \geq 1} |y_n - x_n^k|^2 \leq \sum_{n=1}^{N} |y_n - x_n^k|^2 + \sum_{n>N} |y_n|^2 + \sum_{n>n} |x_n^k|^2$$

and

$$\sum_{n>N} y_n^2 < \frac{\varepsilon}{t} \sum_{n>N} b_n y_n^2 \leq \varepsilon \qquad \text{etc.}$$

Next we want the volume of the ellipsoid with respect to our proposed measures. It should be noted that our proposed measure defines an integral for tame functions restricted to the ellipsoid, where tame means the function depends only on finitely many coordinates. Since tame continuous functions separate pts there should be no difficulty in defining the measure on the compact ellipsoid. The basic ~~[illegible]~~ continuity of the integral under monotone limits should result from Dini's thm.

So the remaining point should be to see that the ellipsoid has positive volume and the volume $\to 1$ as $t \to +\infty$. Set

$$\nu_t = \int_{\frac{1}{2}\sum b_n x_n^2 \le t} d\mu$$

whence

$$\int_0^R e^{-kt}\, d\nu_t = \lim \sum_i e^{-kt_i}(\nu_{t_i+\Delta t} - \nu_{t_i})$$

$$= \int_{\frac{1}{2}\sum b_n x_n^2 \le R} e^{-k(\frac{1}{2}\sum b_n x_n^2)}\, d\mu$$

so

$$\int_0^\infty e^{-kt}\, d\nu_t = \int e^{-k(\frac{1}{2}\sum b_n x_n^2)}\, d\mu .$$

The last integral is a product of

$$\int e^{-k\frac{1}{2}b_n x_n^2 - \frac{1}{2}a_n x_n^2} \frac{\sqrt{a_n}\, dx_n}{\sqrt{2\pi}} = \frac{\sqrt{a_n}}{\sqrt{a_n + k b_n}}$$

so

$$\int_0^\infty e^{-kt}\, d\nu_t = \prod_{n=1}^\infty \left(1 + k\frac{b_n}{a_n}\right)^{-1/2}$$

Now by the assumption that $\sum \frac{b_n}{a_n} < \infty$ this infinite product converges, so $\nu_t > 0$. Also the limit as $k \to 0$ is 1 showing that $\nu_t \uparrow 1$ as $t \to \infty$.

There is a certain amount of work to be done to make the above calculation completely rigorous. The point is that $d\mu$ is defined on the compact ellipsoids, and hence $\int e^{-k(\frac{1}{2}\sum b_n x_n^2)} d\mu$ is defined, and one should be able to evaluate this as a product.

Now let's turn to the more interesting problem, namely how to construct the Hilbert space associated to the measure. This Hilbert space lives independently of the Hilbert space ℓ^2 I start with. All I think I have done is to specify a class of functions which will be integrable.

So let us work from the viewpoint of the Hilbert space. The vacuum state is the function

$$e^{-\frac{1}{4}\sum a_n x_n^2}$$

in some senses. In any there is a vacuum ~~state~~ Φ represented by $1 \in L^2(d\mu)$ and then ~~one~~ spans the Hilbert space using monomials $x^\alpha \cdot \Phi$. This is clearly independent of any topology. So what?

We can ask what sort of functions are in the Hilbert space. Now we know there is a natural grading, so let's look at degree 1 and 2. A typical linear element $\sum c_n x_n$ belongs to $L^2(d\mu)$ when

$$\left\langle \left(\sum c_n x_n\right)^2 \right\rangle = \sum \frac{1}{a_n} c_n^2 < \infty.$$

A quadratic function $\sum c_{mn} x_m x_n$ belongs to $L^2(d\mu)$ when

$$\left\langle \left(\sum c_{mn} x_m x_n\right)^2 \right\rangle = \sum c_{mn} c_{pq} \langle x_m x_n x_p x_q \rangle < \infty.$$

~~One has~~

$$\cancel{\langle x_m x_n x_p x_q \rangle = \delta_{mn}\delta_{pq} + \delta_{mp}\delta_{nq} + \delta_{mq}\delta_{np}}$$

~~so~~ $$\cancel{\left\langle \left(\sum c_{mn} x_m x_n\right)^2 \right\rangle = \left(\sum c_{mm}\right)^2 + 2\sum c_{mn} c_{mn}}$$

December 12, 1984

Equivariant cohomology for S^1-action on $\mathcal{L}BU(1)$.

$\mathcal{L}BG \sim PG \times^G (G)$ where G acts on itself by conjugation. Two proofs.

1) $\mathcal{L}BG$ classifies G-bundles over $S^1 \times Y$:

$$[Y, \mathcal{L}BG] = [S^1 \times Y, BG] = G\text{-bundles over } S^1 \times Y$$

A bundle over $S^1 \times Y$ is up to isomorphism given by a bundle over Y and an automorphism of this bundle up to homotopy. However $PG \times^G (G)$ is easily seen to classify bundles with automorphism. up to homotopy.

2) $\mathcal{L}BG$ is the homotopy-fibre product of

$$\begin{tikzcd} & BG \arrow[d, "\Delta"] \\ BG \arrow[r, "\Delta"] & BG \times BG \end{tikzcd}$$

Another way to compute this is to replace $B(\Delta G)$ by $P(G\times G) \times^{G \times G} (G \times G/\Delta G)$, which is the fibre space over $B(G\times G)$ belonging to G with $G \times G$ acting by left and right mult. Pulling back by Δ we get the fibre space over BG associated to G with G acting by conjugation.

Next, we consider $G = U(1)$

$$\mathcal{L}BU(1) \sim BU(1) \times U(1)$$

since $U(1)$ acts trivially by conjugation on itself.

$$\therefore \quad H^*(\mathcal{L}BU(1)) = \underset{\substack{\uparrow \\ \deg(c)=2}}{k[c]} \otimes \underset{\substack{\uparrow \\ \deg(e)=1.}}{\Lambda[e]}$$

Take $k = \mathbb{Q}$ or $\mathbb{C}$

Set $N = \mathcal{L}BU(1)$ and consider the long exact sequence relating equivariant cohomology $H^*_{S^1}(N)$ to $H^*(N)$. It is the Gysin sequence of the circle bundle $PS^1 \times N \longrightarrow PS^1 \times^{S^1} N$.

$$0 \longrightarrow H^0_S(N) \xrightarrow{p^*} H^0(N) \xrightarrow{p_*} \cancel{H^{-1}_S(N)} \xrightarrow{u} H^1_S(N) \xrightarrow{p^*} \cdots$$

We know
$$\begin{cases} H^{2n}(N) = k \cdot c^n \\ H^{2n-1}(N) = k \cdot c^{n-1} e \end{cases}$$

<u>Lemma</u>: $p^* p_*(c^n) = n c^{n-1} e$ in $H^{2n-1}(N)$.
In particular $p_*(c^n) \in H^{2n-1}_S(N)$ is non-zero.

<u>Proof</u>:

(1)
$$\begin{tikzcd} PS \times (PS \times N) \arrow[r, "\mu"] \arrow[d, "pr_2"] & (PS \times N) \arrow[d, "p"] \\ (PS \times N) \arrow[r, "p"] & PS \times^S N \end{tikzcd}$$
is cartesian

In general if G acts freely on X, then
$$\begin{tikzcd} G \times X \arrow[r, "\mu"] \arrow[d, "pr_2"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & G \backslash X \end{tikzcd}$$
is cartesian, is a pull-back diagram (a principal bundle pulled back over itself is canonically trivial.)

So in (1) we have in cohomology

$$p^* p_*(c^n) = (pr_2)_* \, \mu^*(c^n)$$
$$= (pr_2)_* \, (\mu^*(c))^n .$$

So we need to ask about the action

$$S \times N \xrightarrow{\mu} N$$

and what it does to $c \in H^2(N)$. Now over N is a line bundle obtained using the evaluation at 0 map: $N = \mathcal{L}BU(1) \longrightarrow BU(1)$. ?

I guess the formula I want:

(∗) $$\mu^*(c) = 1 \otimes c + \alpha \otimes e$$ α generates $H^1(S^1)$.

should result from the definition of e. If we define e this way, then one only has to relate e to the monodromy map.

Maybe the simplest way is to define on $\mathcal{L}BU(n)$ odd classes $e_j = p_* (c_j) \in H^{2j-1}$ and state that the e_j, c_j form generators.

In any case from (∗) we get

$$\mu^*(c)^n = 1 \otimes c^n + n \alpha \otimes c^{n-1} e$$

whence $$(pr_2)_* \, \mu^*(c)^n = n c^{n-1} e .$$

<u>Prop:</u>
$$\begin{cases} H_S^{2n}(N) = k \lambda^n & n \geq 0 \\ H_S^{2n-1}(N) = k \cdot p_*(c^n) & n \geq 1 \end{cases}$$

and $p^* : H_S^{2n-1}(N) \xrightarrow{\sim} H^{2n-1}(N)$. Here $\lambda \in H_S^2(pt)$ is the canonical generator.

Proof: $H^0_S = H^0 = k.1$.

$$0 \to H^1_S \xrightarrow{\tilde{p}^*} \underset{\substack{\| \\ ke}}{H^1} \xrightarrow[]{\overset{0}{p_*}} H^0_S \xrightarrow{\tilde{u}} H^2_S \xrightarrow{\overset{0}{p^*}} \underset{\substack{\| \\ kc}}{H^2} \xrightarrow{\tilde{p}_*} H^1_S \xrightarrow{\overset{0}{u}} H^3_S \xrightarrow{p^*} \underset{\substack{\| \\ kec}}{H^3} \xrightarrow{p_*}$$

$$H^1_S \ni p_*(c) \neq 0 \quad \Rightarrow \quad H^1_S = k\, p_*(c) \xrightarrow{\sim} H^1$$

$$\Rightarrow \quad p_* : H^2 \xrightarrow{\sim} H^1_S \quad \Rightarrow \quad H^0_S \overset{u}{\xrightarrow{\sim}} H^2_S$$

$\Rightarrow H^2_S = ku$. Also ——— $\Rightarrow H^1_S \xrightarrow[u]{0} H^3_S$

$\Rightarrow H^3_S \hookrightarrow \underset{\substack{\| \\ kec}}{H^3}$. But $H^3_S \ni p_*(c^2) \neq 0$, ~~so~~ $H^3_S \xrightarrow{\sim} H^3$

etc. etc.

So we know now the equivariant cohomology of $\mathcal{L}BU(1)$. It has the basis λ^n $n \geq 0$, $p_*(c^n)$ $n \geq 1$. Since $\lambda \to 0$ under p^* = the map forgetting the S^1 action, and since $p^* : H^{odd}_S \xrightarrow{\sim} H^{odd}$ it follows that

$$\boxed{\lambda \cdot p_*(c^n) = 0}$$

Also
$$p_*(c^m) . p_*(c^n) = p_*(c^m\, p^* p_*(c^n))$$
$$= p_*(c^m\, n c^{n-1} e)$$

But $p_* : H^{odd} \to H^{even}_S$ is zero as $p^* : H^{odd}_S \xrightarrow{\sim} H^{odd}$ so we see that

$$\boxed{p_*(c^m) \cdot p_*(c^n) = 0}$$

which determines the multiplication structure of $H^*_S(\mathcal{L}BU(1))$

Is there something we can say about the localized theory.

$$H^*\{\underbrace{k[\lambda]\otimes\Omega(N)^S}_{\Omega(N)^S[\lambda]},\ d_\lambda\} = H^*_S(N)$$

Bismut's form is $\underset{\text{map } N\to\Omega(1)}{\text{monodromy}} \times e^{\lambda^{-1}\left\{\substack{\text{averaged}\\ \text{curvature}}\right\}}$

which clearly belongs to

$$\left(\Omega(N)^S[[\lambda^{-1}]]\right)_{\text{degree}=0} = \prod_{k\geq 0}\Omega^{2k}(N)^S\lambda^{-k}$$

Let's adopt the viewpoint that $\mathcal{L}BU(1)$ represents on the category of S^1-manifolds the functor assigning the isomorphism classes of line bundles. Then for N in this category

$$H^*_S(N) = H^*\{\Omega(N)^S[\lambda],\ d_\lambda\}$$

$$H^*_S(N)[\lambda^{-1}] = H^*\{\Omega(N)^S[\lambda,\lambda^{-1}],\ d_\lambda\},$$

Bismut's form belongs to

$$\left(\Omega(N)^S[[\lambda^{-1}]]\right)_{\text{degree } 0} = \prod_{k\geq 0}\Omega^{2k}(N)^S\lambda^{-k}$$

For N finite-diml, this is a polynomial in λ^{-1}, so that this ring is the same as

$$\left(\Omega(N)^S[\lambda^{-1}]\right)_{\text{degree } 0} \subset \Omega(N)^S[\lambda,\lambda^{-1}].$$

However as we let N approach $\mathcal{L}BU(1)$ what happens?

Let's look more generally at the case $N = \mathcal{L}M$, and more precisely let us take finite dimensional manifolds approximating $\mathcal{L}M$. Then we have

$$H^*_S(N) = H^*\{\Omega(N)^S[\lambda], d_\lambda\}$$

which is graded so we can think of it either as polynomial or power series in λ, except that the former allows specialization. Next we can localize

$$H^*_S(N)[\lambda^{-1}] = H^*\{\Omega(N)^S[\lambda,\lambda^{-1}], d_\lambda\}$$

and we know when N is finite dimensional that we get $H^*(N^S) \otimes k[\lambda,\lambda^{-1}]$. If N is infinite-diml. then

December 15, 1984

Let us take up the idea of having a function defined on smooth paths somehow determining an operator on L^2 for the Wiener measure. I will will think of the big Hilbert space $\mathcal{H} = L^2(d\mu)$ as obtained from cylinder functions on a real vector space. To be specific ~~let's~~ suppose that the we are given the real Hilbert space $\ell^2_{\mathbb{R}}$ with coordinate functions x_n $n \geq 1$. Then $\mathcal{H}$ has the orthonormal basis consisting of the Hermite polynomials $H_\alpha(x) = H_{\alpha_1}(x_1) H_{\alpha_2}(x_2) \cdots$. $\mathcal{H}$ is isomorphic in a standard way to the symmetric algebra of $\ell^2_{\mathbb{C}}$ in the Hilbert space sense.

To understand $\mathcal{H}$ best we should think in terms of the holomorphic representation where the orthonormal basis is $\frac{z^\alpha}{\sqrt{\alpha!}}$. I recall that one has the generalized translation operators

$$T_\gamma = e^{-\frac{|\gamma|^2}{2}} e^{\gamma a^*} e^{-\bar{\gamma} a}$$

which are unitary:

$$\underbrace{e^{-\frac{|\gamma|^2}{2}} e^{-\gamma a^*} e^{\gamma a}}_{T_\gamma^*} \underbrace{e^{-\frac{|\gamma|^2}{2}} e^{\gamma a^*} e^{-\bar{\gamma} a}}_{T_\gamma}$$

$$= e^{-|\gamma|^2} e^{[\gamma a, \gamma a^*]} = I.$$

Since $\langle 0 | T_\gamma | 0 \rangle = e^{-\frac{|\gamma|^2}{2}}$ it is clear that $|\gamma|^2 < \infty$, i.e. γ is to lie in the 1 particle Hilbert space.

Now in order to realize $\mathcal{H}$ as L^2 relative to Gaussian (cylinder) measure, we need to take a real subspace V of the 1-particle Hilbert space spanned by the z_n. Then the cylinder measure is to sit on the dual V' of V.

So I think of V' as $\ell^2_{\mathbb{R}}$, so that V is spanned by the linear functions x_n with $v = \sum a_n x_n$ and

$$\langle v^2 \rangle = \sum a_n^2$$

Then $\mathcal{H}$ is obtained by orthogonalizing the polynomials x^α.

Let us define the smooth elements of V' to be vectors (x_n) where only finitely many x_n are $\neq 0$. Let us consider a quadratic function in the space V'_s of smooth elements. This has the form

$$Q(x) = \tfrac{1}{2}\sum c_{mn} x_m x_n$$

where c_{mn} is symmetric. Calculation in the case of finitely many non-zero c's gives

$$\boxed{\langle Q(x)^2 \rangle = \frac{1}{4}\Big(\sum_m c_{mm}\Big)^2 + \frac{1}{2}\sum (c_{mn}^2)}$$

In fact we have an orthogonal sum

$$Q(x) = \tfrac{1}{2}\sum c_{mm}(x_m^2 - 1) + \sum_{m<n} c_{mn} x_m x_n + \tfrac{1}{2}\Big(\sum c_{mm}\Big)$$

so

$$\langle Q(x)^2 \rangle = \frac{1}{4}\sum c_{mm}^2 \underbrace{\langle (x_m^2-1)^2 \rangle} + \frac{1}{2}\sum_{m\neq n} c_{mn} + \frac{1}{4}\Big(\sum c_{mm}\Big)^2$$

$$\underbrace{\langle x^4 \rangle}_{3} - 2 + 1 = 2$$

Let's work in reverse. Suppose we ~~started~~ with an [redacted] even element of $\mathcal{H}$ of degree ≤ 2. Then it can be expressed as an orthogonal sum

$$(\star) \qquad Q(x) = \frac{1}{2}\sum c_{mm}(x_m^2 - 1) + $$ [redacted] $$\sum_{m<n} c_{mn} x_m x_n + c$$

where c is a constant. Since this has a finite norm we must have

$$\sum c_{mn}^2 < \infty,$$

and conversely given such a Hilbert-Schmidt matrix and an arbitrary constant c, the formula $(\star)$ defines an [redacted] even element of $\mathcal{H}$ of degree ≤ 2.

Notice that the value of Q at 0 is not defined in general, nor on the smooth elements, since

$$\sum c_{mm}$$

may diverge.

December 16, 1984

Yesterday I considered the Gaussian cylinder measure on $V' = \ell^2_{\mathbb{R}}$ with basis e_n, and let $\mathcal{H}$ = corresponding Hilbert space $L^2(V')$. Then $\mathcal{H}$ is the symmetric tensor space of $\ell^2_{\mathbb{C}} = V_{\mathbb{C}}$, with the basis (orthonormal) x_n. Then I looked at quadratic functions on smooth vectors versus even elements of degree ≤ 2 in $\mathcal{H}$. Take real-valued elements:

$$Q_s(x) = \frac{1}{2}\sum' c_{mn} x_m x_n$$

$$Q(x) = \frac{1}{2}\sum_m c_{mm}(x_m^2 - 1) + \frac{1}{2}\sum_{m\neq n}' c_{mn} x_m x_n + c$$

One is getting some sort of torsor situation which I would like to understand. Look at Q_s and let's assume it makes sense as a function on $\ell^2_{\mathbb{R}}$. This should be roughly the same as $\sum c_{mn}^2 < \infty$. The argument might go as follows.

$$\left|\sum c_{mn} x_m x_n\right|^2 \leq \sum c_{mn}^2 \cdot \sum' x_m^2 x_n^2$$

$$= \sum' c_{mn}^2 \cdot \|x\|^4$$

More simply, we know that the inner product $\sum c_{mn} b_{mn}$ of two ℓ^2 sequences is finite, so that $Q_s(x)$ is just the effect on decomposable elements of a bdd linear fnl on $S^2(V')$. The real question would be whether a continuous quadratic function on V' extends to a bdd linear fnl. on $S^2(V')$. But this isn't crucial at the beginning.

I conclude that I should just start off with quadratic functions on V' defined by a Hilbert-Schmidt matrix.

Summary: We start with a real Hilbert space V' and consider the polynomial functions on it, i.e. the symmetric tensor Hilbert space $S(V)$. Now we want to attach elements in $L^2(V', d\mu)$, where $d\mu$ is Gaussian cylinder measure to these polynomial functions.

Actually we might first consider whether to try to construct the Hilbert space out of $S(V)$. In this case one would want to use a different measure.

Thus, let us consider $S(V)$ as the polynomial functions on V' and suppose a Gaussian (cylinder) measure is to be found on V'. More precisely I want to give the variance which is a positive definite quadratic form on V, the space of linear functions. Under what conditions can I conclude that the L^2 of this Gaussian measure is a completion of $S(V)$?

Thus we start with $\langle v^2 \rangle$ on V. Suppose the associated self-adjoint operator has discrete spectrum, whence we choose coordinates, i.e. an orthonormal basis of V, call it x_n, such that

$$\langle x_m x_n \rangle = \delta_{mn} \frac{1}{a_n}$$

An obvious necessary condition that $S(V)$ be contained in $L^2(V', d\mu)$ is that the function $\sum x_n^2$ satisfy

$$\left\langle \sum x_n^2 \right\rangle = \sum \frac{1}{a_n} < \infty .$$

I have seen that this condition is necessary + sufficient to produce a Gaussian measure (countably additive) on V'.

Basically I have this problem. I start with V a real topological vector space which to simplify I suppose to be a Hilbert space. Then I get the algebra $S(V_{\mathbb{C}})$ of symmetric tensors which I can think of as functions on V'. Also I am given a positive definite form on V. In finite dimensions this immediately determines a Gaussian probability measure $d\mu$ in V', such that the associated $L^2(V'; d\mu)$ is a completion of $S(V_{\mathbb{C}})$. Specifically one can give a recipe called Wick's theorem for computing the inner products of elements of $S(V)$.

This recipe is really best stated using the exponential functions on V'; to each $v \in V_{\mathbb{C}}$ one has an exponential function e^v on V' and

$$\langle e^v \rangle = e^{\frac{1}{2} Q(v)}$$

where Q is the quadratic extension to $V_{\mathbb{C}}$ of the given quadratic form $\langle v^2 \rangle$ on V. But stated this way it isn't clear there is an obstruction to extending things to infinite dimensions.

So now what happens as we try to compute the inner products of elements of $S(V)$?

First let us get the Hilbert space side straight. We begin with a real vector space V_R equipped with positive quadratic form. Then we make a Fock space which we can identify with the Hilbert space symmetric tensor space of the complex Hilbert space $\hat{V}_{\mathbb{C}}$, obtained by completing $V_R \otimes \mathbb{C} = V_{\mathbb{C}}$ with the inner product, which is the hermitian extension of the given quadratic form on $V_{\mathbb{R}}$. Moreover, on this Fock space we have unitary operators of the type $e^{-\frac{1}{2}|\gamma|^2} e^{\bar{\gamma}a^*} e^{-\gamma a}$ with $\gamma \in \hat{V}_{\mathbb{C}}$ which give a unitary representation of $\hat{V}_{\mathbb{R}}$. Thus to exponential functions $e^{i\gamma}$, $\gamma \in \hat{V}_{\mathbb{R}}$ one has attached unitary operators.

So for the Hilbert space side, one might as well have started with a real Hilbert space $\hat{V}_{\mathbb{R}}$. However we know we are going to have trouble realizing all functions on $\hat{V}_{\mathbb{R}}$ as operators. This is the point of starting with a $V_{\mathbb{R}} \hookrightarrow \hat{V}_{\mathbb{R}}$ which gives a special class of functions on $\hat{V}_{\mathbb{R}}$.

So what next? I think I have to get more specific about <u>Brownian motion</u>.

List the ideas.

1) We have a space $S(V_{\mathbb{C}})$ of polynomial functions on the dual space $V'_{\mathbb{R}}$. A positive definite form on $V_{\mathbb{R}}$ will, under suitable conditions, extend to give an inner product on $S(V_{\mathbb{C}})$. This works in finite dimensions; one gets a Gaussian probability measure on $V'_{\mathbb{R}}$ whose L^2 is the completion of $S(V_{\mathbb{C}})$.

2) Holomorphic repn. picture.

3) Basic torsor picture.

Perhaps the main source of confusion is due to the fact that there are two points of departure. Either I start with a $(V_{\mathbb{R}}, Q)$ and hope for a Gaussian measure on $V'_{\mathbb{R}}$, or I start with a real Hilbert space $\hat{V}_{\mathbb{R}}$ and the Gaussian cylinder measure on $\hat{V}'_{\mathbb{R}}$ in which case I have difficulties with realizing functions on $\hat{V}'_{\mathbb{R}}$ as operators.

I want to regard the Fock space as the basic object. This means that even starting with $(V_{\mathbb{R}}, Q)$ I rapidly get to the Hilbert space $\hat{V}_{\mathbb{R}} \subset \hat{V}_{\mathbb{C}}$ and its symmetric tensor spaces. So now it seems that I am forced to look at the relation between $S(\hat{V}_{\mathbb{C}})$ considered as polynomial functions on $\hat{V}'_{\mathbb{R}}$ and as elements or operators on Fock space.

We suppose to simplify that $V_{\mathbb{R}}$ is a Hilbert space and that we can diagonalize Q relative to the inner product. Use orthonormal coordinates on $\hat{V}'_{\mathbb{R}}$, so that $Q(x) = \sum x_n^2$, and the norms on $V'_{\mathbb{R}}$ is $\sum \frac{x_n^2}{\lambda_n}$ with $0 < \lambda_n \to \infty$. In other words an element of $V_{\mathbb{R}}$ is a linear function $\sum b_n x_n$ with $\sum \lambda_n b_n^2 < \infty$.

Repeat the **idea**: We will be dealing with polynomials in the x_n. A typical polynomial of degree d will be

$$\frac{1}{d!} \sum c_{m_1 \ldots m_d} x_{m_1} \cdots x_{m_d} \quad \text{with} \quad \sum c_{m_1 \ldots m_d}^2 < \infty.$$

This gives a natural polynomial function on $V'_{\mathbb{R}}$ of degree d and an element of Fock space.

I'd like to think of Fock space as consisting of functions on $\hat{V}_{\mathbb{R}}'$. This is what happens in finite dimensions: Fock space is L^2 of Gaussian measure.

In infinite dimensions we run into the problem that polynomial functions on $\hat{V}_{\mathbb{R}}'$ are not in Fock spaces. Only renormalized polys. are. The example is a quadratic polynomial

$$\frac{1}{2}\sum c_{mn}x_m x_n \qquad \text{with} \quad \sum c_{mn}^2 < \infty .$$

If it were in the Fock space, then its inner product with 1, namely

$$\left\langle \frac{1}{2}\sum c_{mn}x_m x_n \right\rangle = \frac{1}{2}\sum c_{mm}$$

would be well-defined. So what we learn from this is the necessity of renormalizing functions on Hilbert space before they become operators.

Now the next point of extreme interest is functions defined almost everywhere. Suppose we have $V_{\mathbb{R}} \subset \hat{V}_{\mathbb{R}}$ consisting of $\sum b_n \frac{x_n}{\sqrt{\lambda_n}}$ with $\sum b_n^2 < \infty$. Then an element of $S^2(V_{\mathbb{R}})$ is of the form $\frac{1}{2}\sum b_{mn}\frac{x_m x_n}{\sqrt{\lambda_m \lambda_n}}$ with $\sum b_{mn}^2 < \infty$. And the expectation is finite as

$$\left(\frac{1}{2}\sum b_{mm}\frac{1}{\lambda_m}\right)^2 \leq \frac{1}{4}\sum b_{mm}^2 \sum \frac{1}{\lambda_m^2}$$

Now I notice that I have made the same mistake as before. The norm on a Hilbert space $\hat{V}_{\mathbb{R}}'$ is indeed a polynomial function $\sum x_n^2$ which is not in $S^2(V_{\mathbb{R}})$. So really this function can't be in the Fock space. Thus the Fock space can, at most, consist of $\sum c_{mn}x_m x_n$ with $\sum c_{mn}^2 < \infty$, and even these have to be renormalized. I am curious about the function $e^{-\frac{t}{2}\sum b_n x_n^2}$ which should satisfy

$$\left\langle e^{-\frac{t}{2}\sum b_n x_n^2}\right\rangle = \prod_n (1+tb_n)^{-1/2}$$

This function makes sense for $|b_n| \leq$ Const., it can be correlated with something in the Fock space when $\sum b_n^2 < \infty$, and finally, it is good enough to define something in the Fock space when $\sum b_n < \infty$. The whole thing is confusing.

Let's consider again a real Hilbert space denoted $\hat{V}_{\mathbb{R}}$ and the Fock space associated to it. Let $\hat{V}_{\mathbb{R}}$ have the orthonormal basis x_n which we interpret as linear functionals on $\hat{V}'_{\mathbb{R}}$. Take a quadratic function diagonal in this basis

$$\sum b_n x_n^2$$

If b_n is a bounded sequence, this is a smooth functions on $\hat{V}'_{\mathbb{R}}$ of degree 2. If $\sum b_n^2 < \infty$, then this function determines an element of Fock space modulo constants. If $\sum |b_n| < \infty$, then this function determines an element of Fock space.

Continuous quadratic functions are of the form $\langle x|B|x\rangle$ with B bdd symmetric. Then B has to be Hilbert-Schmidt before it can be associated to an element of Fock space. The element of Fock space is then unique up to an additive constant which one can fix by requiring the expectation to be zero. But only if B is of trace class can one hope to realize $\langle x|B|x\rangle$ in the obvious way as an element of the Fock space.

Given a real Hilbert space $\hat{V}_{\mathbb{R}}$ we then have two algebras to consider. The algebra of operators on Fock space which commute with multiplication by the x_n, and the algebra of functions on $\hat{V}_{\mathbb{R}}'$. We know that we should replace the latter by $S(\hat{V}_{\mathbb{R}})$ at least if we want things of finite degree. Because Fock space has a cyclic vector for the action of the x_n we can probably speak of finite degree operators. So we have something reminiscent of a deformation of the polynomial ring. Except that both algebras contain polynomials in the x_n densely.

Notice that we do seem to have a map from the operator algebra to the symmetric tensor Hilbert space of $\hat{V}_{\mathbb{R}}$. Namely, given an operator, I apply it to the cyclic vector getting an element of Fock space, which determines the operator, then I use the grading of Fock space. So what this means is that we have some kind of symbol map. Among the quadratic elements what happens

Put $V = \hat{V}_{\mathbb{R}}$. Then

$$\begin{tikzcd} V \arrow[r, "\sim"] & Op_1 \end{tikzcd}$$

$$\begin{tikzcd} V \otimes V \arrow[r, "\sim"] \arrow[d] & Op_1 \otimes Op_1 \arrow[d] \\ S^2(V) & Op_2 \end{tikzcd}$$

This is not very precise. However you ought to be able to define Op_2 by standard symplectic stuff. I ought to be able to rigorously construct 1-parameter groups belonging to quadratic Hamiltonians.

December 17, 1984

Review of yesterday's work. Consider a real Hilbert space, say $\ell^2_{\mathbb{R}}$, consisting of square integrable sequences $x = (x_n)_{n\geq 1}$. Let V be the dual space of linear fns. $\sum b_n x_n$ with b_n square summable. From V one constructs the Fock space which is the Hilbert space symmetric tensor space $\hat{S}(V)$ with operators $x_n = a_n + a_n^*$.

Now the principal theme will be to compare operators on Fock space with functions on $V' = \ell^2_{\mathbb{R}}$. Fock space has the cyclic vector $|0\rangle$ under the operators x_n, and these operators commute, so we have a commutative algebra of operators on Fock space, probably both a C^* and a W^* algebra. In finite dimensions these operators can be identified with functions on V'.

I want to discuss the quadratic case carefully, because I think I can really get my hands on the associated unitary operators by means of symplectic transformations and their unitary implementability.

So what I want to consider is some rigorous version of the 1-parameter unitary group $e^{itQ(x)}$ where Q is a quadratic function of x. I think we only have to make sense of $e^{itQ(x)}|0\rangle$, i.e. to actually define this as an element of Fock space.

A first step will be the formal series in t, and in particular the term of degree 1, $Q(x)|0\rangle$. The latter will have the form

$$Q(x)|0\rangle = \Big(\frac{1}{2}\sum_n c_{nn}(x_n^2-1) + \sum_{m<n} c_{mn}\, x_m x_n + c\Big)|0\rangle$$

$$= \frac{1}{2}\sum_n c_{nn}(a_n^*)^2|0\rangle + \sum_{m<n} c_{mn}\, a_m^* a_n^* |0\rangle + c|0\rangle.$$

This is an orthogonal sum, so $c = \langle 0|Q(x)|0\rangle$ and

$$\left| Q(x)|0\rangle \right|^2 = \frac{1}{2}\sum c_{mn}^2 + c^2$$

So $Q(x)|0\rangle$ is a renormalized version of the quadratic form $\frac{1}{2}\sum c_{mn} x_m x_n$. The operator $Q(x)$ I am after is

$$Q(x) = :\frac{1}{2}\sum c_{mn}(a_m + a_m^*)(a_n + a_n^*): + c$$

$$= \frac{1}{2}\sum_{m\neq n} c_{mn}\, x_m x_n + \frac{1}{2}\sum_{n} c_{nn}\underbrace{\left(a_n^{*2} + 2a_n^* a_n + a_n^2\right)}_{x_n^2 - 1} + \cdots$$

$$x_n^2 = (a_n + a_n^*)^2 = a_n^2 + a_n a_n^* + a_n^* a_n + a_n^{*2} = (a_n^{*2} + 2a_n^* a_n + a_n^2) \cdots$$

So what is it that we learn? The even operators of degree ≤ 2 form a [redacted] one dimensional extension of the [redacted] Hilbert space $\hat{S}^2(V)$, which splits via the normal ordering prescription.

If I were to pursue this line of investigation the obvious next step is to look at the group of implementable symplectic transformations. The Lie algebra should be the Hilbert-Schmidt quadratic functions of the variables x_n, p_n, and there should be an interesting central extension which doesn't split.

Let's leave this for the moment and go back to the case of Brownian motion. The Hilbert space which is L^2 for Wiener measure is the Fock space whose one-particle space $\hat{V}$ consists of real $f(t)$, $t \geq 0$ with

$$\|f\|^2 = \iint G(t,t')\, f(t) f(t')\, dt\, dt' < \infty \qquad G(t,t') = \min\{t,t'\}$$

This is the $\underline{-1}$ norm, since $G = \left(-\frac{d^2}{dt^2}\right)^{-1}$.

My goal will be to pin down as much as possible the significance of the [redacted] parallel transport or Ito type functions in the Hilbert space $L^2(d\mu)$. In the case where one has a countably additive measure we know that somehow the class of polynomial functions has been restricted. ?

Let V be a real Hilbert space and suppose that it is densely embedded in a Hilbert space W in such a way that the Gaussian cylinder measure in V becomes countable additive in W. This means we can find an orthonormal basis e_n in W such that $\frac{1}{\sqrt{a_n}} e_n$ is an orthonormal basis for V where $\sum \frac{1}{a_n} < \infty$. Thus $V = \{\sum x_n e_n \mid \sum a_n x_n^2 < \infty\}$.

So $W = \ell^2_{\mathbb{R}} = \{ x = (x_n) \mid \sum x_n^2 < \infty \}$, and we are dealing with the Gaussian measure

$$\prod_{n=1}^{\infty} \left(e^{-\frac{1}{2} a_n x_n^2} \frac{\sqrt{a_n}\, dx_n}{\sqrt{2\pi}} \right)$$

which I know is countably additive under the hypothesis for $\sum \frac{1}{a_n} < \infty$, i.e. the embedding $V \longrightarrow W$ is Hilbert Schmidt.

Now we know how to describe L^2 of this Gaussian measure as a Fock space based on V. So V itself appears as the 1-particle subspace of the Fock space, which means that pairing with elements of V are almost everywhere defined functions on W.

Look more carefully. An element of V is a sum $v = \sum b_n e_n$ with $\sum a_n b_n^2 < \infty$. Taking inner product with v in V gives the linear function

$$\sum a_n b_n x_n$$

which is defined on $V \subset W$. It seems that this function is almost everywhere defined with respect to the Gaussian measure. That is because

$$\left\langle \left(\sum a_n b_n x_n\right)^2 \right\rangle = \sum a_n^2 b_n^2 \frac{1}{a_n} = \sum a_n b_n^2 < \infty .$$

On the other hand we know that V is of measure zero in W. It is a countable union of balls in V and [redacted] each ball $\sum_1^\infty a_n x_n^2 \leq N$ is an intersection of $\left\{\sum_1^k a_n x_n^2 \leq N\right\}$ whose measure can be shown to go to zero with k. Essentially we are computing the volume of a ball in Hilbert space relative to the Gaussian cylinder measure, and this is zero.

So here we have a peculiar situation. We have a function in the Fock space, so it is somehow defined [redacted] a.e. on W, and it is apparently linear hence defined on some dense subspace.

Let's try to analyze this more carefully. We have a linear function $\sum c_n x_n$ which is densely defined such that

$$\left\langle \left(\sum c_n x_n\right)^2 \right\rangle = \sum \frac{c_n^2}{a_n} < \infty$$

For example we could take all $c_n = 1$. Now this function, or series, should be definable off a set of measure 0!

I seem to have run in the following problem. Suppose x_n are independent Gaussian variables with $\sum \langle x_n^2 \rangle < \infty$. Then $\sum x_n$ is a Gaussian random variable of variance $\sum \langle x_n^2 \rangle$ which means that the series

$\sum x_n$ has some sort of meaning almost everywhere. The problem is to make precise the meaning.

Simple case. Put $\sigma_n^2 = \langle x_n^2 \rangle$ and suppose that σ_n tends rapidly to zero. In this case we can consider the cube $|x_n| \leq b_n$, and we can choose the b_n so that $\sum' b_n < \infty$, so the series $\sum' x_n$ is absolutely convergent on this cube, and also so that the cube has positive measure. Replacing b_n by tb_n and letting $t \to \infty$, we get a set of measure 1 on which the series $\sum x_n$ converges absolutely.

This is not sufficiently interesting. Notice that the function $\sum x_n$ on the cube will be continuous as the convergence is uniform.

A slight variant of the above simple case is to replace the cube $|x_n| \leq b_n \ \forall n$, by the ball $\sum \frac{x_n^2}{b_n^2} \leqslant t$. As long as $b_n^2 \to 0$ this ball will be compact, and I think my earlier arguments show that the measure of the ball will approach 1 as $t \to \infty$ provided that $\sum \frac{\sigma_n^2}{b_n^2} < \infty$. The series will converge absolutely when we can use

$$\sum' |x_n| = \sum \left| \frac{x_n}{b_n} b_n \right| \leq \left(\sum \frac{x_n^2}{b_n^2} \right)^{1/2} \left(\sum b_n^2 \right)^{1/2}$$

so the obvious question is given $\sum \sigma_n^2 < \infty$ can be find b_n such that $\sum b_n^2 < \infty$ and such that $\sum \frac{\sigma_n^2}{b_n^2} < \infty$. These imply $\sum \sigma_n < \infty$

Thus it would seem that in the case $\sum \sigma_n < \infty$ (we take $b_n = \sqrt{\sigma_n}$) and then the series $\sum' x_n$ will converge absolutely a.e.

But really the important case is where the sequence of variances σ_n is square summable, but not summable. Then maybe one has to have cancellation and conditional convergence a.e.

Example: Suppose we take a divergent series of the form $1 + \underbrace{\frac{1}{n_1} + \cdots + \frac{1}{n_1}}_{n_1 \text{ times}} + \underbrace{\frac{1}{n_2} + \cdots + \frac{1}{n_2}}_{n_2 \text{ times}} + \cdots$

which is square-summable: $1 + \overbrace{\frac{1}{n_1}}^{= n_1 \times \frac{1}{n_1^2}} + \frac{1}{n_2} + \frac{1}{n_3} + \cdots < \infty$.

Then we use this divergent series, call it $\sum \sigma_n$, as the sequence of standard deviations: $\langle x_n^2 \rangle = \sigma_n^2$. Now we want to make sense of the sum $\sum x_n$. According to the proof of the Riesz-Fischer thm, one takes a suitable subsequence of the sequence of partial sums $S_N = \sum_{n=1}^{N} x_n$ and shows this converges a.e.

So I want to try this argument using the natural partial sum in the present example. So what I do is to sum up the x_n belonging to a given block of length n_k. Let $y_k = \sum_{\substack{n \in k\text{th} \\ \text{block}}} x_n$. Then y_k is a Gaussian r.v. of variance $\langle y_k^2 \rangle = \frac{1}{n_k}$. But this time we could suppose that $\sum \frac{1}{n_k^{1/2}} = \sum \langle y_k^2 \rangle^{1/2} < \infty$. And I believe in this case I checked that the series $\sum y_k$ converges absolutely a.e.

Let's now try the general case. We are given the sequence of Gaussian r.v. x_n with $\langle x_n^2 \rangle = \sigma_n^2$ and $\sum \sigma_n^2 < \infty$. Then I choose a sequence $0 = n_0 < n_1 <$ such that $\sum\limits_{n_{k-1} < n \le n_k}' \sigma_n^2$ goes to zero rapidly in k. Then put $y_k = \sum\limits_{n_{k-1} < n \le n_k}' x_n$. This is a sequence of Gaussian r.v.'s with $\langle y_k^2 \rangle = \sum\limits_{n_{k-1} < n \le n_k}' \sigma_n^2$ going to zero rapidly.

Therefore I know that $\sum\limits_{k=1}^{\infty} y_k$ converges absolutely, a.e.

Why? [redacted] Because

$$\left(\sum_{k=1}^{\infty} |y_k| \right)^2 \le \sum' \frac{y_k^2}{b_k^2} \sum b_k^2$$

Put $b_k^2 = \langle y_k^2 \rangle^{1/2}$. Then $\sum b_k^2 < \infty$ and

$$\int \sum' \frac{y_k^2}{b_k^2} = \sum' \frac{\langle y_k^2 \rangle}{\langle y_k^2 \rangle^{1/2}} = \sum b_k^2 < \infty .$$

Thus $\sum\limits_{k=1}^{\infty} |y_k|$ is square integrable, and so the series $\sum y_k$ converges abs. a.e.

But actually if the σ_n go to zero fast enough I produced a cube carrying as much of the measure as I want so that the series $\sum y_k$ converges uniformly and absolutely on this cube.

December 19, 1984

On ΛV and why $\det(\omega)\, e^{-\bar{J}\frac{1}{\omega}J}$ is free of denominators. It seem simpler to do this in the orthogonal case, as opposed to the unitary case.

So let V be a finite dimensional vector space. On ΛV we have the operators ι_λ, $\lambda \in V^*$, $\ni \iota_\lambda^2 = 0$ hence there is a unique structure of left module on ΛV over ΛV^* such that $\lambda \in V^*$ acts as ι_λ. This gives a canonical map

$$\Lambda V^* \otimes \Lambda^n V \longrightarrow \Lambda V \qquad n = \dim V$$

which is an isomorphism.

Now let $\omega \in \Lambda^2 V$. Suppose ω is non-degenerate i.e. the map $V^* \longrightarrow V$, $\lambda \longmapsto \iota_\lambda \omega$ is an isomorphism ($\Rightarrow n$ even). Then there is a unique $\hat{\omega} \in \Lambda^2 V^*$ such that $\hat{\omega} \longmapsto \omega$ under this isomorphism.

Let's calculate a formula for $\hat{\omega}$ in terms of a basis $e_1, \dots, e_n$ for V, and the dual basis $e_1^*, \dots, e_n^*$ of V^*. One has $\omega = \frac{1}{2}\omega_{ij} e_i e_j$

$$\iota_\lambda \omega = \omega_{ij}\, \lambda(e_i) e_j \qquad \text{so} \qquad e_k^* \longmapsto \iota_{e_k^*}\omega = \omega_{kj} e_j$$

Put $\hat{\omega} = \frac{1}{2} a_{ij} e_i^* e_j^*$. Then

$$\hat{\omega} \longmapsto \frac{1}{2} a_{ij}\, \omega_{ik} e_k\, \omega_{jl} e_l = \frac{1}{2}(-\omega_{ki} a_{ij} \omega_{jl}) e_k e_l$$

whence we want $-\omega a \omega = \omega$ or $a = -\omega^{-1}$.

Thus

$$\boxed{\omega = \frac{1}{2}\omega_{ij} e_i e_j \implies \hat{\omega} = -\frac{1}{2}(\omega^{-1})_{ij}\, e_i^* e_j^*}$$

For example if $n=2$, $(\omega)_{ij} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $(\omega^{-1})_{ij} = \begin{pmatrix} 0 & -a^{-1} \\ a^{-1} & 0 \end{pmatrix}$, then $\omega = a e_1 e_2$ and $\hat{\omega} = a^{-1} e_1^* e_2^*$

Prop. Under the isomorphism

$$\Lambda V^* \otimes \Lambda^n V \xrightarrow{\sim} \Lambda V$$

one has
$$e^{-\hat{\omega}} \otimes \frac{\omega^{n/2}}{(n/2)!} \longmapsto e^{\omega}$$

Proof: Choose the basis so that $\omega = \sum_{i=1}^{n/2} a_i e_{2i-1} e_{2i}$. It's enough to check when $n=2$. Then

$$e^{-\hat{\omega}} \otimes \frac{\omega^{n/2}}{(n/2)!} = (1 - a^{-1} e_1^* e_2^*) \otimes (a e_1 e_2)$$
$$\longmapsto (1 - a^{-1} a_1 a_2)(a e_1 e_2) = 1 + a e_1 e_2 = e^{\omega}.$$

Next we need to describe e^{ω}. Let's recall the definition of the Pfaffian of a skew symmetric matrix ω_{ij}. To avoid confusion put $\tilde{\omega} = \frac{1}{2} \omega_{ij} e_i e_j$ for the associated element of $\Lambda^2 V$. Then

$$\frac{\tilde{\omega}^{n/2}}{(n/2)!} = Pf(\omega)\, e_1 \wedge \cdots \wedge e_n$$

defines the Pfaffian. More [redacted] detailed formula

$$Pf(\omega) = \sum_{\pi \in \Sigma_n} \frac{1}{2^{n/2}(n/2)!} \operatorname{sgn}(\pi)\, \omega_{\pi 1, \pi 2}\, \omega_{\pi 3, \pi 4} \cdots \omega_{\pi(n-1), \pi(n)}$$

If $n=2$ then $Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$.

If $I \subset \{1, \dots, n\}$ put $e_I = e_{i_1} e_{i_2} \cdots e_{i_p}$ where $I = \{i_1, \dots, i_p\}$, $i_1 < i_2 < \dots < i_p$.

Prop.
$$\boxed{e^{\tilde{\omega}} = \sum_{I \subset \{1, \dots, n\}} Pf(\omega_I)\, e_I}$$

where ω_I is submatrix of ω_{ij} with rows + columns indexed by the set I.

Proof. We consider the projection $V = \bigoplus_{i=1}^{n} Fe_i \longrightarrow V_I = \bigoplus_{i \in I} Fe_i$. This carries $e^{\tilde{\omega}}$ to [redacted] $e^{\tilde{\omega}_I}$ and kills all e_J for $J \not\subset I$, and preserves all e_J for $J \subset I$. Thus if $\mathrm{card}(I) = p$ the coefficient of [redacted] e_I can be found by looking at the degree p coefficient of $e^{\tilde{\omega}_I}$. This is $\dfrac{(\tilde{\omega}_I)^{p/2}}{(p/2)!} = Pf(\omega_I)\, e_I$.

Now we want to apply this to the formula

$$e^{\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} \cdot \frac{\tilde{\omega}^{n/2}}{(n/2)!} = e^{\tilde{\omega}}$$

$$\|$$

$$Pf(\omega) \sum_I Pf((\omega^{-1})_I)\, e_I^*(e_1 \cdots e_n) = \sum_J{}' Pf(\omega_J)\, e_J$$

Now
$$e_{i_1}^* \cdots e_{i_p}^* (e_1 \cdots e_n) = e_{i_1}^* \cdots e_{i_{p-1}}^* (-1)^{i_p - 1} (e_1 \cdots \hat{e}_{i_p} \cdots e_{i_n})$$
$$= (-1)^{\sum (i_j - 1)}\, e_1 \cdots \hat{e}_{i_1} \cdots \hat{e}_{i_p} \cdots e_n$$

So we end up with the formula

$$\boxed{\begin{array}{c} Pf(\omega)\cdot Pf\left((\omega^{-1})_I\right) = (-1)^{\sum_{j=1}^{p}(i_j-1)}\, Pf(\omega_J) \\ \text{if} \quad I = i_1<\cdots<i_p \quad \text{and} \quad J=\{1,\dots,n\}-I \end{array}}$$

We can play with the signs a little bit more as follows. Recall that to the subset $I=\{i_1,\dots,i_p\}$ we have a Schubert cell

$$\begin{array}{cccccc} & & i_1 & & i_2 & \\ * & * & 1 & & & \\ * & * & 0 & * & * & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

whose dimension is $\quad d(I) = (i_1-1)+(i_2-2)+\cdots+(i_p-p)$.

Moreover

$$e^*_{i_p}\cdots e^*_{i_1}(e_1\cdots e_n) = e^*_{i_p}\cdots e^*_{i_2}(e_1\cdots\hat{e}_{i_1}\cdots e_n)\,(-1)^{(i_1-1)}$$
$$= (-1)^{d(I)}\, e_J$$

As a further check we have

$$e_I\, e_J = (-1)^{d(I)}\, e_1\cdots e_n$$

since $(-1)^{d(I)}$ is the sign of the shuffle permutation sending $1,\dots,p$ to $i_1<\cdots<i_p$ and $p+1,\dots,p+q=n$ to $j_1<\cdots<j_q$.

So the above can be written

$$\boxed{Pf(\omega)\; Pf\left((\omega^{-1})_I\right) = (-1)^{\frac{p}{2}+d(I)}\; Pf(\omega_J)}$$

~~But~~ where I note that p is even, so $\frac{p(p-1)}{2} \equiv \frac{p}{2} \mod 2$

Corollary
$$\boxed{Pf(\omega)Pf(\omega^{-1}) = (-1)^{n/2}}$$

$$\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}\cdot\begin{pmatrix} & -a^{-1} \\ a^{-1} & \end{pmatrix}$$

$a\cdot(-a^{-1}) = -1.$
for $n=2$

Now I want to apply this to a Thm

form $Pf(\omega)\, e^{-\frac{1}{2}(x^2 + dx^t \frac{1}{\omega} dx)}$

$$Pf(\omega)\, e^{-\frac{1}{2} dx^i \left(\frac{1}{\omega}\right)_{ij} dx^j} = \sum_I Pf(\omega)\, Pf\left(\left(\frac{-1}{\omega}\right)_I\right) dx^I$$

$$p = |I| \qquad = \sum_I (-1)^{\frac{p}{2}} \underbrace{Pf(\omega)\, Pf\left((\omega^{-1})_I\right)}_{(-1)^{\frac{p}{2}+d(I)}\, Pf(\omega_J)} dx^I$$

So the formula is

$$\boxed{Pf(\omega)\, e^{-\frac{1}{2} dx^t \frac{1}{\omega} dx} = \sum_I (-1)^{d(I)} \underset{J = I'}{Pf(\omega_J)}\, dx^I}$$

$$a\, e^{-(-a^{-1}) dx^1 dx^2} = a + dx^1 dx^2$$

Finally we check the integral over $\mathbb{R}^n$.
Recall that $tr_s(e^{(D+L)^2}) = i^{n/2} Pf(\omega)\, e^{-\frac{1}{2}(x^2 + dx^t \frac{1}{\omega} dx)} \det{}^{1/2}(\text{—})$
(the $\frac{1}{2}$ can be put in by scaling.) Integrate over $\mathbb{R}^n$.
We obtain $\left(\frac{i}{2\pi}\right)^{n/2} \int i^{n/2} e^{-\frac{1}{2}x^2} dx^1 \dots dx^n = \left(\frac{i}{2\pi}\right)^{n/2} i^{n/2} (2\pi)^{n/2} = (-1)^{n/2}$
which is the expected sign.

Another formula for tomorrow $\qquad a_\mu = i(\partial_\mu)$

$$e^{-\frac{1}{2}\omega_{\mu\nu}\, i(\partial_\mu)\, i(\partial_\nu)} dx^1 \dots dx^n = \sum_I Pf(-\omega_I) \underbrace{a_I\, dx^1 \dots dx^n}_{(-1)^{\frac{p}{2}+d(I)} dx^J}$$

$$= \sum_I Pf(\omega_I)\, dx^J (-1)^{d(I)} = Pf(\omega)\, e^{-\frac{1}{2} dx^t \frac{1}{\omega} dx}$$

December 20, 1984

Define structure of left module over ΛV on ΛV^* and consider the isomorphism

$$\Lambda V^{\bullet} \otimes \Lambda^n V^* \xrightarrow{\sim} \Lambda V^*.$$

Given $\omega \in \Lambda^2 V$ non degenerate it determines $V^* \xrightarrow{\sim} V$, $\lambda \mapsto \iota_\lambda(\omega)$ whence ω can be lifted back to $\hat{\omega} \in \Lambda^2 V^*$. Formula.

$$\boxed{\omega = \tfrac{1}{2}\omega_{ij}\, e_i e_j \qquad \hat{\omega} = -\tfrac{1}{2}(\omega^{-1})_{ij}\, e_i^* e_j^*}$$

$n=2 \qquad \omega = a e_1 e_2 \qquad \hat{\omega} = a^{-1} e_1^* e_2^*$

Main Formula:

$$\boxed{e^{-\omega} \cdot \frac{\hat{\omega}^m}{m!} = e^{\hat{\omega}}}$$

$n = \dim V$
$m = n/2$

Proof by reduction to the case $n=2$

$$e^{-a e_1 e_2}(a^{-1} e_1^* e_2^*) = a^{-1} e_1^* e_2^* - (e_1 e_2)(e_1^* e_2^*)$$
$$= 1 + a^{-1} e_1^* e_2^* = e^{a^{-1} e_1^* e_2^*}$$

Next bring in Pfaffian: If ω_{ij} is a skew-symm. matrix its Pfaffian is defined by

$$\frac{\omega^m}{m!} = \mathrm{Pf}(\omega)\, e_1 \cdots e_n$$

where we identify the matrix ω_{ij} with the elt. $\omega = \frac{1}{2}\omega_{ij} e_i e_j$ of $\Lambda^2 V$. Formula:

$$\boxed{e^{\omega} = \sum_S \mathrm{Pf}(\omega_S)\, e_S}$$

where S runs over subsets of $\{1, \dots, n\}$ of even cardinality

$e_S = e_{s_1} \cdots e_{s_p}$ if $S = \{s_1, \ldots, s_p\}$ with $s_1 < \cdots < s_p$, and ω_S is the skew matrix $\omega_{s_i s_j}$.

Look at main formula in degree 0.

$$(-1)^m \frac{\omega^m}{m!} \cdot \frac{\hat{\omega}^m}{m!} = 1$$

$$\|$$

$$(-1)^m \, Pf(\omega)\, e_1 \cdots e_n \; Pf(-\omega^{-1})\, e_1^* \cdots e_n^*$$

$$\|$$

$$Pf(\omega)\, Pf(\omega^{-1}) \underbrace{(e_1 \cdots e_n)(e_1^* \cdots e_n^*)}_{(-1)^{n(n-1)/2} = (-1)^m}$$

as $n = 2m$

whence $\boxed{Pf(\omega)\, Pf(-\omega^{-1}) = 1}$

$$\boxed{Pf(\omega)\, Pf(\omega^{-1}) = (-1)^m}$$

Main formula in general multiplied by $Pf(\omega)$ gives

$$Pf(\omega)\, e^{-\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} = e^{-\frac{1}{2}\omega_{ij} e_i e_j} \cdot \overbrace{(e_1^* \cdots e_n^*)}^{Pf(\omega)Pf(-\omega^{-1})}$$

or

$$\boxed{Pf(\omega)\, e^{-\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} = e^{-\frac{1}{2}\omega_{ij} e_i e_j} \cdot (e_1^* \cdots e_n^*)}$$

$$= \sum_S Pf(-\omega_S)\, e_S (e_1^* \cdots e_n^*)$$

Now $\quad e_{s_1} \cdots e_{s_p} (e_1^* \cdots e_n^*) = (-1)^{\sum (s_j - 1)} e_1^* \cdots \widehat{e_{s_1}^*} \cdots \widehat{e_{s_p}^*} \cdots e_n^*$

Formula: If $S = \{s_1, \ldots, s_p\}$ and $S' = \{1, \ldots, n\} - S = \{s'_1, \ldots, s'_q\}$ where these sequences are in order, then

$$\boxed{e_S^* e_{S'}^* = (-1)^{d(S)} e_1^* \cdots e_n^* \qquad d(S) = \sum_{j=1}^{p} (s_j - j)}$$

where $d(S)$ = length of the shuffle permutation (S, S')
= dimension of Schubert cell.

Actually all I need is the formula for $d(S)$ and

$$(-1)^{d(S)} = (-1)^{\sum(s_j-1) - \sum(j-1)} = (-1)^{\frac{p}{2}}(-1)^{\sum(s_j-1)}$$

since p is even. Then I can write down the final formula

$$\boxed{\begin{aligned} Pf(\omega)\, e^{-\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} &= e^{-\frac{1}{2}\omega_{ij} e_i e_j}(e_1^* \cdots e_n^*) \\ &= \sum_S (-1)^{d(S)} Pf(\omega_S)\, e_{S'}^* \end{aligned}}$$

Next let's work out the formulas in the general linear, as opposed to orthogonal, context. This means we start with $\omega \in V \otimes W^* = Hom(W,V)$

$\omega = \omega_{ij} v_i w_j^*$, where v_i, w_j are bases for V, W resp. Assume ω invertible whence we have

$$\hat{\omega} = (\omega^{-1})_{ji}\, w_j v_i^* \in W \otimes V^* = Hom(V, W)$$

Then we regard $\Lambda W \otimes \Lambda V^*$ as a module over $\Lambda V \otimes \Lambda W^*$ using interior product, and we define thereby the isomorphism

$$\begin{array}{ccc} (\Lambda V \otimes \Lambda W^*) \otimes (\Lambda^m W \otimes \Lambda^n V^*) & \xrightarrow{\sim} & \Lambda W \otimes \Lambda V^* \\ \alpha \quad , \quad \beta & \longmapsto & \alpha \lrcorner \beta \end{array}$$

The main formula is then

$$\boxed{e^{\omega} \lrcorner \frac{\hat{\omega}^m}{m!} = e^{\hat{\omega}}}$$

Check for $m=1$. $\omega = a v_1 w_1^*$, $\hat{\omega} = a^{-1} w_1 v_1^*$

$$e^{a w_1 w_1^*} \lrcorner (a^{-1} w_1 v_1^*) = a^{-1} w_1 v_1^* + \underbrace{v_1 w_1^* w_1 v_1^*}_{1} = e^{a^{-1} w_1 v_1^*}.$$

Now to apply this formula one needs to bring in determinants:

$$\frac{\omega^m}{m!} = \frac{1}{m!} \sum \omega_{i_1 j_1} \omega_{i_2 j_2} \cdots \omega_{i_m j_m} v_{i_1} \cdots v_{i_m} w_{j_m}^* \cdots w_{j_1}^*$$

$$= \sum \omega_{1 j_1} \omega_{2 j_2} \cdots \omega_{m j_m} v_1 \cdots v_m w_{j_m}^* \cdots w_{j_1}^*$$

$$= \det(\omega) \, v_1 \cdots v_m w_m^* \cdots w_1^*$$

reverse order

$$e^{\omega} = \sum_{S,T} \det(\omega_{S,T}) \, v_S (w_T^*)^t \qquad |S| = |T|$$

The main formula multiplied by $\det(\omega)$ says

$$\det(\omega)\, e^{(\omega^{-1})_{ji} w_j v_i^*} = e^{\omega_{ij} v_i w_j^*} \lrcorner w_1 \cdots w_m v_m^* \cdots v_1^*$$

$$= \sum_{S,T} \det(\omega_{S,T}) \, v_S \underbrace{(w_T^*)^t w_1 \cdots w_m}_{(-1)^{d(T)} w_{T'}} v_m^* \cdots v_1^*$$

$$= \sum_{S,T} \det(\omega_{S,T}) \, (-1)^{\bar{d}(S) + d(T)} \, w_{T'} (v_{S'}^*)^t$$

where $\bar{d}(S)$ is the complementary Schubert cell dimension

December 21, 1984:

Prove the formula $\quad \delta \log Pf(\omega) = \frac{1}{2} tr(\omega^{-1}\delta\omega)$

Recall our main formula

$$e^{-\omega} \lrcorner \frac{\hat{\omega}^m}{m!} = e^{\hat{\omega}}$$

where $\quad \omega = \frac{1}{2}\omega_{ij}e_ie_j \qquad \hat{\omega} = -\frac{1}{2}(\omega^{-1})_{ij}e_i^*e_j^*$

$$\frac{\hat{\omega}^m}{m!} = Pf(-\omega^{-1})\, e_1^* \cdots e_n^* = \frac{1}{Pf(\omega)} e_1^* \cdots e_n^*$$

Thus the main formula is equivalent to

$$\frac{(-\omega)^k}{k!} \lrcorner e_1^* \cdots e_n^* = Pf(\omega) \cdot \frac{\hat{\omega}^{m-k}}{(m-k)!}$$

for $k = 0, \ldots, m$. Let's take $k = m-1$ whence

$$\text{A} \qquad (-1)^{m-1}\frac{\omega^{m-1}}{(m-1)!} \lrcorner e_1^* \cdots e_n^* = Pf(\omega)\,\hat{\omega}$$

Now recall that the Pfaffian is defined by

$$Pf(\omega)\, e_1 \cdots e_n = \frac{\omega^m}{m!}$$

or

$$\frac{\omega^m}{m!} \lrcorner e_1^* \cdots e_n^* = (-1)^m Pf(\omega)$$

Actually this is the case of degree 0 of the main formula.

So

$$\delta Pf(\omega) = (-1)^m \frac{\delta(\omega^m)}{m!} \lrcorner e_1^* \cdots e_n^*$$

But we are working in a commutative algebra $\Lambda^{ev} V$,

so $\quad \delta(\omega^m) = (\omega + \delta\omega)^m - \omega^m = m\,\omega^{m-1}\delta\omega = m\,\delta\omega\,\omega^{m-1}$.

$$\therefore \quad \delta Pf(\omega) = (-1)^m \delta\omega \frac{\omega^{m-1}}{(m-1)!} \lrcorner e_1^* \cdots e_n^*$$

which by ✯ is

$$= -\,\delta\omega \lrcorner \hat{\omega}$$ [redacted] $$Pf(\omega)$$

$$= +\frac{1}{2}\,\delta\omega_{ij}\, e_i e_j \lrcorner \left(\frac{1}{2}(\omega^{-1})_{kl}\, e_k^* e_l^* \;\; Pf(\omega)\right)$$

$$= \frac{1}{4}\,\delta\omega_{ij}(\omega^{-1})_{kl}\left\{\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl}\right\} Pf(\omega)$$

$$= \frac{1}{2}\,\mathrm{tr}\,(\delta\omega\cdot\omega^{-1})$$ [redacted] $$Pf(\omega)$$

Thus

$$\boxed{\delta\, Pf(\omega) = \frac{1}{2}\,\mathrm{tr}\,(\omega^{-1}\delta\omega)\;\; Pf(\omega)}$$

Let V be a finite dimensional vector space of dimension n over a field of characteristic zero. To each $v \in V$ we associate the operator ι_v on ΛV^* of contraction with v. As $\iota_v^2 = 0$, the map $V \longrightarrow \mathrm{End}(\Lambda V^*)$, $v \longmapsto \iota_v$ extends to an algebra homomorphism $\Lambda V \to \mathrm{End}(\Lambda V^*)$. This makes ΛV^* into a left module over ΛV. The action of $\alpha \in \Lambda V$ on $\beta \in \Lambda V^*$ will be denoted $i(\alpha)\beta$ or $\alpha \lrcorner \beta$.

<u>Prop. 1</u>; The map

$$(1) \qquad \Lambda V \otimes \Lambda^n V^* \longrightarrow \Lambda V^* \quad , \; \alpha \otimes \beta \longmapsto \alpha \lrcorner \beta$$

is an isomorphism.

Proof. Let $e_1, .., e_n$ be a basis for V and $e_1^*, .., e_n^*$ the dual basis of V^*. If $S \subset \{1, .., n\}$, put $e_S = e_{s_1} e_{s_2} \cdots e_{s_p} \in \Lambda^p V$, where $S = \{s_1, .., s_p\}$ and $s_1 < \cdots < s_p$. ~~The e_S~~ As S runs over the subsets of $\{1, .., n\}$, the e_S (resp. e_S^*) form a basis for ΛV (resp. ΛV^*). We have

$$e_S \lrcorner e_1^* \cdots e_n^* = e_{s_1} \cdots e_{s_p} \lrcorner (e_1^* \cdots e_n^*)$$

$$= e_{s_1} \cdots e_{s_{p-1}} \lrcorner (-1)^{s_p - 1} e_1^* \cdots \widehat{e_{s_p}^*} \cdots e_n^* = \cdots$$

$$= (-1)^{\Sigma(s_j - 1)} e_{S'}^*$$

where $S' = \{1, .., n\} - S$. Put $d(S) = \sum_{j=1}^{p} (s_j - j)$; it is the length of the shuffle permutation (S, S'). Then

$$(2) \qquad e_S \lrcorner\, e_1^* \cdots e_n^* = (-1)^{\frac{p(p-1)}{2} + d(S)} e_{S'}^*$$

where $p = \operatorname{card} S$ and S' is the complement of S. This formula shows the map in the proposition ~~carries~~ map the basis $e_S \otimes (e_1^* \cdots e_n^*)$ for $\Lambda V \otimes \Lambda^n V^*$ ~~1-1 onto~~ bijectively onto a basis for ΛV^*, proving the proposition.

Let $\omega \in \Lambda^2 V$. Suppose ω is nondegenerate in the sense that the map $V^* \longrightarrow V$, $\lambda \mapsto \iota_\lambda \omega$ is an isomorphism. This implies that n is even. This isomorphism induces an isomorphism $\Lambda^2 V^* \xrightarrow{\sim} \Lambda^2 V$, ~~and the inverse image of ω under this~~ under which ω corresponds to an element $\hat{\omega} \in \Lambda^2 V^*$.

Prop. 2: If $\omega = \frac{1}{2}\omega_{ij} e_i e_j$, then $\hat{\omega} = -\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*$

Here and in the following we use the same symbol ω to denote the skew symmetric matrix ω_{ij} and the element $\frac{1}{2}\omega_{ij} e_i e_j$ of $\Lambda^2 V$. Thus $(\omega^{-1})_{ij}$ denotes the inverse ~~of~~ the matrix ω.

~~Proof:~~ $\cancel{\iota_\lambda(\omega) = \frac{1}{2}\omega_{ij}(\lambda(e_i)e_j - e_i\lambda(e_j)) = -e_i\omega_{ij}\lambda(e_j)}$
~~so that under~~ $\cancel{\lambda \mapsto \iota_\lambda(\omega)}$, ~~we have~~ $\cancel{e_j^* \longmapsto -e_i\omega_{ij}}$,
~~hence~~ $\cancel{-e_j^*(\omega^{-1})_{ji} \longmapsto e_i}$. ~~Thus~~

$$\cancel{\hat{\omega} = \frac{1}{2}\omega_{ij}(-e_k^*(\omega^{-1})_{ki})(e_l^*(\omega^{-1})_{lj}) = -\frac{1}{2}(\omega^{-1})_{ki}}$$

Proof: ~~the map~~ One has $\iota_\lambda(\omega) = \frac{1}{2}\omega_{ij}[\lambda(e_i)e_j - e_i\lambda(e_j)] = -e_i\omega_{ij}\lambda(e_j)$, so that under the map $\lambda \longmapsto \iota_\lambda\omega$, one has $e_j^* \longmapsto -e_i\omega_{ij}$. If $\hat{\omega} = \frac{1}{2}a_{ij}e_i^*e_j^*$, then $\hat{\omega} \longmapsto \frac{1}{2}a_{ij}(e_k\omega_{ki})(e_\ell\omega_{\ell j})$ $= -\frac{1}{2}(\omega_{ki}a_{ij}\omega_{j\ell})e_ke_\ell$. Thus $-\omega a\omega = \omega$, so $a = -\omega^{-1}$ proving the proposition.

For example, when $n=2$ and $(\omega_{ij}) = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ $(\omega^{-1}_{ij}) = \begin{pmatrix} 0 & -a^{-1} \\ a^{-1} & 0 \end{pmatrix}$ ~~then~~ $\omega = ae_1e_2$ and $\hat{\omega} = +a^{-1}e_1^*e_2^*$.

Prop. 3: (Main Formula) If $\omega \in \Lambda^2 V$ is non degenerate, then

$$e^{-\omega} \lrcorner \frac{\hat{\omega}^m}{m!} = e^{\hat{\omega}}$$

where $m = n/2$, $n = \dim V$.

Proof: We can choose the basis e_i so that $\omega = a_1e_1e_2 + a_2e_3e_4 + \cdots + a_me_{n-1}e_n$, whence $\hat{\omega} = +a_1^{-1}e_1^*e_2^* + \cdots + a_m^{-1}e_{n-1}^*e_n^*$ and

$$\frac{\hat{\omega}^m}{m!} = \prod_{j=1}^{m}(+a_je_{2j-1}^*e_{2j}^*)$$

So it's clear that it is enough to check the formula when $n=2$. Then

$$e^{-\omega}\lrcorner\hat{\omega} = (1-ae_1e_2)\lrcorner(+a^{-1}e_1^*e_2^*) = a^{-1}e_1^*e_2^* - e_1e_2\lrcorner e_1^*e_2^*$$
$$= 1 + a^{-1}e_1^*e_2^* = e^{\hat{\omega}}.$$

Q.E.D.

Recall that the Pfaffian $Pf(\omega)$ of the skew-symmetric matrix $\omega = (\omega_{ij})$ is defined by

$$\frac{\omega^m}{m!} = Pf(\omega)\, e_1 \cdots e_n$$

where on the left ω means $\frac{1}{2}\omega_{ij} e_i e_j$ in $\Lambda^2 V$. Thus

$$Pf\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$$

Prop. 4: One has $e^{\omega} = \sum_S Pf(\omega_S)\, e_S$ where ω_S is the matrix $\omega_{s_i s_j}$ if $S = \{s_1, \dots, s_p\}$, $s_1 < \cdots < s_p$, and S runs over the subsets with an even number of elements.

Proof: Let V_S be the subspace of V spanned by the e_s, $s \in S$, and consider the projection $V \longrightarrow V_S$ ~~sending e_S~~ which is the identity on the e_s, $s \in S$ and 0 on the $e_{s'}$, $s' \notin S$. This projection extends to an algebra homomorphism $\Lambda V \longrightarrow \Lambda V_S$, which on e_T is 1 for $T \subset S$ and 0 for $T \not\subset S$, and which sends e^{ω} into e^{ω_S} where $\omega_S = \frac{1}{2} \omega_{s_i s_j} e_{s_i} e_{s_j}$. It follows that the coefficient of e_S in e^{ω} is the highest degree term of e^{ω_S}, namely $\frac{\omega_S^{p/2}}{(p/2)!} = Pf(\omega_S)\, e_S$. QED

Prop. 5.
$$Pf(\omega)\, Pf(-\omega^{-1}) = 1$$
$$Pf(\omega)\, Pf(\omega^{-1}) = (-1)^m$$

Proof. Look at the main formula in degree 0, using Prop. 2 to identify $\hat{\omega} \in \Lambda^2 V^*$ with the element belonging to the matrix $-\omega^{-1}$.

$$1 = \frac{(-\omega)^m}{m!} \lrcorner \frac{\hat{\omega}^m}{m!}$$

$$= (-1)^m Pf(\omega) Pf(\omega^{-1}) \{ e_1 \cdots e_n \lrcorner e_1^* \cdots e_n^* \}$$

The last factor is $(-1)^{n(n-1)/2} = (-1)^m$ and the proposition follows.

$$\boxed{\begin{aligned} &\underline{\text{Prop 6.}} \quad Pf(\omega)\, e^{-\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} \\ &= e^{-\frac{1}{2}\omega_{ij} e_i e_j} \lrcorner e_1^* \cdots e_n^* = \sum_S (-1)^{d(S)} Pf(\omega_S)\, e^*_{S'} \end{aligned}}$$

Proof. The main formula says

$$e^{\hat{\omega}} = e^{-\frac{1}{2}(\omega^{-1})_{ij} e_i^* e_j^*} = e^{-\frac{1}{2}\omega_{ij} e_i e_j} \lrcorner Pf(\omega^{-1})\, e_1^* \cdots e_n^*$$

so multiplying by $Pf(\omega)$ one obtains the first equality. Now

$$e^{-\frac{1}{2}\omega_{ij} e_i e_j} = \sum_S (-1)^{\frac{p}{2}} Pf(\omega_S)\, e_S \qquad p = \text{card } S$$

by Prop 4. and by (2)

$$e_S \lrcorner e_1^* \cdots e_n^* = (-1)^{\frac{p}{2} + d(S)} e^*_{S'}$$

since $\frac{p(p-1)}{2} \equiv \frac{p}{2} \pmod 2$ for p even, which proves the second equality.

December 22, 1984

I want to find a denominator free formula for the transgression form

$$Pf(\omega)\frac{\alpha}{d_\omega \alpha} \qquad \Bigg| \qquad \begin{aligned} \alpha &= x^t \omega^{-1} dx \\ d_\omega \alpha &= \underbrace{x^t x}_{x^2} + \underbrace{dx^t \omega^{-1} dx}_{-2\hat{\omega}} \end{aligned}$$

(Recall $\hat{\omega} = -\frac{1}{2}(\omega^{-1})_{ij}\, e_i^* e_j^*$ where e_i^* becomes dx^i)

Use geometric series to obtain

$$Pf(\omega)\frac{\alpha}{x^2 - 2\hat{\omega}} = \sum_{k \geq 0} \frac{2^k}{(x^2)^{k+1}} \underbrace{\left(Pf(\omega)\,\alpha\,\hat{\omega}^k\right)}$$

so we want to show that the term is a polynomial in ω. Consider the derivation given by interior multiplication by $x^i e_i$ on ΛV^*. Then

$$\begin{aligned} i(x^i e_i)\,\hat{\omega} &= -\tfrac{1}{2}(\omega^{-1})_{ij}\, i(x^i e_i)\, e_i^* e_j^* \\ &= -\, x^i (\omega^{-1})_{ij}\, e_j^* \quad = -\alpha \end{aligned}$$

dropped sign

So

$$Pf(\omega)\frac{\alpha}{d_\omega \alpha} = -\sum_{k \geq 0} \frac{2^k k!}{(x^2)^{k+1}}\, i(x^i e_i)\left\{Pf(\omega)\,\frac{\hat{\omega}^{k+1}}{(k+1)!}\right\}$$

$$= -\sum_{k \geq 0} \frac{2^k k!}{(x^2)^{k+1}} \sum_{|S'| = 2k+2} (-1)^{d(S)}\, Pf(\omega_S)\left(i(x^i e_i)\, dx_{S'}\right)$$

is a denominator free expression for the transgression form.

Alternative derivation is to follow through the derivation of the transgression form from the Thom form, which is

$$U = Pf(\omega)\, e^{-\frac{1}{2}(x^2 + dx^t \omega^{-1} dx)} = \sum_S (-1)^{d(S)} Pf(\omega_S)\, dx_{S'}$$

So we pull back U by $x, t \longmapsto xt$ and look for the coefficient of dt which is

$$V_t = + e^{-\frac{1}{2}t^2 x^2} \sum_S (-1)^{d(S)} Pf(\omega_S)\, t^{g-1} \left(x^i e_i \lrcorner\, dx_{S'} \right) \qquad \begin{array}{l} g = n - |S| \\ \text{is even} \end{array}$$

This is to be integrated from 0 to ∞, + we need

$$-\int_0^\infty V_t\, dt = -\int_0^\infty e^{-\frac{1}{2}t^2 x^2}\, t^g \frac{dt}{t} = \frac{1}{2}\int_0^\infty e^{-(\frac{1}{2}x^2)t}\, t^{g/2} \frac{dt}{t} = \frac{\Gamma(g/2)}{2(\frac{1}{2}x^2)^{g/2}}$$

$$= \frac{2^{g/2-1}\left(\frac{g}{2}-1\right)!}{(x^2)^{g/2}}$$

So again we get

$$\boxed{Pf(\omega)\frac{\alpha}{d_\omega \alpha} = -\sum_S (-1)^{d(S)}\ [redacted]\ 2^k k!\ Pf(\omega_S) \left(\frac{x^i e_i \lrcorner\, dx_{S'}}{(x^2)^{k+1}} \right) \quad \text{where } 2k+2 = |S'|}$$

From $\quad e^\omega = \sum_S Pf(\omega_S)\, e_S \quad$ we obtain an addition formula for the Pfaffian.

$$e^{\omega+\eta} = \sum_{S,T} Pf(\omega_S)\, e_S\, Pf(\eta_T)\, e_T$$

so taking terms of highest degree:

$$Pf(\omega+\eta)\, e_1 \cdots e_n = \sum_S Pf(\omega_S)\, Pf(\eta_{S'})\, \underbrace{e_S\, e_{S'}}^{(-1)^{d(S)} e_1 \cdots e_n}$$

or

$$\boxed{Pf(\omega+\eta) = \sum_S (-1)^{d(S)} Pf(\omega_S)\, Pf(\eta_{S'})}$$

Now recall that

$$Pf(\omega)\, e^{-\frac{1}{2}\omega^{-1}_{ij}\, dx^i dx^j} = \sum (-1)^{d(S)} Pf(\omega_S)\, dx_{S'}$$

so the question is whether this might be something like $Pf(\omega_{ij} + dx^i dx^j)$. In any case we can ask what the Pfaffian of the skew symmetric matrix $dx^i dx^j$ is. Note this matrix has values in a commutative ring.

Set $\omega = \frac{1}{2} dx^i dx^j e_i e_j$ and work in the exterior algebra generated by dx^i, e_i. Then

$$\omega = \frac{1}{2} dx^i dx^j e_i e_j = -\frac{1}{2}(dx^i e_i)(dx^j e_j) = -\frac{1}{2}(dx^i e_i)^2$$

so

$$\frac{\omega^m}{m!} = \left(-\frac{1}{2}\right)^m \frac{1}{m!}(dx^i e_i)^n = \frac{n!}{2^m m!}(-1)^m\, dx^1 e_1\, dx^2 e_2 \cdots dx^n e_n$$

$$= \frac{n!}{2^m m!}(-1)^m\, dx^n \cdots dx^1\, e_1 \cdots e_n = \frac{n!}{2^m m!}\, dx^1 \cdots dx^n\, e_1 \cdots e_n$$

so

$$\boxed{Pf(dx^i dx^j) = \frac{n!}{2^m m!}\, dx^1 \cdots dx^n}$$

So $Pf(\omega + dx\, dx^t)$ won't work because of these constants. What we can do is to look for a (signed) measure $d\mu(t)$ so that

$$\int Pf(\omega + t\, dx\, dx^t)\, d\mu(t)$$

converts powers of t into the constants we need:

$$\int t^g\, d\mu(t) = \frac{2^{g/2}(g/2)!}{g!}$$

But it's not clear there is a point to this game.

Next I want to review my ideas on the transgression form for the Euler class of a complex vector bundle V being related to the fundamental relation in $H^*(\mathbb{P}V)$. The starting point is to consider V as an S^1-vector bundle, and compute the Euler class as an equivariant form on the base M, namely

$$\det(u+\Omega) \quad \in \quad S(\mathfrak{g}^*) \otimes \Omega(M)^{S^1}.$$

where u denotes a generic element of $\mathfrak{g} = \mathrm{Lie}(S^1) = i\mathbb{R}$. Then our theory gives a transgression form

$$\tau \in \quad S(\mathfrak{g}^*) \otimes \Omega(SV)^{S^1}$$

However the latter maps to $\Omega(\mathbb{P}V)$ as follows. First we have a connection form in the principal S^1-bundle SV over $\mathbb{P}V$. So we have:

$$\begin{tikzcd}
& W(\mathfrak{g}) \otimes \Omega(SV) \arrow[r] & \Omega(SV) \\
S(\mathfrak{g}^*) \otimes \Omega(SV)^{S^1} & \{W(\mathfrak{g}) \otimes \Omega(SV)\}_{\text{basic}} \arrow[l, "\sim"'] \arrow[u, phantom, "\cup"] \arrow[r] & \Omega(SV)_{\text{basic}} = \Omega(\mathbb{P}V) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

and therefore τ furnishes a form in $\Omega(\mathbb{P}V)$ whose differential will be $\det(\xi + \Omega)$, ξ representing $c_1(\mathcal{O}(1))$.

December 23, 1984

Let $\mathfrak{g}$ be a finite-dimensional vector space, M a vector space equipped with $i : \mathfrak{g} \to \mathrm{End}(M)$ such that $\iota_X^2 = 0$ for all $X \in \mathfrak{g}$. For example, $M = \Lambda\mathfrak{g}^*$. Now equip $\Lambda\mathfrak{g}^* \otimes M$ with ι_X defined by

$$\iota_X(\alpha m) = \iota_X\alpha \cdot m + (-1)^{\deg\alpha}\alpha \cdot \iota_X m$$

whence $\Lambda\mathfrak{g}^* \otimes M$ also becomes a module over $\Lambda\mathfrak{g}$.

Let $\varepsilon : \Lambda\mathfrak{g}^* \to \Lambda^0\mathfrak{g}^* = k$ be the augmentation and define

$$\varepsilon : \Lambda\mathfrak{g}^* \otimes M \longrightarrow M$$
$$\varepsilon(\alpha m) = \varepsilon(\alpha) m .$$

Put $\quad (\Lambda\mathfrak{g}^* \otimes M)_{hor} = \{\gamma \in \Lambda\mathfrak{g}^* \otimes M \mid \iota_X\gamma = 0 \text{ all } X \in \mathfrak{g}\}$.

<u>Prop.</u> The map ε induces an isomorphism of $(\Lambda\mathfrak{g}^* \otimes M)_{hor}$ onto M.

<u>Proof.</u> Let $X_1, \ldots, X_n$ be a basis for $\mathfrak{g}$ and $\theta_1, \ldots, \theta_n$ the dual basis for $\mathfrak{g}^*$, and put $\iota_a = \iota_{X_a}$. An element $\gamma \in \Lambda\mathfrak{g}^* \otimes M$ can be uniquely written

$$\gamma = \gamma_1 + \theta^1\gamma_1' \quad \text{with } \gamma_1, \gamma_1' \in \Lambda[\theta^2, \ldots, \theta^n] \otimes M$$

and one has

$$\iota_1\gamma = \iota_1\gamma_1 + \gamma_1' - \theta^1\iota_1\gamma_1'$$
$$\iota_j\gamma = \iota_j\gamma_1 - \theta^1\iota_j\gamma_1' \qquad j \geq 2.$$

Let $\gamma \in (\Lambda[\mathfrak{g}^*] \otimes M)_{hor}$
As $\Lambda[\theta^2, \ldots, \theta^n] \otimes M$ is closed under the operators $\iota_1, \ldots, \iota_n$ it is clear that γ_1 is

killed by ι_j for $j > 1$ and that $\gamma_1' = -\iota_1 \gamma_1$

or
$$\gamma = (1 - \theta^1 \iota_1)\gamma_1$$

Repeating this argument we see that
$$\gamma = (1-\theta^1\iota_1)\cdots(1-\theta^p\iota_p)\gamma_p$$

where $\gamma_p \in \Lambda[\theta^{p+1}, \ldots, \theta^n] \otimes M$ is killed by ι_j for $j > p$. Taking $p = n$ we have any horizontal γ is of the form
$$\gamma = (1-\theta^1\iota_1)(1-\theta^2\iota_2)\cdots(1-\theta^n\iota_n)m$$
$$= m - \sum_a \theta^a \iota_a m + \frac{1}{2!}\sum_{a,b} \theta^a\theta^b \iota_b \iota_a m - \cdots$$

for some $m \in M$. Then applying ε which kills θ^a we see $m = \varepsilon(\gamma)$. Also it is clear that any γ in the above form is horizontal, so in fact the map $\prod_{a=1}^{n}(1-\theta^a\iota_a)$ from M to $(\Lambda\mathfrak{g}^* \otimes M)_{hor}$ is inverse to ε. Q.E.D.

December 24, 1984

New idea for a proof of
$$(\Lambda \mathfrak{g}^* \otimes A)_{hor} \simeq A$$
goes as follows. First establish an isomorphism
$$\Lambda \mathfrak{g}^* \otimes A \simeq \mathrm{Hom}(\Lambda \mathfrak{g}, A)$$
as modules over $\Lambda \mathfrak{g}$ considered as a Hopf algebra, then take the invariants on both sides.

This approach gets bogged down in signs so it is necessary to proceed carefully. The main point is to identify $\Lambda \mathfrak{g}^*$ with $(\Lambda \mathfrak{g})^*$ as $\Lambda \mathfrak{g}$-modules, where X in $\mathfrak{g}$ acts on $\Lambda \mathfrak{g}^*$ as ι_X. Let us consider the pairing

$$(1) \qquad \begin{aligned} \Lambda \mathfrak{g} \otimes \Lambda \mathfrak{g}^* &\longrightarrow k \\ \omega \otimes \alpha &\longmapsto \varepsilon(\omega \lrcorner \alpha) \end{aligned}$$

where ε = the augmentation in $\Lambda \mathfrak{g}^*$ and $\omega \lrcorner$ denotes the multiplication of $\omega \in \Lambda \mathfrak{g}$ on $\Lambda \mathfrak{g}^*$. Let's see if this pairing, or really this map, is a map of $\Lambda \mathfrak{g}$-modules. If $X \in \mathfrak{g}$

$$\iota_X(\omega \otimes \alpha) = X\omega \otimes \alpha + (-1)^{\omega}\, \omega \otimes \iota_X \alpha$$

$$\longmapsto \varepsilon(X\omega \lrcorner \alpha) + (-1)^{\omega}\, \varepsilon(\underbrace{\omega \lrcorner \iota_X \alpha}_{\omega X \lrcorner \alpha})$$

$$= 2\, \varepsilon(X\omega \lrcorner \alpha)$$

so we see the pairing (1) is wrong. (However, if one writes it as a pairing $\Lambda \mathfrak{g}^* \otimes \Lambda \mathfrak{g} \to k$ it's OKAY. see p. 316)

Let us look at the situation generally from the viewpoint of Hopf algebra. If $H = k[G]$ how does $H^* = Map(G, k)$ become an [redacted] H-module? There are two actions – left + right regular repns.

$$(R_g f)(x) = f(xg) \qquad (L_g f)(x) = f(g^{-1}x)$$

In diagrams R_g is the transpose of

$$\begin{array}{ccccc} H & \longrightarrow & H \otimes H & \xrightarrow{\mu} & H \\ x & \longmapsto & x \otimes g & & \end{array}$$

[redacted] so for $\Lambda \mathfrak{g}$, infinitesimal right translation by X is the transpose of

$$\begin{array}{ccccc} \Lambda\mathfrak{g} & \longrightarrow & \Lambda\mathfrak{g} \otimes \Lambda\mathfrak{g} & \xrightarrow{\mu} & \Lambda\mathfrak{g} \\ \omega & \longmapsto & \omega \otimes X & \longmapsto & \omega X \end{array}$$

hence should be the derivation of $\Lambda\mathfrak{g}^*$ such that $\lambda \longmapsto \lambda(X) = \iota_X \lambda$. Thus ι_X = inf. right translation by X on $(\Lambda\mathfrak{g})^* = \Lambda\mathfrak{g}^*$.

(Digress: What does one mean by dual and transpose in the graded setup? If V is a graded vector space, then $(V^*)_n = (V_{-n})^*$ defines its dual V^*. There is a canonical pairing

$$V^* \otimes V \xrightarrow{c} k$$

[redacted] Given $f: V \longrightarrow W$ one defines $f^t: W^* \to V^*$ how? Use

$$Hom(W, V^*) \longrightarrow Hom(W \otimes V, k)$$

[redacted] So given $f: V \longrightarrow W$ we get

$$\begin{array}{ccccc} W^* \otimes V & \xrightarrow{1 \otimes f} & W^* \otimes W & \xrightarrow{c} & k \\ & {\scriptstyle f^* \otimes 1} \searrow & & & \\ & & V^* \otimes W & \xrightarrow{c} & \end{array}$$

i.e.

$$c(f^* \otimes 1)(w^* \otimes v) = \langle f^* w^*, v \rangle$$

$$= c(1 \otimes f)(w^* \otimes v) = (-1)^{\deg f \cdot \deg w^*} \langle w^*, fv \rangle$$

Next consider the isomorphism with the double dual. The canonical pairing $V^* \otimes V \xrightarrow{c} k$ determines a pairing

$$V \otimes V^* \simeq V^* \otimes V \xrightarrow{c} k$$

and hence a unique map $V \xrightarrow{\psi} (V^*)^*$ such that

$$\begin{tikzcd}
V \otimes V^* \arrow[r, "\simeq"] \arrow[d, "\psi \otimes 1"'] & V^* \otimes V \arrow[d, "c"] \\
V^{**} \otimes V^* \arrow[r, "c"] & k
\end{tikzcd}$$

commutes, or

$$c(\psi \otimes 1)(v \otimes \lambda) = c(-1)^{\deg v \cdot \deg \lambda}(\lambda \otimes v)$$

$$\|$$

$$\langle \psi(v), \lambda \rangle = (-1)^{\deg v \cdot \deg \lambda} \langle \lambda, v \rangle$$

In other words $\psi : V \xrightarrow{\sim} V^{**}$ in degree p is $(-1)^{p^2} = (-1)^p$ the usual isomorphism $V_p \simeq V_p^{**}$. This reminds me a little of the Fourier transform.)

Let $H = \Lambda \mathfrak{g}$ and let us identify $\mathfrak{g}^*$ with $(H^*)^1$ using the pairing $\langle \lambda, X \rangle = \lambda(X)$. Now compute the transpose of right multiplication by X and left multiplication by $-X$ on $\Lambda\mathfrak{g}$. Actually we will compute them on $(H^*)^1 = \mathfrak{g}^*$, using the appropriate sign rules.

The rules are

$$\langle \lambda, R_X 1 \rangle = \langle \lambda, X \rangle$$

$$\langle \lambda, L_X 1 \rangle = -\langle \lambda, X \rangle$$

so that they differ only by sign. This should be the case as the Hopf algebra is commutative in the super sense. ?

Let us start again by setting up a suitable isomorphism [redacted]

$$\Lambda \mathfrak{g}^* \otimes A \simeq \operatorname{Hom}(\Lambda \mathfrak{g}, A)$$

of modules over $\Lambda \mathfrak{g}$. As $\Lambda \mathfrak{g}$ is a Hopf algebra [redacted] the tensor product of $\Lambda\mathfrak{g}$-modules (graded) is defined, so that it suffices to give an isomorphism

$$\Lambda \mathfrak{g}^* \simeq (\Lambda \mathfrak{g})^*$$

of $\Lambda\mathfrak{g}$-modules. On the left the action of $X \in \mathfrak{g}$ on $\Lambda\mathfrak{g}^*$ is ι_X. On the right the action is to be defined so that if one forms the associated pairing

$$\Lambda \mathfrak{g}^* \otimes \Lambda \mathfrak{g} \longrightarrow k$$

it is compatible with the action of $\Lambda\mathfrak{g}$. Let's start with

<u>December 25, 1984</u>

What I want to do is to define an ~~map~~ isom

$$\Lambda\mathfrak{g}^* \otimes A \xrightarrow{\sim} \mathrm{Hom}(\Lambda\mathfrak{g}, A)$$

so that horizontal elements on the left correspond to $\Lambda\mathfrak{g}$ module homomorphisms on the right. It will be easier to first use the canonical isom.

$$\Lambda\mathfrak{g}^* \otimes A \simeq A \otimes \Lambda\mathfrak{g}^*$$

and then to use a canonical pairing

$$\begin{array}{ccc} \Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} & \longrightarrow & k \\ \eta \quad , \quad \omega & \longmapsto & \langle \eta, \omega \rangle \end{array}$$

whose properties are to be determined.

So we define

$$\begin{array}{rcl} f : A \otimes \Lambda\mathfrak{g}^* & \longrightarrow & \mathrm{Hom}(\Lambda\mathfrak{g}, A) \\ a \otimes \eta & \longmapsto & f_{a\otimes\eta} \end{array}$$

where $\quad f_{a\otimes\eta}(\omega) = a\langle \eta, \omega \rangle$. Then

$$f_{\iota_X(a\otimes\eta)}(\omega) = f_{\iota_X a \otimes \eta + (-1)^a a \otimes \iota_X \eta}(\omega)$$

$$= \underbrace{\iota_X a \langle \eta, \omega \rangle}_{\iota_X(f_{a\otimes\eta}(\omega))} + (-1)^a a \langle \iota_X \eta, \omega \rangle$$

Now $f_{a\otimes\eta}$ is of degree $= \deg a + \deg \eta$, so it would be very nice if the last term were

$$-(-1)^a(-1)^\eta a\langle \eta, X\omega \rangle = -(-1)^{a\otimes\eta} f_{a\otimes\eta}(X\omega)$$

because then we would have

$$f_{\iota_X \alpha} = \iota_X \circ f_\alpha - (-1)^\alpha f_\alpha \circ X$$

which yields the desired result that α horizontal $\Rightarrow f_\alpha$ is a $\Lambda\mathfrak{g}$-module homomorphism.

Thus we conclude that the pairing should satisfy

$$\boxed{\langle \iota_X \eta, \omega \rangle = -(-1)^\eta \langle \eta, X\omega \rangle}$$

or $$\langle \iota_X \eta, \omega \rangle + (-1)^\eta \langle \eta, X\omega \rangle = 0.$$

This is the same as asking that the pairing be a pairing of $\Lambda\mathfrak{g}$-modules.

Define the pairing

$$\boxed{\langle \eta, \omega \rangle = \varepsilon(\omega \lrcorner \eta)}$$

$\varepsilon : \Lambda\mathfrak{g}^* \twoheadrightarrow \Lambda^0\mathfrak{g}^* =$

Then
$$\begin{aligned}
\langle \iota_X \eta, \omega \rangle &= \varepsilon(\omega \lrcorner \iota_X \eta) \\
&= \varepsilon(\omega X \lrcorner \eta) \\
&= (-1)^{\deg \omega} \varepsilon(X\omega \lrcorner \eta) \\
&= (-1)^{\deg \omega} \langle \eta, X\omega \rangle
\end{aligned}$$

However if η, ω are homogeneous, then $\langle \iota_X \eta, \omega \rangle$ is zero unless $\deg(\iota_X \eta) = \deg(\eta) - 1 = \deg \omega$. So we have

$$\langle \iota_X \eta, \omega \rangle = -(-1)^{\deg \eta} \langle \eta, \iota_X \omega \rangle$$

as desired.

So the last step is to obtain the formula for the element in $(\Lambda\mathfrak{g}^* \otimes A)_{hor}$ belonging to $a \in A$.

We have the basis $X_{\mu_1} \cdots X_{\mu_p}$ $\quad \mu_1 < \cdots < \mu_p$ for $\Lambda\mathfrak{g}$. The dual basis of $\Lambda\mathfrak{g}^*$ with respect to the canonical pairing defined above is $\theta^{\mu_p} \cdots \theta^{\mu_1}$. Hence the $\Lambda\mathfrak{g}$-module homomorphism $\omega \mapsto \iota_\omega a$ from $\Lambda\mathfrak{g}$ to A corresponds to the element

$$\sum \iota_{\mu_1} \cdots \iota_{\mu_p} a \otimes \theta^{\mu_p} \cdots \theta^{\mu_1} \qquad \in A \otimes \Lambda\mathfrak{g}^*.$$

~~Under the canonical isomorphism~~ ?

Let $\alpha \in (A \otimes \Lambda\mathfrak{g}^*)_{hor}$. Then $f_\alpha(X\omega) = (-1)^{\deg\alpha} \iota_X f_\alpha(\omega)$

so that $$f_\alpha(X_{\mu_1} \cdots X_{\mu_p}) = (-1)^{p \deg\alpha} \iota_{\mu_1} \cdots \iota_{\mu_p} \overbrace{f_\alpha(1)}^{a}$$

Thus $$\alpha = \sum' (-1)^{p \deg a} \iota_{\mu_1} \cdots \iota_{\mu_p} a \otimes \theta^{\mu_p} \cdots \theta^{\mu_1} \quad \in A \otimes \Lambda\mathfrak{g}^*$$

and under the canonical isomorphism $\Lambda\mathfrak{g}^* \otimes A = A \otimes \Lambda\mathfrak{g}^*$ we have

$$\alpha = \sum \underbrace{(-1)^{p \deg a + p(\deg a - p)}}_{(-1)^p} \theta^{\mu_p} \cdots \theta^{\mu_1} \iota_{\mu_1} \cdots \iota_{\mu_p} a$$

which is what I want.

Abstract approach. Work in the category of super $\Lambda\mathfrak{g}$-modules; such a supermodule is a super vector space A equipped with operators $a \mapsto Xa$ of odd degree depending linearly on $X \in \mathfrak{g}$, such that $X^2 = 0$. $\Lambda\mathfrak{g}$ is a Hopf algebra with $\Delta : \Lambda\mathfrak{g} \longrightarrow \Lambda\mathfrak{g} \otimes \Lambda\mathfrak{g}$ determined by $\Delta X = X \otimes 1 + 1 \otimes X$. Hence there is a tensor product operation on $\Lambda\mathfrak{g}$-super modules, namely $A \otimes A'$ equipped with $X(a \otimes a') = Xa \otimes a' + (-1)^{\deg a} a \otimes Xa'$.

Note that $\Lambda\mathfrak{g}^*$ is a supermodule over $\Lambda\mathfrak{g}$ with $X\eta = \iota_X \eta$. Next the pairing

$$\Lambda\mathfrak{g}^* \otimes \Lambda\mathfrak{g} \longrightarrow k \qquad \eta \otimes \omega \longmapsto \overbrace{\varepsilon(\omega \lrcorner \eta)}^{\langle \eta, \omega \rangle \overset{\text{def}}{=}}$$

satisfies $\langle \iota_X \eta, \omega \rangle + (-1)^{\deg \omega} \langle \eta, X\omega \rangle = 0$. This pairing is non-degenerate and identifies $\Lambda\mathfrak{g}^*$ with $(\Lambda\mathfrak{g})^*$ as super $\Lambda\mathfrak{g}$-modules

Now we define canonical isomorphisms

$$\begin{aligned} \Lambda\mathfrak{g}^* \otimes A &\simeq A \otimes \Lambda\mathfrak{g}^* \\ &\simeq A \otimes (\Lambda\mathfrak{g})^* \\ &\simeq \operatorname{Hom}(\Lambda\mathfrak{g}, A) \end{aligned}$$

of $\Lambda(\mathfrak{g})$-super modules, where $\Lambda\mathfrak{g}$ acts on the latter by

$$(Xf)(\omega) = X f(\omega) - (-1)^{\deg f} f(X\omega)$$

i.e. so that evaluation

$$\operatorname{Hom}(\Lambda\mathfrak{g}, A) \otimes \Lambda\mathfrak{g} \longrightarrow A$$

is a supermodule homomorphism.

Thus, to be more accurate, we define X on $Hom(\Lambda\mathfrak{g}, A)$ first and then define

$$A \otimes \Lambda\mathfrak{g}^* \longrightarrow Hom(\Lambda\mathfrak{g}, A)$$
$$a \otimes \eta \longmapsto (\omega \longmapsto a\langle\eta, \omega\rangle)$$

and check it is a map of $\Lambda\mathfrak{g}$-supermodules, which is an isomorphism.

Then one gets an induced isom. on the 'invariants'

$$(\Lambda\mathfrak{g}^* \otimes A)_{hor} \cong Hom_{\Lambda\mathfrak{g}}(\Lambda\mathfrak{g}, A) \simeq A$$

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Let $A = \Omega(M)$, and let X_a be a basis for $\mathfrak{g}$ and let θ^a be the dual basis for $\mathfrak{g}^*$. We identify $\omega \in A$ with $1 \otimes \omega \in \Lambda^0\mathfrak{g}^* \otimes A$ and let $\varepsilon : \Lambda\mathfrak{g}^* \otimes A \longrightarrow A$ be induced by the augmentation $\Lambda\mathfrak{g}^* \longrightarrow \Lambda^0\mathfrak{g}$. Put $\iota_j = \iota_{X_j}$.

Lemma: If $\omega \in A$, then

$$\alpha = \Big(\prod_{j=1}^{n} (1 - \theta^j \iota_j)\Big)\omega$$

$$= \omega - \sum_j \theta^j \iota_j \omega + \frac{1}{2!}\sum_{j,k} \theta^j \theta^k \iota_k \iota_j \omega - \cdots$$

is a horizontal element of $\Lambda\mathfrak{g}^* \otimes A$; it is the unique horizontal element such that $\varepsilon(\alpha) = \omega$ and hence ε induces an isom.

$$(\Lambda\mathfrak{g}^* \otimes A)_{hor} \xrightarrow{\sim} A$$

Proof. The operator $\theta^j \iota_j$ for j fixed is a projector with kernel = $\operatorname{Ker} \iota_j$ and image = $\operatorname{Im} \theta^j$. As the projectors $\theta^j \iota_j$ for $j = 1, \ldots, n$ commute, it follows that $E = \prod (1 - \theta^j \iota_j)$ is a projector with image $= \bigcap \operatorname{Ker} \iota_j = (\Lambda\mathfrak{g}^* \otimes A)_{hor}$. This proves the first statement. As $\operatorname{Ker} \varepsilon = \sum \operatorname{Im} \theta^j$, we have $\varepsilon E = \varepsilon$ and $E(\operatorname{Ker} \varepsilon) = 0$, so $\varepsilon E \omega = \omega$ for all $\omega \in A$. If α is a horizontal element with $\varepsilon\alpha = \omega$, then $\alpha - \omega \in \operatorname{Ker} \varepsilon$, so $\alpha = E\alpha = E\omega$, which concludes the proof.