299-350

Dec. 6, 1983 - Dec. 29, 1983

December 6, 1983

Today I gave the second talk in Jaffe's seminar. In preparing for it, it occurred to me that the restriction to the $\mathcal{G}$-orbit could be done much nicer. The idea is that we have the Chern-Simons form on $\mathcal{A} \times M$

$$\int_0^1 dt\, tr\left(\bar{A}\, e^{tF_{\bar{A}}} + (t^2-t)\bar{A}^2\right)$$

where $\bar{A} = \tilde{A} + \Theta$. Now we restrict to a [redacted] $\mathcal{G}$-orbit $\times M$. We fix a point A of that orbit (say $A = g * A^0$, but from the point of view I want to adopt I can take $A = A^0$). We identify the tangent space to the $\mathcal{G}$-orbit thru A with $\tilde{\mathfrak{g}}$, whence the above becomes an [redacted] element of

$$C^*(\tilde{\mathfrak{g}}, \Omega^*(M))$$

depending on the choice of A. Then we have

$$\bar{A} \xmapsto{rest_A} \underbrace{A}_{0,1} + \underbrace{\Theta}_{1,0}$$

where $A \in C^0(\tilde{\mathfrak{g}}, \Omega^1(M, \mathfrak{u}))$

$\Theta \in C^1(\tilde{\mathfrak{g}}, \Omega^0(M, \mathfrak{u}))$ is the [redacted] obvious identification of $\tilde{\mathfrak{g}} \simeq \Omega^0(M, \mathfrak{u})$. Now I claim that

$$F_{\bar{A}} \xrightarrow{rest} \underset{(0,2)}{F_A}$$

In effect $F_{\bar{A}}$ comes from the curvature of the connection on the base. It is therefore constant in the $\mathcal{G}$-orbit direction i.e. $\iota_v F_{\bar{A}} = 0$ which implies that $F_{\bar{A}}$ [redacted] is of type $(0,2)$, [redacted] when restricted to the gauge orbit.

Look at $F_{\bar{A}}$ in the M-direction

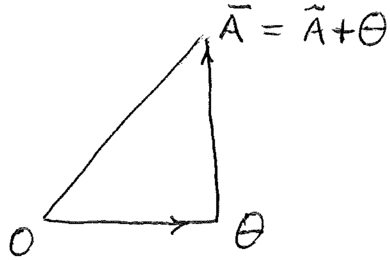
$$F_{\bar{A}} = d(\tilde{A}+\theta) + (\tilde{A}+\theta)^2 = d_M A + A^2 \quad \text{at } A,$$

so it is all clear.

So the Chern-Simons gives the form

$$\int_0^1 dt \, tr\left((\tilde{A}+\theta)\, e^{tF_A + (t^2-t)(A+\theta)^2}\right) \in C^*(\tilde{\mathfrak{g}}, \Omega^*(M)).$$

[redacted] Recall that we can use also the path



which gives instead the [redacted] form

$$\int_0^1 dt \, tr\left(\tilde{A}\, e^{tF_\theta + (1-t)F_{\bar{A}} + (t^2-t)\tilde{A}^2}\right)$$

$$+ \int_0^1 dt \, tr\left(\theta\, e^{tF_\theta + (t^2-t)\theta^2}\right)$$

Upon restricting into $C^*(\tilde{\mathfrak{g}}, \Omega^*(M))$ we [redacted] have

$$\tilde{A} \longmapsto A$$

$$F_\theta = d\theta + \theta^2 \longmapsto d_{\mathfrak{g}}\theta + d_M\theta + \theta^2 = d_M\theta$$

and so ~~the~~ ~~ee~~ we obtain the class

$$\int_0^1 dt \, tr\left(A\, e^{t\,d\theta + (1-t)F_A + (t^2-t)A^2}\right)$$

$$+ \int_0^1 tr\left(\theta\, e^{t\,d\theta + (t^2-t)\theta^2}\right)$$

However I should recall that in the general case I should use the path

$$\tilde{A} \longmapsto \bar{A} = \tilde{A} + \theta$$

which gives the form

$$\int_0^1 dt \ \mathrm{tr}\left(\theta \, e^{tF_{\tilde{A}} + (1-t)F_{\bar{A}} + (t^2-t)\theta^2}\right)$$

The components of sufficiently high degree should be closed, because $F_{\tilde{A}}$ has components of type 1,1 and 0,2 in general, whereas $F_{\bar{A}}$ is of type 0,2 when restricted to a $\mathcal{G}$-orbit.

Notes on conversation with Luis + Ginzparg.

They prove the anomaly formula by mapping a circle $S^1 \longrightarrow \mathcal{G}$ orbit, then contracting it to a point in $\mathcal{A}$, whence one has an $S^2 \longrightarrow \mathcal{A}/\mathcal{G}$. One gets a 2-parameter family of connections $A(t,\theta)$ and they form the 6-dimensional Dirac operator on $S^2 \times S^4$:

$$D^{(2)} + D^{(4)}.$$

[redacted] The thing to be proved is that $\int_{S^2} c_1(\mathcal{L})$ is the index of the 6-diml operator, because one knows already [redacted] by diffl form calculations that $\int_{S^4} ch_3$ leads to the known anomaly formula. (Actually they prefer to work with the winding number $\int_{S^1}\int_{S^4}$ Chern-Simons [redacted] which is the change in phase of the determinant as one goes around the circle.)

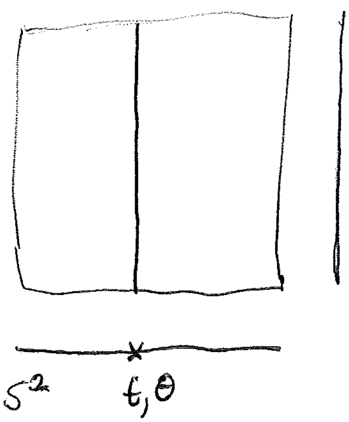
So the problem is to relate the winding number of the phase of the determinant with the index of the 6-diml Dirac operator. They use the Born-Oppenheimer approximation

$$\not{D}_\varepsilon = \text{[redacted]}\, D^{(2)} + \frac{1}{\varepsilon} D^{(4)} \qquad \text{as} \qquad \varepsilon \to 0$$

and the index formula [redacted] on the 6-diml. manifold

$$\text{Index} = \mathrm{Tr}\left(\gamma_5 e^{+\not{D}_\varepsilon^2}\right) = \int ch_3 .$$

The point is that as $\varepsilon \to 0$ the zero modes of $\not{D}_\varepsilon$ are going to concentrate at those t, θ where $D^{(4)}_{t,\theta}$ has a zero mode. We get the picture of a zero mode of $D^{(4)}_{t,\theta}$ running vertically times a very sharply peaked Gaussian in the normal direction. In fact this is what the B-O approx is about. – Recall Coleman's lectures.



Somehow working in this approximation around one of these 'critical' points, one has

$$\not{D}_\varepsilon^2 = \text{[redacted]} -\left(\partial_t^2 + \partial_\theta^2\right) + t^2 + \theta^2 + (\text{linear?})$$

and a careful analysis shows that the zero modes of $D^{(4)}$ at this point contribute to the winding number according to their chirality. (Part of this analysis involves projecting onto the zero modes of the vertical operator.)

December 8, 1983

Let's consider the problems of describing anomalies using the theory of cyclic cohomology. Anomalies are related to index problems. In fact an anomoly is the first Chern class of the index of a family of elliptic operators. Connes relates the index to cyclic cocycles, so it should be possible to relate anomalies to cyclic cohomology.

Let's recall that Connes really looks at the K-homology class of an elliptic operator. He considers the operation of tensoring the operator with a vector bundle, then taking the index. In this way a map $K(M) \longrightarrow \mathbb{Z} = K(pt)$ is obtained which is equivalent to the K-homology class of the operator.

In order to fix the ideas let us start with a Dirac operator $\not{\partial} : \mathcal{H}^+ \longrightarrow \mathcal{H}^-$ over M. If A is a $U(N)$-connection on M, then we can define

$$\not{\partial}_A : \mathcal{H}^+ \otimes \mathbb{C}^N \longrightarrow \mathcal{H}^- \otimes \mathbb{C}^N$$

thereby obtaining a family of elliptic operators over M parametrized by $A \in \mathcal{A}$, equivariant with respect to $\mathcal{G} = \mathrm{Maps}(M, U(N))$. The index of this family is an element of $K^0(\mathcal{A}/\mathcal{G})$, hence its character is an even class in $H^*(\mathcal{A}/\mathcal{G})$. The transgression of this class is an odd class in $H^*(\mathcal{G})$.

I know formulas for $ch(\mathrm{ind}\,\not{\partial}_A) \in H^{ev}(\mathcal{A}/\mathcal{G})$

and its trangression in $H^{odd}(\mathcal{G})$ as diffl forms. Moreover the trangression class

$$e_{2n-1}(\mathrm{ind}) \in H^{2n-1}(\mathcal{G})$$

is realized by an explicit differential form which is (left or right?) invariant provided $n > \dim M$.

On the other hand if the original Dirac operator $\not{D}$ is invertible, then the cyclic cocycles on $C^\infty(M)$ which Connes associates to $\not{D}$ represent invariant differential forms on $\mathcal{G}$. The conjecture is therefore that the cyclic cocycles coincide with the invariant differential forms

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<u>Dec. 11:</u> Witten + Alvarez obtain $\hat{A}$-genus by means of a constant EM field. Is this natural from the Kaluza-Klein viewpoint?

December 11, 1983

Fermion quantum mechanics. Evidently it is possible to do such a theory at least for quadratic Lagrangians. The theory is analogous to what are obtains for ordinary quadratic Lagrangians, namely instead of a path in the meta-plectic group one gets a path in the spinor group. I think I have at one time understood the operator theory. What I want to get an understanding of now is the Lagrangian side, in particular, critical points of the action.

Let's start with the operator formulas in the simplest case, namely, one fermion. Then we have a 2-diml Hilbert space with creation and annihilation operators a^*, a satisfying the comm. relations

$$\{a,a\} = \{a^*,a^*\} = 0 \qquad \{a,a^*\} = 1.$$

The Hamiltonian is $H = \omega a^* a$ + const, so the Green's function is

$$G(t) = \langle 0 | T[a(t)\, a^*] | 0 \rangle.$$

Here $a(t) = e^{iHt} a\, e^{-iHt}$ and

$$[H,a] = [\omega a^* a, a] = -\omega\{a^*,a\}a = -\omega a$$

$$a(t) = e^{-i\omega t} a.$$

Thus

$$G(t) = e^{-i\omega t}\begin{cases} \langle 0 | a a^* | 0 \rangle & t>0 \\ -\langle 0 | a^* a | 0 \rangle & t<0 \end{cases}$$

$$= e^{-i\omega t}\,\theta(t)$$

and this is the Green's function for

$$\partial_t + i\omega .$$

Hence the sort of functional integral that gives rise to these Green's functions is

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{iS} \qquad iS = -\int \bar{\psi}(t)\left[\partial_t + i\omega\right]\psi(t)\, dt$$

$$= -\int \bar{\psi}\left[\frac{\partial}{\partial(it)} + \omega\right]\psi(t)\; i\,dt .$$

So in imaginary time $\tau = it$ we get the Euclidean action

$$\int \bar{\psi}\left[\frac{\partial}{\partial\tau} + \omega\right]\psi \; d\tau .$$

We'll stick to real time where the action is

$$S = \int \bar{\psi}(t)\left[i\partial_t - \omega\right]\psi(t)\, dt$$

and hence the Lagrangian is

$$L = \bar{\psi}\left(i\partial_t - \omega\right)\psi$$

At this point I somehow have to make sense of the idea of expanding around the classical solution. It is probably useful to do the boson case first.

December 13, 1983

Lot's problem: Over $\mathcal{A} \times \mathcal{A}$ we consider the family of elliptic operators

$$\not{D}_{A^+, A^-} = \begin{pmatrix} 0 & -\not{\partial}_{A^-}^{\ *} \\ \not{\partial}_{A^+} & 0 \end{pmatrix} \qquad \text{on } S \otimes E$$

This is just the direct sum of the families $\not{\partial}_{A^+}$ and $\not{\partial}_{A^-}^{*}$ as (A^+, A^-) varies over $\mathcal{A} \times \mathcal{A}$. It's clear that the determinant line bundle of this family is

$$pr_1^*(\mathcal{L}) \otimes pr_2^*(\mathcal{L}^{\text{dual}})$$

where $\mathcal{L}$ is the determinant line bundle [redacted] of the family $\not{\partial}_{\mathcal{A}}$. The first Chern class is

$$pr_1^*(c_1 \operatorname{Ind} \not{\partial}_{\mathcal{A}}) - pr_2^*(c_1 \operatorname{Ind} \not{\partial}_{\mathcal{A}}) \in H^2(\mathcal{A}/\mathcal{G} \times \mathcal{A}/\mathcal{G}).$$

which [redacted] is related by transgression to

$$(*) \qquad pr_1^*(t_1) - pr_2^*(t_1) \in H^1(\mathcal{G} \times \mathcal{G}).$$

Here $t_1 \in H^1(\mathcal{G})$ is the class corresponding to $c_1(\operatorname{Ind} [redacted] \not{\partial}_{\mathcal{A}})$.

Now $(*)$ [redacted] vanishes on the diagonal $\mathcal{G} \xrightarrow{\Delta} \mathcal{G} \times \mathcal{G}$ so if $(*)$ is computed as a differential form, this form is exact. Lot wants to know if it is given by a "local" formula, I think.

I haven't yet got straight his chiral transformations, so let's be more specific. Take a loop $\theta \mapsto g_\theta$, $S^1 \to \mathcal{G}$. Fix $(A_0^+, A_0^-) \in \mathcal{A} \times \mathcal{A}$ and act by the loop in a chiral fashion to get a loop in $\mathcal{A}\times\mathcal{A}$. This means that on the operators we have

$$
\begin{aligned}
D\!\!\!/_{A_\theta^+,A_\theta^-} &= g_\theta^{\gamma_5}\, D\!\!\!/_{A_0^+,A_0^-}\, g_\theta^{\gamma_5} \\
&= \begin{pmatrix} g_\theta & \\ & g_\theta^{-1}\end{pmatrix}\begin{pmatrix} & -\partial\!\!\!/^{*}_{A_0^-} \\ \partial\!\!\!/_{A_0^+} & \end{pmatrix}\begin{pmatrix} g_\theta & \\ & g_\theta^{-1}\end{pmatrix} \\
&= \begin{pmatrix} & -\partial\!\!\!/^{*}_{g_\theta^{-1*}A_0^-} \\ \partial\!\!\!/_{g_\theta^{*}A_0^+} & \end{pmatrix} = D\!\!\!/_{g_\theta^{*}A_0^+,\, g_\theta^{-1*}A_0^-}
\end{aligned}
$$

(Clearer is:

$$
\begin{aligned}
g_\theta\left(-\partial\!\!\!/^{*}_{A_0}\right)g_\theta^{-1} &= g_\theta\,\gamma^\mu(\partial_\mu + A_{0,\mu})\left(\frac{1-\gamma_5}{2}\right)g_\theta^{-1} \\
&= \gamma^\mu(\partial_\mu + (g_\theta^{-1}{*}A_0)_\mu)\left(\frac{1-\gamma_5}{2}\right) = -\partial\!\!\!/^{*}_{g_\theta^{-1*}A_0}\,.\Big)
\end{aligned}
$$

Therefore the loop g_θ in $\mathcal{G}$ becomes the loop $(g_\theta^{*}A_0^+,\, g_\theta^{-1*}A_0^-)$ in $\mathcal{A}\times\mathcal{A}$. This means we are using the map $g\longmapsto(g,g^{-1})$ from $\mathcal{G}$ to $\mathcal{G}\times\mathcal{G}$ not the diagonal. So he must have something slightly different than the determinant line bundle of the family $D\!\!\!/_{A^+,A^-}$ in mind.

~~[illegible]~~ He looks at the analytical expression

$$
\left\{ \mathrm{Tr}\left(\gamma_5 e^{tD\!\!\!/^2_{A^+,A^-}}\right) + \mathrm{Tr}\left(\gamma_5 e^{tD\!\!\!/^2_{A^-,A^+}}\right)\right\}_{\text{coeff of } t^0}
$$

which over the loop $(g_\theta^{*}A_0^+,\, g_\theta^{-1*}A_0^-) = (A_\theta^+, A_\theta^-)$ he can relate to some winding number.

??

December 19, 1983

Fujikawa's approach to anomalies.

1) $$e^{-\Gamma(A)} = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, e^{-\int \bar{\psi} D_A \psi}$$

$S(A) = \int \bar{\psi} D_A \psi$ is the action

$\Gamma(A)$ is the effective action.

The action is invariant under gauge transf.

2) $$\psi \longrightarrow g\psi \qquad \bar{\psi} \longrightarrow \bar{\psi} g^{-1}$$

in the sense that

$$S(A) = \int \bar{\psi} D_A \psi \longrightarrow \int \bar{\psi} g^{-1} D_A g \psi = S(g^* A).$$

If the measure $\mathcal{D}\bar{\psi}\, \mathcal{D}\psi$ were unchanged under the gauge transformation then we could conclude

$$\Gamma(g^* A) = \Gamma(A),$$

so the effective action is gauge-invariant.

Take an infinitesimal gauge transf.

3) $$\psi \longrightarrow \psi + v\psi \qquad \bar{\psi} \longrightarrow \bar{\psi} - \bar{\psi} v$$

and change variables in 1) accordingly

$$e^{-\Gamma(A)} = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \frac{\mathcal{D}(\bar{\psi} - \bar{\psi} v)}{\mathcal{D}\bar{\psi}}\, \frac{\mathcal{D}(\psi + v\psi)}{\mathcal{D}\psi}\, e^{-\int \bar{\psi} \underbrace{\left(D_A + [D_A, v]\right)}_{D_{A + v * A}} \psi}$$

Notice that one is making a <u>linear</u> change of variable in $\psi, \bar{\psi}$ hence the Jacobian is a <u>constant</u> fn. of $\psi, \bar{\psi}$. In fact it is the determinant of the linear transf. Thus

$$\frac{\mathcal{D}(\psi + v\psi)}{\mathcal{D}\psi} = \det(1 + v \text{ on } \psi\text{-space}) = 1 + \mathrm{tr}(v)$$

$$\frac{\mathcal{D}(\bar\psi - \bar\psi v)}{\mathcal{D}(\bar\psi)} = \det(1 - v \text{ on } \bar\psi\text{-space}) = 1 - \mathrm{tr}(\bar v)$$

where $\mathrm{tr}(\bar v)$ denotes the trace of $\bar\psi \to \bar\psi v$. So we get

$$e^{-\Gamma(A)} = e^{-\Gamma(A + v * A)} \left(1 - \mathrm{tr}\,\bar v \text{ [redacted] } + \mathrm{tr}\, v\right)$$

or

$$\Gamma(A + v*A) - \Gamma(A) = \mathrm{tr}\, v - \mathrm{tr}\, \bar v$$

Hence we see that when $D_A = \not{D}_A\left(\frac{1+\gamma_5}{2}\right)$ that the anomaly is formally $\mathrm{tr}(\gamma_5 v)$.

Now take $D_A = i\not{D}_A$ and consider [redacted] a chiral gauge transformation

$$\psi \longrightarrow g^{\gamma_5}\psi \qquad \bar\psi \longrightarrow \bar\psi g^{\gamma_5}$$

$$\psi \longrightarrow \psi + v\gamma_5\psi \qquad \bar\psi \longrightarrow \bar\psi + \bar\psi v \gamma_5$$

Since γ_5 anti-commutes with $\not{D}_A$ this preserves the action in some sense. But the two halves of $\not{D}_A$ transform differently so the family of operators $\not{D}_A$ is not closed under this chiral gauge transformation.

For the moment let us take a $\not{D}_A$ and calculate the change in the the functional integral. Later we can deal with $\not{D}_{A^+,A^-}$. The functional integral is

$$\int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, e^{-\int \bar\psi\, i\not{D}_A \psi} = \det(i\not{D}_A)$$

$$\det\left(e^{v\gamma_5} i \not{D}_A e^{v\gamma_5}\right) = \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\, \frac{\mathcal{D}(\bar\psi e^{-v\gamma_5})}{\mathcal{D}\bar\psi}\frac{\mathcal{D}(e^{-v\gamma_5}\psi)}{\mathcal{D}\psi} e^{-\int \bar\psi i \not{D}_A \psi}$$

$$\det\left(i\left(\not{D}_A + \{v\gamma_5, \not{D}_A\}\right)\right)$$

so that $\quad \delta^{\log}_{v_A} \det(i \not{D}_A) = -\operatorname{tr}(\overline{\gamma_5 v}) - \operatorname{tr}(\gamma_5 v)$

$$= -2\operatorname{tr}(\gamma_5 v)$$

But now we can compare this with what we obtain if $\det(i \not{D}_A)$ is defined using the ζ-function. Now $i\not{D}_A$ is self-adjoint and anti-commutes with γ_5 so that its eigenvalues occur in pairs $\pm\lambda_n$. Then $\Delta = -\not{D}_A^2$ has eigenvalues λ_n^2 of multiplicity 2. So we can define

$$\log \det(i D_A) = -\tfrac{1}{2}\zeta'_\Delta(0).$$

where $\quad \zeta_\Delta(s) = \operatorname{Tr}(\Delta^{-s}).$

Then we have

$$\delta_v \zeta_\Delta(s) = -s \operatorname{Tr}(\Delta^{-s-1}\, \delta_v \Delta)$$

$$= -s \operatorname{Tr}(\Delta^{-s-1}(D\,\delta_v D + \delta_v D\, D)) \qquad D = i\not{D}_A$$

$$= -2s \operatorname{Tr}(\Delta^{-s} D^{-1} \delta_v D) \qquad \delta_v D = \{v\gamma_5, D\}$$

$$= -4s \operatorname{Tr}(\Delta^{-s} v \gamma_5)$$

$$-\tfrac{1}{2}\delta_v \zeta_\Delta(s) = 2s \operatorname{Tr}(\Delta^{-s} v \gamma_5)$$

$$= 2s \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-t\Delta} v \gamma_5)\, t^s \frac{dt}{t}$$

Now we know the structure of the kernel of $e^{-t\Delta}$ along the diagonal. It has an asymptotic expansion in powers of t and no $\log t$ term. Since v is a multiplication operator

<u>December 15, 1983</u>

There is too much confusion connected with this chiral business. The point is that for $(A^+, A^-) \in \mathcal{A} \times \mathcal{A}$ we have the operator

$$i \not{D}_{A^+, A^-} = \begin{pmatrix} 0 & \partial_{A^-}^* \\ \partial_{A^+} & 0 \end{pmatrix}$$

for which we can define a determinant using the ζ-method. We have an action of $\mathcal{G} \times \mathcal{G}$ on $\mathcal{A} \times \mathcal{A}$ and the determinant is invariant under the action of $\Delta \mathcal{G} \subset \mathcal{G} \times \mathcal{G}$. From the topological viewpoint we have the determinant line bundle of the above family of operators over $\mathcal{A}/\mathcal{G} \times \mathcal{A}/\mathcal{G}$, which is $pr_1^*(\mathcal{L}) \otimes pr_2^*(\mathcal{L}^{\text{dual}})$, where $\mathcal{L}$ = det line bundle of $\{\partial_A\}$ over $\mathcal{A}/\mathcal{G}$. So we have the class [redacted]

$$c_1(\mathcal{L}) \otimes 1 - 1 \otimes c_1(\mathcal{L}) \quad \in \quad H^2(\mathcal{A}/\mathcal{G} \times \mathcal{A}/\mathcal{G})$$

which can be pulled back to $H^1(\mathcal{G} \times \mathcal{G})$, and which vanishes on $\Delta \mathcal{G}$.

December 15, 1983

Witten's table:

color	one flavor is the repn.	flavor symm. gp.	unbroken symmetry group = flavor symm. of a generic vacuum
$SU(N)$	$N+\bar{N}$	$SU(F)\times SU(F)$	$SU(F)_{diag}$
$O(N)$	N	$SU(F)$	$O(F)$
$Sp(N)$	$(2n)+(2n)$	$SU(2F)$	$Sp(F)$

multiplicities here are needed to kill anomalies

space of vacuum states	stable homotopy of G/H: π_2	π_3	F large: π_4	π_5
$SU(F)^2/SU(F)_{diag} \simeq SU(F)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$SU(F)/O(F)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$
$SU(2F)/Sp(F)$	0	0	0	$\mathbb{Z}$
physical meaning	confinement of certain color charges	existence of baryon number	ambiguity in quantizing the σ-model	number of colors in the underlying theory is an integer

t'Hooft showed that QCD as $N \to \infty$ becomes equivalent to an effective theory of mesons with a complicated configuration space P. Witten would like to relate the topology of P with the physics.

Now the low energy limit of QCD is supposed to be the σ-model which has the configurations $Map(S^4, G/H)$.

Witten believes that P has the homotopy type of Map$(S^4, G/H)$ in ~~the~~ stable range where F the number of flavors goes to ∞. P exists in any case, but its topology is probably not relevant for small F. For instance, if $F=0$ so we have a pure gauge theory, then [redacted] Witten thinks P is contractible.

(Put another way [redacted] the σ-model makes no reference to quarks and color, so when you say QCD has a low[redacted] energy limit given by the σ-model to have to get rid of both quarks and the intermediate energy. Now what is the low energy situation going to capture the topology? When the higher critical points have lots of ways to decay (i.e. high index in the sense of Morse theory.) Thus you need to have many [redacted] flavors.)

Let me try to explore the [redacted] difference between colors and flavors a bit further. The color symmetry is gauged. So are is really working on a vector bundle of dim ?

In the large N limit QCD becomes an effective theory of mesons with a complicated configuration space P.
In the large F limit P should have the same topology as [redacted] the space Map$(S^4, G/H)$ of configurations of the σ-model

<u>December 16, 1983</u>

Facts about QCD. There are colors and flavors, and both are symmetries, i.e. have groups. But color is gauged and flavor is not, i.e. color is a local symmetry and flavor is a global symmetry.

For historical reasons flavor came first. One first has isospin $SU(2)$ to describe the nucleon and pions. This was enlarged to $SU(3)$ by the "eightfold way" to describe the ~~pion~~ octet and other multiplets. But the fundamental repn. of $SU(3)$ did not occur in the particle spectrum, and so quarks were invented. In trying to make a consistent theory of "quarkonium" i.e. combinations of quarks, one was led to introduce a further degeneracy, namely color.

From the flavor viewpoint there are three kinds of quarks: up, down, strange. These are Dirac spinors, hence each has 4 components. If the quarks are mass-less, then each quark splits into quarks of definite chirality, namely left and right-handed quarks. Each of these has 2 spins, which gives the four components.

So the massless theory of the up, down, and strange quarks admits $SU(3)_L \times SU(3)_R$ as global symmetry group. It is believed that at low energies this symmetry is broken to $SU(3)_{diag}$. The Goldstone bosons are the three pions, and this would explain their relative lightness. Also the effective low energy theory would be described by the σ-model. This is the theory called current algebra.

Notice that color doesn't appear anywhere in the above. Hence my old idea that the σ-model should occur in a gauge theory at low energy seems to be completely wrong.

Let's go over next what Witten had to say about the physical interpretation of the stable homotopy groups.

Lets start with π_4. We are considering the σ-model for which the configurations are the maps S^4 to G/H, say basepoint-preserving. This space of maps has the components $\pi_4(G/H)$. If we take a character α of $\pi_4(G/H)$, then we may modify the path integral by α

$$\int e^{-\text{Action}} \cdot \alpha$$

and get a different quantization. Thus $\pi_4(G/H)$ is related to the ambiguity of quantizing the σ-model.

Another remark: In QCD the path integral is taken over all connection and all $SU(N)$-bundles. Hence it is an integral over $\mathrm{Map}(M, BG)$. The fact that $\pi_0(\mathrm{Map}(S^4, BG)) = \pi_4 BG = \pi_3 G = \mathbb{Z}$ for $G = SU(N)$ explains the different Θ-vacua of QCD.

Further remark: There are two ways to quantize the rigid rotor, which is due to $\pi_1(O(3)) = \mathbb{Z}_2$. The two ways correspond to treating it as a fermion or boson. See Landau-Lifschitz.

Next π_5. General remark. In a path integral

$$\int e^{-\text{Action}}$$

the action A isn't necessarily real. The physical requirement is that $A \to A^*$ under reversal of the orientation of S^4. This is the CPT thm.

Thus the parity-preserving parts of A are real
the parity-reversing parts of A are imaginary.

Also e^{-A} is needed, so A can be defined modulo $2\pi i\mathbb{Z}$. Example of Dirac monopole, Witten's version of Wess-Zumino where the number of colors is the coefficient of the Wess-Zumino term + ∴ an integer.

[redacted] Question: What does the σ-model, which [redacted] is a low energy approximation to QCD (supposedly), remember of the QCD that it is supposed to come from. [redacted] Answer: The number of colors appears as the coefficient of the Wess-Zumino term.

Notice that only in the large F limit do we expect to say things about topology for QCD + the σ-model.

Next: $\pi_3(G/H)$. This time we are interested in the possible [redacted] physical states. Space-time is now Minkowskian $M = \underset{\text{time}}{\mathbb{R}} \times S^3$, and so our configurations are maps $(S^3, G/H)$. The topological classification is $\pi_3(G/H)$ – states evolve in time but stay in the same homotopy class.

This splitting of classical configurations is the counterpart of quantum number sectors of the

Hilbert space. One sees this result by comparing SU(N) and O(N):

$$\left.\begin{array}{ll} \text{meson} & q^i \bar{q}_i \\ \text{baryon} & \epsilon_{i_1 \dots i_N} q^{i_1} \dots q^{i_N} \end{array}\right\} SU(N)$$

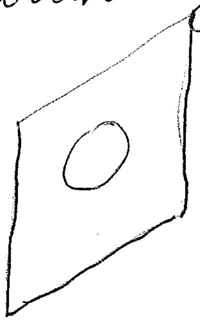
$$\left.\begin{array}{ll} \text{meson} & q^i q^i \\ \text{baryon} & \epsilon_{i_1 \dots i_N} q^{i_1} \dots q^{i_N} \end{array}\right\} O(N)$$

group-theoretically

$$\epsilon_{i_1 \cdots i_N} \epsilon_{j_1 \cdots j_N} = \sum_{\pi \in \Sigma_N} \delta_{i_1 j_{\pi_1}} \cdots \delta_{i_N j_{\pi_N}}$$

so that $\underline{2 \text{ baryons} = N \text{ mesons}}$ in the O(N)-theory.

Finally π_2. Here this has to do with where the equations break down, i.e. impurities. Somehow Witten sees the impurity as giving boundary conditions for QCD. Classically these



t=0 boundary conditions are classified topologically by $\pi_2(G/H)$. The only interesting impurities should transform under a repn. of the gauge gp not included in the repns. generated by quarks + gluons. Not possible for SU(N) since N $\bar{N}$ generate everything, but for O(N) one has spin representation.

Somehow this is related to confinement.

December 17, 1983

Back to the S operator.

Let $\mathfrak{g} \longrightarrow gl_N(A)$ be a Lie algebra homom. Consider the double complex $C^*(\mathfrak{g}, \Omega^*)$ where $\Omega^* = \tilde{\Omega}^*(A^+)$ is the non-commutative DR complex of A. Then the homomorphism is an element

$$\theta \in C^1(\mathfrak{g}, \Omega^0) \otimes gl_N$$

such that $\delta\theta + \theta^2 = 0$. We have the formula

$$tr\left(e^{d\theta}\right) - tr(1) = d\int_0^1 dt \; tr\left(\theta e^{t d\theta + (t^2-t)\theta^2}\right).$$

This enables us to define cocycles

$$u_n = \int_0^1 dt \; tr\left(\theta \frac{[t d\theta + (t^2-t)\theta^2]^{n-1}}{(n-1)!}\right) \in C^{2n-1}(\mathfrak{g}, \Omega^{<n}).$$

Let's start again. To any representation $\rho : \mathfrak{g} \rightarrow gl_N(A)$ of the Lie algebra $\mathfrak{g}$ over A we have associated a cocycle

$$u_{n-1}(\rho) \in C^{2n-1}(\mathfrak{g}, \Omega_A^{<n})$$

as above. This construction is natural and stable so $u_{n-1}(\rho) = \rho^*$ of a universal u_{n-1} which lies in

$$C^{2n-1}\left(gl(A), \Omega_A^{<n}\right) = \mathrm{Hom}\left(\Lambda^{(2n-1)} gl(A), \Omega_A^{<n}\right)$$

From the form of u_{n-1} - it is $tr(\theta \cdots d\theta \cdots)$ we know it is primitive and gl_N-invariant, so comes from an element of

$$\mathrm{Hom}^{(0)}\left(C.(A), \Omega_A^{<n}[-2n+1] \right).$$

One of the first things I could prove is that the diagram

$$\begin{tikzcd} C.(A) \arrow[r, "u_{n-1}"] & \Omega^{<n}[-2n+1] \\ C.(A)[2] \arrow[u, "S"] \arrow[r, "u_n"] & \Omega^{<n+1}[-2n-1][2] \arrow[u, "\text{truncate}"'] \end{tikzcd}$$

commutes. I think this follows rather easily from the formulas for the u_n, and the formula for the S-operator.

More to the point, however, would be to achieve a better understanding of the S-operator. Connes defines it using the homomorphism

$$A^+ \longrightarrow A^+ \otimes ke^+$$

which I find mysterious. (I think most of the [redacted] mystery is due to the fact that you are not used to working in a "locally compact" framework).

We know that $\tilde{\Omega}(ke^+)$ occurs when we take matrix forms. More precisely if e is an idempotent matrix of smooth functions on M, then [redacted] we get a homomorphism

$$\tilde{\Omega}(ke^+) \longrightarrow \Omega(M) \otimes gl_N$$

It might be possible to work with matrix forms instead of non-commutative differential forms. In this way one might preserve the geometric intuition. $\blacksquare$

Let's try to understand the map

$$A^+ \longrightarrow A^+ \otimes ke^+ \qquad a \longmapsto ae$$

on the level of cyclic homology. We are going to think about $\tilde{\Omega}(ke^+)$ as having to do with matrix forms on the Grassmannian. The idea will be to take Connes map

$$\tilde{\Omega}(A^+) \longrightarrow \tilde{\Omega}(A^+) \otimes \tilde{\Omega}(ke^+) \longrightarrow \Omega(A) \otimes \Omega(\text{Grass})$$

and to see if I can interpret the result on cyclic homology geometrically. Now how is this to work?

December 18, 1983

Project: To prove the local index thm. in Euclidean space (or say on a torus) with an arbitrary gauge potential, using path integrals.

The idea will be to start with

$$\text{Index} = \operatorname{Tr}\left(\varepsilon\, e^{-t(i\not{D})^2}\right)$$

express this as a path integral probably involving fermions, then evaluate this path integral in the $t \to 0$ limit by means of a quadratic approximation.

It is possible that in order to get the fermions into the picture correctly we want to use the Kaluza-Klein idea. This means that we look at the Dirac operator on $M \times K$ where the gauge group acts on K and K is positively curved of small radius. In practice for the gauge group $U(N)$ one would take a flag manifold such that the Dirac operator would give back the standard representation on the zero modes.

I am begining to think I ought to understand this whole situation somewhat better, both in order to understand Kaluza-Klein and Alvarez-Gaumé - Ginsparg.

So let's go back to our old notation $X = Y \times M$ where Y is Euclidean space, in this case Euclidean space-time. I consider the Dirac operator in question over $Y \times M$. Then I want to rescale the directions so that M becomes tightly curved.

Actually an interesting point is the following. [redacted] We are really starting with the Dirac operator on Y with coefficients in a vector bundle E over Y. Then our X is the flag manifold of E or at least is very close to it. [redacted] In fact just take X to be the flag manifold and forget the idea that $X =$ [redacted] $Y \times M$ with $U(N)$ acting on M. Now it is clear that the index theorem for E over Y becomes the index thm. over X of some bundle. At this point I should get the K-theory straight to see, namely, if the Dirac operator on a flag manifold gives for its zero modes the standard representation.

On a Kähler manifold there is a close relation between the ∂-operator and Dirac operator. Let's review the ideas.

Given a Riemannian manifold with tangent bundle T one forms the Clifford algebra bundle $C(T)$. Given a vector bundle E [redacted] which is a $C(T)$-module one then has operators γ^μ defined for any orth. frame [redacted] e_μ on the manifold. [redacted] If one is also given a connection $D = D_\mu \omega^\mu$ where ω^μ is the dual frame, then we get a Dirac operator $\gamma^\mu D_\mu$. In practice one wants control over the way D_μ and γ^ν don't commute. So what one does is to take E to be associated to the tangent bundle.

Specifically a spin structure on M is a lifting of the $O(n)$-frame bundle P to a [redacted] $\mathrm{Spin}(n)$-bundle $\tilde{P}$. The Riemann-connection on M induces one in $\tilde{P}$ and hence any associated bundle. The Dirac operator on M is the operator associated to the spin representation.

In the case of an almost-complex structure J, $J^2 = -I$, on M the structural group is reduced from $O(n)$ to $U(\frac{n}{2})$ and there is a unique lifting

$$\begin{tikzcd} & \mathrm{Spin}(n) \arrow[d] \\ U(\frac{n}{2}) \arrow[ur, dashed] \arrow[r, phantom, "\subset"] & O(n). \end{tikzcd}$$

n even

NO see below

The spin representation of $\mathrm{Spin}(n)$ becomes a definite representation of $U(\frac{n}{2})$. I know it has to be the exterior algebra of the standard representation up to a character.

If we take $n=2$, then we expect the characters $z \longmapsto z, z^{-1}$, not $z \longmapsto 1, z$[illegible] which is what one gets for $\Lambda \mathbb{C}$???

You've got Spin and Spin_c mixed up.

Digression on the spinor representation in $2n$ dimensions. The Clifford algebra C_{2n} over $\mathbb{C}$ is generated by $\gamma^1, \ldots, \gamma^{2n}$ of square $+1$. The elements $\gamma^i\gamma^j$ for $i<j$ span the Lie algebra of $SO(2n)$, since

$$[\gamma^i\gamma^j, \gamma^k\gamma^l] = 0 \qquad \text{if } i, j, k, l \text{ are distinct}$$
$$\text{or if the pair } i,j \text{ is the pair } kl$$

And if $i, j=k, l$ are distinct one has

$$[\gamma^i\gamma^j, \gamma^j\gamma^l] = [\gamma^i\gamma^j, \gamma^j]\gamma^l = \gamma^i\{\gamma^j, \gamma^j\}\gamma^l = 2\gamma^i\gamma^l$$

Identify $\gamma^i\gamma^j$ with the element of $O(2n)$ given by bracketing on the degree 1 operators $\gamma^1, \dots, \gamma^{2n}$. Then

$$[\gamma^i\gamma^j, \gamma^k] = \gamma^i\{\gamma^j,\gamma^k\} - \{\gamma^i,\gamma^k\}\gamma^j$$
$$= 2(\gamma^i\delta_{jk} - \delta_{ik}\gamma^j)$$

Now when a complex structure is given we know how to construct an irreducible C_{2n} module. It is the exterior algebra, it has operators a_i, a_i^* as usual. $\gamma_{2i-1}, \gamma_{2i}$ are the real elements of $\mathbb{C}a_i + \mathbb{C}a_i^*$. For $n=1$, the formulas are

$$a = \frac{1}{\sqrt{2}}(\gamma^1 + i\gamma^2)$$
$$a^* = \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2)$$

$$a^*a = \frac{1}{4}(\gamma^1 - i\gamma^2)(\gamma^1 + i\gamma^2) = \frac{1}{4}(1 - i\gamma^2\gamma^1 + i\gamma^1\gamma^2 + 1)$$
$$= \frac{1}{2}(1 + i\gamma^1\gamma^2) \qquad aa^* = \frac{1}{2}(1 - i\gamma^1\gamma^2)$$
$$a^*a + aa^* = 1.$$

So we have $\qquad \gamma^1\gamma^2 = -i\left(a^*a - \frac{1}{2}\right).$

Thus we see that the Lie algebra of $U(n)$ acting on the spinor representation will be given by (the skew-hermitian operators) in the subspace spanned by the operators

$$a_i a_j^* \qquad i \neq j$$
$$a_i^* a_i - \frac{1}{2}$$

In other words we have taking the standard representation of $gl(n)$ on the exterior algebra and have subtracted the character $\frac{1}{2}$ trace, which means that we are considering the representation

$$\lambda(V)^{-1/2} \otimes \Lambda(V)$$

of a certain double cover of $GL(V)$.

So let us now consider a Riemannian manifold M with an almost complex structure, i.e. an orthogonal J operating on T_M such that $J^2 = -I$. Then $T^* = T^{1,0} \oplus T^{0,1}$ where T^* denote complex covectors. In this case the structural group [redacted] of T_M has been reduced to $U(n)$ from $SO(2n)$. [redacted] We have just seen that Spin $(2n)$ induces a double covering of $U(n)$ which amounts to the same thing as taking the square root of the determinant:

$$\begin{tikzcd} \widetilde{U(n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{cart}"] & S^1 \arrow[d, "z \mapsto z^2"] \\ U(n) \arrow[r, "\det"] & S^1 \end{tikzcd}$$

Hence to give a reduction of $V = T_M$ with its $\mathbb{C}$ structure to the group $\widetilde{U(n)}$ means that we have to give a line bundle which is the square root of $\lambda(V)$. Notice that because of the inner product we can identify V with $T^{0,1}$ canonically. Hence the spinor bundle is

$$\lambda(T^{0,n})^{-1/2} \otimes \Lambda(T^{0,1})$$

Next we can do the relation between the index thm. for the Dirac operators and RR. RR says that the index of the Dirac operator (which is the index of the $\bar\partial$-complex for $K^{+1/2}$ where

K is the canonical line bundle) is given by

$$\int_M ch(K^{1/2})\ \mathrm{Todd}(T_M) \ .$$

Now $\quad ch(K^{1/2}) = e^{\frac{1}{2}c_1(K)} = e^{-\frac{1}{2}c_1(\lambda(T))} = e^{-\frac{1}{2}c_1(T_M)}$

so the characteristic class of M being integrated is the multiplicative one associated to the series

$$e^{-\frac{1}{2}x}\ \frac{x}{1-e^{-x}} = \frac{x}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}$$

which of course gives the $\hat{A}$-genus.

Consider complex projective space $\mathbb{C}P^n$. The tangent bundle is

$$T = \mathrm{Hom}\Big(\mathcal{O}(-1), \overbrace{\mathcal{O}\otimes\mathbb{C}^{n+1}}^{W}/\mathcal{O}(-1)\Big)$$

$$= \mathcal{O}(1)\otimes W/\mathcal{O}$$

hence $\quad c_1(T) = (n+1)\,c_1(\mathcal{O}(1))$. Thus $\mathbb{C}P^n$ doesn't have a Dirac operator unless n is odd. The highest exterior power of T is

$$\Lambda^n T = \mathcal{O}(n+1)\otimes(\Lambda^{n+1}W) = \mathcal{O}(n+1)$$

so the canonical bundle is

$$K = \mathcal{O}(-n-1).$$

so

$$K^{1/2} = \mathcal{O}\Big(-\frac{n+1}{2}\Big).$$

Now recall the cohomology of line bundles over projective space $\mathbb{C}P^n$.

$$H^i(\mathbb{C}P^n, \mathcal{O}(d)) = 0 \qquad \text{if } 0 < i < n$$
$$\text{or if } i = 0 \text{ and } d < 0$$
$$\text{or if } i = n \text{ and } d > -n-1$$

so $\quad H^*\Big(\mathbb{C}P^n, \mathcal{O}\big(-\frac{n+1}{2}\big)\Big) = 0.$

December 19, 1983

Rarita-Schwinger. This is the Dirac operator with coefficients in the tangent bundle, or really complexified tangent bundle $T\otimes\mathbb{C}$. When we compute ~~the~~ characteristic classes over $SO(2n)$ we really restrict to the maximal torus. This sits inside $U(n)$. Hence I can pretend T is the direct sum of n complex line bundles with Chern classes x_i, and we have

$$ch(T_{\mathbb{C}}) = \sum_{i=1}^{n} (e^{x_i} + e^{-x_i})$$

Suppose we consider a Kahler manifold and identify the Dirac operator with the operator $\bar\partial + \bar\partial^*$ for the holomorphic bundle $K^{1/2}$. In general suppose given a holom. bundle E with metric I can form the $\bar\partial$ complex

$$E \xrightarrow{\bar\partial} E\otimes T^{0,1} \longrightarrow \cdots \longrightarrow E\otimes T^{0,n}$$

and we have

$$\chi(M,E) = \sum(-1)^q \dim H^q(M, \mathcal{E}_{holom})$$

$$= \operatorname{Ind} \not{\partial} \text{ on Spinors} \otimes (K^{-1/2}\otimes E).$$

Equivalently

$$\chi(M, K^{1/2}\otimes E) = \operatorname{Ind} \not{\partial} \text{ on } (S\otimes E).$$

What does Serre duality say?

$$\chi(M, K\otimes E^*) = (-1)^n \chi(M,E)$$

or

$$\chi(M, K^{1/2}\otimes E^*) = (-1)^n \chi(M, K^{1/2}\otimes E)$$

$$\operatorname{Ind}(\not{\partial} \text{ on } S\otimes E^*) = (-1)^n \operatorname{Ind}(\not{\partial} \text{ on } S\otimes E)$$

Thus if n is odd, which is what Witten considers, then the index of $\not{D}$ on $S \otimes E$ will vanish when E is a holomorphic bundle such that E and E^* have the same character. ~~[illegible]~~ For the Dirac operator on $S \otimes T_{\mathbb{C}}$ we use that $T_{\mathbb{C}} \simeq T \oplus T^*$ as representations of $U(n)$, hence $\not{D}$ on $S \otimes T_{\mathbb{C}}$ will be the sum of $\not{D}$ on $S \otimes T$ and on $S \otimes T^*$. Thus the index of the Rarita-Schwinger operator on a Kähler manifold (of odd dim.) is zero.

Now I have to get started on a proof of the index theorem on a torus where there is no curvature. The idea will be to bring in fermion quantum mechanics. The index is a quantum mechanical expectation value which by its special nature is stable under deformations. Hence it can be evaluated in the classical limit. In the classical limit the fermion operators become something like differential forms.

December 21, 1983

Before we can understand the good proof of the index theorem, we must get straight the Clifford algebra and spinors. This is the fermion version of differential operators, perhaps pseudo-differential operators.

Let's adopt a picture of the Clifford algebra similar to differential operators (with polynomial coefficients). Thus we view the Clifford algebra as the endomorphisms of ΛV where V is a complex vector space. $C = \mathrm{Cliff}(V \oplus V^*)$ is generated by creation and annihilation operators a_i^*, a_i belonging to a basis of V. One has the isomorphism

$$C = \mathrm{Hom}(\Lambda V, \Lambda V) = \Lambda V \otimes (\Lambda V)^*$$

$$\cong \Lambda(V \oplus V^*).$$

One realizes the isomorphism as follows: An elt. of $\Lambda(V \oplus V^*)$ is a Grassman polynomial in the a_i^*, a_i. One interprets it as an operator on ΛV by normal ordering. This is completely analogous to interpreting a polynomial in the q's, p's as a differential operator with polynomial coefficients.

Composition in the Clifford algebra is given by a formula analogous to the composition of diff'l and pseudo-differential operators. Ultimately one uses something like Wick's theorem.

Recall some formulas. Let $A \in \mathrm{End}(V)$. Then A extends to the derivation

$$a_j^* A_{jk} a_k$$

of ΛV. Now I recall the formula

$$\boxed{\Lambda(A) = :e^{a_j^* A_{jk} a_k}:}$$

wrong see 344

because when the exponential is written out and normal-ordered one gets the minors of the matrix A_{jk}. For example in 1-dimension

$$:e^{\lambda a^* a}: = 1 + \lambda a^* a \neq \Lambda(\lambda)$$

is the map on $\Lambda \mathbb{C}$ induced by mult. by λ on $\mathbb{C}$. The above is to be contrasted with the automorphism of ΛV corresponding to the derivation $a_j^* A_{jk} a_k$, which is

$$\boxed{e^{a_j^* A_{jk} a_k} = \Lambda(e^A)}$$

Now the Clifford algebra has on it the reduced trace which gives the trace of the endom. on ΛV.

$$\mathrm{tr}\, \Lambda(A) = \det(1+A)$$

$$\mathrm{tr}_s(\Lambda(A)) = \mathrm{tr}(\Lambda^{ev}(A)) - \mathrm{tr}(\Lambda^{odd}(A))$$

$$= \sum (-1)^g \,\mathrm{tr}(\Lambda^g A) = \det(1-A)$$

Next what is the fermion integral?

$$\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \; e^{-\bar{\psi}_j A_{jk} \psi_k} = \text{const.} \det(A).$$

I am thinking of the fermion integral as selecting out the top component in $\Lambda(V \oplus V^*)$.

Getzler's viewpoint is to use the fact that there is a $O(2n)$-module isomorphism

$$\Lambda(\mathbb{R}^{2n}) \simeq C(\mathbb{R}^{2n})$$

which makes a basis elt $e_{i_1}\cdots e_{i_p}$ $i_1 < \cdots < i_p$ in the exterior algebra correspond to $\gamma^{i_1}\cdots\gamma^{i_p}$. Under this isomorphism projection onto $\Lambda^{2n}(\mathbb{R}^{2n})$ corresponds to $tr(\varepsilon\cdots) = tr_{S_+}(\cdots) - tr_{S_-}(\cdots)$, or really is proportional to this.

This seems strange in contrast with the above.

Let's adopt a more enlightened viewpoint. Suppose we construct the isomorphism $\Lambda(\mathbb{R}^{2n}) \simeq C(\mathbb{R}^{2n})$ in analogy with the Poincaré-Birkhoff-Witt isomorphism

$$S(\mathfrak{g}) \simeq U(\mathfrak{g}).$$

In fact it might be useful to start with the fermion analogue of the Heisenberg Lie algebra which is generated by q_j, p_j, h. The fermion analogue should be a super-Lie-algebra generated by γ^μ with $\{\gamma^\mu, \gamma^\nu\} = 2h\delta^{\mu\nu}$.

The PBW map is just symmetrization in the super sense, i.e.

$$\Lambda^p V \longrightarrow C(V)$$

$$v_1\wedge\cdots\wedge v_p \longmapsto \frac{1}{p!}\sum_{\sigma\in\Sigma_p} (-1)^\sigma v_{\sigma^{-1}1}\cdots\cdots v_{\sigma^{-1}p}$$

For this map one clearly has

$$e_{i_1}\wedge\cdots\wedge e_{i_p} \longmapsto \gamma_{i_1}\cdots\gamma_{i_p}.$$

So I learn that

$$C(V \oplus V^*) = \mathrm{Hom}(\Lambda V, \Lambda V) = \Lambda V \otimes (\Lambda V)^*$$

$$= \Lambda V \otimes \Lambda(V^*) = \Lambda(V \oplus V^*)$$

is not the good isomorphism. In fact the good isomorphism is given by an **anti-symmetrization** process. But maybe this has consequences for the way I have been thinking about fermion path integrals.

[redacted] Probably the way this has to be resolved goes as follows. If I really want the quantum mechanical trace, then I should **expect** to bring in the time coordinate. ??

Another more likely possibility is that the quantization process described for fermions corresponds to the Weyl calculus. I think Weyl's quantization corresponds to interpreting $f(p,q)$ as the corresponding symmetric operator, e.g.

$$pq \longmapsto \frac{pq+qp}{2}$$

In any case a problem arises, namely, how to interpret the fermion integrals.

December 22, 1983

Weyl quantization goes as follows. The problem is to associate an operator to each function $f(p,q)$ on phase space. Using the F.T. it is enough to assign an operator to each exponential function $e^{i(uq+vp)}$. But we have $uq+vp$ given as an operator; it is self-adjoint and so can be exponentiated.

I am used to doing the calculations in the holomorphic representation. This consists of holom. $f(z)$ with

$$\|f\|^2 = \int |f(z)|^2 e^{-|z|^2} \frac{d^2z}{\pi}$$

and $a = \frac{d}{dz}$, $a^* = z$. Then

$$T_\gamma = e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma}a^*} e^{\gamma a} \qquad \gamma \in \mathbb{C}$$

is the operator

$$(T_\gamma f)(z) = e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma}z} f(z+\gamma)$$

which is unitary. In fact because of the fmla

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \qquad \text{when } [A,B] \text{ commutes with } A,B,$$

we have that

$$e^{-\bar{\gamma}a^* + \gamma a} = e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma}a^*} e^{\gamma a} = T_\gamma .$$

So Weyl quantization takes a function $\varphi(a,a^*)$ writes it as a linear combination of the exponential fns. $e^{-\bar{\gamma}a^*+\gamma a}$, and then assigns to φ the operator where $\underbrace{\phantom{e^{-\bar{\gamma}a^*+\gamma a}}}$ is interpreted as T_γ.

Next I want the trace of an operator of the form

$$\int f(\gamma)\, T_\gamma \, \frac{d^2\gamma}{\pi} \qquad f \in C_0^\infty(\mathbb{C}).$$

So we compute the matrix element of T_γ

$$\langle m | T_\gamma | n \rangle \text{[redacted]} = \left\langle \frac{z^m}{\sqrt{m!}} \middle| T_\gamma \middle| \frac{z^n}{\sqrt{n!}} \right\rangle$$

A convenient generating fn. is

$$\sum \frac{t^m}{m!} \langle z^m | T_\gamma | z^n \rangle \frac{u^n}{n!} = \langle 0 | e^{ta} e^{-\frac{|\gamma|^2}{2}} e^{-\bar{\gamma} a^*} \times e^{\gamma a} e^{u a^*} | 0 \rangle$$

$$= e^{-\frac{|\gamma|^2}{2} - t\bar{\gamma} + tu + \gamma u}$$

In order to get the trace of T_γ I can put $u = \bar{t}$ and then integrate wrt $e^{-|t|^2}\frac{d^2 t}{\pi}$; this uses

$$\int t^m \bar{t}^n e^{-|t|^2} \frac{d^2 t}{\pi} = m!\, \delta_{mn}.$$

In fact what we are establishing is the general formula

$$\mathrm{Tr}(T) = \int \langle e^{\bar{t} z} | T | e^{\bar{t} z} \rangle e^{-|t|^2} \frac{d^2 t}{\pi}.$$

So

$$\mathrm{Tr}\left(\int f(\gamma) T_\gamma \frac{d^2\gamma}{\pi} \right) = \int \frac{d^2 t}{\pi} e^{-|t|^2} \int \frac{d^2\gamma}{\pi} \underbrace{\langle e^{\bar{t} z} | T_\gamma | e^{\bar{t} z} \rangle}_{e^{-\frac{|\gamma|^2}{2} - t\bar{\gamma} + \gamma\bar{t} + |t|^2}} f(\gamma)$$

$$= \int \frac{d^2 t}{\pi} \underbrace{\int \frac{d^2\gamma}{\pi} e^{\bar{t}\gamma - \text{[redacted]}\, t\bar{\gamma}} e^{-\frac{|\gamma|^2}{2}} f(\gamma)}_{\text{Fourier transform [redacted] of } e^{-\frac{|\gamma|^2}{2}} f(\gamma).}$$

So by Fourier inversion

$$\operatorname{Tr}\left(\int f(\gamma) T_\gamma \frac{d^2\gamma}{\pi} \right) = e^{-\frac{0^2}{2}} f(0) = f(0).$$

(Check constants: $\bar{t}\gamma - t\bar{\gamma} = (t'-it'')(\gamma'+i\gamma'') - \text{c.c.}$
$= 2i(t'\gamma''-t''\gamma')$. So

$$\int \frac{d^2(2t)}{4\pi^2} e^{2i(t'\gamma''-t''\gamma')} = \frac{(2\pi)^2\delta(\gamma')\,\delta(\gamma'')}{4\pi^2} = \delta(\gamma).\ )$$

Our next project will be to do the fermion version of the above. From the outset it will be necessary to use anti-commuting quantities for γ.

I want to begin by writing down the boson formulas very carefully. Let me denote by $\bar{\psi}$ and ψ the classical quantities corresponding to the quantum quantities a^*, a. Quantizing is therefore a process whereby one goes from a function $f(\psi)$ to an operator. I am going to write $f =$ $f(\bar{\psi}, \psi)$ because if one were to write the Taylor series of f one would need both variables $\bar{\psi}$ and ψ in the Taylor series. Then we define the Fourier transform by

$$F(\bar{\gamma},\gamma) = \int \frac{d^2\psi}{\pi} e^{-\bar{\gamma}\psi + \gamma\bar{\psi}} f(\bar{\psi},\psi)$$

$$f(\bar{\psi},\psi) = \int \frac{d^2\gamma}{\pi} e^{+\gamma\bar{\psi} - \bar{\gamma}\psi} F(\bar{\gamma},\gamma)$$

$$F(\gamma) = \int \frac{d^2\psi}{\pi}\, e^{\bar\gamma\bar\psi \;\; \gamma\psi} f(\psi)$$

$$f(\psi) = \int \frac{d^2\gamma}{\pi}\, e^{-\bar\gamma\bar\psi + \gamma\psi} F(\gamma)$$

When we quantize we have

$$f(\psi) \longmapsto \tilde f(a^*, a) = \int \frac{d^2\gamma}{\pi}\, e^{-\bar\gamma a^* + \gamma a} F(\gamma)$$

Now I want to see if essentially the same formulas work for the fermions. Note that

$$\frac{i}{2\pi}\, dz\, d\bar z = \frac{i}{2\pi}\left[dx\,(-i\,dy) + i\,dy \cdot dx\right] = \frac{d^2 z}{\pi}$$

and that in the formal theory I can expect to leave out the $\frac{i}{2\pi}$. Thus the formulas should be

$$F(\bar\gamma, \gamma) = \int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, e^{-\bar\gamma\bar\psi - \gamma\psi} f(\bar\psi, \psi)$$

$$f(\bar\psi, \psi) = \int \mathcal{D}\bar\gamma\, \mathcal{D}\gamma\, e^{+\bar\gamma\bar\psi + \gamma\psi} F(\bar\gamma, \gamma) \qquad \text{wrong sign}$$

and the ~~quantization~~ should be

$$f(\bar\psi, \psi) \longmapsto \int \mathcal{D}\bar\gamma\, \mathcal{D}\gamma\, e^{+\bar\gamma a^* + \gamma a} F(\bar\gamma, \gamma).$$

Let's check Fourier inversion. Compute F

$$F = \int \mathcal{D}\bar\psi\, \mathcal{D}\psi\, (1 - \bar\gamma\bar\psi)(1 - \gamma\psi) \begin{pmatrix} 1 \\ \psi \\ \bar\psi \\ \bar\psi\psi \end{pmatrix} = \int \mathcal{D}\bar\psi\, \mathcal{D}\psi \begin{pmatrix} \bar\gamma\bar\psi\gamma\psi \\ -\bar\gamma\bar\psi\psi \\ -\gamma\psi\bar\psi \\ \bar\psi\psi \end{pmatrix}$$

$$= \begin{pmatrix} -\bar\gamma\gamma \\ -\bar\gamma \\ \gamma \\ 1 \end{pmatrix}$$

Next compute the transform of F

$$\int D\bar\gamma\, D\gamma\, (1+\bar\gamma\bar\psi)(1+\gamma\psi)\begin{pmatrix}-\bar\gamma\gamma\\ -\bar\gamma\\ \gamma\\ 1\end{pmatrix} = \int D\bar\gamma\, D\gamma \begin{pmatrix}-\bar\gamma\gamma\\ -\gamma\psi\bar\gamma\\ \bar\gamma\bar\psi\gamma\\ \bar\gamma\bar\psi\gamma\psi\end{pmatrix} = \begin{pmatrix}-1\\ -\psi\\ -\bar\psi\\ -\bar\psi\psi\end{pmatrix}$$

So we get $-\mathrm{id}$.

Let's try the formulas you wanted to write down first

$$\boxed{\begin{aligned} F(\bar\gamma,\gamma) &= \int D\bar\psi\, D\psi\; e^{\bar\gamma\bar\psi-\gamma\psi} f(\bar\psi,\psi)\\ f(\bar\psi,\psi) &= \int D\bar\gamma\, D\gamma\; e^{-\bar\gamma\bar\psi+\gamma\psi} F(\bar\gamma,\gamma)\end{aligned}}$$

Then

$$F = \int D\bar\psi\, D\psi\, (1+\bar\gamma\bar\psi)(1-\gamma\psi)\begin{pmatrix}1\\ \psi\\ \bar\psi\\ \bar\psi\psi\end{pmatrix} = \int D\bar\psi\, D\psi \begin{pmatrix}-\bar\gamma\bar\psi\gamma\psi\\ \bar\gamma\bar\psi\psi\\ -\gamma\psi\bar\psi\\ \bar\psi\psi\end{pmatrix}$$

$$= \begin{pmatrix}\bar\gamma\gamma\\ \bar\gamma\\ \gamma\\ 1\end{pmatrix}$$

$$f = \int D\bar\gamma\, D\gamma\, (1+\bar\psi\bar\gamma)(1-\psi\gamma)\begin{pmatrix}\bar\gamma\gamma\\ \bar\gamma\\ \gamma\\ 1\end{pmatrix} = \int D\bar\gamma\, D\gamma \begin{pmatrix}\bar\gamma\gamma\\ -\psi\gamma\bar\gamma\\ +\bar\psi\bar\gamma\gamma\\ -\bar\psi\bar\gamma\psi\gamma\end{pmatrix} = \begin{pmatrix}1\\ \psi\\ \bar\psi\\ \bar\psi\psi\end{pmatrix}$$

Anyway this works. But the reason I wanted to write down what I did was so that $e^{\bar\gamma a^* + \gamma a}$ would be unitary, assuming in some sense that $\bar\gamma$ is the adjoint of γ.

(But one can't have $\gamma^*\gamma + \gamma\gamma^* = 0$ without γ being zero, as $\gamma^*\gamma,\ \gamma\gamma^* \geq 0$. This is why $\bar\gamma$ can't be the adjoint of γ.)

December 23, 1983

[redacted] What I really want to try to develop is a formal theory of [redacted] "path integrals" including fermion integrals, which would admit various realizations. The 2π factors occur in the realization. In the formal theory one has the Legendre transformation, and this becomes the Fourier transform in the realization. [redacted] One should first work out the fermion integration theory [redacted] since this is not affected by convergence questions.

[redacted]

Suppose we calculate the Weyl quantization of the Gaussian function $e^{-\beta\bar\psi\psi}$, where $\psi \in \mathbb{C}$. We first express it as an integral of exponential functions

$$e^{-\beta\bar\psi\psi} = \int \frac{d^2J}{\pi}\, e^{-\bar J\bar\psi + J\psi}\, F(J)$$

where

$$F(J) = \int \frac{d^2\psi}{\pi}\, \underbrace{e^{\bar J\bar\psi - J\psi}\, e^{-\beta\bar\psi\psi}}_{e^{-\beta\bar\psi\psi + \beta\,\text{[redacted]}\,\bar\psi\frac{\bar J}{\beta} - \beta\frac{J}{\beta}\psi}}$$

$$= \int \frac{d^2\psi}{\pi}\, e^{-\beta(\bar\psi + \frac{J}{\beta})(\psi - \frac{\bar J}{\beta}) - \frac{J\bar J}{\beta}} = e^{-\frac{J\bar J}{\beta}} \int \frac{d^2\psi}{\pi}\, e^{-\beta\bar\psi\psi}$$

$$= \frac{1}{\beta}\, e^{-\frac{J\bar J}{\beta}}$$

Thus the operator corresponding to $e^{-\beta\bar\psi\psi}$ is

$$\int \frac{d^2J}{\pi}\, e^{-\bar J a^* + J a}\, \frac{1}{\beta}\, e^{-\frac{J\bar J}{\beta}}$$

We can compute this by using

$$e^{-\bar{J}a^* + Ja} = e^{-\frac{J\bar{J}}{2}} e^{-\bar{J}a^*} e^{Ja} = e^{-\frac{J\bar{J}}{2}} :e^{-\bar{J}a^* + Ja}:$$

So we get

$$\int \frac{d^2J}{\pi} e^{-\bar{J}a^* + Ja} \frac{1}{\beta} e^{-\frac{J\bar{J}}{\beta}} = :\int \frac{d^2J}{\pi} e^{-\bar{J}a^* + Ja} \frac{1}{\beta} e^{-\frac{J\bar{J}}{\beta} - \frac{J\bar{J}}{2}}:$$

$$= :\int \frac{d^2J}{\pi} e^{-\bar{J}a^* + Ja} \frac{1}{\beta} e^{-\frac{J\bar{J}}{\beta_1}}: \qquad \frac{1}{\beta_1} = \frac{1}{\beta} + \frac{1}{2}$$

$$= \frac{\beta_1}{\beta} : e^{-\beta_1 a^* a} :$$

<u>Digression</u>: What is the relation between $:e^{-ta^*a}:$ and e^{-ta^*a}? Calculate on the exponential fn.

$$:e^{-ta^*a}: = \sum \frac{(-t)^n (a^*)^n a^n}{n!}$$

and $e^{\alpha z}$ is an eigenfunction for a, so

$$:e^{-ta^*a}: e^{\alpha z} = \sum \frac{(-t)^n z^n \alpha^n}{n!} e^{\alpha z} = e^{-t\alpha z} e^{\alpha z}$$

$$= e^{(1-t)\alpha z}$$

On the other hand $e^{\alpha z} = \sum \frac{\alpha^n z^n}{n!}$ and z^n is an eigenfunction with eigenvalue n for $a^* a = z\frac{d}{dz}$. So we get

$$e^{-ta^*a} e^{\alpha z} = \sum \frac{\alpha^n}{n!} e^{-ta^*a} z^n = \sum \frac{\alpha^n}{n!} e^{-tn} z^n$$

$$= e^{(e^{-t}\alpha) z}$$

Therefore we have

$$\boxed{:e^{-t a^* a}: \;=\; e^{\log(1-t)\, a^* a}}$$

Finally we get the formula for the Weyl quantization of the function $e^{-\beta \bar{\varphi} \varphi}$

$$\int \frac{d^2 J}{\pi}\, e^{-\bar{J} a^* + J a}\, \frac{1}{\beta} e^{-\frac{|J|^2}{\beta}} = \frac{\beta_1}{\beta} :e^{-\beta_1 a^* a}:$$

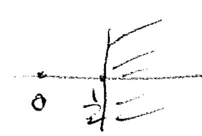
$$= \frac{\beta_1}{\beta}\, e^{\log(1-\beta_1)\, a^* a}$$

where $\beta_1 = \dfrac{1}{\frac{1}{\beta} + \frac{1}{2}} = \dfrac{2\beta}{\beta + 2}$

These formulas are very strange but consistent. One really should write

$$\boxed{\frac{\beta_1}{\beta} \left(1-\beta_1\right)^{a^* a} = \text{Weyl quantization of } e^{-\beta \bar{\varphi} \varphi}}$$

since the eigenvalues of $a^* a$ are $n \geq 0$. Notice that the region $Re(\beta) > 0$ corresponds to



$$Re(\beta) > 0 \iff Re\left(\frac{1}{\beta}\right) > 0 \iff Re\left(\frac{1}{\beta} + \frac{1}{2}\right) \geq \frac{1}{2}$$

$\iff \beta_1 = \dfrac{1}{\frac{1}{\beta} + \frac{1}{2}}$ lies in the circle of rad. 1 around 1.

$$\iff \left|1 - \beta_1\right| < 1$$

This last inequality agrees with the facts that the eigenvalues of $a^* a$ are $n \geq 0$, and so $(1-\beta_1)^{a^* a}$ is bounded for $|(1-\beta_1)| \leq 1$.

Finally let's check the trace

$$\mathrm{Tr}\left(\frac{\beta_1}{\beta}(1-\beta_1)^{a^*a}\right) = \sum_{n\geq 0}\frac{\beta_1}{\beta}(1-\beta_1)^n = \frac{\beta_1}{\beta}\,\frac{1}{1-(1-\beta_1)} = \frac{1}{\beta}$$

$$\int \frac{d^2\psi}{\pi}\, e^{-\beta\bar\psi\psi} = \frac{1}{\beta}.$$

On the other hand we have for the normal ordering quantization

$$\mathrm{Tr}\left(:e^{-\beta a^*a}:\right) = \mathrm{Tr}\,(1-\beta)^{a^*a} = \sum_{n\geq 0}(1-\beta)^n$$

$$= \frac{1}{1-(1-\beta)} = \frac{1}{\beta} = \int\frac{d^2\psi}{\pi}\,e^{-\beta\bar\psi\psi}$$

Question: Suppose you quantize by the Toeplitz process, does this coincide with Weyl quantization?

Thus one takes a function $f(\bar z, z)$ and you first multiply by f then project back onto the holomorphic functions.

Dec. 24, 1983.

Multiplying by $\bar z^k z^l$ on a holom. fn. and then projecting back onto the space of holom. fns. is clearly the operator $a^k (a^*)^l$. Thus Toeplitz quantization corresponds to reverse normal ordering. Here are some formulas.

Given $\varphi(z)$ the corresponding operator is

$$(T_\varphi f)(z) = \sum \langle z|n\rangle\langle n|\varphi f\rangle = \sum z^n \int\frac{d^2w}{\pi}e^{-|w|^2}\,\underset{\varphi(w)}{\bar w^n} f(w)$$

$$= \int \frac{d^2w}{\pi} e^{-|w|^2 + z\bar{w}} \varphi(w) f(w).$$

Now take $\varphi(w) = e^{-\beta|w|^2}$ and compute the effect on an exponential $e^{\alpha z}$.

$$T(e^{\alpha z}) = \int \frac{d^2w}{\pi} e^{-|w|^2 + z\bar{w} - \beta|w|^2 + \alpha w}$$

$$= \int \frac{d^2w}{\pi} e^{-\overbrace{(1+\beta)}^{\beta_2} \bar{w}w + \beta_2 \bar{w}\frac{z}{\beta_2} + \beta_2 \frac{\alpha}{\beta_2} w}$$

$$= \int \frac{d^2w}{\pi} e^{-\beta_2\left(\bar{w} - \frac{\alpha}{\beta_2}\right)\left(w - \frac{z}{\beta_2}\right) + \frac{\alpha z}{\beta_2}}$$

$$= e^{\left(\frac{\alpha}{\beta_2}\right)z} \int \frac{d^2w}{\pi} e^{-\beta_2 \bar{w} w} = \frac{1}{\beta_2} e^{\left(\frac{\alpha}{\beta_2}\right)z} \qquad \beta_2 = 1+\beta$$

$\therefore$ $\boxed{e^{-\beta a^*a} \text{ when reverse normal ordered } = (1+\beta)^{-1-a^*a}}$

Compute the trace

$$\operatorname{tr} (1+\beta)^{-1-a^*a} = \sum_{n\geq 0} \frac{1}{(1+\beta)^{1+n}} = \frac{1}{1+\beta} \frac{1}{1-\left(\frac{1}{1+\beta}\right)} = \frac{1}{\beta}.$$

The interesting fact it seems is that we have discussed three ways to assign an operator to the function $e^{-\beta\bar{\psi}\psi}$. All three give the same trace which is $\int \frac{d^2\psi}{\pi} e^{-\beta\bar{\psi}\psi} = \frac{1}{\beta}$. None of them give the operator $e^{-\beta a^*a}$ associated to the harmonic oscillator whose trace is

$$\operatorname{tr} e^{-\beta a^*a} = \sum_{n\geq 0} e^{-\beta n} = \frac{1}{1-e^{-\beta}}$$

Next I want to discuss fermion analogues of the above ideas. Compute F.T.

$$\int \mathcal{D}\bar\psi\,\mathcal{D}\psi \; e^{\bar J\bar\psi - J\psi}\begin{pmatrix}1\\ \psi\\ \bar\psi\\ \bar\psi\psi\end{pmatrix} = \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\begin{pmatrix}-\bar J\bar\psi J\psi\\ \bar J\bar\psi\psi\\ -J\psi\bar\psi\\ \bar\psi\psi\end{pmatrix} = \begin{pmatrix}\bar J J\\ \bar J\\ J\\ 1\end{pmatrix}$$

Thus we have

$$\int \mathcal{D}\bar J\,\mathcal{D}J \; e^{\bar\phi\bar J - \phi J}\int \mathcal{D}\bar\psi\,\mathcal{D}\psi \; e^{\bar J\bar\psi - J\psi}\begin{pmatrix}1\\ \psi\\ \bar\psi\\ \bar\psi\psi\end{pmatrix} = \begin{pmatrix}1\\ \phi\\ \bar\phi\\ \bar\phi\phi\end{pmatrix}.$$

Here's perhaps a good way to think. The fermion δ-function is

$$\delta(\psi) = \bar\psi\psi$$

since $\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\;\bar\psi\psi\, f(\bar\psi,\psi) = f(0,0)$. The F.T. of 1 is $\delta(J) = \bar J J$ and the F.T. of δ is 1.

The F.T. of the exponential function

$$e^{-\bar\alpha\bar\psi + \alpha\psi} = e^{\bar\psi\bar\alpha - \psi\alpha}$$

is

$$\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{\bar J\bar\psi - J\psi}\, e^{\bar\psi\bar\alpha - \psi\alpha} \overset{\text{direct evaluation by above fmlas}}{=} \bar J J + J\bar\alpha - \bar J\alpha + \bar\alpha\alpha$$

$$= (\bar J - \bar\alpha)(J-\alpha)$$

$$\| \qquad\qquad =$$

$$\int \mathcal{D}\bar\psi\,\mathcal{D}\psi\; e^{(\bar J - \bar\alpha)\bar\psi - (J-\alpha)\psi}$$

Thus we see that if we define $e^{-\bar\alpha\bar\psi + \alpha\psi}$ to be the exponential function with exponent α, then its F.T. is the δ-function $(\bar J - \bar\alpha)(J-\alpha)$ at α. Conversely the F.T. of $\delta(J-\alpha)$ is $e^{-\bar\alpha\bar\psi + \alpha\psi}$.

Next we consider the Gaussian function $e^{\beta\bar\psi\psi}$ and compute its F.T. in two ways

$$\int D\bar\psi D\psi\; e^{\bar J\bar\psi - J\psi}\left(1+\beta\bar\psi\psi\right) = \bar J J + \beta$$

$$\int D\bar\psi D\psi\; e^{\beta\bar\psi\psi - \beta\bar\psi\frac{\bar J}{\beta} - \beta\frac{J}{\beta}\psi + \beta\frac{J}{\beta}\frac{\bar J}{\beta} + \frac{1}{\beta}\bar J J}$$

$$= e^{\frac{1}{\beta}\bar J J}\int D\bar\psi D\psi\; e^{\beta(\bar\psi - \frac{J}{\beta})(\psi - \frac{\bar J}{\beta})} = e^{-\frac{1}{\beta}\bar J J}\int D\bar\psi D\psi\, e^{\beta\bar\psi\psi}$$

$$= e^{\frac{1}{\beta}\bar J J}\beta = \beta\left(1+\frac{1}{\beta}\bar J J\right) = \bar J J + \beta .$$

Finally we can compute the Weyl quantization in two ways. The elementary way is

$$e^{\beta\bar\psi\psi} = 1+\beta\bar\psi\psi \longmapsto 1+\beta\,\tfrac{1}{2}(a^*a \; aa^*) = 1+\beta(a^*a - \tfrac{1}{2})$$

The other way is to use the F.T.

$$e^{\beta\bar\psi\psi} \xmapsto{F.T.} \bar J J + \beta \longmapsto \int D\bar J DJ\; e^{\bar J a^* - Ja}(\bar J J + \beta)$$

$$e^{\bar J a^* - Ja} = 1 + \bar J a^* - Ja + \frac{-1}{2}(\bar J a^* Ja + Ja\,\bar J a^*)$$

$$= 1 + \bar J a^* - Ja + \;\; \bar J J\;\frac{a^*a \; aa^*}{2}$$

So the operator is

$$\int D\bar J DJ \cdot \bar J J\left(1+\beta\,\frac{a^*a - aa^*}{2}\right) = 1+\beta(a^*a - \tfrac{1}{2})$$

In addition to the Weyl quantization we also have the two other possibilities

$$:e^{\beta a^*a}: \;=\; 1+\beta a^* a$$

$$\text{reverse}:e^{\beta a^*a}: \;=\; 1-\beta aa^* = 1+\beta(a^*a - 1).$$

Since
$$\left.\begin{aligned} tr(\varepsilon a^*a) \\ tr(\varepsilon 1) \end{aligned}\right) \begin{aligned} &= 1 \\ &= 0 \end{aligned}$$
it follows that all these operators have the same super-trace:
$$\beta = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\; e^{\beta \bar{\psi}\psi}$$

<u>Correction:</u> Let's compute $:e^{ta^*a}:$ for both bosons and fermions. For bosons

$$:e^{ta^*a}: z^m = \sum \frac{t^n}{n!} \overset{z}{(a^*)}{}^n \overset{\frac{d}{dz}}{a}{}^n z^m = \sum \frac{t^n}{n!} \frac{m!}{(m-n)!} z^m$$
$$= (1+t)^m z^m = (1+t)^{a^*a} z^m .$$

For fermions
$$:e^{ta^*a}: = 1 + t a^* a = \begin{cases} 1 & \text{on } \Lambda^0 \\ 1+t & \text{on } \Lambda^1 \end{cases}$$

and hence we also have
$$:e^{ta^*a}: = (1+t)^{a^*a}$$

Thus the formula on 329 is wrong

Note that fermion integrals rescale differently.
$$1 = \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\; \bar{\psi}\psi \longmapsto \int \mathcal{D}(a\bar{\psi})\, \mathcal{D}(b\psi)\; ab\, \bar{\psi}\psi$$
so $\mathcal{D}(a\bar{\psi}) = \frac{1}{a}\mathcal{D}\bar{\psi}$ $\quad \mathcal{D}(b\psi) = \frac{1}{b}\psi$

December 28, 1983

Interesting problem: We have studied the transgression of the class

$$\int_M ch(E) \in H^*(B\mathcal{G})$$

by computing with differential forms. Is it possible to do the transgression analytically, or perhaps first in K-theory?

This seems to be necessary [redacted] if one wants to relate Connes cocycles, which are analytically defined invariant forms on $\mathcal{G}$ to the index theorem for families of operators on M.

Specific problem: Consider a Dirac operator $i(d+A)$ over the circle. Suppose it is invertible, so that it defines a polarization F [redacted] of $L^2(S^1)^N$ into positive and negative eigenspaces. Then we have a map from $\mathcal{G}$ into the space of polarizations (which is a Grassmannian) and can pull-back the character forms

$$\text{const } Tr\left(F(dF)^{2n}\right)$$

to get invariant forms on $\mathcal{G} = Maps(S^1, U(N))$. On the other we can define by [redacted] differential geometric methods invariant forms on $\mathcal{G}$. The specific problem is now to explicitly relate these two forms.

I want to review the transgression process for constructing diff'l forms in $\mathcal{G}$.

We first construct the characteristic classes on $\mathcal{A}/\mathcal{G}$. These are obtained from the vector bundle $\bar{E}$ over $\mathcal{A}/\mathcal{G} \times M$ which is obtained by descent from $\mathcal{A} \times E$ over $\mathcal{A} \times M$. The bundle $\mathcal{A} \times E = pr_2^*(E)$ has a tautological connection $d + \tilde{A}$ which is modified to $d + \underbrace{\theta + \tilde{A}}_{\bar{A}}$ so that it descends. The transgression class we are interested in comes from the linear path $\bar{A} - t\theta$ joining $\bar{A}$ to $\tilde{A}$.

Next we restrict to a $\mathcal{G}$ orbit, the $\mathcal{G}$-orbit of A^0 to be specific. We then have two connections in $\mathcal{G} \times E = pr_2^*(E)$ over $\mathcal{G} \times M$. One is

$$\delta + d + \tilde{A} = \delta + g^{-1}(d + A^0) g \qquad \begin{matrix} \delta = d_{\mathcal{G}} \\ d = d_M \end{matrix}$$

where $g : \mathcal{G} \times M \longrightarrow U(N)$ is the canonical bundle auto, or evaluation map. The other connection is

$$\delta + d + \bar{A} = g^{-1}(\delta + d + A^0) g$$

$$= [redacted] \; \delta + d + \underbrace{g^{-1}\delta g}_{\theta} + \tilde{A}$$

The trangression form is

$$u(\tilde{A}, \bar{A}) = \int_0^1 dt \; tr\left((-\theta) \, e^{(1-t)F_{\bar{A}} + tF_{\tilde{A}} + (t^2 - t)\theta^2} \right)$$

$$-\theta = -g^{-1}\delta g = g^{-1}\underbrace{(-\delta g \cdot g^{-1})}_{g\delta g^{-1}} g$$

$$F_{\bar{A}} = g^{-1} F_{A^0} g$$

$$F_{\tilde{A}} = [\delta, g^{-1}(d + A^0) g] + g^{-1}(d + A^0)^2 g$$

$$= -g^{-1}\delta g \cdot g^{-1}(d + A^0) g - g^{-1}(d + A^0)\delta g + g^{-1} F_{A^0} g$$

$$F_{\tilde{A}} = g^{-1}\left\{(-\delta g\cdot g^{-1})(d+A^0) - (d+A^0)\,\delta g\cdot g^{-1}\right\}g + g^{-1}F_{A^0}\,g$$

$$(1-t)\,F_{\bar{A}} + t\,F_{\tilde{A}} = g^{-1}\left\{F_{A^0} + t[d+A^0, g\delta g^{-1}]\right\}g \,.$$

Thus on $\mathcal{G}\times M$ we have

$$\boxed{u(\tilde{A},\bar{A}) = \int_0^1 dt \ \mathrm{tr}\left(g\delta g^{-1}\, e^{F_{A^0}+t[d+A^0, g\delta g^{-1}] + (t^2-t)(g\delta g^{-1})^2}\right)}$$

This should be contrasted with the way I proposed to construct left-invariant forms on $\mathcal{G}$, namely to consider the flat partial connection

$$\delta + \theta = g^{-1}\cdot\delta\cdot g = \delta + g^{-1}\delta g$$

extend it to full connections

$$\delta + d + \text{[redacted]}\ g^{-1}\delta g + A^0$$

and to use the path

$$\delta + d + t g^{-1}\delta g + A^0$$

from the extensions of δ to $\delta+\theta = g^{-1}\cdot\delta\cdot g$. Then

$$\begin{aligned}
F_t &= (\delta+d)(tg^{-1}\delta g + A^0) + (t\,g^{-1}\delta g + A^0)^2 \\
&= (\delta+d)(t\theta + A^0) + (t\theta + A^0)^2 \\
&= t(\delta\theta) + t\,d\theta + dA^0 + t^2\theta^2 + t[A^0,\theta] + A^{02} \\
&= F_{A^0} + t\,[d+A^0,\theta] + (t^2-t)\theta^2
\end{aligned}$$

So we get

$$\boxed{u(\theta+A^0, A^0) = \int_0^1 dt \ \mathrm{tr}\left(\theta\, e^{F_{A^0} + t[d+A^0,\theta] + (t^2-t)\theta^2}\right)}$$

December 29, 1983

Consider the family of Dirac operators on M parametrized by $\mathcal{A}/\mathcal{G}$. What does your local index formula give for the index?

I want to think of having Hilbert bundles $\mathcal{H}^\pm$ over $\mathcal{A}/\mathcal{G}$ and an operator between theorem.

Then I need a connection in $\mathcal{H}$ in order to compute the character.

Now $\mathcal{H} = \mathcal{A} \times^{\mathcal{G}} L^2(M, S\otimes E)$ is just the Hilbert bundle over $\mathcal{A}/\mathcal{G}$ associated to the repn. of $\mathcal{G}$ on the Hilbert space $L^2(M, S\otimes E)$. A first problem to look at then is how to construct a connection in the vector bundle $P\times^G V$ associated to a representation $\rho : G \longrightarrow \mathrm{Aut}(V)$. So we start with the G-bundle $P\times V$ over P and its invariant connection d. This is then modified to

$$d + \rho\theta$$

where $\theta \in \Omega^1(P, \mathfrak{g})$ is a connection.

(Formulas: The G-action on sections Ψ of $P\times V$ over P is $(g*\Psi)(p) = \rho(g)\Psi(g*p)$ which infinitesimally is $(v*\Psi)(p) = \rho(v)\Psi(p) + L_v\Psi(p)$. Thus

$$v*\Psi = [L_v + \rho(v)]\Psi$$

so $\rho(v)$ is the Higgs field. Given the connection

$$D\Psi = d\Psi$$

we have $\quad i_v D\Psi = i_v d\Psi = L_v\Psi$

Thus
$$v * \Phi = [\mathcal{L}_v + \rho(v)]\Phi$$
$$= i_v[d + \rho\theta]\Phi$$
where θ is the connection form.)

Now apply this to the representation of $\mathcal{G}$ on $L^2(M, S\otimes E)$. The [redacted] connection on $\mathcal{H} = \mathcal{A} \times^{\mathcal{G}} L^2(M, S\otimes E)$ over $\mathcal{A}/\mathcal{G}$ is obtained by descending the connection $d_{\mathcal{A}} + \theta$ on
$$\mathcal{A} \times L^2(M, S\otimes E) \quad \text{over [redacted]} \quad \mathcal{A}.$$
Here θ is a $\tilde{\mathfrak{g}}$-valued 1-form on $\mathcal{A}$ and it is interpreted as a form with values in Endos of L^2.

At this point we have a connection on the bundle $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. We also need the family L [redacted] of operators parametrized by $\mathcal{A}/\mathcal{G}$, i.e. the operator $L : \mathcal{H}^+ \rightleftarrows \mathcal{H}^-$ on the Hilbert bundle. [redacted] We lift from $\mathcal{A}/\mathcal{G}$ to $\mathcal{A}$ and then want an equivariant family of operators
$$L_A : (L^2)^+ \rightleftarrows (L^2)^- \qquad A \in \mathcal{A}.$$
This comes from the family of Dirac operators
$$+i\not{D}_A = +i[redacted]\,\gamma^\mu(\partial_\mu + A_\mu) \;:\; S^+\otimes E \rightleftarrows S^-\otimes E$$
over M. The [redacted] character of the index of the family is
$$\mathrm{Tr}_{L^2}\left(\varepsilon e^{-tL^2 + \sqrt{t}\,[d_{\mathcal{A}}+\theta, L] + (d_{\mathcal{A}}\theta + \theta^2)}\right)$$
This should be a basic form on $\mathcal{A}$. Hopefully as

$t \downarrow 0$ it approaches the form

$$\int_M \mathrm{tr}\left(e^{F_{\bar{A}}}\right)$$

Now in order to compute the transgression we compared the two connections $\tilde{A}$ and $\bar{A} = \tilde{A} + \Theta$ on $\mathcal{a} \times E / \mathcal{a} \times M$. Corresponding to $\tilde{A}$ is the connection d_a on $\mathcal{a} \times L^2$ and the family $L = \{L_A / A \in \mathcal{a}\}$. The character of the index bundle computed relative to this connection d_a is

$$\mathrm{Tr}_{L^2}\left(\varepsilon\, e^{-tL^2 + \sqrt{t}[d_a, L]}\right)$$

which as $t \downarrow 0$ we expect to approach the form

$$\int_M \mathrm{tr}\left(e^{F_{\tilde{A}}}\right).$$

The next thing is to find the analogue of the form

$$u(\bar{A}, \tilde{A}) = \int_0^1 ds\, \mathrm{tr}\left(\Theta\, e^{(1-s)F_{\tilde{A}} + sF_{\bar{A}} + (s^2-s)\Theta^2}\right)$$

It should be something like

$$\int_0^1 ds\, \mathrm{tr}\left(\varepsilon\Theta\, e^{-tL^2 + \sqrt{t}[d_a + s\Theta, L] + s\, d_a\Theta + s^2\Theta^2}\right)$$