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October 9, 1983:

What is $\mathrm{Ext}^*_\Lambda(k^\natural, A^\natural)$? This is not cyclic homology because the periodicity operator raises degree by 2. Maybe it is a kind of negative K-theory, in the sense that the K-homology is a positive K-theory.

To compute this Ext I need a projective resolution of $k^\natural$, the constant functor k, within the category of cyclic k-modules. Presumably this resolution is obtained already when the homology $H_*(k^\natural \otimes_\Lambda A^\natural)$ is determined. So within Λ, I have the objects $[n]$ and the functor $h_{[n]}$.

Better idea: I feel somewhat happy about the homology, that is, the left-derived functors of the inductive limit. The thing I want to get at is the cohomology, that is, $R^p \varprojlim$. Let's start with $p=0$: $\varprojlim_\Lambda A^\natural$. An element of this inverse limit is a way of choosing elements $\alpha_n \in A^{\otimes(n+1)}$, $n \geq 0$ which are compatible with all the arrows of Λ.

Now suppose we leave out the degeneracies, that is, say A is a ring without unit. Then if I have an idempotent e of A, I can put

$$\alpha_n = e^{\otimes(n+1)} \in A^{\otimes(n+1)}.$$

Suppose I put in the degeneracies. Then I have for $n=0,1$ the maps

$$A \rightrightarrows A\otimes A$$
$$a \longmapsto 1\otimes a,\ a\otimes 1$$

and so the only possibility is for

$$\alpha_n = c(1 \otimes 1 \otimes \cdots \otimes 1) \in A^{\otimes(n+1)}.$$

with $c \in k$.

Let $E = (e_{ij})$ be an idempotent matrix. Check that

$$\alpha_{n+1} = tr\left(E^{\otimes(n+1)}\right) \in A^{\otimes(n+1)}$$

is compatible with the cyclic an multiplication maps.

$$\alpha_1 = e_{ii}$$
$$\alpha_2 = e_{ij} \otimes e_{ji}$$
$$\alpha_3 = e_{ij} \otimes e_{jk} \otimes e_{ki} \qquad \text{Summ. Conv.}$$

Under the cyclic flip

$$\alpha_2 \longmapsto e_{ji} \otimes e_{ij} = \alpha_2$$

because the summing indices are dummy indices. Also consider the multiplication map

$$A^{\otimes 2} \longrightarrow A \qquad\qquad A^{\otimes 3} \longrightarrow A^{\otimes 2}$$

$$e_{ij} \otimes e_{ji} \longmapsto \underbrace{e_{ij}e_{ji}}_{e_{ii}} \qquad\qquad (e_{ij}, e_{jk}, e_{ki}) \longmapsto (\underbrace{e_{ij}e_{jk}}_{e_{ik}}, e_{ki})$$

since $E^2 = E$.

Now an interesting question is to compute

$$\varprojlim_{\Lambda_{nd}} A^{\natural} = \left\{ (\alpha_{n+1} \in A^{\otimes(n+1)} \text{ as above}) \right\}$$

If we taken $n+1$ odd then look at $\alpha_{n+1} \in A^{\otimes(n+1)}$ in an odd column of $C(A)$, say the $p=1$ columns. Then because odd permutations don't have sign we have

$$(1-t)\,\alpha_{n+1} = 0$$

Also we have $b'\alpha_{n+1} = 0$ since in $A^{\otimes(n+1)}$ there are an even number of $\otimes$-signs, hence b' has an even number of identical faces.

Take $n+1 = 3$.

$$\begin{tikzcd}
 & 0 & \alpha_3 \arrow[l, mapsto] \arrow[dd, mapsto] \\
A^{\otimes 3} & A^{\otimes 3} \arrow[l, "1-t"'] \arrow[d, "b'"] & \\
 & A^{\otimes 2} \arrow[d] & 0 \\
 & A &
\end{tikzcd}$$

(This is a boundary.)

So we get a cycle of degree 3 in $\mathcal{C}(A)$. But a more obvious thing to do is to look at the image of α_3 in $A^{\otimes 3}/(1-t)$. The boundary will be the image α_2 in $A^{\otimes 2}/(1-t) = \Lambda^2(A)$ which is zero as [redacted] α_2 is symmetric. Thus we get a class in $HC_2(A) = H_2(\mathcal{C}(A))$.

Let's go back to trying to understand how to compute cohomology for $\Lambda_{nd} = \Lambda'$. The objects are the cyclic-ordered non-empty finite sets

$\langle 0\rangle$ $\langle 1\rangle$ $\langle 2\rangle$

and the morphisms are embeddings. An embedding is determined by its effect on vertices. We are interested in contravariant functors on Λ', such as $\langle n\rangle \mapsto A^{\otimes(n+1)}$.

Hence I need to be concerned with the functor $h_{\langle n\rangle}$ given by

$$h_{\langle n\rangle}(\langle p\rangle) = \mathrm{Hom}_{\Lambda'}(\langle p\rangle, \langle n\rangle).$$

The functors $\mathbb{Z}[h_{\langle n\rangle}]$ are the building blocks for projective resolutions.

What is the category $\Lambda'/\langle n\rangle$. Since the morphisms in Λ' are embeddings, $\Lambda'/\langle n\rangle$ must be equivalent to a poset

$$\begin{tikzcd} \langle p\rangle \arrow[r, "\xi"] & \langle n\rangle \\ \langle p'\rangle \arrow[u, shift left, "\alpha"] \arrow[u, shift right, "\beta"'] \arrow[ur, "\xi'"'] & \end{tikzcd} \qquad \Longrightarrow \quad \alpha = \beta.$$

Given $(\langle p\rangle, \xi : \langle p\rangle \to \langle n\rangle) \in \Lambda'/\langle n\rangle$, we can assign to it the image of ξ which is a non-empty subset of $\langle n\rangle$. Then clearly $\Lambda'/\langle n\rangle$ becomes equivalent to the subobjects of $\langle n\rangle$, i.e. to the subsets $(\neq\phi)$ of $\langle n\rangle$.

What I am trying to do is to compute the cohomology $R^q \varprojlim$ of a contravariant functor on Λ'. But I could ask the same question for Δ'. So how can you compute $R^q\varprojlim$ for a simplicial gadget

$$F_2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; F_1 \;\rightrightarrows\; F_0$$

without the degeneracies. Already the problems of finding $\varprojlim$ is hard, and involves an infinite process, namely

$$F_0,\ \mathrm{Equal}\ F_1 \rightrightarrows F_0,\ \{x\in F_2 \mid d_0x = d_1x = d_2x \in \mathrm{Equal}(F_1 \rightrightarrows F_0)\}, \ldots$$

This means the zeroth term in a projective resolution is infinitely generated.

Return to the problem of letter to Loday.

I want to construct for $\mathfrak{g} \to \mathfrak{gl}_n(A)$ a cocycle of $\mathfrak{g}$ with values in the double complex $C(A)$, and ultimately $B(A)$. First construct the cocycle with values in the Connes complex; this is easy.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\Lambda^3\mathfrak{g} \arrow[r] \arrow[d] & A^{\otimes 3}/1-t \arrow[d, "b"] \\
\Lambda^2\mathfrak{g} \arrow[r] \arrow[d] & A^{\otimes 2}/1-t \arrow[d, "b"] \\
\Lambda^1\mathfrak{g} \arrow[r] & A
\end{tikzcd}$$

We start with $tr(\theta^{\otimes p}) \in C^p(\mathfrak{g}, A^{\otimes p})$ and get the constants to work. Recall $d\theta = -\theta^2$ where d is the Lie alg cochain differential.

$$d\,tr(\theta^{\otimes 2}) = tr(d\theta \otimes \theta - \theta \otimes d\theta)$$
$$= -\,tr(\theta^2 \otimes \theta - \theta \otimes \theta^2)$$
$$b\,tr(\theta^{\otimes 3}) = tr(\theta^2 \otimes \theta - \theta \otimes \theta^2 + \theta^2 \otimes \theta)$$

If we work modulo $1-t$, then

$$d\,tr(\theta^{\otimes 2}) = -2\,tr(\theta^2 \otimes \theta)$$
$$b\,tr(\theta^{\otimes 3}) = 3\,tr(\theta^2 \otimes \theta)$$

which means that if we define

$$u_p : \Lambda^p\mathfrak{g} \longrightarrow A^{\otimes p} \qquad \text{by} \qquad u_p = \frac{1}{p}\,tr(\theta^{\otimes p})$$

then modulo $(1-t)$ we get

$$du_2 + bu_3 = 0$$

I should be more careful about computing the cross-over term in b. Take $p=1$

$$du_1 = d\,\mathrm{tr}(\theta) = d\,\theta_{ii} = -\theta_{ij}\theta_{ji}$$
$$= -\mathrm{tr}(\theta^2)$$

$$bu_2 = \tfrac{1}{2}\, b\, \mathrm{tr}(\theta^{\otimes 2}) = \tfrac{1}{2}\, b(\theta_{ij}\otimes\theta_{ji})$$

$$= \tfrac{1}{2}\left(\theta_{ij}\theta_{ji} + \theta_{ji}\theta_{ij}\right) = \theta_{ij}\theta_{ji} = \mathrm{tr}(\theta^2)$$

One puts in a sign when computing the cross-over term in b.

In general we can drop the trace and suppose $n=1$. This is because if $B = M_nA$, we have a natural transformation

$$B^{\otimes p} \longrightarrow A^{\otimes p}$$

$$X_1\otimes\cdots\otimes X_p \longmapsto \mathrm{tr}(X_1\otimes\cdots\otimes X_p) = (X_1)_{ij}\otimes(X_2)_{jk}\otimes\cdots\otimes(X_p)_{ki}$$

compatible with the operators in $\mathcal{C}(A)$.

In general,

$$du_p = d\,\tfrac{1}{p}\theta^{\otimes p} = \tfrac{1}{p}\Big[\overbrace{(-\theta^2)\otimes\theta^{\otimes(p-1)} + \theta\otimes\theta^2\otimes\theta^{\otimes(p-2)} - \cdots}^{p\text{-terms}}$$
$$= -\tfrac{1}{p}\left[1+t+\cdots+t^{p-1}\right]\left(\theta^2\otimes\theta^{\otimes(p-1)}\right)$$

$$bu_{p+1} = \tfrac{1}{p+1}\Big[\theta^2\otimes\theta^{\otimes(p-1)} - \theta\otimes\theta^2\otimes\theta^{(p-2)} + \cdots + (-1)^{p-1}\theta^{\otimes(p-1)}\otimes\theta^{\otimes 2}$$
$$+\cancel{(-1)^p}\,\theta^2\otimes\theta^{(p-1)}\cancel{(-1)^p}\Big]$$
$$= \tfrac{1}{p+1}\Big[1+t+\cdots+\underset{\overset{\parallel}{1}}{t^p}\Big]\left(\theta^2\otimes\theta^{\otimes(p-1)}\right)$$

so

$$du_p + bu_{p+1} = \underbrace{\left[\tfrac{1}{p+1}(N+1) - \tfrac{1}{p}N\right]}_{}\,\theta^2\otimes\theta^{\otimes(p-1)}$$

this has augmentation 1 in $\mathbb{Z}[t] = \mathbb{Z}[\mathbb{Z}/p]$ so this term disappears mod $1-t$.

$$\left[\frac{N+1}{p+1} - \frac{N}{p}\right] = \frac{p\cancel{N}+p-p\cancel{N}-N}{(p+1)(p)} = \frac{1}{p+1}\left(1-\frac{N}{p}\right)$$

Thus

$$du_p + bu_{p+1} = \left(1-\frac{N}{p}\right)\frac{\theta^2\otimes\theta^{\otimes(p-1)}}{p+1}$$

Now I want to write this in the form of $(1-t)k$. which means I want to divide $1-\frac{N}{p}$ by $1-t$ in the group ring. The division is unique if it lies in the ideal gen. by $1-t$.

$$\frac{1-\frac{N}{p}}{1-t} = \frac{1}{p}\sum_{i=0}^{p-1}\frac{(1-t^i)}{1-t} = \frac{1}{p}\sum_{i=0}^{p-1}(1+t+\dots+t^{i-1})$$

$$= \frac{1}{p}\sum_{i=0}^{p-1}\sum_{0\le j<i} t^j = \frac{1}{p}\sum_{0\le j<p-1} t^j \sum_{j<i\le p-1} 1$$

$$= \frac{1}{p}\sum_{j=0}^{p-2} t^j(p-1-j)$$

This has augmentation $\frac{1}{p}\sum_{j=0}^{p-2}(p-1-j) = \frac{1}{p}\frac{(p-1)p}{2} = \frac{p-1}{2}$ so a better answer is

$$\frac{1}{p}\sum_{j=0}^{p-1} t^j(p-1-j) - \frac{p-1}{2}\frac{1}{p}\sum_{j=0}^{p-1} t^j$$

$$\boxed{\alpha_p = \frac{1}{p}\sum_{j=0}^{p-1} t^j\left(\frac{p-1}{2}-j\right)} \quad \boxed{(1-t)\alpha_p = 1-\frac{N}{p}}$$

Check: $p=2$.

$$\alpha_2 = \frac{1}{2}\left(\frac{1}{2}-\frac{t}{2}\right) = \frac{1}{4}(1-t) \qquad \alpha_2(1-t) = \frac{1}{4}(1-t)^2$$

$$= \frac{1}{4}(1-2t+t^2) = \frac{1}{2}(1-t)$$

$$= 1 - \frac{1+t}{2}$$

Idea

$$\begin{tikzcd}
\Lambda^3 \mathfrak{g} \arrow[r, "u_3"] \arrow[drr, "v_3"] & A^{\otimes 3} & A^{\otimes 3} & \\
\Lambda^2 \mathfrak{g} \arrow[r, "u_2"] \arrow[drr, "v_2"] & A^{\otimes 2} & A^{\otimes 2} \arrow[dr, "w_3"] & \\
 & A & A & A
\end{tikzcd}$$

$$v_{p+1} = \alpha_p \frac{\theta^2 \otimes \theta^{\otimes(p-1)}}{p+1}$$

$$\boxed{\begin{aligned} v_2 &= 0 \\ v_3 &= \tfrac{1}{12}(\theta^2\otimes\theta + \theta\otimes\theta^2) \\ v_4 &= \tfrac{1}{12}(\theta^2\otimes\theta\otimes\theta - \theta\otimes\theta\otimes\theta^2) \end{aligned}}$$

I would like $\quad dv_p = \pm b' v_{p+1}$
but this should hold only modulo the image of N,
in fact we will define w_p by

$$dv_p - (\pm b')v_{p+1} = N w_p$$

Let's work out the signs

$$dv_3 = d\,\tfrac{1}{4}(1-t)\frac{\theta^2\otimes\theta}{3} = d\,\tfrac{1}{12}(\theta^2\otimes\theta + \theta\otimes\theta^2)$$

$$= \tfrac{1}{12}\underbrace{(1-t)\ \theta^2\otimes(-\theta^2)}_{0} \qquad \tfrac{1}{12}\left(\theta^2\otimes(-\theta^2) + (-\theta^2)\otimes\theta^2\right)$$

$$dv_3 = -\tfrac{1}{6}\,\theta^2\otimes\theta^2$$

$$b'v_4 = b'\,\alpha_3 \frac{\theta^2\otimes\theta\otimes\theta}{4} = b'\,\tfrac{1}{3}(1-t^2)\frac{\theta^2\otimes\theta\otimes\theta}{4}$$

$$= \tfrac{1}{12}\,b'\left(\theta^2\otimes\theta\otimes\theta - \theta\otimes\theta\otimes\theta^2\right)$$

$$= \tfrac{1}{12}\left[\theta^3\otimes\theta - \theta^2\otimes\theta^2 - \theta^2\otimes\theta^2 + \theta\otimes\theta^3\right]$$

$$-\left[dv_3 + (-b')v_4\right] = \tfrac{1}{12}\left[\theta^3\otimes\theta + \theta\otimes\theta^3\right]$$

$$= N\cdot\tfrac{1}{24}(\theta^3\otimes\theta + \theta\otimes\theta^3)$$

So $\boxed{\omega_4 = \frac{1}{24}\left(\theta^3 \otimes \theta + \theta \otimes \theta^3\right)}$

I forgot to get $\omega_3 : \Lambda^3 \mathfrak{g} \longrightarrow A$ first.

$$v_2 = 0 \qquad \text{since} \quad \alpha_1 = 0$$

$$\begin{aligned} b' v_3 &= b' \tfrac{1}{12}\left(\theta^2 \otimes \theta + \theta \otimes \theta^2\right) \\ &= \tfrac{1}{12}\left(\theta^3 + \theta^3\right) = \tfrac{1}{6}\theta^3 \end{aligned}$$

$$\therefore \boxed{\omega_3 = \tfrac{1}{6}\theta^3} \qquad \text{up to sign}$$

Now what I would like is nice clean formulas for the ω's, since they should be equivalent to Connes S-operator.

Idea: $\mathcal{C}(A) = \mathcal{B}(\tilde{A})_{\text{norm}}$ and it should be possible to map the latter to the Deligne complex based on $\Omega_{nc}(\tilde{A})$. So I should be able to see the image of the cocycle I want with values in $\mathcal{C}(A)$ down in $\bigoplus \Omega_{nc}^{<i}(\tilde{A})[-2i+1]$ which should appear as a quotient complex of $\mathcal{C}(A)$.

October 10, 1983

Given a repn. V of G and a flat G-bdle P/Y, we can form the associated flat vector bundle $P\times^G V$ over Y, and take the DR complex of forms: $\Omega(Y, P\times^G V)$. Thus from the fibred category of flat principal bundles we have a functor to cochain complexes. Now restrict to flat connections on trivial G-bundles and look for a section of this functor, i.e. a natural way to select a form in $\Omega(Y, P\times^G V)$. Then these sections form a cochain complex which we can identify with $C^*(\mathfrak{g}, V)$, the complex of Lie cochains with values in V considered as a $\mathfrak{g}$-module.

This is clear, because if we ignore D, the trivialization gives
$$\Omega(Y, P\times^G V) = \Omega(Y)\otimes V \qquad \text{corresp. to}$$
$$C^*(\mathfrak{g}, V) = C^*(\mathfrak{g})\otimes V.$$

Next what I want to know is whether this has any bearing on the gauge Lie algebra setup. But I first should work out the differential.

$$\Omega(Y, P\times^G V) \simeq \Omega(Y)\otimes V$$
$$D \longleftrightarrow d+\alpha$$

where α is a zero-th order operator, endomorphism over $\Omega(Y)$ of degree 1 on $\Omega(Y)\otimes V$. So α is determined by
$$\alpha : V \longrightarrow \Omega^1(Y)\otimes V$$
and this must come essentially from $V \longrightarrow \mathfrak{g}^*\otimes V$ belonging to the G-action. Then

$$(d+\alpha)^2 = \quad d\alpha + \alpha^2 = 0$$

Now go to the gauge situation, where $\Omega(Y, P\times^G V)$ becomes the complex $\Omega^{*,0}(Y\times M, pr_2^* E_0)$ and the D is the flat horizontal connection $D' = d' + \theta$. So the only thing new here is the fact I can interpret $D' = d' + \theta$ as the Lie cohomology differential taking the coefficients to be a $\tilde{\mathfrak{g}}$-module.

Why should this be relevant? I ultimately do calculations in the ring $\Omega(M, \operatorname{End} E_0)$ and if I write things as operators on $\Omega(M, E_0)$, the bracket might be unclear. – The usual difficulty in working with the adjoint representation.

However what I am suggesting is that instead of working with

$$d' \quad \text{on} \quad \Omega^*(Y)\otimes\Omega^0(M, \operatorname{End} E_0) \qquad\qquad d' \text{ on } C^*(\mathfrak{g})\otimes A$$

I work with

$$\delta = d' + [\theta, \;]. \quad \text{on } \Omega^*(Y)\otimes\Omega^0(M, \operatorname{End} E_0)$$

This doesn't help $\qquad$ or on $C^*(\mathfrak{g}, A)$.

because $\quad d'\theta = -\theta^2$

$$\delta\theta = -\theta^2 + [\theta,\theta] = \theta^2$$

so all one has changed is a sign.

Key idea: Recall that $\mathcal{C}(A) = \mathcal{B}(\tilde{A})_{red}$.
Thus $\mathcal{C}(A)$ is the complex

$$\begin{tikzcd}
{} \arrow[d] & {} \arrow[d] & \\
\tilde{A}\otimes A^2 \arrow[d, "b"] & \tilde{A}\otimes A^2 \arrow[ul] \arrow[d, "b"] & \\
\tilde{A}\otimes A \arrow[d, "b"] & \tilde{A}\otimes A \arrow[ul, "B"'] \arrow[d, "b"] & {} \arrow[ul] \arrow[d] \\
\tilde{A} & \tilde{A} \arrow[ul, "B"'] & \tilde{A} \arrow[ul]
\end{tikzcd}$$

But the key fact is that the normalized Hochschild complex has a completely different interpretation as the non-commutative differentials.

Usual description. Take an A with 1. Then one can form the A-bimodule

$$\Omega^1 = \operatorname{Ker}\{A\otimes A \longrightarrow A\}$$

and

$$\begin{tikzcd}
\Omega^1 \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & A\otimes A \arrow[r] & A\otimes \bar{A}
\end{tikzcd}$$

Then

$$\Omega^p = \underbrace{\Omega^1 \otimes_A \Omega^1 \otimes_A \cdots \otimes_A \Omega^1}_{p\text{-times}} \xrightarrow{\sim} A\otimes \bar{A}^{\otimes p}.$$

Better description maybe is to get it as a subring of the Amitsur complex

$$A \longrightarrow A\otimes A \longrightarrow A\otimes A\otimes A \longrightarrow \cdots$$

under the AW product.

~~So~~ what this means is that we have a natural product ~~[illegible]~~ to play around with on the normalized

Hochschild complex. For example, what is the Hochschild boundary operator b?

$$b(a_0 da_1) = b(a_0, a_1) = a_0 a_1 - a_1 a_0 = [a_0, a_1]$$

$$b(a_0 da_1 da_2) = (a_0 a_1, a_2) - (a_0, a_1 a_2) + (a_2 a_0, a_1)$$

$$= \cancel{a_0 a_1 da_2} - a_0(\cancel{a_1 da_2} + da_1\, a_2) + a_2 a_0\, da_1$$

$$= -[a_0 da_1, a_2]$$

$$b(a_0 da_1 da_2 da_3) = a_0 a_1\, \cancel{da_2}\, da_3 - a_0\, \overset{\cancel{a_1 da_2} + (da_1)a_2}{d(a_1 a_2)}\, da_3 + a_0\, da_1\, \overset{\cancel{a_2} da_3 + da_2\, a_3}{d(a_2 a_3)}$$
$$- a_3 a_0\, da_1 da_2$$

$$= [a_0 da_1 da_2, a_3]$$

In general

$$\boxed{b(a_0 da_1 \cdots da_p) = (-1)^{p-1}[a_0 da_1 \cdots da_{p-1}, a_p]}$$

Also in this notation B is very simple

$$\boxed{B(a_0\, da_1 \text{[redacted]}\, da_p) = \sum_{i=0}^{p} (-1)^{in}\, da_i \cdots da_n\, da_0 \cdots da_{i-1}}$$

so $B = Nd$ in some sense

October 11, 1983

Karoubi takes an augmented ring $A = k \oplus \bar{A}$, & forms the non-commutative differential algebra of forms $\Omega_n(A) = A \otimes \bar{A}^{\otimes n}$, and the non-commutative DR complex $\bar{\Omega}(A) = \Omega(A)/[\ ,\ ]$. Then [redacted] he proves that

$$H_p(\bar{\Omega}(A)) \xrightarrow{\sim} \mathrm{Ker}\{\bar{HC}_p(A) \longrightarrow H_{p+1}(A)\}.$$

I think this is just Connes' theorem [redacted] for $a = \bar{A}$.

$$H_p(\bar{\Omega}(a^+)) \xrightarrow{\sim} \mathrm{Ker}\{HC_p(a) \longrightarrow H_{p+1}(a)\}$$

[redacted] In view of Connes approach to this whole subject I should have asked long ago the following:

Question: What is $H_*(\bar{\Omega}(A))$?

It seems to me now that I might be able to attack the problem remaining in the Loday letter by [redacted] using Connes theorem. The idea will be to construct [redacted] characteristic class maps from Lie algebra [redacted] homology to non-commutative DR cohomology, and then use the map of the latter to usual DR cohomology. The point is the first map would be equivalent to the [redacted] map $C_*(\mathfrak{g}) \longrightarrow C$[redacted]$(A)$ I am after.

At this point I should go over Connes's approach. Why is cyclic homology related to NC DR coh?

Let's prove Connes starting formula for a ring A with 1.

$$\boxed{\bar{\Omega}^p / d\bar{\Omega}^{p-1} \quad \xrightarrow{\sim} \quad \bar{A}^{\otimes(p+1)} / (1-t)\bar{A}^{\otimes(p+1)} + b\,A^{\otimes(p+2)}}$$

Start by defining

$$\bar{A}^{\otimes(p+1)} \xrightarrow{\varphi} \bar{\Omega}^p / d\bar{\Omega}^{p-1}$$

$$(a_0, \dots, a_p) \longmapsto a_0\, da_1 \cdots da_p$$

Need that if $a_0 = 1$, then $da_1 \cdots da_p = d(a_1\, da_2 \cdots da_p)$. Check $\mathrm{Im}(1-t)$ goes to zero.

$$t(a_0, \dots, a_p) = (-1)^p (a_p, a_0, \dots, a_{p-1})$$

$$\longmapsto (-1)^p\, a_p\, da_0 \cdots da_{p-1}$$

$$\equiv -(-1)^p\, da_p\, a_0\, da_1 \cdots da_{p-1} \qquad (\mathrm{mod}\ \mathrm{Im}\, d)$$

$$\equiv a_0\, da_1 \cdots da_p \qquad (\mathrm{mod}\ [dA, \Omega^{p-1}])$$

Check that $\mathrm{Im}\, b$ goes to zero.

$$b(a_0, \dots, a_{p+1}) = \sum_{i=0}^{p} (-1)^i (\dots a_i a_{i+1} \quad) + (-1)^{p-1}(a_{p+1}a_0, \dots, a_p)$$

$$\longrightarrow (a_0 a_1\, da_2 \cdots da_p) + \sum_{i=1}^{p} (-1)^i \left(\dots \begin{matrix} a_i\, da_{i+1} \dots \\ da_i\, a_{i+1} \end{matrix} \right) + (-1)^{p-1}\, a_{p+1} a_0\, da_1 \cdots da_p$$

$$= (-1)^p\, a_0\, da_1 \cdots da_p\, a_{p+1} + (-1)^{p-1}\, a_{p+1} a_0\, da_1 \cdots da_p$$

$$\equiv 0 \qquad \mathrm{mod}\ [A, \Omega^p].$$

Now define $\underset{\Omega^p}{\overset{\shortparallel}{A \otimes \bar{A}^p}} \xrightarrow{\psi} \bar{A}^{\otimes(p+1)} / \mathrm{Im}(1-t) + \mathrm{Im}(b)$

by $a_0\, da_1 \cdots da_p \longmapsto (a_0, \dots, a_p)$.

If $a_0 = 1$ you get 0 so $d\Omega^{p-1} \in \mathrm{Ker}\,\psi$. Now you can reverse the above arguments to see that $[dA, \Omega^{p-1}] \subset \mathrm{Ker}\,\psi$ follows from mod $1-t$, and $[A, \Omega^p] \subset \mathrm{Ker}\,\psi$

follows from mod b. Finally all one needs to shows is that

$$[\Omega,\Omega]^p = [A,\Omega^0] + [dA,\Omega^1]$$

which follows from the identity

$$\begin{aligned}
[xy,m] &= (xy)m - m(xy) \\
&= x(ym)-(ym)x + y(mx)-(mx)y \\
&= [x,ym] + [y,mx]
\end{aligned}$$

which shows that if B is generated by x_i, then

$$[B,M] = \sum [x_i, M].$$

for any B- bimodule M.

Thus we have

$$\begin{tikzcd}
0 \arrow[d] & & 0 \arrow[d] \\
H^p\bar{\Omega}(A) \arrow[d] & & HC_p(A) \arrow[d] \\
\bar{\Omega}^p/d\bar{\Omega}^{p-1} \arrow[d, "d"] \arrow[rr, "\sim"] & & \bar{A}^{\otimes(p+1)}/\operatorname{Im}(1-t)+\operatorname{Im} b \arrow[d, "b"] \\
\bar{\Omega}^{p+1} & & \bar{A}^{\otimes p}/\operatorname{Im}(1-t)
\end{tikzcd}$$

[redacted] October 13, 1983

I should work out carefully the construction of Chern character classes for $gl_n(A)$ with values in the NCDR cohomology of A. We work in the ring $C^*(g, \tilde{\Omega}^*)$, where $\tilde{\Omega}$ denotes the differential algebra of non-comm. diff. forms on A. $C^*(g, \tilde{\Omega}^*)$ is a bigraded algebra with [redacted] differentials [redacted] d', d'' and total differential $d = d' + d''$. [redacted]

Now we have the basic MC form

$$\theta \in M_n(C^1(g, A))$$

satisfying $d'\theta = -\theta^2$. [redacted] We consider the family of connections $D_t = d + t\theta$ on $C^*(g, \tilde{\Omega}^*)^n$. The curvature is

$$D_t^2 = t d\theta + t^2\theta^2 = t d''\theta + (t^2 - t)\theta^2$$
$$= t d''\theta + (t - t^2) d'\theta.$$

[redacted] We want the character

$$tr\left(e^{\kappa D_t^2}\right)$$

Put κ in later. The key point at the moment is

$$\frac{d}{dt} tr\left(e^{D_t^2}\right) = d [redacted] tr\left(\theta e^{D_t^2}\right)$$

or more generally

$$\frac{d}{dt} tr\left(e^{D^2}\right) = d\, tr\left(\dot{D} e^{D^2}\right).$$

Proof:

$$\overset{\parallel}{tr\left(e^{D^2}(D^2)^{\cdot}\right)} = tr\left([D, \dot{D}] e^{D^2}\right) = tr\left([D, \dot{D} e^{D^2}]\right) \;\; \parallel$$

In the non-commutative setting, how much of this can be proved? I might as well assume $n=1$. Then

[redacted]

$$\frac{d}{dt}\, e^{D^2} = \int_0^1 ds\; e^{sD^2} \underbrace{(D^2)^{\cdot}}_{[D,\dot D]} e^{(1-s)D^2}$$

$$= \left[D, \int_0^1 ds\; e^{sD^2}\, \dot D\, e^{(1-s)D^2}\right]$$

Put in t:

$$\frac{d}{dt}\, e^{D_t^2} = \left[d+t\theta, \int_0^1 ds\; e^{sD_t^2}\, \theta\, e^{(1-s)D_t^2}\right]$$

so
$$e^{D_1^2} - e^{D_0^2} = d\left\{\int_0^1 dt \int_0^1 ds\; e^{sD_t^2}\, \theta\, e^{(1-s)D_t^2}\right\}$$
$$+ \left[\theta, \int_0^1 dt \int_0^1 ds\; e^{sD_t^2}\, t\theta\, e^{(1-s)D_t^2}\right]$$

The only question is why the last term disappears when we take the trace. For example, why is

$$\mathrm{tr}\,[\theta,\theta] = 2\,\mathrm{tr}(\theta^2) = 0 \quad ?$$

Take $X, Y \in \mathfrak{g}$ and compute

$$i(Y)\,i(X)\,\mathrm{tr}(\theta^2) = \mathrm{tr}\; i(Y)(X\theta - \theta X)$$
$$= \mathrm{tr}\,(XY - YX)$$
$$= X_{ij}Y_{ji} - Y_{ji}X_{ij}$$

which will follow if the trace is considered in $A/[A,A]$.

What might be important in the above calculation is that we only have to take the commutator with respect to θ. This means that I take the quotient of the ring $\tilde\Omega^*$ by $[A,\tilde\Omega^*]$, then I get the formula

$$\mathrm{tr}\left(e^{d''\theta}\right) - \mathrm{tr}\left(e^{0}\right) = d\left\{\int_0^1 dt \int_0^1 ds\, \mathrm{tr}\left(e^{sD_t^2}\,\theta\, e^{(1-s)D_t^2}\right)\right\}.$$

This will give me a cocycle modulo the image of b I think.

~~Recall the isom.~~

$$\cancel{\tilde\Omega^p \longleftarrow A\otimes \bar A^{\otimes p}}$$

$$\cancel{a_0\,da_1\dots da_p \longleftrightarrow (a_0,a_1,\dots,a_p)}$$

~~Then~~ $\cancel{b(a_0,a_1,\dots,a_p,a) = a_0a_1a}$

Unfortunately d is not defined modulo $[A,\tilde\Omega^*]$ since

$$d[a,\omega] = [da,\omega] + [a,d\omega]$$

$C_\cdot(A) =$ the Connes complex $\left(A^{\otimes(*+1)}/(1-t), b\right)$.

This complex has a basic "periodicity" endo.

$$S \in \mathrm{Ext}^2(C(A), C(A)).$$

Also there is a map

$$\mu = \sum_{i\geq 0} \mu_i : \quad C(A) \longrightarrow \bigoplus_{i\geq 0} \bar{\Omega}^{\leq i}[-2i]$$

which is defined using the $\mathcal{B}(A)_{\mathrm{norm}}$ model for $C(A)$. We can write that

$$\mu_m \in \mathrm{Ext}^{2m}\left(C(A), \bar{\Omega}^{\leq m}\right)$$

hence

$$\mu_m S \in \mathrm{Ext}^{2m+2}\left(C(A), \bar{\Omega}^{\leq m}\right).$$

But

$$\mu_{m+1} \in \mathrm{Ext}^{2m+2}\left(C(A), \bar{\Omega}^{\leq m+1}\right) \quad \text{and}$$

we have an evident exact sequence

$$0 \longrightarrow \bar{\Omega}^{m+1}[m+1] \longrightarrow \bar{\Omega}^{\leq m+1} \longrightarrow \bar{\Omega}^{\leq m} \longrightarrow 0.$$

From the nature of μ_m it is clear that

$$\mu_{m+1} \longmapsto \mu_m S$$

under the truncation map.

Now when we have a repn. $\mathfrak{g} \longrightarrow \mathfrak{gl}_n(A)$ we can define a class

$$\chi \in H^{+1}(\mathfrak{g}, C(A))$$

(check $\mu_0 : C(A) \to A/[\,,\,]$ gives $\mu_0 \chi \in H^1(\mathfrak{g}, A/[\,,\,])$)

hence we get related classes $S^k \chi \in H^{2k+1}(\mathfrak{g}, C(A))$.

[redacted] I then consider the classes

$$\mu_m \delta^k X \in H^{2m+2k+1}(\mathfrak{g}, \bar{\Omega}^{\leq m})$$

Up to truncation one has $\mu_{m+k} X \longmapsto \mu_m \delta^k X$, so the basic classes are the [redacted]

$$\mu_m X \in H^{2m+1}(\mathfrak{g}, \bar{\Omega}^{\leq m})$$

Now via curvature methods I have defined

$$ch_i \in H^{2i-1}(\mathfrak{g}, \bar{\Omega}^{< i})$$

so the conjecture is that

$$\boxed{\mu_m X = ch_{m+1} \quad ?}$$

October 17, 1983

Formula for the curvature of the Grassmannian connection:

$$D = e \cdot d \cdot e$$

$$\begin{aligned} D^2 &= e \cdot d \cdot e \cdot d \cdot e \\ &= e[e \cdot d + de] \cdot d \cdot e \\ &= e \cdot de \cdot d \cdot e \\ &= e \cdot de \cdot [e \cdot d + de] \\ &= (e \cdot de \cdot e) d + e \cdot de \cdot de \end{aligned}$$

Now $\qquad de = d(e \cdot e) = de \cdot e + e \cdot de$

so $\qquad \begin{aligned} e \cdot de &= de \cdot (1-e) \\ de \cdot e &= (1-e) de \, ; \end{aligned}$

in particular $\qquad e \cdot de \cdot e = e(1-e) \cdot de = 0.$ Thus

$$D^2 = e(de)^2.$$

Now

$$de \cdot de \cdot e = de \cdot (1-e) \cdot de = e \cdot de \cdot de.$$

hence

$$D^4 = e(de)^2 e(de)^2 = e(de)^4$$

$$D^{2n} = e\,(de)^{2n}.$$

so the character forms for the Grassmannian connection are

$$ch_n = \frac{1}{n!}\operatorname{tr}\left(e(de)^{2n}\right)$$

Formula for the index: Let

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- = \begin{matrix}\mathcal{H}^+\\ \oplus \\ \mathcal{H}^-\end{matrix}$$

with the grading given by

$$\varepsilon = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad \text{on} \quad \begin{matrix}\mathcal{H}^+\\ \oplus \\ \mathcal{H}^-\end{matrix}$$

Let F be an operator on $\mathcal{H}$ $\ni$

$$F^2 = I \qquad F\varepsilon = -\varepsilon F$$

i.e.

$$F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix} \quad \text{where} \quad p : \mathcal{H}^+ \xrightarrow{\sim} \mathcal{H}$$

Finally let $e = \begin{pmatrix} e^+ & \\ & e^- \end{pmatrix}$ be an idempotent operator of degree 0 wrt ε. We want a formula for the index of the operator

$$* \qquad e^+\mathcal{H}^+ \xrightarrow{e^- p e^+} e^-\mathcal{H}^- .$$

We assume that $e^+p^{-1}e^-$ is a parametrix for this operator, which means that if we define K by

$$** \quad \begin{cases} e^+ - K^+ = e^+p^{-1}e^-pe^+ \\ e^- - K^- = e^-pe^+p^{-1}e^- \end{cases}$$

then K^n is of trace class for large n.

Then $**$ says that the identity map of the complex is homotopic to the map K given by K^+, K^-. It follows that the identity map is homotopic to K^n so that K^n induces the identity on homology. But if K^n is of trace class one has the Lefschetz formula

$$\mathrm{tr}\left((K)^n \text{ on } H^{+}\right) - \mathrm{tr}\left(K^n \text{ on } H^-\right)$$

$$= \mathrm{tr}\,(K^+)^n - \mathrm{tr}\,(K^-)^n = \mathrm{tr}(\varepsilon K^n).$$

Thus

$$\text{index} = \mathrm{tr}(\varepsilon K^n).$$

for all large enough n such that K^n is of trace class.

~~[illegible]~~

Next from $(**)$ we have

$$e - K = eFeFe$$

or

$$\begin{aligned} K &= e - eFeFe \\ &= eF^2e^2 - eFeFe \\ &= eF(Fe - eF)e \\ &= eF\,[F,e]e \\ &= eF(1-e)[F,e] \\ &= e(eF - Fe)[F,e] \\ &= -e[F,e][F,e] \end{aligned}$$

so

$$\boxed{\text{index} = (-1)^n \,\mathrm{tr}\left(\varepsilon e [F,e]^{2n}\right)}$$

Non-commutative differential forms:

Suppose $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, ε, F are as above. Let $\mathcal{B}$ be the algebra of bounded operators on $\mathcal{H}$. Then $\mathcal{B}$ is a $\mathbb{Z}/2$ graded algebra

$$\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^-$$

where $\mathcal{B}^\pm = \{ B \mid \varepsilon B = \pm B \varepsilon \}$. Introduce

$$d\alpha = [F, \alpha] \qquad \text{on } \mathcal{B}$$

where the commutator is in the graded sense

$$[F, \alpha] = F\alpha - (-1)^{\deg \alpha} \alpha F.$$

Then one has

$$\begin{cases} d(\alpha\beta) = d\alpha . \beta + (-1)^{\deg \alpha} \alpha . d\beta \\ d^2 \alpha = 0 . \end{cases}$$

The latter follows because

$$\begin{aligned} [F, [F, \alpha]] &= [F, F\alpha - (-1)^{\deg\alpha} \alpha F] \\ &= F^2 \alpha - (-1)^{\deg \alpha} F \alpha F \\ &\quad - (-1)^{\deg \alpha + 1} (F\alpha - (-1)^{\deg \alpha} \alpha F) F \\ &= [F^2, \alpha] = 0 \qquad \text{as } F^2 = 1 . \end{aligned}$$

So $\mathcal{B}$ is a $\mathbb{Z}/2$-graded differential algebra.

Now suppose A is an algebra equipped with a homomorphism

$$A \longrightarrow \mathcal{B}^+ .$$

Then we get a homomorphism of graded diff'l algebras

$$\tilde{\Omega}_A \longrightarrow \mathcal{B}$$

$$a_0 da_1 \cdots da_p \longmapsto a_0 [F, a_1] \cdots [F, a_p]$$

What I want to do now is to take the map

$$\alpha \longmapsto tr(\varepsilon \alpha) \qquad \alpha \in \mathcal{B}^*$$

which is defined on the ideal of trace class operators in $\mathcal{B}$. It satisfies

$$tr(\varepsilon\, \alpha\beta) = (-1)^{\deg \alpha \deg \beta}\, tr(\varepsilon \beta \alpha)$$

provided one of α, β is of trace class.

Proof. Assume β is of trace class. Then

$$tr(\varepsilon \alpha \beta) = tr(\beta \varepsilon \alpha)$$

$$= (-1)^{\deg \beta}\, tr(\varepsilon \beta \alpha)$$

which proves the formula, since one gets 0 unless $\alpha\beta$ has <u>degree 0</u>.

[redacted] Now I want to pull this map $tr\, \varepsilon$ back to $\tilde{\Omega}^{ev}$. It is necessary to assume that

$$[F, a_1] \cdots\cdots [F, a_{2n}] \in \text{[redacted] Trace class ops.}$$

If so, we set

$$\tau(a_0 da_1 \cdots da_{2n}) = \tau(a_0, \ldots, a_{2n}) = \operatorname{tr}(\varepsilon\, a_0 [F,a_1] \cdots [F,a_{2n}]).$$

Then we have the identities

$$\tau(\omega\eta) = (-1)^{pq}\, \tau(\eta\omega) \qquad \begin{cases} p = \deg \omega,\ q = \deg \eta \\ p+q \geq 2n. \end{cases}$$

$$\tau(d\eta) = 0$$

Why? $\eta = a_1 da_2 \cdots da_{2n}$

$$\begin{aligned}
\tau(d\eta) &= \operatorname{tr}(\varepsilon [F,a_1] \cdots [F,a_{2n}]) \\
&= \operatorname{tr}(\varepsilon F^2 [F,a_1] \cdots [F,a_{2n}]) \\
&= -\operatorname{tr}(F \varepsilon F [F,a_1] \cdots [F,a_{2n}]) \\
&= -\operatorname{tr}(\varepsilon F [F,a_1] \cdots [F,a_{2n}] F)
\end{aligned}$$

$$\begin{aligned}
\therefore 2\tau(d\eta) &= \operatorname{tr}\big(\varepsilon F \underbrace{[F, [F,a_1] \cdots [F,a_{2n}]]}_{0}\big) \\
&= 0 \qquad \text{as } d^2 = 0,\ d = [F,\ ].
\end{aligned}$$

For the first identity it suffices to consider $\deg \omega = 0, 1$, specifically $\omega = a \in A$ or $\omega = da$. The first case will be clear from the ~~symmetry~~ for the trace, so I should check it for $\eta = a_1 da_2 \cdots da_{2n}$, $\omega = da$.

$$\begin{aligned}
\tau(da\, a_1 da_2 \cdots da_{2n}) &= \tau(d(aa_1) da_2 \cdots da_{2n}) \\
&\quad - \tau(a\, da_1 \cdots da_{2n}) \\
&= -\operatorname{tr}(\varepsilon\, a [F,a_1] \cdots [F,a_{2n}]) \\
&= -\tfrac{1}{2}\operatorname{tr}(\varepsilon F [F,a][F,a_1] \cdots [F,a_{2n}]) = -\tfrac{1}{2}\operatorname{tr}(\varepsilon F [F,a_1] \cdots [F,a]) \\
&= -\operatorname{tr}(\varepsilon a_1 [F,a_2] \cdots [F,a_{2n}]) = -\tau(a_1 da_2 \cdots da_{2n}\, da).
\end{aligned}$$

The basic formula for the index of a Toeplitz type operator is

$$\operatorname{tr}\left(p^{-1}[p,e]\right)^{2n+1}$$

once n is big enough so the product is of trace class. There are various possible ways to write this expression in terms of cyclic cocycles.

Let's adopt the viewpoint that cyclic cocycles come from left-invariant closed forms on a gauge group. The gauge groups act on Hilbert space, hence on corresponding Grassmannians, so invariant forms on the Grassmannian pull-back to left-invariant forms on the gauge group.

There are two kinds of Grassmannians one is concerned with: The ungraded, or ordinary one, which consists of F such that $F^2 = I$. This carries the even-dim'l character forms

$$\frac{1}{2^{2n+1}n!}\operatorname{tr}\left(F(dF)^{2n}\right)$$

which pull back to give cyclic cocycles

$$\text{const.}\ \operatorname{tr}\left(F[F,a_1]\cdots[F,a_{2n}]\right)$$

The graded one consists of operators F in $\mathcal{H}^+\oplus\mathcal{H}^-$ of degree 1 such that $F^2 = I$. Such an F is in the form

$$F = \begin{pmatrix} 0 & p^{-1} \\ p & 0 \end{pmatrix}$$

where $p : \mathcal{H}^+ \xrightarrow{\sim} \mathcal{H}^-$. Thus the Grassmannian is just the space of invertible operators. This space carries odd-dim'l character forms.

$$\frac{(-1)^{n-1}(n-1)!}{(2n-1)!}\operatorname{tr}\left(p^{-1}dp\right)^{2n-1} \qquad \text{rep. } n \text{ by } n+1$$

which pull-back to give cyclic cocycles

$$\text{Const} \quad \operatorname{tr}\left(p^{-1}[p,a_0]\cdots\cdots p^{-1}[p,a_{2n}]\right).$$

Now Connes has another way to write this cocycle, actually two ways.

$$\operatorname{tr}\left(\varepsilon F(dF)^{2n+1}\right) \qquad \text{form on Grass.}$$

$$\operatorname{tr}\left(\varepsilon F\underbrace{[F,a_0]}_{Fa_0-a_0F}\cdots[F,a_{2n}]\right) \qquad \text{cocycle}$$

$$= 2\operatorname{tr}\left(\varepsilon a_0[F,a_1]\cdots[F,a_{2n}]\right)$$

$$\operatorname{tr}\left(\varepsilon F[F,a_0]\cdots[F,a_{2n}]\right)$$

$$= \operatorname{tr}\left(p^{-1}[p,a_0][p^{-1},a_1]\cdots\cdots[p^{-1},a_{2n-1}][p,a_{2n}]\right)$$
$$- \operatorname{tr}\left(p\,[p^{-1},a_0][p,a_1]\cdots\cdots\cdots\cdots[p^{-1},a_{2n}]\right)$$

$$= (-1)^n 2\operatorname{tr}\left(p^{-1}[p,a_0]p^{-1}[p,a_1]\cdots\cdots p^{-1}[p,a_{2n}]\right)$$

$$\boxed{\begin{aligned}\operatorname{tr}\left(\varepsilon F[F,a_0]\cdots[F,a_{2n}]\right) &= 2\operatorname{tr}\left(\varepsilon a_0[F,a_1]\cdots[F,a_{2n}]\right)\\ &= (-1)^n 2\operatorname{tr}\left(p^{-1}[p,a_0]\cdots\cdots p^{-1}[p,a_{2n}]\right)\end{aligned}}$$

$$p^{-1}[p,a] = a - p^{-1}ap$$ is the difference of two homomorphisms, hence is a derivation.

The formula I have for the index
$$tr\left((p^{-1}[p,e])^{odd}\right)$$
gets applied in two cases:

Toeplitz : p, e act on same $\mathcal{H}$.
graded : $p: \mathcal{H}^+ \xrightarrow{\sim} \mathcal{H}^-$ and $e = (e^+, e^-)$.

Both become of the same form namely
$$tr\left((e - p^{-1}ep)^{odd}\right)$$
$$tr\left((e^+ - p^{-1}e^-p)^{odd}\right).$$
So one might in general look at
$$tr\left[(e_1 - e_2)^{\text{[redacted]}\; 2n+1}\right]$$
+ try to show it is an integer independent of n.

Odd case: Here one is given a projector e, or equivalently $F = 2e-1$, satisfying $F^2 = 1$. Then one considers the Toeplitz operator ege on $e\mathcal{H}$, whose index is given by
$$tr\left(g^{-1}[g,e]\right)^{2n+1} \qquad \text{for } n \text{ large}.$$

The even character forms on the space of F are
$$\frac{1}{2^{2n+1}\,\text{[redacted]}\,n!}\, tr\left(F(dF)^{2n}\right).$$
These give rise to cocycles
$$\text{const}\ tr\left(F[F,a_1]\cdots[F,a_{2n}]\right).$$

The index of ege on $e\mathcal{H}$ should be, roughly, the value of this cocycle on $(g^{-1}, g, \dots, g^{-1}, g)$.

$$\operatorname{tr}\left(F([F,g^{-1}][F,g])^{n} \right) = \operatorname{tr}\left((g^{-1} - Fg^{-1}F)[F,g]([F,g^{-1}][F,g])^{n-1}\right)$$

$$= 2\operatorname{tr}\left(g^{-1}[F,g]([F,g^{-1}][F,g])^{n-1}\right)$$

$$= (-1)^{n-1}2 \operatorname{tr}\left((g^{-1}[F,g])^{2n-1}\right)$$

$$= (-1)^{n} 2^{2n} \operatorname{tr}\left(g^{-1}[g,e]\right)^{2n-1}$$

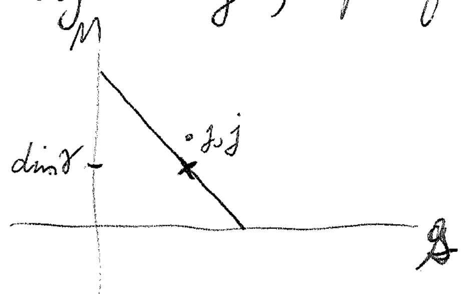
Consider next the question of relating character forms on the Grassmannian $\mathcal{F}$ of all F to the index of operators of the form gFg^{-1}, where g runs over $\mathcal{G}$. One has the map

$$\mathcal{G} \longrightarrow \mathcal{F} \qquad g \longmapsto gFg^{-1}$$

and one can pull-back the character classes on $\mathcal{F}$ to $\mathcal{G}$. This gives even-diml. classes on $\mathcal{G}$. Now I already described the primitive generators for the cohomology of $\mathcal{G}$. One uses the evaluation map

$$\mathcal{G} \times M \longrightarrow U$$

pulls back odd char classes e_j and takes Künneth components. Hence an odd homology class $\gamma \in H_{odd}(M)$ can be capped with e_j so as to obtain even classes in $\mathcal{G}$. If j is sufficiently large, specifically I want $j > \dim \gamma$,



then $\gamma \cap e_j$ will be represented by an invariant diffl' form on $\mathcal{G}$.

Start with a Dirac operator on an odd-dim'l manifold M. This gives me an involution F in a Hilbert space, and if I act on it [redacted] by elts. of $\mathcal{G}$, then I get a map

$$\mathcal{G} \xrightarrow{\varphi} \mathcal{F}$$

where $\mathcal{F}$ is a model for $\mathbb{Z} \times BU$. Now on $\mathcal{F}$ [redacted] we have the Chern character

$$ch_t = \sum_{j \geq 0} ch_j \, t^j$$

cohomology class. This can be pulled back to $\mathcal{G}$ so we obtain a sequence of classes

$$\varphi^*(ch_t) \in H^{ev}(\mathcal{G}).$$

Next we should be able to establish by using the index thm. for families that

$$\varphi^*(ch_t) = \gamma \cap e_t$$

where e_t is the odd character on U_N pulled back to $\mathcal{G} \times M$.