(end of year in England)

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Characteristic classes for $H^*(B\mathcal{G})$. How to realize $ch(E_{univ})$ on $H^*(B\mathcal{G} \times M)$ by equivariant forms on $(\mathcal{G}, \mathcal{A})$

June 30, 1983

The basic program is as follows. I want to see if I can make something out of my diffl' form version of the index theorem for families. The difficulty is that the expression for the character of the index is rather complicated. Atiyah suggests that a simpler formula involving regularizations of the standard character forms should exist. One might expect different regularizations to give cohomologous forms, and to express this in a Connes-type fashion.

The first thing to understand perhaps is the the case where there is no regularization. I think I should therefore start with the case of loop groups.

Fix M a compact manifold, and a vector bundle E with metric over M, and let $\mathcal{G} = \mathrm{Aut}(E)$ be the gauge group, and $\mathcal{A}$ = space of connections on E. We suppose given a metric on M and the Dirac operator on a bundle of spinors S. Then we get a natural family of elliptic operators on M parametrized by $\mathcal{A}$ which is equivariant for $\mathcal{G}$. Namely, on $S \otimes E$ we can take the Dirac operator $\frac{1}{i}\gamma^\mu \nabla_\mu$ where ∇_μ is the connection on E.

Now the question is what does the diffl' form index thm. says for this family.

All the operators $\frac{1}{i}\gamma^\mu \nabla_\mu = \not{D}$ operate on $\mathcal{H} = L^2(M, S\otimes E)$, so in this case we have an equivariant map from $\mathcal{A}$ to Fredholm operators on $\mathcal{H}$. Our first problem is to understand the Chern character of the index of the family. I should have said that the index of the family is this map $\mathcal{A} \longrightarrow \mathrm{Fred}(\mathcal{H})$ equivariant for the action of $\mathcal{G}$. The Chern character of this index is to be some kind of equivariant differential form for the $\mathcal{G}$-action on $\mathcal{A}$.

July 1, 1983:

Dirac operator on $\mathbb{R}^n$ with infinite-diml coefficients. Let $L = L(x)$ be a self-adjoint operator on $\mathcal{H}$ depending on $x \in \mathbb{R}^n$. Let S be the spinors associated to $\mathbb{R}^n$; it has operators γ^μ which are self-adjoint anti-commute and have square one. Now consider

$$\not{D} = \gamma^\mu \frac{h}{i} \nabla_\mu + \varepsilon L$$

acting on $L^2(\mathbb{R}^n, S \otimes \mathcal{H})$. Here $\nabla_\mu = \partial_\mu + A_\mu$ where A_μ are skew-adjoint endos. of $\mathcal{H}$, and $\varepsilon = c\gamma^1 \dots \gamma^n$ gives the grading of S. I am assuming n is even and that L is of odd degree wrt a $\mathbb{Z}/2$ grading of $\mathcal{H}$.

There is another possibility where n is odd and L is ungraded which I will worry about later.

We want to calculate the index of $\not{D}$ assuming the analysis is OK.

$$\text{Index}(\not{D}) = \text{Tr}\left(e^{-t\not{D}^2} \varepsilon_{S\otimes\mathcal{H}}\right)$$

$$= \int_{x\in\mathbb{R}^n} \text{Tr}_{S\otimes\mathcal{H}}\left(\langle x| e^{-t\not{D}^2} |x\rangle \varepsilon_{S\otimes\mathcal{H}}\right)$$

Now

$$\not{D}^2 = \underbrace{-h^2 \nabla_\mu^2 + L^2}_{H} + \underbrace{\frac{h}{i} \gamma^\mu \varepsilon [\nabla_\mu, L] - h^2 \tfrac{1}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}}_{hA + h^2 B}$$

and we have a perturbation expansion for

$$e^{-t(H + hA + h^2 B)}$$

We use the basic cancellation of the γ-matrices which says that $tr_S(\varepsilon\,?)$ applied to a finite product of $h\,\gamma^\mu \varepsilon$ $h^2 \gamma^\mu \gamma^\nu$ will be zero unless it is of order $\geq h^n$. On the other hand the heat kernel for $h^2 \nabla_\mu^2$ starts with something like

$$\frac{e^{-\frac{x^2}{4th^2}}}{\left(\sqrt{4\pi t h^2}\right)^n}.$$

It introduces h^n in the denominator. The point is that

$$tr_S\left(\langle \cancel{x} | e^{-t\not{D}^2} | x\rangle \varepsilon_S\right)$$

has a limit ~~at~~ $h \to 0$ which should be the h^n term of

$$tr_S\left(e^{-t(L^2 + hA + h^2 B)} \varepsilon_S\right)$$

as calculated by the perturbation expansion.

Now my problem is one of algebra. I start with the [redacted] operator L on $\mathcal{H}$ (at $x=0$), and a connection ∇_μ on $\mathcal{H}$, and the variations $[\nabla_\mu, L]$. Then I concoct something looking like a character, except that there is this regularization wrt L. The problem is to find the correct framework for viewing all of this.

Idea: I am working at a point L in the space of Fredholm operators and defining a mixed form made of the tangent space of L and the action of the unitary group of $\mathcal{H}$.

July 2, 1983

Suppose given a graded bundle $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ with a connection D_μ and a odd degree operator L on $\mathcal{H}$. I seem to be able to define differential forms on the base depending as a parameter t. These forms are defined by concocting an index problem, namely, for a Dirac operator on an even diml submanifold of the base.

I am not confident of the analysis behind the derivation of these forms, and therefore want to get the algebra straight.

First suppose $L = 0$. So we just have the connection and we should be calculating the Chern character of this connection. Analytically we have the Dirac operator

$$\mathcal{D} = \frac{h}{i} \gamma^\mu D_\mu \qquad \text{on } S \otimes \mathcal{H}$$

where I put the flat metric on the base since this is a local calculation.

$$\mathcal{D}^2 = \overbrace{-h^2 D_\mu^2}^{H} - h^2 \tfrac{1}{2} \gamma^\mu \gamma^\nu F_{\mu\nu}$$

$$\operatorname{tr}\left(\varepsilon_S\, e^{-t\mathcal{D}^2}\right) = \operatorname{tr} \varepsilon_S \int e^{-t_0 H}\, \tfrac{h^2}{2} \gamma\gamma F\, e^{-t_1 H} \cdots \quad e^{-t_m H}$$
$$+ \cdots$$

Here the base is $n = 2m$ dimensional, and I have used already that $\operatorname{tr} \varepsilon_S$ is non-zero only for a product of at least n γ-matrices. The integration is over the simplex: $t_0 + \cdots + t_m = t$, $t_j \geq 0$. At this point we let $h \to 0$, and I want to assume that H and F commute up to lower terms which don't matter. Thus we should have

$$\langle x|\,\mathrm{tr}(\varepsilon_S\, e^{-tD^2})\,|x\rangle = \int \langle x|e^{-tH}|x\rangle\, \mathrm{tr}\Big(\varepsilon_S\, \overbrace{\tfrac{h^2}{2}\gamma\gamma F\cdots}^{m\text{-times}}\Big)$$

$$= \frac{t^m}{m!}\,\frac{1}{(4\pi t h^2)^m}\left(\frac{h^2}{2}\right)^m \sum_\sigma \underbrace{\mathrm{tr}(\varepsilon_S\gamma^{\sigma_1}\gamma^{\sigma_2}\cdots)}_{2^m i^m\, \mathrm{sgn}(\sigma)}\, F_{\sigma_1\sigma_2}F_{\sigma_3\sigma_4}\cdots F_{\sigma_{n-1}\sigma_n}$$

But
$$\left(\tfrac{1}{2}F_{ij}\,dx^i dx^j\right)^m = \frac{1}{2^m}\sum \mathrm{sgn}(\sigma)\, F_{\sigma_1\sigma_2}\cdots F_{\sigma(n-1)\sigma(n)}\, dx_1\cdots dx_n$$

Thus
$$\lim_{h\to 0}\ \langle x|\,\mathrm{tr}(\varepsilon_S e^{-tD^2})\,|x\rangle_n^{dx^1\cdots dx^n} = \frac{1}{m!}\,\frac{i^m}{(2\pi)^m}\left(\tfrac{1}{2}F_{ij}\,dx^i dx^j\right)^m$$

which is the m-th component of the Chern character after we take the trace over $\mathcal{H}$.

The next thing to do is to bring in the operator L. Now there are some examples, namely the Koszul complex on the coordinates x^i.

In general I should be thinking of L as a Dirac operator on M and the parameters on the base as being deformations of L.

Next consider the case where $D_\mu = \partial_\mu$, i.e. the connection is trivial. (This should be like the case of the family of Dirac ops. parametrized by a.)

$$D = \frac{h}{i}\gamma^\mu\partial_\mu + \varepsilon L$$

$$D^2 = -h^2\partial_\mu^2 + L^2 + \frac{h}{i}\gamma^\mu\varepsilon\,\overbrace{[\partial_\mu, L]}^{L_\mu}$$

$$\mathrm{tr}(\varepsilon_S e^{-tD^2}) = \mathrm{tr}\Big(\varepsilon_S\int e^{-t_0H}(ih\,\gamma^\mu\varepsilon L_\mu)\cdots\cdots e^{-t_nH}\Big) + \cdots$$

$$= (ih)^n\,\underbrace{\mathrm{tr}(\varepsilon\,\gamma^{i_1}\varepsilon\cdots\gamma^{i_n}\varepsilon)}_{(-1)^m\,\mathrm{tr}(\varepsilon\gamma^{i_1}\cdots\gamma^{i_n})}\int e^{-t_0H}L_{i_1}e^{-t_1H}L_{i_2}\cdots e^{-t_nH} + \cdots$$

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[redacted] I feel that as $h \to 0$ I can replace e^{-tH} by $e^{-th^2\partial_\mu^2}\, e^{-tL^2}$ and then it should follow that

$$\langle x | tr_s(\varepsilon\, e^{-tD^2}) | x \rangle = (ih)^n (-1)^m 2^m i^m \, sign(i_1 \cdots i_m)$$
$$\times \frac{1}{(4\pi t h^2)^m} \int e^{-t_0L^2} L_{i_1} e^{-t_1L^2} L_{i_2} \cdots e^{-t_nL^2}$$

$$= \frac{i^m}{(2\pi)^m} \frac{1}{t^m} \sum_{\sigma \in \Sigma_n} sgn(\sigma) \int_{t_0+\cdots+t_n=t} e^{-t_0L^2} L_{\sigma_1} e^{-t_1L^2} \cdots L_{\sigma_n} e^{-t_nL^2}$$

This somehow is a regularized version of

$$(dL)^n = \left(\sum L_i\, dx^i\right)^n = \left(\sum sgn(\sigma)\, L_{\sigma_1} \cdots L_{\sigma_n}\right) dx^1 \cdots dx^n$$

Suppose it should happen that e^{-tL^2} commutes with the $\partial_\mu L$, e.g. where $L = \gamma^i x^i$ comes from a Koszul complex. Then the above becomes

$$\left(\frac{i}{2\pi}\right)^m \frac{1}{t^m}\, e^{-tL^2}\, \frac{t^n}{n!}\, (dL)^n$$

July 3, 1983

I am computing the index of

$$\mathcal{D} = \frac{h}{i}\gamma^\mu D_\mu + \varepsilon L$$

acting on functions in $L^2(\mathbb{R}^n, \Delta\otimes\mathcal{H})$. Here Δ = spinors over $\mathbb{R}^n$ n even, L is an odd degree operator on $\mathcal{H}$ which is $\mathbb{Z}/2$-graded, and $\varepsilon = \varepsilon_\Delta$ gives the grading on Δ. The index is

$$\operatorname{Tr}\left(\varepsilon_\Delta\varepsilon_{\mathcal{H}}\, e^{-t\mathcal{D}^2}\right) = \int_{\mathbb{R}^n} \operatorname{Tr}_{\Delta\otimes\mathcal{H}}\left(\varepsilon_\Delta\varepsilon_{\mathcal{H}}\langle x|e^{-t\mathcal{D}^2}|x\rangle\right)$$

I propose to compute this as

$$\operatorname{Tr}_{\mathcal{H}}\, \varepsilon_{\mathcal{H}} \int_{\mathbb{R}^n} \operatorname{Tr}_\Delta\left(\varepsilon_\Delta \langle x|e^{-t\mathcal{D}^2}|x\rangle\right)$$

which is slightly in the wrong order from the viewpoint of the index of families (but might be closer to the ~~submanifold~~ Thom space idea.)

Next:

$$\mathcal{D}^2 = \underbrace{-h^2 D_\mu^2 + L^2}_{} + \frac{h}{i}\gamma^\mu\varepsilon \overbrace{[D_\mu, L]}^{L_\mu} - \frac{h^2}{2}\gamma^\mu\gamma^\nu \overbrace{[D_\mu, D_\nu]}^{F_{\mu\nu}}$$

$$= \quad H \quad - hA - h^2 B$$

$$A = \gamma^\mu i\varepsilon L_\mu \qquad B = \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}.$$

so we have a perturbation expansion

$$e^{-t\mathcal{D}^2} = e^{-tH} + \int_{t_0+t_1=t} e^{-t_0 H}(hA + h^2 B)e^{-t_1 H} + \dots$$

Now I know that when $Tr_\Delta \varepsilon_\Delta$ is applied to the terms of this expansion only terms of order $\geq n$ in h occur.

Simplification: $(\gamma^\mu i\varepsilon)(\gamma^\nu i\varepsilon) = -\gamma^\mu(\gamma^\nu \varepsilon)\varepsilon = \gamma^\mu\gamma^\nu$ so the $\gamma^\mu i\varepsilon$ anti-commute and have square 1. Thus one can replace $\gamma^\mu i\varepsilon L_\mu$ by $\gamma^\mu L_\mu$. Also one needs at least n gammas to get a non-zero $Tr_\Delta \varepsilon_\Delta$ which is why only h^k with $k \geq n$ occurs.

Let us consider a typical term of degree n in h.

$$\int e^{-t_0 H}\, hA\, e^{-t_1 H} h^2 B\, e^{-t_2 H}\, hA \cdots\cdots e^{-t_n H}$$

p A's
q B's

$p+2q = n$
$p+q = m$

$$= \int e^{-t_0 H}\, h\gamma^{\mu_1} A^L_{\mu_1} \cdots \frac{h^2}{2}\gamma^{\mu_2}\gamma^{\mu_3} F_{\mu_2\mu_3} \cdots h\gamma^{\mu_4} A^L_{\mu_4} \cdots$$

When $Tr_\Delta \varepsilon_\Delta$ is applied we get

$$h^n \int e^{-t_0 H} A^L_{\mu_1} e^{-t_1 H} F_{\mu_2\mu_3} e^{-t_2 H} A^L_{\mu_4} \cdots \frac{1}{2^q} \underbrace{Tr_\Delta\, \varepsilon_\Delta(\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\cdots)}_{2^m i^m\, \varepsilon(\mu_1,\ldots,\mu_n)}$$

where ε is the sign of a permutation and 0 if the sequence $\mu_1,\ldots,\mu_n$ is not distinct.

Next I have to apply $\langle x|\quad|x\rangle$ and let $h \to 0$. The basic idea I have is that

$$e^{-t_j H} = e^{-t_j h^2 \partial_\mu^2} \cdot e^{-t_j L^2}\,(1+O(h))$$

should be true. If this is true which I will suppose, then the above expression becomes

$$h^n e^{-th^2\partial_\mu^2} \int e^{-t_0 L^2} A^L_{\mu_1} e^{-t_1 L^2} \tfrac{1}{2} F_{\mu_2\mu_3} e^{-t_2 L^2} A^L_{\mu_4} \cdots 2^m i^m \varepsilon(\mu).$$

with an error $O(h^{n+1})$. Now

$$\langle x | e^{-th^2\partial_\mu^2} | x \rangle = \frac{1}{(4\pi t h^2)^m} dx^1 \ldots dx^n$$

so we get

$$h^n \frac{1}{(4\pi t h^2)^m} (2i)^m \int e^{-t_0 L^2} A^L_{\mu_1} e^{-t_1 L^2} \tfrac{1}{2} F_{\mu_2\mu_3} \cdots \underbrace{\varepsilon(\mu) dx^1 \ldots dx^n}_{dx^{\mu_1} \ldots dx^{\mu_n}}$$

$$= \left(\frac{i}{2\pi}\right)^m \frac{1}{t^m} \int_{t_0 + \cdots + t_k = t} e^{-t_0 L^2} (L_\mu dx^\mu) e^{-t_1 L^2} \left(\tfrac{1}{2} F_{\mu\nu} dx^\mu dx^\nu\right) \cdots$$

So we get the following formula:

$$\lim_{h \to 0} \operatorname{Tr}_\Delta\left(\varepsilon_\Delta \langle x | e^{-tD^2} | x \rangle\right) = \left(\frac{i}{2\pi}\right)^m \frac{1}{t^m} \times \text{the component}$$

of degree n of $e^{-t\left(L^2 - L_\mu dx^\mu - \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu\right)}$ at x.

$$= \left(\frac{i}{2\pi}\right)^m \times \text{component of degree } n \text{ in } e^{-tL^2 + \sqrt{t} L_\mu dx^\mu + \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu}$$

Conventions:

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$\sigma_1 \sigma_2 \sigma_3 = i$.

Take $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\varepsilon = \sigma_3$. Then

$\varepsilon \gamma^1 \gamma^2 = i$ for $n = 2$

or $\operatorname{Tr}(\varepsilon \gamma^1 \gamma^2) = 2i$ $\quad\therefore \operatorname{Tr}(\varepsilon \gamma^1 \ldots \gamma^n) = (2i)^m$.

I now want to consider the odd case. Then L is a s.a. op on $\mathcal{H}$ (no grading). [redacted] The simplest way to think is that we want to get the Dirac op on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, when $L = \frac{1}{i}\partial_{x_n}$. Thus

$$\mathcal{D} = \frac{h}{i}\gamma^\mu D_\mu + \gamma^n L$$

acts on $L^2(\mathbb{R}^{n-1}, \Delta \otimes \mathcal{H})$, where Δ is the n-diml. spinors, $n = 2m$, and the summation is over $1 \le \mu < n$.

$$\mathcal{D}^2 = -h^2 D_\mu^2 + L^2 + \frac{h}{i}\gamma^\mu\gamma^n \overbrace{[D_\mu, L]}^{L_\mu} - \frac{h^2}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}$$

$$= H - hA - h^2 B \qquad \begin{aligned} A &= \gamma^\mu i\gamma^n L_\mu \\ B &= \tfrac{1}{2}\gamma\gamma F \end{aligned}$$

Arguing as before we need to evaluate

$$Tr_\Delta\left(\varepsilon_\Delta \gamma^{\mu_1}\gamma^{\mu_2}(\gamma^{\mu_3} i\gamma^n)(\gamma^{\mu_4} i\gamma^n)\gamma^{\mu_5}\gamma^{\mu_6}\ldots\right).$$

For this to be nonzero one must have an odd no. of γ^n, otherwise γ^n won't occur. Next note that $i\gamma^n$ commutes with $\gamma^\mu\gamma^\nu$ since $\mu, \nu < n$. Furthermore

$$i\gamma^n\gamma^\mu i\gamma^n = (-1)(-\gamma^\mu\gamma^n)\gamma^n = \gamma^\mu.$$

So we cancel the $i\gamma^n$ in pairs, and because there are an odd number we get

$$Tr_\Delta\left(\varepsilon_\Delta \gamma^{\mu_1}\ldots\gamma^{\mu_{n-1}} i\gamma^n\right)$$

the power of h is the number of γ^μ's which will be h^{n-1}

which is equal to $\varepsilon(\mu_1, \ldots, \mu_{n-1}, n)(2i)^m i$

$$= \varepsilon(\mu_1, \ldots, \mu_{n-1})(2i)^m i$$

The other difference with the even case is that

$$\langle x | e^{-th^2\partial_\mu^2} | x \rangle = \frac{1}{(4\pi t h^2)^{(n-1)/2}} \, dx^1 \cdots dx^{n-1}$$

So we get the formula

$$\lim_{h\to 0} \mathrm{Tr}_\Delta\left(\varepsilon_\Delta \langle x | e^{-tD^2} | x\rangle\right) = \frac{(2i)^m i}{(4\pi)^{m-1/2}} \frac{1}{t^{m-1/2}} \times \text{the component}$$

of degree $n-1$ of $e^{-t(L^2 - L_\mu dx^\mu - \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu)}$

$$= \left(\frac{i}{2\pi}\right)^m i\sqrt{4\pi} \times \text{the degree } (n-1) \text{ comp. of } e^{-tL^2 + \sqrt{t} L_\mu dx^\mu + \frac{1}{2} F dx dx}$$

July 4, 1983:

Check the formula $\left(\frac{i}{2\pi}\right)^m i\sqrt{4\pi} \times$ degree$(n-1)$ part of

$$\mathrm{Tr}\left(e^{-tL^2 + \sqrt{t}\, dL + \Omega}\right).$$

Do this for $D = \frac{1}{i}\gamma^1\partial_x + \gamma^2 x = \frac{1}{i}\begin{pmatrix} 0 & \partial_x + x \\ \partial_x - x & 0 \end{pmatrix}$

so $L = x$ on $\mathcal{H} = \mathbb{C}$. Then

$$\text{index} = \int \frac{i}{2\pi} i\sqrt{4\pi}\, e^{-tx^2} \sqrt{t}\, dx = -\int \sqrt{\frac{t}{\pi}}\, e^{-tx^2} dx$$

$$= -1$$

which is indeed the index of $\partial_x - x$.

Also in general the degree 1 component is the 1-form

$$-\sqrt{\frac{t}{\pi}}\, \mathrm{Tr}(e^{-L^2} dL).$$

Is this the same as $d\eta_L$?

Recall (put $L = A$)

$$\eta(s) = \operatorname{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = \operatorname{Tr}\left(A\,(A^2)^{-\frac{s+1}{2}}\right)$$

$$\delta\eta(s) = -s\operatorname{Tr}\left(\delta A\,(A^2)^{-\frac{s+1}{2}}\right)$$

$$= \frac{-s}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \underbrace{\operatorname{Tr}\left(\delta A\, e^{-tA^2}\right)}_{\text{if } ct^{-1/2} \text{ in the asymp. exp.}} t^{\frac{s+1}{2}}\,\frac{dt}{t}$$

$$\xrightarrow{s\to 0} \frac{-\cancel{s}}{\sqrt{\pi}}\, c\,\frac{2}{\cancel{s}} = -\frac{c}{\sqrt{\pi}}\,2$$

where c is the constant term in the asymptotic expansion of $\sqrt{t}\operatorname{Tr}(\delta A\, e^{-tA^2})$ as $t \downarrow 0$.

Thus we are twice too big, probably because as an eigenvalue of A crosses zero η jumps by 2.

Remark: The above shows that we do not want to take the limit as $t \to 0$ in the form

$$\operatorname{Tr}\left(e^{-tL^2 + \sqrt{t}\,dL + \Omega}\right)$$

but rather the constant term in the asymptotic expansion.

July 5, 1983

[redacted] summarize the situation.

At the moment we have a formula for the character of a Kasparov K-element. More precisely suppose over a space Y I am given a Hilbert bundle $\mathcal{H}$ with connection ∇ and a self-adjoint Fredholm bundle map L on $\mathcal{H}$. Here $\mathcal{H}$ is $\mathbb{Z}_2$-graded or not and L is odd for the grading. Then to this data $(\mathcal{H}, \nabla, L)$ I have attached differential forms on Y by a complicated analytic process.

Consider next the following example. Let M be a compact Riemannian spin manifold, Δ_M the spinor bundle over M, and let $\tilde{E}$ be a vector bundle with metric + connection over $Y \times M$. Then we have the Dirac operator on $\Delta_M \otimes E_y$ on each fibre $y \times M$. Let $\mathcal{H}_y = L^2$-sections of $\Delta_M \otimes E_y$ over $y \times M$. This gives us a Hilbert bundle $\mathcal{H}$ over Y. Because $\tilde{E}$ is given a connection in the Y-directions we get a connection on $\mathcal{H}$ over Y. Finally the Dirac operator on the fibres gives us L.

[redacted] For this example it should be the case that the differential forms $ch(\mathcal{H}, \nabla, L)$ can be calculated by the right side of the index theorem for families:

$$\int_M ch(\tilde{E})\, \hat{A}(M)$$

It seems to be very important to understand what this theorem says for $\dim M = 1, 2$. One side is analytical, involving traces of infinite dimensional operators. The other is geometrical; it depends only on a bundle $\tilde{E}$ over $Y \times M$ with connection.

So let us forget about the operators and concentrate on the geometry. We have a procedure to assign differential forms on the base Y to any bundle with ~~connection~~ E over $Y \times M$, namely

$$\int_M ch(\tilde{E})\, \hat{A}(M).$$

What does such a procedure consist of? A differential form is determined infinitesimally on Y. Locally we can assume $\tilde{E} \approx pr_2^*(E)$. Then the connection ∇ on $\tilde{E}$ splits into horizontal and vertical pieces. The vertical direction is a family of connections on E over M depending on $y \in Y$. The horizontal direction is a map $T_y \longrightarrow \mathrm{End}_{ok}(E) = \mathrm{Lie}(\mathcal{G})$. Finally the isom. $\tilde{E} \approx pr_2^*(E)$ can be changed by a map $Y \longrightarrow \mathcal{G}$.

<u>July 6, 1983:</u>

The problem is to bring some order into all the ideas I have about the cohomology of the gauge group, cyclic homology, etc.

A possible approach might be to list carefully the different objects and their relations.

Fixed will be a compact manifold M and a vector bundle E with metric over it. $\mathcal{G} = \mathrm{Aut}(E)$ is the gauge group. Usually E is the trivial N-diml bundle over M, so $\mathcal{G} = \mathrm{Maps}(M, U_N)$; I am interested in the case where $N \to \infty$ where I know the Lie algebra cohomology and cohomology of $\mathcal{G}$.

Bundle theory: given a variable space Y we can consider vector bundles (with metric) F over $Y \times M$ such that F restricted to each fibre is isomorphic to E. Such bundles over $Y \times M$ are the same as principal $\mathcal{G}$-bundles over Y, hence are classified by maps $Y \longrightarrow B\mathcal{G}$. Now

$$B\mathcal{G} = \underset{\text{belonging to } E.}{\text{component of } \mathrm{Maps}(M, BU_N)}$$

so we know the homotopy groups of $B\mathcal{G}$. I think we also know that we obtain primitive generators for $H^*(B\mathcal{G})$ (rational coefficients) by taking the Künneth components of the Chern character of the tautological bundle $\tilde{E}$ over $B\mathcal{G} \times M$

$$ch(\tilde{E}) \in H^{ev}(B\mathcal{G} \times M) = \mathrm{Hom}^{ev}(H_*(M), H^*(B\mathcal{G})).$$

In other words all characteristic classes for F over $Y \times M$ natural in Y are obtained by Künneth decomposition of

$$ch(F) \in H^{ev}(Y \times M)$$

and capping with cycles in M.

K-theory (alg.) of $C^\infty(M)$: Here one looks at bundles F over $Y\times M$ with a partial flat connection in the Y-direction. Actually it is better to forget the alg. K-theory, and to restrict to unitary connections.

When we compute the characteristic classes of F with such a partial flat Y-connection, we choose an extension of the partial connection to a full connection over $Y\times M$. The curvature is then of type $(1,1)+(0,2)$ so

$$\begin{array}{ccc} H^{2i}(Y\times M) & = & \bigoplus_p H^p(Y)\otimes H^{2i-p}(M) \\ \cup & & \\ ch_i(F) & = & \sum_i ch_{ip}(F) \end{array}$$

will satisfy
$$ch_{ip}(F)\neq 0 \implies p\le 2i-p \text{ or } p\le i.$$
So if I think in K-theory terms I have

$$K_p \xrightarrow{ch_{ip}} H^{2i-p}(M) \quad \text{for } i\ge p$$

or

$$K_p \longrightarrow H^p(M)\oplus H^{p+2}\oplus H^{p+4}\oplus\cdots$$

Now I think it will turn out that if things are analyzed a bit more carefully, one will see that we are using characteristic classes with coefficients in filtered de Rham cohomology

$$ch_i \in H^{2i}(Y\times M, F_i\Omega_M)$$

$$ch_{ip}: K_p(Y)\longrightarrow H^{2i-p}(M, F_i\Omega_M) = \begin{cases} H^{2i-p}(M) & i>p \\ \text{Ker } d \text{ on } \Omega^p & i=p \\ 0 & i<p \end{cases}$$

$2i-p\ge i \iff i\ge p$.

So far we have the bundle theories associated to $\mathcal{G}$ and $\mathcal{G}^\delta$, and it is natural to consider the Lie algebra theory. Recall natural

$$\begin{tikzcd} \text{Fibre} \arrow[r] \arrow[d, "\text{map}"'] & BG^{\delta} \arrow[r] & BG \\ \text{Lie}(G) & & \end{tikzcd}$$

Concretely given a trivial bundle with a flat connection, one gets a connection form which is a globally defined Lie algebra-valued form. It maps tangent vectors ~~on~~ Y to elements of the Lie algebra, so any form on Lie gives a form on Y, and the flatness means this is compatible with d.

Now I can construct characteristic classes for a Y-flat partial connection on $pr_2^*(E)$ over $Y \times M$. This amounts to producing classes with coefficients in the fibre of

$$F_i \Omega_M^{\cdot} \longrightarrow \Omega_M^{\cdot}$$

which is the complex

$$0 \qquad \Omega^0 \qquad\qquad \Omega^{i-1}$$
$$i$$

and leads to map

$$(\text{fibre } K \text{ theory})_p \xrightarrow{ch_{ip}} H^{2i-p}(M, 0 \to \Omega^0 \to \cdots \to \Omega^{i-1} \to 0)$$

$$= \begin{cases} 0 & i > p \\ \Omega^{p-1}/d\Omega^{p-2} & i = p \\ H^{2i-p-1}(M) & i < p \end{cases}$$

This is completely constructive and elementary à la Bott. But the interesting point is that this gives the Lie alg. homology by virtue of the cyclic homology theory.

Conjecture: It should be possible by curvature methods to define the map

$$H_p(gl(C^\infty(M))) \longrightarrow \Omega^{p-1}/d\Omega^{p-2} \oplus H^{p-3}_{DR}(M) \oplus \cdots$$

which the cyclic homology theory proves is an isomorphism.

I think I can already do this map on the level of the fibre from BG^δ to BG. Let us go over the construction a la Bott.

Take E = trivial bundle over M and then on $pr_2^*(E)$ over $Y \times M$ we suppose given a partial connection in the Y direction which is flat. But $pr_2^*(E)$ is trivial so this partial connection is of the form $d_Y + \omega$ where $\omega \in pr_1^*(T_Y^*) \otimes \mathrm{End}(E)$. To compute characteristic classes of $(pr_2^*(E), \nabla_Y)$ we extend ∇_Y to a full connection over $Y \times M$, namely to $\nabla = d + \omega$ where now ω is a form of type $(1,0)$ with values in $\mathrm{End}(E) = gl$.

The characteristic classes we are after result from the fact that $ch_i(pr_2^*(E)) \in H^{2i}(Y \times M)$ are of a filtration having components of bidegree $(p, 2i-p)$ equal to zero for $p \geq i$ (the curvature has components $(1,1)$ and $(0,2)$) on one hand, and because these components are zero because the bundle is trivial. Thus $ch_{i,p}$ for $p > i$ is zero for two reasons, and the difference of these reasons is a class of degree $2i-1$ which we are after.

Two connections $D = d + \omega$, $D = d$. Set $D_t = d + t\omega$, and use the transgression formula

$$tr(e^{D_1^2}) - tr(e^{D_0^2}) = d\int_0^1 tr(e^{D_t^2}\omega)\, dt$$

Thus

$$\mathrm{tr}\left(e^{D^2}\right) = d\int_0^1 \mathrm{tr}\left(e^{D_t^2}\omega\right) dt$$

$$D_t^2 = (d+t\omega)^2 = t\,\underbrace{d\omega}_{\underbrace{d_Y\omega}_{-\omega^2} + d_M\omega} + t^2\omega^2$$

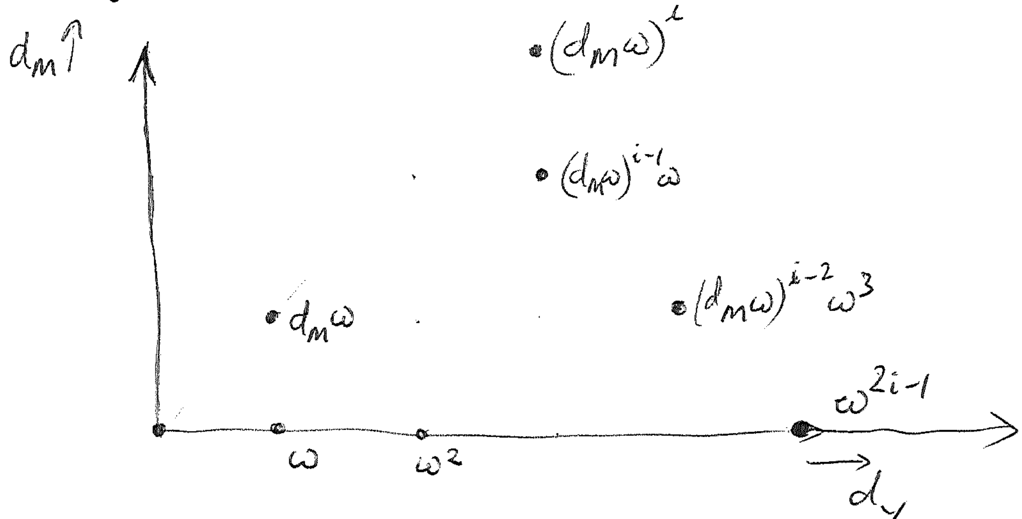
$$D_t^2 = t\,d_M\omega + (t^2-t)\omega^2$$

and so we get

$$(*) \qquad \mathrm{tr}\left(e^{d_M\omega}\right) = d\int_0^1 \mathrm{tr}\left(e^{t d_M\omega + (t^2-t)\omega^2}\omega\right) dt.$$

Now ω is of type $(1,0)$ so $d_M\omega$ is of type $(1,1)$.

Picture of $2i$-th component.



My problem is to see how to get the components of the Chern character; the formula $(*)$ as it stands doesn't show that the individual components on the right side are closed.

July 7, 1983:

Let G be a Lie group and for each Y consider the set of flat connections on the trivial principal bundle $Y \times G$. Such a connection is given by a $\mathfrak{g}$-valued 1-form ω on Y satisfying

$$d\omega + \omega^2 = 0.$$

Now suppose we have a natural transf.

$$\Phi : \{\omega \text{ on } Y\} \longrightarrow \Omega^*(Y).$$

Locally we know any ω on Y is of the form $\omega = g^{-1}dg$ where $g : Y \to G$ is unique up to left multiplication by an elt of G. This shows that Φ is determined by what it does when $Y = G$ and $\omega = \omega_G$ is the Maurer-Cartan form on G. So

$$\Phi(\omega_G \text{ on } G) \in \Omega^*(G)$$

and because left multiplication by an elt of G preserves ω_G, we have that $\Phi(\omega_G \text{ on } G)$ is a left-invariant form on G. Conversely given a left-inv. form on G the fact that $\omega = g^{-1}dg$ can be solved locally and uniquely up a left mult, shows we can define such a Φ.

July 8, 1983

Review of Bott-Chern formulas:

$$D_t = D_0 + tB$$

$$\frac{d}{dt}\operatorname{tr}(e^{D^2}) = \operatorname{tr}(e^{D^2}(DB + BD))$$

$$d\,\operatorname{tr}(e^{D^2}B) = \operatorname{tr}(\underset{\uparrow}{[D, e^{D^2}B]}) = \operatorname{tr}(e^{D^2}(DB+BD))$$

anti-comm. as D, B odd.

$$\therefore \boxed{\operatorname{tr}(e^{D_1^2}) - \operatorname{tr}(e^{D_0^2}) = d\int_0^1 \operatorname{tr}(e^{D_t^2}B)\,dt}$$

In the case of complex manifolds the connection is

$$D = d + \theta = (d' + \theta) + d''$$

where $\theta = N^{-1}d'N$ and N is the hermitian metric. In other words $D = D' + D''$ where $D'' = d''$ and $D' = N^{-1}\circ d'\circ N$. Let $\cdot$ denote derivative wrt changing N. Then

$$\dot{D}' = [D', L] \qquad L = N^{-1}\dot{N}.$$

Now from

$$0 = [D, e^{D^2}] = [D', e^{D^2}] + [D'', e^{D^2}]$$

and the bigrading we see that $[D', e^{D^2}] = [D'', e^{D^2}] = 0$. Thus

$$d'\operatorname{tr}(e^{D^2}L) = \operatorname{tr}([D', e^{D^2}L]) = \operatorname{tr}(e^{D^2}\underbrace{[D', L]}_{\dot{D}' = \dot{D}})$$

and so

$$\boxed{\frac{d}{dt}\operatorname{tr}(e^{D^2}) = d''d'\operatorname{tr}(e^{D^2}L)}$$

$$\frac{d}{dt}\operatorname{tr}(e^{D^2}) = \operatorname{tr}(e^{D^2}(D\dot{D} + \dot{D}D)) = d\,\operatorname{tr}(e^{D^2}\dot{D})$$

Summary: I consider flat connections ω on the trivial principal G-bundle over Y, or equivalently, flat partial Y-connections on the trivial G-bundle over $Y \times M$. I know that a natural transformation from such connections to forms on Y is the same as a left-invariant differential form on G. What I want to do is to use curvature computations to produce such natural transformations.

If F is a bundle over $Y \times M$ with a flat partial Y-connection, then we have refined Chern classes

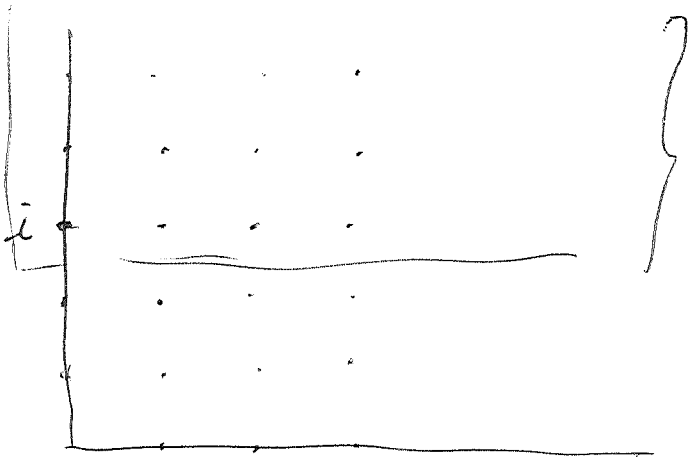
$$(*) \qquad ch_i(F) \in H^{2i}(Y \times M, \widetilde{F_i \Omega_M^{\circ}}).$$

In effect F is a vector bundle on the ringed space given by $Y \times M$ with sheaf of rings $\tilde{\Omega}_M^{\circ} = pr_2^*(\Omega_M^{\circ})$.

We know $\widetilde{\Omega_M^p}$ has a Dolbeault resolution:

$$0 \longrightarrow \tilde{\Omega}_M^p \longrightarrow \Omega_{Y\times M}^{0,p} \xrightarrow{d_Y} \Omega_{Y\times M}^{1,p} \xrightarrow{d_Y} \cdots$$

by fine sheaves over $Y \times M$. Hence the cohomology $(*)$ is given by the forms on $Y \times M$ with M-degree $\geq i$. Picture



gives $H^*(Y \times M, \widetilde{F_i \Omega_M^{\circ}})$

If the given flat Y-connection on F is extended to a connection, the curvature Ω is of type $(1,1)+(0,2)$, and so $tr(\Omega^i)$ lies in $F_i^Y(\Omega^{\bullet}_{Y\times M})$. This is the curvature way to compute the refined Chern classes $(*)$.

Next let F be the trivial bundle over $Y\times M$ with the flat Y-connection $d_Y+\omega$, where ω can be thought of as either a $Lie(\mathcal{G})=\tilde{g}$ valued form on Y, or as a g-valued form on $Y\times M$ of type $(1,0)$. Flatness means

$$(d_Y+\omega)^2 = d_Y\omega+\omega^2=0$$

Extending the Y-connection to a connection is obvious in this case, namely use the trivialization to define the connection in the M-direction:

$$d+\omega = (d_Y+\omega)+d_M$$

Because the bundle is trivial, we know that $ch_i(F)$ which I will now write $ch_i(\omega)$ goes to zero under the map

$$H^{2i}(Y\times M, \widetilde{F_i\Omega_M}) \longrightarrow H^{2i}(Y\times M, \tilde{\Omega}_M)$$

for a canonical reason. So it should be possible to define a refined class:

$$ch_i(\omega)\in H^{2i-1}(Y\times M, \widetilde{\Omega_M/F_i\Omega_M})$$

Now we do this on the level of differential forms as follows. We consider the linear path $d+t\omega$ from the trivial connection d to the extension

$d + \omega$ we have chosen of our given $d_Y + \omega$.
Then

$$(d+t\omega)^2 = t\,d\omega + t^2\omega^2$$
$$= t\,d_M\omega + (t^2-t)\,\omega^2$$

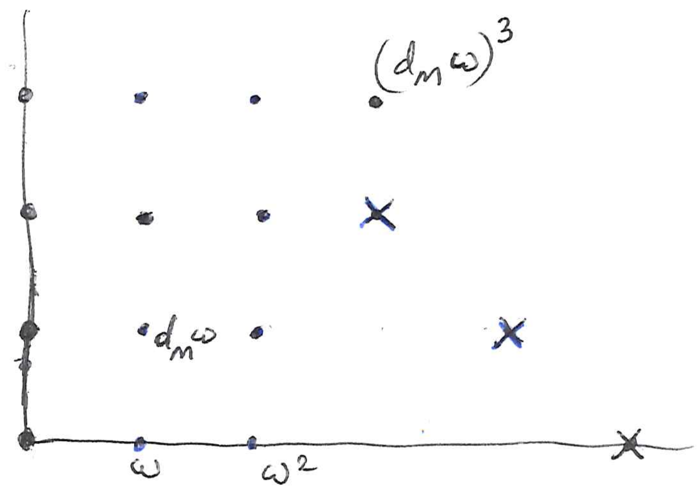
(since $d_Y\omega + \omega^2 = 0$). So the transgression formula gives

$$tr\left(e^{(d+\omega)^2}\right) - tr\left(e^{d^2}\right) = d\int_0^1 dt\; tr\left(e^{(d+t\omega)^2}\omega\right)$$

or

$$\boxed{tr\left(e^{d_M\omega}\right) = d\int_0^1 dt\; tr\left(e^{t d_M\omega + (t^2-t)\omega^2}\omega\right)}$$

Here's the picture for the i-th component: $i = 3$



The three x's mark the components of the above integral which represents

$$ch_i(\omega) \in H^{2i-1}(Y\times M, \Omega/F_i\Omega)$$

Now the problem becomes how to get natural differential forms on Y.

Here is the way to do this. Let us denote by u the explicit form representing $ch_i(\omega)$, namely

$$u = \left[\int_0^1 dt \ \operatorname{tr}\left(e^{t d_M \omega + (t^2-t)\omega^2}\,\omega\right)\right]_{\deg(2i-1)}$$

Then u is a form of degree $2i-1$ on $Y \times M$ such that du has filtration $\geq i$.

Now we use u to map [redacted] currents c on M to forms on Y. We restrict u to $Y \times c$ and then integrate over the fibres of $Y \times c \longrightarrow Y$ to get a form on Y. This gives us a map

$$C_r(M) \longrightarrow \Omega_Y^{2i-1-r}$$

and the thing to check is that it is compatible with d as long as we stay in the range [redacted] where du restricted to $Y \times c$ is zero. This means that $\dim(c) < i$.

Here's a way to see this geometrically. Think of u as a submanifold of $Y \times M$ of codim $2i-1$ with boundary.

$$\begin{tikzcd}
U \cap (Y\times c) \subset & Y \times c \arrow[r] \arrow[d, hook] & c \arrow[d, phantom, "\cap"] \\
U \subset & Y \times M \arrow[r] \arrow[d] & M \\
& Y &
\end{tikzcd}$$

[redacted]

Thus we have a correspondence

$$c \longmapsto \text{[redacted]}\ (pr_1)_*\left(U \cdot pr_2^* c\right).$$

Geometrically $\quad d\left[U \cap (Y \times c)\right] = \left[dU \cap (Y \times c)\right] \cup \left[U \cap (Y \times dc)\right]$

and because we assume du has filtration $\geq i$ it follows that $du \cap (Y \times c) = 0$ when $\dim c < i$.

Up to some sign problems we thus get a map of complexes

$$\begin{array}{ccc} C_{i-1}(M) & \longrightarrow & \Omega_Y^{i} \\ \downarrow & & \downarrow \\ C_{i-2}(M) & \longrightarrow & \Omega_Y^{i+1} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ C_0(M) & \longrightarrow & \Omega_Y^{2i-1} \\ \downarrow & & \downarrow \\ 0 & & \Omega_Y^{2i} \\ & & \downarrow \end{array}$$

This means that to a closed current z in M of dim $r < i$ we have associated a closed form on Y of dim $2i-1-r$, and for $r < i-1$ the DR class of the form on Y depends only on the class of Y.

Conclusion: The explicit Bott-type formula for u will associate to cycles in M certain closed left-invariant forms on $\mathcal{G}$, and as long as we stay away from the edge we can therefore map homology of M ~~into~~ Lie cohomology of $\mathcal{G}$.

Possible generalization. If $\mathcal{G} = \text{Maps}(M, G)$ where G is a compact connected Lie gp., then from the rational viewpt G is a product of odd spheres, hence one knows the rational cohomology of $\mathcal{G}$. The above argument will enable one to realize certain classes for $\mathcal{G}$ in the Lie algebra cohomology.

It might eventually be possible to compute the Lie algebra cohomology for gauge groups.

July 9, 1983

Consider the gauge group $\mathcal{G}$ of all autos. of the bundle E with metric over M. The complexified Lie algebra of $\mathcal{G}$ is $\tilde{\mathfrak{g}} = \Gamma(M, \mathrm{End}\,E)$.

Consider flat partial Y-connections on the pull-back $pr_2^*(E)$. Recall that a left-invariant form on $\mathcal{G}$ can be identified with a natural way to map such connections to forms on Y.

Let's fix a connection on E over M and pull it back to a connection on $pr_2^*(E)$, which is then constant in the Y direction. Call this D_0. Then

$$D_0 = D_0' + D_0''$$

where D_0' is essentially d_Y defined by the constant structure of $pr_2^*(E)$ in the Y-direction, and where D_0'' is the given connection on E on each fibre $\{y\}\times M$. Clearly

$$D_0^2 = \underset{0}{\underset{\shortparallel}{(D_0')^2}} + \underset{0}{\underset{\shortparallel}{(D_0'D_0'' + D_0''D_0')}} + (D_0'')^2$$

where $(D_0'')^2$ is the curvature of the connection on E pulled back to $Y\times M$.

Now we take a flat partial Y-connection on $pr_2^*(E)$ which can be described

$$D' = D_0' + \omega$$

where ω is a $(1,0)$-form on $Y\times M$ with values in

End(E).

Alternative notation: [redacted]

$$D_0 = d_y + D_M \qquad\qquad D_0^2 = D_M^2$$

$$D_y \text{ or } D' = d_y + \omega \qquad\qquad d_y\omega + \omega^2 = 0.$$

Then extend D_y to $D_1 = D_y + D_M = D_0 + \omega$. and use the path of connections.

$$D_t = D_0 + t\omega$$

Calculate curvature

$$\begin{aligned} D_t^2 &= (d_y + D_M + t\omega)^2 \\ &= D_M^2 + t\big(d_y\omega + D_M(\omega)\big) + t^2\omega^2 \\ &= \underset{(0,2)}{D_M^2} + t\,\underset{(1,1)}{D_M(\omega)} + (t^2-t)\,\underset{(2,0)}{\omega^2} \end{aligned}$$

where $D_M(\omega) = [D_M, \omega]$ is an End(E) valued (1,1)-form.

Then we end up with the formular

$$\operatorname{tr}\Big(e^{\overset{(1,1)+(0,2)}{D_1^2}}\Big) - \operatorname{tr}\Big(e^{\overset{(0,2)}{D_0^2}}\Big) = d\int_0^1 \operatorname{tr}\big(e^{D_M^2 + tD_M(\omega) + (t^2-t)\omega^2}\,\omega\big).$$

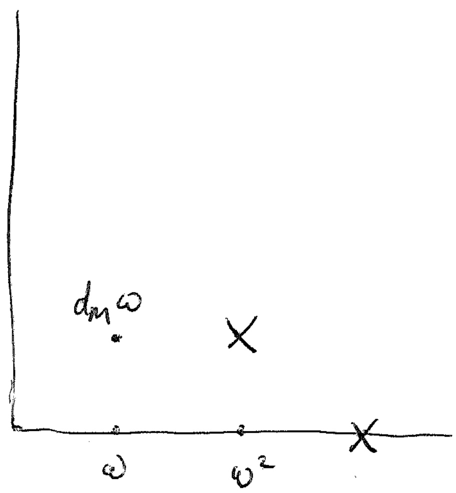
[redacted] So we can argue as before that because the left side has filtration $\geq i$ in degree $2i$, then the $(2i-1)$ component of the integral [redacted] is closed modulo filtration i.

Our next project will be to work out the simple case of [redacted] ch_2 in the case of the trivial bundle E and trivial connection: $D_M = d_M$:

$$\int_0^1 \operatorname{tr}\big((t\,d_M\omega + (t^2-t)\omega^2)\,\omega\big)dt \;=$$

$$\frac{1}{2}\,\mathrm{tr}(d_M\omega\cdot\omega) + \underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{-\frac{1}{6}}\mathrm{tr}(\omega^3)$$

$(2,1)$ $\qquad$ $(3,0)$



$$\begin{aligned} d_Y\,\mathrm{tr}(\omega^3) &= \mathrm{tr}(d_Y\omega\cdot\omega^2 - \omega\cdot d_Y\omega\cdot\omega + \omega^2\cdot d_Y\omega) \\ &= 3\,\mathrm{tr}(d_Y\omega\cdot\omega^2) \\ &= -3\,\mathrm{tr}(\omega^4) = 0 \end{aligned}$$

$$d_M\,\mathrm{tr}(\omega^3) = 3\,\mathrm{tr}(d_M\omega\cdot\omega^2)$$

$$\begin{aligned} d_Y(d_M\omega) &= -d_M(d_Y\omega) = d_M(\omega^2) \\ &= d_M\omega\cdot\omega - \omega\cdot d_M\omega \end{aligned}$$

$$\begin{aligned} \therefore\ d_Y\,\mathrm{tr}(d_M\omega\cdot\omega) &= \mathrm{tr}(d_M\omega\cdot\omega^2 - \omega\cdot d_M\omega\cdot\omega + d_M\omega\,\underbrace{d_Y\omega}_{-\omega^2}) \\ &= \mathrm{tr}(d_M\omega\cdot\omega^2) \end{aligned}$$

$$d_M\,\mathrm{tr}(d_M\omega\cdot\omega) = \mathrm{tr}(d_M\omega)^2$$

Thus one sees that

$$d\Big[\underbrace{\mathrm{tr}(d_M\omega\cdot\omega) - \frac{1}{3}\mathrm{tr}(\omega^3)}_{u}\Big] = \mathrm{tr}(d_M\omega)^2 \qquad \text{type } (2,2).$$

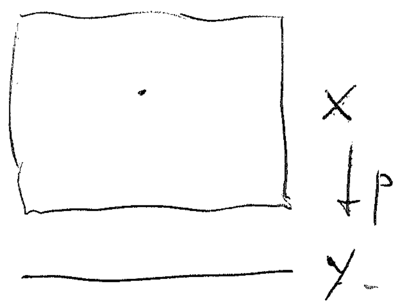
We now want to use u to define maps from currents or chains on M to forms on Y:

$$\begin{array}{ccc} C_0(M) & \longrightarrow & \Omega^3(Y) \\ \uparrow & & \uparrow \\ C_1(M) & \longrightarrow & \Omega^2(Y) \end{array}$$

I would really like to get the signs correct, and this means I have to understand integration over the fibre.

The basic geometric idea is as follows:

One has a submersion, and the tangent bundle satisfy



$$0 \to T_{X/Y} \to T_X \to p^* T_Y \to 0$$

$$0 \to p^* T_Y^* \to T_X^* \to T_{X/Y}^* \to 0$$

If ω is a form on X and I let g generate $\lambda(T_{X/Y})$ at a point, then $i(g)\,\omega$ can be identified with a form on the base. If the fibres of p are oriented, then $\lambda(T_{X/Y})$ is a trivial line bundle so it has such a section g, and on the other hand choosing g gives a measure on the fibres. So we integrate the image forms on $T(Y)$ at y with respect this measure on the fibre. A basic property is that

$$d\, p_*(\omega) = (-1)^{\dim(X/Y)}\, p_*(d\omega)$$

Let's be specific and consider $p : I \times Y \to Y$ where $I = [0,1]$. This time there will be an extra term because of the boundary of I. Given ω on $I \times Y$ we have

$$\omega = dt \cdot \alpha + \beta$$

where α, β are of the form $\alpha(t,y)\, dy^\mu \ldots$. Then

$$p_*(\omega) = \int_0^1 d\mu \cdot \alpha_t$$

where $d\mu$ is Labesque measure.

$$d_y p_*(\omega) = \int_0^1 d\mu \, (d_y \alpha)_t \, .$$

Now

$$d\omega = -dt\, d_y\alpha + \overbrace{d\beta}^{dt\, \partial_t \beta + d_y \beta}$$

$$p_*(d\omega) = \int_0^1 d\mu \, [-d_y\alpha + \partial_t \beta]$$

$$= -d(p_* \omega) + \beta_1 - \beta_0$$

Thus

$$\boxed{p_*(d\omega) = -d(p_*\omega) + i_1^*(\omega) - i_0^*(\omega)}$$

which maybe I should write

$$d(p_*\omega) = -p_*(d\omega) + (\partial p)_*(\omega).$$

In the case I am interested in we have the form $u = tr(d_M\omega.\omega) - \frac{1}{3}tr(\omega^3)$ on $Y \times M$ and I have a path $c : I \longrightarrow M$. Then

$$\begin{tikzcd} Y \times I \arrow[r] \arrow[d, "p"] & Y \times M \ni u \\ Y & \end{tikzcd}$$

so we look at

$$\begin{tikzcd} \partial c \arrow[r, mapsto] & (\partial p)_*(\operatorname{res} u) \\ C_0(M) \arrow[r] & \Omega^3(Y) \\ C_1(M) \arrow[r] \arrow[u, "\partial"] & \Omega^2(Y) \arrow[u, "d"] \\ c \arrow[r, mapsto] \arrow[uuu, bend left, mapsto] & p_*(\operatorname{res} u) \end{tikzcd}$$

Thus it seems that things commute without any sign problems.

Now it is time to tackle the problem of interpreting these classes in Lie algebra cohomology. This means that we take $Y = G$ and also that we convert left-invariant forms on G to alternating forms on the Lie algebra by the rule

$$X_1, \dots, X_p \xrightarrow{\eta} i(X_p)\cdots i(X_1)\eta \qquad \deg(\eta)=p.$$

Let's adopt the following viewpoint. Start with a bundle E over M with connection D and pull-back over $G \times M$ to obtain $pr_2^*(E) = F$ with the connection $D = d_{G} + D_M$. Clearly G acts on $G \times M$, E by left translation and it preserves D, d_{G}, D_M which operate on forms on ~~ ~~ $G \times M$ with values in ~~ ~~ F. Also we have operators $i(X)$ for $X \in \tilde{g}$. Now

$$[i(X), D_M] = i(X)D_M + D_M i(X)$$

is a derivation of degree zero and applied to a 0-form f it gives $i(X)D_M f = 0$, as $D_M f$ is a section of $pr_2^*(E) \otimes \Omega^{0,1}_{G\times M}$. ~~ ~~ Clearly because of the decomposition $G \times M$ and F, D_M come from M one should have

$$[i(X), D_M] = 0.$$

Let ω be the canonical left-invariant $(1,0)$ form on $G \times M$ with values in $\mathrm{End}(pr_2^*(E)) = pr_2^*(\mathrm{End}\, E)$ such that

$$i(X)\omega = X.$$

Then

$$i(X)[D_M, \omega] = -[D_M, i(X)\omega] = -[D_M, X],$$

but I will write this as

$$i(X) D_M \omega = -D_M X$$

interpreting $i(X)$, D_M in the obvious way on $\mathrm{End}(F)$-valued forms. Then we can calculate (say $D_M = d_M$)

$$i(Y)\,i(X)\,\mathrm{tr}\,(d_M\omega\cdot\omega) = i(Y)\,\mathrm{tr}\,(-\overset{1}{d_M}X\cdot\omega + \overset{2}{d_M}\omega\cdot X)$$

$$= \mathrm{tr}\,(d_M X\cdot Y - d_M Y\cdot X)$$

$$i(Z)\,i(Y)\,i(X)\,\mathrm{tr}(\omega^3) = i(Z)\,i(Y)\;3\,\mathrm{tr}\,(X\omega^2)$$

$$= i(Z)\;3\,\mathrm{tr}\,(XY\omega - X\omega Y)$$

$$= 3\,\mathrm{tr}\,(XYZ - XZY)$$

Now suppose $M = S^1$ and let's take c = interval $[a, b]$ and consider $c_*\,\mathrm{tr}(d_M\omega\cdot\omega)$, where c_* denotes pulling back to $G\times I$ and then integrating over the fibre to get a form on G. This will commute with the left G-action, so

$$i(Y)\,i(X)\,c_*\,\mathrm{tr}(d_M\omega\cdot\omega) = \int_a^b dt\;\mathrm{tr}\,(\partial_t X\cdot Y - \partial_t Y\cdot X).$$

This is a left invariant form on G; call it $f(X,Y)$. Its coboundary is

$$i(Z)\,i(Y)\,i(X)\,df = i(Z)\,i(Y)\,[\mathcal{L}(X) - d\,i(X)]\,f$$

$$= \Big(\mathcal{L}(X)\,i(Z)\,i(Y) - i([X,Z])\,i(Y) - i(Z)\,i([X,Y]) - \cdots\Big)f$$

$$= -f(Y,[X,Z]) - f([X,Y],Z) + \cdots$$

$$= -f([X,Y],Z) - f([Y,Z],X) - f([Z,X],Y)$$

We compute

$$\mathrm{tr}(\partial_t[X,Y], Z) - \mathrm{tr}([X,Y]\cdot\partial_t Z) \quad + \text{cyclic perms.}$$

and get

$$\mathrm{tr}(\partial_t X\cdot[Y,Z]) \quad + \text{cyclic perm.}$$

so that

$$\delta f(X,Y,Z) = -\Big[\mathrm{tr}(X[Y,Z])\Big]_a^b$$

Hence it [redacted] follows that

$$d_{\text{[redacted]}}\int_a^b dt\ \mathrm{tr}(\partial_t\omega\cdot\omega) = \Big[-\tfrac{1}{3}\,\mathrm{tr}(\omega^3)\Big]_a^b .$$

Since $u = \mathrm{tr}(\partial_t\omega\cdot\omega) - \frac{1}{3}\mathrm{tr}(\omega^3)$, this is exactly the equality

$$d(c_* u) = (\partial c)_* u .$$

We learn from the above computation that it is a bad idea to work with cochain formulas in [redacted] Lie algebra cohomology. (I didn't copy over the details of the computation, but they are long and uninspiring.) One only has to compare $\mathrm{tr}(\omega^{2p+1})$ with [redacted] the antisymmetrization of $\mathrm{tr}(X_1 \cdots X_{2p+1})$; to show the former is closed is easy whereas with the latter it is very hard.

But [redacted] Connes' cochains are primitive $gl_n(k)$-invariant Lie cochains on $gl_n(A)$ for n large, and the algebra is simpler ~~as~~ it involves cyclic, rather than full, anti-symmetrization. I would like the Connes versions of $\mathrm{tr}(\omega^{2p+1})$ and $\mathrm{tr}(d_M\omega\cdot\omega^{2p})$. Is there a direct way to go from these to Connes' cocycles? Or a Connes' version of these forms, or maybe I should say stable version?

July 10, 1983

I want to work out the characteristic classes defined using the Maurer-Cartan form ω for the case $\dim M = 1$ in terms of Connes cocycles.

$$u = \int_0^1 dt \ \ tr\left(e^{t d''\omega + (t^2-t)\omega^2}\omega\right)$$

Take the component of total degree $2i+1$ and M-degree 1.

[redacted]

$$tr\left([\underbrace{t d''\omega}_{A} + \underbrace{(t^2-t)\omega^2}_{B}]^i \omega\right)$$ [redacted]

We want exactly one A:

$$tr\left(\sum_{j=0}^{i-1} B^j A B^{i-1-j}\omega\right) = \sum_{j=0}^{i-1} t(t^2-t)^{i-1} \underbrace{tr\left(\omega^{2j} d''\omega\, \omega^{2(i-1-j)}\right)}$$

$$\longleftrightarrow i \times tr(\omega^{2i} d''\omega)$$

Need

$$\int_0^1 t(t^2-t)^{i-1} dt = (-1)^{i-1}\int_0^1 (1-t)^{i-1} t^i dt = (-1)^{i-1}\frac{\Gamma(i)\Gamma(i+1)}{\Gamma(2i+1)} = (-1)^{i-1}\frac{(i-1)!\, i!}{(2i)!}$$

$$\int_0^1 (t^2-t)^i dt = (-1)^i \int_0^1 (1-t)^i t^i dt = (-1)^i \frac{(i!)^2}{(2i+1)!}$$

So I get the following form

$$\boxed{(-1)^{i-1}\frac{(i!)^2}{(2i)!}\left[tr(\omega^{2i-1}d''\omega) - \frac{1}{2i+1} tr(\omega^{2i+1})\right]}$$

which d is of filtration ≥ 2. Check

$$d''\, tr(\omega^{2i+1}) = +(2i+1)\, tr(\omega^{2i} d''\omega)$$

$$d'\, tr(\omega^{2i-1}d''\omega) = tr\,[(-\omega^2)\omega^{2i-2} - \omega(-\omega^2)\omega^{2i-3} + \omega^2(-\omega^2)\omega^{\cdots}]\,d''\omega$$

$$- tr(\omega^{2i-1} \underbrace{d'd''\omega}_{-d''(\omega^2)})$$

$$-d''(\omega^2) = d''\omega\cdot\omega - \omega\cdot d''\omega$$

$$= - tr(\omega^{2i}d''\omega) - tr(\omega^{2i-1}d''\omega\cdot\omega) + tr(\omega^{2i-1}\omega\, d''\omega)$$

$$= tr(\omega^{2i}d''\omega) \qquad \text{OK}$$

I now should know that if M is the circle and if I integrate

$$\int_M \operatorname{tr}(\omega^{2i-1} d_M \omega)$$

then I get a closed invariant form on the loop group G. As a form on the Lie algebra level this should be the anti-symmetrization of

$$\int_M \operatorname{tr}(X_1 \cdots X_{2i-1}\, dX_{2i}).$$

Now the cochain interpretation of your theorem with Loday. This gives an equivalence between primitive alternating forms on $gl(A)$ which are $gl(k)$-invariant and Connes cochains.

My first guess for the Connes cochain was

$$f(a_1, \dots, a_{2i}) = \int_M \operatorname{tr}(a_1 \cdots a_{2i-1}\, da_{2i}) \qquad \text{tr is unnecessary.}$$

but this is not cyclically skew-symmetric. So it has to be cyclically skew-symmetrized.

So the Connes cochain should be up to a constant factor

$$(*) \qquad \int_M \sum_{j=1}^{2i} (-1)^j (a_1 \cdots da_j \cdots a_{2i})$$

e.g. If $i=1$ we get

$$\int_M (a_1\, da_2 - a_2\, da_1).$$

Note that $(*)$ is not normalized for $i > 1$ since we get $\int a_1 da_2 \sim da_1 \cdot a_2$ by setting $a_3 = a_4 = \cdots = 1$.

On the other hand I know that the complex (reduced version of the Connes complex)

$$\xrightarrow{b} \bar{A}^3/\sim \xrightarrow{b} \bar{A}^2/\sim \xrightarrow[0]{b} \bar{A}$$

has for its homology k in every odd degree, when $A = k[z, z^{-1}]$. For example we have

$$\begin{array}{ccccccc} A^{\otimes 3} & \xrightarrow{b} & A^{\otimes 2} & \longrightarrow & \Omega^1 & \longrightarrow & 0 \\ & & (f,g) & \longmapsto & f\,dg & & \end{array}$$

and also

$$\begin{array}{ccccccc} \bar{A}^{\otimes 3}/\sim & \xrightarrow{b} & \bar{A}^{\otimes 2}/\sim & \longrightarrow & \Omega^1/d\Omega^0 & \longrightarrow & 0 \\ & & & & \| & & \\ & & (f,g) & \longmapsto & \int f\,dg \in k & & \end{array}$$

Therefore it should be possible to find a normalized Connes cocycle in the same class as $(*)$. This suggests perhaps that the formula $(*)$ is not ultra-canonical.

I learn from the above that it is probably not a good idea to think in terms of Connes cochains, in the same way that it is not a good idea to think in terms of Lie cochains. ▮

New idea: Let

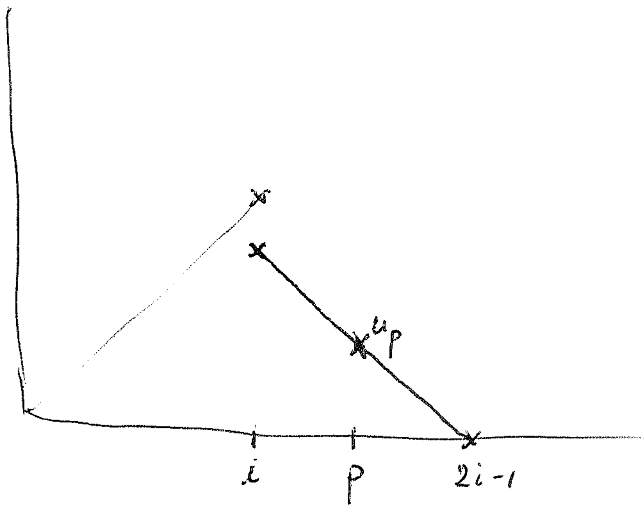
$$u = \int_0^1 tr\left[\left(t d''\omega + (t^2 - t)\omega^2\right)^{i-1}\omega\right]$$

This is a form on $\mathcal{G} \times M$ and I can think of such a thing as a form on $\mathcal{G}$ with values in forms on M.

Thus

$$u \in C^{\circ}(\mathfrak{gl}, \Omega^{\circ}(M))^{2i-1}$$

$$u = \sum_{p \geq i}^{2i-1} u_p \in C^p(\mathfrak{gl}, \Omega^{2i-1-p}(M))$$



But now we can interpret u_p as a homomorphism $\Lambda^p \mathfrak{gl} \longrightarrow \Omega^{2i-1-p}(M)$ and the fact du is of filtration i should mean that we get a map of complexes

$$\Lambda^* \mathfrak{gl} \longrightarrow \begin{cases} \Omega^{2i-1-*}(M) & * \geq i \\ 0 & * < i \end{cases}$$

$$\begin{tikzcd}
\Lambda^{2i} \arrow[r] \arrow[d] & \Lambda^{2i-1} \arrow[r] \arrow[d] & \Lambda^{2i-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Lambda^{i} \arrow[r] \arrow[d] & \Lambda^{i-1} \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & \Omega^0 \arrow[r] & \Omega^1 \arrow[r] & \cdots \arrow[r, "d"] & \Omega^{i-1} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which in Loday's notation is

$$\Lambda^* \mathfrak{g} \longrightarrow \Omega_M^{* \leq i-1}[-2i+1].$$

So we have seen that it is possible to go directly from the Lie algebra complex to the connected Deligne complex. Comments:

1) In the case where M is a point you still need to use invariant theory to see it is a quasi-isomorphism.

2) Can you say anything interesting when M isn't compact? Here $A =$ smooth functions of compact support or rapidly decreasing. The point is that I don't see how to compute the Hochschild cohomology.

3) Let's use the Connes-Karoubi idea of non-commutative differential forms. Here the ring A maps to the degree 0 part of a differential graded algebra (Ω°, d). Then

$$\omega : \tilde{\mathfrak{g}} \longrightarrow M_n(A)$$

is the identity and

$$d\omega : \tilde{\mathfrak{g}} \longrightarrow M_n(A) \longrightarrow M_n(\Omega^1).$$

So things like $d\omega \cdot \omega$ are 2-forms on $\tilde{\mathfrak{g}}$ with values in $M_n(\Omega^\circ)$. Here $d = d''$ previously. Put $\delta\omega = -\omega^2$. Then we compute

$$d\omega \cdot \omega \xmapsto{\ \delta\ } -\omega \cdot d\omega \cdot \omega$$
$$\omega \cdot d\omega \xmapsto{\ \delta\ } -\omega \cdot d\omega \cdot \omega$$

$$\omega^3 \xmapsto{\ d\ } d\omega \cdot \omega^2 - \omega\, d\omega\, \omega + \omega^2 d\omega$$

so if we have a trace

$$tr : \Omega^1 \longrightarrow \Gamma^1$$

factoring through $\Omega^1/[A, \Omega^1]$, then we will have our old formulas

$$\delta\, \mathrm{tr}(d\omega \cdot \omega) - d\, \tfrac{1}{3}\mathrm{tr}(\omega^3) = 0$$

similarly, $\delta \cdot \frac{1}{3}\mathrm{tr}(\omega^3) = 0.$

Thus if

$$\mathrm{tr} : \Omega^\bullet \longrightarrow \Gamma^\bullet$$

denotes dividing out by commutators, then we will have characteristic classes

$$\Lambda \tilde{\mathfrak{g}} \longrightarrow \bigoplus \Gamma^{\bullet \le i-1}[-2i+1].$$

In particular, we have:

$$\begin{tikzcd} C_2(A) \arrow[r] \arrow[d, "b"] & 0 \arrow[d] \\ C_1(A) \arrow[r] & \Gamma^0 \end{tikzcd} \qquad \begin{tikzcd} C_4(A) \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_3(A) \arrow[r] \arrow[d] & \Gamma^0 \arrow[d] \\ C_2(A) \arrow[r] & \Gamma^1 \end{tikzcd} \qquad \begin{tikzcd} C_6(A) \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_5(A) \arrow[r] \arrow[d] & \Gamma^0 \arrow[d] \\ C_4(A) \arrow[r] \arrow[d] & \Gamma^1 \arrow[d] \\ C_3(A) \arrow[r] & \Gamma^2 \end{tikzcd}$$

Call $(\Gamma^\bullet, d)$ the Karoubi complex. The difficulty is that one doesn't see immediately the relation of the Karoubi complex to the complexes and bicomplexes of the Connes theory. According to Connes there is a close relation.

4) If one uses the vanishing of c_i on $\mathfrak{gl}_n$ for $n < i$, one should be able to see that the image of $H_p(\mathfrak{gl}_n)$ in $PH_p(\mathfrak{gl}_\infty) = \Omega^{p-1}/d\Omega^{p-2} \oplus H^{p-3} \oplus \cdots$ is contained in what Loday wants it to be. I don't think one can see it is equal to this without getting a hold on $H_*(\mathfrak{gl}_n)$.

July 14, 1983

Summary: 1) If I use the transgression fmla:

$$\int_0^1 dt \ \operatorname{tr}\left(e^{t d''\omega + (t^2-t)\omega^2}\right)$$

then I get characteristic classes for $gl(A)$ with coefficients the Deligne complex:

$$\begin{tikzcd}
\Omega^2 & & \\
\Omega^1 & \Omega^1 \arrow[ul, "d"'] & \\
\Omega^0 & \Omega^0 \arrow[ul, "d"'] & \Omega^0 \arrow[ul, "d"']
\end{tikzcd}$$

If $C_\cdot(A)$ is the Connes complex we have maps of complexes

$$C_\cdot(A) \xrightarrow{\widetilde{ch}_i} \bigoplus_i \Omega/F_i\Omega\,[-2i+1]$$

The problem will be to show this map is a quis when A is smooth.

2) There should be a theory of char. classes with values in the double complex $\mathcal{C}(A)$ or $\mathcal{B}(A)$, obtained by inverting the quasi-isos.

$$C_\cdot(A) \longleftarrow \mathcal{C}(A) \longleftarrow \mathcal{B}(A).$$

However it can't be a curvature theory, because the map will not be a **sum** of ch_i for different i.

I have the following idea for constructing a map $C_\cdot(A) \longrightarrow \mathcal{C}(A)$. One uses the standard splittings of the rows of $\mathcal{C}(A)$ and then constructs a map by the usual diagram chase:

$$\begin{tikzcd} C_p & A^p \arrow[l] \arrow[d] & \\ & A^{p-1} & A^{p-1} \arrow[l] \arrow[d] \\ & & {} \end{tikzcd}$$

etc.

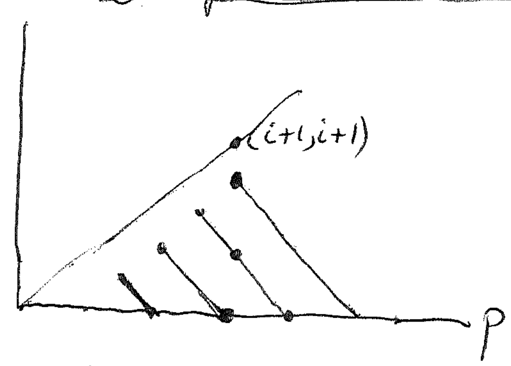
It is not immediately clear that we get a map of complexes, although this ought to be true if the signs are done correctly.

July 15, 1983

I want to prepare a letter to Loday in which I explain the natural transformation from the Connes complex $C(A)$ to the filtered de Rham complex based on the formula

$$u = \int_0^1 tr\left(\omega e^{t\,d\omega + (t^2-t)\omega^2}\right) dt$$

Let u_{pq} denote the (p,q) component: p is the degree in Y and q the degree in M, hence p is the number of occurrences of ω and q is the number of occurrences of $d\omega$. The total degree is $p+q$.



$p+q$ is odd, so one has for a given p the possibilities $q = p-1, p-3, p-5$, etc.

July 17, 1983

Summary: The essential problem is to get an understanding of Connes periodicity operator on the cochain level if possible. It seems to be completely invisible from the Lie algebra viewpoint.

I hope that the different components u_{pq} of the form u with the same q are related by the S-operator on the cochain level. For example, if we take $q=0$ we have the forms

$$u_{2i+1,0} = (-1)^i \frac{i!}{(2i+1)!} \operatorname{tr}(\omega^{2i+1})$$

Idea: We should be able to construct a section of the quis $B(A) \subset C(A) \longrightarrow C(A)$ as follows: Start with

$$C_p(A) \xrightarrow{\eta} A^p$$

represented by the form on $gl(A)$:

$$X_1, \dots, X_p \longmapsto \frac{1}{p} \operatorname{tr}\left(\sum (-1)^\sigma X_{\sigma^{-1}1} \otimes \cdots \otimes X_{\sigma^{-1}p}\right).$$

This is what might be denoted $\frac{1}{p} \operatorname{tr}(\underbrace{\omega \otimes \cdots \otimes \omega}_{p \text{ times}})$

Then if we use $\mu : A^p \rightarrow \Omega^{p-1}$

$$\mu(a_1, \dots, a_p) = \frac{1}{(p-1)!} a_1 da_2 \cdots da_p$$

we get the map

$$C_p(A) \longrightarrow \Omega^{p-1}$$

corresponding to the form $\frac{1}{(p-1)!} \operatorname{tr}(\omega (d\omega)^{p-1})$. Now

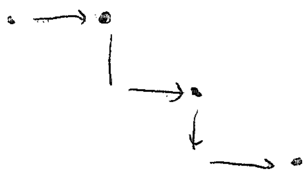
$$u_{p,p-1} = \int_0^1 dt \ \operatorname{tr}\left(\omega \frac{(t d\omega)^{p-1}}{(p-1)!}\right) = \frac{1}{p!} \operatorname{tr}(\omega (d\omega)^{p-1})$$

which is why we divide by $\frac{1}{p}$ in the formula for η.

So now if we combine $\eta : C_p \to A^p$ with the periodicity operator $S : C_p \to C_{p-2}$ we get a map

$$C_p \longrightarrow \mathcal{B}(A)_p = A^p \oplus A^{p-2} \oplus A^{p-4} \oplus \cdots$$

This should be a map of complexes because it ought to correspond to the map of complexes obtained by using canonical liftings:



to $\mathcal{C}(A)$ and then a similar retraction into $\mathcal{B}(A)$.

Lemma: $0 \to I \xrightarrow{i} A \xrightarrow{p} C \longrightarrow 0$ exact sequence of complexes, $u : C \longrightarrow A$ a section of p not-necessarily compatible with d. Let $v : C_n \longrightarrow I_{n-1}$ be the unique map with $iv = [d, u]$. Then v is a map of complexes $\Omega I \xrightarrow{v} I$ and A is canonically isomorphic to the mapping cone on v.

Questions. Can $\eta : C_p \to A^p$ be defined geometrically, i.e. for a flat connection on a trivial bundle over $Y \times M$?

Can one define characteristic classes with values in cyclic homology directly by working over $Y \times \Lambda$, where Λ is Connes' cyclic category?

July 18, 1983

Central problem: Understand Connes S-operator. He approaches it from the index calculation using a parametrix as follows.

Let p denote a fixed invertible operator say from $\mathcal{H}^+$ to $\mathcal{H}^-$ and let e be an idempotent operator of degree zero on $\mathcal{H}^+ \oplus \mathcal{H}^-$. One wants the index of $P = epe : e\mathcal{H}^+ \longrightarrow e\mathcal{H}^-$. Use the parametrix $Q = ep^{-1}e : e\mathcal{H}^- \longrightarrow e\mathcal{H}^+$. We are assuming that Q is a quasi-inverse to P, hence that P is Fredholm. Then if K is defined by

$$\begin{tikzcd} e\mathcal{H}^+ \arrow[r, "P"] \arrow[d, "I-K^+"'] & e\mathcal{H}^- \arrow[d, "I-K^-"] \arrow[dl, dotted, "Q"'] \\ e\mathcal{H}^+ \arrow[r, "P"] & e\mathcal{H}^- \end{tikzcd}$$

we have
$$I \sim K \sim K^2 \sim K^3 \sim \cdots$$

Hence if we know K^n is of trace class for large n we have

$$\mathrm{Index}(P) = tr(\varepsilon K^n)$$

$$K = \begin{pmatrix} K^+ & \\ & K^- \end{pmatrix} = \begin{pmatrix} I - QP & 0 \\ 0 & I - PQ \end{pmatrix}$$

$$= I_{\text{on } e\mathcal{H}} - \begin{pmatrix} 0 & ep^{-1}e \\ epe & 0 \end{pmatrix}\begin{pmatrix} 0 & ep^{-1}e \\ epe & 0 \end{pmatrix}$$

$$= e - eFeFe$$

where $F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$ satisfies $F^2 = I$.

Now Connes' idea is to rewrite K as follows: Think of $dA = [F, A]$ as a non-commutative differ-ential:

$$d^2A = [F[F,A]] = F(\cancel{F}A - AF) + (\cancel{F}A - AF)F$$
$$= [F^2, A] = [I, A] = 0.$$

Here we use the grading of $\mathcal{H}$ so that if A has degree 0, then $dA = [F,A] = FA - AF$, whereas if A has odd degree, then $dA = [F,A] = FA + AF$.

Then

$$K = e - eFeFe$$
$$= eF^2e^2 - eFeFe$$
$$= eF[F,e]e$$
$$= eF(1-e)[F,e] \quad \longleftarrow \text{uses} \quad \begin{aligned}[F,e] &= [F,e^2] \\ &= [F,e]e + e[F,e] \\ \therefore (1-e)[F,e] &= [F,e]e\end{aligned}$$
$$= (eF - eFe)[F,e]$$
$$= e[e,F][F,e]$$
$$= -e(de)^2 \qquad de = [F,e].$$

Thus $-K$ is just the curvature of the Grassmannian connection on the "image of e"

It is perhaps better to avoid the non-commutative mumbo-jumbo and to simply record the formula

$$\text{Index}(epe) = tr(\varepsilon K^n)$$

where $K = -e[F,e][F,e]$ $\qquad F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$

This is applied in two cases:

Even case: Here p is fixed and the projector e varies. The model is the map

$$K_0(M) \longrightarrow \mathbb{Z}$$

defined by an elliptic operator p on M. In this case the interpretation of $-K = e[F,e][F,e]$ as the curvature of the bundle corresponding to e is basic to Connes approach.

Odd case: Here the model is Toeplitz operators. The projector e is fixed and the invertible operator p varies. Usually $\mathcal{H}^+ = \mathcal{H}^-$ and $e^+ = e^-$, and one thinks of p as an automorphism of $\mathcal{H}^+$.

$$\text{Index}(epe) = \operatorname{tr}(\varepsilon K^m) \qquad K = -e[F,e][F,e]$$

$$= (-1)^m \operatorname{tr}(\varepsilon\, e\, [F,e]^{2m}) \qquad F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$$

$$= (-1)^m \Big\{ \operatorname{tr}\big(e\,[p^{-1},e][p,e] \cdots [p,e]\big) \qquad 2m \text{ brackets}$$

$$- \operatorname{tr}\big(e\,[p,e][p^{-1},e] \cdots [p^{-1},e]\big) \Big\}$$

$$= (-1)^m(-1)^m \Big\{ \operatorname{tr}\big(e\, p^{-1}[p,e]\, p^{-1}[p,e] \cdots p^{-1}[p,e]\big) \qquad 2m \text{ brackets}$$

$$- \operatorname{tr}\big(p\,e\,p^{-1}[p,e]\, p^{-1}[p,e] \cdots [p,e]\,p^{-1}\big) \Big\}$$

$$= \operatorname{tr}\big(p^{-1}[p,e]\big)^{2m+1}$$

$$\boxed{\text{Index}(epe) = \operatorname{tr}\big((p^{-1}[p,e])^{2m+1}\big) \quad \text{any } m \text{ such that trace makes sense}}$$

Recall the Lie algebra interpretation.

Even case: $\mathcal{G}$ is acting as unitary transformations of $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ preserving the grading, and we consider an orbit on the set of odd degree

$$F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix} \quad \text{with } F^2 = 1.$$

The natural character forms on the set of such F are the forms

$$(-1)^m \frac{m!}{(2m+1)!} \operatorname{tr} (p^{-1} dp)^{2m+1} .$$

Now $\operatorname{tr}(p^{-1}dp)^{2m+1}$ corresponds to the cyclic cocycle

$$\begin{aligned} \operatorname{tr}\left(p^{-1}[a_0, p] \cdots p^{-1}[a_{2m}, p]\right) &= (-1) \operatorname{tr}\left(p^{-1}[p, a_0] \cdots p^{-1}[p, a_{2m}]\right) \\ &= (-1) \operatorname{tr}\left(\varepsilon a_0 [F, a_1] \cdots [F, a_{2m}]\right) \end{aligned}$$

Odd case: Here $\{F^2 = I\}$ is the Grassmannian and has the character forms

$$\frac{1}{2^{2p+1}} \frac{1}{p!} \operatorname{tr}\left(F (dF)^{2p}\right)$$

where the 2-factors come from the fact that $F = 2e - 1$. This corresponds to the cyclic cocycle

$$\frac{1}{2^{2p+1} p!} \operatorname{tr}\left(F [F, a_1] \cdots [F, a_{2p}]\right)$$

The formula
$$\mathrm{Ind}(epe) = tr\left((p^{-1}[p,e])^{2m+1} \right)$$
is valid for $m=0$, assuming $[p,e]$ is of trace class, but the proof on p. 956 has to be changed

$$\mathrm{Ind}(epe) = tr(\varepsilon K) \qquad \begin{aligned} K &= e - eFeFe \\ &= eF[F,e]e \end{aligned}$$
$$= tr(ep^{-1}[p,e]) - tr(ep[p^{-1},e])$$
$$= tr(ep^{-1}[p,e]) + tr(\underbrace{e[p,e]}_{[p,e](1-e)}p^{-1})$$
$$= tr(e\,p^{-1}[p,e]) + tr((1-e)p^{-1}[p,e])$$
$$= tr(p^{-1}[p,e]).$$

This is nicer because there is only one occurrence of e instead of 3, 5, 7, etc.

But this raises the question of Connes' using $tr(\varepsilon F[F,a_0]\cdots[F,a_{2m}])$ instead of what I was led to, namely $tr(\varepsilon\, a_0[F,a_1]\cdots[F,a_{2m}])$. Compute formally

$$tr(\varepsilon F[F,a_0]\cdots[F,a_{2m}]) = tr(p^{-1}[p,a_0][p^{-1},a_1]\cdots[p^{-1},a_{2m}]) - tr(p[p^{-1},a_0][p,a_1]\cdots[p^{-1},a_{2m}])$$
$$= (-1)^m\, 2\, tr(p^{-1}[p,a_0]\cdots p^{-1}[p,a_{2m}])$$

$$tr(\varepsilon a_0[F,a_1]\cdots[F,a_{2m}]) = tr(a_0[p^{-1},a_1]\cdots[p,a_{2m}]) - tr(a_0[p,a_1]\cdots[p^{-1},a_{2m}])$$
$$= (-1)^m\, tr(p^{-1}[p,a_0]p^{-1}[p,a_1]\cdots p^{-1}[p,a_{2m}])$$

There are the same except for the factor of two. However, Connes' expression is better because it makes sense for smaller m. It has $(2m+1)$ commutators instead of $(2m)$-commutators. For example, $m=0$:

$$\operatorname{tr}\left(\varepsilon F[F,a_0]\right) \qquad \text{versus} \qquad \operatorname{tr}(\varepsilon a_0)$$

Summary:

$$\operatorname{Ind}(epe) = \frac{(-1)^m}{2} \operatorname{tr}\left(\varepsilon F[F,e]^{2m+1}\right)$$

$$= \operatorname{tr}\left(\left(p^{-1}[p,e]\right)^{2m+1}\right)$$

The odd-degree character forms on the F with $\varepsilon F = -F\varepsilon$ and $F^2 = I$ are

$$(-1)^m \frac{m!}{(2m+1)!} \operatorname{tr}\left(p^{-1}dp\right)^{2m+1} = \frac{m!}{(2m+1)!}\,\frac{1}{2} \operatorname{tr}\left(\varepsilon F (dF)^{2m+1}\right)$$

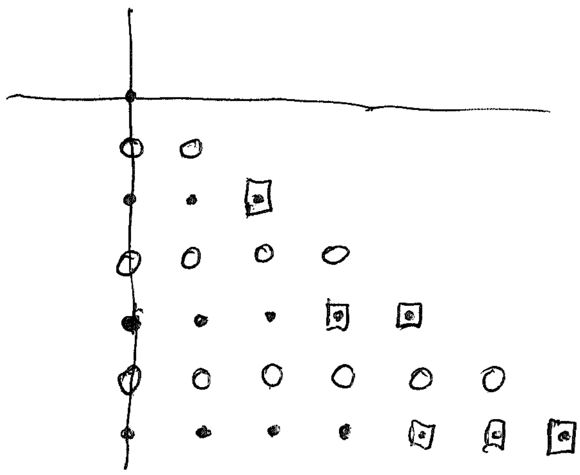
The character forms on the ungraded F with $F^2 = I$ are

$$\frac{1}{2^{2m+1}}\,\frac{1}{m!} \operatorname{tr}\left(F(dF)^{2m}\right)$$

Moral: I was hoping to get a feeling for Connes' S-operator from the fact that $\operatorname{tr}\left(p^{-1}[p,e]\right)^{2m+1}$ is independent of m. But obviously this doesn't work.

July 19, 1983 (Erica is 5)

Let's discuss Loday's conjecture on the filtration of cyclic homology obtained from the gl_n for different n. This conjecture is analogous to one by Soulé for K-groups of a field. As Eric explained it to me one considers the Atiyah Hirzebruch spectral sequence for etale K-theory and looks for the algebraic K-groups inside. Draw the possible groups:



Then the conjecture says that the boxed groups don't occur, i.e. the K-theory sits below the line thru the origin of slope -2. Now ignoring torsion the spectral sequence is degenerate and the pieces in the E^2-term are seen by Chern classes maps

$$K_p \xrightarrow{c_{i,p}} H^{2i-p}(\mu^{\otimes i})$$

So these maps are to be non-zero [redacted] only for $\frac{p}{2} \leq i \leq p$.

It would be clearer to list the possible weights:

K_0	weight	0
K_1		1
K_2		$1, 2$
K_3		$2, 3$
K_4		$2, 3, 4$

and then the [redacted] weights of K_p are i such that $\frac{p}{2} \leq i \leq p$.

Thus c_i on K_p is zero for $i > p$.

But this last statement is clear from stability:

$$\begin{tikzcd} & H_p(GL_p) \arrow[d] \arrow[dr, "0 \text{ for } i>p"] & \\ K_p = \pi_p(BGL^+) \arrow[r] & H_p(GL) \arrow[r, "c_i"] & H^{2i-p}(\mu^{\otimes i}) \end{tikzcd}$$

Now I want to carry out the same argument in the Lie algebra context. It will be necessary probably to work with the Chern character, and the point to check will be that

$$ch_n = \frac{(-1)^{n-1}}{(n-1)!} c_n \qquad \text{on Primitive homology.}$$

(Rapid review of this constant: s_n = power sums, c_i = elem. symm fns.

$$1 + c_1 + c_2 + \cdots = e^{\sum \frac{(-1)^{n-1}}{n} s_n}$$

because $1 + x = e^{\log(1+x)} = e^{\sum \frac{(-1)^{n-1}}{n} x^n}$. Thus mod decomposables:

$$c_n = \frac{(-1)^{n-1}}{n} \underbrace{s_n}_{n!\, ch_n} = (-1)^{n-1}(n-1)!\, ch_n \quad \text{as } ch_n = tr\left(\frac{K^n}{n!}\right).\ )$$

Actually I know for the Lie K-theory that

$$K_p^+ = \Omega^{p-1}/d\Omega^{p-2} \oplus H^{p-3} \oplus H^{p-5} \oplus \cdots$$

where the components are given by ch_p, ch_{p-1}, ch_{p-2}, etc. modulo verifying the Connes isomorphism agrees with these character classes. So I don't yet have the correct version of Loday's conjecture which has to do with the relation of $H(\mathfrak{gl}_n A)$ with the primitive part.

$\tilde{\mathfrak{g}} = \mathfrak{gl}_n(A)$. The goal will be to define ~~characteristic~~ cohomology classes for $\tilde{\mathfrak{g}}$ with values in the filtered De Rham cohomology of A, by the curvature process of Chern-Weil.

Let's work in the ring $C^p(\tilde{\mathfrak{g}}, \Omega^q)$ of forms ~~on~~ $\tilde{\mathfrak{g}}$ with values in forms on A. This is bigraded with $d = d' + d''$.

We have a canonical element

$$\omega \in M_n(C^{\cdot}(\tilde{\mathfrak{g}}, \Omega^{\cdot})) \qquad \text{of type } (1,0)$$

given by the identity map

$$\begin{aligned} M_n(C^1(\tilde{\mathfrak{g}}, \Omega^0_A)) &= M_n \operatorname{Hom}(\tilde{\mathfrak{g}}, A) \\ &= \operatorname{Hom}(\tilde{\mathfrak{g}}, M_n(A)) \end{aligned}$$

and the identification $\mathfrak{gl}_n(A) = M_n(A)$.

By definition of the differential in Lie alg. cohomology we have

$$d'\omega + \omega^2 = 0.$$

We consider the 1-parameter family of "connection" forms $\theta = t\omega$, where t is an indeterminate. The curvature is

$$\begin{aligned} \Omega = d\theta + \theta^2 &= t\,d\omega + t^2\omega^2 \\ &= t\,d''\omega + (t^2 - t)\,\omega^2 \end{aligned}$$

Ω lousy notation as it conflicts with $\Omega^{\cdot}$.

Let $\varphi(X_1, \ldots, X_m)$ be an invariant polynomial on $\mathfrak{gl}_n(k)$. Then we have (see Bott-Chern)

$$\frac{d}{dt}\varphi(\Omega, \ldots, \Omega) = d\sum_{i=1}^{m} \varphi(\Omega, \ldots, \Omega, \underset{\substack{\uparrow \\ i\text{th position}}}{\omega}, \Omega, \ldots, \Omega)$$

(My proof is in terms of $D = d+\theta$:

$$\frac{d}{dt}\varphi(D^2,\dots,D^2) = \sum_1^m \varphi(D^2,\dots,\underbrace{\overset{i}{D\omega+\omega D}}_{[D,\omega]},\dots)$$

$$= d\sum\varphi(D^2,\dots,\omega,\dots,D^2).$$

Here I argue in analogy with $d\,\mathrm{tr}\,A = \mathrm{tr}[D,A]$.
Bott-Chern goes as follows:

$$\frac{d}{dt}\varphi(\Omega,\dots,\Omega) = \sum_1^m \varphi(\Omega,\dots,\overset{i}{\dot\Omega},\dots,\Omega)$$

$$d\sum_i\varphi(\Omega,\dots,\underset{i}{\omega},\dots,\Omega) = \sum_{j\neq i}\varphi(\overset{j}{d\Omega}\quad\overset{i}{\omega}\quad) + \sum_i\varphi(\quad\overset{i}{d\omega}\quad)$$

$$d\Omega = d(d\theta+\theta\theta) = d\theta\cdot\theta-\theta\cdot d\theta = [\Omega,\theta] \quad\Big|\Rightarrow d\Omega+[\theta,\Omega]=0$$

Invariance gives

$$0 = \sum_{j\neq i}\varphi(\cdots\overset{j}{[\theta,\Omega]}\cdots\overset{i}{\omega}\cdots) + \sum_i\varphi(\cdots\overset{\theta,\omega}{[\quad]}\cdots)$$

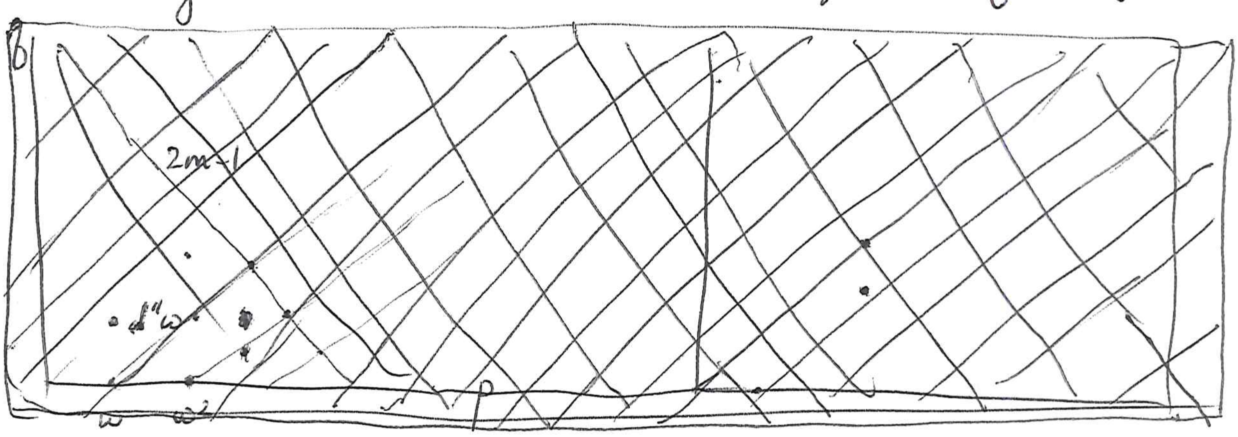
Finally $\dot\Omega = d\omega + 2t\omega^2 = d\omega + \overset{\theta,\omega}{[\quad]}$.)

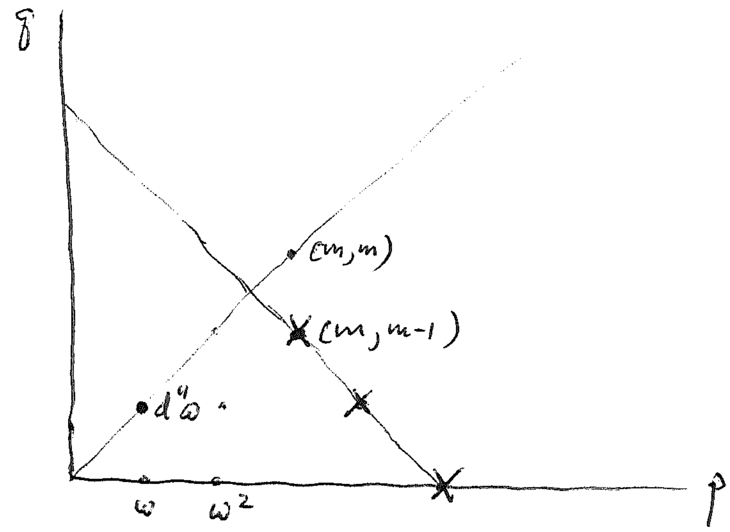
Integrating the Bott-Chern formula gives

$$\varphi(d''\omega,\dots,d''\omega) = d\int_0^1 \sum_{i=1}^m \varphi(\Omega,\dots,\overset{i}{\omega},\dots,\Omega)\,dt$$

$$\Omega = t\,d''\omega + (t^2-t)\omega^2$$

Call the integral u. It is a form of degree $2m-1$.





We have $$u = \sum_{\substack{p>q \\ p+q=2m-1}} u_{pq}$$

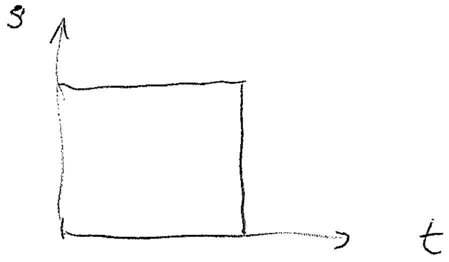
At this point we argue that because du is of type (m,m), the image of u under the map
$$C^{\cdot}(\tilde{g}, \Omega^{\cdot}) \longrightarrow C^{\cdot}(\tilde{g}, \Omega^{\cdot}/F_m\Omega^{\cdot})$$
is closed. Hence we obtain a canonical [redacted] class in
$$H^{2m-1}(\tilde{g}, \Omega^{\cdot}/F_m\Omega^{\cdot}) = H^{2m}(\tilde{g}, \Omega^{<m}[1]).$$

One problem is to show that this construction is multiplicative, i.e. that [redacted] with respect to Deligne's product the u-class belonging to a product $\varphi'\varphi''$ of invariant polys is the product of the corresponding u' and u''.

Let's see if we can see this is true for general reasons.

$$\begin{tikzcd}
 & C(\tilde{g}) \arrow[dl, dotted, "u_\varphi"'] \arrow[d, "\varphi(d''\omega)"] & \\
\Omega^{<m}[1] \arrow[r] & F_m\Omega \arrow[r] & \Omega
\end{tikzcd}$$

Now u_φ is the map which results from the path $\varphi(\Omega_t)$, $\Omega_t = (d+t\omega)^2$, and the fact we have a specific representation for $\frac{d}{dt}\varphi(\Omega_t)$ as a coboundary. Concentrate on the path. When I consider $\varphi(\Omega_t)\bar\varphi(\Omega_t)$, I can consider more generally $\varphi(\Omega_t)\bar\varphi(\Omega_s)$ on the square $[0,1]^2$



and deform the diagonal path to $\longrightarrow\!\uparrow$. This gives

$$\varphi(d''\omega)\bar\varphi(d''\omega) = d\int_0^1 \varphi'(\Omega_t;\omega)\,\bar\varphi(0)\,dt + d\int_0^1 \varphi(d''\omega)\bar\varphi'(\Omega_s;\omega)\,ds$$

$$= d\,u_\varphi \cdot \bar\varphi(0) \mp \varphi(d''\omega)\cdot du_{\bar\varphi},$$

I do assume the degree $\bar\varphi > 0$, so that $\bar\varphi(0)=0$. This the path $\longrightarrow\!\uparrow$ leads to

1) $$\varphi(d''\omega)\bar\varphi(d''\omega) = d\{\varphi(d''\omega)\,u_{\bar\varphi}\}$$

and similarly the path $\uparrow\!\longrightarrow$ leads to

2) $$\varphi(d''\omega)\bar\varphi(d''\omega) = d\{u_\varphi\,\bar\varphi(d''\omega)\},$$

whereas the diagonal path leads to

3) $$\varphi(d''\omega)\bar\varphi(d''\omega) = d\underbrace{\int_0^1 \{\varphi'(\Omega_t;\omega)\bar\varphi(\Omega_t) + \varphi(\Omega_t)\bar\varphi'(\Omega_t;\omega)\}\,dt}_{u_{\varphi\bar\varphi}}$$

It seems likely that either 1) or 2) can be related to the multiplication ~~of~~ the Deligne complexes. Recall how this is defined: One has ~~a~~ homotopy equivalence of complexes

$$\Omega^{<m}[1] \simeq \text{Fibre}\{F_m\Omega \hookrightarrow \Omega\}.$$

Hence a map of a complex K to $\Omega^{<m}[1]$ is represented by a pair consisting of a map $K \xrightarrow{\alpha} F_m\Omega$ and a null-homotopy $\alpha = dh$ with $h : K \to \Omega$ of degree $(+1)$. Such (α, h) are subject to an equivalence relation which I won't write down yet.

So now given also

$$(\beta, k) : L \longrightarrow \text{Fibre}\{F_n\Omega \longrightarrow \Omega\}$$

then the product is represented by

$$\alpha\cdot\beta : K \otimes L \longrightarrow F_{m+n}\Omega$$

and the null-homotopy

$$\alpha\cdot\beta = d(h\beta) \quad \text{or} \quad d(\alpha k).$$

So I am claiming that $(\alpha\beta, h\beta)$ and $(\alpha\beta, \alpha k)$ are equivalent.

In general given $f : X \to Y$ the fibre F of f is the complex

$$F_n = X_n + Y_{n+1}$$

with $\quad d(x, y) = (dx, +fx - dy)$. (Check: $d(dx, fx - dy) = (d^2x, fdx - d(fx - dy)) = 0$.)

For $(\alpha\beta, h\beta)$ and $(\alpha\beta, \alpha k)$ to be equivalent means that their difference $(0, h\beta - \alpha k)$ is d of something. But

$$d(0, hk) = (0, -d(hk)) = (0, -\alpha k + h\beta)$$

Finally the only thing left to do is to show the forms $\varphi(d''\omega)u_{\bar\varphi}$ and $u_{\varphi\bar\varphi}$ differ by a boundary in $\Omega^{<m+n}$. The obvious candidate is

$$d \iint_{\triangle} \varphi'(\Omega_t;\omega)\,\bar{\varphi}'(\Omega_s;\omega)\,dt\,ds$$

$$= \iint \Big[\underbrace{d\varphi'(\Omega_t;\omega)}_{\frac{\partial}{\partial t}\varphi(\Omega_t)}\,\bar{\varphi}'(\Omega_s;\omega) - \varphi'(\Omega_t;\omega)\,\underbrace{d\bar{\varphi}'(\Omega_s;\omega)}_{\frac{\partial}{\partial s}\bar{\varphi}(\Omega_s)}\Big]\,dt\,ds$$

Now use Green's theorem

$$\iint \left(\frac{\partial M}{\partial t} - \frac{\partial N}{\partial s}\right) dt\,ds = \iint d\,(M\,ds + N\,dt)$$

$$= \oint M\,ds + N\,dt$$

and we see the above is equal to

$$\int_{\triangle} \varphi(\Omega_t)\,\bar{\varphi}'(\Omega_s;\omega)\,ds + \varphi'(\Omega_t;\omega)\,\bar{\varphi}(\Omega_s)\,dt$$

which is exactly what we want.

At this point I have multiplicativity of the characteristic classes.

July 21, 1983:

Let's consider Loday's conjecture on the filtration. This time let n denote the homology degree. Consider the invariant theory injection

$$(gl_i^{\otimes n})_{gl_i} \hookrightarrow k(\Sigma_n) \quad = \text{fns. on } \Sigma_n$$

and let the image be $F_i k(\Sigma_n)$:

$$0 \subset F_1 \subset \cdots\cdots \subset F_n = k(\Sigma_n)$$

Next consider the induced filtration on the primitive part:

$$\begin{tikzcd} F_i k(U_n) \arrow[r, hook] \arrow[d, hook] & F_i k(\Sigma_n) \arrow[d, hook] \\ k(U_n) \arrow[r, hook] & k(\Sigma_n). \end{tikzcd}$$

Now tensor with $A^{\otimes n} \otimes (\mathrm{sgn})$ and divide out by Σ_n, and use exactness of taking invariants; we get an intersection

$$\begin{tikzcd} F_i C_n(A) \overset{\text{def.}}{=} \left(F_i k(U_n) \otimes A^n\right) \otimes_{\Sigma_n} (\mathrm{sgn}) \arrow[r, hook] \arrow[d, hook] & \left(\Lambda^n gl_i(A)\right)_{gl_i} \arrow[d, hook] \\ C_n(A) \arrow[r, hook] & \left(\Lambda^n gl(A)\right)_{gl}. \end{tikzcd}$$

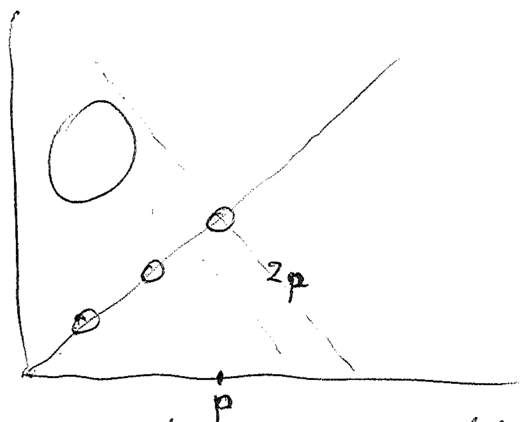
So it is clear that one is taking the induced filtration of the primitive space by the filtration of $gl(A)$ by the $gl_i(A)$. This gives a square

$$\begin{tikzcd} H_n(F_i C_\cdot(A)) \arrow[r] \arrow[d] & H_n(gl_i(A)) \arrow[d] \\ K_n^+(A) \arrow[r] & H_n(gl(A)) \end{tikzcd}$$

Loday considers the spectral sequence obtained by filtering $C_\circ(A)$ by $F_i C_\circ(A)$:

$$E^1_{pq} = H_{p+q}\left(F_p C_\circ / F_{p-1} C_\circ\right) \Longrightarrow K^+_{p+q}$$

His conjecture implies that this spectral sequence lives in the range $p > q \geq 0$



In fact his conjecture is exactly this vanishing range for E^0_{pq}:

$$F_i C_n = 0 \qquad i \leq \frac{n}{2} \qquad n \text{ even}$$
$$= 0 \qquad i < \frac{n}{2} \qquad n \text{ odd}$$

Hence in both cases

$$\boxed{F_i C_n = 0 \quad \text{for} \quad i \leq \frac{n}{2}}$$

$$\Uparrow$$

Actual Loday conjecture $\boxed{F_i\, k(U_n) = k(U_n) \cap F_i\, k(\Sigma_n) = 0 \quad \text{for } i \leq \frac{n}{2}}$

Now let us see what we can prove using Chern classes, assuming the identification of the Chern character map with the known isomorphism:

$$K_n^+ \xrightarrow{\sim} \Omega^{n-1}/d\Omega^{n-2} + H^{n-3} + H^{n-5} + \cdots$$

The components are $ch_{i,n} : K_n^+ \longrightarrow H^{2i-1-n}$ where $i \leq n$. On primitives ch_i, c_i differ by a multiplicative factor.

$$\begin{tikzcd}
H_n(F_p C.) \arrow[r] \arrow[d] & H_n(\mathfrak{gl}_p(A)) \arrow[d] \arrow[dr, "0 \text{ if } i>p"] & \\
K_n^+ \arrow[r] & H_n(\mathfrak{gl}(A)) \arrow[r, "c_{i,n}"'] & H_{DR}^{2i-1-n}
\end{tikzcd}$$

Better diagram

$$\begin{tikzcd}
H_n(F_p C) \arrow[d] \\
K_n^+ \arrow[r, "\sim"] & \underset{ch_{n,n}}{\Omega^{n-1}/d\Omega^{n-2}} \oplus \underset{ch_{n-1,n}}{H^{n-3}} \oplus \underset{ch_{n-2,n}}{H^{n-5}} \oplus \cdots \underset{ch_{i,n}}{H^{2i-1-n}}
\end{tikzcd}$$

The last component is $i = \begin{cases} \frac{n+1}{2} & n \text{ odd} \\ \frac{n}{2}+1 & n \text{ odd} \end{cases}$

Hence for
$$\begin{aligned} p &< \tfrac{n+1}{2} \quad n \text{ odd} \\ &< \tfrac{n}{2}+1 \quad n \text{ even} \end{aligned} \quad \not\Longleftrightarrow \quad p \leq \frac{n}{2}$$

we conclude $H_n(F_p C) \longrightarrow K_n^+$ is zero.

<u>Conclusion:</u> By means of a good theory of Chern classes we can check the following consequence of L-conjecture, namely

$$\operatorname{Im}(H_n(F_p C) \to K_n^+) = 0 \qquad p \leq \frac{n}{2}.$$

On Lie algebra cohomology. Put

$$C^q(\mathfrak{g}, M) = \mathrm{Hom}(\Lambda^q \mathfrak{g}, M)$$

and define i_X, $\mathcal{L}_X$ on $C^\cdot(\mathfrak{g}, M)$ in the straight-forward way. Then d is defined inductively so that

$$d i_X + i_X d = \mathcal{L}_X$$

is satisfied. (This is Bott's approach.) Then one can define the standard formula for d; e.g.

$$i_Y i_X d\omega = i_Y(\mathcal{L}_X - d i_X)\omega$$

$$= (\mathcal{L}_X i_Y - i_{[X,Y]})\omega - i_Y d\,\omega(X)$$

$$= X\omega(Y) - \omega([X,Y]) - Y\omega(X)$$

or $$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

So if $\theta \in C^1(\mathfrak{g};\mathfrak{g})$ is the identity map, and $\mathfrak{g}$ is regarded as a $\mathfrak{g}$-module under the adjoint action, then

$$(d\theta)(X,Y) = XY - YX - [X,Y] = 0,$$

so the MC form is closed. On the other hand if $\mathfrak{g}$ given the trivial $\mathfrak{g}$-action we have

$$(d\theta)(X,Y) = -[X,Y].$$

Now $[\theta,\theta]$ = image of $\theta\otimes\theta \in C^1(\mathfrak{g};\mathfrak{g})^{\otimes 2}$ under cup-product $C^1(\mathfrak{g},\mathfrak{g}) \otimes C^1(\mathfrak{g},\mathfrak{g}) \longrightarrow C^2(\mathfrak{g}, \mathfrak{g}^{\otimes 2})$ followed by $[\,,\,]: \mathfrak{g}^{\otimes 2} \longrightarrow \mathfrak{g}$ satisfies:

$$i(Y).i(X)(\theta\circ\theta) = i(Y)\,(X\otimes\theta - \theta\otimes X)$$
$$= X\otimes Y - Y\otimes X \longmapsto 2[X,Y]$$

Thus we get

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0,$$

i.e. the Maurer-Cartan form is a flat connection form.

For Loday letter I need

$$\theta \in M_n(C^1(\tilde{\mathfrak{g}}, A)) \quad \text{or} \quad M_n(C^\cdot(\tilde{\mathfrak{g}}, \Omega^\cdot))$$

which satisfies $d'\theta + \theta^2 = 0$. Geometrically θ is a connection form on the trivial n-diml bundle over $G \times M$ which is flat in the G-direction. $D = d + \theta$ is a G-inv. connection on the trivial n-dimensional bundle.

$$\varphi(d''\theta) = d \underbrace{\int_0^1 \varphi'(K_t, \theta)\,dt}_{u_\varphi}$$

$$u_\varphi \in \bigoplus_{\substack{p+q=2m-1 \\ p>q}} C^p(\tilde{\mathfrak{g}}, \Omega^q) = C^{2m-1}(\tilde{\mathfrak{g}}, \Omega^{<m})$$

$u_\varphi \in C^{2m-1}(\tilde{\mathfrak{g}}, \Omega^\cdot)$ is not closed but it becomes a cycle in the quotient complex $C^{2m-1}(\tilde{\mathfrak{g}}, \Omega^{<m})$

u_φ gives a cocycle of degree $2m-1$ in $C^\cdot(\tilde{\mathfrak{g}}, \Omega^{<m})$.

Now I have to discuss the character.

$$u_m = \int_0^1 tr\left(\theta \frac{(t d''\theta + (t^2-t)\theta^2)^{m-1}}{(m-1)!}\right) dt$$

Involves terms $tr(\theta\, d''\theta \cdots \theta^2 \cdots)$. Such an expression is primitive.

$$X_1, \dots, X_n \longmapsto tr(X_1 d''X_2 X_3 X_4 \cdots) \quad \text{then}$$

anti-symmetrized. Corresponds to the cyclic cochain

$$a_1(da_2)\,a_3\,a_4\,\cdots$$

and so the point is that $u_m = \sum u_{pq}$ where

$$u_{pq} : C_p(A) \longrightarrow \Omega^q$$

$$\begin{tikzcd}
\Lambda^p \tilde{\mathfrak{g}} \arrow[r, "u_{pq}"] \arrow[d] & \Omega^q \\
C_p(A) \arrow[ur] &
\end{tikzcd}$$

Now assume that

$$H(\tilde{\mathfrak{g}}) \longrightarrow \quad ?$$

Take $\varphi_m = \operatorname{tr}\left(\frac{K^m}{m!}\right)$. The corresponding u_{φ_m} I will denote

$$u_m = \frac{1}{(m-1)!}\int_0^1 \operatorname{tr}\left(\theta\left(t\,d''\theta + (t^2-t)\theta^2\right)^{m-1}\right)dt$$

$$= \sum_{\substack{p+q=2m-1 \\ p>q}} u_{pq} \quad \text{where } u_{pq} = \text{const.} \sum_{\sigma\in\Sigma_m} \operatorname{tr}^\sigma(\theta, \underbrace{d''\theta,\dots,d''\theta}_{q}, \underbrace{\theta^2,\dots,\theta^2}_{r})$$

$$m = 2q+2r+1$$
$$p = q+2r+1$$

From the form of this u_{pq} one sees that it is defined on the Connes complex

$$\begin{tikzcd}
\Lambda^p \tilde{\mathfrak{g}} \arrow[r, "u_{pq}"] \arrow[d] & \Omega^q \\
C_p(A) \arrow[ur, dashed] &
\end{tikzcd}$$

All I need for the purposes of the letter is to restrict u to $C_\cdot(A)$.

July 22, 1983

What do I need to finish the letter to Loday? Discussion of the character and the application to his conjecture.

The remaining main problem is the relation to periodicity. Somehow I want to bring in the Bott element β over S^2. Connes does this as a cup product:

$$H_\lambda^i(A) \otimes H_\lambda^j(B) \longrightarrow H_\lambda^{i+j}(A\otimes B)$$

and then takes $B = \mathbb{C}$ and [redacted] $\beta \in H_\lambda^2(\mathbb{C})$ the canonical generator. What is the mechanism? One takes the tensor product of non-comm. DR complexes

$$\hat{\Omega}_A \otimes \hat{\Omega}_B$$

and then looks at the map

$$\hat{\Omega}_{A\otimes B} \longrightarrow \hat{\Omega}_A \otimes \hat{\Omega}_B$$

coming from the universal property of the former.

When $B = k$ what is $\hat{\Omega}_B$? One adjoins an identity 1 to B, or else one leaves out the relation $d1 = 0$. From my viewpoint it is most natural to adjoin an identity to get

$$\tilde{B} = k \oplus ke \qquad e^2 = e$$

and then $\hat{\Omega}_k$ will have the basis $(de)^p$, $e(de)^p$. Non-commutative because

[redacted] $$de\cdot e + e\,de = d(e^2) = de$$

$$\Longrightarrow \quad de\cdot e = (1-e)\,de$$

This reminds me of Karoubi's way to define a map from K_0 to cyclic homology. Namely a projector in [redacted] $M_n(A)$ defines a [redacted] ring homomorphism

$$k \oplus ke \longrightarrow M_n(A)$$

hence a map

$$HC(k)^2 \cong HC(k \oplus ke) \longrightarrow HC(M_n A) \cong HC(A).$$

Now we think of $B = ke$ ~~so~~ that $A \otimes B$ [redacted] has the elements ae. Then we have the ring homomorphism

$$\begin{array}{ccccc} \hat{\Omega}_A & \cong & \hat{\Omega}_{A \otimes ke} & \longrightarrow & \hat{\Omega}_A \otimes \hat{\Omega}_{ke} \\ a & \longmapsto & ae & \longmapsto & a \otimes e \\ da & \longmapsto & d(ae) & \longmapsto & da \otimes e + a \otimes de \\ bda & \longmapsto & bed(ae) & \longmapsto & (b \otimes e)(da \otimes e + a \otimes de) \\ & & & & bda \otimes e + ba \otimes ede \end{array}$$

etc. So it is more or less clear that by looking at the coefficient of $edede$ we will get a map from $\hat{\Omega}_A$ to $\hat{\Omega}_A$ lowering degree by 2.

Oxford, July 22, 1983

Dear Loday,

Around the time your manuscript arrived I found a simple way using curvature ideas to construct a map of complexes

$$C_\cdot(A) \longrightarrow \bigoplus \Omega_A^{<i}[-2i+1].$$

This can be thought of as the Chern character with values in Deligne cohomology, and hopefully will coincide with the known map

$$C_\cdot(A) \xleftarrow{\text{quis}} \left\{\begin{matrix}\text{Connes' } b, B \\ \text{bicomplex}\end{matrix}\right\} \xrightarrow{\mu} \bigoplus \Omega^{<i}[-2i+1]$$

The construction of this map also seems to shed some light on your conjecture about the filtration of cyclic homology coming from the $\mathfrak{gl}_i(A)$ for different i. I am sending you a sketchy account now, because we move back to Boston at the end of the month and I won't have time to work on these ideas for a while.

Let A be a commutative algebra, $\Omega^\cdot$ its DeRham complex, and $\tilde{\mathfrak{g}} = \mathfrak{gl}_n(A)$. We consider the double complex

$$C^p(\tilde{\mathfrak{g}}, \Omega^q) = \mathrm{Hom}(\Lambda^p \tilde{\mathfrak{g}}, \Omega^q)$$

of cochains on $\tilde{\mathfrak{g}}$ with values in $\Omega^\cdot$. It is a bigraded algebra with $d = d' + d''$, where d' is the Lie cochain differential and d'' comes from the De Rham complex. In the geometric setting: $A = C^\infty(M)$, $\tilde{\mathfrak{g}}$ = Lie algebra of $\mathcal{G} = \mathrm{Maps}(M, GL_n)$, then $C^\cdot(\tilde{\mathfrak{g}}, \Omega^\cdot)$ is the complex of forms on $\mathcal{G} \times M$ which are left-invariant under $\mathcal{G}$.

The identity map of $\tilde{\mathfrak{g}} = \mathfrak{gl}_n(A) = M_n(A)$ gives a canonical matrix-valued form

$$\theta \in M_n(C^\cdot(\tilde{\mathfrak{g}}, \Omega^\cdot))$$

of type $(1,0)$, which satisfies

$$d'\theta + \theta^2 = 0.$$

Geometrically this means that $D = d + \theta$ is a left-invariant connection in the trivial n-dimensional bundle over $G \times M$ which is flat in the G-directions. The curvature is the matrix form

$$K = (d+\theta)^2 = d\theta + \theta^2 = d''\theta$$

of type $(1,1)$.

Let φ denote a homogeneous polynomial function of degree m on $\mathfrak{gl}_n(k)$. For example $\varphi(X) = tr(\Lambda^m X)$ or $tr(\frac{1}{m!}X^m)$, which give rise to the characteristic classes c_m and ch_m respectively, when applied to a curvature form. Since the bundle is trivial, we know $\varphi(d''\theta)$ is cohomologous to zero. We now construct an explicit form u_φ such that $\varphi(d''\theta) = d u_\varphi$.

Consider the 1-parameter family of connections

$$D_t = d + t\theta.$$

$$K_t = D_t^2 = t\, d''\theta + (t^2 - t)\theta^2.$$

Define $\varphi'(X,Y)$ by

$$\varphi(X + \varepsilon Y) = \varphi(X) + \varepsilon\varphi'(X,Y) + O(\varepsilon^2).$$

Then one has the basic formula (Bott-Chern, Acta Math 114 (1965) p. 79)

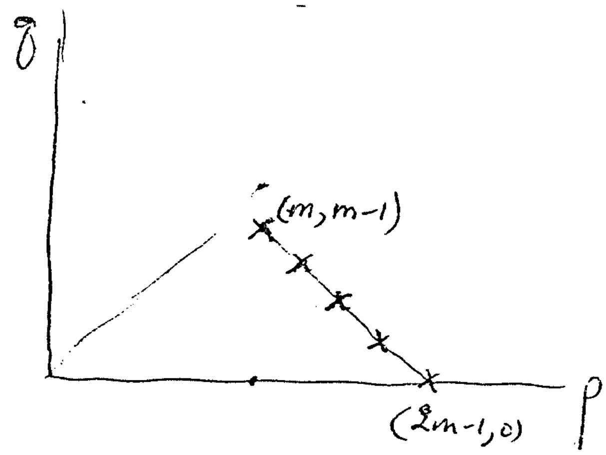
$$\frac{d}{dt}\varphi(K_t) = d\,\varphi'(K_t, \theta).$$

Integrating gives

$$\varphi(d''\theta) = d \underbrace{\int_0^1 \varphi'(K_t, \theta)\, dt}_{u_\varphi} .$$

(Integration of polynomials is possible in characteristic zero.)

Since K_t has components of type $(1,1)$ and $(2,0)$ and θ has type $(1,0)$, u_φ has components of type (p,q) where $p+q = 2m-1$, $p > q$.



Since $du_\varphi = \varphi(d''\theta)$ is of type (m,m), it follows that u_φ is a cocycle of degree $2m-1$ in the complex $C^\bullet(\tilde{\mathfrak{g}}, \Omega^{<m})$, where $\Omega^{<m} = \Omega/\Omega^{\geq m}$ is a quotient complex of Ω. Such a cocycle can also be interpreted as a map of complexes

$$(*) \qquad (\Lambda^\bullet \tilde{\mathfrak{g}}, \partial) \longrightarrow \Omega^{<m}[-2m+1].$$

We next show that this construction is compatible with Deligne's product. Let

$$cl(u_\varphi) \in H^{2m-1}(\tilde{\mathfrak{g}}, \Omega^{<m}) = H^0(\tilde{\mathfrak{g}}, \Omega^{<m}[-2m+1])$$

denote the cohomology class of u_φ, or equivalently the homotopy class of the map $(*)$.

Lemma: $\quad cl\,(u_{\varphi\bar\varphi}) = cl(u_\varphi)\cdot cl(u_{\bar\varphi})$.

Proof: Here $\bar\varphi$ is an invariant polynomial homogeneous of degree n. The product on the right is represented by either of the forms

$$\varphi(d''\theta)\,u_{\bar\varphi} \qquad \text{or} \qquad u_\varphi\,\bar\varphi(d''\theta)$$

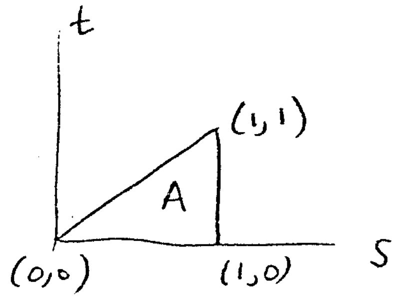
which are cocycles in $C'(\tilde{\mathfrak{g}}, \Omega^{<m+n})$ of degree $2(m+n)-1$, and which differ by $d(u_\varphi u_{\bar\varphi})$. Next

$$u_{\varphi\bar\varphi} = \int_0^1 \left[\varphi'(\Omega_t,\theta)\,\bar\varphi(\Omega_t) + \varphi(\Omega_t)\,\bar\varphi'(\Omega_t,\theta)\right]dt$$

$$d\left[\varphi'(\Omega_s,\theta)\,\varphi'(\Omega_t,\theta)\right] = \partial_s\varphi(\Omega_s)\cdot\varphi'(\Omega_t,\theta) - \varphi'(\Omega_s,\theta)\,\partial_t\bar\varphi(\Omega_t)$$

$$= \partial_s N - \partial_t M \qquad \text{where} \quad \begin{aligned} M &= \varphi'(\Omega_s,\theta)\,\bar\varphi(\Omega_t) \\ N &= \varphi(\Omega_s)\,\bar\varphi'(\Omega_t,\theta) \end{aligned}$$

Now we apply Green's thm. to the triangle



and we get

$$-u_{\varphi\bar\varphi} + \varphi(d''\theta)\,u_{\bar\varphi} = \oint M\,ds + N\,dt = \iint_A (\partial_s N - \partial_t M)\,ds\,dt$$

$$= d\iint_A \varphi'(\Omega_s,\theta)\,\bar\varphi'(\Omega_t,\theta)\,ds\,dt$$

As the last integral has components of type (p,q) with $p>q$ we see $\varphi(d''\theta)u_{\bar\varphi}$ and $u_{\varphi\bar\varphi}$ represent the same class in $H^0(\tilde{\mathfrak{g}}, \Omega^{<m+n}[-2(m+n)+1])$.

Let's now take $\varphi = \frac{1}{m!}\,tr(K^m)$ or $tr(\Lambda^m K)$ in which case the u-classes give maps which will be denoted

$$(\Lambda^{\cdot} gl(A), \partial) \underset{c_m}{\overset{ch_m}{\rightrightarrows}} \Omega^{<m}[-2m+1]$$

Because of the lemma and the standard formulas relating Chern classes and Chern character, one sees that ch_m and $\frac{(-1)^{m-1}}{(m-1)!} c_m$ are homotopic when restricted to the primitive subcomplex $C.(A)$.

One has now the problem of relating the Chern classes defined in the above way to the ones described in §5 of your manuscript. I suspect that

$$ch = \sum ch_m : C.(A) \longrightarrow \bigoplus \Omega^{<m}[-2m+1]$$

will be consistent with the maps

$$C.(A) \xleftarrow{\text{quis}} \left\{ \begin{matrix} \text{Connes } b, B \\ \text{complex} \end{matrix} \right\} \xrightarrow{\mu} \bigoplus \Omega^{<m}[-2m+1].$$

If this is so, then one has

$$\begin{array}{ccccc} H_p\{F_i C.(A)\} & \longrightarrow & H_p(gl_i A) & \xrightarrow{\quad 0 \text{ if } m>i \quad} & \\ \downarrow & & \downarrow & & \\ K_p^+(A) & \longrightarrow & H_p(gl\, A) & \xrightarrow{c_{m,p}} & H_{DR}^{2m-1-p} \end{array}$$

which will imply that for A smooth

$$\operatorname{Im}\{H_p\{F_i C.(A)\} \to K_p^+(A)\} = 0 \quad \text{for} \quad i \le \frac{p}{2}.$$

(Here I am using that $tr(\Lambda^m K)$ is zero on gl_i for $m > i$ and the isomorphism 5.3 of your manuscript.) This is a consequence of your conjecture $F_i\, k[U_p] = 0$ for $i \le \frac{p}{2}$.

I send this letter off now, so that it goes before the weekend. After August 1 I will be back at M.I.T.

Best regards,

Daniel Quillen

Try to get the constants straight in passing between cyclic and Lie cochains. Start with

$$\frac{1}{m!}\, tr\left(\theta (d''\theta)^{m-1}\right)$$

which we will think of as a map

$$\Lambda^m gl(A) \longrightarrow \Omega^{m-1}$$

$$X_1, \ldots, X_m \longmapsto i(X_m)\cdots i(X_1)\, \frac{1}{m!} tr\left(\theta (d''\theta)^{m-1}\right)$$

The signs on this are a bit complicated because $d''\theta$ has degree 2 and $i(X)\, d''\theta = d''X$ has degree 1. However the answer is an anti-symmetrization and so we only have to compute the sign for $tr(X_1 dX_2 \ldots dX_m)$ which is

$$tr(\theta\, d''\theta \ldots d''\theta) \longmapsto tr(X_1 d''\theta \ldots d''\theta)$$

$$\longmapsto (-1)\, tr(X_1 d''X_2\, d''\theta \ldots d''\theta)$$

$$\longmapsto \underbrace{(-1)^{1+2+\cdots+(m-1)}}_{(-1)^{\frac{m(m-1)}{2}}} tr(X_1 dX_2 \ldots dX_m)$$

Thus we get the map

$$(X_1, \ldots, X_m) \longmapsto \frac{(-1)^{\frac{m(m-1)}{2}}}{m!} \sum_{\sigma\in\Sigma_m} (-1)^\sigma\, tr(X_{\sigma 1}\, dX_{\sigma 2} \ldots dX_{\sigma m})$$

Check on the sign

$$i_Y i_X\, tr(\theta\, d''\theta) = i_Y\, tr(X d''\theta + \theta d''X)$$

$$= tr(-X dY + Y dX)$$

where we use that $i_X d'' + d'' i_X = 0$

July 23, 1983

Fadeev-Popov Ansatz:

Start with a 'volume' DA on the space $\mathcal{A}$ of connection and an action $\Gamma_{inv}(A)$ which is invariant for the gauge group $\mathcal{G}$. (In the following I pretend that $\mathcal{A}, \mathcal{G}$ are finite-dim'l.) Then from the measure DA on $\mathcal{A}$ and the Haar measure $D\Omega$ on $\mathcal{G}$ we get an induced measure $D\bar{A}$ on $\mathcal{G}\backslash\mathcal{A}$ such that

$$\int_{\mathcal{G}\backslash\mathcal{A}} D\bar{A}\, e^{i\Gamma_{inv}(\bar{A})} = \frac{\int_{\mathcal{A}} DA\, e^{i\Gamma_{inv}(A)}}{\int_{\mathcal{G}} D\Omega}$$

(Assume the action is free; otherwise the volume of the isotropy group occurs.)

The physics is described by the integral over $\mathcal{G}\backslash\mathcal{A}$. It is somehow the correct sum over classical trajectories. The problem is now to compute this integral.

We suppose that there exists a "gauge function" which is a map

$$g : \mathcal{A} \longrightarrow \mathcal{C}$$

whose fibres $g^{-1}(c)$ (for some c in $\mathcal{C}$) are sections of $\mathcal{A} \to \mathcal{G}\backslash\mathcal{A}$, that is

$$g^{-1}(c) \xrightarrow{\sim} \mathcal{G}\backslash\mathcal{A}.$$

c represents a choice of gauge.

We suppose $\mathcal{C}$ has a measure. Then this measure on $\mathcal{C}$ and $\mathcal{D}A$ on $\mathcal{A}$ give rise to a measure $\mathcal{D}'A$ on $g^{-1}(c)$. In practice this measure in $\mathcal{C}$ ~~is~~ hidden in a δ-function, $\delta(g-c)$. One writes

$$\int_{g^{-1}(c)} \mathcal{D}'A \cdot F(A) = \int_{\mathcal{A}} \mathcal{D}A \;\; \delta(g(A)-c)\, F(A) .$$

Now when we come to describe the integral over $\mathcal{G}\backslash\mathcal{A}$ with the induced measure $\mathcal{D}\bar{A}$ in terms of the integral over $g^{-1}(c)$ with the measure $\mathcal{D}'A$ there is a Jacobian determinant relating the volume $\mathcal{D}\Omega$ on $\mathcal{G}$ to the volume on $\mathcal{C}$. This is denoted

$$\det(\mathcal{M}_A) = \det\left(\frac{\partial g(\Omega A)}{\partial \Omega}\right)\Bigg|_{\Omega=1}$$

So combining we get

$$\int_{\mathcal{G}\backslash\mathcal{A}} \mathcal{D}\bar{A}\, e^{i\Gamma_{inv}(\bar{A})} = \int_{g^{-1}(c)} \mathcal{D}'A \;\; \det(\mathcal{M}_A)\, e^{i\Gamma_{inv}(A)}$$

$$= \int_{\mathcal{A}} \mathcal{D}A \;\; \delta(g(A)-c) \det(\mathcal{M}_A)\, e^{i\Gamma_{inv}(A)}$$

Finally one uses

$$\det(\mathcal{M}_A) = \int \mathcal{D}\bar{\omega}\, \mathcal{D}\omega \; e^{i\bar{\omega}\mathcal{M}_A\omega}$$

$$\delta(g-c) = \int \mathcal{D}\gamma \, e^{i\gamma(g-c)}$$

to rewrite the integral as

$$\int_{\mathcal{G}\backslash\mathcal{A}} \mathcal{D}\bar{A} \, e^{i\Gamma_{inv}(A)} = \int_{\mathcal{A}} \mathcal{D}A \int \mathcal{D}\bar{\omega}\,\mathcal{D}\omega\,\mathcal{D}\gamma \; e^{i(\Gamma_{inv}(A) + \bar{\omega} M_A \omega + \gamma(g(A)-c))}.$$

Thus one obtains a modified action (for $c=0$)

$$\Gamma_{Landau}(A,\omega,\bar{\omega},\gamma) = \Gamma_{inv}(A) + \gamma g(A) + \bar{\omega} M_A \omega.$$

Another possibility is to integrate: $\int e^{iF(c)} \mathcal{D}c$, where $F(c)$ is usually a Gaussian.

$$\int e^{iF(c)} \delta(g(A)-c)\,\mathcal{D}c = e^{F(g(A))}.$$

This leads to a modified action

$$\Gamma_{Feynmann}(A,\omega,\bar{\omega},\gamma) = \Gamma_{inv}(A) + F(g(A)) + \bar{\omega} M_A \omega$$

pertaining to a Feynman type gauge.

July 24, 1983

The following comes from J.C. Taylor's book on gauge theories of the weak interactions.

The Fadeev-Popov trick leads to an effective Lagrangian

$$\mathcal{L} \underbrace{- \frac{1}{2}\xi^{-1}\left(\partial^\lambda W_\lambda^\alpha\right)^2}_{\text{called gauge-fixing term}} - \eta^\alpha \partial^\lambda D_\lambda^{\alpha\beta}\omega^\beta$$

η, ω are fermion fields, the ghosts

The classical $\mathcal{L}$ is invariant under

$$W_\lambda^\alpha \longmapsto W_\lambda^\alpha + D_\lambda^{\alpha\beta}\omega^\beta$$

(ω^β are infinitesimal parameters describing an element of $\mathfrak{g}$). If we try to identify the two ω's, it is necessary to add an infinitesimal anti-commuting quantity ζ:

$$\delta W_\lambda^\alpha = \zeta D_\lambda^{\alpha\beta}\omega^\beta \qquad \zeta^2 = 0$$

Then to preserve the invariance of the effective Lagrangian one puts

$$\delta\eta^\alpha = -\xi^{-1}\zeta\,\partial^\lambda W_\lambda^\alpha$$

$$\delta\omega^\alpha = -\frac{1}{2}\zeta g^\alpha f^{\alpha\beta\gamma}\omega^\beta\omega^\gamma$$

This δ is the BRS transformation. It is a symmetry of the effective Lagrangian of the same character as supersymmetry.

The next problem is to understand the effect of this on vertex functions. The vertex functions starts with the Lagrangian and has quantum corrections given by renormalized Feynman graphs, based on a Lagrangian with counter-terms. If this Lagrangian can be chosen invariant under the BRS or Slavnov symmetry, then the theory turns out to be renormalizable.

I want now to forget the physics and to study a result quoted in the Bonora-Cotta-Ramusino paper due to De Wilde:

<u>Thm</u>: Let E be a vector bundle on M and consider the complex $\mathrm{Diff}(E, \Lambda T^*)$ of diff'l operators from E to forms. Then this complex is acyclic in degree $< n = \dim M$.

Actually the result seems to be stated for maps $L : \Gamma(E) \otimes \cdots \otimes \Gamma(E) \longrightarrow \Omega^{\cdot}_M$ which are local (i.e. given by differential operators).

What is surprising is that if we considered the complex

$$\mathrm{Hom}_{\mathbb{C}}(\Gamma(E), \Gamma(\Lambda^{\cdot} T^*))$$

then the cohomology should be

$$\mathrm{Hom}_{\mathbb{C}}(\Gamma(E), H^{\cdot}_{DR})$$

by exactness of the topological tensor product with $\Gamma(E)'$. Thus the locality requirement changes things.

Another example comes from Alexander-Spanier-Grothendieck. I think I should review this whole business.

Here's the general program: Here is a list of ideas and problems I want to unify:

1) Gelfand-Fuks found it useful to consider the so-called 'diagonal' subcomplex of $C^\cdot(\mathfrak{g})$, $\mathfrak{g}$ = Vect fields on M, consisting of cochains supported on the diagonal of M in M^g. What is the corresponding gadget for the gauge algebra?

2) Relation between cyclic and crystalline cohomology. What do we get if we look only at cyclic cochains supported on the diagonal? It would seem that completing the Hochschild complex along the diagonal wouldn't change its homology. On the other hand completing the Amitsur complex

$$A \xrightarrow{\delta} A^{\otimes 2} \xrightarrow{\delta} A^{\otimes 3} \xrightarrow{\delta}$$

seems to change from trivial to DR cohomology.

3) Whole effect of local condition on cochains, i.e. how about the local subcomplex of $C^\cdot(\tilde{\mathfrak{g}}, \Omega_M^\cdot)$ or of $C^\cdot(\tilde{\mathfrak{g}}, \text{Local maps}(\mathcal{a}, \Omega_M^\cdot))$

4) Ultimately I want to get back to the case of equivariant forms on $(\mathcal{G}, \mathcal{a})$ corresponding to the character of the tautological $\mathcal{G}$-vector bundle over M.

5) $\mathcal{D}$-modules where $\mathcal{D}$ is the sheaf of differential operators.

So let's begin with a review of crystalline cohomology in characteristic zero. X denotes a smooth scheme over some base S with is fixed. We consider nilpotent extensions $U \subset U'$ where U is open in X. Because X is smooth one knows that locally on U there are retractions of U' back on U.

A crystal on X is defined to be a cartesian section of a fibred category over the category of these nilpotent extensions of open sets of X. For example a crystal of vector bundles turns out to be a vector bundle on X with an integrable connection.

Several years ago I ~~[illegible]~~ was led to the idea of a formal category scheme. (Just before going to France in the summer of 68 - right after Adam's lectures at Battelle was when I found the idea of a category scheme). A category affine scheme consists of two rings A, B corresponding to objects + morphisms and corresponding maps

$$A \rightleftarrows B \qquad\qquad B \longrightarrow B \otimes_A B$$

so that one can write down a nerve

$$A \rightrightarrows B \;\equiv\; B \otimes_A B \;\equiv\; B \otimes_A B \otimes_A B \;\cdots$$

which is cosimplicial. In the formal situation B is replaced by its formal completion $\{B/I^{n+1}\}$ where $I = \operatorname{Ker}\{B \to A\}$.

The idea was that in characteristic zero a formal category scheme is equivalent to a De Rham complex $\quad A \xrightarrow{d} \Omega^1 \xrightarrow{d} \Lambda^2_A \Omega^1 \longrightarrow \cdots$

where $\Omega^1 = I/I^2$. [redacted] Put $J_n = B/I^{n+1}$. Then one has an exact sequence (Spencer sequence)

$$0 \longrightarrow A \longrightarrow J_n \longrightarrow J_{n-1}\otimes\Omega^1 \longrightarrow J_{n-2}\otimes\Omega^2 \longrightarrow \cdots$$

which one establishes by induction

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & S^n\Omega \arrow[r] \arrow[d] & S^{n-1}\Omega\otimes\Omega \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & J_n \arrow[r] \arrow[d] & J_{n-1}\otimes\Omega \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & A \arrow[r] & J_{n-1} \arrow[r] \arrow[d] & J_{n-2}\otimes\Omega \arrow[r] \arrow[d] & {} \\
 & & 0 & 0 &
\end{tikzcd}$$

Then starting with the DR complex one could reconstruct B by requiring

$$\begin{tikzcd}
J_n \arrow[r] \arrow[d] & \operatorname{Ker}(J_{n-1}\otimes\Omega \longrightarrow J_{n-2}\otimes\Omega^2) \arrow[d] \\
J_{n-1} \arrow[r] & J_{n-2}\otimes\Omega
\end{tikzcd}$$

to be cartesian. This defines J_n inductively.

Another point which is maybe more interesting is that if we have an A-module M with descent data so that we can form

$$(*) \qquad M \rightrightarrows B\otimes_A M \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; B\otimes_A B\otimes_A M \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}$$

then in fact the cohomology is given by that of the DR complex

$$(**) \qquad M \xrightarrow{d} \Omega^1\otimes M \xrightarrow{d} \Omega^2\otimes M \xrightarrow{d} .$$

Recall that B really stands for the inverse system $\{B/I^{n+1} = J_n\}$, so that $(*)$ is really a complex of pro-objects.

The proof consists in writing down the double complex formed by applying the DR complex $(**)$ to each of the terms in $(*)$. Thus $J_\infty(N) = B\otimes_A N$

comes provided with an integrable connection which allows one to form ~~the~~ sequence

$$0 \to N \longrightarrow J_\infty(N) \longrightarrow \Omega^1 \otimes J_\infty(N) \longrightarrow \Omega^2 \otimes J_\infty(N) \longrightarrow \cdots$$

In fact this is exact because the Spencer sequences

$$0 \longrightarrow N \longrightarrow J_k(N) \longrightarrow \Omega^1 \otimes J_{k-1}(N) \longrightarrow \cdots$$

are exact. So we get the double cx.

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Omega^1 \otimes M \arrow[r] \arrow[u] & \Omega^1 \otimes_A B \otimes_A M \arrow[r] \arrow[u] & \Omega^1 \otimes B \otimes (B \otimes M) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & M \arrow[r] \arrow[u, "d"] & B \otimes_A M \arrow[r] \arrow[u] & B \otimes (B \otimes M) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[r] \arrow[u] & M \arrow[r] \arrow[u] & B \otimes M \arrow[r] \arrow[u] & {}
\end{tikzcd}$$

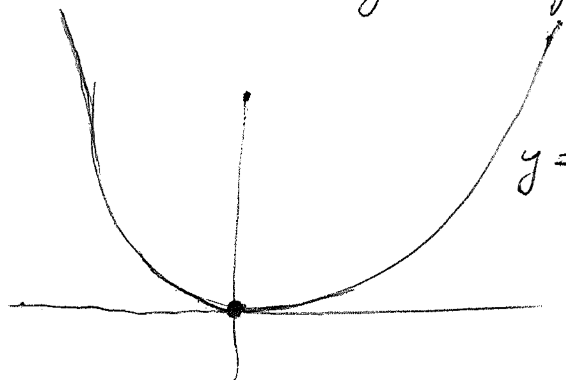
The horizontal rows except the bottom are acyclic because of the cone construction.

Another interpretation of $\underbrace{\text{an } M}_{}$ with a $d : M \longrightarrow \Omega^1 \otimes_A M$ satisfying $d^2 = 0$ is simply a module over the ring $D = \bigcup_n \mathrm{Hom}(B/I^{n+1}, A)$ of differential operators. Clearly from d we get an action of vector fields on M which commutes with $[\,,\,]$, so it extends to differential operators.

A natural question is what sort of cohomology the DR complex gives for a D-module.

$$\mathrm{Ext}_D^*(A, M) \qquad ?$$

Review of Feynman diagrams, effective potential and vertex functions. The model I use is of a real-valued field over 0-diml space. Thus we have a particle on the line with position x and given potential energy $V(x)$.



$$y = V(x) = a\frac{x^2}{2} + b\frac{x^3}{3!} + \dots$$

How can we detect the potential energy $V(x)$? Apply a force J to the particle and it is found at position x such that

$$J = +V'(x)$$

In other words where the combined energy

$$-Jx + V(x)$$

has a minimum.

Now suppose we are at inverse temperature β. Then when the force J is applied the particle is found at the position

$$\langle x \rangle = \frac{\int x e^{-\beta(-Jx+V(x))}dx}{\int e^{-\beta(-Jx+V(x))}dx} = \frac{1}{\beta}\partial_J \log Z$$

where $Z(J) = \int e^{\beta[Jx - V(x)]}dx$. The <u>effective potential</u> is the function $W(x)$ such that

$$J = W'(x) \quad \Longleftrightarrow \quad x = \partial_J\left[\frac{1}{\beta}\log Z\right]$$

One knows that W is the Legendre transform of $\frac{1}{\beta}\log Z(J)$, that is

$$W(x) = Jx - \frac{1}{\beta}\log Z(J)$$

where J is regarded as a fn. of x by

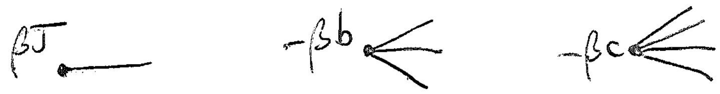
$$x = \partial_J\left(\frac{1}{\beta}\log Z\right).$$

(Actually W is determined up to a constant, so the above formula fixes that constant.)

Diagrams:

$$Z(J) = \int e^{-\beta a\frac{x^2}{2}} \underbrace{e^{\beta J x - \beta b\frac{x^3}{3!} - \cdots}}_{\text{expanded out}} dx$$

Vertices



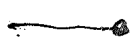
edges have weight $\frac{1}{\beta a}$

Then $Z(J)/\int e^{-\beta a\frac{x^2}{2}}dx$ is a sum over diagrams.

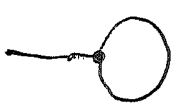
Ignore this last integral. $\log Z(J)$ is the sum over connected diagrams and

$$x = \partial_J\left(\frac{1}{\beta}\log Z\right) = \frac{1}{\beta}\frac{\partial_J Z}{Z}$$

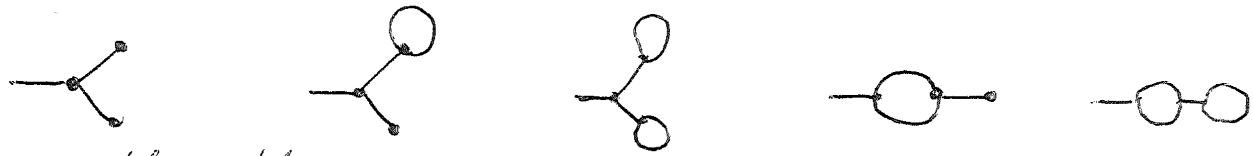
will be a sum over connected diagrams with a single external edge.



$$\frac{\beta J}{\beta a} = \frac{J}{a}$$



$$\frac{-\beta b}{(\beta a)^2 2} = \frac{1}{\beta}\frac{-b}{2a^2}$$



are the third order diagrams.

What is the power of β?

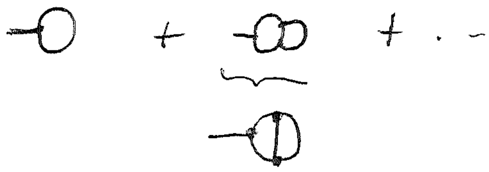
$$\frac{\beta^{v-1}}{\beta^{e}} = \beta^{v-e-1} = \beta^{-\ell}$$

where ℓ is the number of loops.

Finally we take each graph occurring in the sum for x and look at those edges which disconnect it. This leads to the formula

$$x = \frac{J}{a} + \frac{1}{a}\Gamma_1 + \frac{1}{a}\Gamma_2\, x + \frac{1}{a}\Gamma_3 \frac{x^2}{2!} + \cdots$$

where Γ_n is the sum over 1-particle irreducible graphs with n labelled external lines.

Γ_1 : 

Γ_2 :  $+ \cdots$ Γ_3 :  $+ \cdots$

Then one inverts the formula:

$$J = (-\Gamma_1) + (a - \Gamma_2)x + (-\Gamma_3)\frac{x^2}{2} + \cdots$$

so

$$W = (-\Gamma_0) + (-\Gamma_1)x + (a - \Gamma_2)\frac{x^2}{2} + \cdots$$

where $-\Gamma_0$ is a constant which I don't know if it has a diagram interpretation.

Clearly as $\beta \to \infty$, the loop terms go to zero and we have

$$W \longrightarrow \underline{V}$$

At this point I understand a little about diagrams, effective potential, and vertex functions. The above extends formally to fields. Instead of having a single degree of freedom x one has many $\phi(x)$, $x \in$ Space. Still one has the mechanism of an expected value $\langle \phi(x) \rangle$ of the field associated to an external field $J(x)$:

$$\langle \phi(x) \rangle = \frac{\delta}{\delta J(x)} \log Z(J)$$

$$Z(J) = \int \mathcal{D}\phi \, e^{-S(\phi) + \int J(x)\phi(x)\,dx}$$

$$W = \int J(x) \langle \phi(x) \rangle dx - \log Z(J)$$

Now I would like to understand renormalization. This has to be done because of infinities in the above formalism.

The idea I had is the following. First of all I have to understand where the infinities occur. Go back to the one-dimensional case. Here the interesting quantities are the effective position when $J = 0$ and the effective spring constant. These are the critical point of W and W'' there. To make things easy suppose

$$V(x) = a\frac{x^2}{2} + c\frac{x^4}{4!} + \cdots$$

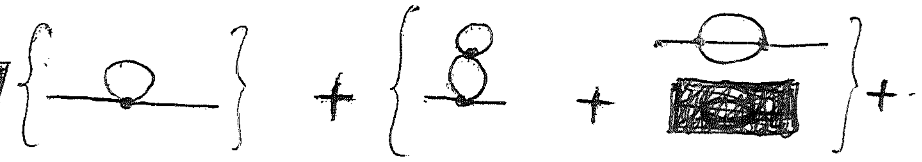
is an even function. Then

$$x = \int x e^{V(x) - Jx} dx \Big/ \int e^{V(x) - Jx} dx$$

is an odd function of J, so W is an fn. of x and we have

$$W(x) = \text{const} + (a - \Gamma_2)\frac{x^2}{2!} + (-\Gamma_4)\frac{x^4}{4!} + \cdots$$

So $(a-\Gamma_2)$ is the effective spring constant. It seems to correspond to the effective mass of an [redacted]. interacting particle.

Γ_2 : [redacted] 

2nd order

July 26, 1983

Review renormalization in the Lee model and then see if one can interpret this as adding counter terms to the Lagrangian. The Lee model has a 1-particle space which is the direct sum of a Hilbert space V, on which H_0 acts with continuous spectrum, and a 1-diml. space $\mathbb{C}\cdot\Phi$. The Hamiltonian is

$$\mathbb{H} = \begin{pmatrix} H_0 & |\gamma\rangle \\ \langle\gamma| & \omega_0 \end{pmatrix}.$$

Assuming $|\gamma\rangle \in V$ is a cyclic vector for H_0, then Φ will be cyclic for $\mathbb{H}$, and $(V + \mathbb{C}\Phi, \mathbb{H})$ can be recovered from [redacted] the function.

$$\left\langle \Phi \left| \frac{1}{\omega - \mathbb{H}} \right| \Phi \right\rangle = \frac{1}{\omega-\omega_0} + \frac{1}{\omega-\omega_0}\langle\gamma|\frac{1}{\omega-H}|\gamma\rangle\frac{1}{\omega-\omega_0} + \cdots$$

$$= \frac{1}{\omega - \omega_0 - g(\omega)} \qquad \text{where } g(\omega) = \langle\gamma|\frac{1}{\omega-H}|\gamma\rangle = \int\frac{\rho(x)\,dx}{\omega - x}$$

and the integral is over the spectrum of H_0 which we assume is $[0,\infty)$.

The function $\rho(x)$ we want to use is such that the integral $g(\omega) = \int \frac{\rho(x)\,dx}{\omega - x}$ is not convergent. So we introduce a cutoff + coupling constant, then let the cutoff + coupling constants change so that in the limit the function $\langle \Phi | \frac{1}{\omega - H} | \Phi \rangle$ remains a Stieltjes transform, hence ~~define~~ a cyclic ~~situation~~. The point is that the end result is what is physically real, but the beginning is fictitious.

ω_0 = bare energy of Φ is fictitious. What is real is a bound state Ψ of energy $\omega_b < 0$. This implies $\omega - \omega_0 - g(\omega)$ vanishes at ω_b determining ω_0:

$$\omega - \omega_0 - g(\omega) = \omega - \omega_b - [g(\omega) - g(\omega_b)]$$

$$= \omega - \omega_b - \int \rho(x)dx \left[\frac{1}{\omega - x} - \frac{1}{\omega_b - x} \right]$$

$$= (\omega - \omega_b)\Bigg(1 + \underbrace{\int \frac{\rho(x)\,dx}{(\omega - x)(\omega_b - x)}}_{f(\omega)} \Bigg)$$

Since the denominator has increased by a factor of x it is possible for f to exist without g existing as the cutoff is removed. This means the ultimate model corresponds to an initial one with $\omega_0 = \infty$.

Next we have

$$\langle \Phi | \frac{1}{\omega - H} | \Phi \rangle = \frac{|\langle \Phi | \Psi \rangle|^2}{\omega - \omega_b} + \text{analytic at } \omega_b$$

where the residue is

$$|\langle \Phi | \Psi \rangle|^2 = \frac{1}{1 + f(\omega_b)} .$$

(Note: $0 < |\langle \Phi | \Psi \rangle|^2 < 1$)

Since Φ is fictitious, we should really be working with the function

$$\frac{\langle \Phi | \frac{1}{\omega - H} | \Phi \rangle}{|\langle \Phi | \Psi \rangle|^2} = \frac{1}{(\omega - \omega_b)\left(\frac{1 + f(\omega)}{1 + f(\omega_b)} \right)}$$

Now
$$\frac{1+f(\omega)}{1+f(\omega_b)} = 1 + \frac{1}{1+f(\omega_b)}\underbrace{\left[f(\omega)-f(\omega_b)\right]}_{\int \frac{\rho(x)\,dx}{(\omega_b-x)}\left[\frac{1}{\omega-x}-\frac{1}{\omega_b-x}\right]}$$

$$= 1 + \frac{(\omega_b-\omega)}{1+f(\omega_b)}\int \frac{\rho(x)\,dx}{(\omega_b-x)^2(\omega-x)}$$

and this ~~[illegible]~~ integral ~~[illegible]~~ can exist even when f doesn't.

The next idea is to replace γ by $e\gamma$ where is a coupling constant, and make e move with the cutoff.

$$\frac{\langle \Phi | \frac{1}{\omega-H} | \Phi \rangle}{|\langle \Phi | \Psi \rangle|^2} = \frac{1}{(\omega-\omega_b)\left[1+(\omega_b-\omega)\underbrace{\frac{e^2}{1+e^2 f_\lambda(\omega_b)}}_{e_R^2}\int_0^\lambda \frac{\rho(x)\,dx}{(\omega_b-x)^2(\omega-x)}\right]}$$

So now let the cutoff $\lambda \to \infty$, choose e^2 so as

$$\frac{e^2}{1+e^2 f_\lambda(\omega_b)} = e_R^2$$

and then you get a limiting theory. The only problem is that because $f_\lambda(\omega_b) \to +\infty$,

$$e^2 = \frac{e_R^2}{1-e_R^2 f_\lambda(\omega_b)}$$

must be < 0 which means that the expected Hilbert space will have states of negative norm.

Now comes the problem of translating this into Lagrangian terms.

July 28, 1983

Problem: $\mathcal{G}$, $\mathcal{A}$, $\mathcal{L}$, M, E as usual: $\mathcal{L}$ is the determinant line bundle of the family of Dirac operator $D_A : S^+ \otimes E \longrightarrow S^- \otimes E$; M is even-dimensional. Suppose we arrange that $\mathcal{G}$ acts freely on $\mathcal{A}$. Then $\mathcal{L}$ descends to $\tilde{\mathcal{L}}$ over $\mathcal{G}\backslash\mathcal{A}$. To trivialize this bundle is equivalent to defining a gauge-invariant $\det(D_A)$.

The topological obstruction is

$$c_1(\tilde{\mathcal{L}}) \in H^2(\mathcal{G}\backslash\mathcal{A}, \mathbb{Z}) = H^2(B\mathcal{G}, \mathbb{Z}).$$

and $c_1(\tilde{\mathcal{L}})$ is computed by the index theorem for families. Now how do we get to differential forms?

We can compute the image of $c_1(\tilde{\mathcal{L}})$ in $H^2_{DR}(\mathcal{G}\backslash\mathcal{A})$ using curvature and differential forms.

The idea is to take the result from the index thm. for families

$$ch(\text{index}) = \int_M \hat{A}(M)\, ch(\tilde{E})$$

where $\tilde{E}$ denotes the bundle over $\mathcal{G}\backslash\mathcal{A} \times M$ obtained from $pr_2^* E$ on $\mathcal{A} \times M$ with its natural $\mathcal{G}$-action. Use the right hand side to produce diff'l forms on $\mathcal{G}\backslash\mathcal{A}$, in particular the ch_1 represents $c_1(\mathcal{L})$.

Now the problem becomes to compute $ch(\tilde{E})$, that is, to produce a connection on the bundle $\tilde{E}$.

Note that the same problem occurs even if I have already a connection on $\mathcal{L}$ which is invariant under G. I still must do something in order to get a connection on $\tilde{L}$ over $G\backslash Q$.

So we end up with the following situation. We have a Lie group G acting freely on P with quotient $G\backslash P = B$ and over P we have an equivariant vector bundle E. Let $\tilde{E}$ be the descended bundle over B. I want a connection on $\tilde{E}$. Such a connection [redacted] can be identified with a connection on E over P which is both G-invariant and 'horizontal' for the G-action. This is essentially the idea that a form on B is the same as a form on P which is invariant under G and killed by $i(X)$, $X \in \mathfrak{g}$.

Now let us suppose that we can find a connection on E which is invariant. Then we have a "moment map" or "Higgs field" which maps $\mathfrak{g}$ to $\mathrm{End}(E)$. Call it $X \mapsto \varphi_X$. It is G-invariant and so descends to a map

$$P \overset{G}{\times} \mathfrak{g} \longrightarrow \mathrm{End}(\tilde{E})$$

Here's the situation:

$$0 \longrightarrow \Omega^1_B \longrightarrow \tilde{\Omega}^1_P \longrightarrow P \overset{G}{\times} \mathfrak{g} \longrightarrow 0$$

$$\mathcal{A}(\tilde{E}) \longrightarrow \mathcal{A}(E)^G \longrightarrow \Gamma \mathrm{Hom}(P \overset{G}{\times} \tilde{\mathfrak{g}}, \mathrm{End}\,\tilde{E})$$

$$0 \longrightarrow \Gamma\{\mathrm{End}(\tilde{E}) \otimes \Omega^1_B\} \longrightarrow \Gamma\{\mathrm{End}(\tilde{E}) \otimes \tilde{\Omega}^1_P\} \longrightarrow \Gamma \mathrm{Hom}(P \times \tilde{\mathfrak{g}}, \mathrm{End}\,\tilde{E}) \longrightarrow 0$$

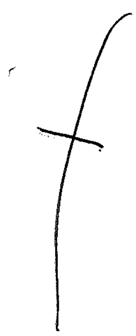
So we start with the invariant connection, take its moment map $\varphi : P \times^G \mathfrak{g} \longrightarrow \mathrm{End}(\tilde{E})$, lift this back to

$\Gamma\left(\mathrm{End}(\tilde{E}) \otimes \tilde{\Omega}^1_P\right)$, modify the connection so that it now has zero moment map, and then it becomes a connection on $\tilde{E}$.

Thus basically what we seem to need is a splitting of

$$0 \longrightarrow \Omega^1_B \longrightarrow \tilde{\Omega}^1_P \longrightarrow P \times^G \mathfrak{g} \longrightarrow 0$$

i.e. a connection in the principal bundle $P \to B$. This makes geometric sense since to obtain a horizontal connection one projects



transversally to the [redacted] G-orbits, and then takes the given connection in the transverse direction.

July 29, 1983

Problem: I know that I get generators for $H^*(B\mathcal{G})$ by taking the Kunneth components of $ch(\tilde{E}) \in H^*(B\mathcal{G}) \otimes H^*(M)$. I want to describe these cohomology classes by "differential forms on $B\mathcal{G}$" i.e. by natural transformations

$$\text{Maps}(Y, B\mathcal{G}) \longrightarrow \Omega^*(Y).$$

Here one has to choose a model for $B\mathcal{G}$, which in practice means taking a $\mathcal{G}$-bundle over Y, i.e. a vector bundle over $Y \times M$, and rigidifying it in some way; for example choosing a connection.

Now assuming M is connected with a basepoint ∞ and that our bundles over $Y \times M$ are trivialized over $Y \times \infty$, then we know that the corresponding $\mathcal{G}$ acts freely on $\mathcal{A}$ and consequently we have a very concrete model $\mathcal{G}\backslash\mathcal{A}$ for $B\mathcal{G}$.

What sort of structure corresponds to a map $Y \xrightarrow{\tilde{f}} \mathcal{G}\backslash\mathcal{A}$. Then we have really a cartesian square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & \mathcal{A} \arrow[d] \\ Y \arrow[r, "\tilde{f}"] & \mathcal{G}\backslash\mathcal{A} \end{tikzcd}$$

because we assume the map $\tilde{f}$ induces a given $\tilde{E}$ over $Y \times M$. Here P is the principal $\mathcal{G}$-bundle over Y, i.e. a point ξ of P over y is an isomorphism of $\tilde{E}|y \times M$ with E over M. The map f assigns to ξ a connection on E, which can be transported to $\tilde{E}|y \times M$.

Thus it would seem that f amounts to giving a vertical connection on $\tilde{E}$ over $Y \times M$.

Given E over $Y \times M$, suppose also given a connection in the vertical direction. Let P be the principal $\mathcal{G}$-bundle over Y consisting over y of isomorphisms $\xi : E \,|\, y \times M \xleftarrow{\sim} E_0$. Then the vertical connection D'' gives a connection on $E / y \times M$ which corresponds under ξ to a connection on E_0, so we get a family D_y of connections on E_0. Hence we get a map $P \longrightarrow \mathcal{A}$ which is canonically defined, so it commutes with the action of $\mathcal{G}$.

In fact $P \times^{\mathcal{G}} \mathcal{A}$ is obviously the fibre bundle over Y of connections in the fibres of E over Y.

Now we have seen that by using transversals to the $\mathcal{G}$-orbits in $\mathcal{A}$ we can define a connection of the tautological bundle $\tilde{E}$ over $\mathcal{G} \backslash \mathcal{A}$. Since E is induced from $\tilde{E}$ by the map

$$Y = P | \mathcal{G} \longrightarrow \mathcal{G} \backslash \mathcal{A}$$

it follows that we get a connection on E.

Thus there has to be a way to go from a vertical connection on E over $Y \times M$ to a full connection. Here is what it ought to be. We work locally on Y and so can suppose at the outset that $E = pr_2^*(E_0)$. Let's fix a connection D_0 on E_0 and then our vertical connection can be described

$$D_0'' + A \qquad A \in \Omega^0(Y, \Omega^1(M, \operatorname{End} E_0)).$$

From the trivial structure in the Y direction on $E = pr_2^*(E_0)$

we get the horizontal connection d'.

So now differentiate the family $D_0'' + A$ in the Y- direction to get

$$d'A \in \Omega^1(Y, \Omega^1(M, \mathrm{End}\, E_0))$$

On the other hand we have

$$\Omega^0(M, \mathrm{End}\, E_0) \xrightarrow{[D''+A_y,\ ]} \Omega^1(M, \mathrm{End}\, E_0)$$

which can be interpreted as the image of $\tilde{\mathfrak{g}}$ in the tangent space to $\mathcal{A}$ at A_y. So what I want to do is to project $d'A$ back into $\tilde{\mathfrak{g}}$ to obtain a

$$\theta \in \Omega^1(Y, \Omega^0(M, \mathrm{End}\, E_0))$$

such that on changing d' to $D' = d' + \theta$ we effectively make $[D', D''] = d'A - D''\theta$ perpendicular to the image of D'' on $\Omega^{1,0}(Y \times M, \mathrm{End}\, E_0)$: In pictures:

$$\begin{array}{ccc} d'A = [d', D''] & \in & \Omega^{1,1}(Y \times M, \mathrm{End}\, E_0) \\ & & \uparrow \; D_A = [D_A'',\ ] \\ & & \Omega^{1,0}(Y \times M, \mathrm{End}\, E_0), \end{array}$$

and our condition defining θ is

$$D_A^* \left(d'A - D_A \theta \right) = 0.$$

This is much too hard - one should try to work in the equivariant differential forms.

Go back to the case of ~~line~~ equivariant line bundles.

My problem is the following: I have the equivariant bundle $pr_2^*(E_0)$ over $\mathcal{A}\times M$ which descends to $(\mathcal{G}\backslash\mathcal{A})\times M$ to give the tautological bundle $\tilde{E}$. I want the characteristic classes of $\tilde{E}$. Similarly I have the equivariant line bundle $\mathcal{L}$ on $\mathcal{A}$ which I descend to $\tilde{\mathcal{L}}$ on $\mathcal{G}\backslash\mathcal{A}$ and I want the curvature of $\tilde{\mathcal{L}}$.

Hence I should consider an equivariant vector bundle, say line bundle L, over a G-space X. Suppose an invariant connection is given; when G is compact one can be produced by averaging in the space of connections.

If ∇ is a G-invariant connection, then the moment map $v \longmapsto \varphi_v$ is defined by

$$\mathcal{L}_v = \nabla_v + \varphi_v \qquad \text{for all } v\in\mathfrak{g}$$

as operators on $\Gamma(L)$. Then one calculates

$$[\mathcal{L}_v, \nabla_{v'}] = \nabla_{[v,v']} \iff d\varphi_v = i(v)\Omega$$

G-invariance of ∇

$$[\mathcal{L}_v, \mathcal{L}_{v'}] = \mathcal{L}_{[v,v']} \implies \varphi_{[v,v']} = v\,\varphi_{v'}.$$

The point is that the pair $\Omega\in\Omega^2(X)^G$, $\varphi\in \mathrm{Hom}_G(\mathfrak{g},\Omega^0(X))$ ~~constitutes~~ a closed equivariant 2-form. The idea I had was that

$$\begin{array}{c}\text{equivariant forms}\\ \text{for } X\end{array} = \left[\Omega^{\cdot}(X)\otimes W(\mathfrak{g})\right]_{basic}$$

$$\cong \left(\Omega^{\cdot}(X)\otimes S(\mathfrak{g}^*)\right)^G.$$

In degree 2 one has $\Omega\in\Omega^2(X)^G$ and $\varphi\in\left(\Omega^0(X)\otimes\mathfrak{g}^*\right)^G$

More generally I should consider an equivariant vector bundle E over X with an invariant connection. Then the equivariant curvature consists of the usual curvature

$$K \in \Omega^2(X, \operatorname{End} E)^G$$

and a moment map

$$\varphi \in \{\mathfrak{g}^* \otimes \Omega^0(X, \operatorname{End} E)\}^G.$$

The next point is to see [redacted] that in the case of free action, if we choose a connection in the principal bundle $X \longrightarrow G \backslash X$, then from the equivariant curvature I actually get the curvature for the descended bundle.

July 30, 1983

Original problem: I know I obtain generators for $H^*(B\mathcal{G})$ by taking the Kunneth components of $ch(\tilde{E})$, where $\tilde{E}$ is the tautological bundle over $B\mathcal{G} \times M$. I want to describe these classes using differential forms, that is, by a method related to Chern-Weil. (I don't believe Chern-Weil gives all these classes.)

The idea will be to think of $B\mathcal{G}$ as $\mathcal{G}\backslash\mathcal{A}$ and to represent $ch(\tilde{E})$ by using a connection on $\tilde{E}$. Now $\tilde{E}$ is obtained by descending $pr_2^*(E_0)$ on $\mathcal{A} \times M$, and $pr_2^*(E_0)$ has a canonical $\mathcal{G}$-invariant connection. Thus we want to descend this connection. I have seen that this involves choosing a connection in $\mathcal{A} \to \mathcal{G}\backslash\mathcal{A}$, which is an unpleasant process. Instead I want to use equivariant forms in order to avoid this unpleasant choice.

Let's isolate the following subproblem. Let a Lie group G act on M and let E be an equivariant vector bdle. Suppose given an invariant connection ∇ on E. Then the Higgs field φ is defined by

$$\mathcal{L}_v = \nabla_v + \varphi_v \qquad\qquad v \in \mathfrak{g}$$

as operators on $\Gamma(E)$. It exists because $\mathcal{L}_v$ and ∇_v are operators with the same symbol; $\varphi \in \mathfrak{g}^* \otimes \Omega^0(M, \mathrm{End}\, E)$.

$$G\text{-Invariance} \iff \underbrace{[\mathcal{L}_v, \nabla_X]} = \nabla_{[v,X]} \qquad\qquad X \in \Gamma(T_M)$$

$$[\nabla_v, \nabla_X] + [\varphi_v, \nabla_X]$$

or $$[\nabla_X, \varphi_\sigma] = [\nabla_\sigma, \nabla_X] - \nabla_{[\sigma, X]}$$
$$= K(\sigma, X) = i(X)\, i(\sigma) K$$

If $\nabla(\varphi)$ is defined to be
$$\Omega^0(\mathrm{End}\, E) \xrightarrow{[\nabla,\ ]} \Omega^1(\mathrm{End}\, E),$$
then this says
$$\boxed{\nabla(\varphi_\sigma) = i(\sigma) K}$$

Next because we have an action of G on $\Gamma(E)$
$$[\mathcal{L}_\sigma, \underbrace{\mathcal{L}_{\sigma'}}_{\nabla_{\sigma'} + \varphi_{\sigma'}}] = \mathcal{L}_{[\sigma, \sigma']} = \nabla_{[\sigma,\sigma']} + \varphi_{[\sigma,\sigma']}$$

so
$$\boxed{[\mathcal{L}_\sigma, \varphi_{\sigma'}] = \varphi_{[\sigma, \sigma']}}$$

which says φ is invariant.

~~Conclusion: The invariant connection gives us~~
$$\cancel{K \in \Omega^2(M, \mathrm{End}\, E)^G}$$
$$\cancel{\varphi \in \{\mathfrak{g}^* \otimes \Omega^0(M, \mathrm{End}\, E)\}^G}$$

Wait: $$[\mathcal{L}_\sigma, \varphi_{\sigma'}] = \varphi_{[\sigma,\sigma']}$$
$$[\nabla_\sigma + \varphi_\sigma, \varphi_{\sigma'}] = \underbrace{i(\sigma)\, i(\sigma') K}_{K(\sigma', \sigma)} + [\varphi_\sigma, \varphi_{\sigma'}]$$

so the ~~second~~ condition can be expressed as
$$\boxed{[\varphi_\sigma, \varphi_{\sigma'}] - \varphi_{[\sigma, \sigma']} = K(\sigma, \sigma')}$$

Summary: The invariant connection gives us invariant forms curvature $\qquad K \in \Omega^2(M, \operatorname{End} E)^G$

Higgs field $\qquad \varphi \in \{\mathfrak{g}^* \otimes \Omega^1(M, \operatorname{End} E)\}^G$

satisfying the above conditions.

Now suppose the action of G on M is free and Θ be a connection in the principal bundle $M \longrightarrow G\backslash M$. Thus

$$\Theta \in \Omega^1(M, \mathfrak{g})^G \qquad\qquad i(v)\Theta = v \quad \text{for } v \in \mathfrak{g}.$$

Now consider

$$\varphi\Theta \in \Omega^1(M, \operatorname{End}(E))^G$$

$$(\varphi\Theta)(X) = \varphi_{\Theta(X)} \quad ;$$

it is G-invariant as both φ, Θ are. Now I can define a new invariant connection on E by

$$\tilde{\nabla} = \nabla + \varphi\Theta .$$

Then
$$\tilde{\nabla}_v = \nabla_v + (\varphi\Theta)(v) = \nabla_v + \varphi_v = \mathcal{L}_v$$
which means that the Higgs map for $\tilde{\nabla}$ is zero, and hence $\tilde{\nabla}$ descends to the orbit space $G\backslash M$.

Sensible questions for what comes next.

1) Why does Chern-Weil break down for $\mathfrak{g}$? What is the new ingredient involved, when we bring in $\mathfrak{a}$?

2) If you have E over $Y \times M$ you could choose a connection to get $ch(E)$, then you have

$$ch(E) \in \Omega^\bullet(Y, \Omega^\bullet(M))$$

so can integrate over cycles to get forms on Y. How does this method compare with what follows?

Consider over $\mathcal{A} \times M$ the bundle $pr_2^*(E_0)$ with its natural $\mathcal{G}$-action. There is a tautological vertical connection on this bundle: over $\{A\} \times M$ one uses the connection A on $E_0 = pr_2^*(E_0) \mid$ fibre. There is also an obvious horizontal connection do to the fact that $pr_2^*(E_0)$ is trivial horizontally. Thus $pr_2^*(E_0)$ has a canonical connection, necessarily $\mathcal{G}$-invariant. This means we can define its characteristic classes as equivariant forms for $(\mathcal{G}, \mathcal{A} \times M)$. If I assume $\mathcal{G}$ acts freely on $\mathcal{A}$, and choose a connection for $\mathcal{A} \longrightarrow \mathcal{G}\backslash\mathcal{A}$, then these equivariant forms will give me forms on $\mathcal{G}\backslash\mathcal{A}$.

(It might be better to think in terms of the curvature: The equivariant curvature of $pr_2^*(E_0)$ is a pair (K, φ) of $pr_2^*(\mathrm{End}\, E)$-valued forms over $\mathcal{A} \times M$. Given a connection θ in $\mathcal{A} \to \mathcal{G}\backslash\mathcal{A}$, one then makes a new invariant connection $\tilde{\nabla} = \nabla + \varphi\theta$ which descends to $\mathcal{G}\backslash\mathcal{A}$.)

At this point we know that $ch(\tilde{E}) \in H^*(B\mathcal{G} \times M)$ is represented by canonical equivariant differential forms on $\mathcal{A} \times M$ for the $\mathcal{G}$-action.

Now let us see what this means for a general family, i.e. a vector bundle E over $Y \times M$, isomorphic to E_0 on each fibres. Let P be the principal $\mathcal{G}$-bundle over Y, whose points ξ over y are isomorphisms of E_y with E_0. Over $P \times M$ E becomes isom. to $pr_2^*(E_0)$, hence the is an obvious horizontal connection.

I want to check carefully that this connection is G-invariant.

$$\begin{tikzcd}
E \arrow[d] & pr_2^*(E_0) \arrow[l] \arrow[r] \arrow[d] & E_0 \arrow[d] \\
Y\times M & P\times M \arrow[l] \arrow[r] & M
\end{tikzcd}$$

G-acts both on P and on E_0, hence sort of diagonally on $pr_2^*(E_0) = P\times E_0$. The P-connection I see on $pr_2^*(E_0)$ is flat and it has leaves $P\times\{e\}$ where e runs over E_0. These are carried into themselves by G, hence the connection is G-invariant.

Next choose a vertical connection in $pr_2^*(E_0)$ which is invariant; this is the same as a vertical connection in E over $Y\times M$. It gives us an equivariant map $P \longrightarrow \mathcal{A}$ such that $pr_2^*(E_0)$ over $P\times M$ is isom. to the pull-back of $pr_2^*(E_0)$ on $\mathcal{A}\times M$, and the isomorphism is compatible with the connections.

Conclusion: Starting from E over $Y\times M$, the only real choice we have to make to get forms realizing the characteristic classes of E is a vertical connection. ?

This is wrong for $M = pt$, so we have to go over the argument to see what breaks down.

Start with E over Y and let P be the principal bundle, so that we have

$$\begin{tikzcd} E & P \times E_0 \arrow[l] \arrow[r] & E_0 \\ Y \arrow[from=1-1] & P \arrow[l] \arrow[r] \arrow[from=1-2] & pt \arrow[from=1-3] \end{tikzcd}$$

Here E_0 is a vector space, and $G = U_n$ is acting diagonally on $P \times E_0$. Clearly $P \times E_0 / P$ has a flat invariant connection; the moment map is non-trivial, being essentially the identity $\mathfrak{g} \xrightarrow{\sim} \mathrm{End}(E_0)$. Choosing a connection for P relative to Y allows us to descend this invariant connection to one on E.

The difference between the case above and the special case $M = pt$. is that we have

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\ Y \arrow[r] & \mathcal{G} \backslash \mathcal{A} \end{tikzcd}$$

and consequently a choice of connection for $\mathcal{A} \to \mathcal{G} \backslash \mathcal{A}$ will give one for $\mathcal{P}$ over Y.

Hence we seem to need $\mathcal{G}$ acting freely on $\mathcal{A}$ which means that we want M connected and the bundle E over $Y \times M$ to be trivialized along $Y \times$ (basepoint of M).

Analytically, suppose the vertical connection D'' on E over $Y \times M$ is given, then there is a good choice for the horizontal connection D' as follows. The $(1,1)$ component of the curvature

$$D'D'' + D''D' \in \Omega^{1,1}(Y \times M, \mathrm{End}\, E)$$

should be $\perp$ to the image of

$$\begin{tikzcd} \Omega^{1,1}(Y\times M, \mathrm{End}\,E) \\ \Omega^{1,0}(Y \times M, \mathrm{End}\, E) \arrow[u, "{[D'',\ ]}"'] \end{tikzcd}.$$

One has to suppose that this $[\theta'', \ ]$ on $\Omega^{1,0}$ is injective, say by [redacted] requiring the form to vanish at a basepoint of M. Also one could require the vertical connections to be very twisted maybe.

<u>August 1, 1983:</u>

The problem is to describe characteristic classes for principal G-bundles, where $G = \mathrm{Aut}(E_0/M)$ is a gauge group, in differential form terms. Thus I want the appropriate extension of the Chern-Weil construction. Recall that this goes as follows.

Given a principal G-bundle P over Y one chooses a connection ~~in~~ it, and then one gets a map from invariant polynomial functions on $\tilde{g}$ into differential forms on Y. The problem is that apparently [redacted] there are not enough invariant polynomials.

A connection in P over Y is the same thing as a horizontal connection in the associated bundle E over $Y \times M$. We regard the choice of this connection as a natural step in the Chern-Weil process, and we concentrate on the remaining steps. It would be nice if our classes came from invariant polys. on $\tilde{g}$, then we would just list the invariant polys. corresponding to the classes we want. Recall that these are the Kunneth components of the Chern character of E.

In order to compute $ch(E)$ we [redacted] pick a connection on E over $Y \times M$. Since we have already chosen the horizontal connection, we therefore have to choose a vertical connection.

Let $\pi : P \longrightarrow Y$; then $\pi^*(E)$ over $P \times M$ is canonically isomorphic to $pr_2^*(E_0) = P \times E_0$ with G-acting diagonally. A vertical connection in E is the same as an invariant vertical connection in $\pi^*(E)$, which is the same thing as an equivariant map $P \longrightarrow \mathcal{A}$.

Now the point is that $pr_2^*(E_0)$ over $P \times M$ (or over $\mathcal{A} \times M$) has an obvious G-invariant flat horizontal connection whose flat sections are $P \times \{e\} \subset P \times E_0$, as $e \in E_0$. So therefore from a choice of vertical connection in E, we obtain an invariant connection on $\pi^*(E)$ over $P \times M$. Hence we obtain an equivariant curvature and characteristic classes as equivariant forms for the G-action on $P \times M$. These descend via a connection in P over Y to give a curvature and character on E over $Y \times M$.

One thing worth checking is that on descending the invariant connection on $\pi^*(E)$ using the connection in P we obtain the connection on E we started with.

Here are some remaining points:

1) What is the relation to the invariant forms on G?

Before when I looked at this I was concerned with the "Maurer-Cartan" connection on

$pr_2^*(E_0) = \mathcal{G} \times E_0$ over $\mathcal{G} \times M$; this is a horizontal connection: $d' + \theta$, and I chose a fixed vertical connection D'' on E_0 to get the connection $d' + \theta + D''$ on $pr_2^*(E_0)$ I was working with. $\mathcal{G}$ acts on $\mathcal{G} \times E_0$ on the left factor.

In the present case I have $\mathcal{G}$ acting diagonally on $\mathcal{G} \times E_0$, the horizontal connection d', and the vertical connection invariant under $\mathcal{G}$, hence obtained from a $\mathcal{G}$-map $\mathcal{G} \to \mathcal{a}$.

The thing to see is that if we use the isomorphism

$$\mathcal{G} \times E_0 \xrightarrow{\sim} \mathcal{G}_0 \times E_0 \qquad (g, e) \mapsto (g, ge)$$

transform the left to the diagonal action, then it transforms $d' + \theta$ to d'. But this follows from

$$d' + \theta = g^{-1} d' g$$

2) What the special feature about bringing in $\mathcal{a}$ that allows the classes in $H^*(B\mathcal{G})$ to be realized?

3) Deformations of a principal G-bundle P_0/M. Such is a P over $Y \times M$ or equivalently a principal $\mathcal{G}$-bundle $\mathcal{P}$ over Y. Formula:

$$\begin{array}{c} P \\ Y \times M \\ \downarrow pr_1 \\ Y \end{array} \qquad (pr_1)_*(P) = \mathcal{P}$$