trace for a 1-seminormal Fredholm module


Connes $S$-operator

Connes-Kaspari thin:

\[ H^0(\mathcal{Q}) = \text{Im} \left\{ S : \overline{HC}_{p+2}(A) \to \overline{HC}_p(A) \right\} \]
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We have certain normalization problems to solve. Precisely, how do we define maps

$$K_0 A \longrightarrow HC_{2n} (A)$$

$$K_1 A \longrightarrow HC_{2n-1} (A)$$

which when mapped to $2^{2n}/d2^{2n-1}$ (resp. $2^{2n-1}/d2^{2n-1}$) coincide with $ch^n$ and $e_{2n}$, respectively.

Suppose an element of $K_0 A$ is given by an idempotent $e$. Then

$$ch^n(e) = \frac{1}{n!} tr(e(de)^{2n})$$

$$= \frac{2n!}{n!} \mu(e \otimes Q^{n+1})$$

Thus it appears that

$$e \rightarrow \frac{2n!}{n!} \mu(e \otimes Q^{n+1})$$

in $A \otimes Q^{2n+1}/(1-t)$ will be the good map.

Let's check that these maps for different $n$ are compatible under Connes $S$ operator.

$$\begin{array}{cccc}
A \otimes (Q^{2n+1})/1-t & \rightarrow & A \otimes (Q^{2n+1}) \\
\downarrow & & \downarrow \\
A \otimes (Q^{2n}) \downarrow 1-t & \rightarrow & A \otimes (Q^{2n}) \\
\downarrow & & \downarrow \\
\frac{1}{2} e \otimes (Q^{2n}) & \rightarrow & \frac{1}{2} e \otimes (Q^{2n}) \\
\downarrow & & \downarrow \\
\frac{1}{2} e \otimes (Q^{2n-1}) & \rightarrow & \frac{1}{2} e \otimes (Q^{2n-1}) \\
\end{array}$$
This calculation shows that
\[ S \exp(2n+1) = \frac{1}{2(2n-1)} \exp(2n-1) \]

hence
\[ S \left( \frac{2n! \exp(2n+1)}{n!} \right) = \frac{1}{2(2n-1)} \frac{2n!}{n!} \exp(2n-1) \]
\[ = \frac{(2n-2)!}{(n-1)!} \exp(2n-1) \]

so it works!

The good map \( K_0 A \to HC_{2n}(A) \) is given by \( e \mapsto \frac{(2n)!}{n!} \exp(2n+1) \mod 1-t \)

Now suppose an element of \( K_1 A \) is given by an invertible \( g \). The odd character \( e_{2n-1}(g) \in \mathbb{Z}^{2n-1} \) is given by
\[ \int_0^T \text{tr}(\Theta, \frac{2n}{(n-1)!} (t^2-t)^{n-1}) \theta \quad \Theta = g^{-1} dg \]

(Recall the transgression formula
\[ \int_0^T \text{tr}(\Theta e^t dt + (t^2-t) e^2) \].

Now
\[ \int_0^T (t^2-t)^{n-1} dt = (-1)^{n-1} \int_0^1 t^{n-1} (1-t)^{n-1} dt \]
\[ = (-1)^{n-1} \beta(n, n) = (-1)^{n-1} \frac{\Gamma(n) \Gamma(n)}{\Gamma(2n)} = (-1)^{n-1} \frac{(n-1)!^2}{(2n)!} \]
\[ c_{2n-1}(g) = (-1)^{n-1} \frac{(n-1)!}{(2n-1)!} \text{tr}(g^{-1} dg)^{2n-1} \]

\[ = \frac{(n-1)!}{(2n-1)!} \text{tr}(g^{-1} dg (dg^{-1} dg)^{n-1}) \]

\[ = \mu\left( (n-1)! \text{tr}(g^{-1} dg)^n \right) \]

Thus it appears that the good map will be

\[ g \rightarrow (n-1)! \text{tr}(g^{-1} dg)^n \in \overline{A}^{\otimes 2n}/(1-c). \]

The reason we take the image in the reduced cyclic homology is so that \((g^{-1} dg)^n\) is a cyclic cyclic, e.g.

\[ b(g^{-1} dg g^{-1} dg) = 1 \otimes g^{-1} dg - g^{-1} \otimes 1 \otimes g + g^{-1} \otimes g \otimes 1 \]

\[ = 2(1 \otimes g^{-1} dg - 1 \otimes g \otimes g^{-1}) \quad \text{mod} \quad (1-c) \]

is not a cyclic cycle.

Let’s check compatibility with \(S\)

\[ (g \otimes g)^{\otimes (n+1)} \]

\[ \downarrow b \]

\[ 1 \otimes (g^{-1} dg)^{\otimes n} - 1 \otimes (g \otimes g^{-1})^{\otimes n} \]

\[ \leftarrow \frac{1}{n} (g \otimes g)^{\otimes n} \]
So we have

\[ S \left( g^{-1} \circ g \right)^{(n+1)} = \frac{1}{n} \left( g^{-1} \circ g \right)^{\otimes n} \]

where

\[ S (n+1)! \left( g^{-1} \circ g \right)^{(n+1)} = n! \left( g^{-1} \circ g \right)^{\otimes n} \]

Hence we see

The good map \( K_1 A \to \overline{HC}_{2n-1} (A) \) is given by

\[ g \mapsto (n-1)! \cdot r \left( g^{-1} \circ g \right)^{\otimes n} \in \overline{A} \otimes^{2n-1} \]

Note that from the exact sequence

\[ 0 \to \overline{HC}_{2n-1} (k) \to \overline{HC}_{2n-1} (A) \to \overline{HC}_{2n-1} (A) \to 0 \]

it would be nicer to define the map \( \overline{A} \) so that it landed in \( \overline{HC}_{2n-1} (A) \).

Let's put \( A = \tilde{A} = k \) the ring obtained by adjoining an identity to \( A \). Then

\[ 0 \to A \xrightarrow{i} A \xrightarrow{\varepsilon} k \to 0 \]

\[ \downarrow \rho \]

\[ A \]

where \( \rho (c_1 + i(a)) = c + a \), \( i, \rho, \varepsilon \) are ring
homomorphisms and \( A \xrightarrow{\psi} \mathbb{Z} \) is a ring isomorphism. Now given any invertible \( g \in A \) we lift it to \( \tilde{A} \), and get the element \( 1 + \tilde{i}(g-1) \). Then under

\[
A \rightarrow \tilde{A} \cong A
\]

\[
1 + \tilde{i}(g-1) \rightarrow g-1.
\]

In other words we have

\[
K_1^g A \rightarrow K_1 A \rightarrow \tilde{H}_{2n-1}(\tilde{A}) = H_{2n-1}^g A
\]

is given by the map

\[
g \mapsto (n-1)! \cdot \text{tr} (g^{-1}) \otimes (g-1) \in A^{\otimes 2n}/(1-t)
\]

Actually even if \( A \) has no unit we can probably define \( K_1 A \) using quasi-inverses, i.e. \( u \) is quasi-invertible when there is a \( v \) with

\[
u + v + uv = 0.
\]

Then the good map

\[
K_1 A \rightarrow \tilde{H}_{2n-1}(A)
\]

is

\[
u \mapsto (n-1)! \cdot \text{tr} (v \otimes u)^{\otimes n} \in A^{\otimes 2n}/(1-t).
\]
For the paper with today I need to prove that \( B(A)_{\text{red}} \) is quasi-isomorphic to \( (A^{\otimes(k+1)}/(1 + t), b) \).

First of all we have an augmentation

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
A^{3/(1-t)} & \leftarrow & A \otimes A^2 \\
\downarrow & & \downarrow \\
A^{2/(1-t)} & \leftarrow & A \otimes A^2 \\
\downarrow & & \downarrow \\
A & \leftarrow & A \\
\end{array}
\]

The point will be to filter \( B(A)_{\text{red}} \) by

\[
\begin{array}{ccc}
1 \otimes A^{8+t} & \leftarrow & A \otimes A^8 \\
\downarrow b & & \downarrow b \\
A \otimes A^8 & \leftarrow & A \otimes A^8 \\
\downarrow b & & \downarrow b \\
A & \leftarrow & A \\
\end{array}
\]

This is a subcomplex because I know that the image of \( B : A \otimes A^8 \rightarrow A \otimes A^{8+t} \) is contained in \( 1 \otimes A^{8+t} \).

What is \( F_8/F_{8-1} \)? It is

\[
\begin{array}{ccc}
1 \otimes A^{8+t} & \leftrightarrow & 1 \otimes A^{8+t} \\
\downarrow & & \downarrow \\
A \otimes A^8 & \leftrightarrow & A \otimes A^8 \\
\downarrow & & \downarrow \\
A & \leftrightarrow & A \\
\end{array}
\]

Here I use the notation \( 1 \otimes A^{8+t} \) to denote a \( k \)-submodule of \( A \otimes A^{8+t} \) which might not be isomorphic to \( A^{8+t} \). In fact from

\[
0 \rightarrow R \rightarrow A \rightarrow A \rightarrow 0
\]

we get
\[ 0 \rightarrow \mathcal{O}_k^*(A, \overline{A}) \rightarrow \mathcal{T}_k^*(A, \overline{A}) \rightarrow k \otimes \overline{A} \rightarrow A \otimes \overline{A} \rightarrow \overline{A} \otimes \overline{A} \rightarrow 0 \]

so if we take \( k = \mathbb{Z}, A = \mathbb{Q} \) then

\[ k \otimes \overline{A} = \mathbb{Q}/\mathbb{Z} \rightarrow A \otimes \overline{A} = \mathbb{Q} \otimes \mathbb{Q}/\mathbb{Z} = 0 \]

and the sort of thing we are trying to prove breaks down. Therefore we need to assume that \( \overline{A} \) is flat over \( k \).

In fact the case \( k = \mathbb{Z}, A = \mathbb{Q} \) is a good example. One can even do it entirely in characteristic zero, i.e. \( \overline{A} = \mathbb{F}_p[t], A = k_0(t) \). Then \( \overline{H}C_*(A) \) will be \( \overline{A} \) in every even dimension, yet the complex \( A^{\otimes(k+1)/2} \) will be just \( A \) in degree 0.

Let's return to the index problem and try to understand why the index, which is a very zero-th order invariant, can be obtained from the Lie algebra cohomology.

I want to consider an odd dimensional manifold \( M \) and a Dirac operator on it with coefficients in \( E \). This gives me a Hilbert space \( H = L^2(M, \mathfrak{so}E) \) and a splitting of it into positive and negative eigenspaces for the Dirac operator. As usual, the splitting is described by an \( F \) in \( H \) such that \( F^2 = I \).

Let \( G \) be the gauge group of \( E \). Then we have a map

\[ \mathcal{G} \rightarrow F \]

\[ g \mapsto gFg^{-1} \]

where \( F \) is a suitable Grassmannian of such splittings.
F is a restricted Grassmannian of some sort and it has the homotopy type of \( \mathbb{Z} \times BU \). On F are the universal Chern character cohomology classes \( c_n \in H^{2n}(F) \).

They are the components of the Chern character of the basic K-element. So I can pull these classes back to even-dimensional classes

\[
\gamma^*(c_n) \in H^{2n}(A).
\]

For example \( \gamma^*(c_0) \) gives the index of the operator.

What is my goal? I want to understand why Lie algebra cohomology of the gauge group is related to index questions. And hence with determinants, since these are next after indices.

Connes' starting point is a p-summable Fredholm module. To fix the ideas we take a Dirac operator on a compact manifold M. Then one gets the Hilbert space \( \mathcal{H} \) of \( L^2 \) spinors which is a \( C^* \)-module over \( A = C^0(M) \). Assuming the Dirac operator is invertible, the Hilbert space \( \mathcal{H} \) breaks up into positive and negative eigenspaces, giving rise to an F such that \( F^2 = I \). Then one gets cyclic cocycles

\[
\tau(a_0, \cdots, a_{2n}) = \text{tr} \left( e^{a_0 [F_{a_1} \cdots F_{a_{2n}}]} \right)
\]

\[
\tau(a_0, \cdots, a_{2n-1}) = \text{tr} \left( F [F_{a_0} \cdots F_{a_{2n-1}}] \right)
\]

even case

odd case
which I sort of understand how to interpret in terms of Lie algebra cohomology for $\mathfrak{g} = \text{Map}(M, \mathbb{U}_N)$.

Also these cocycles are directly related to indices. Hence the question of relating Lie algebra cohomology to indices.

Let's look at the index problem belonging to this original Dirac operator, or $F$. Topologically the operator is a $K$-homology class, $\tau \in K^*(M)$. The index problems associated to $\Phi$ or $F$ are values of the map

$$K^*(M) \to K^*(pt).$$

So we should think of the operator $\Phi$ or $F$ as giving us a way to go from the $K$-theory of $M$ to that of a point. Since the operator is given one should think of the operator as a $K$-cycle, not just a homology class.

Similarly one should think of $K^*(M)$, $K^*(pt)$ as described by actual $K$-cocycles, and the map $\tau$ as given concretely. This is what happens when we represent elements of $K^0(M)$ by idempotent matrices $e$ (resp. elements of $K^1(M)$ by invertible matrices $g$).

Note: $e$ (resp. $g$) is the same as a map $M \to \text{Grass}$, (resp. $M \to \mathbb{U}_N$).

So what I really have is a map from a space such as $\text{Map}(M, \mathbb{Z} \times \mathbb{U})$, $\text{Map}(M, \mathbb{U})$ into the space of Fredholm operators. This is...
how I should think of the $K$-cycle given by the operator.

Now one has another way to do this thing in the even case. Instead of maps into a Grassmannian $\text{Map}(M, \text{BUN})$ we can get an equivalent space using $B_G = \mathbb{A} \setminus \mathbb{G}$. Strictly one has to pass to a component of this mapping space.

So my idea now is to replace Connes' use of $c$ by a connection $A$. Then I have the map

$$A \mapsto D_A$$

from $A$ to a space of Fredholm operators, which is equivariant for $G$.

In more detail we are considering the family of Dirac operators

$$A \mapsto D_A \text{ on } L^2(S \otimes E) = \mathcal{H}$$

where $S$ is the vector bundle of spinors. The gauge $g$ acts on $\mathcal{H}$ and this family is equivariant. Thus

$$P_G \times A \to F(P_G \times B_G \mathcal{H})$$

over $B_G$ we have a Hilbert bundle with fibre $\mathcal{H}$ and a map from $P_G \times A$ to a Fredholm bundle map.

If we trivialize $P_G \times B_G \mathcal{H}$ by Kumpera's trick, then we get a map from $B_G$ to the space of Fredholm
operators \( F \) well-defined up to homotopy:

\[ \mathbb{B} \mathcal{D} \rightarrow F \sim \mathbb{Z} \times \mathbb{B}U. \]

This map corresponds to the \( K \)-element of \( \mathbb{B} \mathcal{D} \) given by the index of the family of Dirac operators.

Now the index theorem for families tells us what the above map is. The cohomological version would describe exactly what the pull-back of the Chern character classes are. Where does the cyclic cohomology enter?

The point now should be that the character classes on \( \mathbb{B} \mathcal{D} \) can be suspended to odd classes on \( F \) which in high dimensions are realized by invariant differential forms. And the point is that the corresponding cyclic cocycles should be the ones that Connes describes. This will be the point that I have to make clear.
Yesterday I started work on the problem of relating Lie algebra cohomology of gauge groups to indices and determinants.

The setting: One is given an operator over M. There is the basic operation of taking a vector bundle E over M, tensoring the operator with it, and taking the index. This gives a map

\[ K^*(M) \longrightarrow K^*(pt). \]

But one should think more generally as having a map of "theories"

\[ K(M) \longrightarrow K(pt). \]

One way to make this more precise is that the operator on M gives a natural transformation

\[ \mathbb{A} : K^*(Y \times M) \longrightarrow K^*(Y). \]

Now we have

\[ \begin{array}{ccc}
K^*(Y \times M) & \mathbb{A} & \longrightarrow & K^*(Y) \\
\downarrow \text{ch} & & & \downarrow \text{ch} \\
H^*(Y \times M) & \tau & \longrightarrow & H^*(Y)
\end{array} \]

where \( \tau \in H^*_x(M) \) depends on the operator. (This should be a consequence of the fact that ch is an
isomorphism naturally, and \( \ast \) is a \( K(Y) \)-module homomorphism.

The point seems to be that if I want to find \( T \), then I can use various \( Y \)'s.

The simplest \( Y \) to use is \( Y = \text{pt} \).

So far I am missing the role of periodicity, which makes other \( Y \)'s turn out in some way to be simpler than \( Y = \text{pt} \). (?)

(The last statement doesn't make sense. What is at stake has probably to do with the situation in which we want differential forms representing \( T \).)
Annex S-Operator: The basic idea here is to consider the map $a \mapsto ae$ from $A$ to $A \otimes ke$ where $A$ is a non-unital algebra. Then we have a map from $A$ to the degree zero part of the DG algebra

$$\tilde{\Omega}^*(a^+) \otimes \tilde{\Omega}^*(k+ke).$$

In fact

$$\tilde{\Omega}^*(a^+) \otimes \tilde{\Omega}^*(k+ke) = (k+a) \otimes (k+ke) = k + ke + a + ae$$

We consider the map $a \mapsto ae$ from $A$ to $A^+ \otimes (ke)^+$. This gives a homomorphism

$$a \mapsto a^+ \otimes (ke)^+ \subset \tilde{\Omega}^*(a^+) \otimes \tilde{\Omega}^*(ke)^+)$$

and the dotted arrow exists by the universal property of $\tilde{\Omega}^*(a^+)$. The induced map on homology is

$$H^*_{ck}(a^+) \longrightarrow H^*_{ck}(a^+) \otimes H^*_{ck}(ke)^+ \begin{cases} k \text{ even} \\ 0 \text{ odd} \end{cases}$$

so in particular corresponding to $e(dk)^2$ in $H^*_{ck}(ke)^+$ we get a map

$$S: H^p_{ck}(a^+) \longrightarrow H^{p-2}_{ck}(a^+)$$
\[ \overline{\Omega}(A^+) \longrightarrow \overline{\Omega}(A^+) \otimes \overline{\Omega}((ke)^t) \] is the unique map of differential graded algebras such that
\[ \alpha \mapsto ac \] for \( a \in A \).

Then \( a_0 da_1 \ldots da_n \longrightarrow a_0 e[da_1 e + a_1 de][da_2 e + a_2 de] \ldots \).

To find the \( S \)-map we want the coefficient of \( e de^2 \).

In the tensor product \( \overline{\Omega}(A^+) \otimes \overline{\Omega}((ke)^t) \) we are allowed to move \( a_j de \) past \( e, de \) subject to picking up a sign. Recall also that \( e.d.e.e = 0 \). This means that if we take terms \( a_j de, a_j de \) in the product that \( i, j \) must be consecutive, otherwise we get zero.

Hence we get
\[
(a_0 a_1 a_2 da_3 \ldots da_n + a_0 da_1 a_2 a_3 da_4 \ldots da_n + \ldots
\]
\[
\ldots + a_0 da_1 \ldots da_{n-2} a_{n-1} a_n ) e de^2
\]
for the term involving \( e de^2 \). Hence we conclude that the map
\[ S(a_0 da_1 \ldots da_n) = \]
\[
a_0 a_1 a_2 da_3 \ldots da_n + a_0 da_1 a_2 a_3 da_4 \ldots da_n + \ldots
\]
\[ + a_0 da_1 \ldots da_{n-2} a_{n-1} a_n \]
induces a map on Connes-Kreimer cohomology. I can be more precise. We have a map of complexes
\[
\overline{\Omega}(A^+) \longrightarrow \overline{\Omega}(A^+) \otimes \overline{\Omega}((ke)^t) \longrightarrow \overline{\Omega}(A^+) \otimes \overline{\Omega}((ke)^t)/[e, e]
\]
k in even degrees.
and hence we get a map of complexes

$$S: \tilde{\Omega}^*(a^+) \longrightarrow \tilde{\Omega}^{*-2}(a^+)$$

with

$$S(a_0 da_1 \ldots da_n) \text{ defined as above and }$$

$$S(da_1 \ldots da_n) = d(S(a_1 da_2 \ldots da_n)) = d a_1 da_2 \ldots da_n$$

One could define any map of complexes this way, but the point is that $S$ will induce a map on commutator quotients

$$\tilde{\Omega}^*(a^+)/[ , ] \longrightarrow \tilde{\Omega}^{*-2}(a^+)/[ , ]$$

This is because

$$\tilde{\Omega}(A) \otimes \tilde{\Omega}(B)/[ , ] = \tilde{\Omega}(A)/[ , ] \otimes \tilde{\Omega}(B)/[ , ]$$

From the formula for $S$ one sees that it is not defined on $\tilde{\Omega}(A)$ for a non-augmented ring since

$$S(a_0 dl da_2 \ldots da_n) = a_0 a_2 da_3 \ldots da_n + a_0 dl a_2 \ldots$$

so one can't set $dl = 0$.

I think the next thing to do is to take $A = C^\infty(M)$. Then show how a closed current $\omega$ of dimension $p$ on $M$ furnishes cyclic cocycles of degrees $p, p+2, p+4, \ldots$ on $A$.
New problem. Let's go back to the original index motivation for Lannes' theory. Let's try to avoid the assumption that the original operator is invertible. So we begin with

\[ H^+ \xrightarrow{P} H^- \]

where \( P \) is Fredholm, \( H^\pm \) are \( A \)-modules, and \( [P,a] \) is compact, say in some Schatten class. Then replacing \( P \) by \( P^n \), we get an action of the gauge group \( \mathcal{G} \) on \( H^\pm \), and hence a map

\[ \mathcal{G} \rightarrow \text{Aut}(H^+, H^-) \]
\[ g \mapsto gPg^{-1} \]

Before it was a map into invertible operators from \( H^+ \) to \( H^- \) which carries odd character forms \( h \cdot (P^{-1}dP)^{2k+1} \). The problem is now to construct somehow odd forms in \( H \) defined formally for all degrees, but which will make sense for large degrees because the trace of a product of enough commutators is well-defined.
Let me consider the space of Fredholm operators \( P : \mathcal{H}^+ \to \mathcal{H}^- \), call it \( \mathcal{F} \) and the larger space \( \mathcal{F} \) of \( F = (\phi, \Psi) \) such that \( F^2 - I \) is compact. Over these spaces we have the determinant line bundle \( L \), and this line bundle is equivariant for the action of \( \mathcal{G} = \mathcal{H}^+ \times \mathcal{H}^- \) of automorphisms of \( \mathcal{H} \) of degree zero.

In the case of interest \( \mathcal{H} \) is an \( \mathcal{A} \)-module so we have \( \mathcal{A} \) mapping to \( \mathcal{A} \) and a Lie map of \( \mathcal{A} \) to \( \text{Lie}(\mathcal{G}) \). Connes is able to define a trace on \( \mathcal{A} \) associated to \( \mathcal{F} \) assuming that \( [F, a] \) and \( a(F^2 - I) \) are trace class for all \( a \in \mathcal{A} \).

The problem I want to look at is to make sense of his construction, hopefully using the determinant line bundle \( L \). An element \( a \in \mathcal{A} \) will give rise to a vector field on \( P \) and on \( L \). If I have an invariant connection on \( L \), then from the \( \mathcal{A}^*\)-orbit we get an invariant connection on the trivial line bundle over \( \mathcal{A}^* \), which corresponds to a 1-form on \( \mathcal{A}^* \) which is left-invariant. If the connection is flat, then we would have a trace on \( \mathcal{A} \).

But I should begin with the topology first. The space \( P \) has the homotopy type \( \mathbb{Z} \times BU \), and the first Chern class of \( L \) lies in \( H_2^P(\mathbb{Z}) \).
as $\mathcal{D}$ is contractible by the Kuiper theorem. I am trying to produce an elt. of $\text{H}^1(A^*)$. Can I do this topologically? The idea would be to construct maps

$$
\begin{align*}
A^* & \longrightarrow U \\
\downarrow & \downarrow \\
CA^* & \longrightarrow X \\
\downarrow & \downarrow \\
\Sigma A^* & \longrightarrow BU.
\end{align*}
$$

Here $U$ is invertibles which are of the form $1 +$ compact.

Let's go back to the old notation and try to see what the story is in the case of Dirac operators.

$$
\mathcal{D} \longrightarrow A \longrightarrow P_{\mathcal{D}} \times \mathcal{A} \sim \mathcal{B}_\mathcal{A}
$$

$$
\begin{array}{c}
\text{Describing } \Psi: \text{ Over } \mathcal{B}_\mathcal{A} \text{ one has the Hilbert bundles } P_{\mathcal{D}} \times \mathcal{H}^+ \times \mathcal{H}^{-} \\
\text{and the equivariant map } A \longrightarrow D_A : \mathcal{H}^+ \longrightarrow \mathcal{H}^-
\end{array}
$$

from $A$ to $\mathcal{F}(\mathcal{H}^+ , \mathcal{H}^-)$ gives a map

$$
\begin{array}{c}
P_{\mathcal{D}} \times \mathcal{A} \longrightarrow \text{Fred}(P_{\mathcal{D}} \times \mathcal{H}^+ , P_{\mathcal{D}} \times \mathcal{H}^-) \\
\downarrow & \\
\mathcal{B}_\mathcal{D} \times \mathcal{A} \longrightarrow \mathcal{D}(\mathcal{H}^+ , \mathcal{H}^-)
\end{array}
$$

(*) which are trivial by Kuiper's thm. Thus we get a
map \( B\mathfrak{g} \to \mathcal{F}(\mathcal{H}^+, \mathcal{H}^-) \cong \mathbb{Z} \mathfrak{g} \mathfrak{u} \) which is well-defined up to homotopy.

In other words, \( \mathfrak{g} \) is the map which gives the index of the family of Dirac operators over \( PG \times \mathfrak{g} \mathfrak{a} \). It is useful to assume \( \mathfrak{g} \) acts freely, so that \( B\mathfrak{g} = \mathfrak{g} \mathfrak{a} \).

Next, we want to describe the map \( \Sigma \mathfrak{g} \to B\mathfrak{g} \) belonging to \( \mathfrak{g} \to \mathfrak{a}, g \to g(A) \). This uses the contractibility of \( A \).

\[
\begin{align*}
\mathfrak{g} & \rightarrow A \\
\downarrow & \\
\mathfrak{g} & \rightarrow C\mathfrak{g} \\
\downarrow & \\
\Sigma \mathfrak{g} & \rightarrow \mathfrak{g} \mathfrak{a} = B\mathfrak{g}
\end{align*}
\]

Thus, if I take \( L \) over \( B\mathfrak{g} \) and pull it back to \( \mathfrak{g} \), this line bundle becomes trivial for two reasons:
1) \( \mathfrak{g} \)-invariance,
2) \( L \) becomes trivial over \( A \).

This shows that in order to construct a 1-form on \( \mathfrak{g} \) corresponding to \( c_1(L) \), I seem to have to build \( \mathfrak{g} \) in the contractibility of \( A \).

Let us now try to work directly with the operators \( \mathcal{P} : \mathcal{H}^+ \to \mathcal{H}^- \) and call the space \( \mathcal{F} \) of these operators \( \mathcal{F}(\mathcal{H}^+, \mathcal{H}^-) \) or just \( \mathcal{F} \). We have the group \( \mathfrak{g} \) acting on \( \mathcal{F} \) by \( g(P) = gPg^{-1} \), and the equivariant line bundle \( L \) over \( \mathcal{F} \). The line bundle represents the 2-dim cohomology class on \( \mathcal{F} \). I want to construct a 1-dim class on \( \mathfrak{g} \). Map \( \mathfrak{g} \to \mathcal{F} \) by \( g \to g(P) \); then the pull-back of \( L \) has the trivialization
because of the $G$-action. On the other hand I need another trivialization which should come by using the linear structure, that is, the linear path $tP_1 + (1-t)P_0$, joining $P_0$ to a typical point $P_1 = gP_0g^{-1}$ of the $G$-orbit of $P_0$.

Interesting point. It just occurred to me that a trace on the Lie algebra of $G$ is equivalent to a homomorphism $G \to \mathbb{C}^*$. In other words one has some sort of determinant homomorphism defined on the invertible orbits.

The first point is that the operators $P$ that we are considering all differ from $P_0$ by a trace class operator. For example

$$P_0 - gP_0g^{-1} = P_0g^{-1} - gP_0^{-1} = [P_0g]g^{-1}$$

and we are assuming that $[P_0g]$ is of trace class.

On the other hand Fredholm operators are stable under compact perturbations: if $P$ is Fredholm and $K$ is compact, then $P + K$ is Fredholm.

So now what we can do is the following in the case of index zero. Fix a $P_0$ which is invertible and then define a determinant function

$$\Phi(P_0 + B) = \det(1 + P_0^{-1}B)$$

on the set of all operators differing from $P_0$ by a trace class operator. This function is non-vanishing exactly when $P_0 + B$ is invertible, so it probably is equivalent to a trivialization of $L$ over $\{P_0 + B | B \text{ of trace class}\}$.  

A better way to describe this function is to introduce the function
\[ \Phi(P_1, P_2) = \det(P_1^{-1}P_2) \]
defined on \((P_0 + \text{trace class}) \cap \text{invertible}\). Then
\[
\Phi(gP_1, gP_2) = \det((gP_1g^{-1})(gP_2g^{-1}))
\]
\[= \det(gP_1^{-1}P_2g^{-1}) = \det(P_1^{-1}P_2)\]
\[= \Phi(P_1, P_2)\]
so that \(\Phi\) is invariant under the \(g\) action.

Also we have
\[
\Phi(P_1, P_2) \ast \Phi(P_2, P_3)
\]
\[= \det(P_1^{-1}P_2)\det(P_2^{-1}P_3) = \det(P_1^{-1}P_3)\]
\[= \Phi(P_1, P_3).\]

Thus
\[
\Phi(P_0, g_1g_2P_0) = \Phi(P_0, g_1P_0) \Phi(g_1P_0, g_1g_2P_0)
\]
\[= \Phi(P_0, g_1P_0) \Phi(P_0, g_2P_0),\]
which shows that we really have a determinant function.
Yesterday I considered the problem of constructing a trace for a 1-summable Fredholm module. More precisely I consider a coet $P_0 + L^1$ of Fredholm operators

$$P : \mathcal{H}^+ \longrightarrow \mathcal{H}^-$$

module trace class operators. Over this coet the determinant line bundle $L$ appears to have a canonical flat connection. Evidence: suppose the index of the coet is zero. Then the coet contains invertible elements: fix one $P_0$ and define

$$\varphi(P) = \det(P_0^{-1}P)$$

on the coet. This function is analytic and vanishes exactly where $P$ is singular. This suggests that $\varphi$ is obtained from a trivialization of $L$ by taking the image of the canonical section under this trivialization.

Independence of the choice of $P_0$:

$$\det(P_0^{-1}P) = \det(P_0^{-1}P_0) \det(P_0^{-1}P)$$

Because the determinant line bundle has this canonical flat connection, the flat sections form a 1-diml repren. of the group $G$ of automos of $\mathcal{H}$ preserving the coet.

In fact

$$\varphi(gPg^{-1}) = \det(P_0^{-1}gPg^{-1}) = \det(P_0^{-1}gP_0g^{-1}) \det(gP_0^{-1}g^{-1})$$

$$= \varphi(gP_0g^{-1}) \varphi(P)$$
whence \[
\frac{\varphi(gPq^{-1})}{\varphi(p)} = \frac{\varphi(gPq^{-1})}{\varphi(p)}
\]
is independent of the choice of \(p_0\). From this it follows that we have a character of \(G\) given by
\[
g \mapsto \frac{\varphi(gPq^{-1})}{\varphi(p)} = \frac{\det(p_0^{-1}gPq^{-1})}{\det(p_0^{-1}p)} = \det(p_0^{-1}gPq^{-1})
\]
for any invertible \(P\) in the coset. The corresponding character on the Lie algebra is
\[
x \mapsto \text{tr}(P^{-1}[x,P])
\]
and it is independent of the choice of \(P\) in the coset.

Check: Let \(P \mapsto P + \delta P\) with \(\delta P\) of trace class. Then
\[
\text{tr}(P^{-1}[x,P]) = \text{tr}(-P^{-1}\delta P \text{tr}(xP - \overline{P}x) + P^{-1}(x\delta P - \overline{\delta P}x))
\]
\[
= \text{tr}(-P^{-1}xpP^{-1}\delta P + P^{-1}x\delta P) = 0
\]

Let's see if we can work out a formula for the character on the Lie algebra which works in general. The idea will be to start with
\[
\text{tr}(P^{-1}[x,P])
\]
and then replace \(P^{-1}\) by a parametric \(Q\) for \(P\), i.e., \(P^{-1} = Q + J\) where \(J : \mathcal{H}^--\mathcal{H}^+\) is of trace class. So we get
\[ \text{tr} \left( P^{-1} [x, P] \right) = \text{tr} \left( Q [x, P] + J [x, P] \right) \]
\[ = \text{tr} (Q [x, P]) + \text{tr} (J [x, P]) \]

Now I want to proceed as follows:
\[ = \text{tr} (Q [x, P]) + \text{tr} ( [J x, P] - [J, P] x ) \]
\[ + \text{tr} ( [J x, P] ) - \text{tr} ( [J, P] x ) \]
\[ = 0 \text{ as } J x \]
\[ \text{is of trace class} \]
\[ = \text{tr} (Q [x, P]) + \text{tr} ( [Q, P] x ) \]
\[ = \text{tr} ( [Q x, P] ) \]

This calculation should be valid for operators on the same space, so I have to check it works in the graded case. The point is that \([x, P] = x^{-p} - px^+\).

\[ \text{tr} (P^{-1} [x, P]) = \text{tr} \left( Q [x, P] \right) + \text{tr} (J [x, P]) \]

Now
\[ \text{tr}_A^+ (\mathcal{A} [x, P]) = \text{tr}_A^+ (\mathcal{A} x^- p) - \text{tr}_A^+ (J p x^+) \]
\[ = \text{tr}_A^+ (p j x^-) - \text{tr}_A^+ (j p x^+) \]

Now
\[ p j = p (p^{-1} - q) = I - pq \]
\[ j p = (p^{-1} - q) p = I - q p \]

So
\[ \text{tr}_A^+ (J [x, P]) = \text{tr}_A^+ ( (I - pq) x^- ) - \text{tr}_A^+ ( (I - q p) x^+ ) \]
\[ \text{tr}_A^+ (Q [x, P]) = \text{tr}_A^+ (Q (x^- p - p x^+)) \]
So adding we get that
\[
- (I - QP)x^+ + Q(x^+ P - Px^+)
\]
\[
\text{tr}_+ (P^{-1} [x, P]) = \text{tr}_+ (- (I - PQ)x^-) + \text{tr}_+ (Qx^- P - x^+)
\]
The right side makes sense in general. Moreover, taking the variation w.r.t. $SQ$ of trace class gives
\[
- \text{tr}_- (PSQx^-) + \frac{1}{\text{tr}_+ (PSQx^-)} \text{tr}_+ (SQx^- P) = 0
\]
Also taking the variation w.r.t $SQ$ gives
\[
- \text{tr}_- (SPQx^-) + \text{tr}_+ (Qx^- SP)
\]
which is also zero.

Now we can argue this function has to be a trace by continuity from the invertible case. This handles index 0 and the general case follows by adding a trivial piece left fixed by $x$ in order to cancel the index.

Now I want to understand better what the determinant line bundle is doing. Let $B = C_0 (\mathbb{H})$ and let $I$ be the ideal of trace class operators. Any Fredholm operator is invertible mod $I$, hence
\[
F/I = (B/I)^ injecting
\]

\[
\frac{F}{I} = (B/I)^ injecting
\]
We have an extension of groups

\[ 0 \to (1+\mathfrak{d})^* \to B^* \to (B/\mathfrak{d})^* \to 0 \]

This group is mapping into \( B^* \) and the element
\( P_0 \) of \( B^* \) such that \( g P_0 g^{-1} \equiv P_0 \mod \mathfrak{d} \).

We have also a sequence of monoids

\[ 1+\mathfrak{d} \to F \to (B/\mathfrak{d})^* \]

but it is not clear that one can say anything interesting.

At this point I have explored the 1-dim Lie cocycles on \( G \) belonging to a Fredholm \( P \)

preserved by \( \tilde{G} \) modulo trace class.

The next case is to look at 2-dim Lie cocycles on \( \tilde{G} \). This should connect up with the loop group theory.

Let's begin as before with the topological analysis in the Dirac operator situation. We have an odd manifold like the circle and a family of self-adjoint Fredholm operators over \( A \) equivariant for \( \tilde{G} \).
cohomology class on $G$, in fact a left-invariant 2 form. This corresponds to a three-dimensional cohomology class on $BG \cong G/A$.

**Question**: Can one find an infinite dimensional Azumaya algebra in the $\mathbb{Z}$ group case? This should be an algebra with center the functions on $G/A$, hence might be the functions on $A$ crossed product with $G$. Somehow this algebra when restricted or "localized" near an orbit of $G$ on $A$ should have an irreducible representation related to the line bundle over the orbit.

The standard way to proceed to describe an invariant 2-form in $G$ is to consider the maps $g \mapsto g e g^{-1}$ to the Grassmannians and pull back the first Chern form $\text{tr} e(\text{de})^2 = \frac{1}{8} \text{tr} (F(Fe)^2)$ where $F = 2e - 1$. The corresponding two-form $\tilde{\omega} = A$ is

$$a, b \mapsto \text{tr} \left( e[a, e][b, e] - e \left[ \left[ b, e \right][a, e] \right] \right)$$

$$= \text{tr} \left( (2e-1)[a, e][b, e] \right) = \frac{1}{4} \text{tr} (F[5a][5b])$$

$$= \frac{1}{4} \text{tr} \left( (a - Fa F)[5b] \right) = \frac{1}{2} \text{tr} (a[5b])$$

$$= \text{tr} (a[e, b])$$

Little idea: If $P, Q$ are inverses mod $\mathbb{Z}$, then one can see directly that $\text{Ind}(P) = -\text{Ind}(Q)$. But then this implies that $\text{Ind}(P + K) = \text{Ind}(P)$ for $K$ of trace class.
Recall the definition of the $S$ operator on $\tilde{\Omega}(a^+)$.

By the universal property of $\tilde{\Omega}(a^+)$

the homomorphism

\[
\begin{align*}
\alpha & \mapsto \alpha e \\
\alpha & \mapsto \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+)
\end{align*}
\]

extends uniquely to a morphism of differential graded algebras. $\tilde{\Omega}(ke^+)$ has the basis $e_1(ke_1^+), e_2(ke_2^+)$ so we can take the coefficient of $e(de)^2$.

Question: Does $S^2: \tilde{\Omega}^* \rightarrow \tilde{\Omega}^{*+4}$ give the coefficient of $e(de)^4$?

To compute $S^2$, we need

\[
\begin{align*}
\tilde{\Omega}(a^+) & \rightarrow \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \\
S \quad & \rightarrow \tilde{\Omega}(a^+) \\
\text{coeff of } e_{1de_2^2} & \rightarrow \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke_1^+) \\
S \quad & \rightarrow \tilde{\Omega}(a^+) \\
\text{coeff of } e_1de_1^2 & \rightarrow \tilde{\Omega}(a^+)
\end{align*}
\]

Thus we see that $S^2$ gives the coefficient of $e_1de_1^2 e_2de_2^2$ under the map

\[
\tilde{\Omega}(a^+) \rightarrow \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke_1^+) \otimes \tilde{\Omega}(ke_2^+)
\]
\[
\hat{\Omega}(a^+) \longrightarrow \hat{\Omega}(a^+) \otimes \hat{\Omega}(ke_1^+) \otimes \hat{\Omega}(ke_2^+)
\]
\[
\hat{\Omega}(a^+) \otimes \hat{\Omega}(ke^+) \quad \quad \xrightarrow{a} e, e_2.
\]

So all we have to do is to see what linear functional on \(\hat{\Omega}(ke^+)\) is obtained by using the homomorphism \(\hat{\Omega}(ke^+) \longrightarrow \hat{\Omega}(ke_1^+) \otimes \hat{\Omega}(ke_2^+)\) and then taking the coefficient of \(e_1 de_1^2 e_2 de_2^2\).

\[
(de)^4 \longmapsto (e_1 de_2 + de_1 e_2)^4
\]

This should go to zero because our "integral" should kill \(\text{Im } d\). Let's check it. There are 16 terms, but we want two \(de_1\) and two \(de_2\), which reduces us to 6 terms:

\[
(e_1 de_2)^2 (de_1 e_2)^2 \longmapsto e_1 de_1^2 e_2 de_2^2
\]

\[
0 = (e_1 de_2)(de_1 e_2)(e_1 de_2)(de_1 e_2) \quad \quad (e_1 de_2)(de_1 e_2)^2 (de_1 e_2) \longmapsto e_1 de_1^2 de_2 e_2 de_2
\]

\[
0 = (de_1 e_2)(e_1 de_2)(de_1 e_2)(e_1 de_2) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (de_1 e_2)^2 (e_1 de_2)^2 \longmapsto e_1 de_1^2 e_2 de_2^2
\]

\[e_1 de_1^2 + de_1 e_1 de_1 = de_1^2\]

\[\frac{1}{(1-e_1) de_1}\]

\[de_1^2 e_2 de_2^2 + e_1 de_1^2 de_2^2\]

Which does give zero.
$(de)^4$ will involve six terms also but because we multiply by $e_1e_2$ those having $de, e_1de$, or $de_2 e_2 de_2$ will give zero. Thus we are going to get

$$e (de)^4 \longrightarrow 2 \; e_1 de_1^2 e_2 de_2$$

It seems worthwhile to describe carefully the coalgebra $\tilde{\Omega}(ke^+)$ since this acts on $\tilde{\Omega}(a^+)$ for any $\alpha$. We have

$$\tilde{\Omega}(ke^+) \longrightarrow \tilde{\Omega}(ke^+) \otimes \tilde{\Omega}(ke^+)$$

given by $e \longrightarrow e_1 e_2$ where $e_1 = e \otimes 1$, $e_2 = 1 \otimes e$. It looks like $\tilde{\Omega}(ke^+)$ is a Hopf algebra, except for the fact that it isn't reduced to $k$ in degree zero.

We have

$$e (de)^2 \longrightarrow e_1 e_2 (de_1 e_2 + e_1 de_2)^2$$

$$= e_1 e_2 (de_1 e_2 + de_1 e_2 + de e_1 de_2 + e_1 de_2 de_1 e_2 + e_1 de_2 e_1 de_2)$$

$$= (e_1 de_1^2) e_2 + e_1 (e_2 de_2^2)$$

Consequently, we use the basis $\frac{e (de)^2}{n!}$ for $H^{2n}(ke^+)$.?
Let's go over the $S$-operator. There is a unique morphism of DG algebras
\[
\tilde{\Omega}(a^+) \xrightarrow{\Delta} \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+)
\]
such that $a \mapsto ae$. Moreover, $\Delta$ is associative in the sense that
\[
\begin{array}{ccc}
\tilde{\Omega}(a^+) & \xrightarrow{\Delta} & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \\
\downarrow & & \downarrow \Delta \otimes \text{id} \\
\tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) & \xrightarrow{\text{id} \otimes \Delta} & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \otimes \tilde{\Omega}(ke^+)
\end{array}
\]
commutes. Using $\Delta$ one makes $\tilde{\Omega}(ke^+)^*$ into a DG algebra which is commutative, and $\tilde{\Omega}(a^+)$ becomes a module over it. I should check that $\tilde{\Omega}(ke^+)^*$ is unital. Define a counit
\[
\tilde{\Omega}(ke^+) \longrightarrow k \\
e \mapsto 1
\]
and it is then clear that
\[
\begin{array}{ccc}
\tilde{\Omega}(a^+) & \xrightarrow{\Delta} & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \\
\downarrow \text{id} & & \downarrow \\
\tilde{\Omega}(a^+) & \otimes & \tilde{\Omega}(ke^+) \otimes k
\end{array}
\]
commutes: $a \mapsto ae$.
I computed the structure of \( \hat{\Omega}(ke^*)^* \) and got the following. Define \( 1, \varepsilon \in \hat{\Omega}^0 \) by
\[
\langle 1, (e^1) \rangle = (1) \\
\langle \varepsilon, (e^1) \rangle = (0)
\]
and \( \gamma, \delta \in (\hat{\Omega}^1)^* \), \( \beta \in (\hat{\Omega}^2)^* \) by
\[
\langle \gamma, (de) \rangle = (1) \\
\langle \delta, (ede) \rangle = (0) \\
\langle \beta, (ede^2) \rangle = (1) 
\]
where \( de \) denotes the divided power algebra.

Then
\[
\hat{\Omega}(ke^*)^* = ke \oplus (\Lambda [\gamma, \delta] \otimes \Gamma(\beta))
\]
where \( \Gamma \) denotes the divided power algebra. Here \( \varepsilon \) kills \( \gamma, \delta, \beta \), so that it would appear very natural to replace \( \hat{\Omega}(ke^*) = [\bullet] \oplus ke \) by just \( ke \). This kills \( \varepsilon \) in the dual.

The differential we computed to be
\[
\begin{align*}
d \delta &= 0 \\
d \gamma &= -1 + \varepsilon \quad \text{(hence } d \gamma = -1 \text{ if } \varepsilon = 0) \\
d \beta &= 0
\end{align*}
\]
I should check the divided power structure.
\[
\langle \beta^n, e(e^2)^n \rangle = \langle \beta^n, e_1 \cdots e_n (de_1 \cdots e_n)^2 \rangle
\]
Better is to use $\Delta(e(de)^2) = e(de) e = e_1 e e_2$. Hence $\Delta^n(e(de)^2) = (\sum e_i (de_i)^2) e_1 \cdots e_n$ so

$$\langle \beta^n, (e(de)^2)^n \rangle = \langle \beta^n, (\sum e_i (de_i)^2)^n e_1 \cdots e_n \rangle = n!$$

Summary. Introduce $\Omega(A)$ as Cannas does, so that $\Omega(A)$ is non-unital. Then we have

$$\Omega(A) \rightarrow \Omega(A) \otimes \Omega(k(e))$$

$$\tilde{\Omega}(a^+) \rightarrow \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(k(e^+))$$

and we have determined the following structure for the DG algebra $\Omega(k(e))^*$:

$\Omega(k(e))^*$ basis: \begin{array}{ccc}
e & de & de^2 \\
\end{array}

$\Omega(k(e))^*$ basis: \begin{array}{ccc}
1 & +\eta & -\eta \tilde{\eta} \\
\tilde{\eta} & \beta & \\
\end{array}

$$d\eta = -1, \quad d\tilde{s} = 0, \quad d\beta = 0$$

Note: because $\tilde{s}$ is a cycle it induces a morphism of $\Omega(A)$ to $\Omega(A)$ of degree $-1$ which commutes with $\Delta$. 


Let us now try to relate what we know about the $S$ operator to the Lie algebra cohomology of $\mathfrak{gl}_n(A)$. Recall that we have a canonical element

$$\Theta \in C^1(\mathfrak{g}, \mathbb{M}_n(A)) = \mathbb{M}_n(C^1(\mathfrak{g}, A))$$

where $\mathfrak{g} = \mathfrak{gl}_n(A)$ given by the identity map. This Maurer-Cartan form satisfies

$$\delta \Theta + \Theta^2 = 0$$

where $\delta$ is the differential as Lie cochains. Now suppose I have a homomorphism

$$\mathbb{M}_n(A) \rightarrow \mathbb{R}^*$$

where $\mathbb{R}^*$ is a DG algebra. Then we get a bigraded differential algebra $(\mathfrak{g}, \mathbb{R}^*)$ and $\Theta \in C(\mathfrak{g}, \mathbb{R}^*)$ satisfies $d'\Theta + \Theta^2 = 0$.

**Example:** What is the largest possible $\mathbb{R}^*$? The important thing is that the homomorphism $\Theta: \mathfrak{g} \rightarrow \mathbb{R}^*$ be a Lie homomorphism. So therefore we want $\mathbb{R}^*$ to be $\mathcal{U}(\mathfrak{g})$ and $\mathbb{R}^*$ to be $\tilde{\Omega}(\mathcal{U}(\mathfrak{g}))$.

When $\mathfrak{g} = \mathfrak{gl}_n(A)$ some other possibilities are $\mathbb{R}^* = \mathbb{M}_n(A)$ or $\mathbb{R}^* = \mathbb{M}_n(A^+)$. And then we have a choice of $\mathbb{R}^* = \mathbb{M}_n \tilde{\Omega}(A)$ or $\tilde{\Omega}(\mathbb{M}_n A)$. This raises the
Question: Is the complex $\tilde{\Omega}(\Lambda)/[\Lambda]$ Morita invariant?

Anyway, let us now construct other classes from $\Theta \in C^1(\partial, R^n)$ using the 1-parameter family of connections $D_t = d + t\Theta$, where $d = d' + d''$. One has

$$D_t^2 = (d + t\Theta)^2 = t d'' \Theta + (t^2 - t) \Theta^2$$

Hence

$$\frac{d}{dt} e^{D_t^2} = \int_0^1 ds \ e^{(1-s)D_t^2} \left( \int_0^1 e^{(1-s)D_t^2} \Theta e^{sD_t^2} \right)$$

At this point if we project modulo commutators we get

$$\frac{d}{dt} e^{D_t^2} = d \ \Theta e^{D_t^2}$$

or

$$e^{D_t^2} - e^{D_0^2} = d \int_0^1 dt \ \Theta e^{t d'' \Theta + (t^2 - t) \Theta^2}$$

$$e^{d'' \Theta} - 1$$

One thing obtained in this way are forms

$$\Theta (d'' \Theta)^n \in C^{n+1}(\partial, R^n/dR^{n-1})$$

where $R = R/[R, R]$. These are Liecocycles.
\[ C_p = A^{p+1}/(1-t). \]

Define
\[ C_p \rightarrow \bar{\Omega}_p / d\bar{\Omega}_r \]+ [dA, \bar{\Omega}_r].\]

by \((a_0, \ldots, a_p) \mapsto a_0 da_1 \ldots da_p.\) In fact we have
\[ A \otimes \bar{A}^p \twoheadrightarrow \bar{\Omega}_p \]
\[ \downarrow \]
\[ \bar{A}^{p+1} \twoheadrightarrow \bar{\Omega}_p / d\bar{\Omega}_r \]
\[ \downarrow \]
\[ C_p \rightarrow \bar{\Omega}_p / d\bar{\Omega}_r \]+ [dA, \bar{\Omega}_r].\]

To see the dotted arrow exists:
\[ a_0 da_1 \ldots da_p \equiv (-1)^p da_p a_0 da_1 \ldots da_p \quad \text{mod} \ [dA, \bar{\Omega}_r]. \]
\[ \equiv (-1)^p a_0 da_1 \ldots da_p \quad \text{mod} \ [d\bar{\Omega}_r]. \]

Conversely, the reverse map \( a_0 da_1 \ldots da_p \rightarrow (a_0, \ldots, a_p) \) sends \( d\bar{\Omega}_r \) to zero, and tells
\[ a_0 da_1 \ldots da_p \equiv (-1)^p a_0 da_1 \ldots da_p + \ldots + (-1)^p da_p a_0 da_1 \ldots da_p \]
\[ = a_0 da_1 \ldots da_p - (-1)^p da_p a_0 da_1 \ldots da_p \]
\[ = [a_0 da_1 \ldots da_{p-1}, da_p]. \]

Thus I conclude
\[ C_p = \bar{\Omega}_p / d\bar{\Omega}_r \]+ [dA, \bar{\Omega}_r].\]
Next we have

\[ b(a_0, \ldots, a_{p+1}) = (a_0 a_1 \ldots a_{p+1}) - (a_0 a_2 \ldots a_{p+1}) + \ldots + (-1)^p (a_{p+1} a_0 a_1 \ldots a_p) \]

\[ + (-1)^p a_0 d(a_2 \ldots a_{p+1}) \]

Thus we conclude

\[ A \otimes \bar{A}^p = \tilde{\Omega}^p \]

\[ \bar{A}^{p+1} = \tilde{\Omega}^p / d\tilde{\Omega}^{p-1} \]

requires \( p > 0 \)

\[ \bar{C}_p = \bar{A}^{p+1} / (1 - t) = \tilde{\Omega}^p / d\tilde{\Omega}^{p-1} + [dA, \tilde{\Omega}^{p-1}] \]

\[ \bar{C}_p / b\bar{C}_{p+1} = \tilde{\Omega}^p / d\tilde{\Omega}^{p-1} + [dA, \tilde{\Omega}^{p-1}] + [A, \tilde{\Omega}^p] \]

From this we conclude that
$H\overline{C}_p(A)$ and $H\overline{\Omega}^p(A)$ both inject into $\overline{\Omega}^p/d\overline{\Omega}^{p+1}$.

What I want to do now is to show that, viewed as subspaces of $\overline{\Omega}^p/d\overline{\Omega}^{p+1}$, the Connes Karoubi homology $H\overline{\Omega}^p(A)$ is contained in the reduced cyclic homology, in fact it coincides with the kernel of $B: H\overline{C}_p(A) \to H^{p+1}_p(A)$.

First of all recall that $B: A\otimes \overline{A}^p \to A\otimes \overline{A}^{p+1}$ is the composition

$$A\otimes \overline{A}^p \xrightarrow{N} \overline{A}^{p+1} \xrightarrow{\iota \otimes ?} A\otimes \overline{A}^{p+1}$$

so that we can view it as the composition

$$A\otimes \overline{A}^p \xrightarrow{\iota \otimes ?} \overline{C}_p \xrightarrow{B = 1 \otimes N} A\otimes \overline{A}^{p+1}.$$ 

Secondly $Bb + bB = 0$. Hence $B$ induces a map

$$\overline{C}_p/b\overline{C}_{p+1} \xrightarrow{B} A\otimes \overline{A}^{p+1} \big/ b(A\otimes \overline{A}^{p+2})$$

$$H^p(\overline{C}(A)) \xrightarrow{B} H^{p+1}_p(A)$$

Notice also that if $\alpha \in \overline{C}_p$ satisfies $B\alpha \in \text{Im} b$, then $B(b\alpha) = -bB\alpha \in b\text{Im} b = 0$, so $b\alpha = 0$ as $B: \overline{C}_p \to A\otimes \overline{A}^{p+1}$ is injective.
Thus we have

$$\text{Ker} \left( \frac{C_p / b C_{p+1}}{B} \rightarrow A \otimes \bar{A}^{p+1} \bigg/ b(A \otimes \bar{A}^{p+2}) \right)$$

$$= \text{Ker} \left( H \bar{C}_p(A) \xrightarrow{B} \bar{H}_{p+1}(A) \right)$$

On the other hand

$$H \bar{\Omega}_p(A) = \text{Ker} \left( \frac{\bar{\Omega}^p / d \bar{\Omega}^{p-1}}{d} \rightarrow \bar{\Omega}^{p+1} \right)$$

so the only thing to do is correlate $B, d$:

$$\frac{C_p / b C_{p+1}}{B} \xrightarrow{\|} \frac{A \otimes \bar{A}^{p+1} / b(A \otimes \bar{A}^{p+2})}{\|}$$

$$\frac{\bar{\Omega}^p / d \bar{\Omega}^{p-1}}{d} \xrightarrow{\|} \frac{\bar{\Omega}^{p+1} / [A, \bar{\Omega}^{p+1}]}{[i]} \xrightarrow{\text{quotient by } \mathbb{Z}/p^2 \text{ action}}$$

Define $i$ to be the inclusion of the invariants.

Claim then $B = \begin{pmatrix} \mathbb{Q} \\ \mathbb{Q} \end{pmatrix} \oplus \text{id}$

$$\left( a_0, a_1, \ldots, a_p \right) \xrightarrow{B} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \left( a_0 a_1 a_p, a_{i+1}\right)$$

$$\frac{1}{p!} a_0 da_1 \ldots da_p \xrightarrow{d} \frac{1}{p!} da_0 da_1 \ldots da_p \xrightarrow{i} \frac{1}{(p+1)!} \sum_{i=0}^{p} (-1)^i \binom{p}{i} \ldots$$
In more detail we define

\[ i : \tilde{\Omega}^{p+1} \rightarrow \tilde{\Omega}^{p+1} / [A, \tilde{\Omega}^{p+1}] \]

by

\[ \omega_0 \ldots \omega_p \mapsto \left( \sum_{i=0}^{p} (-1)^i \omega_i \ldots \omega_p \omega_0 \ldots \omega_{i-1} \right) / (\rho+1) \]

Because \( i \) is injective one has then \( \ker B = \ker d \).