July 12, 1984

Let's consider a connection \( D \) on \( p^*(E) \) over \( \mathbb{R} \times M \). It can be written

\[
D = d' + \theta + D''
\]

where \( D'' \) is a family of connections in \( E \) parametrized by \( \mathbb{R} \), and where \( \theta \) is a 1-form on \( \mathbb{R} \) with values in \( \text{End} \, E \). The curvature is

\[
D^2 = \frac{d'\theta + \theta^2 + [d'\theta, D'']}{2} + (D'')^2
\]

I want to consider now the case where \( m = S^1 \). In this case \( (D'')^2 = 0 \) and so the odd forms on \( \mathbb{R} \) obtained by integrating \( \text{tr}(e^{\Omega}) \) over \( M \) are very simple, namely

\[
\int_M \text{tr}(e^{\Omega} \, [d'\theta, D''])
\]

I know I can write this as an equivariant form for \( G \) acting on \( A \), \( \mathbb{R} \) and \( A \) being associated to the trivial rank 1 bundle \( E \) over \( S^1 \).

I now want to relate these odd character forms to two different things. First of all there is the family of left-invariant even forms on \( G \) one obtains by the transgression process. Secondly there is the map \( A \to U \) given by monodromy which we know is equivariant with \( \mathbb{R} \to U \) given by evaluation at the basepoint, where \( U \) acts on itself by conjugation. On \( (U, U) \) are odd equivariant forms, it seems. These can be pulled back to \( (\mathbb{R}, A) \).
Let's describe the transgression process. We consider the $G$-map $G \to A$ given by a point $D_0$ of $A$. Thus we have the family $g \mapsto gD_0g^{-1}$ of connections on $A$ parametrized by elts of $G$. The transgression of an equivariant form is computed as follows. One first lifts to $A$, this means forget the $G$-part, write as a coboundary and restrict to $G$. One also restricts to $G$ taking the universal $\Theta$ to be the $\Theta$ on $A$ that descends; this means $\Theta = gdg^{-1}$ where $d = dg$. In the present case, the equivariant forms come from connections on $pr_2^*(E)$, so we might as well look at the connections on $pr_2^*(E)$ over $G \times M$.

So the connection which corresponds to lifting to $A$ and restricting to $G$ is

$$d' + gD_0g^{-1} = g(d' + g^{-1}dg + D_0)g^{-1}$$

The connection which descends is

$$d' + gdg^{-1} + gD_0g^{-1} = g(d' + D_0)g^{-1}$$

To express the fact that both connections compute the same classes we use the linear path between them:

$$g(d' + tw + D_0)g^{-1}$$

where $\omega = g^{-1}dg$

and the standard formula (need curvature

$$(d' + tw + D_0)^2 = D_0^2 + t[D_0, w] + (t^2 - t)w^2$$

and the fact that $D_0^2 = 0$ over $S^1$. 
\[ \text{tr} \, e^{[D_0, \omega]} - \text{tr} \, 1 = d' \int_0^1 dt \, \text{tr} \left( e^{tD_0} \omega + (t^2-t)\omega^2 \right) \]

Now \( [D_0, \omega]^2 = 0 \) and \( \omega^2 \) commutes with \( \omega \), so taking the part of degree 1 in \( M \) on both sides we have

\[
\text{tr} \, [D_0, \omega] = d' \int_0^1 dt \, \text{tr} \left( e^{(t^2-t)\omega^2} \omega \, t \, [D_0, \omega] \right) + d'' \int_0^1 dt \, \text{tr} \left( e^{(t^2-t)\omega^2} \omega \right)
\]

Now integrate over \( S^1 \) and one gets

\[
d' \left\{ \int_M \int_0^1 dt \, \text{tr} \left( e^{(t^2-t)\omega^2} \omega \, t \, [D_0, \omega] \right) \right\} = 0
\]

because

\[
\int_M \text{tr} \, [D_0, \omega] = \int_M d'' \, \text{tr} \, (\omega) = 0.
\]

Thus we find that the good lift invariant 2k-forms on \( G \) are

\[
\text{Const} \int_M \text{tr} \left( \omega^{2k-1} [D_0, \omega] \right)
\]

for \( D_0 \) any point in \( A \). The constant is

\[
\int_0^1 dt \, \frac{(t^2-t)^{k-1}}{(k-1)!} t = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 t^k \, (1-t)^{k-1} \, dt
\]

\[
= \frac{(-1)^{k-1}}{(k-1)!} \frac{k! \, (k-1)!}{(2k)!} = \frac{(-1)^{k-1} \, k!}{(2k)!}
\]
Thus for $k=1$ and $D_0 = 1$ we get

$$\frac{1}{2} \int_{S^1} \text{tr} (\omega d\omega)$$

Recall the constant for basic forms on $\mathbb{G}_m$

$$\int_0^1 dt \, \text{tr} \left( e^{(t^2-t)\omega^2} \right)$$

In degree $2k-1$ we get

$$\text{tr}(\omega^{2k-1}) \int_0^1 dt \, \frac{(t^2-t)^{k-1}}{(k-1)!} \frac{(-1)^{k-1} (k-1)!}{(2k-1)!}$$

These two constants differ by a factor of $2$.

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**Digression:** The van Est formalism suggests there is a multiplicative $K$-theory fitting into a long exact sequence

$$\rightarrow K_{n+1}^{\text{top}}(A) \rightarrow HC_n(A) \rightarrow K_n(A) \rightarrow$$

which should somehow be related to Borel cohomology of $\text{GL}(A)$. The above should be viewed as a kind of analogue of the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 0$$

Also there should be a map

$$K_{n+1}^{\text{alg}}(A) \rightarrow K_n^X(A),$$

What should the relation be with complex coefficients. Taking primitives
in the van Est spectral sequence gives a long exact sequence

\[ \text{Hom}(K_{n+2}^* A, \mathbb{C}) \leftarrow HC^n(A) \leftarrow \text{Hom}(K_n^* A, \mathbb{C}) \]

(Curious: \( HC^0(A) \) is related to \( K_2 \) directly rather than \( K_0 \). A trace on \( A \) determines a closed 1-form on \( GL(A) \), hence a linear functional on \( \pi_1 GL(A) = \pi_2 BGL(A) = K_2 A \).)

Problem: Find the best way to construct the regulator maps \( K_{2n-1} C \rightarrow C^\times \).

Method of Deligne cohomology, Karoubi's method using relative K-theory.

\[
\begin{align*}
K_{n+1}^\text{top} A & \rightarrow K_n^\text{rel} A \rightarrow K_n^\text{alg} A \rightarrow K_n^\text{top} A \\
& \downarrow \\
& HC_{n-1} A
\end{align*}
\]

Check dimensions.

\[
\begin{align*}
K_n^\text{rel} &= \text{Fiber } \{ BGL(A)^+ \rightarrow BGL(A) \} \\
HC_{n-1}(A) &= \text{Prim } \{ H_n(\text{Gr}^0(A)) \}
\end{align*}
\]
One conjectures that the map \( \alpha \) coincides with the map defined by Connes. Connes' definition involves directly defining sequences, stable under \( S \) in the cyclic cohomology attached to generators for \( K_{\text{top}} \). The map \( \alpha \) defined by the above diagram is transverse to the process of taking a cyclic cocycle, interpreting it as a differential form on \( G \), and then integrating over spherical cycles.

Possibility: One might define the accessible algebraic \( K \)-theory \( K_{\text{acc}} \) as the fibre of the map \( \alpha \). The correct degrees are

\[
K_{n}^\text{acc} = \text{Fibre theory of } \{ K_{n} \rightarrow HC_{n-2} \}
\]

This is in the spirit of flat bundles — an element of \( K_{n}^\text{acc} \) will be a element in \( K_{n} \top \) with a trivialization of "order" depending on \( n \). To see what this means, recall

\[
HC_{n-2} = \Omega^{n-2}/d \oplus H^{n-4} \oplus H^{n-6} \oplus ...
\]

and the map \( \alpha \) associates to an element of \( K \) its Chern character. Thus \( K_{n}^\text{acc} \) consists of \( K \)-elements equipped with a reason why the classes \( \text{ch}_{0}(i), \text{ch}_{1}(i), \ldots, \text{ch}_{n-1}(i) \) are zero, assuming \( n \) is even.
Discussion of some difficulties with defining regulator maps \( K^{\text{alg}}_{2m+1} \mathbb{C} \to \mathbb{C}^\times \) and various generalizations.

These regulator maps can be defined simply on the \( K \)-group level as follows. Use Max's relative sequence (n even)

\[
\begin{array}{c}
K^{\text{top}}_{2m+1} \mathbb{C} \\
\downarrow \\
H^2_{2m}(\mathbb{C}) \simeq \mathbb{C}.
\end{array}
\]

In more elementary terms, an element of \( K^{\text{alg}}_{2m+1} \mathbb{C} \) can be realized by a flat vector bundle over a homotopy \( S^{2m+1} \). An invariant form on \( GL_n \mathbb{C} \) then induces via the flat connection a closed form on the principal bundle of the vector bundle. The bundle is topologically trivial as \( \pi_{2m+1}(SU) = 0 \), and choosing a section we can pull back the form on the principal bundle to the homotopy sphere where it can be integrated to get a \( \mathbb{C}^\times \)-number. The ambiguity in the trivialization means the no.

One sees that using the sphere \( SU \) in the above way is analogous to the way Witten constructs the Wess-Zumino action.

\[
\text{Map}(S^{2m}, U) \longrightarrow S^1
\]

However we know that there is a generalization.
of the regulator map just defined on K-groups. (Although the generalization might be off by the usual integers, \((-1)^{k-1}(k-1)\))

Given a flat vector bundle \(\xi\), we know that it is possible to define characteristic classes in \(H^{2k-1}(M, \mathbb{C}^k)\). These are the Chern classes defined via Deligne cohomology. Hence there is a universal class in

\[(*) \quad c_k \in H^{2k-1}(\text{GL}(\mathbb{C}), \mathbb{C}^k)\]

which maps under Bockstein to \(c_k\) in integral cohomology.

The idea behind the Deligne cohomology is that Chern classes are integral classes which when viewed in DR cohomology have a certain filtration. Thus given a flat bundle \(\xi\) over \(M\) one builds some sort of associated fibre bundle and does constructions with differential forms inside, somehow tied to the holomorphic structure of the projective bundle.

Presumably it is possible to define the class \((*)\) in the continuous or Borel cohomology

\[H^{2k-1}(\text{GL}(\mathbb{C}), \mathbb{C}^k)\]

I tried to recall Graeme's construction of these. From the exponential sequence

\[H^{2k-1}(G, \mathbb{Z}) \rightarrow H^{2k-1}_{\text{Borel}}(G, \mathbb{C}) \rightarrow H^{2k-1}_{\text{Borel}}(G, \mathbb{C}) \rightarrow H^{2k}_{\text{Borel}}(G, \mathbb{Z}) \]

\[H^{2k-1}_{\text{Borel}}(BG, \mathbb{Z}) \rightarrow H^{2k-1}_{\text{Borel}}(G, \mathbb{Z}) \rightarrow H^{2k}_{\text{Borel}}(G, \mathbb{Z}) \]
one can't produce the desired class in $G(U)$, but if $G = U$, then we get an isomorphism

$$H_{\text{Bor}}^k(U, \mathbb{C}) \cong H_{\text{Bor}}^{k-1}(U, \mathbb{C}^*) \cong H^k(BU, \mathbb{Z})$$

so it looks like one might easily define $S^1$-valued odd characteristic classes for a flat unitary bundle.

Possible procedure: I want to use the idea that emerged when I looked at the Witten construction, namely, to replace the sphere by any manifold $M$ and then to use a Dirac operator on $M$ as realizing the basic homology class which gives the ultimate number.

In this case we have a flat unitary bundle $E$ over an odd-dimensional manifold $M$ and instead of getting a number out of a typical odd homology class, we will want to get a number out of a Dirac operator on $M$. The number should be independent of the connections used in the Dirac operator. So we should get an actual function on the space of connections $A$, which is constant. The thing to take then is the difference of the $\eta$-invariant for the operator tensored with the flat bundle and the $\eta$-invariant of the operator tensored with a trivial bundle of the same rank.
Let's return to the map which goes from the space of connections on the trivial bundle over $S^1$ to the unitary group $U = U_1$ obtained by taking parallel transport around the circle starting at the basepoint $0$.

$$A \rightarrow \tau \left\{ e^{-i\int_0^{2\pi} A \, dx} \right\}$$

This map identifies $A/\Omega U$ with $U$. Now I want to descend the trivial bundle over $A \times S^1$ to a bundle over $U \times S^1$. This means that given a point of $U \times S^1$, I have to give an $r$-dimensional vector space together with an identification of this vector space with the fibre of $\phi_1^*(E)$ over any $A, x \in A \times S^1$ mapping to $j$ and the whole business should be $U_1$-equivariant.

It would be better to think if $E$ as a given vector bundle over $S^1$ without a trivialization. But one is given a trivialization over the basepoint $0 \in S^1$, so that the monodromy is defined for any connection. Now what is needed is a vector space attached to each $(\theta, x) \in U \times S^1$. Let's think of $g$ as the class of connections with monodromy $g$.

Given such a connection I can follow it back to the basepoint and so identify the fibre of $\phi_1^*(E)$ at $(A, x)$ with the fibre $E_0$ equivariantly under $\Omega U = \Omega U_0$. There is a problem as $x$ crosses $2\pi$ but then the given $g$ tells how to make the identification smoothly.

So it is fairly clear how the bundle over $A \times S^1$ is to be descended. One uses the connection to push backward to the basepoint and one obtains the same clutching construction needed to define the
descended bundle over $U \times S^1$.

The next point is to take the connection defined on $E$ lift it back to $pr_1^*(E)$ over $A \times S^1$. In terms of the trivialization of $pr_1^*(E)$ which we have given over $A \times (0, 2\pi)$ it should be easy to write this connection down, so now I have a connection that descends and I can go thru the transgression process by joining this connection to the tautological connection on $pr_2^*(E)$. By the time one restricts to a $G$-orbit, the results ought to be the same as before.

Another point: Compare the invariant forms found on the loop group with the character forms on the Grassmannian under the "scattering map".

\[
\int_0^1 e^{(t^2 - t)x^2} dt = e^{-\frac{1}{4}x^2} \int_0^1 e^{(t - \frac{1}{2})x^2} dt
\]

\[
= e^{-\frac{1}{4}x^2} \int_{-\frac{1}{2}}^{1/2} e^{t^2x^2} dt = e^{-\frac{x^2}{4}} \int_{-\frac{1}{2}}^{1/2} e^{t^2} dt
\]

\[
\sim \begin{cases} 
  e^{\frac{1}{4}y} (i \sqrt{\pi}) & \text{if } x = \frac{4}{y}, \quad y \to +\infty \\
  e^{\frac{1}{4}y} (-i \sqrt{\pi}) & \text{if } \quad y \to -\infty 
\end{cases}
\]
We are considering the trivial bundle $\tilde{M} \to M$ over $\mathbb{S}^1$ with fibre $\mathbb{C}^n$, $\pi: \mathbb{C}^n \to M$. The covers $\mathbb{S}^1$ are indexed by $x \in \mathbb{C}^n$. We consider the trivial bundle $\tilde{M}$ with fibre $\mathbb{C}^n$ over $\mathbb{S}^1$.

Consider the action of $\mathbb{S}^1$ on the bundle $\pi: \tilde{M} \to M$, the action $g \cdot \tilde{x} = g \cdot x$. We want to lift this action back to $\mathbb{S}^1$.

Now consider the action of $\mathbb{S}^1$ on $\mathbb{S}^1$ given by $g \cdot x = g \cdot x$. Then we know that there is an isomorphism given by $A \to \frac{1}{i} \epsilon_{-iA}$.
use the given basis. Thus everywhere sections are defined by the function (invertible metric function)

\[ \psi(A, x) = T \{ e^{-\int_0^x A dx} \} \]

which satisfies

\[ \psi(A, 0) = I \]

\[ (\partial_x + A) \psi(A, x) = 0 \]

The second trivialization is obtained by parallel transporting from \( x \) to \( 2\pi \):

\[ \psi(A, x) = T \{ e^{-\int_{2\pi}^x A dx} \} \]

(Think intrinsically: Suppose \( E \) is a v.b. over \( S^1 \) equipped with a fixed frame at \( 0 \in S^1 \). Then over \( S^1 \times I \) we construct these two trivializations. The trivializations are evidently equivariant under the action of \( S^0 \).)

The connection on the trivial bundle over \( \mathbb{A} \times I \) associated to the \( \varphi \) - trivialization (resp. \( \psi \)) is \( \varphi \cdot d \varphi^{-1} \) (resp. \( \psi \cdot d \psi^{-1} \)) where \( \varphi = \varphi_0 + d_0 \). We define a connection on the trivial bundle over \( \mathbb{A} \times I \):

\[ (1 - \frac{x}{2\pi}) \varphi \cdot d \varphi^{-1} + \frac{x}{2\pi} \psi \cdot d \psi^{-1} \]

Example: Let us take the trivial line bundle over the circle and the 1-parameter family of connections

\[ dx (\partial_x + iy) \]

Then

\[ \psi(y, x) = e^{-iyx} \]

\[ \psi(y, x) = e^{iy(2\pi - x)} \]
Then the connection I am interested in is
\[
(1 - \frac{x}{2\pi}) \, d(yx) + \frac{x}{2\pi} \, d(y(x - 2\pi))
\]
\[= \quad i \, (d(yx) - x \, dy) = i \, y \, dx\]

Notice that this agrees with the tautological connection
\[d + A dx = dy \, \frac{\partial}{\partial y} + dx \, (\partial_x + iy),\]
probably because the gauge transformation group is discrete in this case.

What we've done so far is to define a connection on the trivial bundle over \(\mathbb{A} \times [0, 2\pi]\)
which is \(G_0\)-invariant. The connection is
\[
(1 - \frac{x}{2\pi}) \, \varphi \, d \, \varphi^{-1} + \frac{x}{2\pi} \, \varphi \, d \, \varphi^{-1}
\]
where 
\[
\varphi(A, \phi) = T \{ e^{-\int_0^x A dx} \}, \quad \varphi(A, x) = T \{ e^{\frac{x}{2\pi} A dx} \}
\]
so that 
\[
\varphi^{-1} \varphi = T \{ e^{-\int_0^x A dx} \} = \text{parallel transport from } 2\pi \text{ to } 0. \quad \text{Let's denote this } \tau^{-1}.
\]
so that 
\[
\varphi = \varphi \tau^{-1}.
\]

Then our connection is
\[
\varphi \left\{ (1 - \frac{x}{2\pi}) \, d + \frac{x}{2\pi} \, \tau^{-1} d \, \tau^{-1} \right\} \varphi^{-1}
\]
\[= \quad \varphi \left\{ d + \frac{x}{2\pi} \, \tau^{-1} d(e^\phi) \right\} \varphi^{-1}
\]
which, shows more or less that the connection we have is the inverse of the one
\[d + \frac{x}{2\pi} \, g^{-1} dg \quad \text{over } U \times [0, 2\pi]
\]
under the quotient by \(G_0\).

Now that we have our descendable connection
we compare it to the tautological one
\[ d + A_\nu \]
except that we now want to restrict to a \( \mathcal{H}_0 \)-orbit, or rather pull-back by the map
\[ \mathcal{H}_0 \to A \quad g \to g \partial_x g^{-1} + g A_\nu g^{-1} \]

What is \( \psi(g, x) \)?
\[ g (\partial_x + A_\nu) g^{-1} \psi = (\partial_x + A) \psi = 0 \]
\[ \psi(g, x) = g \psi_0(x) \quad \psi_0(x) = T \{ e^{-\frac{t^2}{2}} e^0 dx \} \]
so that \( (\partial_x + A_\nu) \psi_0 = 0 \) or \( \psi_0 \partial_x \psi_0^{-1} = A_\nu \).

What is \( \psi(d + \frac{x}{2\pi} t dt^{-1}) \psi^{-1} \) restricted to \( \mathcal{H}_0 \)?
The monodromy is constant on the \( \mathcal{H}_0 \)-orbit \( \Rightarrow dt = 0 \).
So we have the connection
\[ \psi \cdot d \psi^{-1} = g \cdot \psi_0 \cdot d \cdot \psi_0^{-1} \cdot g^{-1} \]
\[ = g \cdot (d + \psi_0 \partial_x \psi_0^{-1}) \cdot g^{-1} \]
\[ = g \cdot (d + A_\nu) g^{-1} \]
The tautological connection restricted to \( \mathcal{H}_0 \) is
\[ d + d_x (g \partial_x g^{-1} + g A_\nu g^{-1}) \]
\[ = d'_x + g \cdot (d_x \cdot \partial_x + A_\nu) g^{-1} \]
\[ = g \cdot (d + g^{-1} dg + A_\nu) g^{-1} \]

and so we are in the usual transgression situation.
Thus I have checked the fact, which are known for general reasons, that upon restricting the descendable connection to the \( \mathcal{H}_0 \)-orbit, one
gets the connection $g (d + A_0) g^{-1}$, so the end result of transgressing the odd forms on $\mathcal{U}$ to $\Omega \mathcal{U}$ is the same as calculated earlier.

Now one thing that is striking about this calculation is that we go to a lot of work to pass from the character forms on $BU_n$ to forms on $\Omega BU_n$. The obvious method is to use the evaluation map

$$\Omega^2 BU \times S^2 \longrightarrow BU$$

$$\downarrow$$

$$\Omega^2 \mathcal{U}$$

pull back and integrate over $S^2$. Let's break this into 2 steps.

$$U_n \times S^1 \longrightarrow BU_n$$

$$\Omega U_n \times S^1 \text{ ev} \longrightarrow U_n$$

The first map comes from assigning to $g \in U$ the graph $(t^1) V^0 \subset V^0 \oplus V^1 \quad (V^0 = C_n)$ and then letting $t$ go from 0 to $\infty$. Strictly speaking I then get paths from $V^0$ to $V^1$ in the Grassmannian, and I have to pick a fixed path back.

The connection form for the graph embedding is

$$\frac{1}{1 + T^*T} \quad T^* dT$$

so that if I take $T = t g$, I get

$$\frac{1}{1 + |t|^2} \quad E dt + \frac{|t|^2}{1 + |t|^2} \quad g^{-1} dg$$
for the connection form over this family. Let's just concentrate on the 1-parameter family of connections on the trivial bundle over the unitary group. Then we get the family
\[ \frac{t^2}{1+t^2} g^{-1} dg \quad \text{for} \quad 0 \leq t \leq \infty \Rightarrow 0 \leq \frac{t^2}{1+t^2} \leq 1 , \]
and so we get the usual odd forms on the unitary group. (Note that
\[ \int tr (e^{(t\omega - \omega) \omega^2} \omega) du \]
is independent of the function \( \omega \).)

**Remark:** This description of \( \omega \) how to obtain the odd forms on \( U \) suggests that the parameter \( t \) in the connection family \( d + t \omega \) is sort of a "moment map" associated to a circle action on the Grassmannian. So it is very natural to think of it in terms of convex linear combinations.

**Important idea:** Take \( G_{2n}^\omega (C) \) and introduce a circle action acting on the last \( n \) coordinates. Think in terms of the moment map for this circle action.

The next idea will be to look at the evaluation map \( \Omega^1 U_n \times S' \rightarrow U_n \). I know the forms defined on \( \Omega^1 U_n \) by pulling back and integrating are not left invariant. My guess is that these forms are ones obtained by the following naive process. One has the
\[ \text{map } A \times [0, 2\pi] \rightarrow U \]
\[ A, x \mapsto \varphi(A, x) \]

which is like taking a path starting at the identity and evaluating at \( x \). If we pull-back a form \( \omega \) and integrate over \([0, 2\pi]\), this gives a cobounding form for the pull-back of \( \omega \) under monodromy. So if we restrict to \( A, x \rightarrow A \)

thought of as closed loops, we are transgressing \( \omega \) on \( U \) to a form on \( SU \).

My feeling is that if this is phrased in terms of the connections over \( U \times I \), then one is using two connections such as \( d \) and \( g \cdot dg^{-1} \) over \( G \), not the restriction of the tautological connection.

The next project is to find how character forms on the Grassmannian look on the loop group.
Let's work out the Dirac operator and the Weitzenböck formula, etc., using the principal frame bundle.

Let $M$ be a Riemannian $n$-manifold and let $P$ be the principal bundle of its tangent bundle. A point $p$ of $P$ over $m \in M$ is an isomorphism $R^n \rightarrow T_{m,j}$ compatible with inner products. Let $\omega^k \in \Omega^1(P)$ be the one form whose effect on a tangent vector at $u$ projects this vector down to $M$ and gives the $\mu^k$ coord. of the projected vector relative to the framing $u$.

Alternatively, we have

$$0 \rightarrow \pi^*(T^*_M) \rightarrow T^*_P \rightarrow T^*_\pi \rightarrow 0$$

and a canonical trivialization $\pi^*(T^*_M) \cong P \times (R^n)^m$, so the basis sections of $\pi^*(T^*_M)$ relative to this trivialization can be viewed as 1-forms on $P$.

Now suppose we have a connection in $T^*_M$. Then when lifted back to $\pi^*(T^*_M)$ it can be written relative to the trivialization in the form

$$D = d + \Theta$$

or equivalently

$$D \omega^k = \Theta^k_v \wedge \omega^v$$

where $\Theta^k_v \in \Omega^1(P)$. Recall

$$D : \Gamma(\pi^*T^*_M) \rightarrow \Gamma(T^*_P \otimes \pi^*T^*_M)$$

so $D\omega^k$ is indeed uniquely expressible in the above form.

Because the connection $D = d + \Theta$ on $\pi^*T^*_M$ descends it follows that $\Theta^k_v = -\Theta^k_v$ and that the $\Theta^k_v$ with $k < v$ form a basis for the vertical 1-forms. We can
easily work out \( \omega^* = (R_g)^* \) on the \( \omega^* \) and \( \Theta^* \), and also \( i_X \) for \( X \in \text{Lie} \, SO(n) \) if we want.

So we obtain a framing of \( \Lambda^1(P) \). Define vector fields \( X_p \) on \( P \) to be the horizontal vector fields which project to the \( \mu \)-th basis vector of the frame are is at: \( i_X \omega^* = \delta^*_\mu \), \( i_X \Theta^* = 0 \).

Next let's consider the process of covariant differentiation on tensors or spinors, i.e. sections of a vector bundle associated to the principal frame bundle. This means we have a representation \( S \) of \( \mathfrak{g} = \text{so}(n) \) in a vector space \( V \) and the vector bundle is \( P \times^S V \). Its sections are invariant elements of \( \Lambda^0(P) \otimes V \) and the connection will be the operator

\[
D = d + \hat{\rho}(\Theta) \quad \text{on} \quad (\Lambda^0(P) \otimes V)^{\text{basic}}.
\]
Recall the check that the curvature of the jet line bundle given by superconnection formalism agrees with that given by the $S$ formalism.

$$L = i(\circ D^{*})$$

Then the degree 2 component of $\text{tr}_S e^{K(t^2 + dt)}$ gives a form which represents $n \cdot d\mathbf{A}$. Compute the degree 2 component by the perturbation series.

$$\int_0^N \int_0^{t_1} dt_1 \int_0^{t_2} dt_2 \text{tr}_S \left\{ e^{(K-t_1)L^2} e^{(t_1-t_2)L^2} e^{t_2L^2} \right\}$$

$$= \int_0^N dt_1 \int_0^{t_1} dt_2 \text{tr}_S \left\{ e^{(t_1-t_2)L^2} e^{(t_1-L)L^2} e^{t_2L^2} \right\}$$

$$= \int_0^N dt_1 \int_0^{t_1} ds \text{tr}_S \left\{ e^{(t_1-s)L^2} e^{sL^2} e^{t_2L^2} \right\}$$

$$= \int_0^N ds (n-s) \text{tr}_S \left\{ \right\}$$

By symmetry the supertrace term is invariant under $s \leftrightarrow n-s$. So we get

$$\frac{n}{2} \int_0^N ds \text{tr}_S \left\{ e^{(n-s)L^2} e^{sL^2} e^{t_2L^2} \right\}$$

and so the $d\mathbf{A}$ is represented by

$$\int_0^t dt_1 \frac{1}{2} \text{tr}_S \left\{ e^{(t-t_1)L^2} e^{t_1L^2} \right\}$$

for any $t$. Actually we have proved this:

$$\text{tr}_S e^{(tL^2 + t^2 dt)\mathbf{A}} = \int_0^t dt_1 \frac{1}{2} \text{tr}_S \left\{ e^{(t-t_1)L^2} e^{t_1L^2} \right\}$$

Next consider the $S$-approach. The curvature
is \( -\delta \mathcal{J}(0) \) and we have

\[
-\delta \mathcal{J}(s) = s \operatorname{Tr} \left( (D^* D)^s D^{-1} \delta D \right)
\]

\[
= s \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr} \left( e^{-t D^* D} D^{-1} \delta D \right) t^{s-1} \frac{dt}{t}
\]

So

\[
-\delta \mathcal{J}(s) = \frac{s}{\Gamma(s)} \int_0^\infty \operatorname{Tr} \left\{ \int_0^t e^{-(t-t_1) D^* D} D^{-1} \delta D e^{-t_1 D^* D} D^{-1} \delta D \right\} t^{s-1} \frac{dt}{t}
\]

\[
= \frac{s}{\Gamma(s)} \int_0^\infty \operatorname{Tr} \left\{ \int_0^t e^{-(t-t_1) D^* D} D^{-1} \delta D e^{-t_1 D^* D} D^{-1} \delta D \right\} t^{s-1} \frac{dt}{t}
\]

Assuming the limit exists, we get

\[
-\text{curvature} = -\delta \mathcal{J}(0) = \lim_{t \to 0} \int_0^t \operatorname{Tr} \left( e^{-(t-t_1) D^* D} D^{-1} \delta D e^{-t_1 D^* D} D^{-1} \delta D \right)
\]

which is clearly consistent with the previous formula.

Now in the \( \mathcal{J} \)-approach, one requires not the limit as \( t \to 0 \), but only the zero-th coefficient in the asymptotic expansion. However, if the local index formula for families works, then the actual limit as \( t \to 0 \) should exist.

\[\]

I want to do the odd version of Grassmannian graph. Take a skew-adjoint operator \( T \) and put \( L = T \delta \sigma \), then the super-connection gives the odd forms

\[
\operatorname{Tr}_\sigma \left( e^{(L^2 + [D, L] + D^2)} \right)
\]

However, I don't know yet how these are to be normalized. So let's compute in the case of the Clifford multiplication \( T = i \gamma^I x^I \) over \( \mathbb{R}^{n-1} \), \( n = 2m \).
According to my convention, $T = i\gamma^\mu x^\mu$, where $\gamma^\mu$ are the generators of $C_n$ for $\mu < n$, and $\text{tr}_\sigma(...) \equiv \frac{1}{2\pi i} \text{tr}_S(...)$. So with this we have

$$\frac{1}{2\pi i} \text{tr}_S\left( e^{\mu\left(-|x|^2 - i\mu^\alpha dx^\alpha\right)} y_n\right)$$

$$= \frac{1}{2\pi i} e^{-|x|^2 \left(-i\mu\right)^{n-1}} \text{tr}_S\left(y^1 dx^1 \cdots y^{n-1} dx^{n-1} y^n\right) \frac{\text{tr}_S(y^n \cdots y^1) dx^1 \cdots dx^{n-1}}{(-1)^{\frac{n(n-1)}{2}} (2i)^n} = (-2i)^m$$

$$= \frac{1}{2\pi i} n^{n-1} e^{-|x|^2 (2i)^m} dx^1 \cdots dx^{n-1}$$

Now the convention about integrating an odd form over a $(n-1)$-cycle is to multiply by the factor

$$\left(\frac{i}{2\pi}\right)^m 2\sqrt{\pi}$$

which gives

$$\frac{1}{2\pi i} \int \left(\frac{i}{2\pi}\right)^m 2\sqrt{\pi} i n^{n-1} e^{-|x|^2 (2i)^m} \frac{dx^1 \cdots dx^{n-1}}{(2\pi)^{n-1}}$$

$$= 2\pi (-1)^m n^{n-1} \frac{n-1}{2}$$

Subexample: $T = i\chi$

$$\text{tr}_\sigma\left( e^{\mu\left(-|x|^2 + i\Delta x^\alpha\right)}\right) = e^{-\mu^2 x^2} i \mu dx$$

$$\int \frac{i}{2\pi} 2\sqrt{\pi} e^{-\mu^2 x^2} i \mu dx = -\mu^{1/2}$$

Anyway up to signs we should have

$$\text{tr}_\sigma\left( e^{\mu\left(L^2 + D^2\right) + \Delta^2}\right) \sim \sum_{m=1}^{\infty} n^{-\frac{1}{2}} \text{ch}_m$$
\[
\int_0^\infty \text{tr}_\sigma \left( e^{u(L^2+\lambda L)} \right) e^{-\lambda u} \frac{du}{u} \sim \sum_{m \geq 1} \frac{\Gamma(m+\frac{1}{2})}{\lambda^{m-\frac{1}{2}}} c_m^{2k+1}
\]

Instead of \(c_m\) but \(c_{2k+1}\) and the above becomes

\[
\sim \sum_{k \geq 0} \frac{\Gamma(k+\frac{1}{2})}{\lambda^{k+\frac{1}{2}}} c_{2k+1}
\]

Now consider \(D = d\), whence

\[
\int_0^\infty \text{tr}_\sigma \left( e^{u(L^2+dL)} \right) e^{-\lambda u} \frac{du}{u}
= \text{tr}_\sigma \log \left( 1 - \frac{1}{\lambda - L^2} dL \right)
= \sum_{k \geq 1} \frac{1}{k} \text{tr}_\sigma \left( \frac{1}{\lambda - L^2} dL \right)^k
\]

Let's recall \(L = \Sigma\), so

\[
\left( \frac{1}{\lambda - L^2} dL \right)^2 = \frac{1}{\lambda - T^2} (dT)^2 - \frac{1}{\lambda - T^2} (dT)^2
\]

\[
= - \left( \frac{1}{\lambda - T^2} dT \right)^2.
\]

Thus

\[
\int_0^\infty \text{tr}_\sigma \left( e^{u(L^2+dL)} \right) e^{-\lambda u} \frac{du}{u} = \sum_{k \geq 1} \frac{(-1)^k}{2k+1} \text{tr} \left( \frac{1}{\lambda - T^2} dT \right)^{2k+1}
\]
Now consider the map
\[ T \mapsto g = (I - T)^{-1}(I + T) \]
from skew-adjoint to unitary operators. Then
\[
g^{-1}dg = (I + T)^{-1}(I - T) \left\{ \frac{1}{1 + T} dT (I - T)^{-1}(I + T) + (I - T)^{-1} \right\}
\]
\[ = (I + T)^{-1} \left\{ dT \ (I - T)^{-1}(I + T) + dT \right\}
\]
\[ = (I + T)^{-1} dT \ (I - T)^{-1} \left\{ (I + T) + (I - T) \right\}
\]
\[ = 2 \frac{1}{1 + T} dT \frac{1}{1 - T}
\]
and so
\[
\text{tr} \ (g^{-1}dg)^{2k+1} = 2^{2k+1} \text{tr} \left( \frac{1}{1 - T^2} \ dT \right)^{2k+1}
\]
Rescaling \( T \mapsto \lambda^{1/2} T \) gives the forms
\[ 2^{2k+1} \text{tr} \left( \frac{\lambda^{1/2}}{\lambda - T^2} \ dT \right)^{2k+1}
\]
But recall that
\[
c_{2k+1} = \int_0^1 dt \ \frac{(t^2 - t)^k}{k!} \text{tr} \ (\omega^{2k+1})
\]
\[ \text{tr} \ (\omega^{2k+1}) = (-1)^k \frac{\beta(k+1)}{k!} = (-1)^k \frac{k!}{(2k+1)!} \]

Thus
\[
c_{2k+1} = (-1)^k \frac{k!}{k!} \frac{2^{2k+1}}{(2k+1)!} \lambda^{k+\frac{1}{2}} \text{tr} \left( \frac{1}{\lambda - T^2} \ dT \right)^{2k+1}
\]
But
\[
\Gamma(k + \frac{1}{2}) = \Gamma(\frac{1}{2}) \frac{1}{2} \frac{3}{2} \cdots \frac{2k-1}{2} = \Gamma(\frac{1}{2}) \frac{(2k)!}{k! \ 2^k \ 2^k}
\]
\[ = \sqrt{\pi} \frac{(2k)!}{k! \ 2^{2k}}
\]
Hence
\[
\frac{\Gamma(k+\frac{1}{2})}{\lambda^{k+\frac{1}{2}}} e_{2k+1} = 2\sqrt{\pi} \frac{(-1)^k}{2k+1} \text{tr} \left( \frac{1}{\lambda - \lambda^2} d\lambda \right)^{2k+1}
\]

and everything cross-checks very nicely.

For \( e_1 \), the above is as follows:
\[
e_1 = \text{tr} \left( g^{-1} dg \right) = 2 \text{tr}_\tau \left( \frac{\sqrt{\lambda}}{\lambda - \lambda^2} dL \right)
\]
\[
= 2 \text{tr} \left( \frac{\sqrt{\lambda}}{\lambda - \lambda^2} d\tau \right)
\]

and
\[
\text{tr}_\tau (e^{uL^2} u^{1/2} dL) \sim c e_1 \quad \text{for some constant } c.
\]

Apply \( \int e^{-\lambda u} u^{1/2} du \)?
\[
\frac{e_1}{2\sqrt{\pi}} = \text{tr}_\tau \left( \frac{1}{\lambda - \lambda^2} dL \right) \sim c e_1 \frac{\sqrt{\pi}}{\sqrt{\lambda}}
\]
so
\[
e_1 = 2\sqrt{\pi} \text{tr}_\tau (e^{uL^2} u^{1/2} dL)
\]
\[
= 2\sqrt{\pi} \text{tr} (e^{uT^2} u^{1/2} dT)
\]
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The transgression process for the \( e_1 \) class: (This is somewhat degenerate and has to be handled carefully so as to gain experience with the calculations, but yet not to get the wrong intuition).

We have \( a \) 1-diml \( \text{class} \) on \( B G = A/\mathbb{G} \) which transgresses to a 0-diml reduced \( \text{class} \) on \( A \).

(Recall transgression goes from \( \ker \{ H^*(B) \to H^*(E) \} \to \coker \{ H^*(E) \to H^{*+1}(F) \}, \) so in the present case from

\[ H^*(B_G) \to \coker \{ H^0(A) \to H^0(\mathbb{G}) \} = \tilde{H}^0(\mathbb{G}). \]

Geometrically one takes a 1-cocycle \( \omega \) on \( A/\mathbb{G} \) lifts it up to \( A \) and writes it \( \delta f \); then given a basepoint \( A_0 \in A \) and \( g \in G \) one assigns to \( g \) the number \( f(gA_0) - f(A_0) \). This gives one a function on \( G \) vanishing at the identity.

If \( \gamma \) is a path from \( A_0 \) to \( gA_0 \), then

\[
f(gA_0) - f(A_0) = \int_\gamma f = \int_\gamma \delta f = \int_\gamma \pi^*(\omega) = \int_{\pi(\gamma)} \omega
\]

and \( \pi(\gamma) \) is a loop in \( A/\mathbb{G} \), whose homotopy class depends only on the component of \( \mathbb{G} \) to which \( g \) belongs.

So this is the topology of the situation. Now we want to place ourselves in the situation when we take the form \( \omega \) representing \( e_1 \) on \( A/\mathbb{G} \), it happens that \( \pi^*\omega = df \) within equivariant forms. This means in this case that \( f \) is constant on \( G \)-orbits.
Note that this implies that \( e_1 = 0 \) and so the setup is very degenerate topologically, but nevertheless we might learn something from looking at what happens.

An equivariant 1-form on \( A \) is the same thing as a basic 1-form. Let's consider the possible forms representing \( e_1 \), or rather \( (2\pi i)^{-1} \).

There are the forms

\[
\alpha_t = \text{tr}(e^{t^2T^2} t \, dT)
\]

for different \( t \). If

\[
\beta_t = \text{tr}(e^{t^2T^2} T)
\]

then

\[
d\beta_t = \text{tr}(e^{t^2T^2} t^2 (\frac{dT}{T}) T) + \text{tr}(e^{t^2T^2} dT) = \text{tr}(e^{t^2T^2} 2t^2T^2 \, dT) + \text{tr}(e^{t^2T^2} dT)
\]

\[
\partial_t \alpha = \text{tr}(e^{t^2T^2} 2tT^2 \, dT) + \text{tr}(e^{t^2T^2} dT)
\]

so we have \[ \partial_t \alpha = d\beta_t \]

We can also combine the forms \( \alpha_t \) by integration. Thus \( \int f(t) \alpha_t \, dt \) should be cohomologous to \((\int f(t) \, dt) \alpha_s\) for any \( s \). Specifically if

\[
g(t) = \int_s^t f(t) \, dt
\]

Then

\[
\int_a^b f(t) \alpha_t \, dt = \int_a^b g'(t) \alpha_t \, dt
\]
\[ \int_0^1 f(t) \alpha_t \, dt = \left( \int_0^1 f(t) \, dt \right) \alpha_0 - d \int_0^1 g(t) \beta_t \, dt \]

One thing I want to use is that \( \alpha_0 = 0 \), as this corresponds to what we know about the classical limit of Dirac operators. Then

\[ \alpha_1 = \int_0^1 \beta_t \, dt = d \int_0^1 \beta_t \, dt \]

\[ = d \int_0^1 \text{tr} \left( e^{t^2 T^2} T \right) \, dt \]

So if we take \( \pi \omega \) to be \( \alpha_1 \), then we get the \( \Gamma \)-invariant function on \( T \)

\[ f(T) = \int_0^1 \text{tr} \left( e^{t^2 T^2} T \right) \, dt \]

Natural question is how this compares to \( \eta(A) \) where \( A = i T \)?

\[ \eta_A(S) = \text{tr} \left( \frac{A}{1A1} 1A1^{-S} \right) \]

\[ = \text{tr} (A (A^2)^{-\frac{S+1}{2}}) = \frac{1}{\Gamma\left(\frac{S+1}{2}\right)} \int_0^\infty \text{tr} (A e^{-tA^2}) \left( t^{\frac{S+1}{2}} \frac{dt}{t} \right) \]

\[ = \frac{2}{\Gamma\left(\frac{S+1}{2}\right)} \int_0^\infty \text{tr} (e^{-t^2 A^2} A) \left( t^{\frac{S+1}{2}} \frac{dt}{t} \right) \]

\[ \eta(A) = \frac{2}{\sqrt{\pi}} \int_0^\infty \text{tr} (e^{-t^2 A^2} A) \, dt \]

\[ i \pi \eta(A) = 2 \sqrt{\pi} \int_0^\infty \text{tr} \left( e^{-t^2 T^2} T \right) \, dt \]
so we see again, without understanding why, that the form of interest is \[ \int_0^\infty \beta_t \, dt. \]

This form has for boundary \( \pm \infty \) which is zero at least when \( T \) is invertible.

The fundamental problem therefore seems to be why the deformation \( t \to \infty \) should be equivalent topologically to the trivialization of the character form over \( \mathcal{D} \) which results from the character form being an equivariant form.

Bott periodicity theorem from the differential form viewpoint.

Let's consider the Bott map which associates to a subspace a path from \( I \) to \( -I \) in the unitary group. Let \( F = 1 \) in the subspace and \( -1 \) in the orthogonal complement, whence \( F = 2e - 1 \), where \( e \) is the projector on the subspace. The path will then be

\[ g = \frac{1 + itF}{1 - itF} \quad 0 \leq t \leq \infty. \]

and we have

\[ g^{-1} dg \cong 2 \frac{1}{1 + t^2} d(itF) \]

So

\[ \text{tr} (g^{-1} dg)^{2k+1} = 2 \frac{(i)^{2k+1}}{(1 + t^2)^{2k+1}} \text{tr} \left( dt \frac{F + t dF}{1 + t^2} \right)^{2k+1} \]

\[ = \left( \frac{2i}{1 + t^2} \right)^{2k+1} (2k+1) dt \cdot t^{2k} \text{ tr} F(dF)^{2k} + \text{ term not involving } dt. \]
Check the constants. I need
\[ \int_0^\infty \frac{t^{2k} dt}{(1+t^2)^{2k+1}} = \frac{1}{\Gamma(2k+1)} \int_0^\infty t^{2k} dt \int_0^\infty e^{-(1+t^2)u} u^{2k+1} \frac{du}{u} \]
\[ = \frac{1}{\Gamma(2k+1)} \int_0^\infty e^{-u} u^{2k+1} \frac{du}{u} \int_0^\infty e^{-ut^2} t^{2k+1} \frac{dt}{2t} \]
\[ = \frac{1}{2 \Gamma(2k+1)} \int_0^\infty e^{-u} u^{2k+1/2} \frac{du}{u} \frac{\Gamma(k+1/2)}{\sqrt{2 \pi}} \frac{\Gamma(k+1/2)}{\sqrt{2 \pi}} \]
\[ = \frac{\Gamma(k+1/2)^2}{2 \Gamma(2k+1)} \Gamma(k+1/2) = \sqrt{\pi} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k} = \sqrt{\pi} \frac{2k!}{2^{2k} k!} \]

Next recall \( e_k = \frac{(-1)^k k!}{(2k+1)!} \) \( \text{tr} \left( (g^{-1} dg)^{2k+1} \right) \) and \( c_{\chi_k} = \frac{1}{k! 2^{2k+1}} \text{tr} \left( dF \right)^{2k} \).

Thus \( e_k \) pulls back to
\[ \frac{(-1)^k k!}{(2k+1)!} \cdot \frac{2^{2k+1} k! \Gamma(k+1/2)^2}{\Gamma(2k+1)^2} \cdot \frac{1}{k! 2^{2k+1}} \text{tr} \left( dF \right)^{2k} \]
\[ = \left( 2i \pi \right) c_{\chi_k} \]
Consider now the maps occurring in the above version of periodicity, but let's avoid assuming $L^2 = I$. Consider invertible $L = i(T^*T)$ and the pull-back of $\chi_1$ under the map $(t,T) \mapsto$ graph $tT$. One gets the form

$$ \frac{1}{2} \operatorname{tr}_S \left( \frac{1}{1-t^2L^2} d(tL) \right)^2. $$

Then

$$ \int_0^\infty \frac{1}{2} \operatorname{tr}_S \left( \frac{1}{1-t^2L^2} (dtL + t dL) \frac{1}{1-t^2L^2} (dtL + t dL) \right) dt = \int_0^\infty dt \left\{ \frac{1}{(1-t^2L^2)^2} tL \right\} dL \right\} + \left( \frac{1}{2} \operatorname{tr}_S \left( \frac{1}{1-t^2L^2} L^{-1} dL \right) \right)_{x=0}^4 

= \left[ \frac{1}{2} \operatorname{tr}_S \left( \frac{1}{1-t^2L^2} L^{-1} dL \right) \right]_{x=0}^4 

= -\frac{1}{2} \operatorname{tr}_S \left( L^{-1} dL \right). 

The minus sign is correct because

$$ L = i( T^* + T^* a) \quad L^{-1} = (-i)( T^* a^* + T^* a) $$

$$ dL = i( dT a^* + dT^* a) $$

$$ L^{-1} dL = -T^{-1} dT a a^* - (T^*)^{-1} a^* a $$

$$ = -(T^* T) \left( \begin{array}{c} 0 \\ \psi(t, \eta) \end{array} \right) $$

So

$$ -\frac{1}{2} \operatorname{tr}_S \left( L^{-1} dL \right) = \frac{1}{2} \left( \operatorname{tr} (T^* T) - \operatorname{tr} (T^* T^* T) \right). $$

Here's the same calculation done with the Gaussian version

$$ \operatorname{tr}_S \left( e^{\frac{t^2}{2} + t dL + dL} \right) = \operatorname{tr}_S \left( e^{\frac{t^2}{2} + dL} \right) e^{t dL} + \operatorname{tr}_S \left( e^{\frac{t^2}{2} + dL} \right) e^{t dL} + \operatorname{tr}_S \left( e^{\frac{t^2}{2} + dL} \right) e^{t dL} + \operatorname{tr}_S \left( e^{\frac{t^2}{2} + dL} \right) e^{t dL} $$
Then
\[ \int_0^\infty dt \beta_t = \int_0^\infty dt \operatorname{tr} \left\{ \frac{e^{\frac{1}{2} \hat{L}^2} \hat{L}^2 \hat{L}^{-1} d\hat{L}}{\frac{1}{2} \partial_t e^{\frac{1}{2} \hat{L}^2}} \right\} = -\frac{1}{2} \operatorname{tr} (\hat{L}^{-1} d\hat{L}) \]

Here's the odd version treated above in Gaussian fashion, but now using the Cayley map:

\[ t, T \mapsto \frac{1 + tT}{1 - tT} = g \]

The pull back of \( \operatorname{tr} (g^{-1} dg) \) is

\[ \operatorname{tr} \left\{ 2 \frac{1}{1 - t^2 T^2} d(tT) \right\} \]

So

\[ \int_0^\infty \operatorname{tr} \left\{ 2 \frac{1}{1 - t^2 T^2} dt T \right\} = \int_0^\infty dt \operatorname{tr} \left\{ \frac{1}{1 + tT} + \frac{1}{1 - tT} T \right\} \]

\[ = \left[ \operatorname{tr} \left\{ \log \left( \frac{1 + tT}{1 - tT} \right) \right\} \right]_0^\infty \]
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So far we understand the behavior of periodicity and differential forms provided we stick to models of $U$ and $2 \times BU$ using involutions and projectors. This is why Connes theory reduces to $F$ satisfying $F^2 = 1$.

But it is natural to expect that if we enlarge attention from unitaries to invertibles, i.e. from $U$ to $GL(C)$, then perhaps the continuous cohomology of $GL(C)$ enters into the behavior of differential forms. This in turns suggests looking at groups of complex gauge transformations, and to get started, with complex gauge transformations over the circle and a Riemann surface.

Let's consider the example of $\tilde{\mathcal{D}}$ operators of index 0 on $S^2$. The group $\tilde{\mathcal{G}}_c$ of complex gauge transformations acts transitively on the open set of invertible $\tilde{\mathcal{D}}$-operators. Since $\tilde{\mathcal{G}}_c$ is connected, questions about determinants should be answerable using the anomaly formulas and then integrating.

Let's restrict gauge transformations to be fixed at $\infty$, and then consider the following setup. We consider operators $\tilde{\partial}_\epsilon + \chi(z)$ over $C$ with $\chi$ decaying rapidly at $\infty$. $\tilde{\mathcal{G}}_c$ consists of matrix functions $\phi(z)$ which approach 1 rapidly at $\infty$. Then the operators we consider are those which are gauge-equivalent to $\tilde{\partial}_\epsilon$:

$$\tilde{\partial}_\epsilon + \chi = \phi \tilde{\partial}_\epsilon \phi^{-1}$$

What are the general considerations? Let's go back to $S^2$ with $A = \text{all Dirac ops of index zero}$ and $\tilde{\mathcal{G}} = \text{all complex gauge transformations fixing the fibre}$
at \( \infty \). We then know that \( \mathcal{S}_c \) acts simply transitively on the invertible Dirac operators.

Now we consider the determinant line bundle \( L \) over \( A \) with its canonical section and \( \mathcal{G}_c \)-action. Topologically this line bundle represents an element of \( H^1(\mathcal{G}_c, \mathbb{C}) \cong H^1(\mathcal{G}_c, \mathbb{Z}) \). To realize this class on \( \mathcal{G}_c \) we trivialize the line bundle \( L \) over \( A \); then over a \( \mathcal{G}_c \)-orbit one has two trivializations – more precisely given \( A_0 \in A \) and \( g \in \mathcal{G}_c \) one has two ways to go from \( L_{A_0} \) to \( L_{gA_0} \), and hence an element \( f(g) \in \mathbb{C}^* \). Thus one gets a map 
\[
f: \mathcal{G}_c \longrightarrow \mathbb{C}^* \]
realizing the class in \( H^1(\mathcal{G}_c, \mathbb{C}) \). If \( A_0 \) is in the invertible set, then the trivialization of \( L \) maps the canonical section \( \sigma \) into a determinant function \( \det(A) \), and 
\[
f(g) = \frac{\det(gA_0)}{\det(A_0)}.
\]

Question: How do we produce a differential 1-form on \( \mathcal{G}_c \) realizing the class in \( H^1(\mathcal{G}_c, \mathbb{C}) \)?

To answer this we can use the Atiyah–Singer formula. Consider over \( \mathcal{G} \times M \) the connection on \( \mathfrak{g}_*^*(E) \) obtained from the map 
\[
\mathcal{G} \to A, \quad g \mapsto gD_{A_0}g^{-1}
\]
and the connection over \( \mathcal{G} \times M \) which descends. This connection is 
\[
\delta + gDg^{-1} + gD_{A_0}g^{-1} = g(\delta + D_{A_0})g^{-1}
\]
and its curvature is 
\[
g : F_{A_0}g^{-1}.
\]
Better put 
\[
D_{A_0} = d_M + A_0,
\]
so that we are dealing with the connection 
\[
\delta + d_M + A = \delta + d_M + gDg^{-1} + d_M g^{-1} + gA_0 g^{-1}
\]
We want to join this to the connection 
\[
d = \delta + d_M
\]
which will give the following transgression formula
\[ \frac{1}{2} \text{tr} \left( F_A^2 \right) = \int_0^1 dt \text{ tr} \left( F_{tA} A \right) \]

0 because \( F_A^2 = g_A A g^{-1} \) has degree \( > \text{deg} M \).

Instead of passing \( A \) to \( g^{-1} d A_0 g^{-1} \) it might be easier to join \( g^{-1} d g \) to \( d A_0 \).

\[ F_t = t F_A + (t^2 - t) A_0 - \int g^{-1} d g \]

Now
\[ F_{tA} = t F_A + (t^2 - t) A^2 \]

A = g d g^{-1} + g A g^{-1}

so
\[ \int_0^1 dt \text{ tr} \left( F_{tA} A \right) = \frac{1}{2} \text{ tr} \left( F_A A \right) - \frac{1}{6} \text{ tr} \left( A^3 \right) \]

However it seems to me that I want to take \( A_0 = 0 \) in which case we are looking at \( A = g d g^{-1} = -d g g^{-1} \) and the three form over \( G \times M \) is \( \frac{1}{6} \text{ tr} \left( d g^{-1} \right)^3 \). This is then integrated over \( M = S^2 \) to get the 1-form on \( G \).

But what confuses me is the calculations which produce the \( S \)-model kinetic energy in the determinant. This is present even in the \( U(1) \) case where the topology is trivial.
General problem: Continuous cohomology \& periodicity.

One knows that the continuous cohomology of \( GL_n, \mathbb{C} \) is periodic, and given by the familiar differential forms of cyclic theory. The differential form theory should be more than just the topological K-theory.

Let's try to unify what we know about the dilogarithm. It occurs in two forms described to me by Atiyah. The imaginary part of the dilog gives the volume of a tetrahedron in hyperbolic 3-space. The real part has to do with \( \eta \)-invariants and Chern-Simons terms for 3-manifolds.

Also in Helfand-Macpherson the dilog occurs relative to \( G_2^2(\mathbb{R}) \).

Let's start with the simplest case we understand. Let us consider a flat \( G = SL(2, \mathbb{C}) \)-bundle over \( M \). Let \( P \) be the principal bundle and form the hyperbolic space bundle \( P \times \mathbb{H}^3 = X \), where \( \mathbb{H}^3 = SL(2, \mathbb{C})/SU(2) \).

Because of the flat connection, the volume form on \( \mathbb{H}^3 \) gives rise under the Weil homomorphism to a closed 3-form on \( X \). As the fibre is contractible, \( X \) has a section \( \sigma \). The bundle \( X/\sigma = \sigma \times \mathbb{H}^3 \) and the convex structure of \( \mathbb{H}^3 \) there is a natural way to map \( \sigma \) to \( \mathbb{H}^3 \) given where the vertices are supposed to go.
The actual 3-cochain on $M$ we get this way assigns to a 3-simplex $\sigma = (v_0, v_1, v_2, v_3)$, the volume of the tetrahedron in hyperbolic space with the vertices where $v_0, v_1, v_2, v_3$ go under the trivialization of $X$ over $\sigma$. The cocycle condition will follow from the invariance of the volume and the fact the form is closed:

\[
\int_{\sigma} f^* \omega = \int_{\sigma} df^* \omega = \int_{\sigma} f^* d\omega = 0
\]

Now the volume of a tetrahedron with vertices at the 2-sphere boundary of $H^3$ is defined. One should start by saying that $H^3$ has a compactification by $\mathbb{P}^1(\mathbb{C}) = S^2$ just as the upper half plane is compactified by adding $\mathbb{P}^1(\mathbb{R})$.

Hence if we started constructing our section by choosing the vertices to lie at the boundary, then we get a much simpler formula for the cocycle.

What is happening I think is the following. The group $G$ acts on the 3-disk obtained by adding $S^2$ to $H^3$. (This action should be continuous at least.) We are constructing a section of the disk bundle with fibre $H^3$. Now we get a 3-cocycle which assigns to each 3-simplex $(v_0, ..., v_3)$, the volume of the tetrahedron with vertices on the boundary of $H^3$ associated to the trivialization of $X|_{\sigma}$.  

\[
\int_{\sigma} f^* \omega = \int_{\sigma} df^* \omega = \int_{\sigma} f^* d\omega = 0
\]
Recall that given 4 distinct points \((z_0, \ldots, z_4)\) in \(S^2 = \mathbb{CP}^1\) they have a cross-ratio which determines completely the orbit of \(SL(2, \mathbb{C})\) acting on 4-tuples of distinct points. The volume of the tetrahedron in \(H^3\) with the vertices \((z_0, \ldots, z_4)\) must therefore be a function of the cross-ratio. This is supposedly the Lobachevsky function.
Cone's description of the multiplicative map on $K_1 \text{alg}(a)$ associated to a trace $\tau$ in $A$:

$$K_2^{top}(a) \to K_1^{rel}(a) \to K_1^{alg}(a) \to K_1^{top}(a)$$

A trace induced a homomorphism $K_1^{rel}(a) \to \mathbb{C}$ as follows. Given an element of $K_1^{rel}(a)$, it is represented by a unit $u \in GL(a)$ together with a path $u(t)$ joining 1 to $u$. Then one obtains a number

$$\int_0^1 \tau(u_t^{-1}u) \, dt \in \mathbb{C}.$$  

If we compare the effect of two different paths joining 1 to $u$, i.e. take a loop $u(t)$ in $GL(a)$, then we obtain the ambiguity

$$\oint \tau(u_t^{-1}du).$$

By Bott periodicity this number is $2\pi i \tau(\alpha)$ where $\alpha \in K_0^A$ is the element which corresponds to the loop $u = (u_t)$.

(Here we see again how a trace determines an element of $(K_2^{top}(a))^*$ in the van Est picture, rather than an element of $(K_0^A)^*$ as one would expect.)

On the Movshev conjecture: Movshev's basic idea is to generalize the concept of representation of a discrete group $\Gamma$ as follows. One considers a pair of Hilbert space representations $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ together with a Fredholm $F: \mathcal{H}^+ \to \mathcal{H}^-$ which commutes with the $\Gamma$ action modulo compacts. Such data $(\rho^+, \rho^-, F)$ define...
in element of $K^0(B\Gamma)$ for which the Novikov property holds: Given a map $M \rightarrow B\Gamma$ of $M$ compact and orientable, then the image of $\mathcal{L}(M)$ in $H^\ast(M)$ under the map paired with $ch(x)$ is a homotopy invariant of $M$. I still don't see this last property clearly.

If $\Gamma$ is the fundamental group of a negatively curved manifold, then Mishchenko proves that all of $K^0(B\Gamma)$ can be so represented. The idea is to use the Hilbert space of $L^2$-forms on $\widetilde{M}$ and the "Gaussian de Rham" complex.

So let's try to understand this in the case of a torus $\mathbb{R}^n/\Gamma$, say $n = 2$. Then in order to define the Gaussian de Rham complex I have to choose a base point of $C = \mathbb{R}^2$, any translation commutes with the operator up to a bounded function, so we clearly get the Mishchenko situation, although unbounded.

Now it is supposedly possible to show that any element of $K^0(\text{torus})$ arises from such a Mishchenko representation. It's clear that the ring of functions on the torus acts on the Hilbert space of $L^2$-forms upstairs and commutes with the Gaussian part of the operator. A function commutes with the de Rham part $d + \delta$ modulo a bounded function, which should be compact. So there is a Kasparov cup product setup — given a bundle $E$ on the torus one can tensor with the Gaussian OR complex to obtain a Mishchenko representation.

It must be then easy to see that upon going back to $K^0(B\Gamma)$ we get the class of $E$. 