§1. Generalization of Thm of Narasimhan–Seshadri:

\[ Y \text{ alg. curve, } s : \pi_1(Y) \to SU(2) \text{ irreducible, } \] get holomorphic v.r. a \ Y

Thm: This bundle is stable and all stable bundles of rank 2 and deg 0 arise from a unique repn. up to equivalence.

Def: \( E \) is stable \((\text{rank } 2, \deg E = 0) \Leftrightarrow \) for all \( L \) with \( L \to E \) one has \( \deg L < 0 \).

Prop (i) Stable, parametrized by a Hausdorff moduli space

(ii) Stability an open condition

For line bundles, Jacobian = reps. \( \pi_1(Y) \to U(1) \).

§2. Extension to alg. surfaces.

\[ X \hookrightarrow \mathbb{CP}^N \]

Again have notion of stable bundle over \( X \), same definition with degree defined by restricting to a curve, which is a generic hyperplane section.

Given any holm. v.r. with metric \( F \) unique connection: \( \bar{\partial}_A = \partial_A + \bar{\partial}_A \quad \bar{\partial}_A s = 0 \Leftrightarrow s \text{ holm.} \)

Give \( X \) a Kähler metric consistent with the embedding, e.g. induced from \( \mathbb{CP}^N \).

Thm: A stable bundle \( E \) over \( X \) \((\text{rank } 2, \Lambda^2 E = 0) \) has a unique consistent anti-self dual \( SU(2) \)-connection w.r.t. the Kähler metric.

\[ \Lambda^2 T_X^+ = \Lambda_+^2 \oplus \Lambda_-^2 \text{ so any connection } \]

For Riemann \( M^4 \) \( SO(4) \sim SO(3) \times SO(3) \).
\[ \text{So}(4) \sim \text{So}(3) \times \text{So}(3) \]
\[ \uparrow \quad \uparrow \quad \rightarrow \quad \rightarrow \]
\[ \text{U}(2) \sim \text{U}(1) \times \text{So}(3) \]

\( \Lambda^2 \) splits into Kahler form + canonical bide.

In Kahler case
\[ F_\pm = \hat{F} \omega + \left( F_0^{0,2} + F_0^{0,0} \right) \]
\[ \text{these vanish for a holom. bundle} \]

Thus thm. says you can kill \( \hat{F} \).

Equiv to \( F_\pm \omega = 0 \).

General conjecture relating stable bundles to hermitian-Einstein connections. *essentially Ricci zero.*

**Example:** On CP\(^2\), moduli space of bundles as above and \( c_2 = 2 \) is equivalent to space of non-singular conics in \( P^*_2 \).

Compactify by adding pairs of lines to \( P^*_2 \) (= degen. conics) or pairs of points in \( P^*_2 \).

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**§3. General theory of stability**

\[ G, G^c \text{ on } A \subset CP^n \]

via a repn. \( G \longrightarrow \text{SL}_{n+1}(\mathbb{C}) \).
Given $x \in \mathbb{P}^n$ choose $x'$ over it $\in C^{n+1}$. $x$ stable $\iff G^c \rightarrow C^{n+1}$ is proper.

Criteria of Kempf + Ness: Choose norm on $C^{n+1}$. Then an orbit in $C^{n+1}$ is stable iff it contains a point of minimal norm, (and then this is unique up to the action of the compact gp. $G$, assuming metric fixed under $G$.)

Metric gives symplectic structure on $\mathbb{CP}^n$ hence there are moment maps. The gradient of $\log |x|^2$ given by the moment map. That is, if $u \in G^c$

$$\mu_p \log |x|^2 = (\mu, iu)$$

Hence a point of minimal norm $= \text{zero of moment map}$. Thus in f.d. case $\exists$ good representative in the orbit picked out by the orbit.

§ 4. Application to our problem

Fixed $C^\infty$ bln. + metric $E$ over $X$.

$A$ = all connections on $E$.

$
\text{Subset } A^{(1,1)} \text{ of ones with (1,1)-curvature.}$

I.e. $\nabla_A$ part has square zero $\Rightarrow$ holom. structure.

Gauge gp. $G^c$ acts on $A^{(1,1)}$.

$G^c$ acts on holom. structures.

$$g : \nabla_A \mapsto g \nabla_A g^{-1}, \quad \nabla_A \mapsto (g^*)^{-1} \nabla_A g^+$$
Two integrable connections give same holom. bdls if lie in same $\mathbb{G}_c$ orbit.

Now carry over the f.d. machinery.

Symplectic structure on $\mathcal{A}^{(\mathbb{S})}$. Tangent vectors are $\text{End}(E)$ valued $1$-forms:

$$(a, b) = \int_X \text{Tr} (ab) \wedge \omega$$

The moment map for action of $\mathbb{G}$ is

$$A \mapsto F_A \wedge \omega$$

so that anti-self-duality condition $\Leftrightarrow$ this is zero.

Want analogue of $\mathbb{C}^{n+1}$ = a line bundle $L$ over $\mathcal{A}^{(\mathbb{S})}$ acted on by $\mathbb{G}_c$ with metric preserved by $\mathbb{G}$. Then the connections we want lie under points in the orbits of minimal distance from $0$ section.

Note everything goes fairly for Riemann surface, even simpler.

All we really want is the "height" functional. that is, given $A, B$ in same orbit want $M(A, B)$ such that $e^{M(A, B)}$ is the multiplier of the lengths

Familiar picture. You have a fixed holom. bundle and are varying the metric. Thus are working with space $\mathbb{G}_c/\mathbb{G}$ which parametrizes the connections.
up to isomorphism on a given holom. ball.

\[ M(H,K) = \text{functional of two metrics} \]
\[ \text{on same holom. ball } E. \]

\( M(H,K) \) is a convex functional on the space of metrics, so one deduces the uniqueness of a minimum pt. if it exists.

§5. Method of proof.

To find critical points of height is to follow gradient flow. Also gives path for the Yang-Mills flow.

\[ \frac{\partial M(H_t,H_0)}{\partial t} = \mathbf{0} - \|F\|^2 \]

So if \( M \) is bounded below, then \( F \rightarrow 0 \).

Then use analysis of last term to get critical point.

Criterion: \( E \) over \( X \) is stable \( \Rightarrow \) \( E|Y \) is semi-stable for \( Y \in \ln H! \).

\[ M(H,K) = M(H|Y, K|Y) + \int \frac{1}{2} \text{Tr} (F_H^2 - F_K^2) X \]

\( Y = n\omega + i\partial \bar{\partial} \psi \)

\( \psi \) has a mild singularity along \( Y \).