How can we find a link between convex bodies and algebraic varieties to prove theorems about convex bodies.

Isoperimetric inequalities

In $\mathbb{R}^2$

$$L^2 \geq 4\pi A$$

-shows suffices for convex

In $\mathbb{R}^n$ K bounded convex

$$(\text{vol } \partial K)^n \geq n^n (\text{vol } B^n) (\text{vol } K)^{n-1}$$

-unit ball in $\mathbb{R}^n$.

In and out radius

inequalities

All these can be proved by methods in algebraic geometry.

1. Mixed Volumes $K_1, K_2 \subset \mathbb{R}^2$ convex

$$\lambda K_1 + \lambda_2 K_2 = \{ \lambda_1 x_1 + \lambda_2 x_2, \ x_1 \in K_1 \}$$

E.g.

$$\frac{2}{3} K_1 + \frac{1}{3} K_2 \quad \frac{1}{3} K_1 + \frac{2}{3} K_2$$
Fact
\[ \text{vol}(\lambda_1 K_1 + \lambda_2 K_2) = \lambda_1^{\frac{n}{n-1}} \text{vol}(K_1) + \lambda_2^{\frac{n}{n-1}} \text{vol}(K_2) + \lambda_1^{\frac{1}{n-1}} \lambda_2^{\frac{1}{n-1}} \text{vol}(K_1, K_2) \]

Now look at
\[ \text{vol}(K + \varepsilon B_2) = \text{vol}(K) + \varepsilon \text{vol}(K, B_2) + O(\varepsilon^2) \]
\[ \Rightarrow \text{vol}(K + \varepsilon B_2) = \text{vol}(\partial K) \]

In \( \mathbb{R}^n \)
\[ \text{vol}(\lambda_1 K_1 + \lambda_2 K_2) = \lambda_1^{\frac{n}{n-1}} \text{vol}(K_1) + \lambda_2^{\frac{n}{n-1}} \text{vol}(K_1, K_2) + \lambda_1^{\frac{1}{n-1}} \lambda_2^{\frac{1}{n-1}} \text{vol}(K_1, K_2) + \cdots \]

Then \( n v_1(K, B_n) = \text{vol}(\partial K) \)

Hence iso-perimetric inequality is equivalent to
\[ v_1(K, B_n)^n \geq \text{vol}(B_n) \text{vol}(K)^{n-1} \]

Shall prove \( \forall K_1, K_2 < \mathbb{R}^n \), we have
if \( v_j(K_1, K_2), \quad 0 \leq j \leq n \)
\[ v_0/v_1 \leq v_1/v_2 \leq \cdots \leq v_{n-1}/v_n \]
(Alexandroff-Fenchel inequalities 1930's)

These imply \( (v_0/v_1)^n \leq v_0/v_n \) i.e. \( v_0^{n-1}v_n \leq v_1^n \)

Proof: Enough to prove for convex polyhedra with integral vertices, i.e. \( K_i = S_i \), \( S_i \) finite
\( \chi \in Z^n \iff \text{character of } T^*_C = (C^*)^n \)

\( V_s = \bigoplus_{\chi \in S} V_\chi \), \( T^*_C \) acts on \( P_S = P(V_s) \)

We can assume orbit of \( \lambda, \ldots, \lambda \) is faithful; otherwise \( \hat{S} \) has 0 vol.

Assume differences \( x - \beta \), \( \gamma \in S \) span \( Z^n \).

Get \( T^*_C \hookrightarrow P_{S_1} \times P_{S_2} \) wi Kahler forms on \( P_S \).

call the image \( \overline{X} \); \( \overline{X} \) is proj. var. Moment makes

\[ \mu_i : \overline{X} \longrightarrow \mathbb{R}^n \]

\[ \text{Im}(\mu_i) = \hat{S}_i = K_i \]

Used this to get

\[ \text{vol} (\lambda_1 K_1 + \lambda_2 K_2) = \frac{1}{n!} (\lambda_1 \omega_1 + \lambda_2 \omega_2)^n [X] \]

So

\[ \Rightarrow \quad V_j (K_1, K_2) = \frac{1}{n!} \omega_1^{n-j} \omega_2^j [X] \]

AF ineqs \( \iff \quad V_j^2 \geq V_{j+1}, V_{j-1} \).

Can reduce to a surface

\[ S = \overline{X} \cap n-j-1 \text{ hyperplanes generic in } P_{S_1} \cap j-1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad P_{S_2} \]

Will assume for the moment that \( \overline{X} \) is non-sing.

Bertini \( \Rightarrow \) \( S \) non-singular surface.

\[ [S] = \omega_1^{n-j-1} \omega_2^{j-1} [X] \]

\[ V_j = \omega_1 \omega_2 [S], \quad V_{j-1} = \omega_1^2 [S], \quad V_{j+1} = \omega_2^2 [S] \]

Take \( C_i = S \cap \text{gen. hyperplane of } P_{S_i} \) curves on \( S \)
Then \[ v_j = C_1 C_2 \quad v_{j-1} = C_1^2 \quad v_{j+1} = C_2^2 \]

and so we want \[ (C_1 C_2)^2 \geq C_1^2 C_2^2. \]

Consider quadratic form \[ (C_1 + \lambda C_2)^2 = C_1^2 + 2\lambda C_1 C_2 + \lambda^2 C_2^2 \]

Derived ineq. \( \Leftrightarrow \) form is indefinite \( \Leftrightarrow \) form not pos. def. (since \( C_i > 0 \))

Precisely the Hodge index thm implies this.

Can desingularize \( S \). But actually the singularities of \( \overline{X} \) can be resolved by a combinatorial process. However can chop off corners to reduce to a non-singular case.

When is \( \overline{X} \) non-singular? \( n = 2 \)

Example:

\[
\begin{array}{c}
1,1 \\
0,0 \\
2,0
\end{array}
\]

\[
\begin{array}{c}
1,1 \\
2,1
\end{array}
\]

orbit \( (s, t, s^2t, s^2) \quad s, t \in C^* \)

\[ 2x^2 = y^2 \] defines \( \overline{X} \)

If you add in \( 0 \) then \( \overline{X} \) is nonsing.

Faces of \( K = \hat{S} \leftrightarrow \) orbits in \( \overline{X} \)

vertices \( \leftrightarrow \) fixpts.

Enough to show all fixpoints are non-singular.