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Problem: Consider a graded vector bundle $E = E^+ \oplus E^-$ with a connection D preserving the grading. If L is an odd degree endom. of E we can form the differential form

(1) $$\operatorname{tr}_E \left(\varepsilon_E \, e^{L^2 + [D,L] + D^2} \right)$$

The problem is to decide whether the cohomology class of this form is independent of L. In particular is this form closed?

Why do I think this should be true? Suppose the base M is a $2n$ torus. Then we can form the Dirac operator on sections of $S \otimes E$:

$$\not{D} = \frac{h}{i} \gamma^\mu D_\mu + \varepsilon_S L$$

and compute the index

(2) $$\operatorname{tr}\left(\varepsilon \, e^{-\not{D}^2} \right)$$

$$\not{D}^2 = -h^2 D_\mu^2 - \frac{h^2}{2} \gamma^\mu \gamma^\nu [D_\mu, D_\nu] + \frac{h}{i} \gamma^\mu \varepsilon_S [D_\mu, L] + L^2$$

We know the index is independent of h. As $h \to 0$ we think it should be possible to show that the trace tends to the integral of the form (1) over M, up to some constant. Since the index is unchanged under a scaling $L \longrightarrow tL$, one could hopes that the cohomology class of (1) is unchanged under this scaling.

Why is (2) stable under perturbations of $\not{D}$? We consider a variation $\delta \not{D}$. Then

$$-\delta \operatorname{tr}\left(\varepsilon e^{-\not{D}^2} \right) = \operatorname{tr}\left(e^{-\not{D}^2} \varepsilon \; \delta(\not{D}^2) \right)$$

as ε commutes with $\not{D}^2$.

$$= \mathrm{tr}\left(e^{-\not{D}^2} \varepsilon \{\not{D}, \delta\not{D}\}\right) = \mathrm{tr}\left(e^{-\not{D}^2}[\not{D}, \varepsilon\delta\not{D}]\right)$$

$$= \mathrm{tr}\left([\not{D}, e^{-\not{D}^2}\varepsilon\,\delta\not{D}]\right) = 0$$

The last step results from the cyclic symmetry of the trace on operators, but must correspond to the formula

$$d\,\mathrm{tr}(A) = \mathrm{tr}([D, A])$$

on differential forms.

Now the idea will follow through the above proof on the differential form level.

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Super-trace. In super mathematics we work with $\mathbb{Z}_2$-graded algebras $A = A^+ \oplus A^-$ and a different notion of commutative. A module over such an A is a graded module $M = M^+ \oplus M^-$. Free modules have generators in both even and odd degrees. [redacted] An endomorphism of a free module should have a trace with values in $A/[A,A]$.

Already the above sentence [redacted] raises questions. What do I mean by endomorphisms? Degree zero or [redacted] both odd + even?

Formal calculations: Let D be an operator and consider a change $D + \delta D = D + \varepsilon X$ where ε anti-commutes with D, ε commutes with X, and $\varepsilon^2 = 0$. Then

$$\delta e^{D^2} = \int_0^1 ds\, e^{(1-s)D^2}\, \delta(D^2)\, e^{sD^2}$$

Better

$$e^{(D+\varepsilon X)^2} = e^{D^2 + D\varepsilon X + \varepsilon X D + \overbrace{\varepsilon \overbrace{X\varepsilon}^{\varepsilon X} X}^{0}}$$

$$= e^{D^2 + [D,X]\varepsilon} = e^{D^2} + \int_0^1 dt\, e^{(1-t)D^2}[D,X]\varepsilon\, e^{tD^2}$$

$$= e^{D^2} + \left[D, \int_0^1 dt\, e^{(1-t)D^2} X e^{tD^2}\right]\varepsilon$$

$$= e^{D^2} + \left\{D, \int_0^1 dt\, e^{(1-t)D^2} X \varepsilon e^{tD^2}\right\}$$

$$= e^{D^2} + \{D, \ e^{D^2 + X\varepsilon} - e^{D^2}\} \qquad ?$$

What I would like to duplicate on the diffl form level somehow is the following

$$\delta \, tr(\varepsilon \, e^{\not{D}^2}) = tr\left(\varepsilon \int_0^1 ds \, e^{(1-s)\not{D}^2} \, \delta\not{D}^2 \, e^{s\not{D}^2}\right)$$

$$= tr\left(\varepsilon \, e^{\not{D}^2} \delta\not{D}^2\right) \qquad \text{as } \not{D}^2 \text{ commutes with } \varepsilon$$

$$= tr\left(\varepsilon \, e^{\not{D}^2}(\not{D}\,\delta\not{D} + \delta\not{D}\,\not{D})\right)$$

$$= tr\left(\varepsilon \, \{\not{D}, e^{\not{D}^2}\delta\not{D}\}\right) = 0$$

where the last comes from the property of $tr\,\varepsilon$. This is very close to the diffl form calculation

$$\delta \, tr(e^{D^2}) = tr\left(e^{D^2}\delta(D^2)\right)$$

$$= tr\left(e^{D^2}(D\,\delta D + \delta D\,D)\right)$$

$$= tr\left([D, e^{D^2}\delta D]\right) \qquad \text{here } [\ \ ] = \text{graded comm.}$$

$$= d \, tr\left(e^{D^2}\delta D\right)$$