\[ K^*(X) \xrightarrow{\text{ch}} H^*(X) \]
any elliptic class
\[ \mathbb{Z} \]
any homology class
\[ \mathbb{C} \]
problem is that when you push forward you get something in open sets, which then has to be compared with Bott element.

\[ -\frac{d^2}{dx^2} + V(x) = D \]
\[ T^2 \]
quasi periodic

\[ D \text{ not invertible mod smooth kernel} \]
\[ D \text{ is invertible mod the algebra of the foliation} \]
\[ DQ - 1 = S_0 \]
\[ QD - 1 = S_1 \]

\[ \mathbb{C}^*(V,F) \]

properties describing \( \mathbb{C}^*(V,F) \):
1) filtration: \( \mathbb{C}^*(V,F) \sim \mathbb{C}^*(V/F) \oplus k \)
2) local: \( W \subset V \Rightarrow \mathbb{C}^*(W,F_W) \subset \mathbb{C}^*(V,F) \)
3) functoriality.

Index: \( D \in K^0(\mathbb{C}^*(V,F)) \)

Index then for mean foliations
transverse measure \( \to \) \( \text{tr} \) on \( \mathbb{C}^*(V,F) \to \text{Ham}(K_0(\mathbb{C}^*(V,F),\mathbb{C})) \)

\[ \langle \text{ch} \sigma_D, T_d \alpha V, [\alpha] \rangle \]
Ruelle-Sullivan current
two obvious questions arise.

If scalar curvature of leaves $> 0$ \( D = \text{Dirac} \Rightarrow \text{Index} \, D = 0 \)

leaf-wise h. e.g. \( \text{Signature of leaf} \in K_0(C^*(V, F)) \)

1) How to get info on \( K(C^*(V, F)) \)
2) How to pass from \( K \) to cohomology.

\[ \frac{V}{F} \xleftarrow{\text{ch} \times p_i} \frac{P(c_i)}{Q(c_i)} \text{ GF classes, } [\gamma_i] \xrightarrow{\text{trans. oriented}} \]

Idea: to integrate \( \gamma \).

Results: If \( A(V) \neq 0 \), can't have foliation \( R > 0 \)

some signature result.

\[ G \text{ graph of foliation } \rightarrow BG \]

\[ \Gamma, V \rightarrow V \times E \Gamma = V_\Gamma \]

Main conjecture

\[ K_{* 1} \left( BG \right) \xrightarrow{\mu} K(C^*(V, F)) \]

\[ H^* (BG) \xrightarrow{\text{ch}} K_0(C^*(V, F)) \]

Thus: says \( \gamma \in \Gamma \) an
This theorem says you can integrate over the leaf space both in K-theory and cohomology.

Proof involves three steps:

1. \[ C^0(V) \int f_0 df_1 \cdots df_n = \tau(f_0, \ldots, f_n) \]
   
   Based on Grassmann alg. — doesn’t work
   
   So need cyclic cohomology

   **Lemma:** \( \forall \tau \in \text{cyclic cocycle}, \text{then } e \in \text{Proj } K(A) \)
   
   \[ \tau(e, \ldots, e) \text{ is a map } K(A) \to \mathbb{C}. \]

   Cyclicity in cohomology is the linearization of algebra
   
   \[ A \xrightarrow{\text{Ext}} \text{Cyclic theory of } A. \]

   Derived functor of traces.

2. \( C^* \text{alg } A \supset A \rightarrow \text{cyclic cocycle } \tau \text{ defined on } A \)
   
   Problem is to show that map defined by cyclic cocycle on \( A \) extend to \( K(A) \)

   - cocycle: \( \text{When meaningful on } A \)?
     
     \[ \tau(f_0, f_0) = \langle S(f_0), f_0 \rangle \]
     
     \( S: A \otimes A^* \to A^* \)

   \( S \) is a derivation

   \[ \langle S(f_0), f_0 \rangle = -\langle d(f_0), f_1 \rangle \]

   So \( S(f) \) is closable

   Basic lemma says closure has same K-theory

   Banach alg. Thm.

   So a cyclic cocycle satisfying a weak continuity condition will pair with topology

   \[ \| \tau(x^0 da_1 x^1 da_2 \cdots x^n da_n) \| \leq C_a^1 \cdots a^n \| x \| \]
(3) completely new non-comm. phenomenon

\[ V \times \Gamma \rightarrow \sum f_g U_g \quad \sum \omega \hat{U}_g \quad \hat{U}_g \omega \hat{U}_g^{-1} = g^* \]

\[ d \sum f_g U_g = \sum d f_g U_g \]

\[ \int \sum \omega \hat{U}_g = \int \omega \hat{U}_g \]

like Tomita modular theory.

\[ \omega \hat{U}_g \]

basis

\[ f \omega \hat{U}_g = \sum \omega \hat{\Theta}_i (f) \]

turns out to be a dual action of GL(n, R)

But not a unitary repn. no invariant Heim. metric

General idea: Hyperbolic \( \rightarrow \) parabolic

\[ \Gamma, V \quad \text{reduction of } TV \quad \text{to } \left[ \begin{array}{cc} \mathfrak{so}(p) & 0 \\ \ast & \mathfrak{so}(q) \end{array} \right] \]

\[ \Gamma \text{-invar. reduction} \]

Put ideas of Mackey, Zeeman, Kasparov

\[ V \times \Gamma \rightarrow GL(n, R) \quad \text{1-frame bundle} \quad \mathcal{J}(V) / \mathfrak{so}(n) \rightarrow V \]

fibres are symmetric spaces GLn/\mathfrak{so}n

negative or zero curvature, so can apply Kasparov

\[ 1 \]

geodesics

angle goes to zero

\[ \int \in KK_{\Gamma}(V, W) \]

action of \( \Gamma \) on \( W \) is

distal
\[ K(V/\Gamma) \longrightarrow K(W/\Gamma) \]

This is exactly like passing from $\text{III}$ to $\text{II}$ by taking cross product with the modular auto group.

Where Gelf-Fuks classes come from. Consider

\[ J_k(V)^+/SO(n). \]

Generalization of Tônita Takesaki modular theory to cyclic cocycles involving a dual action of $GL_n^+$. Physics
two new theorems about cocycles over amenable groups, applications to characteristic classes for foliations.

A foliated set whose leaves are $C^\infty$ obstruction to foliated spaces having certain measure properties.

Problem: Classify all $C^\infty$ foliations on a compact manifold

Def: Say two codim n foliations are concordant if $\exists$ fol. a $\mathbb{R}^n$

Haefliger-Thurston: concordance classes = lifts $M \overset{\nu}{\to} \mathbb{R}^n \overset{\nu}{\to} S^{2n}$

with given $\nu$

Remark 1: The direction $\nu$ is a great existence thm.

2. Uncountable types

Problem: Find another equiv. relation

(Corres: Are you interested in the longitudinal or transverse aspects.)

Hudson wants to pass to foliated space, and find equiv. relations.

What are the invariants of a foliation as a dynamical system?

So let $X$ be a measurable union of leaves. $C^\infty$.

For $F/X$ ergodic have invariants

A. von Neumann type

I. $X/F = \text{standard}$

II. no type I component but has invariant transverse measure

III. what's left

B. Amenability

Thm. (Connes-Feldman-Weiss) $F/X$ amenable $\iff F/X$ hyperfinite $\iff F/X$ is a limit of type I

Ex: Solvable Lie gp. action.
C. Growth rate of leaves. 

D. Representations of virtual group \( (X, F, \Gamma) \).

E. Properties of measure + C* algebra.

Secondary classes of \( F \):

\[
\text{WO}(n) \to A^*(M) \\
H^*(g_{L_\ell}, O_\ell) \otimes (R[Z_{-1}^n]/\text{deg} \geq 2n) \\
\otimes \mathcal{C}_f
\]

(Cinese - \( \text{WO}(n) \) is invariant differential forms on higher jet bundles)

If \( M \) is oriented, then \( \Delta (y_{\mathcal{E}_f}) \in H^p(M) = H^{n-p}(M)^* \)

Thm. (Heitsch - Huisken) If \( X \) is saturated, then \( y_{\mathcal{E}_f} \) gets full on \( H^{n-p}(M) \), \( \{ y_{\mathcal{E}_f} \} \to \int_X \Delta (y_{\mathcal{E}_f}) \wedge \phi \)

Thm: There is an operator \( X(y_{\mathcal{E}_f}) \) on \( X \) s.t. \( \Delta_k (y_{\mathcal{E}_f}) = X(y_{\mathcal{E}_f}) [e_f] \) and the operator \( X_k (y_{\mathcal{E}_f}) \) depends only on measurable coh. class of the normal \( GL_n \)-cocycle to \( F \)

The goal: There characteristic classes distinguish the different types A, B, C, D, E.