August 18 – Sept 18, 1984

August 18, 1984

Berligne-Vergne proof of index thm.

A key idea is to work with the Laplacean Δ on the principal Spin(n)-bundle P/M, that is, the Laplacean on functions. One can identify $\Gamma(M, S^{\pm})$ with $(\Omega^0(P)\otimes S^{\pm})^G$, $G = \text{Spin}(n)$, and the operator $-\not{D}^2$ differs from Δ on $(\Omega^0(P)\otimes S^{\pm})^G$ by a scalar, which is something like $-R/4$ + eigenvalue of Casimir in $S^{\pm}$. Consequently the index is

$$\lim_{t\to 0} tr_s\left(e^{-t\Delta}\pi\right)$$

where π projects ~~onto~~ G-invariants in $\Omega^0(P)\otimes S^{\pm}$, and this can be written

$$\lim_{t\to 0}\int_M\int_G k_t(u, ug^{-1})\,(\chi^+-\chi^-)(g)$$

where $k_t(u,u')$ is the kernel of $e^{-t\Delta}$ in P.

Now $\chi^+ - \chi^-$ vanishes to the same order at $g = id$ as the pole of $k_t(u, ug^{-1})$, so we can calculate the limit as $t\to 0$ of the integral over G as an integral over the Lie algebra. The point is that there is a Jacobian factor in the leading term of the asymptotics of $k_t(u,u')$, the Jacobian factor being related to the exponential map in P. This contributes the A-hat genus.

I want to try to understand the significance of their approach in the case of a Dirac operator over the forms with coefficients in a bundle equipped with connection. In this case the principal frame bundle is trivial $P = M \times Spin(n)$ and so the heat kernel for P should be the product of the two heat kernels. Therefore passing to P and its Laplacean shouldn't give anything new.

Instead one wants to introduce the frame bundle P' of the coefficient bundle ξ, and to replace the covariant derivative Laplacean $-D_\mu^2$ on P' by the Laplacean Δ on functions on P'. I don't quite see why this is any better.

August 19, 1984

Problem: To construct 'the' Dirac operator on a Riemann surface.

M is a 2-manifold with metric and orientation. Let P be the bundle of oriented orthonormal frames in M; it is a principal $SO(2)$ bundle which can be identified with the unit tangent bundle.

The group $SO(2)$ acts on $\mathbb{R}^2$ hence on the Clifford algebra C_2. In general the action of Lie $SO(n)$ on C_n is given by:

$$\left[\frac{1}{4} a_{\mu\nu} \gamma^\mu \gamma^\nu, \gamma^\lambda x_\lambda\right] = \gamma^\mu (a_{\mu\nu} x_\nu)$$

The generator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of Lie $SO(2)$ which generates counterclockwise rotation induces on C_2 the inner derivation given by $\frac{1}{2}\gamma^2\gamma^1 = -\frac{1}{2} i\varepsilon$. (Recall $\gamma^1, \gamma^2, \varepsilon$ are the three Pauli matrices and $\gamma^1\gamma^2 = i\varepsilon$). Thus the spin representation is on the Lie alg. level

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longmapsto -\frac{1}{2} i\varepsilon = \begin{pmatrix} -\frac{1}{2}i & 0 \\ 0 & \frac{1}{2}i \end{pmatrix} \quad \text{on } S_2$$

and on the group level

$$(\star) \qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longmapsto \begin{pmatrix} e^{-\frac{1}{2}i\theta} & 0 \\ 0 & e^{\frac{1}{2}i\theta} \end{pmatrix}$$

which means one has to pass to the double covering of $SO(2)$, called $\mathrm{Spin}(2)$ and consisting of the elements

$$\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\varepsilon = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\gamma^1\gamma^2$$

in C_2.

Now from (A) we see that once a lifting of P to a principal $Spin(2)$ bundle $\tilde{P}$ is given, then the associated vector bundles $S^{\pm}$ on M are complex line bundles such that S^- is a square root of T_M regarded as a complex line bundle, and such that S^+ is dual to S^-. Thus S^+ is a square root of the canonical line bundle $T^{1,0} = K$

In the case of $M = \mathbb{C}$ with $ds^2 = |dz|^2$ the Dirac operator is

$$\gamma^\mu \partial_\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \partial_2 = \begin{pmatrix} 0 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & 0 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 0 & \partial_z \\ \partial_{\bar{z}} & 0 \end{pmatrix}.$$

This shows us that the Dirac operator from S^+ to S^- is to be a $\bar{\partial}$-operator, that is, we have

$$\begin{array}{ccc} T^* \otimes S^+ & \xrightarrow{\text{Cliff mult}} & S^- \\ & \searrow \qquad \nearrow{\scriptstyle \wr} & \\ & T^{0,1} \otimes S^+ & \end{array}$$

Let's check the degrees in the case of compact M, genus g. Then

$$\deg(T_M) = 2-2g \qquad \deg(S^-) = 1-g$$

$$\deg(\underset{K}{\overset{\|}{T^{1,0}}}) = 2g-2 \qquad \deg(S^+) = g-1$$

$$\deg(T^{0,1}) = 2-2g$$

(Note $T^{1,0} \otimes T^{0,1} = T^{1,1}$ which is trivial - Kahler form)

so indeed $T^{0,1} \otimes S^+ \simeq S^-$ checks out. Also we see that as S^+ has degree $g-1$, the Dirac or $\bar{\partial}$-operator

$$S^+ \longrightarrow T^{0,1} \otimes S^+$$

has index 0 as it should

So far we have constructed the spinor bundle over our Riemann surface. One starts with the tangent bundle which is a complex line bundle with connection and inner product, the connection being the Levi-Civita connection. The fact that the complex structure is preserved by the LC connection is clear, as it is defined by the orientation and metric, and this is the reason Riemann surfaces are Kähler.

One chooses a square root of the tangent bdle, and sets $S^- =$ this square root and $S^+ =$ dual of S^-. This is possible as $\deg(T_M) = 2-2g$ is even. S^- will be unique up to a free action of $H^1(M, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{2g}$, for the following reasons. By a square root of T_M we mean a line bundle L together with an isomorphism $L^{\otimes 2} \simeq T_M$. Two square roots differ by a line bundle L together with an isom $L^{\otimes 2} = \mathbb{1}$, which is the same thing as a $\mathbb{Z}/2\mathbb{Z}$ torsor over M.

Because S^- is a square root of T_M it inherits a metric and connection. So the holomorphic structure on S^- which is induced by the connection is compatible with the holom. structure on T_M. Thus we conclude that 'the' Dirac operator on a Riemann surface is the sum of the $\bar\partial$-operator on a square root S^+ of K and minus its adjoint

$$D : \quad S^+ \underset{-\bar\partial^*}{\overset{\bar\partial}{\rightleftarrows}} S^- \left(= T^{0,1} \otimes S^+\right)$$

Example: $M = S^2 = \mathbb{CP}^1$. Here $K = \mathcal{O}(-2)$ and so $S^+ = \mathcal{O}(-1)$. We wish to describe

$$\bar\partial : \mathcal{O}(-1) \longrightarrow T^{0,1} \otimes \mathcal{O}(-1)$$

using trivalyations over $\mathbb{C} = M - \{\infty\}$. $\mathcal{O}(-1)$ has the sections $s_0 = (1, z)$, $s_\infty = (\frac{1}{z}, 1)$ defined outside of ∞ and 0 resp. with the ratio

$$\frac{s_\infty}{s_0} = \frac{1}{z} \qquad (\deg = -1)$$

$T^{0,1}$ has the sections

$d\bar z$ defined on $M - \{\infty\}$

$d\left(\frac{1}{\bar z}\right) = -\frac{1}{\bar z^2} d\bar z$ ——— $M - \{0\}$

with the ratio

$$\frac{d\left(\frac{1}{\bar z}\right)}{d\bar z} = -\frac{1}{\bar z^2} \qquad (\deg = +2)$$

We can identify $\Gamma(S^2, \mathcal{O}(-1))$ with smooth functions f over $\mathbb{C}$ such that $f s_0 = (zf) s_\infty$ is smooth at ∞. Thus $\Gamma(S^2, \mathcal{O}(-1))$ = space of smooth fns. for $\mathbb{C}$ such that at ∞

$$f(z) \sim \sum_{\substack{m \leq -1 \\ n \leq 0}} a_{mn} z^m \bar z^n \qquad |z| \to \infty$$

Similarly, using $d\bar z \otimes s_0$ to trivialize $T^{0,1} \otimes \mathcal{O}(-1)$ over $\mathbb{C}$, we can identify $\Gamma(S^2, T^{0,1} \otimes \mathcal{O}(-1))$ with smooth g on $\mathbb{C}$ such that

$$g \, d\bar z \otimes s_0 = g\left(-\bar z^2 d\tfrac{1}{\bar z} \otimes z s_\infty\right) = -(z \bar z^2 g) \, d\left(\tfrac{1}{\bar z}\right) \otimes s_\infty$$

is smooth at ∞. Thus $\Gamma(S^2, T^{0,1} \otimes \mathcal{O}(-1))$ = space of smooth g on $\mathbb{C}$ such that

$$g(z) \sim \sum_{\substack{m \leq -1 \\ n \leq -2}}' b_{mn} z^m \bar{z}^n$$

Then the $\bar{\partial}$-operator is given by

$$f \longmapsto g = \partial_{\bar{z}} f \quad \blacksquare$$

One can see that $\bar{\partial}$ is an isomorphism as follows. Injectivity, because if f is holom. and zf is bdd, then f has to be zero by Liouville. Surjectivity; it's enough to ~~assum~~ treat the case where $g \equiv 0$ near ∞. Then f is

$$f(z) = \int \frac{d^2 z'}{\pi} \frac{g(z')}{z - z'}$$

and it is holomorphic near ∞ and $O(\frac{1}{z})$ as $z \longrightarrow \infty$, hence f is a smooth section of $\mathcal{O}(-1)$ as required.

August 20, 1984

Bismut's theorem (better - construction). Let E be a vector bundle equipped with connection over the manifold M. We wish to construct an even differential form on $\mathcal{L}M$ and will proceed to construct it first in $\mathcal{L}P$, where P is the principal bundle of E. Thus we can suppose E trivial to begin with, and we let $A = dx^\mu A_\mu$ be the connection form.

(Digression: Consider a circle bundle Q over Y and a vector bundle E with connection over Q. Is it possible to do Bismut's construction to define a differential form on Y? Presumably one can define an even form on Q killed by $d - \iota_X$. However as S^1 acts freely on Q, the Witten cohomology of Q is trivial.)

Let's review what we know in the case of line bundles. The form in question in general is

$$\operatorname{tr} T\{e^{\int_0^1 [-\dot{x}^\mu A_\mu(x) + \frac{1}{2}\delta x^\mu \delta x^\nu F_{\mu\nu}(x)] dt}\}$$

and we want to prove it is killed by $\delta - \iota_X$, where δ denotes d in $\Omega^*(\mathcal{L}M)$. If

$$U_t = T\{e^{\int_0^t [\quad] dt}\}$$

then U_t satisfies

$$\begin{cases} \partial_t U_t = \left(-\dot{x}_t^\mu A_\mu(x_t) + \frac{1}{2}\delta x_t^\mu \delta x_t^\nu F_{\mu\nu}(x_t)\right) U_t \\ U_0 = I. \end{cases}$$

Let's try to understand the meaning of these equations. We start with the connection form $A = dx^\mu A_\mu$ which is a matrix-valued form on M. If $ev : \mathcal{L}M \times S^1 \longrightarrow M$ is the evaluation map, then

$$ev^*(A)_{x,t} = \delta x_t^\mu A_\mu(x_t) + dt\, \dot{x}_t^\mu A_\mu(x_t).$$

Note that if X is the time translation vector field on $\mathcal{L}M$, we have

$$\iota_X \delta x_t^\mu = \dot{x}_t^\mu .$$

Let $ev_t : \mathcal{L}M \longrightarrow M$ be evaluation at t and

$$A_t = (ev_t)^*(A) = \delta x_t^\mu A_\mu(x_t)$$

Then A_t is a family of 1-forms on $\mathcal{L}M$ depending on t and

$$ev^*(A) = A_t + dt\, \iota_X A_t$$

So far the matrix 1-form A on M gives rise to the family of 1-forms A_t and the family of 0-forms $\iota_X A_t$. Let

$$F = dA + A^2 = \tfrac{1}{2} dx^\mu dx^\nu F_{\mu\nu}$$

be the curvature; it is a matrix 2 form on M, and gives rise to a family of 2-forms on $\mathcal{L}M$:

$$F_t = \frac{1}{2} \delta x_t^\mu \delta x_t^\nu F_{\mu\nu}(x_t).$$

U_t is the family of even forms on $\mathcal{L}M$ defined by

$$\partial_t U_t = (-\iota_X A_t + F_t) U_t \qquad U_0 = I.$$

We want to show that $\mathrm{tr}(U_1)$ is killed by $\delta - \iota_X$.

First the abelian case: $F = dA \Rightarrow F_t = \delta A_t$ so that

$$-\iota_X A_t + F_t = (\delta - \iota_X) A_t$$

and

$$(\delta - i_X)(F_t - \iota_X A_t) = \underbrace{(\delta - \iota_X)^2}_{-\mathcal{L}_X} A_t$$

This should be $-\partial_t A_t$. Check:

$$A_t = \delta x_t^\mu A_\mu(x_t)$$

$$\partial_t A_t = \delta \dot{x}_t^\mu A_\mu(x_t) + \delta x_t^\mu \dot{x}_t^\nu \partial_\nu A_\mu(x_t)$$

$$(\delta - i_X)(F_t - \iota_X A_t) = -\dot{x}_t^\mu \delta x_t^\nu F_{\mu\nu}(x_t) - \delta(\dot{x}_t^\mu A_\mu(x_t))$$

$$= \delta x_t^\mu \dot{x}_t^\nu F_{\mu\nu}(x_t) - \delta \dot{x}_t^\mu A_\mu(x_t) - \dot{x}_t^\nu \delta x_t^\mu \partial_\mu A_\nu(x_t)$$

so it's OK.

Now what I need is the formula

$$\boxed{(\delta - i_X)(F_t - \iota_X A_t) = -\partial_t A_t + [F_t - i_X A_t, A_t]}$$

August 22, 1984

More on Bismut's construction. Let us adopt his viewpoint. We start with E over M, and suppose E equipped with unitary connection. Let P be the principal bundle of E; it is a principal bundle for $U = U_r$. Then $\mathcal{L}P$ is a principal bundle over $\mathcal{L}M$ with the group $\mathcal{L}U$.

Now this is a typical gauge transf. gp. situation. One looks at the bundle $\tilde{E} = ev^*(E)$ [redacted] over $\mathcal{L}M \times S^1$, where $ev : \mathcal{L}M \times S^1 \to M$ is the evaluation map. We get a family of vector bundles over S^1 parametrized by $\mathcal{L}M$. The natural thing is to introduce over $\mathcal{L}M$ the principal bundle $\mathcal{P}$ for the group $\mathcal{G} = \mathcal{L}U$, whose fibre at a loop x is an isomorphism of the vector bundle over S^1 given by the restriction of E to the loop x with the trivial bundle of rank r over S^1. Clearly $\mathcal{P}$ can be identified with $\mathcal{L}$ [redacted] P.

So we are in a gauge situation. Better, we have a family of vector bundles over S^1, and so we can try the things which are obvious from this viewpoint. We know that the connection on E lifts to a connection on $\tilde{E}$ over $Y \times S^1$, where $Y = \mathcal{L}M$, and that the partial connection in the Y-direction gives a connection in $\mathcal{P}$ over Y. Too abstract.

Try this. Let A be the connection form in P; it is a $\mathrm{Lie}(U)$-valued 1-form. Then for each t we have $ev_t : \mathcal{L}P \longrightarrow P$ and so get $A_t = ev_t^*(A)$ which is a $\mathrm{Lie}(U)$-valued 1-form on $\mathcal{L}P$ depending on t

which we can interpret as a 1-form $\mathcal{L}P$ with values in the loops in $Lie(U)$, i.e. the Lie algebra of $\mathcal{G}$. So what I denoted

$$A_t = \delta x_t^\mu A_\mu(x_t)$$

should be identified with $\tilde{A} \in \Omega^1(\mathcal{L}P, \tilde{\mathfrak{g}}) = \Omega^{1,0}(\mathcal{L}P \times S^1, \mathfrak{g})$, where $\mathfrak{g} = Lie(U)$, $\tilde{\mathfrak{g}} = Lie(\mathcal{G}) = \mathcal{L}\mathfrak{g}$.

Next we have to go over the equivariant bundle formalism. Suppose we have an equivariant bundle E/M with respect to a circle action, and we are given an invariant connection D on E. Let P be the principal bundle, where S^1 acts on P, and we have $\pi^*(E) \simeq P \times V$ with S^1 acting trivially on V. Then D lifts to $d + A$ acting on $\Omega^*(P) \otimes V$, where $A \in \Omega^1(P) \otimes \mathfrak{g}$ is the connection form.

Recall the formulas I used for calculating equivariant curvature

$$\begin{cases} D - u\iota_X \quad \text{acting on } k[u] \otimes \Omega(M, E)^{S^1} \\ (D - u\iota_X)^2 = D^2 - u[\iota_X, D] = D^2 + u\varphi \\ \qquad \in k[u] \otimes \Omega(M, \operatorname{End} E)^{S^1} \end{cases}$$

Here φ is the Higgs field, defined by

$$\mathcal{L}_X = [\iota_X, D] + \varphi_X.$$

Now how do these formulas look up in P ? The equivariant connection is

$$d + A - u\iota_X \quad \text{acting on } k[u] \otimes \Omega(P)^{S^1} \otimes V$$

and the equivariant curvature is

$$dA + A^2 - u[d+A, i_X] = (dA+A^2) - u(i_X A)$$
$$\in k[u] \otimes \Omega(P)^{S^1} \otimes \mathfrak{g}$$

Let's now go back to our bundle over $\mathcal{L}M$ whose fibre at a loop x is the space of sections of E over the restriction $x^*(E)$; $x : S^1 \to M$ of E to the loop. $\mathcal{L}P$ is the principal bundle and we have given the connection form $\tilde{A}$. Time translation gives a vector field X in $\mathcal{L}P$ compatible with the same vector field in $\mathcal{L}M$, however X also acts on the group $\mathcal{L}U$.

I don't see any advantage to working in the principal bundle $\mathcal{L}P$ except that the basic bundle becomes trivial. It should be possible to get the same results by starting with a trivial E over M and connection $d+A$, then considering the trivial bundle over $\mathcal{L}M$, whose fibre is $\Gamma(S^1, 1^{\oplus n}) = a^n$, $a =$ fns. on S^1. On this trivial bundle we have the connection $d+\tilde{A}$, where if $A = dx^\mu A_\mu$ in $\Omega^1(M)\otimes\mathfrak{g}$, then $\tilde{A} = \delta x_t^\mu A_\mu(x_t)$ in $\Omega^1(\mathcal{L}M, \tilde{\mathfrak{g}}) = \Omega^{1,0}(\mathcal{L}M \times S^1, \mathfrak{g})$. So now suppose we try to carry out the equivariant form arguments.

So we look at the operator

$$d + \tilde{A} - u\, i_X \quad \text{acting on} \quad \Omega^*(\mathcal{L}M, a^n) = \Omega^{*,0}(\mathcal{L}M \times S^1)^n$$

whose square is (here u is a parameter)

$$(d+\tilde{A}-u i_X)^2 = \widetilde{dA+A^2} - u[d+\tilde{A}, i_X]$$

$$= \tilde{F} - u i_X \tilde{A} - u \mathcal{L}_X$$

I would like to identify this with an element of $\Omega^*(\mathcal{L}M, \tilde{\mathfrak{g}}) = \Omega^{*,0}(\mathcal{L}M \times S^1, \mathfrak{g})$.

More precisely, I have this bundle $\mathcal{E}$ over $\mathcal{L}M$ which is a sort of vector bundle with fibre a finite projective $\mathcal{A}$-module. The curvature should be a two form on $\mathcal{L}M$ with values in End $\mathcal{E}$, and it should be some sort of equivariant 2-form. Then you want to exponentiate this and take the trace.

So the naive picture is that of an $\mathcal{A}$-vector bundle

<u>August 23, 1984</u>

Let X/Y be [redacted] a principal S^1-bundle, and let L be a line bundle on X. Consider the Gysin sequence in $\mathbb{Z}$-cohomology (Y conn.)

$$0 \to H^0(Y) \xrightarrow{\sim} H^0(X) \to 0 \to H^1(Y) \to H^1(X) \to \underset{\mathbb{Z}}{\overset{}{H^0(Y)}}$$

$$\to H^2(Y) \to H^2(X) \to H^1(Y)$$

which gives

$$\mathbb{Z} \xrightarrow{\text{class of } X} H^2(Y) \longrightarrow H^2(X) \xrightarrow{\text{integrating over the fibre}} H^1(Y)$$

$$H^2(Y) = \mathrm{Pic}(Y) \qquad H^2(X) = \mathrm{Pic}(X)$$

Geometrically, the image of the class of L in $H^1(Y)$ under integration-over-the-fibre should be obtained by choosing a connection in L and taking the monodromy around the fibres to obtain a map $\tau: Y \to S^1$. The exact sequence [redacted] confirms the result that the bundle L descends to Y $\iff$ the monodromy map has a logarithm, in which case the descended bundle is unique up to multiplication by [redacted] the line bundle L_X associated to X.

Now I would like to find a geometric object on Y which ~~correspond~~ is associated to L on X. The thing that comes to mind is some kind of Azumaya algebra over Y, because it is some sort of gadget which is specified by giving locally on Y a line bundle unique up to tensoring with a power of L_X. The thing that comes to mind is some kind of Azumaya algebra over Y, because it ~~because~~ is locally isomorphic to $\mathrm{End}(V)$, where V is a vector bundle unique up to tensoring with a line bundle.

Certainly we should be able to define an Azumaya algebra on Y associated to L over X. There is an element of $H^3(Y, \mathbb{Z})$ obtained by integrating $c_1(L)^2$ over the fibre.

Let us see if we can construct differential forms on X satisfying $(d - \iota_X) \cdots = 0$ from a vector bundle E over X equipped with a connection. Start with a line bundle to see what is needed in this case. The line bundle L gives us a curvature which is a closed 2 form F on X. Multiplying by dt and integrating gives a 2 form over Y. dt must be a connection form θ in X/Y. But then $f_*(\theta F)$ won't be closed:

$$d\, f_*(\theta F) = -f_*(d(\theta F))$$

$$= -f_*(f^* u . F) + \sout{f_*(\theta dF)}$$

$$= -u\, f_*(F)$$

So we have the wrong operation.

Instead what we need ~~to do~~ is to average the curvature over the compact gp. S^1.

$$\bar{F} = \int_0^1 dt\, R_t^* F \qquad\qquad \iota_X dA = \mathcal{L}_X A - d\iota_X A$$

Then $$\iota_X \bar{F} = \int_0^1 dt\, \iota_X R_t^* F = \int_0^1 dt\, R_t^* \iota_X F \qquad ?$$

$$= \int_0^1 dt\, \partial_t [R_t^* A] - d \int_0^1 dt\, R_t^* \iota_X A$$

$$= 0 - d\, f_* A. \qquad\qquad f: X \to Y$$

Suppose given L with connection over a circle bdle X over Y. Let's see that Bismut's form is well-defined on X. Working locally on Y we can suppose $X = Y \times S^1$ and that L is trivial. The connection on L is then given by a 1-form

$$A = dy^\mu A_\mu + dt\, A_0$$

and the curvature by the 2-form

$$F = dA = \tfrac{1}{2} dy^\mu dy^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + dy^\mu dt (\partial_\mu A_0 - \partial_0 A_\mu)$$

Here A_μ, A_0 are functions of the y^μ, t. Now the S^1-action on X given by translation on the fibre can be used to average F vertically to obtain the S^1-invariant 2 form

$$\bar{F} = \tfrac{1}{2} dy^\mu dy^\nu \int_0^1 dt\, (\partial_\mu A_\nu - \partial_\nu A_\mu) + dy^\mu dt \int_0^1 dt (\partial_\mu A_0 - \cancel{\partial_0 A_\mu})$$

The last term integrates to zero as $\partial_0 = \partial/\partial t$. Note that the dt in $\int_0^1 dt$ is a measure not a differential, in fact $\int_0^1 dt$ is just the averaging operation.

Then we have

$$\iota_X \bar{F} = -dy^\mu \int_0^1 dt\ \partial_\mu A_0$$

$$= -dy^\mu \partial_\mu \int_0^1 dt\, A_0$$

Since $\int_0^1 dt\, A_0 = f_* A$ is a function on Y it follows that $\iota_X \bar{F}$ is the basic 1-form on X which is minus $d(f_* A)$ on Y.

Recall that

$$\tau = e^{-\int_0^1 dt\, A_0} \quad : Y \longrightarrow \mathbb{C}^\times$$

is the monodromy of the connection in the fibres.

Now the monodromy is intrinsic, but $\int_0^1 dt\, A_0 = f_* A$ depends upon the trivialization of L on the fibres, since a gauge transformation changes A to $A + g^{-1}dg$ and

$$\int_0^1 dt\, g^{-1}\partial_t g = 2\pi i \text{ degree } g.$$

Summarizing: Given L with connection over a circle bundle X with base Y. Then the curvature of F can be averaged with respect to the S^1-action so as to obtain an invariant 2-form $\bar{F}$. Formula:

$$\iota_X \bar{F} = f^* d\log\tau$$

where $\tau : Y \longrightarrow \mathbb{C}^\times$ is the monodromy function. If τ is null-homotopic so that $\log\tau$ is defined globally on Y, and if θ is a connection form in X/Y, then

$$\bar{F} + d(\theta \cdot \log\tau)$$

is basic so defines a closed 2-form on Y.

Because S^1 acts freely on X the Witten cohomology of X should be trivial. Here's a proof: We have

$$\Omega^*(X)^{S^1} = \Omega^*(Y) \oplus \Omega^*(Y)\theta$$

where θ is a connection form in X/Y. Now consider the operator of multiplication by θ on $\Omega^*(X)^{S^1}$. We have

$$[d + a\iota_X, \theta] = d\theta + a$$

and this is invertible, as $d\theta$ is nilpotent, if $a \neq 0$. Thus multiplication by θ is a homotopy from 0 to an invertible operator, so the Witten cohomology has to be zero.

In the remaining minutes I would like to see if I can construct the Bismut form in the case where one has a **vector bundle** E of rank > 1. Again we have a connection on E and E is over a circle bundle X/Y. What we are trying to do is to produce a differential form on X by the formula

$$\operatorname{tr} T\left\{ e^{\int_0^1 dt\,[-\iota_X A + F]} \right\}.$$

Is it possible to view this as a kind of Chern character where one exponentiates $\partial_t + \cdots$ and takes the trace? Notice that the operator

$$\mathcal{L}_X + \iota_X A - F$$

operates on forms on X with values in E. Now what happens as we exponentiate an operator of the form

$$\partial_t + W(t)$$

where $W(t)$ is a periodic matrix function of t

August 24, 1984

Consider the operator $\partial_x + a(x)$ acting on functions on the line. We want to exponentiate it:

$$e^{t(\partial_x + a)}.$$

If g satisfies $\quad \partial_x + a = g^{-1}\partial_x g$

then $$e^{t(\partial_x + a)} = g^{-1} e^{t\partial_x} g$$

and we know that

$$\left(e^{t\partial_x} f\right)(x) = f(x+t)$$

Thus

$$\left(e^{t(\partial_x + a)} f\right)(x) = g^{-1}(x)\; g(x+t) f(x+t)$$

Think of $D_x = \partial_x + a$ as a connection on the trivial bundle over the line. Then $e^{t(\partial_x + a)}$ applied to the section f is the backwards translation of f through the distance t. ($(e^{t\partial_x} f)(x) = f(x+t)$ represents a wave travelling backwards with unit speed.)

Next we consider the periodic case, where a is periodic and $\partial_x + a(x)$ is operating on functions on the circle. It's clear that $e^{t(\partial_x + a)}$ is the operator which parallel transports backwards thru a distance t.

At this point I have to decide what I want to do with this operator. Some possibilities are:

$tr\left(e^{t(\partial_x+a)}\right)$, $\det(\partial_x+a)$ and fermion integrals associated to the action $\psi(\partial_x+a)\psi$

The expression $tr\, e^{t(\partial_x+a)}$ can be understood in the same sense as Hörmander's $tr\left(e^{-itP}\right)$, namely, as a distribution on the t-line. The trace is obviously zero when t is not a period. When t is a period one has that $e^{t(\partial_x+a)}$ is a multiplication operator, specifically, multiplication by

$$g^{-1}(x)\,g(x+t) = \vec{T}\left\{e^{\int_x^{x+t} a\,dx}\right\}$$

where the arrow means later times multiply on the right. (Thus we can take

$$g(x) = \vec{T}\left\{e^{\int_0^x a\,dx}\right\}$$

which satisfies $$g(x+dx) = g(x)\underbrace{e^{a(x)dx}}_{1+a(x)dx}$$

or $$\partial_x g = g\cdot a \quad \text{as required.})$$

Get the notation straight. The connection is $D_x = \partial_x + a(x)$, hence the parallel transport from x' to x is

$$T_x^{x'} = T\left\{e^{-\int_{x'}^{x} a\,dx}\right\}$$

Then $$g(x)^{-1} = T_x^{0} \quad \text{and so}$$

$$e^{t(\partial_x+a)} = g^{-1}(x)\,g(x+t) = T_x^{0}\,T_0^{x+t}$$

$$e^{t(\partial_x+a)} = T^{x+t}$$

$$g^{-1}(x)\, g(x+t) = T_x^0\, T_0^{x+t} = T_x^{x+t}$$

Thus we see that

$$\left(e^{t(\partial_x + a)} f\right)(x) = T_x^{x+t} f(x+t)$$

is the backwards parallel transport operator on sections of the vector bundle.

For later reference I want formulas for the case where a is a constant number. The eigenvalues of $\partial_x + a$ acting on functions with period L are $\frac{2\pi i n}{L} + a$ so that

$$\operatorname{tr} e^{t(\partial_x + a)} = \sum_{n\in\mathbb{Z}} e^{t\left(\frac{2\pi}{L} i n + a\right)}$$

$$= e^{at} \sum_{n\in\mathbb{Z}} e^{2\pi i n \frac{t}{L}} = e^{at} \sum_{k = \frac{2\pi}{L} n \in \frac{2\pi}{L}\mathbb{Z}} e^{ikt}$$

$$= L e^{at} \sum_{m\in\mathbb{Z}} \delta(t - mL)$$

(Here have used $\langle x | k \rangle = \frac{e^{ikx}}{\sqrt{L}}$ $k \in \frac{2\pi}{L}\mathbb{Z}$ as an orth. basis for $L^2(\mathbb{R}/L\mathbb{Z})$ and so

$$\sum_k \langle x|k\rangle\langle k|x'\rangle = \frac{1}{L} \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} e^{ik(x-x')} = \sum_{m\in\mathbb{Z}} \delta(x - x' - mL)$$

In addition to this trace there is also the determinant.

$$\det(\partial_x + a) = \text{"}\prod_{n\in\mathbb{Z}} \left(\tfrac{2\pi}{L} n i + a\right)\text{"}$$

$$= \text{const} \cdot a \cdot \prod{}' \left(1 + \frac{aL}{2\pi n i}\right)$$

$$= \text{const} \cdot \sin\left(\frac{aL}{2i}\right) \sim e^{aL} - 1$$

$$\text{monodromy} = e^{-\int_0^L a\,dx} = e^{-aL}$$

Now let's consider the Bismut setup. One has a circle bundle X/Y and a v. bundle E with connection over X. The circle action gives us a vector field, also denoted X. Change X to M

On $\Omega^*(M,E)$ we have the operator D from the connection, and ι_X from the circle action, so we can consider $D + \lambda \iota_X$, where λ is a constant. The square is

$$(D + \lambda \iota_X)^2 = \lambda[\iota_X, D] + D^2$$

and $[\iota_X, D] = D_X$ is just covariant differentiation in the X-direction.

Next we exponentiate this operator

$$e^{t(D+\lambda\iota_X)^2} = e^{t(\lambda D_X + D^2)}$$

and take the trace, which is a distribution on the line with values in the forms on M. Similarly, I should ask about the determinant.

Let's look at the operator $\int dt\, \varphi(t)\, e^{t(\lambda D_X + D^2)}$ with $\varphi(t) \in C_0^\infty(\mathbb{R})$. Then

$$\mathrm{tr}\left\{\int dt\, \varphi(t)\, e^{t(\lambda D_X + D^2)}\right\} \quad \in \quad \Omega^{ev}(M).$$

satisfies

$$(d + \lambda \iota_X)\ \mathrm{tr}\left\{\int dt\, \varphi(t)\, e^{t(D+\lambda\iota_X)^2}\right\}$$

$$= \mathrm{tr}\left\{\int dt\, \varphi(t)\ [D+\lambda\iota_X, e^{t(D+\lambda\iota_X)^2}]\right\} = 0$$

I want to get this whole business a bit clearer. Suppose we have a vector field X on a manifold M whose flow is globally defined, i.e. there exists the one-parameter group e^{tX} on the functions. Let A be a matrix function on M and let's try to show the operator $e^{t(X+A)}$ exists. The idea is that

$$\partial_t\left(e^{-tX} e^{t(X+A)}\right) = \left(e^{-tX} A e^{tX}\right) e^{-tX} e^{t(X+A)}$$

Thus if we put $A_t = e^{-tX} A e^{tX}$ so that A_t is a matrix fn. on M, we only have to solve

$$\partial_t V_t = A_t V_t$$

which has the solution

$$V_t = T\left\{e^{\int_0^t dt\, A_t}\right\}.$$

Note that V_t is a multiplication operator on vector functions. We have

$$e^{t(X+A)} = e^{tX} V_t$$

Now suppose X generates a circle action, say $e^X = I$. Then $e^{(X+A)}$ is the multiplication operator V_1.

Let's apply this in the case of a circle action on a manifold M on which we have a vector bundle E with connection D, but not an equivariant bundle. Then we have the operator $D + \lambda \iota_X$ on $\Omega^*(M, E)$ which has odd degree. Its square is

$$(D + \lambda \iota_X)^2 = \lambda D_X + D^2$$

and relative to a local trivialization, it has the form $\lambda X +$ multiplication operator. (In fact if we introduce the principal bundle P of E, then the vector field X on M lifts to a vector field on P thanks to the connection. Thus one has up on P that relative to the canonical trivialization:

$$D_X = [\iota_X, d + A] = X + \iota_X A = X$$

as X has been lifted horizontally. So up on P we have

$$(D + \lambda \iota_X)^2 = \lambda X + D^2$$

which is exactly in the form I have discussed.)

In any case we know that we can form

$$U_t = e^{t(\lambda D_X + D^2)}$$

as an operator on $\Omega^*(M, E)$ and that this operator looks like the translation $e^{t\lambda X}$ times a multiplication operator. Now it is obvious that

$$[D + \lambda \iota_X, U_t] = 0$$

for any t.

Now if $t_0 \lambda = 1$, then $e^{t_0 \lambda X} = I$, so that U_{t_0} is a multiplication operator, i.e. an element of $\Omega^*(M, \mathrm{End}\, E)$. (One should keep track of the $\Omega^*(M)$-module structure. $U_t(\omega\alpha) = (e^{tX}\omega) U_t \alpha$ should be true, whence for $t = t_0$, U_{t_0} is linear over the forms, hence lies in $\Omega^{ev}(M, \mathrm{End}\, E)$.)

Finally one has

$$(d + \lambda \iota_X)\, \mathrm{tr}\, U_{t_0} = \mathrm{tr}\, [D + \lambda \iota_X, U_{t_0}] = 0$$

Summary: For any circle action Bismut's form is defined.

Let's review the analogies between cyclic theory and equivariant cohomology of the free loop space.

First of all, there is the relation between cyclic cohomology of A and the Hochschild cohomology. This takes the form of an exact sequence (and spec. seq. if one wants)

$$\longrightarrow HC^{n-2}(A) \xrightarrow{S} HC^n(A) \longrightarrow H^n(A) \longrightarrow HC^{n-1}(A) \longrightarrow \cdots$$

which resembles the sequence for a circle action

$$\longrightarrow H_S^{n-2}(M) \xrightarrow{u} H_S^n(M) \longrightarrow H^n(M) \longrightarrow H_S^{n-1}(M) \rightarrow \cdots$$

Secondly there is Connes cyclic object category Λ which has the homotopy type of $BS^1 = \mathbb{C}P^\infty$. We saw how the category Λ approximates the monoid of degree 1 maps from S^1 to itself.

Thirdly, there is Bott's (or Morse's) idea [redacted] of [redacted] approximating $\mathcal{L}M$ by a nbd. of the diagonal in M^k. This begins to connect up the cyclic objects category Λ with the loop space $\mathcal{L}M$.

However, there are problems with variance. Let $A = C^\infty(M)$. The cyclic cohomology of [redacted] A, which is the thing such that S increases degree by 2, gives currents on M, or the DR homology in the limit. It is therefore covariant in M.

On the other hand if we were to hope for a good version of the equivariant cohomology of the loop space $\mathcal{L}M$, then it should be contravariant in M. So what seems to be needed to go with the loop space is something like Connes

$$\operatorname{Ext}^*_\Lambda(k^\natural, A^\natural).$$

(This reminds me that Connes mentioned that for $A = C^\infty(M)$, the S-localized version of the above coincides with the S-localized version of $HC(A)$. The reason is the Poincaré duality proof in K-theory and the fact that one can map Kasparov[redacted] theory to localized $\operatorname{Ext}^*_\Lambda$ preserving [redacted] cup products.)

It seems clear that the basic construction one wants to use is the time-ordered product, meaning that if one has a path a_t of matrices in A, then one has the product

$$T\{e^{\int a_t dt}\}.$$

So what does this tell us? I don't see where to go from here. Connes' basic map

$$K_0 A \longrightarrow S^{-1}\mathrm{Ext}^*_\Lambda(k^\natural, A^\natural)$$

is defined by associating to an idempotent e the element $e^{\otimes p}$ of $A^{\otimes p}$. There ought to be some relationship between differential forms on $\mathcal{L}M$ and $A^{\otimes p} = C^\infty(M^p)$.

Let's pose the following problem: Is there a good definition of equivariant cohomology for $\mathcal{L}M$ which would be a $\mathbb{Z}$-graded theory having an S-operator increasing degree by 2? One would like to localize it so as to obtain the Witten cohomology.

Return to the determinant of $\partial_x + a$ on $\mathbb{R}/L\mathbb{Z}$. Use the formal formula

$$\log\det A = -\int_0^\infty tr(e^{tA})\,\frac{dt}{t}$$

$$\log\det(\partial_x + a) = -\int_0^\infty \underbrace{tr\left(e^{t(\partial_x+a)}\right)}_{L e^{at}\sum_{m\in\mathbb{Z}}\delta(t-mL)}\,\frac{dt}{t}$$

$$= -L\sum_{m\geq 1}\frac{e^{aLm}}{mL} = -\sum_{m\geq 1}\frac{1}{m}\left(e^{aL}\right)^m$$

$$= \log\left(1-e^{aL}\right)$$ which is reasonable.

So if I consider the Bismut form in the case of [redacted] a line bundle, it is

$$\tau e^{\bar{F}}$$

where τ is the monodromy and $\bar{F}$ is the averaged curvature. This is already $\exp \int a\,dt$ where $\partial_x + a = \partial_t - A_0 + F$. For a line bundle there is no trace, so the determinant is just

$$\tau e^{\bar{F}} - 1$$

up to normalizations. [redacted] This doesn't look interesting.

Let's go back to $\mathcal{L}M$

August 29, 1984

Summary of ideas the past few days.

The Chern character is the basic transformation from K-theory to cohomology. The form of the Chern character $\operatorname{tr} e^{D^2}$ supports the use of the heat operator in index questions.

An annoying problem is how to see simply the existence of heat operators. It would be nice to construct the heat operator from the Dirac operator viewpoint. This possibility is supported by the standard form of the Ito equation

$$dy = a\,dw + b\,dt$$

where a is a "square root" of the variance.

In Bismut's construction of a form on $\mathcal{L}M$, he uses a variant the Chern character, where the connection is $D + i_X$, and the curvature is

$$(D+i_X)^2 = D_X + D^2$$

a kind of one-dimensional Dirac operator, and where one takes a suitable kind of trace. Corresponding to this new kind of Chern character, is there a new heat operator, or a new Dirac operator on $\mathcal{L}M$? Could this Bismut construction be part of a new theory of characteristic classes related to Kac-Moody Lie algebras? The idea is that the circle action is to enter, so that one has a kind of gravity game occurring.

August 30, 1984

Gaussian processes. Given a f.d. real vector space V and a positive-definite q.f. [redacted] Q on V, there is a unique Gaussian probability measure $d\mu$ on the dual V^* such that $Q(v) = \langle v^2 \rangle$, where we think of v as a function on V^*. The symmetric algebra $S(V)$ is dense in $L^2(V^*, d\mu)$, in fact, we can identify $L^2(V^*, d\mu)$ with the [redacted] Hilbert space symmetric tensor space

$$S\mathcal{H} = \hat{\bigoplus_n} \mathcal{H}^{\hat{\otimes} n}/\Sigma_n$$

where $\mathcal{H} = V \otimes \mathbb{C}$ equipped with the hermitian inner product extending Q.

There are ways to extend this to [redacted] infinite dimensions. Two examples:

1) Kolmogoroff: Let X be a set and let Q be a matrix $Q(x, x')$ which is positive definite. Let V be the real vector space with basis X, i.e. $\bigoplus_X \mathbb{R}$. The dual V^* is then the product $V^* = \prod_X \mathbb{R}$. Kolmogorov says that one has a probability measure on this product which induces on each finite product $\prod_S \mathbb{R}$ the Gaussian measure [redacted] with variance $Q(x, x')$ restricted to the finite subset S.

2) Let $V = C_0^\infty(\mathbb{R}^n)$ (real valued fns) and let $V^* =$ distributions. One supposes given [redacted] an [redacted] inner product on V:

$$\|v\|^2 = \int dx\, dx' \, K(x,x')\, v(x)\, v(x')$$

[redacted] where K is a distribution on the product. Then I think the Gelfand-Shilov books on distributions assert that one gets a Gaussian probability measure on $V^* =$ space of distributions.

We now consider the Lie superalgebra with one odd generator and no relations. A repn. of this consists of a super vector space $\mathcal{H}^{\pm}$ with an odd operator $\not{D}$. The corresponding 1-parameter subgroup is

$$e^{\theta \not{D} + t \not{D}^2}$$

In the case where $\not{D}$ is a Dirac operator, Friedan-Windey show how to represent this super heat operator using an "integral" over superpath configurations, where the superpath is

$$X^{\mu}_{(t,\theta)} = x^{\mu}_t + \theta \psi^{\mu}_t$$

Now I would like to use the fact that $\not{D}$ generates the Lie super algebra, so that the supergroup is generated by elements $e^{\theta_i \not{D}}$ for different θ_i. Here θ_i are anti-commuting quantities of square zero. Note that

$$T\left\{\prod e^{\theta_i \not{D}}\right\} = e^{\left(\sum \theta_i\right)\not{D} - \left(\sum_{i>j} \theta_i \theta_j\right)\not{D}^2}$$

since

$$e^{\theta_2 \not{D}} e^{\theta_1 \not{D}} = (1+\theta_2 \not{D})(1+\theta_1 \not{D})$$

$$= 1 + (\theta_1+\theta_2)\not{D} - \theta_2\theta_1 \not{D}^2 = \left(1+(\theta_1+\theta_2)\not{D}\right)\left(1-\theta_2\theta_1\not{D}^2\right)$$

$$= e^{(\theta_1+\theta_2)\not{D}}\, e^{-\theta_2\theta_1 \not{D}^2}$$

August 31, 1984

Idea: When one tries to construct the heat kernel e^{-tH} from a path $\gamma(t)$, $t \geq 0$ starting at I with tangent vector $-H$, one should use a product

$$\prod_{j=1}^{N} \gamma(t_j)$$

where $t_j \geq 0$ and $\sum t_j = t$. Here the idea is analogous to Riemann integration, where one allows arbitrary subdivisions instead of the standard subdivision $t_j = \frac{t}{N}$.

The nice point is that one is now over a product space, namely the set of all such subdivisions. One can hope then to [redacted] find a general existence argument.

I still feel [redacted] that this time-ordered product game represented by path integrals is fundamentally sound.

September 1, 1984

I am investigating the question of whether I can generate the super heat kernel by using a product of operators $e^{\theta_j \not{D}}$ where θ_j are Grassmann variables. We have

$$e^{\theta_1 \not{D}} e^{\theta_2 \not{D}} = e^{(\theta_1+\theta_2)\not{D}} e^{\frac{1}{2}[\theta_1\not{D}, \theta_2\not{D}]}$$
$$= e^{(\theta_1+\theta_2)\not{D}} e^{-\theta_1\theta_2 \not{D}^2}$$

and more generally

$$e^{\theta_1 \not{D}} e^{\theta_2 \not{D}} \cdots e^{\theta_N \not{D}} = e^{(\sum \theta_j)\not{D}} e^{(\sum_{i>j} \theta_i\theta_j)\not{D}^2}.$$

I guess we want to use a one parameter family θ_t of Grassman variables eventually. In this case the formula is

$$\boxed{T\left\{e^{\int dt\, \theta_t \not{D}}\right\} = e^{\left(\int dt\, \theta_t\right)\not{D} - \left(\int_{t>t'} dt\, dt'\, \theta_t \theta_{t'}\right)\not{D}^2}}$$

In order to use this formula the idea is to go from the Hilbert space $\mathcal{H}$ on which $\not{D}$ operates, and to extend the base ring from $\mathbb{C}$ to a commutative superalgebra R containing the variables θ_t. Then the elements $\theta_t \not{D}$ belong to the superalgebra

$$R \hat{\otimes} \operatorname{End} \mathcal{H},$$

and so does the above boxed expression. Next one sees that if we put

$$\hat{\theta} = \int dt\, \theta_t, \quad \hat{t} = -\int_{t>t'} dt\, dt'\, \theta_t \theta_{t'}$$

then these elements generate a subalgebra of R, call it S, and we have

$$T\{e^{\int dt\, \theta_t \not{D}}\} = e^{\hat{\theta}\not{D} + \hat{t}\not{D}^2} \quad \in S \hat{\otimes} \operatorname{End} \mathcal{H}.$$

The hope would be that θ_t and R can be chosen so that S turns out to be the algebra of smooth functions on the super line $\mathbb{R}^{1,1}$, actually, half-line $t > 0$.

It seems to me that the central problem with all this algebraic manipulation is how to incorporate the <u>positivity</u> which must be present before the heat operator can exist. Algebraically I can't see the difference between $e^{\theta \not{D}}$ and $e^{i\theta\not{D}}$ yet the heat operator $e^{\hat{t}\not{D}^2}$ can exist only for $\hat{t} \geq 0$.

Thus what should R be? It is supposed to contain Grassmann variables θ_t for t real. In the weakest sense this means that we can map test functions $f(t)$ to elements of R by $f \longmapsto \int dt\, f(t)\, \theta_t$. Thus θ_t is a distribution with values in R. The universal R from this viewpt is the exterior algebra on the space of test functions

Now suppose we can show the existence of

$$T\{e^{\int dt\, \theta_t \not{D}}\} = e^{\hat{\theta}\not{D} + \hat{t}\not{D}^2}$$

in the super algebra $R \hat{\otimes} \operatorname{End}(\mathcal{H})$. Then it seems to me that I can change θ_t to $i\theta_t$ whence

$$\hat{t} = \int_{t>t'} dt\, dt'\, \theta_t \theta_{t'}$$ changes sign. But this algebraic automorphism ought to mean that $\hat{t}$ can't be specialized to a real number.

September 3, 1984

The problem is how to [redacted] do the section on Connes–Getzler theory. Let's go over the logical structure of the arguments.

One has the supertrace of a heat operator

$$\operatorname{tr}_s\left(e^{h^2L^2 + h\,dy^a[D_a,L] + \frac{1}{2}dy^a dy^b F_{ab}}\right)$$

depending on a parameter $h > 0$. [redacted]

[redacted] The assertion is that the limit as $h \to 0$ exists and can be evaluated in a certain way. This way involves replacing the operator trace by an integral and the algebra of differential operators on $S \otimes E$ with its associated graded algebra with respect to the Getzler filtration.

Ultimately I must give the formula in my situation, which is more complicated than Getzler's. So let's explain my setup and how it differs from Getzler's. Start with Getzler: He works with operators on sections of $S \otimes E$ over M. The operator $L^2 = \not{D}^2$ [redacted] has negative elliptic symbol so $e^{h^2L^2}$ is an operator with smooth Schwartz kernel. Hence its ordinary supertrace is defined. By ordinary I mean the trace over $\mathbb{C}$ of a smooth kernel operator on $\Gamma(S \otimes E_0)$.

[redacted] Now in my setup the operator

$$h^2L^2 + h\,dy^a[D_a,L] + \frac{1}{2}dy^a dy^b F_{ab}$$

belongs to the algebra $\operatorname{End}(\mathcal{H}_0)\,\hat{\otimes}\,\mathcal{A}$, where $\mathcal{A} = \Lambda[dy^a]$. We exponentiate it, which is possible as it has negative elliptic symbol, and we obtain an element of $\mathrm{End}'(\mathcal{H}_0)\hat{\otimes}\mathcal{A}$, where End' denotes smooth kernel operators on $\mathcal{H}_0 = \Gamma(M, S\otimes E_0)$. Then we use the $\mathcal{A}$-valued s-trace

$$\mathrm{tr}_s^{\mathcal{H}_0} : \mathrm{End}'(\mathcal{H}_0)\otimes\mathcal{A} \longrightarrow \mathcal{A}$$
$$T\otimes a \longmapsto \mathrm{tr}_s^{\mathcal{H}_0}(T)\, a$$

So it seems that I might think in terms of an operator on sections of $S\otimes E_0\otimes\mathcal{A}$ which commutes with right multiplication by elements of $\mathcal{A}$. Corresponding to this 'symmetry' we have that the supertrace of a smooth kernel endom. has values in $\mathcal{A}$. In effect one restricts the kernel to the diagonal and thereby obtains at each x an endomorphism of $(S\otimes E_0)_x\otimes\mathcal{A}$ commuting with right multiplication by $\mathcal{A}$.

There are two ways in which my setup is more complicated than Getzler's. First of all, there is the $\mathcal{A}$-trace. Secondly my Laplacean operator

$$h^2 L^2 + h\, dy^a [D_a, L] + \tfrac{1}{2} dy^a dy^b F_{ab}$$

$$= h^2\Big\{\big(D_\mu^2 + \Gamma^\alpha_{\mu\mu} D_\alpha\big) + \frac{R}{4} + \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}\Big\}$$

$$+ h\, dy^a \gamma^\mu F_{a\mu} + \frac{1}{2} dy^a dy^b F_{ab}$$

is more complicated in that the "potential"

$$h^2\frac{R}{4} + \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} + h\, dy^a\gamma^\mu F_{a\mu} + \frac{1}{2} dy^a dy^b F_{ab}$$

is not just homogeneous of degree 2 in h. What this means from a practical viewpoint is that I have to distinguish between t and h in the same way that [redacted] in ordinary QM for the Hamiltonian $\hat{H} = p^2 + \hat{V}$ one has to distinguish between β, h ~~so~~ as to get the correct correspondence:

$$\operatorname{tr}\left(e^{-\beta H}\right) = \frac{1}{h^n} \int_{T^*M} e^{-\beta(p^2+V)} \frac{d^n x\, d^n p}{(2\pi)^n} \left(1 + O(h)\right)$$

Wait: A careful reading of Getzler's paper shows that his t is always Planck's constant, i.e. he uses $t^2 D^2$ and shows

$$(\text{his } a(t)) \quad \sigma(t^2 D^2)_{t^{-1}} \longrightarrow -|\xi|^2 + \tfrac{1}{2}F$$

$$(\text{his } r_\lambda(t)) = \sigma\left(\frac{1}{\lambda + t^2 D^2}\right)_{t^{-1}} \longrightarrow \frac{1}{\lambda - |\xi|^2 + \frac{1}{2}F}$$

September 5, 1984

First we want to present a way to think of the Clifford algebra as a deformation of the exterior algebra. We use the Clifford multiplication on the exterior algebra: $\omega \in V$ acts on ΛV by

$$\omega * \alpha = (e_\omega + \iota_\omega)\alpha \qquad [\iota_\omega, e_{\omega_1}] = (\omega|\omega_1)$$

This extends to a left-module structure on ΛV over $C(V)$, and one has an isomorphism

$$C(V) \xrightarrow{\sim} \Lambda V \qquad \alpha \longmapsto \alpha \cdot 1$$

We can define another action of $C(V)$ on ΛV by

$$c'(\omega) = e_\omega - \iota_\omega$$

Then
$$[e_\omega + \iota_\omega, e_{\omega_1} - \iota_{\omega_1}] = (\omega|\omega_1) - (\omega_1|\omega) = 0.$$

If we convert this second action into a right action:

$$\alpha * \omega_1 = (-1)^{\deg\alpha}(e_{\omega_1} - \iota_{\omega_1})\alpha$$

then we have

$$\left\{\begin{array}{l} \omega * (\alpha * \omega_1) \\ -(\omega * \alpha) * \omega_1 \end{array}\right\} = \left\{\begin{array}{l} (-1)^{\deg\alpha}(e_\omega + \iota_\omega)(e_{\omega_1} - \iota_{\omega_1})\alpha \\ \quad -(-1)^{\deg(\omega * \alpha)}(e_{\omega_1} - \iota_{\omega_1})(e_\omega + \iota_\omega)\alpha \end{array}\right\} = 0$$

Summary: Given V with quadratic form $(\omega|\omega_1)$ one can define left and right actions of $C(V)$ on ΛV by the formulas above. Each action is the commutant of the other.

The [redacted] supertrace of the left action of $C(V)$ on $\Lambda(V)$ doesn't seem to be interesting.

Now let us consider the 1-parameter family of Clifford algebras with generators ψ^μ $\mu = 1, \ldots, n$ satisfying

$$[\psi^\mu, \psi^\nu] = h^2 \, 2\delta^{\mu\nu}$$

where h is the parameter, say $h \in \mathbb{R}$. We can think of having a bundle [redacted] of algebras, which are finite-diml, over the real line. The [redacted] sections of this algebra bundle is the superalgebra over the smooth functions on $\mathbb{R}$ with odd generators ψ^μ and the above relations.

Conforming to the above description of $C(V)$ as operators on ΛV, we can [redacted] identify the Clifford algebra corresponding to h with the operator algebra on $\Lambda[v_1^0, \ldots, v_n^0]$ generated by

$$\psi^\mu = e_{v_\mu} + h^2 \iota_{v_\mu}$$

I want to concentrate on the idea that the algebras for different h are all operating in the same space. This will enable me pin down elements of this algebra in some sense.

For $h \neq 0$ one sees that $[C, C] = F_{n-1} C$ so there is a unique possible supertrace up to a scalar. For n even one has the supertrace on the spinor module which satisfies

$$tr_s(\psi^{\mu_1} \ldots \psi^{\mu_k}) = 0 \qquad k < n$$

$$tr_s(\psi^1 \ldots \psi^n) = (2i)^{n/2} h^n$$

Here I am using the isomorphism (for $h \neq 0$)

$$C \longrightarrow C_n \qquad \psi^\mu \longmapsto h\gamma^\mu .$$

So for all h we can define a supertrace on C which I will denote

$$\int : C \longrightarrow \mathbb{C}$$

by

$$C \xrightarrow{\sim} \Lambda[v_1, \ldots, v_n] \xrightarrow{\text{proj.}} \Lambda^n[v_1, \ldots, v_n] \xrightarrow{\sim} \mathbb{C}$$

$$v_1 \cdots v_n \longleftrightarrow 1 .$$

Thus

$$\int \psi^{\mu_1} \cdots \psi^{\mu_k} = 0 \qquad k < n$$

$$\int \psi^{\mu_1} \cdots \psi^{\mu_n} = \varepsilon^{\mu_1 \cdots \mu_n}$$

and we have the formula

$$\boxed{tr_s(\alpha) = h^n (2i)^{n/2} \int \alpha}$$

for $h \neq 0$.

The next step will be to pass to the Weyl algebra side, trying to follow the above Clifford situation as much as possible.

This time we start with a real vector space $V = \mathbb{R}^n$ having a (purely imaginary valued) skew-symmetric form $F_{\mu\nu}$ and we look at the covariant derivative operators for the line bundle with connection having this curvature. I know that we can find a connection $D_\mu = \partial_\mu + A_\mu$ in the trivial line bundle having curvature F such that A_μ depends

linearly on x, for example $A_\mu = -\frac{1}{2} F_{\mu\alpha} x^\alpha$.
$\left(\partial_\mu A_\nu - \partial_\nu A_\mu = -\frac{1}{2} F_{\nu\mu} + \frac{1}{2} F_{\mu\nu} = F_{\mu\nu} \right)$. Then

$$[D_\mu, D_\nu] = F_{\mu\nu}.$$

The Weyl algebra is the [redacted] algebra generated by the operators D_μ with the above relations. It is a twisted polynomial algebra and operates on the smooth functions on $\mathbb{R}^n = V$.

Before [redacted] I get involved with computations I really ought to understand what the goal is. I ultimately want to consider the family of twisted polynomial algebras depending on $h \in \mathbb{R}$ described by the CCR

$$[p_\mu, x^\nu] = \frac{h}{i} \delta_\mu^\nu \qquad [p_\mu, p_\nu] = 0 \qquad [x^\mu, x^\nu] = 0.$$

There is no trace defined on this algebra the way there is in the Clifford algebra case. Instead one must enlarge the twisted polynomial algebra to an algebra of "functions" $f(x,p)$, and then find inside [redacted] a suitable family of "trace class" operators. These things should be defined for all values of h including zero. On the trace class elements should be an integral, and one should have for $h \neq 0$ a formula of the form

$$\mathrm{tr} = \frac{1}{h^n} \int$$

[redacted] It appears to be a bad idea to think in terms of covariant derivatives with respect to a connection as the generators on the twisted polynomial algebra, since as $h \to 0$ we want the limiting algebra to be a polynomial[redacted] algebra.

One problem consists of the following. In the case of the Clifford algebra, we can define $C(V)$ as the algebra with certain generators and relations. Here, if we take the algebra generated by the p_μ[redacted], x^μ[redacted] we get a twisted polynomial algebra, and the trace isn't defined on this algebra.

An alternative might be to take the algebra of smooth, or perhaps Schwartz, linear combinations of elts. of the many-parameter group $e^{i(a_\mu p_\mu + b_\mu x^\mu)}$ [redacted]

September 6, 1984

Consider the twisted polynomial algebra with generators p_μ satisfying $[p_\mu, p_\nu] = R_{\mu\nu}$, [redacted] where for the moment we suppose $R_{\mu\nu}$ is a skew matrix of scalars. Let T be the real vector space spanned by the p_μ, which we think of as linear functions on T^*.

[redacted] Now I have run up against the standard problem, namely, how to speak of the smooth Weyl algebra in the same [redacted] generator and relation language that one uses with the Weyl and Clifford algebras. Presumably, Irving Segal has gotten this straight.

We can [redacted] maybe look at the algebra of smooth linear combinations of [redacted] operators

$$e^{iap} \qquad a = a^\mu \in T$$

subject to the relations

$$e^{iap} \cdot e^{ibp} = e^{i(a+b)p}\, e^{-\frac{1}{2}aRb}$$

September 7, 1984:

Let's consider the twisted polynomial algebra with generators p_μ and the relations

$$[p_\mu, p_\nu] = R_{\mu\nu}$$

where R is a skew-symmetric matrix, say purely imaginary to begin with. Following the Weyl idea, one [redacted] wants [redacted] the integrated form of these relations. One wants an algebra which is in some sense generated by exponentials

$$e^{ipa} \qquad pa = p_\mu a^\mu \qquad a = (a^\mu) \in \mathbb{R}^n$$

with the [redacted] multiplication

$$e^{ipa} e^{ipb} = e^{ip(a+b)} e^{-\frac{1}{2} aRb}$$

(Note $[pa, pb] = [p_\mu a^\mu, p_\nu b^\nu] = a^\mu R_{\mu\nu} b^\nu$.) So the algebra we are after is a sort of a group algebra. [redacted] It will consist of

$$(*) \qquad \int d^n a \; f(a)\, e^{i(pa)}$$

and the product will amount to convolution of the functions f twisted by R. We will want to consider the algebra consisting of $(*)$ where $f(a) \in \mathcal{S}(\mathbb{R}^n)$, the smooth convolution algebra. The problem is now to construct the heat [redacted] operator:

$$e^{-tp^2} = \int d^n a \; f(t,a)\, e^{ipa}$$

within this algebra.

First determine what we can by using the fact that bracketing with p^2 preserves the space of $p_\mu a^\mu$: Set

$$e^{tp^2}(pa)e^{-tp^2} = pa_t$$

$$[p^2, pa_t] = p\dot{a}_t$$

$$[p_\mu^2, p_\nu a_t^\nu] = 2p_\mu R_{\mu\nu} a_t^\nu$$

$$\Rightarrow \quad \dot{a}_t = 2Ra_t \quad \Longrightarrow \quad \boxed{a_t = e^{2Rt} a}$$

Thus

$$e^{tp^2} e^{ipa} e^{-tp^2} = e^{ipa_t}$$

or

$$e^{ipa} e^{-tp^2} = e^{-tp^2} e^{ipa_t}$$

The left side is

$$\int d^n b\, f(t,b)\, e^{ipa} e^{ipb} = \int d^n b\, f(t,b)\, e^{ip(a+b)} e^{-\frac{1}{2}aRb}$$

$$= \int d^n b\, f(t, b-a)\, e^{ipb} e^{-\frac{1}{2}aR(b-a)}$$

$$= \int d^n b\, f(t, b-a) e^{-\frac{1}{2}aRb} e^{ipb}$$

The right side is

$$\int d^n f\, f(t,b)\, e^{ip(b+a_t)} e^{-\frac{1}{2}bRa_t}$$

$$= \int d^n f\, f(t, b-a_t) e^{-\frac{1}{2}(b-a_t)Ra_t} e^{ipb}$$

so we find

$$f(t, b-a)\, e^{-\frac{1}{2} a R b} = f(t, b-a_t)\, e^{\frac{1}{2} a_t R b}$$

$$f(t, a_t - a) = f(t, 0)\, e^{\frac{1}{2} \underbrace{a R a_t}_{a R (a_t - a)}}$$

Let $a_t = u a$, so that $u = e^{2Rt}$, and suppose R generic so that $u-1$ is invertible. Then

$$f(t, a) = f(t, 0)\, e^{\frac{1}{2}[(u-1)^{-1} a] R a}$$

and

$$\frac{1}{2}\left(\frac{1}{u-1} a\right) R a = -\frac{1}{2} a R \frac{1}{u-1} a$$

$$= -\frac{1}{2} a R \cdot \left(\frac{1}{u-1} + \frac{1}{2}\right) a$$

$$= -\frac{1}{4} a R \left(\frac{u+1}{u-1}\right) a$$

so that we have

$$\boxed{f(t, a) = f(t, 0) \cdot e^{-\frac{1}{4} a \left(R \frac{e^{tR} + e^{-tR}}{e^{tR} - e^{-tR}}\right) a}}$$

Notice that as $R \to 0$, the Gaussian factor becomes

$$e^{-\frac{a^2}{4t}}$$

We are now going to try to find the heat type equation satisfied by $f(t, a)$, where

$$e^{-tp^2} = \int d^n a \; f(t, a)\, e^{ipa}$$

For this I need to know have to compute

$$p^2 e^{ipa}$$

as differential operator in a applied e^{ipa}. First we compute $\partial_\mu e^{ipa}$ where $\partial_\mu = \partial/\partial a^\mu$.

$$e^{ip(a+\delta a)} = e^{ip\delta a} \; e^{ipa} \; e^{\frac{1}{2}\delta a R a}$$

$$= (1 + \delta a \cdot ip)(1 + \delta a \tfrac{1}{2} R a)\, e^{ipa}$$

$$= \left(1 + \delta a (ip + \tfrac{1}{2} R a)\right) e^{ipa}$$

$$\therefore \quad \partial_\mu e^{ipa} = \left(ip_\mu + \tfrac{1}{2} R_{\mu\nu} a^\nu\right) e^{ipa}$$

or

$$\left(\partial_\mu - \tfrac{1}{2} R_{\mu\nu} a^\nu\right) e^{ipa} = (ip_\mu) e^{ipa}$$

Thus

$$-p_\mu^2 e^{ipa} = \left(\partial_\mu - \tfrac{1}{2} R_{\mu\nu} a^\nu\right)^2 e^{ipa}.$$

so

$$\int d^n a \; \partial_t f(t,a)\, e^{ipa} = -p^2 \int d^n a \; f(t,a)\, e^{ipa}$$

$$= \int d^n a \; f(t,a) \left(\partial_\mu - \tfrac{1}{2} R_{\mu\nu} a^\nu\right)^2 e^{ipa}$$

$$= \int d^n a \left\{ \left(\partial_\mu + \tfrac{1}{2} R_{\mu\nu} a^\nu\right)^2 f(t,a) \right\} e^{ipa}$$

and so we want

$$\left[\partial_t - \left(\partial_\mu + \tfrac{1}{2} R_{\mu\nu} a^\nu\right)^2\right] f(t,a) = 0$$

Let's look for a solution in the form

$$f(t,a) = m_t \, e^{-\frac{1}{4} a Q_t a}$$

where Q_t is a symm. matrix and m_t is a positive function of t. Then we want

$$\left\{-\frac{1}{4} a\dot{Q}a + \frac{\dot{m}}{m} - \left(\partial_\mu - \frac{1}{2}Q_{\mu\nu}a^\nu + \frac{1}{2}R_{\mu\nu}a^\nu\right)^2\right\}1 = 0$$

Just looking at the purely quadratic terms gives

$$a\dot{Q}a + |(-Q+R)a|^2 = 0$$

Previous calculation shows Q is to be a function of R^2, whence

$$|(-Q+R)a|^2 = |Qa|^2 + |Ra|^2 - (Qa, Ra) - (Ra, Qa)$$

$$(Qa, Ra) = (a, QRa) = (a, RQa)$$

$$= |Qa|^2 + |Ra|^2$$

$$= a(Q^2 - R^2)a$$

(Actually one should do this as follows

$$|(-Q+R)a|^2 = a^t \underbrace{(-Q-R)(-Q+R)}_{Q^2 - QR + RQ - R^2} a \quad .)$$

Thus

$$\dot{Q} + Q^2 - R^2 = 0 .$$

This is the Riccati equation associated to

$$\ddot{y} - R^2 y = 0$$

so it has solutions $y = R\dfrac{c_1 e^{tR} - c_2 e^{-tR}}{c_1 e^{tR} + c_2 e^{-tR}}$, in particular the solution

$$R\,\frac{e^{tR} + e^{-tR}}{e^{tR} - e^{-tR}} \sim \frac{1}{t} \quad \text{as } t \to 0$$

which we found earlier.

Now go back to

$$\left\{ \frac{\dot m}{m} - \frac{1}{4} a^t \dot Q a - \left(\partial + \tfrac{1}{2}(R-Q)a\right)^2 \right\} 1 = 0$$

which becomes two equations

$$\dot Q + Q^2 - R^2 = 0$$

$$\frac{\dot m}{m} - \frac{1}{2}(R-Q)_{\mu\mu} = 0 \qquad \text{or} \qquad \frac{\dot m}{m} = -\frac{1}{2} \operatorname{tr} Q$$

$$\frac{\dot m}{m} = -\frac{1}{2} \operatorname{tr} \frac{\dot y}{y} \qquad\qquad y = e^{tR} - e^{-tR}$$

$$= -\frac{1}{2} \frac{d}{dt} \log \det y$$

$$\therefore \; m = \text{const.} \left[\det\left(e^{tR} - e^{-tR}\right)\right]^{-1/2}$$

In order to evaluate the constant we use that we want

$$m_t \sim \frac{1}{(4\pi t)^{n/2}} \qquad \text{as} \quad t \to 0$$

because

$$e^{-tp^2} = \int d^n a \; m_t \; e^{-\frac{1}{4} a^t Q_t a} \; e^{ipa}$$

is supposed to approach the identity as $t \to 0$, and so we want

$$m_t \, e^{-\frac{1}{4} a^t Q_t a} \sim \frac{1}{(4\pi t)^{n/2}} e^{-\frac{a^2}{4t}}$$

Now
$$\frac{1}{(4\pi)^{n/2}} \left(\det \frac{e^{tR} - e^{-tR}}{2R}\right)^{-1/2} = \frac{1}{(4\pi t)^{n/2}} \Bigg(\underbrace{\det \frac{e^{tR} - e^{-tR}}{2tR}}_{\to 1}\Bigg)^{-1/2}$$

and so we have

$$m_t = \frac{1}{(4\pi)^{n/2}} \left[\det \left(\frac{e^{tR} - e^{-tR}}{2R} \right) \right]^{-1/2}$$

Final[redacted] formula is that if $[p_\mu, p_\nu] = R_{\mu\nu}$ then

$$\boxed{e^{-tp^2} = \int d^n a \, \frac{1}{(4\pi t)^{n/2}} \det\left(\frac{\sinh tR}{tR}\right)^{-1/2} e^{-\frac{1}{4} a R \frac{\cosh tR}{\sinh tR} a} \, e^{ipa}}$$

Next I want the trace for this algebra. [redacted] What I expect in this situation is to have an "integral" defined by

$$\int d^n a \, f(a) e^{ipa} \longmapsto \text{const } f(0)$$

and [redacted] then a comparison of this "integral" with the trace on the metaplectic representation when R is non-degenerate.

Suppose $n = 2$ and that $[p_1, p_2] = R_{12}$ is $\neq 0$. R_{12} ″ $i\omega$
The classical motion with $H = p^2$ is described by

$$a_t = e^{2tR} a$$

so that the [redacted] angular frequency is 2ω, and the eigenvalues of p^2 are $(n + \frac{1}{2}) 2\omega$ for $n \geq 0$. We can check this another way:

$$p_\mu^2 = p_1^2 + p_2^2 \qquad [p_1, p_2] = -i\omega$$

$$= \underbrace{(p_1 + ip_2)}_{c^*} \underbrace{(p_1 - ip_2)}_{c} + \underbrace{i[p_1, p_2]}_{\omega}$$

$$[c, c^*] = [p_1 - ip_2, p_1 + ip_2]$$
$$= 2i[p_1, p_2] = 2i(-i\omega) = 2\omega$$

so the ground eigenvalue is ω and the difference between eigenvalues is 2ω. Thus

$$\operatorname{tr} e^{-tp^2} = \sum_{n \geq 0} e^{-t(2n+1)\omega} = \frac{e^{-t\omega}}{1-e^{-2t\omega}} = \frac{1}{e^{t\omega} - e^{-t\omega}}$$

Now $R = \begin{pmatrix} 0 & -i\omega \\ i\omega & 0 \end{pmatrix}$ has eigenvalues $\pm\omega$ so

$$\det\left(\frac{\sinh tR}{tR}\right)^{-1/2} = \frac{t\omega}{\sinh t\omega} = \frac{2t\omega}{e^{t\omega} - e^{-t\omega}}$$

We want a formula of the form

$$\operatorname{tr}\left\{\int d^n a \, f(a) e^{ipa}\right\} = C \cdot f(0)$$

(with "const" written under the C)

In our example this gives

$$\frac{1}{e^{t\omega} - e^{-t\omega}} = C \cdot \frac{1}{4\pi t} \frac{2t\omega}{e^{t\omega} - e^{-t\omega}}$$

$$\therefore C = \frac{2\pi}{\omega}$$

So the general formula must be something like

$$\boxed{\operatorname{tr}\left\{\int d^n a \, f(a) e^{ipa}\right\} = \frac{(2\pi)^{n/2}}{\det\left(\frac{1}{i}R\right)^{1/2}} \cdot f(0)}$$

Digression: We are considering the twisted polynomial ring generated by variables p_μ ∋ $[p_\mu, p_\nu] = R_{\mu\nu}$. The corresponding smooth algebra consists of group-ring-like elements

$$\int d^n a \;\; f(a)\, e^{ipa}$$

where $f \in \mathcal{S}(T)$, T = real linear combinations of the p_μ.

Getzler prefers to realize this convolution algebra as operating on the functions on T^* by the assignment

$$p_\mu = \xi_\mu + \tfrac{1}{2} R_{\mu\nu}\partial_\nu \qquad\qquad \partial_\nu = \partial/\partial\xi_\nu$$

(This is the exact analogy of the Clifford alg. formula

$$\gamma^\mu = e_{\omega^\mu} + i_{\omega^\mu}\,.)$$

Notice that under this representation

$$e^{i a^\mu p_\mu} = e^{ia\xi}\; e^{a^\mu \frac{i}{2} R_{\mu\nu}\partial_\nu}$$

so that

$$\left(\int d^n a \;\; f(a)\, e^{ipa}\right) 1 = \int d^n a \; f(a)\, e^{ia\xi}$$

is just the Fourier transform of f.

The heat operator e^{-tp^2} is such that if we put $g(t,\xi) = e^{-tp^2}1$, then

$$\left\{ \partial_t + \left(\xi_\mu + \tfrac{1}{2} R_{\mu\nu}\partial_\nu\right)^2 \right\} g = 0$$

We can solve this by

$$g = m_t\, e^{-\xi Q_t \xi}$$

and we get

$$\frac{\mathring{m}}{m} + \left(\xi + \tfrac{1}{2}R[\partial - 2Q\xi]\right)^2 - \xi\mathring{Q}\xi = 0$$

$$\frac{\mathring{m}}{m} - \xi\mathring{Q}\xi + \left((1-RQ)\xi + \tfrac{1}{2}R\partial\right)^2 = 0$$

$$-\xi\mathring{Q}\xi + \xi(1+QR)(1-RQ)\xi = 0 \qquad \text{assume } [Q,R]=0$$

or $$-\mathring{Q} + 1 - Q^2R^2 = 0$$

But this is satisfied by

$$Q = \frac{y}{y'} \quad \text{where} \quad y'' - R^2 y = 0$$

since $$\mathring{Q} = 1 - \frac{yy''}{(y')^2} = 1 - \underbrace{\left(\frac{y}{y'}\right)^2}_{Q^2} R^2 , \quad \text{etc.}$$

Thus we should be able to derive the formula for the heat operator in this way.

Now we must turn to the super situation where there is a good trace even though R is degenerate.

Review: The problem is to evaluate $\mathrm{tr}_s\, e^{h^2 \not{D}^2}$ as $h \to 0$, and we want to explain Getzler's machine. Steps: Filter the diffl operators on $S \otimes E$ and identify the associated graded with sections of $S(T) \otimes \Lambda(T^*) \otimes (\mathrm{End}\, E)$ with twisted product structure, twisted by the curvature viewed as a skew-form with values in $\Lambda^2(T^*)$. The image of $\not{D}^2$ in the associated graded algebra is $-\rho^2 + F$.

Construct e^{-p^2+F} in the convolution algebra associated to this twisted polynomial ring. Then we must identify a trace on this convolution algebra which I will denote by $\int$. Then state Getzler's thm. that

$$\lim_{h\to 0} \operatorname{tr}_s\left(e^{h^2 D\!\!\!/^2}\right) = \left(\int e^{-p^2} [redacted] \operatorname{tr}_E(e^F)\right)$$

At the moment [redacted] I don't understand the [redacted] appropriate trace on the convolution algebra.

~~Let's consider the obvious possibility which is to take the heat operator $e^{t(-p^2+F)}$ which we have seen lies in $S(T_x)\otimes\Lambda T_x^*\otimes \operatorname{End} E_x$ and send it to the right~~

Obvious possibility. Let

$$\int d^na\; f(a)\, e^{ipa} \in S(T_x)\otimes \Lambda T_x^*$$

be an element of our convolution algebra at the point x. Here $f: T_x \longrightarrow \Lambda T_x^*$ is a Schwartz function. The obvious thing to do is to take

$$(2i)^{n/2}\left[f(0)\right]_{(n)} \in \Lambda^n T_x^*$$

Let's see how this works for the index thm. We have

$$e^{-tp^2} = \int d^na \underbrace{\frac{1}{(4\pi)^{n/2}} \det\left(\frac{R}{\sinh tR}\right)^{1/2} e^{-\frac{1}{4}a\,\cdots\, a}}_{f(a)} e^{ipa}$$

So what we do is to take

$$(2i)^{n/2}\left[f(0)\right]_{(n)} = (2i)^{n/2}\left[\frac{1}{(4\pi t)^{n/2}}\left(\det \frac{tR}{\sinh tR}\right)^{1/2}\right]_{(n)}$$

where I still must say what R is.

$$-[p_\mu, p_\nu] = R_{\mu\nu k l} \tfrac{1}{4}\omega^k \omega^l .$$

What is the $\hat{A}$ genus? We take the curvature of the tangent bundle which is the two form

$$\tfrac{1}{2}\omega^\mu \omega^\nu R_{\mu\nu k}{}^{l}$$

with values in Lie $SO(n)$ = skew-symm. matrices. Then we substitute the curvature in the power series

$$\hat{a}(x) = \frac{x/2}{\sinh x/2} = \frac{x}{e^{x/2}-e^{-x/2}} = e^{-\frac{x}{2}}\frac{x}{1-e^{-x}}$$

and take the $(\text{determinant})^{1/2}$. Thus

$$\hat{A}(M) = \det{}^{1/2}\left(\hat{a}\left(\tfrac{1}{4}\omega^\mu\omega^\nu \underbrace{R_{\mu\nu k}{}^{l}}_{R_{kl\mu}{}^{\nu}}\right)\right)$$

$$= \det{}^{1/2}\left(\hat{a}(-[p_\mu, p_\nu])\right)$$

and so it really does work.

Thus we conclude that the appropriate trace on the convolution algebra $S(T)\otimes \Lambda T^*$ is

$$\int d^n a\, f(a)\, e^{ipa} \longmapsto (2i)^{n/2}\left[f(0)\right]_{(n)}$$

As another example let us consider the

case of a flat metric but non-trivial gauge field.

$$e^{t(-p^2+F)} = \int d^n a \frac{1}{(4\pi t)^{n/2}} e^{-\frac{a^2}{4t}} e^{tF}$$

and

$$\text{trace} = (2i)^{n/2} \frac{1}{(4\pi t)^{n/2}} \operatorname{tr}_E\left(e^{tF}\right)_{[n]}$$

$$= \left(\frac{2i}{4\pi}\right)^{n/2} \operatorname{tr}_E \frac{F^{n/2}}{(n/2)!}$$

Everything should now work, but to make it really convincing you should setup the tangent groupoid

September 9, 1984

Standard notation of tensor calculus. Let e_i be a frame and ω^i the dual coframe. It appears from Weinberg's book that the covariant derivative of a contravariant vector v^i, i.e. vector field $v^i e_i$ is defined by

$$(*) \qquad v^i{}_{;j} = \underbrace{\partial_j}\, v^i + \Gamma^i_{jk}\, v^k$$

stands for the vector field e_j acting on the function v^i.

I want to think of the connection on T as

$$D = \omega^i D_i \;:\; \Gamma(T) \longrightarrow \Gamma(T^* \otimes T)$$

where D_i is an operator on $\Gamma(T)$. One has

~~$D_i(v^j e_j) = (\partial_i v^j) e_j + v^j D_i e_j$~~

$$D_j(v^i e_i) = (\partial_j v^i) e_i + v^k D_j e_k$$

$$= \left(\partial_j v^i + v^k \langle D_j e_k, \omega^i \rangle\right) e_i$$

Comparing with $(*)$ we see that

$$\boxed{\langle D_j e_k, \omega^i \rangle = \Gamma^i_{jk}}$$

i.e.

$$\boxed{\nabla_{e_j}(e_k) = \Gamma^i_{jk} \cdot e_i}$$

The next point that I want emphasized is that the connection form is the matrix 1-form

$$\theta^i_k = \omega^j \Gamma_j{}^i{}_k \qquad \begin{array}{ll} i & \text{row index} \\ k & \text{column index} \end{array}$$

since

$$D(v^i) = (dv^i + \theta^i_k v^k).$$

In particular the curvature is the matrix of 2 forms

$$R^i_k = d\theta^i_k + \theta^i_j \theta^j_k$$

or

$$R_{\mu\nu}{}^i{}_k = \tfrac{1}{2}\omega^\mu\omega^\nu\Big[-C^\alpha_{\mu\nu}\Gamma_\alpha{}^i{}_k - \partial_\nu\Gamma_\mu{}^i{}_k + \partial_\mu\Gamma_\nu{}^i{}_k + \Gamma_\mu{}^i{}_j\Gamma_\nu{}^j{}_k - \Gamma_\nu{}^i{}_j\Gamma_\mu{}^j{}_k\Big]$$

where

$$d\omega^\alpha = -\tfrac{1}{2}\omega^\mu\omega^\nu C^\alpha_{\mu\nu}$$

$$\text{i.e.}\quad [e_\mu, e_\nu] = C^\alpha_{\mu\nu} e_\alpha$$

Actually this is the negative of what one sees in the tensor calculus formulas since curvature $R_{\mu\nu}$ is defined by

$$v^a{}_{;\mu;\nu} - v^a{}_{;\nu;\mu} = R_{\mu\nu}{}^a{}_b v^b$$

which means the order of covariant differentiation is reversed.

For my purposes it is enough to have a consistent set of formulas. I will therefore want to change sign:

$$D_\mu \omega^\alpha = -\Gamma^\alpha_{\mu\nu}\omega^\nu$$

whence

$$[D_\mu, \gamma^\alpha] = -\Gamma^\alpha_{\mu\nu}\gamma^\nu$$

Question: If $T \simeq T^*$ by the metric why is there a change in sign?

Probably one has to keep track of the row + column indices; one must think of Γ as a matrix:

$$D_\mu (a^i e_i) = (\partial_\mu a^i + \Gamma_{\mu}{}^{i}{}_{j}\, a^j)\, e_i$$

$$D_\mu (b_i \omega^i) = (\partial_\mu b_i)\, \omega^i - b_j \Gamma_{\mu}{}^{j}{}_{k}\, \omega^k$$

$$= (\partial_\mu b_i - \Gamma_{\mu}{}^{j}{}_{i}\, b_j)\, \omega^i$$

In the case of an orthonormal frame, these formulas are the same under the identification $a^i = b_i$, since $\Gamma_{\mu}{}^{i}{}_{j} = -\Gamma_{\mu}{}^{j}{}_{i}$

$$\begin{array}{c} \partial_\mu a^i + \Gamma_{\mu}{}^{i}{}_{j}\, a^j \\ \parallel \qquad\qquad \parallel \\ \partial_\mu b_i - \Gamma_{\mu}{}^{j}{}_{i}\, b_j \end{array}$$

So there's a single formula for covariant diffn.

$$D_\mu (a_i e_i) = (\partial_\mu a_i + \Gamma_{\mu,ij}\, a_j)\, e_i$$

Then the curvature is

$$\left([D_\mu, D_\nu] - D_{[e_\mu, e_\nu]}\right)(a_i e_i) = \Big(\big(\partial_\mu \Gamma_{\nu,ij} - \partial_\nu \Gamma_{\mu,ij} - c^\alpha_{\mu\nu} \Gamma_{\alpha,ij}$$
$$+ \Gamma_{\mu,ik}\Gamma_{\nu,kj} - \Gamma_{\nu,ik}\Gamma_{\mu,kj}\big)\, a_j\Big)\, e_i$$

Now I should be in a position to describe the Riemannian geometry. We start with an orthonormal frame X_i of sections of T, and let ω^i be the dual frame of sections of T^*. Then

$$d\omega^i = -\tfrac{1}{2}\,\omega^j\omega^k\, c^i_{jk} \qquad \text{where}$$

$$[X_j, X_k] = c^i_{jk} X_i .$$

Let the Levi Civita connection be given by

$$D_\mu(f_i\omega^i) = \omega^i\left(X_\mu f_i + \Gamma_{\mu,ij} f_j\right)$$

that is, relative to the trivialization defined by the frame ω^i we have

$$D = d + \theta \qquad \theta_{ij} = \omega^\mu \Gamma_{\mu,ij} .$$

It would be better to start with the LC connection on the tangent bundle

$$D_\mu(f_i X_i) = \left(X_\mu f_i + \Gamma_{\mu,ij} f_j\right) X_i$$

so that

$$\Gamma_{\mu,ij} = \langle \omega^i | D_\mu(X_j) \rangle .$$

Torsion-zero amounts to

$$D_\mu(X_j) - D_j(X_\mu) = [X_\mu, X_j]$$

or

$$\Gamma_{\mu,ij} - \Gamma_{j,i\mu} = c^i_{\mu j}$$

and preserving the metric to $\Gamma_{\mu,ij} + \Gamma_{\mu,ji} = 0$. The

LC connection is the unique one with these properties and is given by

$$\Gamma_{\mu,ij} = \frac{1}{2}\left(c^i_{\mu j} - c^j_{\mu i} - c^\mu_{ji}\right)$$

(Alternative version:

$$\Gamma_{\mu,ij} = \langle X_i , D_\mu(\omega^j) \rangle$$

so that $D_\mu \omega^j = \omega^i \Gamma_{\mu,ij}$. Torsion zero

means $d\omega^j = \omega^\mu D_\mu \omega^j = \omega^\mu \omega^i \Gamma_{\mu,ij}$

$$= \frac{1}{2}\omega^\mu \omega^i \left(\Gamma_{\mu,ij} - \Gamma_{i,\mu j}\right)$$

or that

$$\Gamma_{\mu,ij} - \Gamma_{i,\mu j} = -c^j_{\mu i}$$

or

$$\Gamma_{\mu,ji} - \Gamma_{ij\mu} = c^j_{\mu i}$$

which is the same as the above.)

Next consider the curvature

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

$$R(X_\mu, X_\nu) = [D_\mu, D_\nu] - c^i_{\mu\nu} D_i$$

and the curvature matrix

$$R_{\mu\nu,ij} = \langle \omega^i , R(X_\mu, X_\nu) X_j \rangle$$

[illegible]

$$R(X_\mu, X_\nu) = [X_\mu + \Gamma_\mu, X_\nu + \Gamma_\nu] - c^\alpha_{\mu\nu}(X_\alpha + \Gamma_\alpha)$$

$$= X_\mu \Gamma_\nu - X_\nu \Gamma_\mu - c^\alpha_{\mu\nu} \Gamma_\alpha + [\Gamma_\mu, \Gamma_\nu]$$

whence

$$R_{\mu\nu, ij} = X_\mu \Gamma_{\nu, ij} - X_\nu \Gamma_{\mu, ij} - c^\alpha_{\mu\nu} \Gamma_{\alpha, ij} + \Gamma_{\mu, ik} \Gamma_{\nu, kj} - \Gamma_{\nu, ik} \Gamma_{\mu, kj}$$

Next we consider the spinor bundle, which because of the frame X_i can be identified with the trivial bundle having fibre the vector space of spinors S_n for the n diml Clifford algebra C_n. The spinor bundle is associated to the spinor representation, which on the Lie alg. level is

$$(a_{ij}) \longmapsto a_{ij} \tfrac{1}{4} \gamma^i \gamma^j$$

Thanks to the trivialization of T given by (X_i) sections of the spinor bundle can be local identified with functions with values in the standard spinors S_n. The connection on S is then

$$D^S_\mu = X_\mu + \Gamma_{\mu, ij} \tfrac{1}{4} \gamma^i \gamma^j$$

and the curvature is

$$R^S_{\mu\nu} = R_{\mu\nu, ij} \tfrac{1}{4} \gamma^i \gamma^j$$

September 17, 1984

Singer's suggestion – the form on $\mathcal{L}M$

$$\operatorname{tr} T\left\{e^{\int\left(-\dot{x}^\mu A_\mu(x) + \frac{1}{2}\psi^\mu\psi^\nu F_{\mu\nu}(x)\right)dt}\right\}$$

should be viewed as the trace of super parallel transport. Hence it is killed by $d - \iota_X$ which is the super time translation operator.

Recall the setting I worked with. One has a manifold M with circle action and a bundle with connection D over it. One $\Omega^*(M, E)$ we have the operator $D - \iota_X$, more generally $D + \lambda\iota_X$, with square

$$(D + \lambda\iota_X)^2 = \lambda D_X + D^2$$

Then $e^{t(\lambda D_X + D^2)}$ is a parallel transport operator in the sense that it covers, or is compatible with, $e^{t\lambda\mathcal{L}_X}$ on forms. So if $t\lambda$ is a period, then it is in $\Omega^*(M, \operatorname{End} E)$ and one can form

$$\operatorname{tr} e^{t(\lambda D_X + D^2)} \qquad t\lambda \text{ period}$$

This form is killed by $d + \lambda\iota_X$:

$$(d + \lambda\iota_X) \operatorname{tr} e^{t(\lambda D_X + D^2)} = \operatorname{tr}\left[D + \lambda\iota_X, e^{t(\lambda D_X + D^2)}\right] = 0$$

My idea yesterday is that just as one forms the super heat operator

$$e^{\theta \not{D} + t\not{D}^2}$$

one can form the super parallel transport operator

$$e^{\theta(D + \lambda\iota_X) + t(\lambda D_X + D^2)}$$

Hence Singer's suggestion makes sense.

September 18, 1984

Before the Berkeley trip I worked out the following formulas pertaining to Getzler's proof of the index theorem.

Outline:

1. Connection and curvature on T
2. Connection and curvature on S = spinor bundle
3. Dirac operator $\not{D}$ and $\not{D}^2$
4. [redacted] $\mathrm{Diff}(S \otimes E)$ and Getzler's filtration
5. Weyl algebra and computation of e^{-tp^2} in the convolution algebra associated to the Weyl algebra.
6. Index formula.

The Riemannian structure on M will be described locally using an orthonormal frame $X_i \in \Gamma(T)$, $i = 1, \ldots, n$:

$$\langle X_i | X_j \rangle = \delta_{ij}$$

and the dual frame $\omega_i \in \Gamma(T^*)$:

$$\iota_{X_j}(\omega^i) = \delta_{ij}$$

A connection on the tangent bundle

$$D : \Gamma(T) \longrightarrow \Gamma(T^* \otimes T)$$

is of the form $D = \omega^\mu D_\mu$, where $D_\mu = \iota_{X_\mu} D$ is the covariant derivative operator in the direction X_μ. Let

$$\Gamma_{\mu ij} = \langle X_i | D_\mu X_j \rangle$$

Relative to the trivialization of T given by the X_i we have

$$D_\mu = X_\mu + \Gamma_\mu \qquad \text{i.e.}$$

$$D_\mu(f_i X_i) = (X_\mu f_i + \Gamma_{\mu ij} f_j) X_i$$

The <u>Levi-Civita connection</u> is the one preserving the metric:

1) $$\Gamma_{\mu ij} = -\Gamma_{\mu ji}$$

and having torsion-zero:

$$D_\mu X_\nu - D_\nu X_\mu = [X_\mu, X_\nu] \qquad \text{i.e.}$$

2) $$\Gamma_{\mu i \nu} - \Gamma_{\nu i \mu} = c^i_{\mu\nu}$$

where the c's are determined by the equivalent formulas

$$[X_\mu, X_\nu] = c^i_{\mu\nu} X_i$$

$$d\omega^i = -\tfrac{1}{2}\omega^\mu \omega^\nu c^i_{\mu\nu}$$

The solution of 1) and 2) is

$$\Gamma_{\mu i \nu} = \tfrac{1}{2}\left(c^i_{\mu\nu} - c^\nu_{\mu i} - c^\mu_{\nu i}\right)$$

The <u>curvature of T</u> is

$$D^2 = \omega^\mu D_\mu \, \omega^\nu D_\nu$$
$$= \tfrac{1}{2}\omega^\mu \omega^\nu \left([D_\mu, D_\nu] - c^i_{\mu\nu} D_i\right)$$
$$= \tfrac{1}{2}\omega^\mu \omega^\nu \, R(X_\mu, X_\nu)$$

where
$$R(X_\mu, X_\nu) = X_\mu \Gamma_\nu - X_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu] - c^i_{\mu\nu}\Gamma_i$$
is the skew symmetric matrix with components
$$R_{\mu\nu ij} = \langle X_i \,|\, R(X_\mu, X_\nu) X_j \rangle$$

The <u>Clifford algebra</u> C_n has generators γ^μ $\mu = 1, \dots, n$ which are anti-commuting involutions. It is $\mathbb{Z}_2$-graded [redacted] by requiring the γ^μ to be of odd degree. Then
$$[\gamma^\mu, \gamma^\nu] = 2\delta^{\mu\nu}$$
where the bracket is a graded (or super-) commutator. Supposing $n = 2m$, C_n has a unique (up to isomorphism) irreducible module S_n of dimension 2^m, called the <u>spinor space</u>. It is $\mathbb{Z}_2$-graded via the involution
$$\varepsilon = i^{-m}\gamma^1 \cdots \gamma^n$$
The super-trace of $\alpha \in C_n$ acting on S_n is
$$tr_s(\alpha) = tr(\varepsilon\alpha \text{ on } S_n)$$
and satisfies
$$tr_s(\gamma^{i_1} \cdots \gamma^{i_p}) = 0 \qquad p < n$$
$$tr_s(\gamma^{i_1} \cdots \gamma^{i_n}) = (2i)^m \, \varepsilon_{i_1 \dots i_n}$$ [redacted]

(Recall $S_n = S_2^{\otimes m}$, $C_n = (C_2)^{\hat{\otimes} m}$ and that things are normalized so that $S_2 = \mathbb{C}^2$ with
$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
the three Pauli matrices so that $\gamma^1\gamma^2 = i\varepsilon$.)

The Lie algebra Lie $SO(n)$ of skew-symmetric matrices embeds in C_n:

$$\begin{array}{lccc} & \mathrm{Lie}\ SO(n) & \hookrightarrow & C_n \\ (*) & a_{ij} & \longmapsto & a_{ij}\tfrac{1}{4}\gamma^i\gamma^j \end{array}$$

One has

$$\left[a_{ij}\tfrac{1}{4}\gamma^i\gamma^j \, , \, x_k\gamma^k \right] = (a_{ij}x_j)\gamma^i .$$

$\mathrm{Spin}(n)$ is the connected Lie subgroup of $C_n^\times$ with this Lie algebra. The infinitesimal <u>spin representation</u> is given by $(*)$.

The <u>spinor bundle</u> S on M is the vector bundle associated to the spin representation by the principal frame bundle. It is necessary to suppose a reduction of the structural group of the principal frame bundle from $O(n)$ to $\mathrm{Spin}(n)$; this is what one means by a spin structure. The LC connection in T induced one D^S in S.

Because T is trivialized, so is S, so sections of S can be identified with S_n-valued functions. The connection D^S is then given by applying the spin representation on the Lie algebra level

$$D^S_\mu = X_\mu + \Gamma_{\mu ij}\tfrac{1}{4}\gamma^i\gamma^j$$

and similarly the curvature of D^S is

$$F^S_{\mu\nu} = R_{\mu\nu ij}\tfrac{1}{4}\gamma^i\gamma^j$$

With respect to a local trivialization of the coefficient bundle E one can write its connection and curvature respectively

$$D_\mu^E = X_\mu + A_\mu$$

$$F_{\mu\nu}^E = X_\mu A_\nu - X_\nu A_\mu + [A_\mu, A_\nu] - c_{\mu\nu}^i A_i$$

The connection and curvature in $S \otimes E$ are

$$D_\mu^{S\otimes E} = X_\mu + \left(\Gamma_{\mu ij} \tfrac{1}{4} \gamma^i \gamma^j\right) \otimes 1 + 1 \otimes A_\mu$$

$$F_{\mu\nu}^{S\otimes E} = \left(R_{\mu\nu ij} \tfrac{1}{4} \gamma^i \gamma^j\right) \otimes 1 + 1 \otimes F_{\mu\nu}^{S\otimes E}.$$

We drop $\otimes 1$, $1 \otimes$ to simplify the notation.

The algebra $\mathrm{Diff}(S \otimes E)$ contains the algebra $\mathcal{E} = \Gamma(\mathrm{End}\, E)$ as a subalgebra and also the operators γ^μ, $D_\mu = D_\mu^{S\otimes E}$. These satisfy the <u>relations</u>

$$[\gamma^\mu, \mathcal{E}] = 0 \quad , \quad [D_\mu, \mathcal{E}] \subset \mathcal{E}$$

$$[\gamma^\mu, \gamma^\nu] = 2\delta_{\mu\nu} \qquad \text{(super commutator)}$$

~~$[D_\mu, \gamma^\nu] = \Gamma_{\mu\nu\lambda} \gamma^\lambda$~~

$$[D_\mu, \gamma^\nu] = \gamma^\lambda \Gamma_{\mu\lambda\nu}$$

$$[D_\mu, D_\nu] = c_{\mu\nu}^i D_i + F_{\mu\nu}^{S\otimes E}$$

$$= c_{\mu\nu}^i D_i + R_{\mu\nu ij} \tfrac{1}{4} \gamma^i \gamma^j + F_{\mu\nu}^E$$

The Dirac operator is $\not{D} = \gamma^\mu D_\mu$ and its square is computed to be

$$\not{D}^2 = \gamma^\mu D_\mu \gamma^\nu D_\nu$$

$$= \gamma^\mu \gamma^\nu D_\mu D_\nu + \gamma^\mu [D_\mu, \gamma^\alpha] D_\alpha$$

$$= \gamma^\mu \gamma^\nu \left(D_\mu D_\nu + \Gamma_{\mu\nu\alpha} D_\alpha \right)$$

$$= D_\mu^2 + \Gamma_{\mu\mu\alpha} D_\alpha + \tfrac{1}{2} \gamma^\mu \gamma^\nu \left([D_\mu, D_\nu] + \underbrace{(\Gamma_{\mu\nu\alpha} - \Gamma_{\nu\mu\alpha})}_{} D_\alpha \right)$$

$$-C^\alpha_{\mu\nu} = C^\alpha_{\nu\mu} = -\Gamma_{\mu\alpha\nu} + \Gamma_{\nu\alpha\mu}$$

$$= \left(D_\mu^2 + \Gamma_{\mu\mu\alpha} D_\alpha \right) + \tfrac{1}{2} \gamma^\mu \gamma^\nu F^{S\otimes E}_{\mu\nu}$$

$$= \left(\qquad \right) + \tfrac{1}{8} R_{\mu\nu ij} \gamma^\mu \gamma^\nu \gamma^i \gamma^j + \tfrac{1}{2} \gamma^\mu \gamma^\nu F^E_{\mu\nu}$$

In virtue of the curvature identities $R_{[ijk]l} = 0$ etc. the middle term is $-\frac{1}{4} R$, $R = R_{\mu\nu\mu\nu}$ = the scalar curvature.

$$\boxed{\not{D}^2 = \left(D_\mu^2 + \Gamma_{\mu\mu\alpha} D_\alpha \right) - \frac{R}{4} + \frac{1}{4} \gamma^\mu \gamma^\nu F^E_{\mu\nu}}$$

We know that $C_n = \mathrm{End}(S_n)$ and that C_n has the basis $\gamma^{i_1} \cdots \gamma^{i_p}$ where $1 \le i_1 < \cdots < i_p \le n$ describes the 2^n-subsets of $\{1, \ldots, n\}$. Hence every element of $\Gamma(\mathrm{End}(S \otimes E))$ can be written uniquely in the form

$$\sum_{\underline{i}} \varphi_{\underline{i}} \gamma^{i_1} \cdots \gamma^{i_p}$$

with $\varphi_{\underline{i}}$ in $\mathcal{E}$. It follows that every element of $\mathrm{Diff}(S\otimes E)$ is uniquely expressible as a finite sum of the form

$$\sum_{\underline{\iota},\underline{\mu}} \varphi_{\underline{\iota},\underline{\mu}}\, \gamma^{\iota_1}\cdots\gamma^{\iota_p} D_{\mu_1}\cdots D_{\mu_q}$$

where $\underline{\iota}$ runs over strictly increasing sequences and $\underline{\mu} = \{\mu_1 \leq \cdots \leq \mu_q\}$ runs over weakly increasing sequences in $\{1, 2, \ldots, n\}$.

Following Getzler define $F_k = F_k\, \mathrm{Diff}(S\otimes E)$ to be the subspace spanned by operators in the above form where the sum is taken over $\underline{\iota}, \underline{\mu}$ with $p+q \leq k$. To show this is an algebra filtration

$$F_k F_l \subset F_{k+l}$$

let us adjoin an indeterminate h to $\mathrm{Diff}(S\otimes E)$ and form the subring $\mathcal{F}$ of $\mathrm{Diff}(S\otimes E)\otimes\mathbb{C}[h]$ generated by $\mathcal{E}$, $h\gamma^\mu$, hD_μ. Since these generators are homogeneous we know $\mathcal{F}$ is a graded subalgebra

$$\mathcal{F} = \bigoplus_{k\geq 0} \mathcal{F}_n h^k \quad , \quad \mathcal{F}_k \subset \mathrm{Diff}(S\otimes E)$$

where $\mathcal{F}_k \mathcal{F}_l \subset \mathcal{F}_{k+l}$. Also the associated graded ring

$$\mathcal{F}/h\mathcal{F} \cong \bigoplus_{k\geq 0} \mathcal{F}_k/\mathcal{F}_{k-1}$$

is generated by $\mathcal{E}$, and the images of $h\gamma^\mu$, hD_μ in $gr_1 = \mathcal{F}_1/\mathcal{F}_0$. Clearly $F_k \subset \mathcal{F}_k$.

Structure of the graded ring: Set $\psi^\mu = \overline{h\gamma^\mu}$ and $ip_\mu = \overline{D_\mu}$ in gr_1. Then the associated graded ring is generated by $\mathcal{E}, \psi^\mu, \iota p_\mu$ and these satisfy the relations

$$\begin{cases} [\psi^\mu, \mathcal{E}] = [\iota p_\mu, \mathcal{E}] = 0 \\ [\psi^\mu_{\text{[redacted]}}, \psi^\nu] = 0 \qquad (\text{supercomm.}) \\ [\iota p_\mu, \psi^\nu] = 0 \\ [\iota p_\mu, \iota p_\nu] = \frac{1}{4} R_{\mu\nu ij} \psi^i \psi^j \end{cases}$$

from which one can see that any element of gr Diff$(S \otimes E)$ is a sum of terms

$$\text{[redacted]} \; \varphi_{\underline{i},\underline{\mu}} \;\; \psi^{i_1} \cdots \psi^{i_p} \;\; \iota p_{\mu_1} \cdots \iota p_{\mu_q} \; \text{[redacted]}$$

It follows that F_k maps onto $\mathcal{F}_k / \mathcal{F}_{k-1}$.
Then one sees by induction on k:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_{k-1} & \longrightarrow & F_k & \longrightarrow & F_k / F_{k-1} & \longrightarrow & 0 \\ & & \cap & & \cap & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_{k-1} & \longrightarrow & \mathcal{F}_k & \longrightarrow & \mathcal{F}_k / \mathcal{F}_{k-1} & \longrightarrow & 0 \end{array}$$

(isom. by induction hyp. — marking the left-hand $\cap$)

that $F_k = \mathcal{F}_k$ for all k.

It follows that gr Diff$(S \otimes E)$ is the algebra of sections of a bundle of algebras of the form

$$(\text{End } E) \otimes \mathcal{W}$$

where at a point x of M, $\mathcal{W}_x$ is the algebra [redacted] over the exterior algebra [redacted] $\Lambda[\psi^\mu]$ generated by elements [redacted] p_μ

subject to the relations

$$[p_\mu, \psi^\nu] = 0 \qquad \Longrightarrow \mathcal{W}_x \text{ is an alg. over } \Lambda[\psi^\mu]$$

and $+[p_\mu, p_\nu] = -\frac{1}{4} R_{\mu\nu ij}(x)\, \psi^i \psi^j$. Thus $\mathcal{W}_x$ is a twisted polynomial algebra over $\Lambda[\psi^\mu]$, or Weyl algebra, associated to the skew matrix

$$R'_{\mu\nu} = -\frac{1}{4} R_{\mu\nu ij}(x)\, \psi^i \psi^j$$

The image of $\not{D}^2$ in $\operatorname{gr} \operatorname{Diff}(S \otimes E)$ is

$$\overline{\not{D}^2} = -p_\mu^2 + \frac{1}{2} \psi^\mu \psi^\nu F^E_{\mu\nu}.$$

According to Getzler's analysis, we can evaluate the analytical index

$$\lim_{h \to 0} \operatorname{tr}_s e^{h^2 \not{D}^2}$$

by first computing $e^{\overline{\not{D}^2}} = e^{-p_\mu^2}\, e^{\frac{1}{2}\psi^\mu\psi^\nu F^E_{\mu\nu}}$ and then taking a suitable trace or integral.

The heat operator $e^{-tp_\mu^2}$ does not exist in the Weyl algebra, which consists of polynomials, ~~so we must introduce the~~ rather it belongs to a convolution algebra associated to the Weyl algebra.

Notation: $\mathcal{a} = \Lambda[\psi^\mu]$, $\mathcal{W}$ = twisted polynomial algebra generated by p_μ over $\mathcal{a}$ with relations

$$[p_\mu, p_\nu] = R'_{\mu\nu} \qquad R'_{\mu\nu} \text{ as above.}$$

(Precisely, $\mathcal{W}$ is a quotient of $\mathcal{a} \otimes$ (tensor algebra of $T = \{a_\mu p_\mu \mid a_\mu \in \mathbb{R}^n\}$), etc.) The convolution algebra consists of Schwartz functions $f : T \longrightarrow \mathcal{a}$ with $f * g$ defined as follows.

First write the commutation relations in integrated, or Weyl, form

$$e^{iap}\, e^{ibp} = e^{i(a+b)p}\, e^{-\frac{1}{2} aR'b}$$

where we are thinking in terms of a representation where $ia_\mu p_\mu = iap$ can be exponentiated. To the Schwartz fn. f we associate the operator

$$\tilde{f} = \int d^n a\; f(a)\, e^{iap}$$

and then convolution is defined so that we get a representation of the convolution algebra:

$$\int d^n a\; f(a)\, e^{iap} \int d^n b\; g(b)\, e^{ibp}$$

$$= \iint d^n a\, d^n b\;\; f(a)\, g(b)\;\; e^{-\frac{1}{2} aR'b}\;\; e^{i\overbrace{(a+b)}^{c}p} \qquad \begin{matrix} c = a+b \\ a = c-b \end{matrix}$$

$$= \int d^n c \left\{ \int d^n b\;\; f(c-b)\, g(b)\, e^{-\frac{1}{2}(c-b)R'b} \right\} e^{icp}$$

Thus

$$\boxed{(f*g)(c) = \int d^n b\;\; f(c-b)\, g(b)\, e^{-\frac{1}{2} cR'b}}$$

At this point one sees that it is necessary to make a hypothesis on R' that convolution be defined. If $\mathcal{A} = \mathbb{C}$ we would want R' to be purely imaginary. When $\mathcal{A} = \Lambda[\psi^\mu]$ and R' is as on p. 73 it's all right as R' is nilpotent. ($\Rightarrow e^{-\frac{1}{2} cR'b}$ is a polynomial in c, b; this aspect is clearer from Getzler's T^* picture.)

Take $f(a) = \delta_x(a) = \delta(a-x)$ and the operator is

$$\int d^n a\; \delta(a-x)\, e^{iap} = e^{ixp}$$

Also
$$(\delta_x * g)(a) = \int d^n b \;\, \delta(a-b-x)\, g(b)\, e^{-\frac{1}{2} a R' b}$$
$$= \qquad g(a-x) \;\; e^{-\frac{1}{2} a R'(a-x)}$$
$$= \; e^{-\frac{1}{2} x R' a} \; g(a-x)$$
is essentially translation thru x. Thus infinitesimal translation is
$$(i p_\mu) * g \;=\; \partial_{x^\mu} (\delta_x * g)$$
$$\boxed{((i p_\mu) * g)(a) \;=\; -\left(\partial_{a_\mu} + \tfrac{1}{2} R'_{\mu\nu} a_\nu\right) g(a)}$$

Now we want ~~the~~ to find the heat operator with this convolution alg.
$$e^{-tp^2} = \int d^n a \;\; f(t,a)\, e^{iap}$$
Then f satisfies
$$\left[\partial_t + (p_\mu *)(p_\mu *)\right] f = 0$$
$$\left[\partial_t - \sum_\mu \left(\partial_\mu + \tfrac{1}{2} R'_{\mu\nu} a_\nu\right)^2\right] f = 0$$

Look for f in the form
$$f = m(t)\, e^{-\frac{1}{4} a Q(t) a}$$
$$\left[\frac{\mathring{m}}{m} - \frac{1}{4} a \mathring{Q} a - \sum_\mu \left(\partial_\mu - \tfrac{1}{2} Q_{\mu\nu} a + \tfrac{1}{2} R'_{\mu\nu} a\right)^2\right] 1 = 0$$
$$\frac{\mathring{m}}{m} - \frac{1}{4} a \mathring{Q} a - \frac{1}{4} \sum_\mu \left[(Q - R')_{\mu\nu} a_\nu\right]^2 + \frac{1}{2} (Q - R')_{\mu\mu} = 0$$

$$\therefore \quad \frac{\mathring{m}}{m} + \frac{1}{2} Q_{\mu\mu} = 0$$

$$\mathring{Q} + \underbrace{(Q-R')^t (Q-R')}_{Q^2 - (R')^2 + \sout{R'Q - QR'}} = 0$$

assuming Q commutes with R'.

So we get a Riccati equation

$$\mathring{Q} + Q^2 = (R')^2$$

which has the solution $Q = \mathring{Y} Y^{-1}$ where Y is a matrix solution of $\ddot{Y} = (R')^2 Y$.

Now we want $f(t, a) \to \delta(a)$ as $t \to 0$ which means we would like $Q \sim \frac{1}{t}$ and $m(t) \sim \frac{1}{(4\pi t)^{n/2}}$. $Y = e^{R't} c_1 + e^{-R't} c_2$ is the general solution, so clearly

$$Y = \frac{e^{R't} - e^{-R't}}{2R'} = \frac{\sinh R't}{R'}$$

$$\boxed{Q = R' \frac{\cosh R't}{\sinh R't}}$$

works. Then

$$\frac{\mathring{m}}{m} + \frac{1}{2} \operatorname{tr} \mathring{Y} Y^{-1} = \frac{d}{dt}\left(\log m + \frac{1}{2} \log \det Y\right) = 0$$

so

$$m(t) = \text{const} \, \det{}^{-1/2}(Y)$$

$$= \text{const} \, \det{}^{-1/2}\left(\frac{\sinh R't}{R'}\right)$$

$$\boxed{m(t) = \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{R't}{\sinh R't}\right)}$$

Therefore the heat kernel is

$$\boxed{e^{-tp_\mu^2} = \int d^n a \left\{ \frac{1}{(4\pi t)^{n/2}} \det{}^{1/2}\left(\frac{R't}{\sinh R't}\right) e^{-\frac{1}{4} a R' \frac{\cosh R't}{\sinh R't} a} \right\} e^{iap}}$$

Now to get the index formula one needs the trace or integral on the convolution algebra belonging to

$$gr\, Diff(S \otimes E) = \Gamma(End\, E \otimes \mathcal{W})$$

Here $\mathcal{W}$ is the bundle of Weyl algebras over $\Lambda(T^*)$ generated by T with the relations determined by the skew-form

$$R'(p_X, p_Y) = -\frac{1}{2} R(X,Y) \in \Gamma(\Lambda^2 T^*)$$

Here for $X, Y \in \Gamma(T)$ the Riemannian curvature $R(X,Y)$ is a skew-symmetric endomorphism of T which we identify with a 2-form via the formula

$$R(X_\mu, X_\nu) = \frac{1}{2} R_{\mu\nu ij}\, \omega^i \omega^j$$

(Here ip_X denotes the image of D_X in $gr(Diff(S \otimes E))$

Review formulas: $[p_\mu, p_\nu] = R'_{\mu\nu} = -\frac{1}{4} R_{\mu\nu ij} \psi^i \psi^j$

but now we are identifying ψ^μ with ω^μ whence

$$[p_\mu, p_\nu] = -\frac{1}{2} R(X_\mu, X_\nu) \quad \text{interpreted in } \Gamma(\Lambda^2 T^*).$$

The trace or integral on the convolution algebra is defined as follows. One one hand we evaluate the kernel at $a=0$:

$$\int d^n a \, f(a) e^{ipa} \longmapsto f(0)$$

and on the other hand, corresponding to the super trace on spinors we have the map on forms which takes the highest degree (n) component times $(2i)^m$:

$$\omega \longmapsto (2i)^m [\omega]_{(n)}$$

Finally one takes the trace over E and integral over M: Thus applied to

$$e^{-tp_\mu^2 + t\frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu}}$$

we get using the formula on p. 77.

$$\int_M \frac{(2i)^m}{(4\pi t)^m} \left[\det{}^{1/2}\left(\frac{R' t}{\sinh R' t}\right) \operatorname{tr}_E e^{t\frac{1}{2}\omega^\mu\omega^\nu F_{\mu\nu}} \right]_{(n)} \qquad F = F^E$$

where $R'_{\mu\nu} = -\frac{1}{2}\left(\frac{1}{2} R_{\mu\nu ij}\, \omega^i \omega^j\right)$. By the curvature identity $R_{\mu\nu ij} = R_{ij\mu\nu}$ this is the same as

$$R'_{\mu\nu} = -\frac{1}{2}\left(\frac{1}{2}\omega^i\omega^j R_{ij\mu\nu}\right) = -\frac{1}{2} R$$

where $R = R_{\mu\nu}$ is the curvature matrix of 2 forms, i.e. End(T)-valued 2 form. Since $\frac{x}{\sinh x}$ is even we get the formula

$$\int_M \left(\frac{i}{2\pi}\right)^m \det{}^{1/2}\left(\frac{R/2}{\sinh R/2}\right) tr\left(e^F\right)$$

which is the Atiyah-Singer formula for the index.