Atiyah Nov. 29

In finite dimensions $\exists$ formulas which enable one to evaluate integrals over fixpoint submanifolds.

Finite-dim: symplectic manifold M^{2n} compact, action of S^1
(Duistermaat - Heckman)

[redacted] ω symplectic form, $\frac{\omega^n}{n!} \neq 0$ volume
X generator of S^1 action.

$$dH = i(X)\omega$$

H Hamiltonian

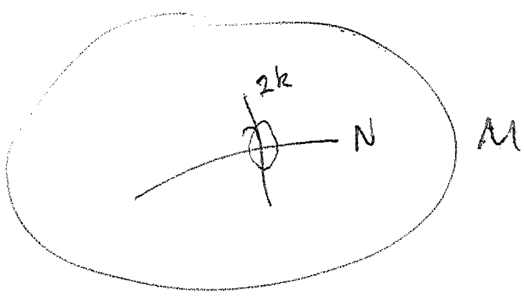
$$\int e^{-tH} \frac{\omega^n}{n!}$$

(Think of this formally as $\int_M e^{\omega - tH}$)

Thm:
$$\int_M e^{-tH} \frac{\omega^n}{n!} = \sum_P \frac{e^{-tH(P)}}{t^n \prod_{j=1}^n m_j^P}$$ isolated fixpts.

where $z_1^{m_1}, \dots, z_n^{m_n}$ are the weight of S^1 on $T_P(M)$.

Atiyah-Bott gave a cohomology proof.



$$c(N) = \prod_j (1 + \alpha_j)$$

where the α_j are formal roots.

$$\int_M e^{-tH} \frac{\omega^n}{n!} = \sum_N \int_N \frac{e^{-tH(N)} e^{\omega}}{\prod_{j=1}^k (t m_j + \alpha_j)}$$

ω can be allowed to degenerate, except $\omega^n = 0$ is an $S \subset M$, and defines $[S] \in H^1(M, \mathbb{Z}_2)$.
Thus this represents w_1 of M.

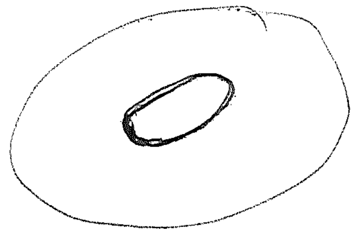
Assume M orientable.

Given a Riemannian metric then ω defines skew-adjoint $A = -A^*$. $\det A = (\mathrm{Pf} A)^n$.

$$\frac{\omega^n}{n!} = (\mathrm{Pf}\, A) \cdot \text{Riemannian volume}$$

$\mathcal{M} = \Omega M = \mathrm{Map}(S^1, M)$ $\dim M = 2n$ compact oriented with Riem. metric

$\varphi : S^1 \hookrightarrow M$



T_φ = all vector fields along $S^1 = \Gamma(\varphi^* T_M)$

has inner product $\int_{S^1} \xi \cdot \eta$

$\exists$ natural skew-adjoint operator ∇ cov. deriv.

This form can be degenerate e.g. if geodesic.

S^1 acts on ΩM. preserves ω.

Hamiltonian = Energy of loop $= \frac{1}{2} \int |\nabla \varphi|^2$

When is ΩM orientable? Answer $\Leftrightarrow$ M is a spin manifold.

Explanation: look where the form ω become degenerate and calculate the element of $H^1(\Omega M, \mathbb{Z}_2)$ and you get $w_2 \in H^2(M, \mathbb{Z}_2)$. Alternatively can look at the $\parallel$ transport or monodromy along φ, call it T_φ, determined up to conjugacy in $SO(2n)$. $\alpha_1, \dots, \alpha_n$ angles of T_φ

Lemma: $|\det(\nabla_\varphi)| = \det(1 - T) = \prod (1 - e^{i\alpha_j})(1 - e^{-i\alpha_j})$
$= \prod_j \left[2 \sin(\alpha_j/2)\right]^2$

$$\therefore \quad Pf(\nabla_\varphi) = \sqrt{\det(1-T)}$$

so the manifold **will** be orientable when this can be chosen **coherently**.

$$\chi(S^+) - \chi(S^-)$$
$$\shortparallel$$

$$\begin{array}{ccc} Spin(2n) & \sqrt{\det(1-T)} & \text{is defined on } Spin(2n) \\ \downarrow & & \\ SO(2n) & \det(1-T) & \end{array}$$

$$\shortparallel$$
$$\sum (-1)^q \, Tr(\Lambda^q T) = \chi(\Lambda^{ev}) - \chi(\Lambda^{odd})$$

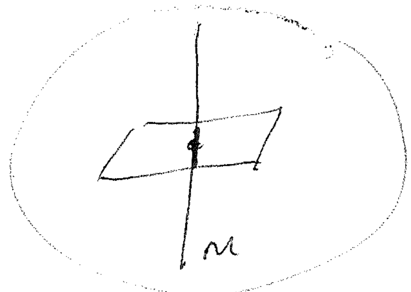
Thus M spin $\iff$ all monodromy can be lifted to spinor gp.
$\iff$ $Pf(\nabla_\varphi)$ defined.

So everything is now set up to apply formula

Fixpts of S^1 on ΩM are constant loops.

(a) Fixpt contribution

$$\int_M \frac{\cancel{e^{-tH(M)}}\, 1 \; e^{\omega \cdot 1}}{\prod (t m_j + \alpha_j)}$$



ΩM

normal directions are non-constant Fourier exp.

so denom. is

$$\text{"} \; \frac{\text{fn on } S^1}{\text{const}} \otimes TM$$

$$\prod_{n=1}^{\infty} \prod_j (tn + \alpha_j) = \frac{\sin \alpha_j/2}{\alpha_j/2}$$

$$\text{fixpt contribution} = \int_M \prod \frac{\alpha_j/2}{\sin \alpha_j/2} \; \hat{A}(M) \quad \text{where} \quad Pont(M) = \prod (1-\alpha_j)^2$$

= index of Dirac operator

(b) Integration over ΩM.

$$\int_{\Omega M} e^{-tH} \ \text{Pfaff} \cdot \text{Riem volume}$$

$$= \int_{\Omega M} e^{-tH(\varphi)} \sqrt{1 - T_\varphi} \, d\varphi$$

$$= \int_{\Omega M} e^{-tH(\varphi)} \left\{ \mathrm{Tr}\left(S_\varphi^+\right) - \mathrm{Tr}\left(S_\varphi^-\right) \right\} \, d\varphi$$

Now remember $\quad \text{index}(D) = \mathrm{Tr}\left(e^{-tD^*D}\right) - \mathrm{Tr}\left(e^{-tDD^*}\right)$

and

$$\langle x | e^{-t\Delta} | y \rangle = \int_{\substack{\text{paths} \\ \Omega_t(x,y)}} e^{-H(\varphi)} \, d\varphi$$

Now when a vector bundle V is around with connection, then you expect $\quad \Delta_V = \nabla_V^* \nabla_V$

$$\underset{\in \mathrm{Hom}(V_y, V_x)}{\langle x | e^{-t\Delta_V} | y \rangle} \approx \int e^{-H(\varphi)} \, T_\varphi \, d\varphi.$$

fudge factors $\quad \nabla^*\nabla \neq D^*D \quad$ by scalar curvature.

so this scalar has to be put in.