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August 21, 1983

Consider the motion of a particle on the line governed by the Hamiltonian (anharmonic oscillator)

$$H = \frac{p^2}{2} + \frac{\omega_0^2}{2} x^2 + \frac{\lambda}{4!} x^4 .$$

One can think of the imaginary time version of this motion, i.e. the thermal behavior of this particle, as being mathematically the same as the field theory of a real function $x(t)$ of one variable $t \in \mathbb{R}$ with the action

$$S(x) = \int \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} \omega_0^2 x^2 + \frac{\lambda}{4!} x^4 \right) dt$$

Specifically we have identities like

$$(*) \qquad \langle 0 | T[x(t) x(t')] | 0 \rangle = \frac{\int \mathcal{D}x \, e^{-S(x)} x(t) x(t')}{\int \mathcal{D}x \, e^{-S(x)}}$$

where $x(t) = e^{tH} x e^{-tH}$ as an operator. More generally if S_J denotes the scattering operator for the perturbation

$$H + H_{int} = H - J(t) x$$

where $J \in C_0^\infty(\mathbb{R})$, then

$$\langle 0 | S_J | 0 \rangle = \frac{\int \mathcal{D}x \, e^{-S(x) + \int Jx \, dt}}{\int \mathcal{D}x \, e^{-S(x)}} .$$

The identities (*) follow by taking the Taylor-Volterra series in J, and using Dyson's expansion for S_J.

The point however is that the field theory is completely computed in terms of the operators x, H on $L^2(\mathbb{R})$. So one has exact results which one can compare with perturbation results, i.e. formal series in λ.

Let's compute the two point function

$$G(t-t') = \langle 0 | T[x(t)x(t')] | 0 \rangle$$

and its Fourier transform $G(\omega)$ exactly using a basis of eigenfunctions for H. If $t > 0$

$$G(t) = \langle 0 | \, x(t) \, x \, | 0 \rangle = \langle 0 | e^{+tH} x e^{-tH} x | 0 \rangle$$

$$= e^{t\mathcal{E}_0} \sum_n \langle 0 | x | n \rangle e^{-t\mathcal{E}_n} \langle n | x | 0 \rangle$$

$$= \sum_n |\langle n | x | 0 \rangle|^2 e^{-t(\mathcal{E}_n - \mathcal{E}_0)}$$

and if $t < 0$

$$G(t) = \langle 0 | x \, x(t) | 0 \rangle = \langle 0 | x e^{tH} x e^{-tH} | 0 \rangle$$

$$= \sum_n |\langle n | x | 0 \rangle|^2 e^{t(\mathcal{E}_n - \mathcal{E}_0)}$$

so

$$\boxed{\begin{aligned} G(t) &= \sum_n |\langle n | x | 0 \rangle|^2 e^{-(\mathcal{E}_n - \mathcal{E}_0)|t|} \\ G(\omega) &= \sum_n |\langle n | x | 0 \rangle|^2 \frac{1}{\omega^2 + (\mathcal{E}_n - \mathcal{E}_0)^2} \end{aligned}}$$

In particular when $\lambda = 0$ we have

$$\boxed{G_0(t) = \frac{e^{-\omega_0 |t|}}{2\omega_0} \qquad G_0(\omega) = \frac{1}{\omega^2 + \omega_0^2}}$$

This formula shows that $G(\omega)$ is [redacted] pretty complicated. It is determined by the relative energies $\varepsilon_n - \varepsilon_0$ and the amplitude $\langle n|x|0\rangle$ which measures the transition between $|0\rangle, |n\rangle$ produced by x. In the anharmonic oscillator case one knows that the spectrum is simple and the eigenfunctions are alternately even and odd. Hence

$$\langle n|x|0\rangle = 0 \qquad \text{for } n \text{ [redacted] even}$$

so the leading term is

$$G(\omega) = \frac{|\langle 1|x|0\rangle|^2}{\omega^2 + (\varepsilon_1 - \varepsilon_0)^2} + \cdots$$

Now we do things perturbatively.

$$G(\omega) = \frac{1}{\omega^2 + \omega_0^2 - \Gamma_2}$$

$$\Gamma_2 = \mathcal{Q} + \cdots = -\frac{\lambda}{2}\int \frac{dp}{2\pi}\frac{1}{p^2+\omega_0^2} = -\frac{\lambda}{2}\frac{1}{\omega_0}\frac{1}{2}$$

so to first order

$$G(\omega) = \frac{1}{\omega^2 + \omega_0^2 + \frac{\lambda}{4\omega_0} + \cdots}$$

Hence

$$\varepsilon_1 - \varepsilon_0 = \left(\omega_0^2 + \frac{\lambda}{4\omega_0} + \cdots\right)^{1/2} = \omega_0\left(1 + \frac{\lambda}{4\omega_0^3} + \cdots\right)^{1/2}$$

$$= \omega_0 + \frac{\lambda}{8\omega_0^2} + \cdots$$

Next compute $\varepsilon_1, \varepsilon_0$ to first order using

$$\delta\lambda = \frac{\langle\psi|\delta H|\psi\rangle}{\langle\psi|\psi\rangle}$$

$$\delta\varepsilon_0 = \frac{\lambda}{4!}\langle 0|x^4|0\rangle = \frac{\lambda}{4!}\,3\cdot 1\,\frac{1}{(2\omega_0)^2}$$

$$\delta\varepsilon_1 = \frac{\lambda}{4!}\,\frac{\langle 0|x\,x^4\,x|0\rangle}{\langle 0|x\,x|0\rangle} = \frac{\lambda}{4!}\,\frac{5\cdot 3\cdot 1}{1}\left(\frac{1}{2\omega_0}\right)^{\cancel{3}\,2}$$

$$\therefore\ \delta(\varepsilon_1-\varepsilon_0) = \frac{\lambda}{4!}\,\frac{1}{(2\omega_0)^2}\,(15-3) = \frac{\lambda}{8\omega_0^2}$$

which checks.

Let's try to summarize what we have. We are interested in the one-dim'l field $x(t)$ with the action

$$S(x) = \int\left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\omega_0^2 x^2 + \frac{\lambda}{4!}x^4\right)dt.$$

We have a physical interpretation of this field theory in terms of the quantum mechanics of the anharmonic oscillator.

We think of the anharmonic oscillator as a perturbation of the harmonic oscillator. In particular for the low eigenstates and small λ there is a 1-1 correspondence between eigenstates.

On the other the harmonic oscillator has an interpretation involving particles: The n-th eigenstate has n particles present. So we carry this interpretation over the anharmonic oscillator at least for small n. The n-th eigenstate has n particles present, and the energy $\varepsilon_n-\varepsilon_0$ is not exactly $n(\varepsilon_1-\varepsilon_0)$ because the particles are interacting.

However $\varepsilon_1-\varepsilon_0$ is the energy of the 1-particle state and it can be found by looking at the low pole of $G(\omega)$.

Change notation $\omega \longmapsto k$, $\omega_0 \longmapsto m_0$ "bare mass", $\varepsilon_1 - \varepsilon_0 \longmapsto m$ actual mass: Then

$$G(k) = \frac{1}{k^2 + m_0^2 - \Gamma_2(k)}$$

and the denominator will vanish at $k = im$:

$$-m^2 + m_0^2 - \Gamma_2(im) = 0$$

Now we move on to a 2-diml theory where some renormalization problems arise. But we want to preserve the above picture as much as possible. We know the free theory has a nice particle interpretation and we again assume a correspondence between free and interacting eigenstates. (In practice this is carried out by scattering techniques.)

We know the free theory has a single 1-particle state for each momentum $\underline{k}$ and that the energy of this state is $\sqrt{\underline{k}^2 + m_0^2}$. Note that

$$G_0(k) = G_0(\omega, \underline{k}) = \frac{1}{k^2 + m_0^2} = \frac{1}{\omega^2 + \underline{k}^2 + m_0^2}$$

has a pole at $\omega = i\sqrt{\underline{k}^2 + m_0^2}$. We assume that the interacting theory has a corresponding state which is a dressed version of the free state, and which results by "turning on the interaction adiabatically". The problem is to compute the energy of the one-particle state of momentum $\underline{k}$.

This should be given by the pole of

$$G(k) = \frac{1}{k^2 + m_0^2 - \Gamma_2(k)}$$

which is "near" $i\sqrt{\underline{k}^2 + m_0^2}$. I think that because of Lorentz-invariance of the theory, the vertex fn. $\Gamma_2(k)$ is actually a function of $k^2 = \omega^2 + \underline{k}^2$. Hence the equation

$$k^2 + m_0^2 - \Gamma_2(k) = 0$$

has solutions on a hyperboloid

$$k^2 = -m^2 \qquad \text{i.e.} \qquad \omega = \pm i\sqrt{\underline{k}^2 + m^2}$$

where

$$-m^2 + m_0^2 - \Gamma_2(im) = 0 .$$

Now the problem which I run into in 2 dims. is the divergence of the diagram  to Γ_2. We can put in a cutoff, remove it and let $m_0 \longrightarrow \infty$ so as to achieve a given effective mass m. I want to see that this process is all that one has to do. It is a bit complicated because the various terms in Γ_2 depend upon m_0.

August 29, 1983

Recall the way to think about a magnet, at least classically. At each site x is a spin variable $\phi(x)$ which is a unit vector in a vector space ($= \mathbb{R}$ if we assume the spin is ± 1, $= \mathbb{R}^M$ in Kac's spherical model.). Then as usual we have an external field $J(x)$ and an energy $S(\phi)$ for the spin configuration $\phi = (\phi(x))$, and we form the partition function

$$(1) \qquad Z(J) = \int \mathcal{D}\phi \; e^{-\beta[S(\phi) - \int_x J\phi]}.$$

I have written things so they look like a field theory.

Now in magnetism one is interested in the response to a uniform applied field, and one assumes translation invariance, say the sites form a periodic lattice or torus. Then the field $\langle \phi(x) \rangle$ corresponding to a uniform J is independent of x and we have an M-dimensional situation: $\langle \phi \rangle \in \mathbb{R}^M$, $J \in (\mathbb{R}^M)^* = \mathbb{R}^M$. To fix the ideas suppose $M = 1$.

We therefore have a 1-diml. problem: To each $J \in \mathbb{R}$ we get a magnetization $\varphi = \langle \phi(x) \rangle$ for any x. We want to explain this response by means of an effective potential $W(\varphi)$, i.e. the response is given by

$$J = W'(\varphi).$$

Then we also want the susceptibility

$$\chi = \frac{d\varphi}{dJ} = \frac{1}{W''(\varphi)}$$

which is the obvious physical quantity of interest.

The problem is how to get these "bulk" quantities out of the field theory. So we go back to the partition function (1) and suppose J is constant

$$Z(J) = \int \mathcal{D}\phi \; e^{-\beta\left[S(\phi) - J\int_x \phi(x)\right]}$$

Then

$$\frac{1}{\beta}\frac{\partial}{\partial J}\log Z(J) = \left\langle \int_x \phi(x) \right\rangle = \int_x \langle \phi(x) \rangle$$

$$= V\varphi$$

and so we have

$$\varphi = \frac{\partial}{\partial J}\;\frac{\log Z(J)}{\beta V}$$

hence $W(\varphi)$ is the Legendre transform of $\dfrac{\log Z(J)}{\beta V}$.

We can also consider the map

$$\phi \longmapsto \int_x \phi(x)$$

which we will denote p. Then we have

$$Z(J) = \int e^{\beta J\varphi} \cdot p_*\left\{ e^{-\beta S(\phi)} \mathcal{D}\phi \right\}.$$

This shows that $Z(J)$ is a "Laplace" transform of a measure. However the measure has a complicated β-dependence, probably not of the form $e^{-\beta V(\varphi)}d\varphi$.

The next question is how to go about computing the susceptibility, and more generally $W(\varphi)$ in terms of the vertex functions of the field theory.

August 30, 1983

Consider a translationally-invariant field theory like ϕ^4, and think in terms of magnetism. One is then interested in the response to a uniform external field.

In general the response to a general external field $J(x)$ is given by

$$(1) \qquad \phi(x) = \frac{\delta}{\delta J(x)} \log Z(J)$$

where

$$Z(J) = \int \mathcal{D}\phi \; e^{-S(\phi) + \int_x J\phi}$$

Take $J(x)$ to be a constant field $J(x) = J_0$. Then the response is constant: $\phi(x) = \phi_0$ for all x, and

$$\frac{\partial}{\partial J_0} \log Z(J_0) = \left\langle \int_x \phi(x) \right\rangle$$

$$= V \phi_0$$

where V is the volume. Hence

$$\phi_0 = \frac{\partial}{\partial J_0} \frac{\log Z(J_0)}{V}$$

Now the relation between $J(x)$ and $\phi(x)$ can also be expressed

$$J(x) = \frac{\delta}{\delta \phi(x)} W(\phi)$$

where

$$W(\phi) = \int_x J(x)\phi(x) - \log Z(J)$$

is the Legendre transform. Further, in perturbation theory we have

$$J(x) = (-\Gamma_1)(x) + \int (a-\Gamma_2)(x,y)\,\phi(y)$$

$$+ \frac{1}{2}\iint (-\Gamma_3)(x,y,z)\,\phi(y)\,\phi(z) + \cdots$$

Taking $\phi(x) = \phi_0$ to be a constant field gives

$$J_0 = (-\Gamma_1)(0) + \left[\int (a-\Gamma_2)(0,y)\right]\phi_0$$

$$+ \frac{1}{2}\left[\iint (-\Gamma_3)(0,x,y)\right]\phi_0^2 + \cdots$$

where 0 is a basepoint. Also if we started with

$$W(\phi) = \text{const} + \int (-\Gamma_1)(x)\phi(x) + \frac{1}{2}\int (a-\Gamma_2)(x,y)\,\phi(x)\phi(y) + \cdots$$

and put $\phi(x) = \phi_0$, then

$$W(\phi_0) = \text{const} + V\,(-\Gamma_1)(0) + \frac{1}{2}V\left[\int (a-\Gamma_2)(0,y)\right]\phi_0^2 + \cdots$$

This shows that if we define

$$W_0 = \text{Leg. transf. of } \frac{\log Z(J_0)}{V}$$

$$= J_0\,\phi_0 - \frac{\log Z(J_0)}{V}$$

then

$$W(\phi_0) = \int J_0\phi_0 - \log Z(J_0) = V \cdot W_0(\phi_0)$$

<u>Conclusion</u>: For a translation-invariant theory the response to a constant external field is given

by
$$J_0 = \frac{\partial}{\partial \phi_0} W_0(\phi_0)$$

where
$$W_0(\phi_0) = \frac{W(\phi_0)}{V} = \text{const} + (-\Gamma_1)(0)$$
$$+ \frac{1}{2}\left[\int (a-\Gamma_2)(0,y)\right]\phi_0^2 + \frac{1}{3!}\left[\iint (-\Gamma_3)(0,y,z)\right]\phi_0^3 + \cdots$$

In other words we take the average of the vertex functions:

$$\frac{1}{V}\int (-\Gamma_n)(x_1,\ldots,x_n)\,dx_1\ldots dx_n = \int (-\Gamma_n)(0,x_1,\ldots,x_{n-1})\,dx_1\ldots dx_{n-1}.$$

In the momentum picture integrating over space corresponds to setting $k_i = 0$.

Let's now do the calculations in the ϕ^4 theory. In the tree approximation the only 1PI vertex is

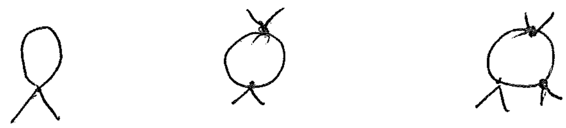
and so
$$\Gamma_4 = \frac{-\lambda}{4!}\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4)$$

which gives
$$W(\varphi) = \int\left[\tfrac{1}{2}\phi(-\Delta+m_0^2)\phi + \frac{\lambda}{4!}\phi^4\right]dx$$
in the tree approximation. Hence

$$W_0(\varphi_0) = \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda}{4!}\phi_0^4.$$

Next consider the 1-loop vertices. On one hand we have $v - e = 1 - \ell = 0$, and on

the other
$$E + 2e = 4v \qquad E = \text{no. of external lines}$$
so we see $E = 2v$. It is easily seen that the 1-loop 1PI diagrams are


etc.

Hence the 1-loop contribution to W is

$$-\frac{(-\lambda)}{4}\int G_0(x,x)\,\phi(x)^2 \;-\; \frac{(-\lambda)^2}{16}\int G_0(x,y)\,\phi(y)^2\, G_0(y,x)\,\phi(x)^2 + \cdots$$

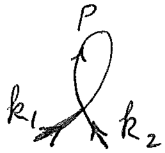
$$= -\frac{1}{2}\log\det\left(1+\frac{\lambda}{2}G_0\phi^2\right)$$

where $G_0\phi^2$ stands for the operator with kernel $G_0(x,y)\,\phi(y)^2$.

So for the effective potential we get in the 1-loop approximation:

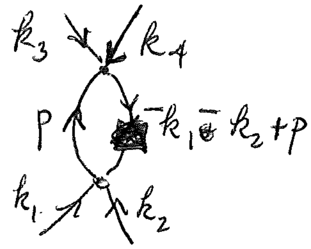
$$W_0(\varphi_0) = \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda}{4!}\phi_0^4 + \frac{\lambda}{4}G_0(0,0)\,\phi_0^2 - \frac{\lambda^2}{16}\left(\int G_0(0,y)\,G_0(y,0)\,dy\right)\phi_0^4 + \cdots$$

Next do the calculations in momentum coords.



$$\Gamma_2(k_1,k_2) = \frac{-\lambda}{2}\sum_p \frac{1}{p^2+m_0^2}\,\delta(k_1+k_2)$$

$$\Gamma_2(0,0) = -\frac{\lambda}{2}\sum_p \frac{1}{p^2+m_0^2}$$



$$\Gamma_4(k_1,k_2,k_3,k_4) = \delta(k_1+k_2+k_3+k_4)\,\frac{(-\lambda)^2}{2} \times \sum_p \frac{1}{p^2+m_0^2}\,\frac{1}{(p-k_1-k_2)^2+m_0^2}$$

+ two similar terms which arise when k_1 meets k_3, or k_1 meets k_4 at a vertex.

Now
$$G_0(x,y) = \frac{1}{V}\sum_p \frac{e^{ip(x-y)}}{p^2+m_0^2}$$

$$G_0(0,0) = \frac{1}{V}\sum_p \frac{1}{p^2+m_0^2} \xrightarrow{V\to\infty} \int \left(\frac{dp}{2\pi}\right)^D \frac{1}{p^2+m_0^2}$$

$$\int G_0(0,y)\,G_0(y,0)\,dy = \frac{1}{V^2}\,V\sum_p \frac{1}{(p^2+m_0^2)^2} \longrightarrow \int\left(\frac{dp}{2\pi}\right)^D \frac{1}{(p^2+m_0^2)^2}$$

Thus
$$\Gamma_2(0,0) = -\frac{\lambda}{2}\int \frac{1}{p^2+m_0^2} = \frac{-\lambda}{2}\,G_0(0,0)$$

$$\Gamma_4(0,0,0,0) = \frac{3\lambda^2}{2}\int\frac{1}{(p^2+m_0^2)^2} = \frac{3\lambda^2}{2}\int G_0(0,y)\,G_0(y,0)\,dy$$

and we have have at the 1-loop level the following corrections to m_0^2 and λ, the bare mass and coupling constant

$$m_0^2 \longmapsto m_0^2 + \frac{\lambda}{2}\int\left(\frac{dp}{2\pi}\right)^D \frac{1}{p^2+m_0^2}$$

$$\lambda \longmapsto \lambda - \frac{3\lambda^2}{2}\int\left(\frac{dp}{2\pi}\right)^D \frac{1}{(p^2+m_0^2)^2}$$

September 1, 1983

What Jackiw told me today.

1) On anomalies: The question was why anomalies, which occur in QED, don't imply inconsistency of QED. Because the anomaly discovered by Adler, Bell, Jackiw was an axial current anomaly, i.e. the current is $\bar{\psi}\gamma^\mu\gamma_5\psi$ which is not conserved. The actual current used in QED is $\bar{\psi}\gamma^\mu\psi$ and this is conserved. At the time people were claiming all kinds of currents were conserved in the quantum theory, because they were conserved classically.

2) On the σ-model approx. to low energy QCD. QCD involves fermions $\psi_a^{(i)}$ where $a = 1, 2, 3$ is the [redacted] color index and [redacted] i is the flavor. The Lagrangian is something like

$$\frac{1}{4} F^2 + \sum_i \bar{\psi}^{(i)}_{[redacted]} (\not{\partial} + \not{A}) \psi^{(i)}$$

where A is an $SU(3)$-gauge field. For low energy i goes from 1 to 3 and one gets Gell-Mann's $SU(3)$. For just neutrons + protons $i = 1, 2$ is enough.

The σ-model approx. consists of the octet of fields

$$\phi^a = (\bar{\psi}\ \lambda^a\ \psi)\gamma_5$$

where λ^a is a basis for $SU(3)$. $(3\pi, 2K, 2\bar{K}, 1\eta)$

The σ-model Lagrangian + Wess-Zumino term describes the dynamics of this model. One believes this model is a consequence of QCD.

Recently Witten has identified solitons in the ϕ^a-theory with baryons, in the same spirit as Coleman finding massive Thirring fermions in sine-Gordon.

Let's try to make sense out of BRS and Dixon's work. I shall begin with a review of the Fadeev-Popov ansatz.

We have the gauge group $\mathcal{G}$ acting on the space of connections $\mathcal{A}$. The action S is a $\mathcal{G}$-invariant function on $\mathcal{A}$, and we are interested in the integral (of gauge-invariant quantities like $\mathrm{tr}(e^F)$, $F=$ curvature) with respect to $\mathcal{D}A \cdot e^{iS}$.

Assuming $\mathcal{D}A$ is invariant, and that $\mathcal{G}$ acts freely on $\mathcal{A}$, one has

$$\int_{\mathcal{A}} \mathcal{D}A \cdot e^{iS} \cdot \Phi = \int_{\mathcal{A}/\mathcal{G}} \overline{\mathcal{D}A}\, e^{iS} \Phi \cdot \mathrm{vol}(\mathcal{G})$$

where $\overline{\mathcal{D}A}$ is the 'measure' on $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{G}$ belonging to $\mathcal{D}A$ and the 'Haar measure' on $\mathcal{G}$. We ignore the constant $\mathrm{vol}(\mathcal{G})$ and concentrate on how to do the integral over $\overline{\mathcal{A}}$.

We suppose given a gauge-fixing function,

$$g : \mathcal{A} \longrightarrow \mathcal{C}$$

whose fibres $g^{-1}(c)$ are transversals to the $\mathcal{G}$-orbits:

$$g^{-1}(c) \xrightarrow{\sim} \mathcal{A}/\mathcal{G}.$$

We suppose $\mathcal{C}$ comes with a 'measure' $\mathcal{D}C$ and thus reach the problem of comparing the measure $\overline{\mathcal{D}A}$ on $\mathcal{G}\backslash\mathcal{A}$ with the measure on $g^{-1}(C)$ obtained from $\mathcal{D}A$ on $\mathcal{A}$ and $\mathcal{D}C$ on $\mathcal{C}$.

How is this latter measure on $g^{-1}(C)$ defined? The idea is that the measure $\mathcal{D}C$ allows us to define a δ-function at the point C on $\mathcal{C}$ whose inverse image under g is a function

$$A \longmapsto \delta\big(g(A) - C\big)$$

on $\mathcal{A}$ peaking on $g^{-1}(C)$; one has the formula

$$\int_{g^{-1}(C)} \left(\frac{\mathcal{D}A}{\mathcal{D}C}\right)\Phi = \int_{\mathcal{A}} \mathcal{D}A\cdot\, \delta\big(g(A) - C\big)\, \Phi(A)$$

which defines the measure on $g^{-1}(C)$ obtained from $\mathcal{D}A$ and $\mathcal{D}C$.

Next we must compare the measure on a $\mathcal{G}$-orbit obtained from the Haar measure on $\mathcal{G}$ with the measure $\mathcal{D}C$. This gives us a Jacobian transf.

$$M_A : \tilde{g} \longrightarrow \text{tangent space to } \mathcal{C} \text{ at } g(A)$$

$$\delta g \longmapsto g(A + \delta g\cdot A) - g(A)$$

or $\qquad M_A = \dfrac{\delta g}{\delta g}$. $\quad$ Then we can take the determinant $\det(M)$ with respect to the measures on $\mathcal{G}$ and $\mathcal{C}$.

Let $\Psi = e^{iS}\Phi$ be the gauge-invariant function we wish to integrate over $\bar{\mathcal{A}}$. Then we wish to do the integral as follows:

$$\int_{\bar{\mathcal{A}}} \mathcal{D}\bar{A}\,\Psi = \int_{\mathcal{A}} \mathcal{D}A \cdot \Psi \cdot \underbrace{\text{suitable } \delta\text{-fn. concentrated on } g^{-1}(c)}_{\text{call this } \delta^*(A).}$$

Then $\delta^*(A)$ must integrate to 1 with respect to the Haar measure on the orbits. Thus

$$\delta^*(A) \cdot = \delta(g(A)-c)\,\det(M_A)$$

since $M = \frac{\delta g}{\delta g}$. So we get

$$\int_{\bar{\mathcal{A}}} \mathcal{D}\bar{A}\;\Psi = \int_{\mathcal{A}} \mathcal{D}A \cdot \det(M_A) \cdot \delta(g(A)-c)\,\Psi(A)$$

Finally I can bring the determinant of M_A into the exponent by introducing ghost fields

$$\det(M) = \int \mathcal{D}\bar{\omega}\,\mathcal{D}\omega\; e^{-\bar{\omega}M\omega}.$$

The δ-function can be brought into the exponent using F.T. on $\mathcal{C}$

$$\delta(g-c) = \int \mathcal{D}\gamma\, e^{i\gamma(g-c)}.$$

Or, one can integrate over C:

$$\int \mathcal{D}c \; e^{-F(c)} \, \delta(q-c) = e^{-F(q)}$$

September 10, 1983:

Let's review the problem left over from the last Loday letter.

Let E_0 be the trivial n-dimensional bundle over M, so $G_0 = U(n)$ and $\mathcal{G} = \mathrm{Maps}(M, U(n))$, $\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathfrak{gl}_n(C^\infty(M))$. Recall that we think of a left-invariant diff'l form on $\mathcal{G}$ as a natural transf. from flat connections on the trivial principal $\mathcal{G}$-bundle to forms on the base. Now a flat connection on $Y \times \mathcal{G}$ over Y can be identified with a flat partial connection in the Y-direction for the trivial n-dim'l bundle over $Y \times M$. Hence it is of the form

$$D' = d' + \theta \qquad \theta \in \underset{\displaystyle \Omega^1(Y, \tilde{\mathfrak{g}})}{\underset{\parallel}{\Omega^{1,0}(Y \times M, \mathfrak{gl}_n)}}$$

with

$$(D')^2 = d'\theta + \theta^2 = 0.$$

In the universal case we take $Y = \mathcal{G}$ and then $\theta \in \Omega^1(\mathcal{G}, \tilde{\mathfrak{g}})$ is the Maurer-Cartan form for $\mathcal{G}$.

This isn't very clear. The point of working with flat connections on $Y \times \mathcal{G}$ over Y is that we don't have to get involved with Lie algebra cochains. So things should be clearer this way.

Let us explain this more carefully. Given a Lie group G and a flat connection in the trivial principal bundle $Y \times G$ over Y, we know locally that there are flat sections

$$s(y) = (y, g(y)^{-1})$$

where $g : Y \longrightarrow G$ is a map. If I think of G as being a matrix group, the flat connection can be described by an operator

$$D = d + \theta, \qquad \theta \in \Omega^1(Y, \mathfrak{g}).$$

The fact s is flat says that

$$(d + \theta)(g^{-1}) = 0$$

or

$$\theta = -d(g^{-1})g = g^{-1}dg.$$

Thus θ is the pull-back of the Maurer-Cartan form on G via the mapping $g(y)$. The form θ determines g up to left mult. by a constant function from Y to G.

I want to think of a flat connection in $Y \times G \longrightarrow Y$ as being a 1-form

$$\theta \in \Omega^1(Y, \mathfrak{g})$$

satisfying $d\theta + \theta^2 = 0$ where $\theta^2 = \frac{1}{2}[\theta, \theta]$ as usual. Locally $\theta = g^{-1}dg$ where $g : Y \longrightarrow G$ is unique up to left multiplication by elts. of G.

The M-C form is the element of

$$\Omega^1(G, \mathfrak{g})^G = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$$

belonging to the identity map of G. Note that G

does not act on $\mathfrak{g}$, but only on G by left mult. Thus the MC form is in $C^1(\mathfrak{g}, \mathfrak{g})$ where the action on $\mathfrak{g}$ is trivial.

So now let's return to the case of $\mathcal{G}$ and $\tilde{\mathfrak{g}} = \Omega^0(M, \mathfrak{g})$. In this case

$$\theta \in \Omega^{1,0}(Y \times M, \mathfrak{g}) = \Omega^1(Y, \tilde{\mathfrak{g}})$$

satisfies $d'\theta + \theta^2 = 0$, where d' = exterior derivative in the Y-direction.

To define our characteristic classes what we do is to extend the flat Y-connection $D' = d' + \theta$ to a full connection, and the simplest way is to take

$$D = d + \theta.$$

Then we need a one-parameter family of connections

$$D_t = d + t\theta$$

and we use the standard formula

$$\varphi(D_1^2) - \varphi(D_0^2) = d \int_0^1 \varphi'(D_t^2, \theta)\, dt$$

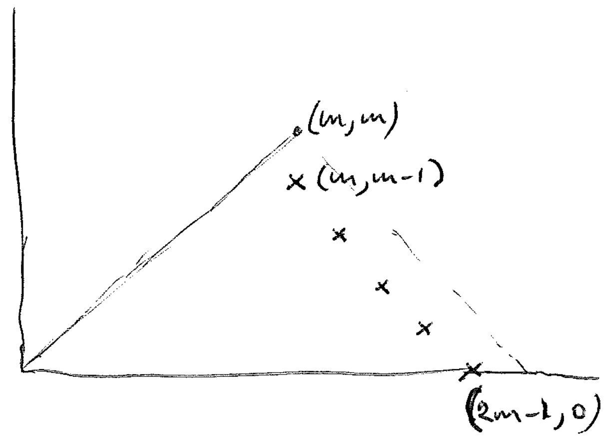
in order to define our odd diml. classes.

$$D_t^2 = (d + t\theta)^2 = td\theta + t^2 \underbrace{\theta^2}_{-d'\theta} = \underbrace{td''\theta}_{1,1} + \underbrace{(t^2 - t)\theta^2}_{(2,0)}$$

$$\varphi(K) = \operatorname{tr}\left(\frac{K^m}{m!}\right).$$

$$\operatorname{tr}\frac{(d''\theta)^m}{m!} = d \int_0^1 \operatorname{tr}\left(\frac{(td''\theta + (t^2 - t)\theta^2)^{m-1}}{(m-1)!}\,\theta\right) dt$$

Call this last integral u_m. It has components pictured as follows:



Because du_m is of type m,m it represents a cycle in the complex

$$\Omega^{\cdot}(G, \Omega_M^{<m})^G = C^{\cdot}(\tilde{g}, \Omega_M^{<m})$$

of degree $2m-1$. So it gives rise to a map of complexes

$$C_{\cdot}(\tilde{g}) \longrightarrow \Omega_M^{<m}[-2m+1]$$

Further steps involve showing that because u_m involves terms like $tr(\theta^2 \cdots d''\theta \cdots \theta^2 \cdots d''\theta \cdots \theta)$, it is primitive and corresponds to a specific map from the Connes complex $C_{\cdot}(A)$ to $\Omega_M^{<m}$. Then the problem becomes one of identifying, or showing compatibility, of this map with the one given by the cyclic homology theory.

Let's review determinant line bundles. Over the space $\mathcal{A}$ of connections we have the family of Dirac operators $A \mapsto \not{D}_A$ which is equivariant for the action of the gauge gp. $\mathcal{G}$. The index of this family is a well-defined K-element [redacted] on $B\mathcal{G}$ ($= \mathcal{G}\backslash\mathcal{A}$ when $\mathcal{G}$ acts freely). The index thm. for families gives a formula for this index, whose cohomological form is

$$ch\,(\text{index}) = \int_M ch(\tilde{E})\cdot T(M).$$

Here $\tilde{E}$ denotes the tautological bundle over $B\mathcal{G}\times M$ [redacted] corresponding to $\mathcal{G}$ acting on E over M.

Now we can compute $ch(\tilde{E})$ as an equivariant diff'l form on $\mathcal{A}\times M$ for the $\mathcal{G}$-action, since $\tilde{E} = pr_2^*(E)$ on $\mathcal{A}\times M$ has a canonical connection (vertically, it is tautological since a point A of $\mathcal{A}$ is a connection on E, and horizontally it is the obvious flat connection). So ch(index) is an equiv. form for $\mathcal{A}, \mathcal{G}$.

We are interested primarily in the highest exterior power line bundle belonging to the index virtual bundle. We want to [redacted] trivialize this determinant line bundle $\mathcal{L}$ so as to have a gauge-invariant determinant for the Dirac operator $\not{D}_A$.

The above index thm. gives the [redacted] first Chern class of $\mathcal{L}$ as an equivariant 2-form on $\mathcal{A}, \mathcal{G}$. Such a form consists of a closed invariant form $\omega \in \Omega^2(\mathcal{A})$, and a Higgs field $\varphi: \tilde{\mathfrak{g}} \longrightarrow \Omega^0(\mathcal{A})$, also

$\mathcal{G}$-invariant, such that

$$i(X)\omega = d\varphi_X.$$

An equivariant 1-form is simply an invariant 1-form $\eta \in \Omega^1(\mathcal{A})$. It's differential is the pair $d\eta \in \Omega^2(\mathcal{A})$, $\varphi_X = i(X)\eta \in \Omega^0(\mathcal{A})$.

But again I make the mistake of not concentrating on the topology first. This first Chern class lies in $H^2(B\mathcal{G})$, in fact, in $H^2(B\mathcal{G}, \mathbb{Z})$. Under suspension it corresponds to an element of $H^1(\mathcal{G}, \mathbb{Z}) = [\mathcal{G}, S^1]$. Geometrically we have this line bundle $\mathcal{L}$ over $B\mathcal{G}$, and the suspension of $\mathcal{L}$ sits inside $B\mathcal{G}$. So then comparing the two possible trivializations over the halves of the suspension gives the desired maps from $\mathcal{G}$ to S^1!

Let's try to understand this better from the viewpoint of the space $\mathcal{A}$. Fix a basepoint A_0, then the gauge gp orbit $\{g^*A_0 \mid g \in \mathcal{G}\}$ is the fibre of the map $\mathcal{A} \longrightarrow \mathcal{G}\backslash\mathcal{A} = B\mathcal{G}$, where I assume the action is free. The map $\mathrm{Susp}(\mathcal{G}) \to B\mathcal{G}$ arises from the two reasons this fibre is zero in $B\mathcal{G}$: because it maps to a point, and because $\mathcal{A}$ is contractible.

So we take $I \times \mathcal{G}$ and map it to $\mathcal{A}$ by sending $t, g \longmapsto tA_0 + (1-t)\, g^*A_0$. Then use the gauge isomorphism at the ends to defines the family over the suspension.

<u>September 15, 1983</u>

Two problems:

1) Reconcile the Lie algebra approach to cyclic cohomology with the index approach of Connes. Connes associate to an operator F (some kind of Kasparov gadget) a family of cyclic cocycles which I can interpret as Lie algebra classes. However his construction comes from using idempotents and calculating indices.

2) Is there any direct connection between cyclic cohomology and anomalies?

Let's consider the problem of why there should be explicit anomaly formulas.

Start with $\mathcal{L}$ over $\mathcal{A}$ equivariant for $\mathcal{G}$. A gauge-invariant determinant exists iff $\mathcal{L}$ is equiv. trivial, and this is measured by $c_1(\mathcal{L}) \in H^2(B\mathcal{G}, \mathbb{Z})$. The image in real cohomology is given by the index thm. for families:

$$c_1(\mathcal{L}) = \Big[f_* \big(ch(\tilde{E}) \cdot Todd(M) \big) \Big]_{\text{deg 2 component}}$$

where $f = pr_1 : B\mathcal{G} \times M \longrightarrow B\mathcal{G}$, and $\tilde{E}$ is the tautological bundle over $B\mathcal{G} \times M$. Use the model for forms on $B\mathcal{G}$ given by equivariant forms on $(\mathcal{A}, \mathcal{G})$. Then $\tilde{E} = pr_2^*(E) = \mathcal{A} \times E$ over $\mathcal{A} \times M$ has a tautological connection, and so $ch(\tilde{E})$ is given by explicit equiv. forms. Therefore $c_1(\mathcal{L})$ will be realized by an explicit equivariant 2-form.

Now it is essential for me to start producing formulas at some stage, so [redacted] that I can compare with physicists formulas.

Two procedures to compare: 1) Mine: I assume that the 2-form on $B\mathcal{G}$, i.e. the equivariant 2-form on $\mathcal{A}, \mathcal{G}$ comes from an invariant connection on $\mathcal{L}$. Then the equivariant 2-form consists of the curvature of $\mathcal{L}$ and a Higgs map. Now to construct a determinant I need a flat connection on $\mathcal{L}$, so I take the curvature ω of $\mathcal{L}$ and write it $d\eta$ where η is a 1-form on $\mathcal{A}$, then I use η to modify the given connection on $\mathcal{L}$ so as to make it flat. Now I have to worry about gauge invariance of the determinant, namely I restrict the determinant function to a gauge orbit. This gives a map $\mathcal{G} \longrightarrow \mathbb{C}^*$ which I can convert to a 1-form on $\mathcal{G}$.

2) Singer's: Let's assume $\mathcal{G}$ acts freely, so that $B\mathcal{G} = \mathcal{G}\backslash\mathcal{A}$. [redacted] The idea is $c_1(\mathcal{L})$ is the transgression of an element of $H^1(\mathcal{G})$. So you take the 2-form on $B\mathcal{G}$, lift it to $\mathcal{A}$, it becomes d of something α, and then we restrict α to a $\mathcal{G}$-orbit $\cong \mathcal{G}$.

Let's compare these. Let (ω, φ) be the equiv. 2-form giving $c_1(\mathcal{L})$. In order to get a 2-form down on $\mathcal{G}\backslash\mathcal{A}$, we need a connection form for $\mathcal{A} \longrightarrow \mathcal{G}\backslash\mathcal{A}$, call it θ. θ is a 1-form on $\mathcal{A}$ with values in $\tilde{\mathfrak{g}}$, φ maps $\tilde{\mathfrak{g}}$ to $\Omega^0(\mathcal{A})$, so $\varphi\theta$ is a 1-form on $\mathcal{A}$, which is $\mathcal{G}$-invariant as both φ, θ are. The 2-form on $\mathcal{A}$ which descends to $\mathcal{G}\backslash\mathcal{A}$ is $\omega + d(\varphi\theta)$.

Then we choose α so that

$$\omega + d(\varphi\theta) = d\alpha$$

so my η satisfying $d\eta = \omega$ is just

$$\eta = \alpha - \varphi\theta .$$

I should get the formulas better. Then connection in $\mathcal{L}$ is ∇ and the $\mathcal{G}$-action is

$$\mathcal{L}_X = \nabla_X + \varphi_X .$$

Then

$$\begin{aligned} [\mathcal{L}_X, \nabla_Y] &= \nabla_{[X,Y]} \\ \| \\ [\nabla_X, \nabla_Y] + [\varphi_X, \nabla_Y] \end{aligned}$$

or $Y\varphi_X = \omega(X,Y)$
$= i(Y)i(X)\omega$

or

$$d\varphi_X = i(X)\omega .$$

Check

$$\begin{aligned} i(X)(\omega + d(\varphi\theta)) &= i(X)\omega + i(X)\, d(\varphi\theta) \\ &= i(X)\omega - d\, \underbrace{i(X)(\varphi\theta)}_{\varphi_X} = 0 . \end{aligned}$$

Let's suppose the index is zero so that we have a canonical section s. Then where $s \neq 0$

$$\nabla s = s\lambda$$

and

$$0 = \mathcal{L}_X s = (\nabla_X + \varphi_X)s \implies i(X)\lambda + \varphi_X = 0$$

or

$$\varphi_X = -i(X)\lambda .$$

I choose η with $d\eta = \omega$ and then define $\log\det W$ by

$$dW = \lambda - \eta.$$

In other words, the new connection is $\nabla - \eta$ and I want

$$(\nabla - \eta)\left(e^{-W}s\right) = 0,$$

so that under a flat trivialization of $\mathcal{L}$ we have $s \longleftrightarrow e^{W}$. Thus

$$i(X)\,dW = \overbrace{i(X)\lambda}^{-\varphi_X} - i(X)\left[\alpha - \varphi\theta\right]$$

$$= -i(X)\,\alpha$$

which shows that the actual 1-form on $\mathcal{G}$ obtained from my procedure agrees with Singer's up to sign.

Now things like φ, ω are canonical, but η has to be chosen, so it is still not clear why there should be an anomaly formula.

It appears that the anomaly

$$i(X)\,dW = -i(X)\,\alpha = -i(X)\left[\eta + \varphi\theta\right]$$

$$= -i(X)\,\eta - \varphi_X$$

has two terms, one coming from the curvature ω and the other thru the Higgs field φ.

At this point it becomes necessary to get our hands on these forms.

I want to consider the case where $M = \mathbb{R}^4$ and the connections and gauge transformations are trivial near ∞. The bundle E_0 is trivial, say of rank N, so that $\mathcal{G} = \Omega^4 U_N$.

We consider $E = pr_2^*(E_0) = \mathcal{A} \times E_0$ over $\mathcal{A} \times M$. This is a trivial bundle, but $\mathcal{G}$ acts diagonally. The tautological connection on E is

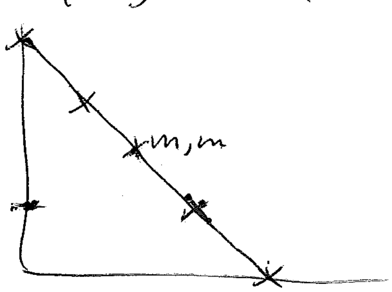
$$D = d + \theta$$

where θ is a $(0,1)$ form on $\mathcal{A} \times M$ which at a point (A, m) takes a tangent vector to M into the endo of E_0 at m given by A. The curvature of this connection

$$D^2 = (d+\theta)^2 = d\theta + \theta^2$$

has components of types $1,1$ and $0,2$. So the m-th character class has Kunneth components of degree (m,m) to $(0,2m)$; i.e. above the diagonal.

However we know that $ch_m(\tilde{E})$ on $B\mathcal{G} \times M$ has components of all degrees where M has cohomology. More precisely it has a component in degree $(m-j, m+j)$ provided $H^{m+j}(M) \neq 0$.



In order to get the components below the diagonal, the Higgs field φ must enter.

September 17, 1983:

Could Connes Λ-interpretation of cyclic homology help in establishing the compatibility of the two maps from Lie algebra homology to connected Deligne cohomology?

Start with E_0 over M, $\mathcal{G}$, $\mathcal{a}$ as usual. The end problem is to get a hold on the Chern character of $\tilde{E}_0$ over $B\mathcal{G} \times M$. The answer should ultimately be given in terms of equivariant forms on $\mathcal{a} \times M$ for the $\mathcal{G}$-action, using the fact that $pr_2^*(E_0) = \mathcal{a} \times E_0$ over $\mathcal{a} \times M$ has a tautological $\mathcal{G}$-invariant connection. At the moment I am lacking a facility with equivariant forms, but I feel that I should be able to obtain the facility by using the idea that equivariant forms are what descend to the base of a principal bundles once one is given a connection form.

Let's discuss this more. Suppose given a Lie group G acting on a manifold $\mathcal{a}$. Equivariant cohomology for the action is the cohomology of $PG \times^G \mathcal{a}$. Such cohomology can be interpreted as characteristic classes for maps $Y \to PG \times^G \mathcal{a}$, i.e. a principal G-bundle P/Y and a G map $P \to \mathcal{a}$.

Repeat: A cohomology class on $PG \times^G A$ is a characteristic class for pairs consisting of a principal G-bundle P/Y and a G-map $P \to A$.

Now when A is a point, Chern-Weil tells us how to construct characteristic classes in DR cohomology [redacted] for principal G-bundles. One chooses a connection in P/Y, this is equivalent to a map

$$W(\mathfrak{g}) \longrightarrow \Omega^{\cdot}(P)$$

and then one passes to basic elements getting

$$S(\mathfrak{g}^*)^G \longrightarrow \Omega^{\cdot}(Y).$$

In general when A is arbitrary, we can do the same thing. Choose a connection and combine it with the map on forms induced by the G-map $P \to A$ to get a map

$$W(\mathfrak{g}) \otimes \Omega(A) \longrightarrow \Omega(P)$$

then pass to basic elements to get

$$\left[S(\mathfrak{g}^*) \otimes \Omega(A)\right]^G \longrightarrow \Omega(Y)$$

But now the problem is to make this explicit in the case of interest.

Again we take the Chern-Weil example. If I want [redacted] the form in $\Omega(Y)$ belonging to an invariant polynomial $\phi \in S(\mathfrak{g}^*)^G$ what I need is the curvature of the connection. The ~~[illegible]~~ connection and curvature are $\mathfrak{g}$-valued forms on P, i.e. elements of $\mathfrak{g} \otimes \Omega(P)$, or equivalently, maps $\mathfrak{g}^* \longrightarrow \Omega(P)$.

In practice one usually operates locally on Y and chooses sections of P, whence the connect. + curvature become $\mathfrak{g}$-valued forms on Y, namely

$$\theta \quad \text{and} \quad d\theta + \tfrac{1}{2}[\theta,\theta] .$$

I have a more specific problem, namely I have an equivariant v. bundle over $\mathcal{A}$ and I want to compute its Chern classes in $H^*(PG \times^G \mathcal{A})$. I assume that there is an invariant connection in the bundle given.

Let's consider the problem of constructing the Chern character classes for an equivariant bundle E over a G-manifold M. Suppose one is given an invariant connection D in E; such a connection exists when G is compact by averaging.

Suppose G acts freely on M, and let $\theta \in \mathfrak{g} \otimes \Omega^1(M) = \Omega^1(M, \mathfrak{g})$ be a connection form for the principal fibr. $M \to G\backslash M = \overline{M}$. Then I know that $D + \varphi\theta$ is another invariant connection which descends to a connection on $\overline{E} = G\backslash E$ over $\overline{M}$. So the character of $\overline{E}$ is given by the form $tr\left(e^{(D+\varphi\theta)^2}\right)$, which I know descends to $\overline{M}$.

[redacted] The idea I have is that it should be possible to eliminate [redacted] θ by using equivariant forms. More precisely, I would like to [redacted] express $tr\left(e^{(D+\varphi\theta)^2}\right)$ as obtained by first forming the character in equivariant cohomology, then using [redacted] the connection to go from equivariant forms on M to forms on $\bar{M}$.

The latter process works as follows. The connection gives us a $G, \mathfrak{g}$ map

$$W(\mathfrak{g}) \longrightarrow \Omega(M)$$

which we can use to form a $G,\mathfrak{g}$ map

$$W(\mathfrak{g}) \otimes \Omega(M) \longrightarrow \Omega(M).$$

Taking basic forms gives the desired map from equivariant forms to forms on $\bar{M}$, or better from equivariant forms to basic forms.

Review equivariant forms in the case of a circle action. Generator for Lie algebra $\mathfrak{g}$ will be denoted X and the dual generator for $\mathfrak{g}^*$ will be denoted θ. The Weil algebra is

$$W(\mathfrak{g}) = k[u] \otimes \Lambda[\theta]$$

with u, θ invariant under $\mathfrak{g}$

$$\iota_X \theta = 1 \qquad \iota_X u = 0$$
$$d\theta = u \qquad du = 0.$$

The way to think is that a $G, \mathfrak{g}$ homomorphism $W(\mathfrak{g}) \longrightarrow \Omega(P)$ is the same as a connection in P.

And I know that a connection in a principal S^1-bundle is given by a connection form $\theta \in \Omega^1(P)$ invariant under S^1 with $\iota_X \theta = 1$. The curvature is then $d\theta = u$.

Now let's calculate the equivariant forms, i.e. basic elements in

$$W(\mathfrak{g}) \otimes \Omega(M) = k[u, \theta] \otimes \Omega(M).$$

The typical element is of the form

$$u^m \alpha_m + u^m \theta \beta_m$$

and

$$\iota_X(u^m \alpha_m + u^m \theta \beta_m) = u^m(\iota_X \alpha_m + \beta_m - \theta \iota_X \beta_m) = 0$$

implies $\beta_m = -\iota_X \alpha_m$, and conversely if this holds then $\iota_X \beta_m = -\iota_X^2 \alpha_m = 0$. So the horizontal elements are

$$u^m(\alpha_m - \theta i_X \alpha_m) = (id - \theta \iota_X)\left(\sum u^m \alpha_m\right)$$

and we may identify

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{horiz} \xleftarrow[\sim]{id - \theta\iota_X} k[u] \otimes \Omega(M).$$

Then (★) $\quad [W(\mathfrak{g}) \otimes \Omega(M)]_{basic} \xleftarrow{\sim} k[u] \otimes \Omega(M)^G.$

Finally we must calculate the differential

$$d\,(id - \theta i_X)(\alpha) = d\alpha - u i_X \alpha + \theta \underbrace{d i_X \alpha}_{-\iota_X d\alpha}$$

provided $\mathcal{L}_X \alpha = 0$. Also

$$(id - \theta \iota_X)(d - u i_X)\alpha = (d - u i_X)\alpha - u i_X \alpha$$

so $\boxed{d\,(id - \theta i_X) = (id - \theta i_X)(d - u i_X)}$

which shows that under the isom. (A) the differential is given by $d - u i_X$ on $k[u] \otimes \Omega(M)^G$.

Now let us suppose that we have a connection form η for the circle action on M. This means $\mathcal{L}_X \eta = 0$ and $\iota_X \eta = 1$. Then we get a homomorphism

$$
\begin{array}{ccc}
W(\mathfrak{g}) \otimes \Omega(M) & \longrightarrow & \Omega(M) \\
\| & & \| \\
k[u,\theta] \otimes \Omega(M) & \longrightarrow & \Omega(M) \\
\theta & \longmapsto & \eta \\
u & \longmapsto & d\eta \\
\alpha & \longmapsto & \alpha
\end{array}
$$

and I want to compute what happens to a basic element using the formula (A). On the right side the basic element is $u^m \alpha \in k[u] \otimes \Omega(M)^G$ and this goes to

$$(id - \theta \iota_X)(u^m \alpha) = u^m(\alpha - \theta \iota_X \alpha)$$

which gets mapped to

$$(d\eta)^m(\alpha - \eta i_X \alpha).$$

(It might be useful to note that $id - \theta\iota_X = \iota_X \theta$. Hence the d-calculation goes:

$$d\,\iota_X\theta = -\iota_X d\theta \qquad \text{on invariants}$$

$$= -\iota_X[u - \theta d] = \iota_X\theta[d - u\iota_X].\,)$$

Better: Let us start with an elt of $k[u]\otimes\Omega(M)^G$ of degree $2m$:

(1) $\alpha_{2m} + u\alpha_{2m-2} + \cdots + u^m \alpha_0$.

This corresponds to the following basic elt of $W(\mathfrak{g})\otimes\Omega(M)$:

(2) $\alpha_{2m} - \theta\iota_X\alpha_{2m} + u\,\alpha_{2m-2} - u\,\theta\,\iota_X\alpha_{2m-2} + \cdots$

which gets mapped to the following basic elt of $\Omega(M)$:

(3) $\alpha_{2m} - \eta\,\iota_X\alpha_{2m} + d\eta\cdot\alpha_{2m-2} - d\eta\cdot\eta\cdot\iota_X\alpha_{2m-2} + \cdots$

For the element (1) to be closed means

$$\begin{cases} d\alpha_{2m} = 0 \\ i_X\alpha_{2m} = d\alpha_{2m-2} \\ i_X\alpha_{2m-2} = d\alpha_{2m-4} \qquad \text{etc.} \\ \cdots\cdots \end{cases}$$

and one can check this implies (2) + (3) are closed.

Now the next step will be to take an equivariant vector bundle E over M for the circle action.

<u>September 18, 1983:</u>

Situation: E equivariant bundle over a G-manifold M, D an invariant connection on E. D is an operator

$$\Omega^0(M,E) \xrightarrow{D} \Omega^1(M,E) \xrightarrow{D} \cdots$$

where $\Omega^q(M,E) = \Gamma(M, \Lambda^q T^* \otimes E)$.

Assume G acts freely on M, so that we can descend E to $\bar{E} = G\backslash E$ over $\bar{M} = G\backslash M$. I want to construct $ch(\bar{E})$ in DR coh. of $\bar{M}$, hence I need a connection that descends. Choose a connection θ is the principal bundle $M \to \bar{M}$

$$\theta \in \mathfrak{g} \otimes \Omega^1(M).$$

Then $D + \varphi\theta$ descends.

I should think of θ as the alg. analogue of choosing a classifying map $M \longrightarrow PG$. (This is because a connection is the same as a $G, \mathfrak{g}$ map $W(\mathfrak{g}) \to \Omega(M)$, and $W(\mathfrak{g})$ is a model for the forms on PG.)

What I am after is the following. I know starting from $(D+\varphi\theta)^2 \in \Omega^2(M, \operatorname{End} E)$ how to construct character forms for $\bar{E}$. I want to obtain these character forms in two steps — first construct them as equivariant forms, then use the connection θ to take equiv. forms into basic forms in $\Omega(M)$.

Here's the geometric situation:

$$\begin{tikzcd} \overset{E}{M} \arrow[r] \arrow[d] & \overset{pr_2^*(E)}{PG \times M} \arrow[r] \arrow[d] & PG \arrow[d] \\ \bar{M} \arrow[r] & PG \times^G M \arrow[r] & BG \end{tikzcd}$$

I am thinking of PG/BG as having a canonical connection, and the classifying map $M \to PG$ as inducing Θ from this canonical connection. Algebraically this translates to

$$\Omega(M) \longleftarrow W(\mathfrak{g}) \otimes \Omega(M) \longleftarrow W(\mathfrak{g})$$

so what I see is that the only difference between working with $D + \varphi\Theta$ over M, and the corresponding thing in equivariant cohomology is that the $\Theta \in \mathfrak{g} \otimes \Omega^1(M)$ is to be replaced by the universal Θ in $\mathfrak{g} \otimes W^1(\mathfrak{g})$.

At this point we digress to describe $W(\mathfrak{g})$. The main idea is that $W(\mathfrak{g})$ represents connection forms. Thus there is a canonical element

$$\Theta \in \mathfrak{g} \otimes W^1(\mathfrak{g})$$

and W is generated by Θ and by the curvature

$$\Omega = d\Theta + \tfrac{1}{2}[\Theta, \Theta].$$

Formulas: Let λ_a be a basis for $\mathfrak{g}$. Then

$$\Theta = \lambda_a \Theta^a \qquad \Theta^a \in W^1(\mathfrak{g}).$$

I need also structure ~~constants~~

$$[\lambda_a, \lambda_b] = f_{ab}^c \lambda_c .$$

Then

$$\Omega = d\Theta + \tfrac{1}{2}[\Theta, \Theta]$$

$$= \lambda_a d\Theta^a + \tfrac{1}{2}[\lambda_b \Theta^b, \lambda_c \Theta^c]$$

$$= \lambda_a d\Theta^a + \tfrac{1}{2}\left(\lambda_b \lambda_c \Theta^b \Theta^c + \lambda_c \lambda_b \Theta^c \Theta^b\right)$$

$$\lambda_a \Omega^a = \lambda_a d\Theta^a + \tfrac{1}{2}\underbrace{[\lambda_b, \lambda_c]}_{f_{bc}^{a}\lambda_a} \Theta^b \Theta^c$$

or

$$\Omega^a = d\Theta^a + \tfrac{1}{2} f_{bc}^{a} \Theta^b \Theta^c$$

So $W(\mathfrak{g}) = S(\Omega^a) \otimes \Lambda(\Theta^a)$ with d defined in this way. Finally we need to know

$$\iota_X \Theta = X$$

or

$$\iota_{\lambda_a} \Theta^b = \delta_a^b,$$

and also that the action $\mathcal{L}_X$ on $W(\mathfrak{g})$ is the obvious one for $W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$.

At this point I have a complete description of $W(\mathfrak{g})$ given by the universal property of representing connection forms. It would seem then that [redacted] working with $D + \varphi\Theta$ has to be equivalent to a computation with equivariant forms.

Idea: Because we assume G acts freely [redacted] on M, we can work locally on $\bar{M}$ and trivialize $\bar{E}$. Thus we can assume that $E = M \times V$ where V is a vector space on which G acts trivially. Then an invariant connection has the form

$$D = d + A$$

where A is an End(V)-valued 1-form on M which is G-invariant. Now the connection descends iff $i_X(A) = 0$ for all $X \in \mathfrak{g}$. The connection for Θ allows me to split the tangent bundle to M into the tangent spaces to the G-orbits (longitudinal) and normal spaces (transversal). Thus A descends when its transverse component is zero. When we write

$$D + \varphi\Theta \quad (= d + A - A_L)$$

we are getting $d + A_T$, since $\varphi_X = -i_X A$ for $X \in \mathfrak{g}$.

It seems that I can simplify $(D + \varphi\Theta)^2$ somewhat if I use the identities connecting D, φ. I want to express things as operators on $\Omega(M, E)$.

Recall that D is a degree 1 operator on $\Omega(M, E)$ which is a derivation relative to $\Omega(M)$-module structure. Invariance under G means that

$$[\mathcal{L}_X, D] = 0 \qquad \text{for } X \in \mathfrak{g}.$$

We have

$$\mathcal{L}_X = [i_X, D] + \varphi_X$$

because both $\mathcal{L}_X$ and $[i_X, D]$ are degree 0 derivations, hence their difference will be $\Omega(M)$-linear of degree 0. Combining the above two formulas we get

$$0 = [D, \mathcal{L}_X] = \underbrace{[D,[\iota_X, D]]}_{D(\cancel{\iota_X D} + D\iota_X) - (\iota_X D + \cancel{D\iota_X})D \;=\; [D^2, \iota_X]} + [D, \varphi_X]$$

$$\therefore \quad [D, \varphi_X] = [\iota_X, D^2] \quad \text{as ops. on } \Omega(M,E)$$

hence in degree 1 this is just

$$\boxed{D(\varphi_X) = \iota_X \cdot D^2}$$

Also we have

$$[\mathcal{L}_X, \mathcal{L}_Y] = [\mathcal{L}_X, [\iota_Y, D] + \varphi_Y]$$

$$\| $$

$$\mathcal{L}_{[X,Y]} = [\iota_{[X,Y]}, D] + \varphi_{[X,Y]}$$

hence φ is invariant:

$$[\mathcal{L}_X, \varphi_Y] = \varphi_{[X,Y]}$$

This can be written

$$\varphi_{[X,Y]} = [[\iota_X, D], \varphi_Y] + [\varphi_X, \varphi_Y]$$

$$= \underbrace{[[\iota_X, \varphi_Y], D]}_{\text{zero as it (is } \Omega(M) \text{ linear of degree } -1)\,?} + [\iota_X, \underbrace{[D, \varphi_Y]}_{[\iota_Y, D^2]}] + [\varphi_X, \varphi_Y]$$

I have to argue more carefully. An element of $\Omega(M,E) = \Gamma(M, \Lambda T^* \otimes E)$ is a sum of terms of the form $\alpha \cdot s$ with α a form and $s \in \Gamma(E)$. Then

$$\iota_X \varphi_Y (\alpha \cdot s) = \iota_X(\alpha \cdot \overbrace{\varphi_Y s}^{\deg 0}) = \iota_X \alpha \cdot \varphi_Y s$$

$$\varphi_Y \iota_X (\alpha \cdot s) = \varphi_Y(\iota_X \alpha \cdot s) = \iota_X \alpha \cdot \varphi_Y s \qquad \text{etc.}$$

So we get

$$\varphi[x,y] - [\varphi_x, \varphi_y] = [\iota_x, [\iota_y, D^2]] \quad \text{as ops.}$$

or simply

$$\boxed{[\varphi_x, \varphi_y] - \varphi[x,y] = -\iota_x \iota_y D^2 \quad \text{in } \Omega^0(\mathrm{End}\,E)}$$

Now I want to see the implications of these formulas for

$$(D+\varphi\theta)^2 = D^2 + [D, \varphi\theta] + \varphi\theta\varphi\theta$$

Here $\varphi\theta$ stands for $\varphi_a\theta^a$ where $\varphi_a \in \Omega^0(\mathrm{End}\,E)$ and $\theta^a \in \Omega^1(M)$. Thus

$$[D, \varphi\theta] = \underbrace{[D, \varphi_a]}_{\iota_a(D^2)}\theta^a + \varphi_a d\theta^a$$

$$\begin{aligned}
\varphi\theta\varphi\theta &= \varphi_b\varphi_c\theta^b\theta^c = \tfrac{1}{2}[\varphi_b, \varphi_c]\theta^b\theta^c \\
&= \tfrac{1}{2}\left\{\varphi[\lambda_b, \lambda_c] - i_b i_c D^2\right\}\theta^b\theta^c \\
&= \tfrac{1}{2}\left\{\varphi_a f^a_{bc} - \iota_b\iota_c D^2\right\}\theta^b\theta^c
\end{aligned}$$

$$\therefore (D+\varphi\theta)^2 = D^2 + (i_a D^2)\cdot\theta^a + \varphi_a\left\{d\theta^a + \tfrac{1}{2}f^a_{bc}\theta^b\theta^c\right\} - \tfrac{1}{2}(i_b i_c D^2)\cdot\theta^b\theta^c$$

$$\boxed{(D+\varphi\theta)^2 = D^2 + (i_a D^2)\theta^a + \varphi_a\Omega^a - \tfrac{1}{2}(\iota_b\iota_c D^2)\theta^b\theta^c}$$