Witten 12/13/83

Caricature of particle physics throw away 99%

$$G \supset SU(3) \times SU(2) \times U(1)/\mathbb{Z}_6$$

"massless" fermions of spin $\frac{1}{2}$ (1 pt in 10^6)

"

repn. H of group G. (one particle space of fermions of momentum k)

$\xrightarrow{G}$ $\qquad J_z = \begin{cases} \hbar/2 \\ -\hbar/2 \end{cases}$

$H = H_L \oplus H_R$ $\qquad H_L, H_R$ are separately repns of G.

An essential further fact — an anti unitary transf

CPT $\qquad \Theta: H_L \longleftrightarrow H_R$.

[H_L is 45 dim'l in nature

Better description: A real vector space H with a complex structure

Existence of Θ guarantees that if H_L is a rep Q of G then H_R is the complex conjugate representation.

Experimental Fact.

Expand $Q = \oplus Q_i$ 15

$\bar{Q} = \oplus \bar{Q}_j$ $\qquad Q_i \not\equiv \bar{Q}_j$

When does such a thing arise?

Solution of an ~~xxxx~~ equivariant index problem.

This assumption "solve" one classical problem: origin of G: If B admits G-action, the effective 4-dim'l space will have a gauge group G.

This explanation doesn't have much competition Other possibility is super-gravity which can be combined with K-K.

Now for fermion quantum nos.

We assume in 4+n dimensions that we have fermions behaving a Dirac equation

$$\not D \psi = 0$$

$$\not D = \gamma^i D_i = \not D^{(4)} + \not D^{(n)} \qquad\qquad D^{(4)}D^{(n)} + D^{(n)}D^{(4)} = 0$$

$$\not D^2 = (\not D^{(4)})^2 + (\not D^{(n)})^2$$

$$\not D^2 \psi = 0 \implies \not D^{(4)}\psi = 0 \quad \text{and} \quad \not D^{(n)}\psi = 0$$

So a) 4-dim'l Dirac equation.

b) zero mode for Dirac eqn. on B

$D^{(n)} : S_L \rightleftarrows S_R$ $\qquad$ $\exists$ anti-unitary Θ which

In 4 dims. $\quad \text{Ker } \not D_-^{(4)} \simeq H_L$ left handed massless ferms.

$\text{Ker } \not D_+^{(4)} \simeq H_R$

We can't solve the problem of fermion quantum numbers in this way because of a thm. of Atiyah-Hirzebruch which says that the equivariant index of Dirac $= 0$. (M admits an action of a cont. group)

Another thm: Equivariant index of the Signature Operator is always equal to zero except for the trivial representation (Follows because G acts trivially on H^*_{DR})

Manifold of dim $4k+2$

Spinor $\quad S = S_L \oplus S_R \quad$ related by an anti-unitary transf.

$$\not{D}: \quad S_L \longrightarrow S_R \qquad\qquad S_R \longrightarrow S_L$$

S_L, S_R any Hilbert spaces

$$\text{Index} = \underbrace{\dim(\text{Ker}\,\not{D}_+)}_{A} - \underbrace{\dim(\text{Ker}\,D_-)}_{B}$$

Generically A or $B = 0$ if there are no special principles.

A group action G, can give a special reason: We can define a character-valued index.

$$\text{Ker}\,\not{D}_+ = V_+ = \oplus\, \alpha_i Q_i$$
$$\text{Ker}\,\not{D}_- = V_- = \oplus\, \beta_i Q_i$$

Then $\alpha_i - \beta_i$ is invariant under G-perturbations.

The point I want to stress, then generically, with no other principles α_i or $\beta_i = 0$ for each i.

Now the game is to find an equivariant operator such that V_+, V_- can be identified with H_L, H_R

Spinor repns are complex in $4k+2$ dimensions so there is a CPT.

To realize this idea start with the assumption that the real world has dim $d = 4+n$, $n = 4k+2$ some k.

General Relativity in $4+n$ dim

$$\left(\mathcal{L} = \int d^{4+n}x \times \sqrt{g}\,R \;\cdots\cdots\right)$$

"Ground State"

$$\underset{\substack{\text{4diml} \\ \text{Minkowski}}}{M^4} \times \underset{\substack{\text{compact } n\text{-diml. manifold} \\ \text{radius}}}{B}$$

Third thm. Rarita-Schwinger operator has equivariant index which is an integer (i.e. trivial repn.) (partial proof by Witten)

Mathematicians suggest ~~one~~ could get around the index $= 0$ thm. by considering a twisted Dirac equ.^{n} $S \otimes E$. Can get non-zero index. Problem is that where does the connection on E from? E should come from the Riemannian geometry. So it comes from a repn. R of $O(d)$

Physical criteria

(i) spin-statistics $\Rightarrow$ R is a rep of $O(n)$ not just its double covering

(ii) for suitable quantization only allowed choice is standard repn hence $E = T$.

As far as the index problem is concerned the Rarita-Schwinger operator is a twisted Dirac operator

$$\not{D} : S_L \otimes T \longrightarrow S_R \otimes T$$

Witten's proof: Verify for homogeneous spaces, use spin-cobordism invariant. If spin-cobordism (equivariant) like equivariant unitary cobordism) is gen. by homog. manifolds + fibre bundles, then one is done.

Why R-S equation makes sense in physics.

Quantization $\qquad \mathcal{L} = i \int d^n x \, \bar{\psi} \not{D} \psi$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta} \, \delta(x-y) \qquad \psi_\alpha \text{ spinor index}$$

Naive RS op. comes from

$$\mathcal{L} = \int d^n x \, \bar{\psi}_\mu \not{D} \psi_\mu$$

$\psi_{\alpha\mu}$ (α spinor, μ vector) $\qquad \{\psi_{\alpha\mu}, \psi_{\beta\nu}\} = \delta_{\alpha\beta} \, g_{\mu\nu}$

ψ supposed to be hermitian operators and $g_{\mu\nu}$ has -1. So it is inconsistent.

Naive R-S.	Real
$\not{D}\,\psi_\mu = 0$ $\gamma^\nu D_\nu \psi_\mu = 0$	$\gamma^\nu (D_\nu \psi_\mu - D_\mu \psi_\nu) = 0$

We want to find a gauge invariance that will render "unphysical" the components of ψ for which the algebra has the wrong sign.

The R.S. equation is obeyed automatically if $\psi_\mu = D_\mu \varepsilon$ for any ε. So can use $\psi_\mu \to \psi_\mu + D_\mu \varepsilon$ to fix a gauge such as $\psi_0 = 0$.

$$\delta(\gamma^\nu D_\nu \psi_\mu - D_\mu \psi_\nu)$$
$$= \gamma^\nu (D_\nu D_\mu - D_\mu D_\nu)\epsilon$$
$$= \frac{1}{8} \gamma^\nu R_{\mu\nu\alpha\beta} [\gamma_\alpha, \gamma_\beta] \epsilon$$

Use $R_{\mu\nu\alpha\beta} + R_{\mu\alpha\beta\nu} + R_{\mu\beta\nu\alpha} = 0$

$R_{\mu\nu} = 0 \longleftarrow$ Ricci tensor

Conclusion: The R-S equation has this gauge invariance when $R_{\mu\nu} = 0$ and so can be quantized.

$R_{\mu\nu} = 0$ are Einstein's equations in absence of matter.

$$0 \longrightarrow S \xrightarrow{\alpha} S \otimes T \xrightarrow{\beta} S \otimes T \xrightarrow{\gamma} S \rightarrow 0$$

$$\alpha : \psi \longrightarrow D_\mu \psi$$

$$\beta : \psi_\mu \longrightarrow \Gamma_{\mu\nu\lambda} D_\nu \psi_\lambda$$

$$\gamma : \psi_\mu \longrightarrow D_\mu \psi_\mu$$

$$\Gamma_{\mu\nu\lambda} = \Gamma_\mu \Gamma_\nu \cdot \Gamma_\lambda = \frac{1}{6}\left(\Gamma_\mu \Gamma_\nu \Gamma_\lambda - \Gamma_\mu \Gamma_\nu \Gamma_\lambda \pm \text{perms}\right)$$

Thm: $\beta\alpha = \gamma\beta = 0$ iff $R_{\mu\nu} = 0$

Witten feels above should be useful in studying Ricci flat (or Einstein) manifolds