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$$H^p(\overline{\Omega}) = \operatorname{Im}\{S : \overline{HC}_{p+2}(A) \longrightarrow \overline{HC}_p(A)\}$$

October 20, 1983

We have certain normalization problems to solve. Precisely, how do we define maps

$$K_0 A \longrightarrow HC_{2n}(A)$$
$$K_1 A \longrightarrow HC_{2n-1}(A)$$

which when mapped to $\Omega^{2n}/d\Omega^{2n-1}$ (resp. $\Omega^{2n-1}/d\Omega^{2n-1}$) coincide with ch_n and e_{2n-1} respectively.

Suppose an element of $K_0 A$ is given by an idempotent e. Then

$$ch_n(e) = \frac{1}{n!} tr\left(e(de)^{2n}\right)$$
$$= \frac{2n!}{n!} \mu\left(\overset{tr}{\wedge} e^{\otimes(2n+1)}\right).$$

Thus it appears that

$$e \longmapsto \frac{2n!}{n!} tr\left(e^{\otimes(2n+1)}\right) \qquad \text{in } A^{\otimes(2n+1)}/(1-t)$$

will be the good map.

Let's check that these maps for different n are compatible under Connes S operator.

$$\begin{tikzcd}
 & A^{\otimes(2n+1)}/1-t & A^{\otimes(2n+1)} \arrow[l] \arrow[dd, "b"] & & & \\
e^{\otimes(2n+1)} & e^{\otimes(2n+1)} \arrow[l, mapsto] \arrow[dd, mapsto] & & & & \\
 & & A^{\otimes 2n} & & A^{\otimes 2n} \arrow[ll, "1-t"'] \arrow[dd, "b'"] & \\
 & e^{\otimes(2n)} & & \frac{1}{2}e^{\otimes 2n} \arrow[ll, mapsto] \arrow[dd, mapsto] & & \\
 & & & & A^{\otimes(2n-1)} & A^{\otimes(2n-1)}/(1-t) \arrow[l, "N"'] \\
 & & & \frac{1}{2}e^{\otimes(2n-1)} & & \frac{1}{2(2n-1)}e^{\otimes(2n-1)} \arrow[ll, mapsto]
\end{tikzcd}$$

This calculation shows that

$$S\, e^{\otimes(2n+1)} = \frac{1}{2(2n-1)}\, e^{\otimes(2n-1)}$$

hence

$$S\left(\frac{2n!}{n!}\, e^{\otimes(2n+1)}\right) = \frac{1}{2(2n-1)}\,\frac{2n!}{n!}\, e^{\otimes(2n-1)}$$

$$= \frac{(2n-2)!}{(n-1)!}\, e^{\otimes(2n-1)}$$

so it works!

> The good map $K_0 A \longrightarrow HC_{2n}(A)$ is given by $\quad e \longmapsto \frac{(2n)!}{n!}\operatorname{tr}\left(e^{\otimes 2n+1}\right) \mod 1-t$

Now suppose an element of $K_1 A$ is given by and invertible g. The odd character $e_{2n-1}(g) \in \Omega^{2n-1}$ is given by

$$\int_0^1 dt\, \operatorname{tr}\left(\theta \cdot \frac{\theta^{2(n-1)}}{(n-1)!}\right)(t^2-t)^{n-1} \qquad \theta = g^{-1}dg$$

(Recall the transgression formula

$$\int_0^1 dt \quad \operatorname{tr}\left(\theta\, e^{t\,d\theta + (t^2-t)\theta^2}\right). \qquad)$$

Now

$$\int_0^1 (t^2-t)^{n-1}\, dt = (-1)^{n-1}\int_0^1 t^{n-1}(1-t)^{n-1}\, dt$$

$$= (-1)^{n-1}\beta(n,n) = (-1)^{n-1}\frac{\Gamma(n)\Gamma(n)}{\Gamma(2n)} = (-1)^{n-1}\frac{(n-1)!^2}{(2n-1)!}$$

So

$$e_{2n-1}(g) = (-1)^{n-1}\frac{(n-1)!}{(2n-1)!}\, tr\left((g^{-1}dg)^{2n-1}\right)$$

$$= \frac{(n-1)!}{(2n-1)!}\, tr\left(g^{-1}dg(dg^{-1}dg)^{n-1}\right)$$

$$= \mu\left((n-1)!\, tr\left((g^{-1}\otimes g)^{\otimes n}\right)\right)$$

Thus it appears that the good map will be

$$g \longmapsto (n-1)!\ tr\,(g^{-1}\otimes g)^{\otimes n} \quad \in \bar{A}^{\otimes 2n}/(1-t).$$

The reason we take the image in the reduced cyclic homology is so that $(g^{-1}\otimes g)^{\otimes n}$ is a cyclic cyclic, e.g.

$$b(g^{-1}\otimes g\otimes g^{-1}\otimes g) = 1\otimes g^{-1}\otimes g - g^{-1}\otimes 1\otimes g + g^{-1}\otimes g\otimes 1 - 1\otimes g\otimes g^{-1}$$

$$\equiv 2\left(1\otimes g^{-1}\otimes g - 1\otimes g\otimes g^{-1}\right) \quad \mathrm{mod}\ (1-t)$$

is not a cyclic cycle.

Let's check compatibility with S

$$\begin{tikzcd}
(g^{-1}\otimes g)^{\otimes(n+1)} & (g^{-1}\otimes g)^{\otimes(n+1)} \in A\otimes\bar{A}^{2n+1} \arrow[l, mapsto] \arrow[d, mapsto, "b"] & \\
& 1\otimes(g^{-1}\otimes g)^{\otimes n} - 1\otimes(g\otimes g^{-1})^{\otimes n} & \\
& & \frac{1}{n}(g^{-1}\otimes g)^{\otimes n} \arrow[ul, mapsto, "B"']
\end{tikzcd}$$

So we have

$$S\,(g^{-1}\otimes g)^{\otimes(n+1)} = \frac{1}{n}(g^{-1}\otimes g)^{\otimes n}$$

whence

$$\boxed{S\ (n+1)!\,(g^{-1}\otimes g)^{\otimes(n+1)} = n!\,(g^{-1}\otimes g)^{\otimes n}}$$

Hence we see

> The good map $K_1A \longrightarrow H\bar{C}_{2n-1}(A)$ is given by
>
> $$g \longmapsto (n-1)!\ tr\,(g^{-1}\otimes g)^{\otimes n} \quad \in \bar{A}^{\otimes 2n}/1-t$$

Note that from the exact sequence

$$\longrightarrow HC_{2n-1}(k) \longrightarrow HC_{2n-1}(A) \longrightarrow H\bar{C}_{2n-1}(A) \longrightarrow$$
$$HC_{2n-1}(k) = 0$$

it would be nicer to define the map from K_1A so that it landed in $HC_{2n-1}(A)$.

Let's put $\mathcal{a} = \tilde{A}$ = the ring obtained by adjoining an identity to A. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & \mathcal{a} & \xrightarrow{\varepsilon} & k & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle p} & & & & \\ & & & & A & & & & \end{array}$$

where $p(c1 + i(a)) = c + a$. i, p, ε are ring homomorphisms and $\mathfrak{a} \xrightarrow[\cong]{(\varepsilon, p)} k \times A$ is a ring isomorphism. Now given an invertible $g \in A$ we lift it to $\mathfrak{a}$ with augmentation $= 1$, and get the element $1 + i(g-1)$. Then under

$$\mathfrak{a} \longrightarrow \bar{\mathfrak{a}} \simeq A$$

$$1 + i(g-1) \longmapsto g - 1.$$

In other words we have

$$\overset{g}{K_1 A} \longrightarrow \overset{1+i(g-1)}{K_1 \mathfrak{a}} \longrightarrow \overline{HC}_{2n-1}(\bar{\mathfrak{a}}) = HC_{2n-1}(A)$$

is given by the map

$$g \longmapsto (n-1)! \; tr\left(\left((g^{-1}-1)\otimes(g-1)\right)^{\otimes n}\right) \in A^{\otimes 2n}/1-t$$

Actually even if A has no unit we can probably define $K_1 A$ using quasi-inverses, i.e. u is quasi-invertible when there is a v with

$$u + v + uv = 0.$$

Then the good map

$$K_1 A \longrightarrow HC_{2n-1}(A)$$

is $$u \longmapsto (n-1)! \; tr\left((v \otimes u)^{\otimes n}\right) \in A^{\otimes 2n}/(1-t).$$

October 22, 1983

For the paper with Loday I need to prove that $B(A)_{red}$ is quasi-isomorphic to $(\bar{A}^{\otimes(*+1)}/(1-t), b)$. First of all we have an augmentation

$$\begin{tikzcd}
\downarrow & \downarrow & & \\
\bar{A}^3/(1-t) \arrow[d] & A\otimes\bar{A}^2 \arrow[l] \arrow[d, "b"] & A\otimes\bar{A}^2 \arrow[d] \arrow[dl, "B"'] & \\
\bar{A}^2/(1-t) \arrow[d] & A\otimes\bar{A} \arrow[l] \arrow[d] & A\otimes\bar{A} \arrow[d] \arrow[dl, "B"'] & \\
\bar{A} & \bar{A} \arrow[l] & \bar{A} & \bar{A} \arrow[ul]
\end{tikzcd}$$

The point will be to filter $B(A)_{red}$ by

$$F_q\, B(A)_{red} \qquad \begin{tikzcd}
1\otimes\bar{A}^{q+1} \arrow[d, "b"] & 1\otimes\bar{A}^{q+1} \arrow[d, "b"] & \\
A\otimes\bar{A}^q \arrow[d, "b"] \arrow[u, phantom] & A\otimes\bar{A}^q \arrow[d, "b"] \arrow[ul, "B"'] & \arrow[ul, "B"'] \arrow[d] \\
{} \arrow[u, phantom] & \arrow[ul, "B"'] & \arrow[ul, "B"']
\end{tikzcd}$$

This is a subcomplex because I know that the image of $B : A\otimes\bar{A}^q \longrightarrow A\otimes\bar{A}^{q+1}$ is contained in $1\otimes\bar{A}^{q+1}$.

What is F_q/F_{q-1}? It is

$$\begin{tikzcd}
1\otimes\bar{A}^{q+1} \arrow[d] & 1\otimes\bar{A}^{q+1} \arrow[d] & \downarrow \\
\bar{A}\otimes\bar{A}^q & \bar{A}\otimes\bar{A}^q \arrow[ul] & \bar{A}\otimes\bar{A}^q \arrow[ul]
\end{tikzcd}$$

Here I use the notation $1\otimes\bar{A}^{q+1}$ to denote a ~~subsp~~ k-submodule of $A\otimes\bar{A}^{q+1}$, which might not be isomorphic to $\bar{A}^{q+1}$. In fact from

$$0 \longrightarrow k \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0$$

we get

$$0 \to \mathrm{Tor}_1^k(A,\bar{A}) \to \mathrm{Tor}_1^k(\bar{A},\bar{A}) \to k\otimes_k \bar{A} \to A\otimes\bar{A} \to \bar{A}\otimes\bar{A} \to 0$$

so if we take $k=\mathbb{Z}$, $A=\mathbb{Q}$ then

$$k\otimes_k\bar{A} = \mathbb{Q}/\mathbb{Z} \longrightarrow A\otimes\bar{A} = \mathbb{Q}\otimes\mathbb{Q}/\mathbb{Z} = 0$$

and the sort of thing we are trying to prove breaks down. Therefore we need to assume that $\bar{A}$ is flat over k.

In fact the case $k=\mathbb{Z}$, $A=\mathbb{Q}$ is a good example. One can even do it entirely in characteristic zero, i.e. $k = k_0[t]$, $A = k_0(t)$. Then $\overline{HC}_*(A)$ will be $\bar{A}$ in every even dimension, yet the complex $\bar{A}^{\otimes(*+1)}/1-t$ will be just $\bar{A}$ in degree 0.

Let's return to the index problem and try to understand why the index which is a very zero-th order invariant can be obtained from the Lie algebra cohomology.

I want to consider an odd dimensional manifold M and a Dirac operator on it with coefficients in E. This gives me a Hilbert space $\mathcal{H} = L^2(M, S\otimes E)$ and a splitting of it into positive and negative eigenspaces for the Dirac operator. As usual, the splitting is described by an F on $\mathcal{H}$ such that $F^2 = I$.

Let $\mathcal{G}$ = the gauge group of E. Then we have a map

$$\mathcal{G} \xrightarrow{\gamma} \mathcal{F} \qquad g \longmapsto gFg^{-1}$$

where $\mathcal{F}$ is a suitable Grassmannian of such splittings.

$\mathcal{F}$ is a restricted Grassmannian of some sort, and it has the homotopy type of $\mathbb{Z} \times BU$. On $\mathcal{F}$ are the universal Chern character cohomology classes

$$ch_n \in H^{2n}(\mathcal{F}).$$

They are the components of the Chern character of the basic K-element. So I can pull these classes back to even-dimensional classes

$$\gamma^*(ch_n) \in H^{2n}(\mathcal{G}).$$

For example $\gamma^*(ch_0)$ gives the index of the [redacted] operator.

What is my goal? I want to understand why Lie algebra cohomology of the gauge group [redacted] is related to [redacted] index questions. [redacted] And hence with determinants, since these are next after indices.

Connes' ~~starting~~ point is a p-summable Fredholm module. To fix the ideas we take a Dirac operator on a compact manifold M. Then one gets the Hilbert space $\mathcal{H}$ of L^2 spinors which is a C^*-module over $A = C^\infty(M)$. Assuming the Dirac operator is invertible, the Hilbert space $\mathcal{H}$ breaks up into positive and negative eigenspaces, giving rise to an F such that $F^2 = I$. Then one gets cyclic cocycles

$$\tau(a_0, \ldots, a_{2n}) = tr(\varepsilon a_0 [F, a_1] \cdots [F, a_{2n}]) \qquad \text{even case}$$

$$\tau(a_0, \ldots, a_{2n-1}) = tr(F[F, a_0] \cdots [F, a_{2n-1}]) \qquad \text{odd case}$$

which I sort of understand how to interpret in terms of Lie algebra cohomology for $\mathcal{G} = Map(M, U_N)$.

Also these cocycles are directly related to [redacted] indices. Hence the question of relating Lie algebra cohomology to indices.

Let's look at the index problem belonging to this original Dirac operator $\not{D}$, or F. Topologically the operator is a K-homology class, $\tau \in K_*(M)$. The index problems [redacted] associated to $\not{D}$ or F are values of the map

$$(\star) \qquad K^*(M) \xrightarrow{\ \cap \tau\ } K^*(pt).$$

So we should think of the operator $\not{D}$ or F as giving us a way to go from the K-theory of M to that of a point. Since the operator is given one should think of the operator ~~as~~ a K-cycle, not just a homology class.

Similarly one should think of $K^*(M)$, $K^*(pt)$ as described by actual "K-cocycles," and the map $\star$ as given concretely. This is what happens[redacted] when we represent elements of $K^0(M)$ by idempotent matrices e (resp.[redacted] elements of $K^1(M)$ by invertible matrices g). Note: e (resp. g) is the same as a map $M \to$ Grass, (resp. $M \to U_N$.)

So what I really have is a map from a space such ~~at~~ $Map(M,$ [redacted] $\mathbb{Z} \times BU)$, $Map(M, U)$ into the space of Fredholm operators. This is how I should think of the K-cycle given by the operator.

Now one has another way to do this thing in the even case. Instead of maps into a Grassmannian $\mathrm{Map}(M, BU_N)$ we can get an equivalent space using $B\mathcal{G} = $ "$\mathcal{G}\backslash\mathcal{A}$". Strictly one has to pass to a component of this mapping space.

So my idea now is to replace Connes' use of e by a connection A. Then I have the map

$$A \longmapsto \not{D}_A$$

from $\mathcal{A}$ to a space of Fredholm operators, which is equivariant for $\mathcal{G}$.

In more detail we are considering the family of Dirac operators

$$A \longmapsto \not{D}_A \quad \text{on} \quad L^2(S\otimes E) = \mathcal{H}$$

where S is the vector bundle of spinors. The gauge gp acts on $\mathcal{H}$ + this family is equivariant. Thus

$$\begin{tikzcd} P\mathcal{G}\times^{\mathcal{G}}\mathcal{A} \arrow[r, dashed] \arrow[dr, "\sim"'] & \mathcal{F}(P\mathcal{G}\times^{\mathcal{G}}\mathcal{H}) \arrow[d] \\ & B\mathcal{G} \end{tikzcd}$$

over $B\mathcal{G}$ we have a Hilbert bundle with fibre $\mathcal{H}$ and a map from $P\mathcal{G}\times^{\mathcal{G}}\mathcal{A}$ to a Fredholm bundle map.

If we trivialize $P\mathcal{G}\times^{\mathcal{G}}\mathcal{H}$ by Kuiper's thm., then we get a map from $B\mathcal{G}$ to the space of Fredholm operators $\mathcal{F}$ well-defined up to homotopy:

$$B\mathcal{G} \longrightarrow \mathcal{F} \sim \mathbb{Z} \times BU.$$

This map corresponds to the K-element of $B\mathcal{G}$ given by the index of the family of Dirac operators.

[redacted] Now the index theorem for families tells [redacted] what the above map is. The cohomological version would describe exactly what the pull-back of the chern character classes are. [redacted] So where does the cyclic cohomology enter?

The point now [redacted] should be that [redacted] the character classes on $B\mathcal{G}$ can be suspended to odd classes on $\mathcal{G}$ which in high dimensions are realized by invariant differential forms. And the point is that the corresponding cyclic cocycles [redacted] should be the ones that Connes [redacted] describes. This will be the point that I have to make clear.

October 23, 1983

Yesterday I started work on the problem of relating Lie algebra cohomology of gauge groups to indices and determinants. [redacted]

The setting: One is given an operator over M. There is the basic operation of taking a vector bundle E over M, tensoring the operator with it, and taking the index. This gives a map

$$K^*(M) \longrightarrow K^*(pt).$$

But one should think more generally as having a map of "theories"

$$\underline{\underline{K}}(M) \longrightarrow \underline{\underline{K}}(pt).$$

One way to make this more precise is that the operator on M gives a natural transformation

$$\star \qquad K^*(Y \times M) \longrightarrow K^*(Y).$$

Now we have

$$\begin{tikzcd} K^*(Y\times M) \arrow[r, "\star"] \arrow[d, "ch"] & K^*(Y) \arrow[d, "ch"] \\ H^*(Y\times M) \arrow[r, "\cap \tau"] & H^*(Y) \end{tikzcd}$$

where $\tau \in H_*(M)$ depends on the operator. (This should be a consequence of the fact that ch is an isomorphism rationally and A is a $K^*(Y)$-module homomorphism.)

The point seems to be that if I want to find τ, then I can use various Y's. The simplest Y to use is $Y = pt$.

So far I am missing the role of periodicity, which makes other Y's turn out in some way to be simpler than $Y = pt$. (?)

(The last statement doesn't make much sense. What is at stake has probably to do with the situation in which we want differential forms representing τ.)

October 26, 1983

Connes S-operator: The basic idea here is to consider the map $a \mapsto ae$ from $\mathcal{A}$ to $\mathcal{A} \otimes ke$ where $\mathcal{A}$ is a non-unital algebra. Then we have a map from $\mathcal{A}$ to the degree zero part of the DG algebra

$$\tilde{\Omega}^*(\mathcal{A}^+) \otimes \tilde{\Omega}^*(k+ke).$$

In fact $\tilde{\Omega}^0(\mathcal{A}^+) \otimes \tilde{\Omega}^0(k+ke) = (k+\mathcal{A}) \otimes (k+ke)$

$$= k + ke + \mathcal{A} + \mathcal{A}e$$

We consider the map $a \mapsto ae$ from $\mathcal{A}$ to $\mathcal{A}^+ \otimes (ke)^+$. This gives a homomorphism

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, phantom, "\cap" description] & \mathcal{A}^+ \otimes (ke)^+ \subset \tilde{\Omega}^*(\mathcal{A}^+) \otimes \tilde{\Omega}^*((ke)^+) \\ \tilde{\Omega}^*(\mathcal{A}^+) \arrow[ur, dashed] & \end{tikzcd}$$

and the dotted arrow exists by the universal property of $\tilde{\Omega}^*(\mathcal{A}^+)$. The induced map on homology is

$$H^*_{CK}(\mathcal{A}^+) \longrightarrow H^*_{CK}(\mathcal{A}^+) \otimes \underbrace{H^*_{CK}(ke)^+}_{\begin{cases} k & * \text{ even} \\ 0 & * \text{ odd} \end{cases}}$$

so in particular corresponding to $e(de)^2$ in $H^*_{CK}(ke)^+$ we get a map

$$S : H^p_{CK}(\mathcal{A}^+) \longrightarrow H^{p-2}_{CK}(\mathcal{A}^+)$$

October 27, 1983

$\tilde{\Omega}(A^+) \longrightarrow \tilde{\Omega}(A^+) \otimes \tilde{\Omega}((ke)^+)$ is the unique map of differential graded algebras such that

$$a \longmapsto ae \qquad \text{for } a \in A.$$

Then $\quad a_0 da_1 \cdots da_n \longmapsto a_0 e[da_1 e + a_1 de][da_2 e + a_2 de] \cdots$

To find the S-map we want the coefficient of ede^2. In the tensor product $\tilde{\Omega}(A^+) \otimes \tilde{\Omega}((ke)^+)$ we are allowed to move a, da past e, de subject to picking up a sign. Recall also that $e.de.e = 0$. This means that if we take terms $a_i de$, $a_j de$ in the product that i, j must be consecutive, otherwise we get zero. Hence we get

$$(a_0 a_1 a_2 \, da_3 \cdots da_n + a_0 da_1 \, a_2 a_3 \, da_4 \cdots da_n + \cdots$$
$$\cdots + a_0 da_1 \cdots da_{n-2} \, a_{n-1} a_n) \, ede^2$$

for the term involving ede^2. Hence we conclude that the map

$$S(a_0 da_1 \cdots da_n) =$$
$$a_0 a_1 a_2 \, da_3 \cdots da_n + a_0 da_1 \, a_2 a_3 \, da_4 \cdots da_n + \cdots$$
$$+ a_0 da_1 \cdots da_{n-2} \, a_{n-1} a_n$$

induces a map on Connes-Karoubi cohomology.

I can be more precise. We have a map of complexes

$$\tilde{\Omega}(A^+) \longrightarrow \tilde{\Omega}(A^+) \otimes \tilde{\Omega}((ke)^+) \longrightarrow \tilde{\Omega}(A^+) \otimes \underbrace{\tilde{\Omega}((ke)^+)/[\ ,\ ]}_{k \text{ in even degrees.}}$$

and hence we get a map of complexes

$$S: \tilde{\Omega}^*(a^+) \longrightarrow \tilde{\Omega}^{*-2}(a^+)$$

with $\quad S(a_0 da_1 \cdots da_n)$ defined as above and

$$S(da_1 \cdots da_n) = S(d(a_1 da_2 \cdots da_n))$$
$$= d\, S(a_1 da_2 \cdots da_n)$$

[redacted] One could define any map of complexes this way, but the point is that S will induce a map on commutator ~~quotients~~

$$\tilde{\Omega}^*(a^+)/[\ ,\ ] \longrightarrow \tilde{\Omega}^{*-2}(a^+)/[\ ,\ ].$$

This is because

$$\tilde{\Omega}(A) \otimes \tilde{\Omega}(B)/[\ ,\ ] = \tilde{\Omega}(A)/[\ ,\ ] \otimes \tilde{\Omega}(B)/[\ ,\ ]$$

From the formula for S one sees that it is not defined on $\tilde{\Omega}(A)$ for a non-augmented ring since $S(a_0 d1\, da_2 \cdots da_n) = a_0 a_2\, da_3 \cdots da_n + a_0\, d1\, a_2 a_3 \cdots$ so one can't set $d1 = 0$.

I think the next thing to do is to take $a = C^\infty(M)$. Then show how a closed current γ of dimension p on M [redacted] furnishes cyclic cocycles of degrees $p, p+2, p+4, \ldots$ on a

New problem: Let's go back to the original index motivation for Connes' theory. [redacted] Let's try to avoid the assumption that the original operator is invertible. So we begin with

$$\mathcal{H}^+ \xrightarrow{\ P\ } \mathcal{H}^-$$

where P is Fredholm, $\mathcal{H}^\pm$ are A-modules, and $[P, a]$ is compact, say in some Schatten class. Then replacing P by P^N we get an action of the gauge group $\mathcal{G}$ on $\mathcal{H}^\pm$, and hence a map

$$\begin{aligned} \mathcal{G} &\longrightarrow \mathcal{F}(\mathcal{H}^+, \mathcal{H}^-) \\ g &\longmapsto gPg^{-1} \end{aligned}$$

Before it was a map into invertible operators from $\mathcal{H}^+$ to $\mathcal{H}^-$ which carries odd [redacted] diml character forms $\operatorname{tr}(P^{-1}dP)^{2k+1}$. [redacted] The problem [redacted] is now to construct somehow odd left-invariant forms on $\mathcal{G}$ defined formally for all odd degrees, but which will make sense for large degrees because the trace of a product of enough commutators is well-defined.

October 29, 1983

Let me consider the space of Fredholm operators $P: \mathcal{H}^+ \longrightarrow \mathcal{H}^-$, call it $\mathcal{P}$ and the larger space $\mathcal{F}$ of $F = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix}$ such that $F^2 - I$ is compact. Over these spaces we have the determinant line bundle $\mathcal{L}$, and this line bundle is equivariant for the action of $\mathcal{G} = \mathcal{G}^+ \times \mathcal{G}^-$ of autos of $\mathcal{H}$ of degree zero.

In the case of interest $\mathcal{H}$ is an A-module so we have A^* mapping to $\mathcal{G}$ and a Lie map of A to $\mathrm{Lie}(\mathcal{G})$. Connes is able to define a trace on A associated to F assuming that $[F, a]$ and $a(F^2 - I)$ are trace class for all $a \in A$.

The problem I want to look at is to make sense of his construction, hopefully using the determinant line bundle $\mathcal{L}$. An element $a \in A$ will give rise to a vector field on $\mathcal{P}$ and on $\mathcal{L}$. If I have an invariant connection on $\mathcal{L}$, then pulling back ~~from~~ the A^*-orbit

$$\begin{tikzcd} & \mathcal{L} \arrow[d] \\ A^* \arrow[r] & A^*(P) \end{tikzcd}$$

we get a left-invariant connection on the trivial line bundle over A^*, which corresponds to a 1-form on A^* which is left-invariant. If the connection is flat, then we would have a trace on A.

But I should begin with the topology first. The space $\mathcal{P}$ has the homotopy type $\mathbb{Z} \times BU$, and the first Chern class of $\mathcal{L}$ lies in $H^2_{\mathcal{G}}(\mathcal{P}) \xrightarrow{\sim} H^2(\mathcal{P})$

as $\mathcal{G}$ is contractible by the Kuiper theorem. I am trying to produce an elt. of $H^1(A^*)$. Can I do this topologically? The idea would be to construct maps

$$\begin{tikzcd} A^* \arrow[r, dashed] \arrow[d] & U \arrow[d] \\ CA^* \arrow[r, dashed] \arrow[d] & * \arrow[d] \\ \Sigma A^* \arrow[r, dashed] & BU. \end{tikzcd}$$

Here U is invertibles which are of the form $1+$ compact.

Let's go back to the old notation and try to see what the story is in the case of Dirac operators.

$$\begin{tikzcd} \mathcal{G} \arrow[r] & \mathcal{A} \arrow[r] & P\mathcal{G}\times^{\mathcal{G}}\mathcal{A} \sim B\mathcal{G} \arrow[d, "\Phi"] \\ & & BU \end{tikzcd}$$

Description of Φ: Over $B\mathcal{G}$ one has the Hilbert bundles $P\mathcal{G}\times^{\mathcal{G}}\mathcal{H}^{\pm}$ (*) and the equivariant map $A \longmapsto D_A : \mathcal{H}^+ \to \mathcal{H}^-$ from $\mathcal{A}$ to $\mathcal{F}(\mathcal{H}^+,\mathcal{H}^-)$ gives a map

$$\begin{tikzcd} P\mathcal{G}\times^{\mathcal{G}}\mathcal{A} \arrow[r] \arrow[d, phantom, "\wr"] & \underset{B\mathcal{G}}{\underline{\mathrm{Fred}}}\left(P\mathcal{G}\times^{\mathcal{G}}\mathcal{H}^+, P\mathcal{G}\times^{\mathcal{G}}\mathcal{H}^-\right) \arrow[d, phantom, "\wr\!\!\wr \text{ Kuiper}"] \\ B\mathcal{G} & B\mathcal{G}\times \mathcal{F}(\mathcal{H}^+,\mathcal{H}^-) \end{tikzcd}$$

(*) which are trivial by Kuiper's thm. Thus we get a map $BG \longrightarrow \mathcal{F}(\mathcal{H}^+, \mathcal{H}^-) \sim \mathbb{Z} \times BU$ which is well-defined up to homotopy.

In other words Φ is the map which gives the index of the family of Dirac operators over $P\mathcal{G} \times^{\mathcal{G}} \mathcal{A}$. It is useful to assume $\mathcal{G}$ acts freely, so that $B\mathcal{G} = \mathcal{G}\backslash\mathcal{A}$

Next we want to describe the map $\Sigma \mathcal{G} \longrightarrow B\mathcal{G}$ belonging to $\mathcal{G} \to \mathcal{A}$, $g \mapsto g(A_0)$. This uses the contractibility of $\mathcal{A}$

$$
\begin{tikzcd}
\mathcal{G} \arrow[r] \arrow[d, equal] & \mathcal{A} \arrow[r] & \mathcal{G}\backslash\mathcal{A} = B\mathcal{G} \\
\mathcal{G} \arrow[r] & C\mathcal{G} \arrow[r] \arrow[u, dashed, "\exists \text{ by contract of } \mathcal{A}"'] & \Sigma \mathcal{G} \arrow[u, dashed]
\end{tikzcd}
$$

Thus if I take $\mathcal{L}$ over $B\mathcal{G}$ and pull it back to $\mathcal{G}$, this line bundle becomes trivial for two reasons. 1) $\mathcal{G}$-invariance, 2) $\mathcal{L}$ becomes trivial over $\mathcal{A}$.

This shows that in order to construct a 1-form on $\mathcal{G}$ corresponding to $c_1(\mathcal{L})$ I seem to have to build in the <u>contractibility</u> of $\mathcal{A}$.

Let's us now try to work directly with the operators $P: \mathcal{H}^+ \longrightarrow \mathcal{H}^-$ and call the space of these operators $\mathcal{F}(\mathcal{H}^+, \mathcal{H}^-)$, or just $\mathcal{F}$. We have the group $\mathcal{G}$ acting on $\mathcal{F}$ by $g(P) = gPg^{-1}$ and the equivariant line bundle $\mathcal{L}$ over $\mathcal{F}$. The line bundle represents the 2 diml cohomology class on $\mathcal{F}$. I want to construct a 1-diml class on $\mathcal{G}$. Map $\mathcal{G} \longrightarrow \mathcal{F}$ by $g \longmapsto g(P_0)$; then the pull-back of $\mathcal{L}$ has the trivialization because of the G-action. On the other hand I need another trivialization which should come by using the linear structure, that is, the linear path $tP_1 + (1-t)P_0$ joining P_0 to a typical point $P_1 = gP_0g^{-1}$ of the G-orbit of P_0.

Interesting point: It just occurred to me that a trace on the Lie algebra of G is roughly equivalent to a homomorphism $G \to \mathbb{C}^*$. In other words one has some sort of determinant homomorphism [redacted] defined on the invertible orbits.

[redacted] The first point is that the operators P that we are considering all differ from P_0 by a trace class operator. For example

$$
\begin{aligned}
P_0 - gP_0g^{-1} &= P_0gg^{-1} - gP_0g^{-1} \\
&= [P_0, g]g^{-1}
\end{aligned}
$$

and we are assuming that $[P_0, g]$ is of trace class. On the other hand Fredholm operators are stable under compact perturbations: If P is Fredholm and K is compact, then $P+K$ is Fredholm.

So now what we can do is the following in the case of index zero. [redacted] Fix a P_0 which is invertible and then define a determinant function

$$
\Phi(P_0 + B) = \det(1 + P_0^{-1}B)
$$

on the set of all operators differing from P_0 by a trace class operator. This function is non-vanishing exactly when $P_0 + B$ is invertible, so it probably is equivalent to a trivialization of $\mathcal{L}$ over $\{P_0 + B \mid B \text{ of trace class}\}$.

A better way to describe this function is to introduce the function

$$\Phi(P_1, P_2) = \det(P_1^{-1}P_2)$$

defined on $(P_0 + \text{trace class}) \cap$ invertible. Then

$$\begin{aligned}\Phi(g(P_1), g(P_2)) &= \det\left((gP_1g^{-1})^{-1}(gP_2g^{-1})\right)\\ &= \det(gP_1^{-1}P_2g^{-1}) = \det(P_1^{-1}P_2)\\ &= \Phi(P_1, P_2)\end{aligned}$$

so that Φ is invariant under the $\mathcal{G}$-action. Also we have

$$\begin{aligned}&\Phi(P_1, P_2)\,\Phi(P_2, P_3)\\ &= \det(P_1^{-1}P_2)\det(P_2^{-1}P_3) = \det(P_1^{-1}P_3)\\ &= \Phi(P_1, P_3).\end{aligned}$$

Thus

$$\begin{aligned}\Phi(P_0, g_1g_2P_0) &= \Phi(P_0, g_1P_0)\,\Phi(g_1P_0, g_1g_2P_0)\\ &= \Phi(P_0, g_1P_0)\,\Phi(P_0, g_2P_0)\end{aligned}$$

which shows that we really have a determinant function.

Yesterday I considered the problem of constructing a trace for a 1-summable Fredholm module. More precisely I consider a coset $P_0 + \mathcal{L}^1$ of Fredholm operators

$$P : \mathcal{H}^+ \longrightarrow \mathcal{H}^-$$

modulo trace class operators. Over this coset the determinant line bundle $\mathcal{L}$ appears to have a canonical flat connection. Evidence: suppose the index of the coset is zero. Then the coset contains invertible elements; fix one P_0 and define

$$\varphi(P) = \det(P_0^{-1}P)$$

on the coset. This function is analytic and vanishes exactly where P is singular. This suggests that φ is obtained from a trivialization of $\mathcal{L}$ by taking the image of the canonical section under this trivialization.

Independence of the choice of P_0:

$$\det(P_0^{-1}P) = \det(P_0^{-1}P_1)\det(P_1^{-1}P)$$

Because the determinant line bundle has this can. flat connection, the flat sections form a 1-diml repn. of the group G of autos of $\mathcal{H}$ preserving the coset. In fact

$$\varphi(gPg^{-1}) = \det(P_0^{-1}gPg^{-1}) = \det(P_0^{-1}gP_0g^{-1})\det(gP_0^{-1}Pg^{-1})$$
$$= \varphi(gP_0g^{-1})\varphi(P)$$

whence
$$\frac{\varphi(gPg^{-1})}{\varphi(P)} = \frac{\varphi(gP_0g^{-1})}{\varphi(P_0)}$$
is independent of the choice of P_0. From this it follows that we have a character of $\mathcal{G}$ given by
$$g \longmapsto \frac{\varphi(gPg^{-1})}{\varphi(P)} = \frac{\det(P_0^{-1}gPg^{-1})}{\det(P_0^{-1}P)} = \det(P^{-1}gPg^{-1})$$

for any invertible P in the coset. The corresponding character on the Lie algebra is
$$X \longmapsto tr\left(P^{-1}[X,P]\right)$$
and it is independent of the choice of P in the coset.

Check: Let $P \longmapsto P+\delta P$ with δP of trace class. Then
$$\delta\, tr\left(P^{-1}[X,P]\right) = tr\left(-P^{-1}\delta P\, P^{-1}(XP-PX) + P^{-1}(X\delta P-\delta P X)\right)$$
$$= tr\left(-P^{-1}XPP^{-1}\delta P + P^{-1}X\delta P\right) = 0$$

Let's see if we can work out a formula for the character on the Lie algebra which works in general. The idea will be to start with
$$tr\left(P^{-1}[X,P]\right)$$
and then replace P^{-1} by a parametrix Q for P, i.e. $P^{-1} = Q + J$ where $J: \mathcal{H}^- \to \mathcal{H}^+$ is of trace class. So we get

$$\mathrm{tr}(P^{-1}[X,P]) = \mathrm{tr}(Q[X,P] + J[X,P])$$
$$= \mathrm{tr}(Q[X,P]) + \mathrm{tr}(J[X,P])$$

Now I want to proceed as follows:

$$= \mathrm{tr}(Q[X,P]) + \mathrm{tr}([JX,P] - [J,P]X)$$
$$+ \underbrace{\mathrm{tr}([JX,P])}_{\substack{=0 \text{ as } JX \\ \text{is of trace class}}} - \mathrm{tr}([J,P]X)$$

$$= \mathrm{tr}(Q[X,P]) + \mathrm{tr}([Q,P]X)$$
$$= \mathrm{tr}([QX,P])$$

This calculation should be valid for operators on the same space, so I have to check it works in the graded case. The point is that $[X,P] = X^-P - PX^+$.

$$\mathrm{tr}(P^{-1}[X,P]) = \mathrm{tr}_{\mathcal{H}^+}(Q[X,P]) + \mathrm{tr}_{\mathcal{H}^+}(J[X,P])$$

Now $\mathrm{tr}_{\mathcal{H}^+}(J[X,P]) = \mathrm{tr}_{\mathcal{H}^+}(JX^-P) - \mathrm{tr}_{\mathcal{H}^+}(JPX^+)$

$$= \mathrm{tr}_{\mathcal{H}^-}(PJX^-) - \mathrm{tr}_{\mathcal{H}^+}(JPX^+)$$

Now
$$PJ = P(P^{-1} - Q) = I - PQ$$
$$JP = (P^{-1} - Q)P = I - QP$$

So
$$\mathrm{tr}_{\mathcal{H}^+}(J[X,P]) = \mathrm{tr}_{\mathcal{H}^-}((I-PQ)X^-) - \mathrm{tr}_{\mathcal{H}^+}((I-QP)X^+)$$
$$\mathrm{tr}_{\mathcal{H}^+}(Q[X,P]) = \mathrm{tr}_{\mathcal{H}^+}(Q(X^-P - PX^+))$$

So adding we get that

$$\operatorname{tr}_{\mathcal{H}^+}\left(P^{-1}[X,P]\right) = \operatorname{tr}_{\mathcal{H}^-}\left((I-PQ)X^-\right) + \operatorname{tr}_{\mathcal{H}^+}\Big(\overbrace{QX^-P - X^+}^{-(I-QP)X^+ + Q(X^-P - PX^+)}\Big)$$

The right side makes sense in general. Moreover taking the variation w.r.t. δQ of trace class gives

$$-\operatorname{tr}_{\mathcal{H}^-}(P\,\delta Q\, X^-) + \underbrace{\operatorname{tr}_{\mathcal{H}^+}(\delta Q X^- P)}_{\operatorname{tr}_{\mathcal{H}^-}(P\,\delta Q\, X^-)} = 0$$

Also taking the variation wrt δP gives

$$-\operatorname{tr}_{\mathcal{H}^-}(\delta P\, Q X^-) + \operatorname{tr}_{\mathcal{H}^+}(Q X^- \delta P)$$

which is also zero.

Now we can argue this function has to be a trace by continuity from the invertible case. This handles index 0 and the general case follows by adding a trivial piece left fixed by X in order to cancel the index.

Now I want to understand better what the determinant line bundle is doing. Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ and let $\mathcal{I}$ be the ideal of trace class operators, $\mathcal{K}$ the ideal of compacts. Any Fredholm operator is invertible mod $\mathcal{I}$, hence

$$\mathcal{F}/\mathcal{I} = (\mathcal{B}/\mathcal{I})^\times$$

We have an extension of groups

$$\begin{tikzcd} 0 \arrow[r] & (1+\mathcal{I})^{*} \arrow[r] \arrow[d, "\det"] & \mathcal{B}^{*} \arrow[r] & (\mathcal{B}/\mathcal{I})^{*}_{0} \arrow[r] & 0 \\ & \mathbb{C}^{*} & & & \end{tikzcd}$$

$(1+\mathcal{I})^* \sim U$, $\quad \mathcal{B}^* \sim pt$ by Kuiper, $\quad (\mathcal{B}/\mathcal{I})^*_0 \sim BU$

Now we have this group G mapping into $\mathcal{B}^\times$ and the element P_0 of $\mathcal{B}^\times$ such that $gP_0g^{-1} \equiv P_0 \mod \mathcal{I}$.

We have also a sequence of monoids

$$\begin{tikzcd} 1+\mathcal{I} \arrow[r] \arrow[d, "\det"] & \mathcal{F} \arrow[r] & (\mathcal{B}/\mathcal{I})^{\times} \\ \mathbb{C} & & \end{tikzcd}$$

but it is not clear that one can say anything interesting.

At this point I have explored the 1-diml Lie cocycle on G belonging to a Fredholm P preserved by G modulo trace class.

The next case is to look at 2-diml cocycles on $\bar{\mathfrak{g}}$. This should connect up with the loop group theory.

Let's begin as before with the topological analysis in the Dirac operator situation. We have an odd manifold like the circle and a family of self-adjoint Fredholm operators over $\mathcal{A}$ equivariant for G. We are after a two-dimensional cohomology class on $\mathcal{G}$, in fact a left-invariant 2 form. This corresponds to a three-dimensional cohomology class on $B\mathcal{G} \sim \mathcal{G}\backslash\mathcal{A}$.

<u>Question</u>: Can one find an infinite dimensional Azumaya algebra in the loop group case? This should be an algebra with center the functions on $\mathcal{G}\backslash\mathcal{A}$, hence might be the functions on $\mathcal{A}$ crossed product with $\mathcal{G}$. Somehow this algebra when restricted or "localized" near an orbit of $\mathcal{G}$ on $\mathcal{A}$ should have an irreducible representation related to the line bundle over the orbit.

The standard way to proceed to describe an invariant 2-form on $\mathcal{G}$ is to consider the map $g \longmapsto geg^{-1}$ to the Grassmannian and pull back the first Chern form $\operatorname{tr} e(de)^2 = \frac{1}{8}\operatorname{tr}(F(dF)^2)$ where $F = 2e-1$. The corresponding two-form on $\tilde{\mathfrak{g}} = A$ is

$$a, b \longmapsto \operatorname{tr}\big(e[a,e][b,e] - \underbrace{e\,[b,e]}_{[b,e](1-e)}[a,e]\big)$$

$$= \operatorname{tr}\big((2e-1)[a,e][b,e]\big) = \frac{1}{4}\operatorname{tr}\big(F[F,a][F,b]\big)$$

$$= \frac{1}{4}\operatorname{tr}\big((a - FaF)[F,b]\big) = \frac{1}{2}\operatorname{tr}\big(a[F,b]\big)$$

$$= \operatorname{tr}\big(a[e,b]\big)$$

Little idea: If P, Q are inverses mod trace class, then one can see directly that $\operatorname{Ind}(P) = -\operatorname{Ind}(Q)$. But then this implies that $\operatorname{Ind}(P+K) = \operatorname{Ind}(P)$ for K of trace class. as Q doesn't change.

<u>October 31, 1983</u>

Recall the definition of the S operator on $\tilde\Omega(a^+)$. By the universal property of $\tilde\Omega(a^+)$ the homomorphism

$$\begin{tikzcd}
a \arrow[r, "a \mapsto ae"] \arrow[d, phantom, "\cap"] & \tilde\Omega(a^+) \otimes \tilde\Omega(ke^+) \\
\tilde\Omega(a^+) \arrow[ur, dashed] &
\end{tikzcd}$$

extends uniquely to a morphism of diff'l graded algebras. $\tilde\Omega(ke^+)$ has the basis $(de)^n$, $e(de)^n$ so we can take the coefficient of $e(de)^2$.

<u>Question</u>: Does $S^2 : \tilde\Omega^* \to \tilde\Omega^{*-4}$ give the coefficient of $e(de)^4$?

To compute S^2 we need

$$\begin{tikzcd}
\tilde\Omega(a^+) \arrow[r] \arrow[dr, "S"'] & \tilde\Omega(a^+) \otimes \tilde\Omega(ke^+) \arrow[r] \arrow[d, "\text{coeff of } ede^2"] & \tilde\Omega(a^+) \otimes \tilde\Omega(ke_1^+) \otimes \tilde\Omega(ke^+) \arrow[d, "\text{coeff of } ede^2"] \\
& \tilde\Omega(a^+) \arrow[r] \arrow[dr, "S"'] & \tilde\Omega(a^+) \otimes \tilde\Omega(ke_1^+) \arrow[d, "\text{coeff of } e_1de_1^2"] \\
& & \tilde\Omega(a^+)
\end{tikzcd}$$

Thus we see that S^2 gives the coeff of $e_1 de_1^2\, e_2 de_2^2$ under the map

$$\tilde\Omega(a^+) \longrightarrow \tilde\Omega(a^+) \otimes \tilde\Omega(ke_1^+) \otimes \tilde\Omega(ke_2^+)$$

induced by $a \longmapsto a e_1 e_2$. But this map factors

$$\begin{tikzcd}
\tilde{\Omega}(a^+) \arrow[rr] \arrow[dr] & & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke_1^+) \otimes \tilde{\Omega}(ke_2^+) \\
& \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{array}{l} a \longmapsto a \\ e \longmapsto e_1 e_2 \end{array}$$

So all we have to do is to see what linear functional on $\tilde{\Omega}^4(ke^+)$ is obtained by using the homomorphism $\tilde{\Omega}(ke^+) \longrightarrow \tilde{\Omega}(ke_1^+) \otimes \tilde{\Omega}(ke_2^+)$ and then taking the coefficient of $e_1 de_1^2 e_2 de_2^2$.

$$(de)^4 \longmapsto (e_1 de_2 + de_1 e_2)^4$$

This should go to zero because our "integral" should kill Im d. Let's check it. There are 16 terms, but we want two de_1 and two de_2, which reduces us to 6 terms

$$\begin{aligned}
& (e_1 de_2)^2 (de_1 e_2)^2 && \longrightarrow e_1 de_1^2 e_2 de_2^2 \ \checkmark \\
0 = \ & (e_1 de_2)(de_1 e_2)(e_1 de_2)(de_1 e_2) && \\
& (e_1 de_2)(de_1 e_2)^2 (e_1 de_2) && \longrightarrow e_1 de_1^2 \, de_2 e_2 de_2 \\
& (de_1 e_2)(e_1 de_2)^2 (de_1 e_2) && \longrightarrow de_1 e_1 de_1 \, e_2 de_2^2 \ \checkmark \\
0 = \ & (de_1 e_2)(e_1 de_2)(de_1 e_2)(e_1 de_2) && \\
& (de_1 e_2)^2 (e_1 de_2)^2 && \longrightarrow \underline{e_1 de_1^2 e_2 de_2^2}
\end{aligned}$$

$$de_1^2 e_2 de_2^2 + e_1 de_1^2 de_2^2$$

which does give zero.

$$e_1 de_1^2 + \underbrace{de_1 e_1 de_1}_{(1-e_1)de_1} = de_1^2$$

$e(de$[redacted]$)^4$ will involve six terms also but because we multiply by $e_1 e_2$ those [redacted] having $de_1 e_1 de_1$ or $de_2 e_2 de_2$ will give zero. Thus we are going to get

$$e(de)^4 \longmapsto 2\, e_1 de_1^2\, e_2 de_2^2$$

It seems worthwhile to describe carefully the coalgebra $\tilde{\Omega}(ke^+)$ since this acts on $\tilde{\Omega}(a^+)$ for any a. We have

$$\tilde{\Omega}(ke^+) \longrightarrow \tilde{\Omega}(ke^+) \otimes \tilde{\Omega}(ke^+)$$

given by $\quad e \longmapsto e_1 e_2 \quad$ where $e_1 = e \otimes 1$, $e_2 = 1 \otimes e$. It looks like $\tilde{\Omega}(ke^+)$ is a Hopf algebra, except for the fact that it isn't reduced to k in degree zero.

We have

$$\begin{aligned} e(de)^2 &\longmapsto e_1 e_2 (de_1\, e_2 + e_1\, de_2)^2 \\ &= e_1 e_2 \big(de_1\, e_2\, de_1\, e_2 + de_1 \cancel{e_2 e_1} de_2 \\ &\qquad + e_1 \cancel{de_2\, de_1} e_2 + e_1 de_2\, e_1 de_2 \big) \\ &= (e_1 de_1^2) e_2 + e_1 (e_2 de_2^2) \end{aligned}$$

Consequently [redacted] we use the basis $\dfrac{e(de)^{2n}}{n!}$ for $H^{2n}_{CK}(ke^+)$.

?

November 1, 1983

Let's go over the S-operator. There is a unique morphism of DG algebras

$$\tilde{\Omega}(a^+) \xrightarrow{\Delta} \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+)$$

such that $a \longmapsto ae$. Moreover Δ is associative in the sense that

$$\begin{tikzcd} \tilde{\Omega}(a^+) \arrow[r, "\Delta"] \arrow[d, "\Delta"] & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \arrow[d, "\Delta \otimes id"] \\ \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \arrow[r, "id \otimes \Delta"'] & \Omega(a^+) \otimes \tilde{\Omega}(ke_1^+) \otimes \tilde{\Omega}(ke_2^+) \end{tikzcd}$$

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & ae \arrow[d, mapsto] \\ ae \arrow[r, mapsto] & (ae_1)e_2 \\ & a(e_1 e_2) \end{tikzcd}$$

commutes. Using Δ one makes $\tilde{\Omega}(ke^+)^*$ into a DG algebra which is commutative, and $\tilde{\Omega}(a^+)$ becomes a module over it. I should check that $\tilde{\Omega}(ke^+)^*$ is unital. Define a counit

$$\tilde{\Omega}(ke^+) \longrightarrow k \qquad e \longmapsto 1$$

and it is then clear that

$$\begin{tikzcd} \tilde{\Omega}(a^+) \arrow[r, "\Delta"] \arrow[dr, "id"'] & \tilde{\Omega}(a^+) \otimes \tilde{\Omega}(ke^+) \arrow[d] \\ & \tilde{\Omega}(a^+) \otimes k \end{tikzcd}$$

commutes: $a \longmapsto ae \longmapsto a$

I computed the structure of $\tilde\Omega(ke^+)^*$ and got the following. Define $1, \varepsilon \in (\tilde\Omega^0)^*$ by

$$\langle 1, \begin{pmatrix}1\\ e\end{pmatrix}\rangle = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$\langle \varepsilon, \begin{pmatrix}1\\ e\end{pmatrix}\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$$

and $\eta, \zeta \in (\tilde\Omega^1)^*$, $\beta \in (\tilde\Omega^2)^*$ by

$$\langle \eta, \begin{pmatrix}de\\ ede\end{pmatrix}\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} \qquad \langle \zeta, \begin{pmatrix}de\\ ede\end{pmatrix}\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\langle \beta, \begin{pmatrix}de^2\\ ede^2\end{pmatrix}\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}.$$ $\zeta \cdot \eta$ sees de^2

~~Then~~ $\cancel{\tilde\Omega(ke^+)^* = k[\varepsilon]/\varepsilon^2 \otimes \Lambda[\eta,\zeta] \otimes \Gamma[\beta]}$
~~where Γ denotes the divided power algebra~~

Then

$$\tilde\Omega(ke^+)^* = k\varepsilon \oplus \left(\Lambda[\eta,\zeta] \otimes \Gamma(\beta)\right)$$

where Γ denotes the divided power algebra. Here ε kills η, ζ, β, so that it would appear very natural to replace $\tilde\Omega^0(ke^+) = k + ke$ by just ke. This kills ε in the dual.

The differential we computed to be

$$d\zeta = 0, \qquad d\eta = -1 + \varepsilon \qquad (\text{hence } d\eta = -1 \text{ if } \varepsilon = 0)$$

$$d\beta = 0$$

I should check the divided power structure.

$$\langle \beta^n, e(de)^{2n} \rangle = \langle \beta^{\otimes n} \mid e_1 \cdots e_n (de_1 \cdots e_n))^{2n} \rangle$$

Better is to use $\Delta\, e(de)^2 = e(de_1)^2 e_2 + e_1 e_2 de_2^2$.

Hence $\Delta^{(n)} e(de)^2 = \left(\sum e_i (de_i)^2\right) e_1 \cdots e_n$ so

$$\langle \beta^n, (e(de)^2)^n \rangle = \langle \beta^{\otimes n}, \left(\sum e_i de_i^2\right)^n e_1 \cdots e_n \rangle$$
$$= n!.$$

Summary. Introduce $\Omega(a)$ as Connes does, so that $\Omega(a)$ is non-unital. Then we have

$$\begin{array}{ccc} \Omega(a) & \longrightarrow & \Omega(a) \otimes \Omega(ke) \\ \cap & & \cap \\ \tilde\Omega(a^+) & \xrightarrow{\Delta} & \tilde\Omega(a^+) \otimes \tilde\Omega(ke^+) \end{array}$$

and we have determined the following structure for the DG algebra $\Omega(ke)^*$:

$$\Omega(ke) \text{ basis: } \quad e \quad \begin{array}{cc} de & de^2 \\ ede & ede^2 \end{array}$$

$$\Omega(ke)^* \text{ basis: } \quad 1 \quad \begin{array}{cc} +\eta & -\eta\zeta \\ \zeta & \beta \end{array}$$

$$d\eta = -1, \quad d\zeta = 0, \quad d\beta = 0$$

Note: because ζ is a cycle it induces a morphism of $\Omega(a)$ to $\Omega(a)$ of degree -1 which commutes with d.

Let us now try to relate what we know about the S operator to the Lie algebra cohomology of $gl_n(A)$. Recall that we have a canonical element

$$\theta \in C^1(\mathfrak{g}, M_n(A)) = M_n(C^1(\mathfrak{g}, A))$$

where $\mathfrak{g} = gl_n(A)$ given by the identity map. This Maurer-Cartan form satisfies

$$\delta\theta + \theta^2 = 0$$

where δ is the differential on Lie cochains. Now suppose I have a homomorphism

$$M_n(A) \longrightarrow R^0$$

where R^* is a DG algebra. Then we get a bigraded diff algebra $C^*(\mathfrak{g}, R^*)$ and $\theta \in C^1(\mathfrak{g}, R^0)$ satisfies $d'\theta + \theta^2 = 0$.

<u>Example</u>: What is the largest possible R? The important thing is that the homomorphism $\theta: \mathfrak{g} \longrightarrow R^0$ be a Lie homomorphism. So therefore we want R^0 to be $U(\mathfrak{g})$ and R^* to be $\tilde{\Omega}(U(\mathfrak{g}))$.

When $\mathfrak{g} = gl_n(A)$ some other possibilities are $R^0 = M_n(A)$ or $R^0 = M_n(A^+)$ and then we have a choice of $R^* = M_n\,\tilde{\Omega}(A)$ or $\tilde{\Omega}(M_n A)$. This raises the

Question: ~~If~~ the complex $\tilde{\Omega}(A)/[\,,\,]$ Morita invariant?

Anyway let us now construct other classes from $\theta \in C^1(\mathfrak{g}, R^0)$, using the 1-parameter family of connections $D_t = d + t\theta$, where $d = d' + d''$. One has

$$D_t^2 = (d + t\theta)^2 = t\,d''\theta + (t^2 - t)\theta^2$$

hence

$$\frac{d}{dt} e^{D_t^2} = \int_0^1 ds\; e^{(1-s)D_t^2} \underbrace{(D_t^2)^{\cdot}}_{[D_t,\theta]} e^{sD_t^2}$$

$$= \left[D_t, \int_0^1 ds\; e^{(1-s)D_t^2}\,\theta\, e^{sD_t^2}\right]$$

At this point if we project modulo commutators we get

$$\frac{d}{dt} e^{D_t^2} = d\;\theta e^{D_t^2}$$

or

$$\underbrace{e^{D_1^2} - e^{D_0^2}}_{e^{d''\theta} - 1} = d \int_0^1 dt\;\theta e^{t d''\theta + (t^2 - t)\theta^2}$$

One thing obtained in this way are forms

$$\theta (d''\theta)^n \in C^{n+1}(\mathfrak{g}, \bar{R}^n / d\bar{R}^{n-1})$$

where $\bar{R} = R/[R,R]$. These are Lie cocycles.

November 2, 1983

$\bar{C}_p = \bar{A}^{\otimes(p+1)}/(1-t)$. Define

$$\bar{C}_p \longrightarrow \tilde{\Omega}^p / d\tilde{\Omega}^{p-1} + [dA, \tilde{\Omega}^{p-1}]$$

by $(a_0,\dots,a_p) \longmapsto a_0\, da_1 \dots da_p$. In fact we have

$$\begin{tikzcd} A\otimes\bar{A}^p \arrow[r, "\sim"] \arrow[d] & \tilde{\Omega}^p \arrow[d] \\ \bar{A}^{p+1} \arrow[r, "\sim"] \arrow[d] & \tilde{\Omega}^p/d\tilde{\Omega}^{p-1} \arrow[d] \\ \bar{C}_p \arrow[r, dashed] & \tilde{\Omega}^p/d\tilde{\Omega}^{p-1} + [dA, \tilde{\Omega}^{p-1}] \end{tikzcd}$$

To see the dotted arrow exists:

$$\begin{aligned} a_0\, da_1 \dots da_p &\equiv (-1)^{p-1} da_p\, a_0\, da_1 \dots da_p && \text{mod } [dA, \tilde{\Omega}^{p-1}] \\ &\equiv (-1)^p a_p\, da_0\, \cancel{da_1} \dots da_p && \text{mod } [d\tilde{\Omega}^p]. \end{aligned}$$

Conversely ~~[illegible]~~ the reverse map $a_0 da_1 \dots da_p \to (a_0,\dots,a_p)$ in $\bar{C}_p$ sends $d\tilde{\Omega}^{p-1}$ to zero, and kills

$$a_0\, da_1 \dots da_p - (-1)^p a_p\, da_0 \dots da_{p-1} + (-1)^p d(a_p a_0)\, da_1 \dots da_{p-1}$$

$$= a_0 da_1 \dots da_p - (-1)^{p-1} da_p\, a_0\, da_1 \dots da_{p-1}$$

$$= [a_0 da_1 \dots da_{p-1}, da_p].$$

Thus I conclude

$$\bar{C}_p = \tilde{\Omega}^p / d\tilde{\Omega}^{p-1} + [dA, \tilde{\Omega}^{p-1}]$$

Next we have

$$b(a_0,\ldots,a_{p+1}) = (a_0a_1,\ldots,a_{p+1}) - (a_0,a_1a_2,\ldots,a_{p+1}) + \cdots + \cdots + (-1)^{p+1}(a_{p+1}a_0,a_1,\ldots,a_{p-1})$$

$$\longmapsto \left.\begin{array}{l} a_0a_1\,da_2\cdots da_{p+1} \\ -\,a_0\,d(a_1a_2)\cdots da_{p+1} \\ +\cdots\cdots \\ +(-1)^p a_0\,da_1\cdots d(a_pa_{p+1}) \\ +(-1)^{p+1}a_{p+1}a_0\,da_1\cdots da_p \end{array}\right\} = (-1)^p[a_0da_1\cdots da_p, a_{p+1}]$$

Thus we conclude

$$\begin{tikzcd}
A\otimes\bar{A}^p \arrow[d, two heads] \arrow[r, phantom, "\simeq"] & \tilde{\Omega}^p \arrow[d] \\
\bar{A}^{p+1} \arrow[d, two heads] \arrow[r, phantom, "\simeq"] & \tilde{\Omega}^p/d\tilde{\Omega}^{p-1} \arrow[d] & \text{requires } p>0 \\
\bar{C}_p = \bar{A}^{p+1}/(1-t) \arrow[d, two heads] \arrow[r, phantom, "\simeq"] & \tilde{\Omega}^p/d\tilde{\Omega}^{p-1}+[dA,\tilde{\Omega}^{p-1}] \arrow[d, two heads] \\
\bar{C}_p/b\bar{C}_{p+1} \arrow[r, phantom, "\simeq"] & \tilde{\Omega}^p/d\tilde{\Omega}^{p-1}+[dA,\tilde{\Omega}^{p-1}]+[A,\tilde{\Omega}^p]
\end{tikzcd}$$

$$\tilde{\Omega}^p/d\tilde{\Omega}^{p-1}+[dA,\tilde{\Omega}^{p-1}]+[A,\tilde{\Omega}^p] = \bar{\Omega}^p/d\bar{\Omega}^{p-1}!$$

From this we conclude that

$H\bar{C}_p(A)$ and $H\bar{\Omega}^p(A)$ both inject into $\bar{\Omega}^p / d\bar{\Omega}^{p+1}$.

What I want to do now is to show that, viewed as subspaces of $\bar{\Omega}^p/d\bar{\Omega}^{p+1}$, the Connes Karoubi homology $H\bar{\Omega}^p(A)$ is contained in the reduced cyclic homology, in fact it coincides with the kernel of $B : H\bar{C}_p(A) \longrightarrow \bar{H}_{p+1}(A)$.

First of all recall that $B : A\otimes\bar{A}^p \longrightarrow A\otimes\bar{A}^{p+1}$ is the composition

$$\begin{tikzcd}
A\otimes\bar{A}^p \arrow[r] & \bar{A}^{p+1} \arrow[r, "N"] \arrow[dr, two heads] & \bar{A}^{p+1} \arrow[r, "1\otimes ?"] & A\otimes\bar{A}^{p+1} \\
& & \bar{C}_p \arrow[u, hook] &
\end{tikzcd}$$

so that we can view it as the composition

$$A\otimes\bar{A}^p \longrightarrow \bar{C}_p \xhookrightarrow{B = 1\otimes N} A\otimes\bar{A}^{p+1}.$$

Secondly $Bb + bB = 0$. Hence B induces a map

$$\begin{array}{ccc}
\bar{C}_p / b\bar{C}_{p+1} & \longrightarrow & A\otimes\bar{A}^{p+1} / b(A\otimes\bar{A}^{p+2}) \\
\cup & & \cup \\
H_p\bar{C}(A) & \xrightarrow{B} & \tilde{H}_{p+1}(A)
\end{array}$$

Notice also that if $\alpha \in \bar{C}_p$ satisfies $B\alpha \in \operatorname{Im} b$, then $B(b\alpha) = -bB\alpha \in b\operatorname{Im} b = 0$, so $b\alpha = 0$ as $B : \bar{C}_p \longrightarrow A\otimes\bar{A}^{p+1}$ is injective.

Thus we have

$$\mathrm{Ker}\left(\bar{C}_p / b\bar{C}_{p+1} \xrightarrow{B} A\otimes\bar{A}^{p+1} / b(A\otimes\bar{A}^{p+2}) \right)$$

$$= \mathrm{Ker}\left(H\bar{C}_p(A) \xrightarrow{B} \bar{H}_{p+1}(A) \right)$$

On the other hand

$$H\bar{\Omega}_p(A) = \mathrm{Ker}\left(\bar{\Omega}^p / d\bar{\Omega}^{p-1} \xrightarrow{d} \bar{\Omega}^{p+1} \right)$$

so the only thing to do is correlate B, d:

$$\begin{tikzcd}
\bar{C}_p / b\bar{C}_{p+1} \arrow[r, "B"] \arrow[d, equal] & A\otimes\bar{A}^{p+1} / b(A\otimes\bar{A}^{p+2}) \arrow[d, equal] \\
\bar{\Omega}^p / d\bar{\Omega}^{p-1} \arrow[dr, "d"'] & \tilde{\Omega}^{p+1} / [A, \tilde{\Omega}^{p+1}] \arrow[d, shift left, "\text{quotient by } \mathbb{Z}/p+1 \text{ action}"] \\
 & \bar{\Omega}^{p+1} \arrow[u, shift left, "i"]
\end{tikzcd}$$

Define i to be the inclusion of the invariants.
Claim then $B = i\,d$

$$(a_0, a_1, \ldots, a_p) \xmapsto{B} \sum_{i=0}^{p} (-1)^{ip} (1, a_i, \ldots, a_p, a_0, \ldots, a_{i-1})$$

$$\downarrow$$

$$\frac{1}{p!} a_0 da_1 \cdots da_p \xmapsto{d} \frac{1}{p!} da_0 da_1 \cdots da_p \mapsto \frac{1}{(p+1)!} \sum_i (-1)^{ip} \cdots$$

In more detail we define

$$i : \bar{\Omega}^{p+1} \longrightarrow \tilde{\Omega}^{p+1}/[A, \tilde{\Omega}^{p+1}]$$

by $$\omega_0 \cdots \omega_p \longmapsto \frac{1}{p+1} \sum_{i=0}^{p} (-1)^{ip}\, \omega_i \cdots \omega_p\, \omega_0 \cdots \omega_{i-1}$$

Because i is injective one has then $\operatorname{Ker} B = \operatorname{Ker} d$.