May 25 – June 2, 1984 851–884

May 25, 1984

$F = 2e - 1$. Then the first Character form

$$\operatorname{tr} e(de)^2 = \frac{1}{8} \operatorname{tr}(F(dF)^2)$$

gives rise to the cyclic cocycle

$$F[F,b] = -[F,b]F$$

$$\frac{1}{8}\operatorname{tr}(F[F,a][F,b]) = \frac{1}{8}\operatorname{tr}((F^2a - FaF)[F,b])$$

$$= \frac{1}{4}\operatorname{tr}(a[F,b]) = \frac{1}{2}\operatorname{tr}(a[e,b])$$

which we simplify to $\operatorname{tr}(a[e,b])$.

I want to calculate this 1-cocycle in various examples.

First consider the case of the circle where e is the projector on $L^2(S^1)$ with image the Hardy space spanned by the z^n $n \geq 0$. Then

$$e(z,z') = \frac{1}{2\pi}\sum_{n\geq 0} z^n \overline{(z')^n} = \frac{1}{2\pi}\frac{1}{1-\frac{z}{z'}}$$

analytically continued from $|z| < |z'| = 1$.

so the kernel of $[e,b]$ is

$$[e,b](z,z') = \frac{1}{2\pi}\frac{b(z') - b(z)}{1 - \frac{z}{z'}}$$

$$z = e^{i\theta} \qquad \frac{dz}{iz} = d\theta$$

~~so~~

$$\operatorname{tr} a[e,b] = \int \frac{d\theta}{2\pi}\, a(z)\, z\frac{d}{dz} b(z)$$

$$= \frac{1}{2\pi i}\oint a\, db$$

Second do the same computation on the line where

$$(ef)(x) = \int_0^\infty \frac{d\xi}{2\pi}\, e^{ix\xi}\, \hat f(\xi) = \int dy \left(\int_0^\infty \frac{d\xi}{2\pi}\, e^{i\xi(x-y)} \right) f(y)$$

$$= \int dy \; \frac{i}{2\pi}\, \frac{1}{x-y+i0^+}\; f(y)$$

Then $\displaystyle ([e,b])(x,y) = \frac{1}{2\pi i}\, \frac{b(x)-b(y)}{x-y}$ so

$$\operatorname{tr} a[e,b] = \int \frac{dx}{2\pi i}\, a(x)\, b'(x)$$

Thirdly I want to consider the holomorphic representation.

Digression to calculate the analogue for the plane of the Hilbert operator. This is multiplication by $\frac{w}{|w|}$ on the Fourier transform side. So I need the Fourier transform of $\frac{w}{|w|}$. Put

$$f(z,k,s) = \int \frac{d^2w}{\pi}\, e^{\bar w z - w \bar z}\, \frac{1}{w^k |w|^{2s}}$$

Then

$$f(tz,k,s) = \int \frac{d^2 w}{\pi}\, e^{\bar w t z - w \bar t \bar z}\, \frac{1}{w^k |w|^{2s}}$$

$$= \frac{1}{|t|^2} \int \frac{d^2w}{\pi}\, e^{\bar w z - w \bar z} \left(\frac{\bar t}{w}\right)^k \left|\frac{\bar t}{w}\right|^{2s}$$

$$= \frac{t^k}{t^k}\, \bar t^k\, |t|^{2s-2}\, f(z,k,s) = \frac{1}{t^k |t|^{2(1-s-k)}}\, f(z,k,s).$$

Thus setting $t \to z$, $z \to 1$

$$f(z,k,s) = \frac{f(1,k,s)}{z^k\,|z|^{2(1-s-k)}}$$

(As a check recall that for $k=1$, $s=0$ we have that $\frac{1}{w} \longmapsto \text{const}\ \frac{1}{z}$ under F.T.)

To determine the constant $f(1,k,s)$ first take $k=0$:

$$\int \frac{d^2w}{\pi}\, e^{\bar{w}z - w\bar{z}}\, \frac{1}{|w|^{2s}} = \int \frac{d^2w}{\pi}\, e^{\bar{w}z - w\bar{z}}\, \frac{1}{\Gamma(s)} \int_0^\infty e^{-t|w|^2}\, t^s \frac{dt}{t}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{dt}{t} \int \frac{d^2w}{\pi}\, e^{\bar{w}z - w\bar{z} - t|w|^2}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty t^s\, \frac{dt}{t}\, \frac{1}{t}\, e^{-\frac{|z|^2}{t}} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{dt}{t}\, t^{1-s} e^{-t|z|^2}$$

$$= \frac{1}{\Gamma(s)}\, \frac{\Gamma(1-s)}{|z|^{2(1-s)}} \qquad \therefore\ f(1,0,s) = \frac{\Gamma(1-s)}{\Gamma(s)}$$

Next differentiate to get the other values of k.

$$\int \frac{d^2w}{\pi}\, e^{\bar{w}z - w\bar{z}}\, \frac{(+w)^\ell}{|w|^{2s}} = (-1)^\ell\, \partial_{\bar{z}}^\ell\, (\bar{z}^{s-1})\, z^{s-1}\, \frac{\Gamma(1-s)}{\Gamma(s)}$$

$$= (-1)^\ell (s-1)\cdots(s-\ell)\, \bar{z}^{\,s-1-\ell}\, \frac{z^{s-1-\ell}}{z^{-\ell}}\, \frac{\Gamma(1-s)}{\Gamma(s)}$$

$$= \frac{\Gamma(1-s)(1-s)(2-s)\cdots(\ell-s)}{\Gamma(s)}\, \frac{1}{z^{-\ell}\,|z|^{2(1-s+\ell)}}$$

or

$$\boxed{\int \frac{d^2 w}{\pi}\, e^{\bar{w} z - w \bar{z}}\, \frac{1}{w^{-k}\,|w|^{2s}} = \frac{\Gamma(1-s-k)}{\Gamma(s)}\, \frac{1}{z^{k}\,|z|^{2(1-s-k)}}}$$

Check: Take $s = 0$, $k = 1$. Recall that

$$\partial_{\bar z} \frac{1}{\pi z} = \delta(z)$$

and that

$$\partial_{\bar z} \int \frac{d^2 w}{\pi}\, e^{\bar w z - w \bar z}\, \frac{1}{w} = -\int \frac{d^2 w}{\pi}\, e^{\bar w z - w \bar z} = -\pi\,\delta(z)$$

the point being that $\bar w z - w \bar z = 2i(w_x z_y - w_y z_y)$ and

$$\delta(z) = \int \frac{d^2 w}{(2\pi)^2}\, e^{i(w_x z_x + w_y z_y)} = \int \frac{d^2 w}{\pi^2}\, e^{\bar w z - w \bar z}$$

Thus

$$\int \frac{d^2 w}{\pi}\, e^{\bar w z - w \bar z}\, \frac{1}{w} = -\frac{1}{z}$$

which agrees with the above.

May 26, 1984

I have decided it is important to get some feeling for the operators in Kasparov theory. This theory is modelled on ψDO's of order 0 in several ways. In particular, the natural way in that theory to think of the fundamental K-homology class of $\mathbb{C}$ is using the ψDO $P = |\partial_{\bar{z}}|^{-1}\partial_{\bar{z}}$ of order zero instead of the Dirac operator $\partial_{\bar{z}}$. (Other possibilities are $\left(1+(\partial_{\bar{z}})^*\partial_{z}\right)^{-1/2}\partial_{\bar{z}}$.)

So it is important to calculate the cyclic cocycles attached by Connes to these operators. Yesterday it occurred to me to use the Fourier transform to essentially simplify P. Then mult. by functions becomes convolution, and so can be dealt with in terms of translations. The a cyclic cocycle on a convolution algebra is some sort of distribution on the group, maybe a group cocycle, and this will explain some more examples in Connes theory.

In any case the idea of using the F.T. to calculate the cocycles seems to be very good. It seems close to physicists index calculations, and hopefully will lead to ~~[illegible]~~ a better understanding of asymptotic evaluations of cocycles.

So the general problem now is cocycle computation for constant coefficient operators on $\mathbb{R}^n$. We work in the dual picture. Multiplication ~~on the F~~ by e^{ikx} corresponds on the F.T. side to the translation operator τ_k given by

$$(\tau_k f)(\xi) = f(\xi - k).$$

Check:

$$e^{ikx}\tilde{f}(x) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i(k+\xi)x} f(\xi) = \int \frac{d^n\xi}{(2\pi)^n}\, e^{i\xi x} f(\xi-k)$$

We have the following relations between translation + mult.

$$\tau_k \circ g \circ \tau_{-k} = (\tau_k g)$$

which I will write

$$\tau_k \cdot g = g_k \cdot \tau_k \qquad g_k = \tau_k g.$$

This is because τ_k is an algebra homomorphism.

Let's look at the cocycle for the Hilbert projector in this notation, call it e; then e is given on the F.T. side by multiplying by the Heaviside fn $\Theta(\xi)$.

It just occurred to me that I am working in the usual Weyl algebra situation where the alg. is generated by the operators $\tau_k e^{ib\xi} = e^{ikx} e^{ib\xi}$. What is the trace on this algebra?

$$\operatorname{tr}\left(\int dk\, f(k)\, \tau_k\right) g(\xi)$$

$$= \int dx \left(\int dk\, e^{ikx} f(k)\right) \int \frac{d\xi}{2\pi} g(\xi)$$

$$= 2\pi f(0) \int \frac{d\xi}{2\pi} g(\xi) = f(0)\int d\xi\, g(\xi)$$

Thus

$$\operatorname{tr}\left(\tau_k \cdot g(\xi)\right) = \delta(k) \int d\xi\, g(\xi)$$

So now let's use this to compute $\operatorname{tr}(a[e,b])$ where a, b are these translation operators.

$$\tau_k [\theta(\xi), \tau_\ell] = \tau_k \theta \tau_\ell - \tau_{k+\ell}\theta$$
$$= \tau_{k+\ell}(\theta_{-\ell} - \theta)$$

so
$$\operatorname{tr} \tau_k [\theta(\xi), \tau_\ell] = \delta(k+\ell)\int \frac{d\xi}{} (\theta(\xi+\ell) - \theta(\xi))$$
$$= \delta(k+\ell)\frac{\ell}{}$$

which yields
$$\operatorname{tr} a[e,b] = \int \frac{dk}{2\pi}\hat{a}(k)\tau_k \left[e, \int\frac{d\ell}{2\pi}\hat{b}(\ell)\tau_\ell\right]$$
$$= \int \frac{dk}{2\pi}\frac{d\ell}{2\pi} \hat{a}(k)\hat{b}(\ell) \; \delta(k+\ell)\frac{\ell}{}$$
$$= \frac{1}{2\pi i}\int \frac{d\ell}{2\pi}\hat{a}(-\ell)\, i\ell\, \hat{b}(\ell) = \frac{1}{2\pi i}\int a b' dx$$

Next I want to use this in n-dimensions where e is a ψDO of zero-th order, and where there are more commutators. Take $n=2$ and let P be multiplication by $\rho(\xi)$ where ρ is $\xi/|\xi|$, or $\xi/\sqrt{1+|\xi|^2}$, or something similar. We want to calculate

$$\phi(a,b,c) = \operatorname{tr} P^{-1}[P,a]\, P^{-1}[P,b]\, P^{-1}[P,c].$$

Now taking a to be an exponential e^{ikx}, then one has

$$P^{-1}[P, a] = \rho^{-1}[\rho, \tau_k] = \tau_k - \rho^{-1}\tau_k \rho$$

$$= \tau_k \underbrace{\left(\rho^{-1} - \rho_{-k}^{-1}\right)\rho}_{\sigma(k,\,\xi)}$$

so

$$P^{-1}[P,a] \cdots P^{-1}[P,c] = \tau_{k+l+m}\, \sigma(k, l+m+\xi)\, \sigma(l, m+\xi)\, \sigma(m, \xi)$$

and so the trace we are interested in is

$$\delta(k+l+m) \int d^2\xi \;\; \sigma(k, l+m+\xi)\, \sigma(l, m+\xi)\, \sigma(m, \xi)$$

where

$$\sigma(k, \xi) = 1 - \frac{\rho(\xi)}{\rho(\xi+k)} .$$

Is it clear the integral makes sense? We have to understand the behavior of σ for large ξ.

To simplify notation let's change ρ into ρ^{-1}. The problem is now the behavior of

$$\frac{\rho(\xi+k)}{\rho(\xi)} - 1 \qquad \text{as} \quad |\xi| \longrightarrow \infty .$$

Assume ρ is a smooth homogeneous function of degree zero outside a compact, i.e. essentially a smooth function on the sphere extended constant in the radial direction. Notice that then

$$\rho(\xi+k) = \rho\left(\frac{\xi+k}{|\xi|}\right) = \rho\left(\frac{\xi}{|\xi|} + \frac{k}{|\xi|}\right)$$

and now one can use the Taylor series of ρ to get a complete asymptotic expansion:

$$\rho(\xi+k) = \rho\left(\frac{\xi}{|\xi|}\right) + \rho^{(j)}\left(\frac{\xi}{|\xi|}\right)\frac{k_j}{|\xi|} + \cdots\cdots$$

In fact the Taylor series,

$$\rho(\xi+k) = \rho(\xi) + \rho^{(j)}(\xi)\, k^j + \cdots$$

is the required asymptotic expansion, since $\rho^{(\alpha)}(\xi)$ is homogeneous of degree $-|\alpha|$.

Thus I know that

$$\frac{\rho(\xi+k)}{\rho(\xi)} - 1 = k^\mu \frac{\partial}{\partial \xi^\mu} \log \rho + O\left(\frac{1}{|\xi|^2}\right)$$

and so the integrand is $O\left(\frac{1}{|\xi|^3}\right)$, and the integral converges.

Next, I want to put in a parameter and perform the asymptotic evaluation

May 27, 1984

Today I want to see how much I can understand about the family of $\bar\partial$-operators on a torus $\mathbb{C}/\Gamma$.

May 28, 1984

Let's discuss the general picture. Start with a Dirac operator on M an even dim'l manifold which is compact, e.g. $\mathbb{C}/\Gamma$. Think of this operator as a cycle for K-homology: It can be paired with a vector bundle over M to get a Fredholm operator, i.e. an element of the K-theory of a point.

More precisely given a v.b. E we get a family of Dirac operators with coeffs in E where the parameter space is the space $\mathcal{A}$ of connections on E. The ~~whole~~ group $\mathcal{G}$ of gauge transformations acts on the Hilbert space where the operators live.

Better: Think of the given operator as a cycle in K-theory: any vector bundle over M can be "integrated" to get a K-class over a point. More generally any family of vector bundles over M can be integrated to get a ~~[illegible]~~ K-class on the parameter space.

The universal family of vector bundles over M all bundles of which are isomorphic to E has parameter space $B\mathcal{G}$. There is a tautological bundle $\tilde{E}$ over $B\mathcal{G}\times M$ and hence a canonical element of $K(B\mathcal{G})$.

$\mathcal{G}$ = group of gauge transformations of the bundle E; it acts as unitary operators on $L^2(M, S\otimes E) = \mathcal{H}$. The Dirac operator of interest is an odd degree operator on $\mathcal{H}$ and we consider its $\mathcal{G}$-orbit.

Let L be the Dirac operator. If L^2 were to be I, then $[L,\ ]$ would be a differential and one would have Connes situation. In general L^2 is like a curvature that probably has to be incorporated into the calculations. If I pass to the classical limit, then L^2 becomes $-p^2+F$, where p^2 behaves like a scalar; it sort of passes out in front to give the $\frac{1}{2\pi}$ factors.

So let me try to do some calculations. I want to concentrate on the Dirac operator over $\mathbb{C}/\Gamma$ and the second character class

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Review the cohomological ideas. Over $\mathcal{A}\times M$ is the bundle $pr_2^*(E)$ with its tautological connection which is flat in the $\mathcal{A}$-direction. Tensoring with the Dirac operator on M gives the family of operators on M parametrized by $\mathcal{A}$. Now the group $\mathcal{G}$ acts on the situation so that we have an equivariant family. The character of the index of the family is computed as an equivariant form on $(\mathcal{A}, \mathcal{G})$.

If $\dim M = 2m = n$, then the index thm. for families says that $ch_{m+1}(\text{index}) = \int_M \left(\hat{A}(M)\, ch(\,.\,\tilde{E})\right)_{n+1}$. Now if we forget the $\mathcal{G}$-action, i.e. lift from "$\mathcal{A}/\mathcal{G}$" to $\mathcal{A}$, then we can use the tautological connection to compute the character and we know by Bott's thm that we get the zero form. Thus ch_{n+1}(index) lifted to $\mathcal{A}$ is the coboundary of an invariant form of degree $n+1$. Restricting to a $\mathcal{G}$-orbit gives an invariant form on $\mathcal{G}$ of degree $n+1$, hence a cyclic n-cocycle.

So what I want to do is to take the form on $\mathcal{A}$ representing the character of the index computed using the tautological connection on $pr_2^*(E)$ over $\mathcal{A} \times M$, and somehow write the components of degrees $n+2, n+4, \dots$ (or perhaps some combination of these representing $ch_{m+1}, ch_{m+2}, \dots$) as coboundaries of equivariant forms on $\mathcal{A}$.

The character form is

$$tr_s \left\{ e^{u(L^2 + dL)} \right\}$$

where u is a parameter so I can select the forms representing ch_k. Here L is the given Dirac operator, and dL denotes its variation considered as a 1-form on $\mathcal{A}$ with values in $\mathrm{End}(\mathcal{H})$. Recall

$$L = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + A_\mu)$$

$$dL = \gamma^\mu \cdot \delta A_\mu$$

In the classical limit $\gamma^\mu \longmapsto dx^\mu$ so the above form on $\mathcal{A}$ has components of degree $\leq n = \dim M$.

May 29, 1984

I am studying the form

$$\mathrm{tr}_s\ e^{u(L^2+dL)}$$

where L is a family of odd degree, skew-adjoint endos of a super vector space $\mathcal{H}$. In practice L is the family of Dirac operators $D\!\!\!/_A$ as A ranges over the space of connections; here $\mathcal{H}$ is $L^2(M, S\otimes E)$.

A simpler example I have studied is the family $L = i\gamma^\mu x^\mu$ on S as x^μ ranges over $\mathbb{R}^n$. Then I get the form

$$\mathrm{tr}_s\, e^{u(-|x|^2 + i\, dx^\mu \gamma^\mu)} = e^{-u|x|^2}(ui)^n\, \mathrm{tr}_s\,(dx^1\gamma^1 \cdots dx^n\gamma^n)$$

$$= e^{-u|x|^2}(ui)^n\, dx^1 \cdots dx^n\, \mathrm{tr}_s(\gamma^n \cdots \gamma^1)$$

$$= e^{-u|x|^2} u^n dx^1 \cdots dx^n\ i^n (-1)^{\cancel{n(n-1)/2}\ m}(2i)^m$$

$$= (2i)^m e^{-u|x|^2} u^n dx^1 \cdots dx^n$$

which when integrated over $\mathbb{R}^n$ gives

$$(2i\pi)^m u^m$$

A related example is the Chern character form for $i_!1$ where $i: M \to E$ is the zero section in a complex line bundle. I recall that this character form has the shape

$$\mathrm{ch}(i_!1) = i_*1 \cdot \pi^*\left(\frac{1-e^{u\omega}}{u\omega}\right)$$

where ω is the curvature of E and the Thom form i_*1 is given by

$$u \cdot d\left\{e^{-u|z|^2}\left(\frac{dz}{z} + \theta\right)\right\}$$

where z is a fibre coordinate on E and $\frac{dz}{z}+\theta$ is the connection form on E-(0 section).

$$d\left\{e^{-u|z|^2}\left(\frac{dz}{z}+\theta\right)\right\} = e^{-u|z|^2}d\theta + e^{-u|z|^2}(-u)(\bar{z}dz + z d\bar{z})\left(\frac{dz}{z}+\theta\right)$$

$$= e^{-u|z|^2}\left\{ d\theta - u\left[d\bar{z}dz + \bar{z}dz\theta + z d\bar{z}\theta\right]\right\}$$

$$= e^{-u|z|^2}\left\{\omega + u(dz+\theta z)(d\bar{z}-\bar{z}\theta)\right\}$$

(This should be multiplied by u to fit with the character form.)

So what I learn is that when dealing with these character forms, the forms themselves have complicated u-dependence. However the cohomology classes have a simple polynomial dependence. The same sort of behavior will appear when one takes a transform with respect to u. For example, take the Laplace transform

$$\int_0^\infty e^{-\lambda u}\, tr_s\, e^{u(L^2+dL)}\,du = tr_s\left(\frac{1}{\lambda - L^2 - dL}\right)$$

e.g.

$$\int_0^\infty du\, e^{-\lambda u}\, e^{-u|x|^2} u^n d^n x = \frac{\Gamma(n+1)}{(\lambda+|x|^2)^{n+1}}\, d^n x$$

As a form this has complicated λ dependence, but as a cohomology class it is a multiple of $\frac{1}{\lambda^{n+1}}$.

Now one thing nice about the Clifford mult. example is that L is unbounded in some sense at ∞, hence u is in some sense forced to stay in $Re(u) > 0$.

Let's return to the differential form

$$\operatorname{tr}_s e^{u(L^2+dL)}$$

and take the Laplace transform

$$\operatorname{tr}_s \frac{1}{\lambda - L^2 - dL}$$

which can then be expanded to gives forms of each degree

$$\operatorname{tr}_s \frac{1}{\lambda - L^2} + \operatorname{tr}_s\left(\frac{1}{\lambda-L^2}\, dL \,\frac{1}{\lambda-L^2}\right) + \cdots$$

Only the even degree terms can be $\neq 0$ since L is of odd degree. These forms are closed and their DR class are independent of L.

But maybe an important point is that the DR class of

$$\operatorname{tr}_s\left\{\left(\frac{1}{\lambda-L^2}\, dL\right)^{2k} \frac{1}{\lambda-L^2}\right\}$$

is a constant times $\frac{1}{\lambda^{k+1}}$. This is clear from the fact we know the DR class is ind. of L as follows.

$$\operatorname{tr}_s\left\{\left(\frac{1}{\lambda-L^2}\, dL\right)^{2k} \frac{1}{\lambda-L^2}\right\} \equiv \operatorname{tr}_s\left\{\left(\frac{1}{\lambda-\lambda L^2}\, d\sqrt{\lambda} L\right)^{2k} \frac{1}{\lambda-\lambda L^2}\right\}$$

$$= \operatorname{tr}\left\{\left(\frac{1}{1-L^2}\, dL\right)^{2k} \frac{1}{1-L^2}\right\} \underbrace{\frac{\lambda^k}{\lambda^{2k+1}}}_{\frac{1}{\lambda^{k+1}}}$$

Now how can I use this to advantage? By using the homotopy from L to $\sqrt{\lambda}\, h L$ and letting $h \to 0$ I obviously get some sort of coboundary expression for $tr_s\left\{\left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} \frac{1}{\lambda - L^2}\right\}$ in the case where the classical limit gives zero.

Review the ideas behind the Grassmannian graph construction. Let $T : M \to Hom(V^0, V^1)$, whence we get $\Gamma_T : M \to Grass^d(V^0 \oplus V^1)$, $d = \dim V^0$. Pull back the Grassmannian connection on the subbundle to get a connection on the trivial bundle with fibre V^0 over M. Let's compute the curvature.

$$\Gamma_T = Im\begin{pmatrix} 1 \\ T \end{pmatrix}$$

orth. proj. on Γ_T $\qquad e = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}$

$$1 - e = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} \frac{1}{1+TT^*} \begin{pmatrix} -T & 1 \end{pmatrix}$$

$$de = \begin{pmatrix} 0 \\ dT \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix} + \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T} \begin{pmatrix} 0 & dT^* \end{pmatrix}$$
$$+ \begin{pmatrix} 1 \\ T \end{pmatrix} d\left(\frac{1}{1+T^*T}\right) \begin{pmatrix} 1 & T^* \end{pmatrix}$$

$$de(1-e) = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T}\, dT^*\, \frac{1}{1+TT^*} \begin{pmatrix} -T & 1 \end{pmatrix}$$

$$de(1-e)de = \begin{pmatrix} 1 \\ T \end{pmatrix} \frac{1}{1+T^*T}\, dT^* \frac{1}{1+TT^*}\, dT\, \frac{1}{1+T^*T} \begin{pmatrix} 1 & T^* \end{pmatrix}$$

This is an operator on Γ_T, but if we use the isom $\Gamma_T \simeq V^0$, $\begin{pmatrix} 1 \\ T \end{pmatrix} v \longleftrightarrow v$, then we get the operator

$$\frac{1}{1+T^*T}\, dT^* \,\frac{1}{1+TT^*}\, dT$$

on V^0 for the curvature.

Next let $L = i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ on the trivial bundle with fibre $V^0 \oplus V^1$. I should write this $L = i(a^*T + aT^*)$, in order to do the calculations in the superalgebra $\Omega(M)\hat{\otimes}\operatorname{End}(V)$.

$$L^2 = -(a^*a\, TT^* + aa^*\, T^*T) = \begin{pmatrix} -T^*T & 0 \\ 0 & -TT^* \end{pmatrix}$$

so $\quad 1-L^2 = \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}$. Now

$$dL = -i(a^*\, dT + a\, dT^*)$$

where the sign comes as d, a, a^* are of odd degree.

$$dL\,\frac{1}{1-L^2}\,dL = (-1)(a^*dT + a\,dT^*)\left(a^*a(1+TT^*)^{-1} + aa^*(1+T^*T)^{-1}\right)(a^*dT + a\,dT^*)$$

$$= -a\,dT^*\, a^*a\,(1+TT^*)^{-1}\, a^*dT$$

$$- a^*\,dT\; aa^*(1+T^*T)^{-1}\, a\,dT^*$$

$$= aa^*\; dT^*(1+TT^*)^{-1}dT \;+\; a^*a\; dT(1+T^*T)^{-1}dT^*$$

Thus we see that

$$\frac{1}{1-L^2}\,dL\,\frac{1}{1-L^2}\,dL = \cancel{(1+TT^*)^{-1}dT\,(1+T^*T)^{-1}dT^*}$$

$$= \begin{pmatrix} (1+T^*T)^{-1}dT^*(1+TT^*)^{-1}dT & 0 \\ 0 & (1+TT^*)^{-1}dT(1+T^*T)^{-1}dT^* \end{pmatrix}$$

gives the curvature for Γ_T and $(\Gamma_T)^\perp$.

Now I want to check that the factorials work out all right as far as the cohomology goes. This means we have to put in an ε and define $\Gamma_T = \binom{\varepsilon}{T} V^0$. The curvature of E^0 is then

$$\frac{\varepsilon}{\varepsilon^2 + T^*T}\, dT^* \,\frac{\varepsilon}{\varepsilon^2 + TT^*}\, dT$$

which will be the V^0 part of

$$\lambda \cdot \frac{1}{\lambda - L^2}\, dL \,\frac{1}{\lambda - L^2}\, dL \qquad \text{where } \lambda = L^2.$$

Thus $ch_R E^0 - ch_R E^1$ will be represented by

?

Let's start over again to see what we are missing. Let us start with $T: E^0 \to E^1$ and its adjoint where E^0, E^1 are trivial bundles. Then we use the embedding

$$(\Gamma_T)^{\perp} : E^1 \hookrightarrow E^0 \oplus E^1$$

$$v \longmapsto \begin{pmatrix} -T^* \\ \varepsilon \end{pmatrix} v$$

so as to get a new connection on E^1. Need

$$1 - e = \begin{pmatrix} -T^* \\ \varepsilon \end{pmatrix} \frac{1}{\varepsilon^2 + TT^*} \begin{pmatrix} -T & \varepsilon \end{pmatrix} \qquad e = \begin{pmatrix} \varepsilon \\ T \end{pmatrix} \frac{1}{\varepsilon + T^*T} \begin{pmatrix} \varepsilon & T^* \end{pmatrix}$$

and then

$$de(1-e) = \begin{pmatrix} \varepsilon \\ T \end{pmatrix} \frac{1}{\varepsilon + T^*T}\, dT^* \varepsilon \,\frac{1}{\varepsilon^2 + TT^*} \begin{pmatrix} -T & \varepsilon \end{pmatrix}$$

$$d(1-e)\,de\,(1-e) = \begin{pmatrix} -T^* \\ \varepsilon \end{pmatrix} \frac{1}{\varepsilon^2+TT^*}(-dT)\varepsilon \frac{1}{\varepsilon^2+T^*T}\, dT^* \frac{\varepsilon}{\varepsilon^2+TT^*}(-T\ \varepsilon)$$

$$\therefore (1-e)\,de\,de = \begin{pmatrix} -T^* \\ \varepsilon \end{pmatrix} \frac{1}{\varepsilon^2+TT^*}(\varepsilon\, dT) \frac{1}{\varepsilon^2+T^*T}(\varepsilon\, dT^*) \frac{1}{\varepsilon^2+TT^*}(-T\ \varepsilon)$$

Thus the ~~[illegible]~~ k-th character form for E' is

$$\frac{1}{k!}\operatorname{tr}\left[\frac{1}{\varepsilon^2+TT^*}\varepsilon\, dT \frac{1}{\varepsilon^2+T^*T}\varepsilon\, dT^*\right]^k$$

and for E^0 it is

$$\frac{1}{k!}\operatorname{tr}\left[\frac{1}{\varepsilon^2+T^*T}\varepsilon\, dT^* \frac{1}{\varepsilon^2+TT^*}\varepsilon\, dT\right]^k.$$

As a check note that these two add to give zero.

The basic problem occurs already for the Bott class over $\mathbb{C}$. Suppose we apply Grass. graph to the map $\mathbb{1} \longrightarrow \mathbb{1}$ given by z. Thus we map $\mathbb{C}$ into $\mathbb{P}^1$ by $z \longmapsto (\varepsilon, z)$ and get the curvature

$$d''d' \log\left(\varepsilon^2+|z|^2\right) = d''\left\{\frac{\bar{z}\,dz}{\varepsilon^2+|z|^2}\right\}$$

$$= \frac{d\bar{z}\,dz}{\varepsilon^2+|z|^2} - \frac{z\,d\bar{z}\,\bar{z}\,dz}{(\varepsilon^2+|z|^2)^2} = \frac{\varepsilon^2\,d\bar{z}\,dz}{(\varepsilon^2+|z|^2)^2}$$

Notice that $\displaystyle\int_{\mathbb{C}} \frac{\varepsilon^2\,d\bar{z}\,dz}{(\varepsilon^2+|z|^2)^2} = \frac{\varepsilon^2\cdot\varepsilon^2}{\varepsilon^4}\int_{\mathbb{C}}\frac{d\bar{z}\,dz}{(1+|z|^2)^2}$

is independent of ε as it should be.

On the other hand, the Gaussian form is

$\operatorname{tr}_s e^{u(L^2+dL)} = e^{-u|z|^2}u^2\, d^2z\ 2i$ which integrates to

$u\cdot 2\pi i$ not independent of u.

Discussion: I know my chern character form is a good way to [redacted] do index theory for Dirac operators, and I feel that Grassmannian graph should also work well for this purpose. Moreover it would be nice if Connes cocycles fit with the Grassmannian approach. Then the problem is to relate both Gaussian + Grassmannian approaches.

In both cases [redacted] one starts with a family $L = i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ of odd operators. [redacted] We consider the map which sends $T \in \mathrm{Hom}(V^0, V^1)$ into the graph of $\varepsilon^{-1}T$, i.e. the image of $\begin{pmatrix} \varepsilon \\ T \end{pmatrix}$. Thus over the space of T we have this subbundle of $V^0 \oplus V^1$ and its curvature we found to be

$$K^0 = \frac{1}{\varepsilon^2 + T^*T}\,\varepsilon\, dT^*\,\frac{1}{\varepsilon^2 + TT^*}\,\varepsilon\, dT$$

using the isomorphism $\begin{pmatrix} \varepsilon \\ T \end{pmatrix}$ of V^0 with this subbundle. Similarly the orthogonal subbundle $\begin{pmatrix} -T^* \\ \varepsilon \end{pmatrix} V^1$ has curvature

$$K^1 = \frac{1}{\varepsilon^2 + TT^*}\,\varepsilon\, dT\,\frac{1}{\varepsilon^2 + T^*T}\,\varepsilon\, dT^*.$$

Now I also computed that

$$\lambda\left(\frac{1}{\lambda - L^2}\, dL\right)^2 = \begin{pmatrix} K^0 & 0 \\ 0 & K^1 \end{pmatrix} \qquad \text{where } \lambda = \varepsilon^2$$

Better:

$$\left(\frac{1}{\varepsilon^2 - L^2}\, dL\right)^2 = \begin{pmatrix} \frac{1}{\varepsilon^2 + T^*T}\, dT^*\,\frac{1}{\varepsilon^2 + TT^*}\, dT & 0 \\ 0 & \frac{1}{\varepsilon^2 + TT^*}\, dT\,\frac{1}{\varepsilon^2 + T^*T}\, dT^* \end{pmatrix}$$

Now my point is that I can compute the Chern character classes over the parameter space using the curvatures K^0, K^1. In this case we know that character classes of two bundles are of opposite sign. Thus

$$\mathrm{tr}\,(K^0)^k + \mathrm{tr}\,(K^1)^k = 0$$

and so

$$\begin{aligned}\lambda^k \,\mathrm{tr}_s\left(\frac{1}{\lambda - L^2}\,dL\right)^{2k} &= \mathrm{tr}\,(K^0)^k - \mathrm{tr}\,(K^1)^k \\ &= 2\,\mathrm{tr}\,(K^0)^k\end{aligned}$$

represents the ch_k class times $k!$ It's cohomology class is independent of λ

May 30, 1984

I have a new approach to a local index theorem which is based on the Grassmannian graph idea of MacPherson, et. al.. Take two trivial bundle E^0, E^1 over M and a map $T: E^0 \to E^1$ and put $L = i(T+T^*)$ as usual. We define a new connection on E^0 [redacted] by using the identification $v \mapsto \begin{pmatrix}\varepsilon \\ T\end{pmatrix} v$ of E^0 with the graph of $\varepsilon^{-1}T$ in $E^0 \oplus E^1$, and taking the induced connection [redacted] from d on $E^0 \oplus E^1$. Thus

$$D^0 v = \frac{1}{\varepsilon^2 + T^*T}(\varepsilon \;\; T^*)\, d \begin{pmatrix}\varepsilon \\ T\end{pmatrix} v$$

$$= \left\{ d + \frac{1}{\varepsilon^2+T^*T} T^* dT \right\} v$$

so the connection form is

$$\theta^0 = \frac{1}{\varepsilon^2 + T^*T} T^* dT = T^* \frac{1}{\varepsilon^2 + TT^*} dT$$

and the curvature is

$$K^0 = d\theta^0 + \theta^{0\,2} = \varepsilon^2 \frac{1}{\varepsilon^2+T^*T} dT^* \frac{1}{\varepsilon + TT^*} dT$$

Similarly we use the identification $w \mapsto \begin{pmatrix}-T^* \\ \varepsilon\end{pmatrix} w$ from E^1 to the orthogonal complement of $\begin{pmatrix}\varepsilon \\ T\end{pmatrix} E^0$, to obtained an induced connection on E^1.

$$D^1 w = \frac{1}{\varepsilon^2 + TT^*}(-T \;\; \varepsilon)\, d \begin{pmatrix}-T^* \\ \varepsilon\end{pmatrix} w$$

$$= \left\{ d + \frac{1}{\varepsilon^2 + TT^*} T\, dT^* \right\}$$

with the connection form

$$\theta' = \frac{1}{\varepsilon^2 + TT^*} T\,dT^* = T \frac{1}{\varepsilon^2 + T^*T} dT^*$$

and curvature

$$K' = \varepsilon^2 \frac{1}{\varepsilon^2 + TT^*} dT \frac{1}{\varepsilon^2 + T^*T} dT^*$$

~~So on $E^0 \oplus E^1$ we have the new connection which we can describe as follows. We perform the [illegible] automorphism $\begin{pmatrix} \varepsilon & -T^* \\ T & \varepsilon \end{pmatrix}$, then apply d and then the inverse automorphism~~

~~$$D = \begin{pmatrix} \varepsilon^2 + T^*T & 0 \\ 0 & \varepsilon^2 + TT^* \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon & T^* \\ -T & \varepsilon \end{pmatrix} d \begin{pmatrix} \varepsilon & -T^* \\ T & \varepsilon \end{pmatrix}$$~~

~~No - this is obviously wrong since it gives a flat connection again.~~

Let's rewrite this using $L = i(T + T^*) \in \mathrm{End}^1(E)$.
Then $L^2 = -(TT^* + T^*T)$, $dL = i(dT + dT^*) \in \Omega^1(M) \otimes \mathrm{End}^1(E)$

$L\,dL = -(T\,dT^* + T^*dT)$, then

$$\theta = \theta^0 \oplus \theta' = -\frac{1}{\varepsilon^2 - L^2} L\,dL$$

So here arises the problems with the sign conventions: $L \in \mathrm{End}^1(E)$, $dL \in \Omega^1(M) \otimes \mathrm{End}^1(E)$, then $L\,dL$ has to be computed in the algebra $\Omega(M) \hat{\otimes} \mathrm{End}(E)$. But when I write $T\,dT^*$ I compute in $\Omega(M) \otimes \mathrm{End}\,E$. Thus

$$L\,dL = T\,dT^* + T^*dT$$

is the correct thing. So

$$\theta = \frac{1}{\varepsilon^2 - L^2} L\,dL = L \frac{1}{\varepsilon^2 - L^2} dL$$

and the curvature is

$$d\theta + \theta^2 = dL \frac{1}{\varepsilon^2-L^2} dL - L\Big[+ \frac{1}{\varepsilon^2-L^2} \overset{dL\cdot L - L\,dL}{d(\cancel{\varepsilon^2}+L^2)} \frac{1}{\varepsilon^2-L^2}\Big] dL$$

$$\overbrace{\qquad + L \frac{1}{\varepsilon^2-L^2} dL \quad L \frac{1}{\varepsilon^2-L^2} dL}^{\text{cancel}}$$

$$= \Big[1 + L \frac{1}{\varepsilon^2-L^2} L \Big] dL \frac{1}{\varepsilon^2-L^2} dL$$

$$= \varepsilon^2 \frac{1}{\varepsilon^2-L^2} dL \frac{1}{\varepsilon^2-L^2} dL .$$

Next I want to check this out for the case where $T = z$ over $\mathbb{C}$, whence $L = i\gamma^\mu x^\mu$ $= i(z a^* + \bar{z} a)$.

$$tr_s\ \varepsilon^2 \Big(\frac{1}{\varepsilon^2-L^2} dL\Big)^2 = \frac{\varepsilon^2}{(\varepsilon^2+|z|^2)^2}\ tr_s \big(i\, dx^\mu \gamma^\mu\big)^2$$

$$= \frac{\varepsilon^2}{\varepsilon^2+|z|^2}\ 2i\ \overset{d^2 z}{2!}$$

(where $\sum_\mu i\, dx^\mu \gamma^\mu = n! \prod_\mu i\, dx^\mu \gamma^\mu$ in general

and we know $tr_s \big(\prod_\mu i\, dx^\mu \gamma^\mu\big) = i^n d^n x \underbrace{tr_s(\gamma^n \cdots \gamma^1)}_{(-1)^{\frac{n(n-1)}{2}} (2i)^m}$

$= d^n x\, (2i)^m$.)

At this point, it seems that the important question concerns how to sort out the different representations for the Chern character classes. From the above Grassmannian business we learn that the k-th Chern character class is represented by the

form
$$\frac{1}{2}\frac{1}{k!}\lambda^k tr_s\left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} \qquad ?$$

I am confused about the $\frac{1}{2}$. Let us start with the idea that we have a map $T : E^0 \longrightarrow E^1$ between trivial bundles over M such that T becomes infinite as we head toward ∞. I am supposing E^0, E^1 have the same dimension, that T is invertible outside of a compact Z and that $T^{-1} \longrightarrow 0$ as we approach ∞. Then I have defined an element of $K_c(M)$, in fact, in $K(M, M-Z)$.

Now I want to correlate this with the map to the Grassmannian which is given by sending $x \in M$ to $\mathrm{Im}\begin{pmatrix}\varepsilon \\ T(x)\end{pmatrix}$ in $E^0 \oplus E^1$. Where T is invertible this is the same as $\begin{pmatrix}\varepsilon T^{-1} \\ I\end{pmatrix} E^1$ and so by assumption the map sends infinity to the point of the Grassmannian represented by E^1. Thus we have the situation

$$\begin{array}{ccc} \{\infty\} & \longmapsto & [E^1] \\ M \cup \infty & \longrightarrow & \mathrm{Grass}_d(V^0 \oplus V^1) \\ x & \longmapsto & \left[\begin{pmatrix}\varepsilon \\ T(x)\end{pmatrix} V^0\right] \end{array}$$

So what do I want? The K-class is realized by the two trivial bundles and the isomorphism given by T outside of Z. I want the Chern character of this K-class realized as a differential form on M decaying sufficiently fast at ∞. The idea is that the Grassmannian carries such differential forms.

Let's make it clearer. What we are interested in is the bundle E^0 trivialized via T at ∞. Thus to compute its character we need a connection on E^0 which is consistent with this trivialization at ∞. Now via the graph mapping we identify E^0 with the pull-back of the subbundle S on the Grassmannian.

$$E^0 \xrightarrow{\sim} \begin{pmatrix} \varepsilon \\ T \end{pmatrix} V^0 \longrightarrow V'$$

$$v \longmapsto \begin{pmatrix} \varepsilon v \\ T(v) \end{pmatrix} \longmapsto T(v)$$

Thus the identification of E^0 with f^*S and the isomorphism of $S \simeq V'$ near the point $[V']$ are consistent with the identification of E^0 with $E' = f^*(V')$ near ∞.

Let us take now $L = i x^\mu \gamma^\mu$ over $\mathbb{R}^n$. I want to calculate

$$\frac{1}{2}\frac{1}{m!}\lambda^m \operatorname{tr}_s\left(\frac{1}{\lambda - L^2}\, dL\right)^n$$

which I believe represents the character of this Bott element, and hence should integrate over $\mathbb{R}^n$ to give $\left(\frac{2\pi}{i}\right)^m (-1)^m = (2\pi i)^m$. The answer is

$$\frac{1}{2}\frac{1}{m!}\frac{\lambda^m}{(\lambda+|x|^2)^n}\, n!\,(2i)^m\, d^n x$$

Now integrate over $\mathbb{R}^n$

$$\int \frac{\lambda^m}{(\lambda+|x|^2)^n}\, d^n x = \lambda^m \int d^n x \int_0^\infty e^{-t\lambda - t|x|^2}\, \frac{t^n}{\Gamma(n)}\, \frac{dt}{t}$$

$$= \lambda^m \int_0^\infty e^{-t\lambda} \left(\frac{\pi}{t}\right)^m \frac{t^n}{\Gamma(n)} \frac{dt}{t}$$

$$= \frac{\cancel{\lambda^m} \pi^m}{\Gamma(n)} \frac{\Gamma(m)}{\cancel{\lambda^m}} = \frac{\pi^m (m-1)!}{(n-1)!}$$

Thus $\displaystyle\int_{\mathbb{R}^n} \frac{1}{2} \frac{1}{m!} \frac{\lambda^m}{(\lambda+|x|^2)^n} n!\,(2i)^m d^n x$

$$= \frac{1}{2} \frac{1}{\underset{m}{\cancel{m!}}} \frac{\pi^m \cancel{(m-1)!}}{\cancel{(n-1)!}} \overset{n}{\cancel{n!}} (2i)^m = (2\pi i)^m$$

which checks.

<u>Conclude</u>: ch_k of the K-class represented by this T is represented by

$$\frac{1}{2} \frac{1}{k!} \lambda^k \, tr_s \left(\frac{1}{\lambda - L^2} dL\right)^{2k}$$

Now supposedly the form

$$tr_s \left\{ e^{u(L^2+dL)} \right\}$$

represents $\displaystyle\sum_{k=0}^\infty u^k ch_k$ and hence taking Laplace transform we see that

$$tr_s \left\{ \frac{1}{\lambda - L^2 - dL} \right\} = \sum_{k=0}^\infty tr_s \left\{ \left(\frac{1}{\lambda - L^2} dL\right)^{2k} \frac{1}{\lambda - L^2} \right\}$$

represents $\displaystyle\sum \frac{k!}{\lambda^{k+1}} ch_k$. Comparing degrees we see that

ch_k is also represented by

$$\frac{\lambda^{k+1}}{k!}\, tr_s\left\{\left(\frac{1}{\lambda-L^2}\,dL\right)^{2k}\frac{1}{\lambda-L^2}\right\}$$

Let's check this again for $L = i x^\mu \gamma^\mu$

$$\frac{\lambda^{m+1}}{m!}\,\frac{1}{(\lambda+|x|^2)^{n+1}}\,\underbrace{tr\,(dL)^n}_{n!\,(2i)^m\,d^n x}$$

$$\int d^n x\,\frac{1}{(\lambda+|x|^2)^s} = \int d^n x \int_0^\infty e^{-t\lambda - t|x|^2}\,\frac{t^s}{\Gamma(s)}\,\frac{dt}{t}$$

$$= \int_0^\infty e^{-t\lambda}\left(\frac{\pi}{t}\right)^m \frac{t^s}{\Gamma(s)}\,\frac{dt}{t} = \pi^m\,\frac{\Gamma(s-m)}{\Gamma(s)}\,\frac{1}{\lambda^{s-m}}$$

So the $\int_{\mathbb{R}^n}$ gives

$$\frac{\lambda^{m+1}}{\cancel{m!}}\,\pi^m\,\frac{\Gamma(\cancel{m+1}-m)}{\Gamma(\cancel{n+1})\,\lambda^{m+1}}\,\cancel{n!}\,(2i)^m = (2\pi i)^m$$

so it checks.

Concludes that

$$\frac{1}{2}\,tr_s\left(\frac{1}{\lambda-L^2}\,dL\right)^{2k} \qquad\qquad \lambda\, tr_s\left\{\left(\frac{1}{\lambda-L^2}\,dL\right)^{2k}\frac{1}{\lambda-L^2}\right\}$$

represent the same class.

May 31, 1984

Summary: I am considering an odd degree map $L = i\begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}$ on ~~between~~ a trivial bundle $E = E^0 \oplus E'$ such that L goes to infinity as one goes to infinity on the manifold. Then I get a K class with compact support on M, which I can think of in terms of the map from $M \cup \{\infty\}$ to the Grassmannian given by the graph of $\varepsilon^{-1}T$. I calculated that when the form $k!\, ch_k = tr\, e(de)^{2k}$ on the Grassmannian is pulled back one obtains the form

$$(1) \qquad \frac{1}{2}\lambda^k tr_s \left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} \qquad\qquad \lambda = \varepsilon^2$$

On the other hand if I start with the character form computed from the super-connection formalism

$$tr_s\; e^{u(L^2 + dL)}$$

and take the Laplace transform, I find that $k!\, ch_k$ is represented by the form

$$(2) \qquad \lambda^{k+1} tr_s \left\{ \left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} \frac{1}{\lambda - L^2} \right\}$$

The problem is to show that (1) (2) ~~represent~~ are cohomologous.

We know that the cohomology class represented by (1) is independent of λ, hence its derivative is a coboundary. The derivative is

$$\frac{1}{2}k\lambda^{k-1} tr_s \left(\frac{1}{\lambda - L^2}\, dL\right)^{2k} + \frac{1}{2}\lambda^k tr_s \Big(\frac{-1}{(\lambda - L^2)^2}\, dL\, \frac{1}{\lambda - L^2}\, dL \cdots$$

$$+ \frac{1}{\lambda - L^2}\, dL\, \frac{-1}{(\lambda - L^2)^2}\, dL \cdots$$

$$(2k \text{ terms}) \quad + \cdots \Big)$$

so we get that the following is a coboundary

$$k\frac{1}{2}\lambda^{k-1}\,tr_s\left(\frac{1}{\lambda-L^2}dL\right)^{2k}-\frac{1}{2}\lambda^k\,2k\;tr_s\left\{\left(\frac{1}{\lambda-L^2}dL\right)^{2k}\frac{1}{\lambda-L^2}\right\}$$

which is exactly what we want at least for $k>0$.

If $k=0$ one has

$$\frac{1}{2}\,tr_s(id) = \lambda\;tr_s\left(\frac{1}{\lambda-L^2}\right)$$

which off-hand looks contradictory, however both numbers are zero under our assumptions.

At this point I have managed, partially at least, to link the heat kernel and the resolvent approaches to the Chern character.

The following seemed to be an interesting calculation. The Chern character form

$$tr_s\; e^{u(L^2+dL)}$$

has for its cohomology class $\sum_{k\geq 0} u^k ch_k$. It makes sense at least when L is invertible to take the Mellin transform of this differential form as a function of u: (Change tr_s to tr_ε)

$$\int_0^\infty tr_\varepsilon\left\{e^{u(L^2+dL)}\right\}u^s\frac{du}{u} = tr_\varepsilon\left\{\frac{1}{(-L^2-dL)^s}\right\}\Gamma(s)$$

For example, go back to $L = ix^\mu\gamma^\mu$ on $\mathbb{R}^n$ where $tr_\varepsilon\left\{e^{u(L^2+dL)}\right\} = e^{-u|x|^2}u^n d^nx\,(2i)^m$

and then the Mellin transform is

$$\frac{\Gamma(s+n)}{|x|^{2(s+n)}} \; d^n x \; (2i)^m$$

It is not clear what to make of this, but there appears to be something interesting happening. Let's discuss this.

First note if we pass from forms to cohomology then the character form becomes $\sum_{k\geq 0} u^k ch_k$, and u^k doesn't have a Mellin transform. One has in some sense

$$\int_0^\infty u^{k+s} \frac{du}{u} = 2\pi\delta(t)$$

if $s = -k+it$. Thus ch_k, which is cohomology of dimension $2k$, is associated to the point $s=-k$.

The next point is that the Mellin transform is defined on the set where L is invertible, but not in an obvious way on the [redacted] singular set. [redacted] This suggests that the Mellin transform is probing the [redacted] sort of information one obtains in the B.F.M. game by letting the parameter multiplying L go to ∞. [redacted]

June 2, 1984

I am now looking at the transgression problem from the viewpoint of the Grassmannian graph construction. In this case we start with a family of operators parametrized by a group G, like the group of gauge transformation. The family is the orbit under the group of a fixed operator.

The first thing I thought one should do is to look at the simplest transgression situation, namely

$$U(n) \subset Grass_n(\mathbb{C}^n \oplus \mathbb{C}^n)$$

where to a group element g one assigns the image of $\binom{1}{g}$. (More generally to $g \in GL(n)$ one can assign the projector $e = \binom{1}{g} \frac{1}{2} (1 \ \ g^{-1})$ and so get character forms on $GL(n)$.)

Recall the formula that given $T \in \Gamma\, End(E^0, E^1)$, where E^0, E^1 are trivial we have the connection form

$$\theta = \frac{1}{1+T^*T} T^* dT = T^* \frac{1}{1+TT^*} dT$$

and curvature

$$F = d\theta + \theta^2 = \frac{1}{1+T^*T} dT^* \frac{1}{1+TT^*} dT .$$

Now apply this to $T = tg$, $T^* = tg^{-1}$ and one gets

$$F = \frac{t^2}{(1+t^2)^2} dg^{-1} dg = \frac{-t^2}{(1+t^2)^2} (g^{-1}dg)^2 . \text{ and}$$

$$\theta = \frac{t^2}{1+t^2} g^{-1} dg$$

The character forms $\frac{1}{k!} tr\, F^k$ are zero in this case even though the ~~curvature~~ connection isn't flat.

However in the transgression I think we are interested in the path from $t=0$ to $t=\infty$, i.e. one has a 1-parameter family of connections, and one uses the formula

$$\frac{d}{dt}\, tr\left(e^{F_t}\right) = d\, tr\left(e^{F_t}\dot{\theta}_t\right)$$

which one recalls comes from the connection $d+\theta_t$ over $\mathbb{R}\times G$. In our case

$$\frac{1}{(n-1)!}\, tr\left((F_t)^{n-1}\dot{\theta}_t\right) = \frac{1}{(n-1)!}\, tr\, (dg^{-1}dg)^{n-1} g^{-1}dg \left[\frac{t^2}{(1+t^2)^2}\right]^{n-1} \frac{d}{dt}\left(\frac{t^2}{1+t^2}\right)$$

So the transgression form is

$$\frac{(-1)^{n-1}}{(n-1)!}\, tr\left(g^{-1}dg\right)^{2n-1} \cdot \text{const}$$

where

$$\text{const} = \int_0^\infty \left\{\frac{t^2}{(1+t^2)^2}\right\}^{n-1} d\left(\frac{t^2}{1+t^2}\right)$$

$$= \int_0^\infty \left(\frac{t}{(1+t)^2}\right)^{n-1} d\left(\frac{t}{1+t}\right) \qquad u = \frac{t}{1+t}, \quad 1-u = \frac{1}{1+t}$$

$$= \int_0^1 \left[u(1-u)\right]^{n-1} du = \frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = \frac{(n-1)!\,(n-1)!}{(2n-1)!}$$

Thus the transgression form is

$$(-1)^{n-1}\frac{(n-1)!}{(2n-1)!}\, tr\left(g^{-1}dg\right)^{2n-1}$$

which is what one obtains from Chern-Simons. This is not too surprising because after all we are looking at the family $\quad \theta = \frac{t^2}{1+t^2}\, g^{-1}dg$, i.e.

rescaling the <u>flat MC</u> connection $g^{-1}dg$.

Now