Ginsparg 12/13/83.

fermion generating fnl.

Dirac fermions $e^{-\Gamma(A)} = \int_{\bar\psi\psi} e^{-\int \bar\psi i \not{D} \psi} = \det(i\not{D})$ $\quad \not{D} = \gamma^\mu(\partial_\mu + A_\mu)$

classically this has invariance under $\psi \to e^{i\alpha\gamma_5}\psi$, $\quad \bar\psi \to \bar\psi e^{i\gamma_5\alpha}$

leads to conserved current $\partial_\mu(\bar\psi \gamma_\mu \gamma_5 \psi) = 0$

Fujikawa: Change in measure corresp. to this gauge transf.

$$\sum_n \bar\psi_n \gamma_5 \psi_n e^{-\lambda_n^2/M^2} = \mathrm{tr}\left(\gamma_5 e^{-(\not{D})^2/M^2}\right)$$

Thus $\partial_\mu j_5^\mu = \quad = \left(\frac{i}{2\pi}\right)^n \frac{1}{2^n n!} \mathrm{tr}(F^n)$ (axial anomaly)

Now for non-abelian anomaly

$$\frac{1 \pm \gamma_5}{2} = P_\pm \qquad \not{D}_\pm = \not{D} P_\pm \qquad D_+ : \Delta_+ \to \Delta_-$$

Weyl fermions

$$\int e^{-S} = \int e^{-\int \bar\psi i \not{D}_+ \psi} \overset{?}{=} \det i\not{D}$$

doesn't give a eigenvalue problem

Instead put $\hat{D} = \not{D}_+ + \not\partial_- = \not\partial + \not{A}_+ = \begin{pmatrix} & \not\partial_- \\ \not{D}_+ & \end{pmatrix}$

$\hat{D}$ non-hermitian so has cx eigenvalues

$$|\det \hat{D}^2| = \det(\not{D}_- \not{D}_+) \det(\not\partial \not\partial) = \text{const}\sqrt{\det \not{D}^2}$$

so $|\det \hat{\not{D}}^2|$ is gauge-invariant ($\exists$ gauge-invariant PV regularization procedure). Also eigenvalues of $\not{D}$ occur in pairs $\pm\lambda$ because $\not{D}$ anti-commutes with γ_5.

gauge transf. $\quad \psi_+ \longrightarrow g\psi_+$
$\qquad\qquad\qquad \psi \longrightarrow \psi - v\psi_+ \qquad\qquad g = e^{-v}$

$$\Gamma(A - Dv) = \Gamma(A) + \int v^a D_\mu \frac{\delta\Gamma}{\delta A^a_\mu}$$

Again anomalous contribution from the measure

$$\int v^a D_\mu \frac{\delta\Gamma}{\delta A^a_\mu} = \int dx\ \mathrm{tr}\ v\,\gamma_5\, e^{-\frac{1}{M^2}\hat{D}^2}$$

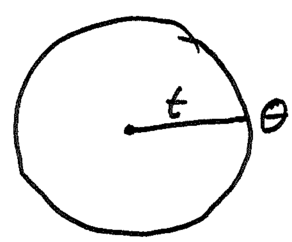
4 dims
$$= \frac{1}{24\pi^2}\int \mathrm{tr}\ v\, d\left(A\,dA + \tfrac{1}{2}A^3\right)$$

Interesting question is why does one obtain the Chern-Simons form out of the analytical procedure.

S^{2n}, x^μ coords., A fixed gauge field no zero modes

$$A^\theta = g^{-1}(d_x + A)\,g \qquad\qquad g(\theta, x) : S^1 \times S^{2n} \longrightarrow G$$

2 parameter family
$$A^{t,\theta} = t A^\theta$$



How does $\det(\hat{D}^{t,\theta})$ behave?

$$\det(\hat{D}^{\theta}) = \underbrace{\sqrt{\det \not{D}(A)}}_{\text{gauge-inv.}}\ e^{iW(A,\theta)}$$

Winding no. of $W(\theta) = \frac{1}{2\pi}\int d\theta \, \frac{dW}{d\theta}$

To derive a formula for $\frac{dW}{d\theta}$ which uses the Chern-Simons form. This then explains the connection between anomalies (phase of W) and Chern-Simons.

Look at the points (t,θ) in the disk where $\hat{D}^{t,\theta}$ has zero. Have a winding number of W at each such point. This local winding number depends only on the winding number of the smallest eigenvalue around the points - Can assume only one zero mode (or maybe 2 ?) by genericalness.

$$2n+2 \qquad x^{\mu}, t, \theta \qquad \not{D}_{2n+2} = \sum_{a=1}^{2n+1} (\partial_a + A_a)\Gamma^a$$

$$\Gamma^{\mu} = \sigma_1 \otimes \gamma^{\mu}$$

$$\Gamma^{2n+1} = \sigma_1 \otimes \gamma_5$$

$$\Gamma^{2n} = \sigma_2 \otimes 1$$

$$A^{t,\theta}_{\mu} = A_{\mu}(t,\theta,x)\,dx^{\mu}$$

$$A_{\theta} = A_t = 0$$

Actually one completes the disk $D^2 \times S^{2n}$ to $S^2 \times S^{2n}$ using $g(t,\theta)$. Get non-trivial bdle

$\therefore \; \mathrm{ind}\left(\not{D}_{2n+2}\right) \neq 0$

Consider $\not{D}^{\varepsilon}_{2n+2} = \frac{1}{\varepsilon} D_{\mu}\Gamma^{\mu} + D_i \Gamma^i$

Take ε being very small $\Rightarrow$ S^{2n} small + S^2 large

The index is invariant under deformation.
To analyze zero modes I define a Hamiltonian

$$H_\varepsilon = \left(i \not{D}^{\varepsilon}_{2n+2}\right)^2 = \frac{1}{\varepsilon^2} \mathbb{1} \times (i\not{D}_{2n})^2 + \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial t^2} + \frac{1}{\varepsilon}\Gamma^i \Gamma^\mu \partial_i A_\mu$$

$$\int dt \int dx\, \mathcal{F}^{n+1} = \int dx\ \omega^1_{2n}(A^\theta, F^\theta)$$ 1-form on S^1

$$\mathcal{F} = (d_x + d_t + d_\theta) A^{t\theta} + (A^{t\theta})^2$$

In adiabatic approximation using ψ_+ ψ_- for zero modes of $\not{D}_{2n}$.

$$H_\varepsilon = -\frac{\partial^2}{\partial\varphi_1^2} - \frac{\partial^2}{\partial\varphi_2^2} + \frac{1}{\varepsilon^2}\left(|z_1|^2\varphi_1^2 + |z_2|\varphi_2^2\right) + \frac{1}{\varepsilon}\begin{pmatrix} & x z_1^* + i z_2^* \\ x z_1 - i z_2 & \end{pmatrix}$$

<u>Principle:</u> $\int_{S^2} q(\mathcal{L})$ = winding no. of W

$$= \sum_{\substack{\text{critical} \\ \text{points}}} \text{local winding no} \overset{\text{molecular physics}}{=} \operatorname{ind}(D^{2n+2}) \overset{\text{old index thm.}}{=} \int_{S^2\times S^n} \mathcal{F}^{n+1}$$

$$= \int_{S^1}\int_{S^n} \text{Ch-Simons}$$

Prepare graduate class: The goal is the <u>proposition</u>: that an invariant diffl. form on G is the same thing as a natural transf. from flat connections on the trivial bundle to forms on the base.

Instead of saying trivial bundles, I could say principal bundles P with section s.

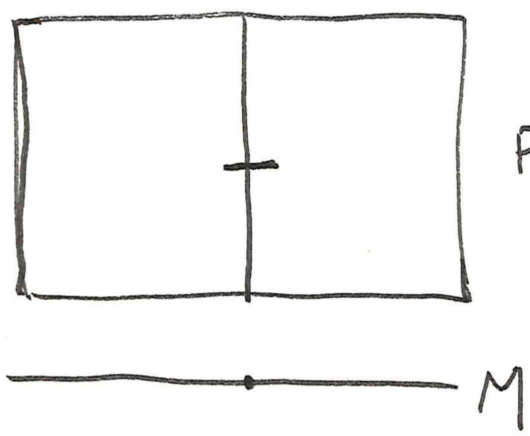
The process goes as follows:

$$\boxed{\Lambda \mathfrak{g}^* \xrightarrow[\text{flat connection}]{} \Omega(P) \xrightarrow{s^*} \Omega(Y).}$$

Then you could say that ~~it suffices to prove~~ one gets a 1-1 corresp. without actually formulating the prop.

For today's lecture I want to define d on $\Lambda \mathfrak{g}^*$.

Describe connection geometrically as a selection of horizontal space



$$\begin{tikzcd} 0 \arrow[r] & T_\pi \arrow[r] & T_P \arrow[r] & \pi^* T_M \arrow[r] & 0 \\ & \mathfrak{g} \arrow[u, "s"] & & & \end{tikzcd}$$

So we get $\Omega^1_P \longrightarrow \mathfrak{g}^* \otimes \Omega^0_P$

$\omega \longmapsto (X \mapsto \iota_X \omega)$

Define connection to be a

map $\mathfrak{g}^* \longrightarrow \Omega^1_P$ or $\theta \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$

Such that right-invariant: $\mathcal{L}_X \theta + [X, \theta] = 0$

and $\iota_X \theta = X$.