In finite dimensions, formulas which enable us to evaluate integrals over fixed-point submanifolds.

Finite-dim: symplectic manifold $M^{2n}$ compact, action of $S^1$ (Duistermaat - Heckman)

A symplectic form, $\frac{\omega^n}{n!} \neq 0$ volume

$x$ generator of $S^1$ action.

$$dH = i(x) \omega$$

$H$ Hamiltonian

$$\int e^{-tH} \frac{\omega^n}{n!}$$

(Think of this formally as $\int_M e^{-\omega - tH}$)

Thm: $\int_M e^{-tH} \frac{\omega^n}{n!} = \sum_P \frac{e^{-tH(P)}}{T^n \prod_{j=1}^{m} |m_j|}$ isolated fixed points.

where $\mathbf{z}_1, \ldots, \mathbf{z}_m$ are the weight of $S^1$ on $T_p(M)$.

Atiyah - Bott gave a cohomology proof.

$$c(M) = \prod (1 + x_j)$$

when the $x_j$ are formal roots.

$$\int_M e^{-tH} \frac{\omega^n}{n!} = \sum_P \int_N \frac{e^{-tH(N)}}{\prod (t m_j + x_j)}$$
$\omega$ can be allowed to degenerate, except $\omega^n = 0$ if $S \subset M$, and defines $[S] \in H^1(M, \mathbb{Z}_2)$. Thus this represents $\omega_1$ of $M$.

Assume $M$ orientable.

Given a Riemannian metric then $\omega$ defines skew-adjoint $A = -A^*$. \begin{equation*} \det A = (\text{Pf}A)^n, \end{equation*} \begin{equation*} \frac{\omega^n}{n!} = \text{Pf}A : \text{Riemannian volume} \end{equation*}

\begin{equation*} M = \Omega M = \text{Map}(S^1, M) \end{equation*}

\begin{equation*} \dim M = 2n \quad \text{compact oriented with Riem. metric} \end{equation*}

\begin{equation*} \varphi : S^1 \rightarrow M \end{equation*}

\begin{equation*} T\varphi = \text{all vector fields along } S^1 = \Gamma(\varphi^*TM) \end{equation*}

\begin{equation*} \text{has inner product } \int_{S^1} \langle \vec{\gamma} \rangle \end{equation*}

\begin{equation*} \nabla \text{ natural skew-adjoint operator } \nabla \text{ cov. deriv.} \end{equation*}

This form can be degenerate e.g. if geodesic.

$S^1$ acts on $\Omega M$. preserves $\omega$.

Hamiltonian = Energy of loop $= \frac{1}{2} \int |T\varphi|^2$

What is $\Omega M$ orientable? Answer: $\iff M$ is a spin manifold.

Explanation: Look where the form becomes degenerate and calculate the element of $H^1(\Omega M, \mathbb{Z}_2)$ and you get $\omega_2 \in H^2(M, \mathbb{Z}_2)$. Alternatively can look at the transport or monodromy along $\varphi$, call it $T\varphi$, determined up to conjugacy in $SO(2n)$.\begin{align*} \text{Lemma: } \det (T\varphi) &= \det (1 - T) = \prod (1 - e^{i\alpha})_j (1 - e^{-i\alpha})_j \\ &= \prod \left[ 2 \sin (\alpha/2) \right]^2 \end{align*}
\[ Pf \left( \nabla_\theta \right) = \sqrt{\text{det}(1-T)} \]

so the manifold will be orientable when this can be chosen coherently.

\[ X(\delta^+) - X(\delta^-) \]

\[ \text{Spin}(2n) \quad \sqrt{\text{det}(1-T)} \quad \text{is defined on} \quad \text{Spin}(2n) \]

\[ \text{SO}(2n) \quad \text{det}(1-T) \quad \sum \left[ \delta \text{Tr}(\Lambda \delta T) \right] = X(\Lambda^{(0)}) - X(\Lambda^{(0)}) \]

Thus \( M \) spin \( \Leftrightarrow \) all monodromy can be lifted to spinor gb.

\[ \Leftrightarrow Pf \left( \nabla_\theta \right) \text{ defined.} \]

so everything is now set up to apply formula

Fixpts of \( S' \) on \( \Sigma M \) are constant before.

(a) Fixpt Contribution

\[ \sum_{M} \frac{e^{i\pi \gamma} \cdot \epsilon \cdot \omega}{\prod (t^{m} + x_{j})} \]

normal directions are non-constant Fourier exp.

so denom. is

\[ \prod_{n=1}^{\infty} \prod_{j} (t^{n} + x_{j}) = \prod_{n=1}^{\infty} \frac{\sin \frac{\alpha_{j}/2}{\alpha_{j}/2}}{\frac{\alpha_{j}/2}{\alpha_{j}/2}} \]

\[ \text{fixpt contribution} = \sum_{M} \prod_{n} \frac{\alpha_{j}/2}{\sin \frac{\alpha_{j}/2}{\alpha_{j}/2}} \hat{A}(M) \]

\[ = \text{index of Dirac operator} \]
(b) Integration over $\Omega M$.

$$
\int_{\Omega M} e^{-tH(\Psi)} \sqrt{1 - T_{\Psi}} \ d\Psi
$$

$$
= \int_{\Omega M} e^{-tH(\Psi)} \left\{ \text{Tr}(S^+_{\Psi}) - \text{Tr}(S^-_{\Psi}) \right\} \ d\Psi
$$

Now remember $\text{index}(D) = \text{Tr}(e^{-tD^*D}) - \text{Tr}(e^{-tDD^*})$

and

$$
\langle x | e^{-t\Delta} | y \rangle = \int_{\Omega_4(x,y)} e^{-H(\Psi)} \ d\Psi
$$

Now when a vector bundle $V$ is around with connection, then you expect $\Delta_V = \nabla^* V \nabla_V$

$$
\langle x | e^{-t\Delta_V} | y \rangle \leq \int_{\text{Hom}(V_y, V_x)} e^{-H(\Psi)} \frac{1}{T_{\Psi}} \ d\Psi.
$$

Fudge factors $\nabla^* V \neq D^* D$ by scalar curvature.

So this has to be put in.