$\boxed{\text{Cuntz}}$

quasi-homomorphism from A to B

is a triple $(\varphi, \bar\varphi, \mu)$, $A \underset{\bar\varphi}{\overset{\varphi}{\rightrightarrows}} \mathcal{E} \rhd \underset{\text{ideal}}{J} \xrightarrow{\mu} B$

$\varphi(x) - \bar\varphi(x) \in J$ $\qquad$ $\varphi, \bar\varphi, \mu$ homomorphisms.

(can always assume J is the ideal generated by the differences, $\mathcal{E}$ gen. by $\bar\varphi(\mathcal{A}), \varphi(\mathcal{A})$

μ injective (for C^* alg. true that $\operatorname{Ker}\mu$ is an ideal in $\mathcal{E}$)

$$D(x) = \mu(\varphi(x) - \bar\varphi(x)) : A \longrightarrow B \qquad \text{linear}$$

$$Q(x,y) = \mu\big(\bar\varphi(x)(\varphi(y) - \bar\varphi(y))\big) : A \times A \longrightarrow B \qquad \text{bilinear}$$

Essential information for a quasi-homom. is contained in D, Q. These can be axiomatized.

From now on assume μ injective, and drop.

$$\varphi_0, \bar\varphi_0 \sim \varphi_1, \bar\varphi_1 \iff \exists\ \psi_t, \bar\psi_t \qquad t \in [0,1]$$

D_t, Q_t continuous

Prop: $(\varphi_0, \bar\varphi_0)$, $(\varphi_1, \bar\varphi_1)$ are homotopic $\iff (\psi, \bar\psi) : A \to B[0,1] \ni$

$$KK(A,B) = \{\text{homotopy classes of } (\varphi, \bar\varphi) : A \to \mathcal{K} \otimes B\}$$

uses $\mathcal{K} \oplus \mathcal{K} \subset M_2(\mathcal{K}) \cong \mathcal{K}$. to define $+$ and to show it is a group.

$$\begin{tikzcd}
K_0(\mathcal{A}) \arrow[r, "\varphi_* - \bar\varphi_*"] \arrow[d, dotted] & K_0(\mathcal{E}) \arrow[r] & K_0(\mathcal{E}/J) \\
K_0(J) \arrow[ur] \arrow[r, "\mu"] & K_0(\mathcal{B}).
\end{tikzcd}$$

$$A \longrightarrow \mathcal{L}(H_+) \times \mathcal{L}(H_-)$$

~~[illegible]~~ $F^2 = 1 \Longrightarrow H_+ \simeq H_-$

so we get two homomorphisms $A \rightrightarrows \mathcal{L}(H)$

and we assume $[F, a] \in \mathcal{K} \otimes B$

<u>Thm:</u> (Pedersen) $B \xrightarrow{\pi} B/J$ quotient map, B sep.

Then every derivation δ of B/J lifts to a deriv. of B.

$\mathcal{A}$, $(H^{\pm}, F)$ defines character τ_n $H_\lambda^n(\mathcal{A})$

1) $H_\lambda^*(\mathbb{C}) = \mathbb{C}[\sigma]$
cup product with σ defines $H_\lambda^n(\mathcal{A}) \xrightarrow{S} H_\lambda^{n+2}(\mathcal{A})$
gives relation between τ_n, τ_{n+2}.

2) $\langle K_0(\mathcal{A}), H_\lambda^{\text{pair}}(\mathcal{A}) \rangle \longrightarrow \mathbb{C}$

gives $e \longmapsto \mathrm{Ind}(ePe)$

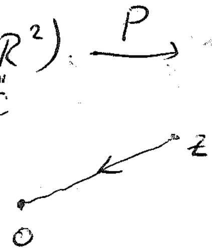
Cuntz: Γ = groupe libre deux gen
On a module de Fredholm $\ell^2(\Gamma) \underset{Q}{\overset{P}{\rightleftarrows}} \ell^2(\Gamma \cup \Gamma \cup pt)$
1-summable.
τ trace canonique sur $H_\lambda^0(\mathcal{A})$
$$\mathcal{A} = \{x \mid [F, x] \in \mathcal{L}^1\}$$

$\therefore$ τ has $\mathbb{Z}$-values on $K^0(\mathcal{A})$
$\Rightarrow$ no non-trivial idempotents in $\mathbb{C}[\Gamma]$

$SL(2, \mathbb{R})$ non-Euclidean

$\mathbb{R}^2$ Euclidean
$\mathcal{A} = \mathcal{S}(\mathbb{R}^2)$ muni de la convolution
$\wr\!\wr$
$\mathcal{S}(\hat{\mathbb{R}}^2)$ produit
$H_\lambda^2(\mathcal{A}) \ni \varphi : \hat{f}^0 \cdot \hat{f}^2 = \int_{\mathbb{R}^2} \hat{f}^0 \, d\hat{f}^1 \, d\hat{f}^2$
$K_0(\hat{\mathbb{R}}^2)$

module de Fredholm tres simple a decrire.
$H^+ = L^2(\mathbb{R}^2) \xrightarrow{P} H^- = L^2(\mathbb{R}^2)$
$\overset{\shortparallel}{\mathbb{C}}$



$\dfrac{z}{|z|}$ $\qquad p(z) = \dfrac{z}{|z|}$

$$F = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}$$

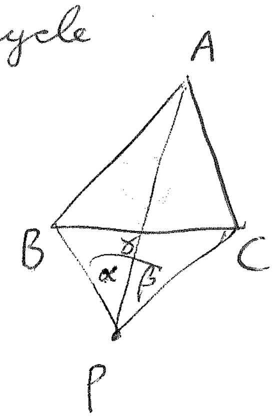
F commute presque avec les translation

module de Fredholm $(2+\varepsilon)$-summable.

calculer character

$$\varphi(f^0, f^1, f^2) = \int_{x^0+x^1+x^2=0} f^0(x^0)\, f(x^1)\, f(x^2)\; c(x^1, x^2)\, dx^1 dx^2$$

2-cocycle



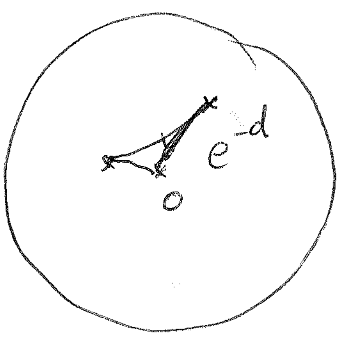
$\alpha+\beta+\gamma = 0$ implies convergence in

$$c(A,B,C) = \int_{\text{Plan}} (\sin\alpha + \sin\beta + \sin\gamma)\, dP$$

$$\underset{\text{claim.}}{=} 4\ \overset{(or.)}{\text{Aire}}\,(A,B,C)$$

Demonstration: pos. if pos. oriented  additive $\Rightarrow$ [redacted] it is a measure QED.

Geometric non-Euclidean



$$L^2(U)$$

$$p(g) = k^2 \qquad G = KAK$$
$$\qquad\qquad\qquad g = k'ak$$

$$L^2(G) \oplus L^2(G)$$

2-summable $\quad \mathcal{S}(G) \subset C_c^\infty(G)$

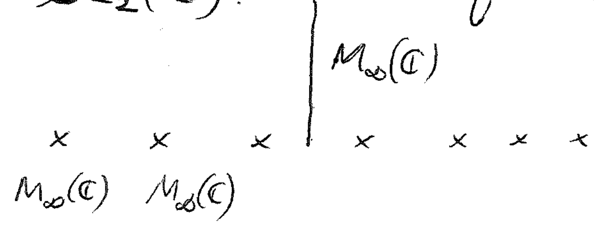
$$\varphi(f^0, f^1, f^2) = \int f^0(g^0)\, f(g^1)\, f^2(g^2)\, c(g^1, g^2)$$

gen. de $H^2(G/K, \mathbb{R})$ demonstrates integrality

(3) Premier question: $\varphi \in H^n_\lambda(\mathcal{A})$ est-ce que $\varphi \in \operatorname{Im} S$

Lemme: $\varphi \in \operatorname{Im} S \iff \varphi = 0$ dans $H^n(\mathcal{A}, \mathcal{A}^*)$

Ex. $SL_2(\mathbb{C})$: transform. de Fourier gives description of $\mathcal{S}(G)$ as sum of matrix rings.



no Hochschild cohomology in dim 2, so φ above comes from degree 0. i.e. a trace on $\mathcal{A}$ $= \mathcal{S}(G)$.

Decompose $H = \bigoplus_n H_n$. Then $\varphi = \sum \varphi_n$.

$$\varphi_n = \pm S \underset{\substack{\uparrow \\ \text{trace of discrete series.}}}{\sigma_m}$$

Thm: deux cycles sont cobordant $\iff$ ses traces τ, τ' verifient $\tau - \tau' \in \operatorname{Im} B$ where

$$H^{n+1}(\mathcal{A}, \mathcal{A}^*) \xrightarrow{B} H^n_\lambda(\mathcal{A}) \xrightarrow{S} H^{n+2}_\lambda(\mathcal{A}) \xrightarrow{I} H^{n+2}(\mathcal{A}, \mathcal{A}^*)$$

Def: $B\varphi = A B_0 \varphi$ $\qquad B_0\varphi(x^0, \dots, x^n) = \varphi(1, x^0, \dots, x^n) - (-1)^n \varphi(x^0, \dots, x^n, 1)$.

Crucial lemma: $B^2 = 0$ $\qquad Bb + bB = 0$

$\operatorname{Im} B$ on $C^{n+1}(\mathcal{A}, \mathcal{A}^*)$ is exactly $C^n_\lambda(\mathcal{A})$.

Thm: suite exacte.

(4) $\mathcal{A} \longrightarrow \mathcal{L}(H)/\mathcal{L}^p(H)$

$\mathcal{A} = C^\infty(S(T_V^*)) \longrightarrow$ pseudo-diffl

usual character $2n+1$ OK but
$2n-1$ has $\log \rho$ divergence. But τ_{2n+1} has Hochschild part 0, so comes from τ_{2n-1}.

Corollaire: $\begin{pmatrix}\text{Cobordism de cycles} \\ \text{sur } \mathcal{A}\end{pmatrix} \otimes_{\mathrm{Cob.}(\mathbb{C})} \mathbb{C} = \lim_{S} H^n_\lambda(\mathcal{A})$

(Conner-Floyd business)

Cor: $H\left[\left(\Omega(\mathcal{A})/[\Omega(\mathcal{A}),\Omega(\mathcal{A})]\right)^*, d\right] = H^*_\lambda(\mathcal{A})/\mathrm{Im}\, B$

Deux Lemmes Imp.

Lemma: $\underbrace{\mathrm{Im}\, B \cap \mathrm{Ker}\, b / b(\mathrm{Im}\, B)}_{H^n_\lambda(\mathcal{A})} \longrightarrow \mathrm{Ker}\, B \cap \mathrm{Ker}\, b / b(\mathrm{Ker}\, B)$ est un [redacted] isom.

($\Leftrightarrow$ one spec. seq. of bicomplex degenerates)

Lemma: $bB^{-1} = S$

$\varphi(a^0 \cdots a^n) = \sum_{\prod g^i = 1} a^0(g^0) \cdots a^n(g^n)\, c(g_0, \ldots, g_n)$

$\mathbb{C}(\Gamma)$

c bien-normalized cocycle

~~[illegible]~~ maps $H^*(\Gamma)$ into $\mathrm{Ker}\left(H^*(\mathcal{A}) \xrightarrow{I}\ \right)$

Module de Fredholm p-summable

$H^+ \oplus H^-$ sur $\mathcal{A}$, $F^2 = 1$, $[F, a] \in \mathcal{L}^p(H)$

$\mathcal{A} \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \cdots \xrightarrow{d} \Omega^{2n}$

$da = i[F, a]$ $\qquad\qquad$ Ω^1 = space gen by $a^0 da^1$

$$\tilde{P} = \begin{bmatrix} P & 1 - QP \\ 1 - PQ & Q(2 - PQ) \end{bmatrix}$$

agit sur $\tilde{H}^+ = H^+ + \underset{\substack{\uparrow \\ a = 0 \text{ here}}}{H^-}$

<u>Def</u>: Soit $\mathcal{A}$ une algèbre, un cycle (dim n) sur $\mathcal{A}$ est donné par 1) une algebra graduee $\Omega = \Omega^0 \oplus \cdots \oplus \Omega^n$

$d^2 = 0$ $\qquad$ $d(\omega_1 \omega_2) = d\omega_1\, \omega_2 + (-1)^{\partial\omega_1} \omega_1\, d\omega_2$

2) un homomorphisme $\rho : \mathcal{A} \to \Omega^0$

3) une trace graduee ferme $\Omega^n \xrightarrow{\int} \mathbb{C}$

$$\int d\omega = 0 \qquad \int \omega_1 \omega_2 = (-1)^{\partial\omega_1 \partial\omega_2} \int \omega_2\, \omega_1$$

<u>Ex</u>: X Lipschitz space (metric space)

$V \xrightarrow{f} X$ $\qquad$ f Lipschitz

Complex of ~~[illegible]~~ Whitney forms on V - Lip coeffs whose d is also Lip. Then

$$\begin{array}{l} \mathcal{A} \xrightarrow{f^*} \Omega^0_V \to \Omega^1_V \to \cdots\cdots \to \Omega^n \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \int_V \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbb{C} \end{array}$$

gives a cycle sur $\mathcal{A}$ = Lip fns. on X.

Given a cycle consider its

$$\varphi(f^0, \dots, f^n) = \int f^0 df^1 \cdots df^n \qquad f = \rho(f)$$

<u>Prop</u>: Pour que $\varphi \in C^n(a, a^*)$ soit le caractère d'un cycle il faut et il suffit que

a) $\varphi(f^0 f^1, f^2, \dots f^{n+1}) - \varphi(f^0, f^1 f^2, \dots) + \cdots$

$$+ (-1)^{n+1} \varphi(f^{n+1} f^0, f^1, \dots) = 0$$

(Here $T(f^1 \;\; f^n) \in a^* \longleftrightarrow \varphi \in (a^{\otimes n+1})^*$

$$T(f^1, \dots, f^n)(f^0) = \varphi(f^0, \dots, f^n)$$

$(a\psi)(x) = \psi(xa)$ left action of a on a^*

$$0 = f^1 T(f^2, \dots) - T(f^1 f^2, \dots) + \cdots + (-1)^{n+1} T(f^1, \dots, f^n) f^{n+1})$$

b) $\varphi(f^1, \dots, f^n, f^0) = (-1)^n \varphi(f^0, \dots, f^n)$

($\cancel{(f^0 df^1)(df^2 \cdots df^n) = (-1)^{1(n-1)} \int df^2 \cdots df^n f^0 df^1}$

$$\int f^0 (df^1 \cdots df^n) = \int (df^1 \cdots df^n) f^0$$

$$= \int d\left[(f^1 df^2 \, df^n) f^0\right] + (-1)^{n-1} \int f^1 df_2 \, df^n df^0)$$

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universal diff^l alg. $\Omega(a)$ $\qquad d(ab) = da \cdot b + a \cdot db$

$\cancel{d1 = 0}$

$$\Omega^1(a) = \tilde{a} \otimes a = a \otimes a + 1 \otimes a$$

$\varphi \in C^{n-1}(\mathcal{A}, \mathcal{A}^*)$ $\qquad \varphi^\lambda = \varepsilon(\lambda)\varphi$ $\qquad \lambda$ cyclic

on a $\quad (b\varphi)^\lambda = \varepsilon(\lambda)\, b\varphi$

Connes prefers to write this

$$b \circ A = A \circ b' \qquad A\psi = \sum_{\text{cyclic}} \varepsilon(\lambda)\psi^\lambda$$

$$(b'\psi)(a^0, \dots, a^{n+1}) = \psi(a^0a^1, a^2, \dots) + \dots\dots + (-1)^n \psi(a^0, \dots, a^n a^{n+1})$$

(note last term missing)

b' given by cup product with 1-cochain $\psi(1,1) = 1$.

$(b')^2 = 0$ and the complex for b' is trivial

Lemma: need C matrices infinies $\{a_{ij}\}$ finite rows + columns only finitely many elts from $\mathcal{A}$. $\exists\, \rho : C \to C$

$$u \begin{bmatrix} 0 & 0 \\ 0 & \rho(x) \end{bmatrix} u^{-1} = \begin{bmatrix} x & 0 \\ 0 & \rho(x) \end{bmatrix}$$

Then no matter what cohomology theory you consider $\mathcal{A} \otimes C$ gives 0.

Also interior derivations of $\mathcal{A}$ give identity on $H^*_\lambda(\mathcal{A})$. $\qquad \delta x = xa - ax$

$$\psi(a^0 \dots a^n) = \sum \psi(a^0 \quad \delta a^i \dots a^n)$$

Lemma: Pour qu'un cycle $\varphi \in Z^n_\lambda(\mathcal{A})$ soit un bord iff et if s. que φ soit le caractere d'un cycle Ω avec Ω^0 flasque

Thm: Le cup produit donne une app. bilineaire

$$\varphi, \psi \longmapsto \varphi \# \psi$$

$$H^n_\lambda(\mathcal{A}) \otimes H^m_\lambda(\mathcal{B}) \longrightarrow H^{n+m}_\lambda(\mathcal{A} \otimes \mathcal{B})$$

$\mathcal{A} = \mathbb{C}$ $\qquad \sigma(1,1,1) = 1$

$$\begin{array}{ccc} \mathbb{C} & \Omega^1 & \Omega^2 \\ 1 & d1 & d1d1 \\ & 1d1 & 1d1d1 \end{array} \qquad \int 1d1d1 = 1$$

$$\mathcal{A} \xrightarrow{\mathcal{L}} \Omega^0 \to \Omega^1 \to \cdots$$

$$\Omega^i \otimes C \oplus \Omega^i \Sigma \oplus \Omega^i \Sigma^2$$

$$\int (f^0 \otimes 1)\, d(f^1 \otimes 1) \cdots d(f^{n+2} \otimes 1)$$

since $1^2 = 1 \qquad 1d1\,1 = 0 \qquad 1d1d1 = d1d1\,1$

$$d(f \otimes 1) = df \otimes 1 + f \otimes d1$$

$$(\varphi \# \sigma)(f^0, \ldots, f^{n+2}) = \varphi(f^0 f^1 f^2, f^3, \ldots) + \cdots\cdots$$

messy to write in terms of φ because of the terms $f^0 df^1 f^2 f^3$

Cor: $H^*_\lambda(\mathcal{A})$ est un module sur $H^*_\lambda(\mathbb{C}) = \mathbb{C}[\sigma]$

$(2\pi i)^m m!\, \mathrm{Tr}(\varepsilon\omega)$ is the good normalization $m = \frac{n}{2}$ to relate $2m$-trace with $2m+2$-trace etc. coming from a p-summable Fredholm module.

Ex. $\mathbb{C}/\Gamma \qquad \bar\partial$

$$\mathrm{Tr}\ \varepsilon f^0 \underbrace{[F,f^1]}_{\mathcal{L}^{2+\varepsilon}} [F,f^2] \overset{\mathrm{def}}{=} \frac{1}{2}\mathrm{Tr}\left(\varepsilon F [F,f^0][F,f^1][F,f^2]\right)$$

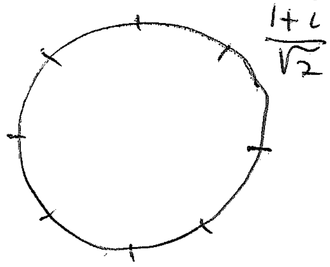
non $\mathcal{L}^2$ so one makes

$$\mathrm{Tr}\ \varepsilon f^0 df^1 df^2 = \frac{1}{2i\pi}\int f^0 df^1 df^2$$

The good justification for the 2π is when one does cup product of two degree 1 cycles in Kasparov

$M, N \qquad M^2+N^2=1$

introduces the circle



$$\int \sigma^{-1} d\sigma = 2\pi i$$

$$K_{odd} \otimes K_{odd} \xrightarrow{\cup\ \mathrm{Kasp}} K_{even}$$

$$H_\lambda^{odd}(A) \otimes H_\lambda^{odd}(B) \longrightarrow H_\lambda^{even}(A \otimes B)$$

If you want compatibility with chern character you must introduce $2\pi i$.

Thm: $\mathrm{Ind}(ePe) = \frac{1}{m!}\frac{1}{(2\pi i)^m}\tau_{2m}(e,\dots,e)$

où τ_{2m} est le caractère de $F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$.