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Jan. 1 '83 - Jan 26, '83

December 31, 1982:

Connes cocycles. Consider an invertible elliptic first order operator D, and set

$$F = \begin{pmatrix} 0 & D^{-1} \\ D & 0 \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $F^2 = 1$, $\varepsilon^2 = 1$, $\varepsilon F \varepsilon^{-1} = -F$. The key object in Connes theory is his

[redacted] $$\int a^0 da^1 \dots da^n = Tr(\varepsilon a^0 [F, a^1] \cdots [F, a^n])$$

where the a_i are functions on the manifold. Note that this is zero if n is odd, because F hence $[F, a]$ is odd relative to the $\mathbb{Z}_2$-grading.

January 1, 1983:

On Connes theory:

1) He defines a coho. and homo. theory for algebras which is unchanged under replacing an alg. A by the matrix algebra $M_n(A)$. This has the advantage of allowing him to deal with nasty equivalence relations on manifolds, such as [redacted] foliations (and ergodic group actions).

In what follows I limit myself to $A = C^\infty(M)$ (although this is a lousy restriction for K-theory purposes.)

2) There is a natural character map from bundles and bundle autos. to the Connes cohomology of M. There is a map from [redacted] $K_0(M)$, described as operators, and $K_1(M)$, described as BDF extensions, to the Connes homology of M. Things are arranged so that the pairing

$$K_*(M) \otimes K^*(M) \longrightarrow K^*(pt)$$

given by the index is compatible with the natural pairing of Connes homology with cohomology. (One can view this as an index thm., but of a formal sort, since one has yet to link [redacted] Connes groups with the DR cohomology.)

3) My goal is to obtain a specific, diff'l form [redacted] index formula. Precise version: Fix an operator, say the Dirac operator, on M and then for any vector bundle E over M I get a family of operators parametrized by the space of connections on E which is equivariant for the gauge group. I want the [redacted] character of the index of this family [redacted] computed as an equivariant form over the space of connections.

Could Connes theory serve this purpose? Various interpretations are possible for the idea of a differential form index formula.

i) for a [redacted] single operator Connes defines its character as a multilinear function on $C^\infty(M)$, hence as a distribution on M^p where p is any odd, large enough relative to dim M, integer. Presumably this distribution is supported on the diagonal. Does this distribution have a simple structure one could easily describe, say by using invariant theory?

ii) Take a family of elliptic operators of simple type [redacted] parametrized by Y. Then the index of this family has a character in the Connes cohomology of Y. Can one get a specific Connes cycle? This would mean a function on [redacted] Y^p, possibly only a jet along the diagonal, for each odd p. For $p=1$ this gives the constant function with value the index. For $p=3$ it might give the curvature form for the determinant line bundle.

I now want to take a simple example of a family of elliptic operators, [redacted] I want to really understand the index of this family from different viewpoints. So let us take [redacted] a family of Dirac operators on a fixed Riem spin manifold $M_{[redacted]}$ of even dimension. Denote the family $y \mapsto D_y$ and suppose the [redacted] operator D_y is generically invertible. All these operators are working on the same Hilbert spaces, consequently what we have is a map

$$Y \longrightarrow \begin{array}{c}\text{Fredholm operators}\\ \text{of index } 0.\end{array}$$

$$y \longmapsto D_0^{-1}D_y = D_0^{-1}(D_0+B_y) = 1+D_0^{-1}B_y$$

Now we want the index of this family in various senses. Connes, I believe, likes to use the tensor product of functions on Y with k = compact (or maybe trace class) operators:

$$C^\infty(Y)\otimes k.$$

Questions: 1) Can the index of the family be defined naturally as an idempotent in this algebra? 2) Can the character of the index be defined naturally as a Connes cycle in $C^\infty(Y)$?

Now 1) is false because the index might be <0 and the trace of an idempotent is always ≥ 0. But this suggests that we consider a map

$$Y \longrightarrow \begin{array}{c}\text{surjective Fredholm}\\ \text{operators}\end{array} \qquad y \mapsto A_y$$

Then for each y we have a finite-dimensional subspace of Hilbert space, namely $\operatorname{Ker} A_y$, and hence we get a map

$$y \longmapsto e_y = \text{orth. proj. on } \operatorname{Ker} A_y$$

into finite rank projection operators. But such a thing is an idempotent in

$$C^\infty(Y) \otimes k = \{\text{smooth maps } Y \longrightarrow k\}$$

I think this makes clear what to expect in the general case. Given the map $y \longmapsto D_y$, let us choose a parametrix Q_y for D_y in a nice way as follows. Let's arrange that

$$e_y^+ = I - Q_y D_y \qquad e_y^- = I - D_y Q_y$$

are projection operators. Then we get Connes cycles

$$(*) \qquad (e_y^+)^{\otimes p} - (e_y^-)^{\otimes p}$$

for each odd p which represent the character of the index bundle. One can hope that the cycles $(*)$ have a simple extension to the case where Q_y is an arbitrary parametrix, and that the resulting Connes homology class on $C^\infty(Y) \otimes k$ is independent of the choice of Q_y.

The first non-trivial case occurs when $p = 3$. It would be nice to have an analogous ~~problem involving~~ odd K-classes which would lead to calculations for $p = 2$. Two possibilities:

a) $K_1(M) \times K^0(M \times Y) \longrightarrow K^{-1}(Y)$ where $Y \to B\mathcal{G}$, so that Y gives a family of vector bundles over M. Represent an element of $K_1(M)$ by a self-adjoint elliptic operator, twist by the vector bundle, and you get a map of Y to self-adjoint Fredholm ops. [Don't see how to do this with BDF extensions].

Let's consider then the family of self-adjoint elliptic Dirac operators on an odd-dim'l M with coeffs. in a vector bundle E, that one obtains by varying the connection on E. Then we will have a natural homotopy class of maps from the parameter space Y to S^1.

b) $K_0(M) \times K^{-1}(M \times Y) \longrightarrow K^{-1}(Y)$ where $Y \to \mathcal{G}$, so that Y gives a family of autos g_y of the bundle E. Represent an element of $K_0(M)$ by an operator over M, tensor with E using a connection in E; then the auto g_y of E gives a new connection which can be joined linearly to the old. The result is a family of operators parameterized by $Y \times S^1$. So one gets $Y \times S^1 \longrightarrow$ Fredholm operators, and this gives rise to an element of $K^{-1}(Y)$, in fact an actual map of Y to U if one knows how to lift in the S^1-direction. So in particular if we have a connection on the determinant line bundle for the family we would get a map $Y \longrightarrow S^1$.

Thus quite generally, assuming I can put a connection on the determinant line bundle of the family of Dirac operators over M (even-diml), I will get a map

$$\mathcal{G} \longrightarrow S^1,$$

depending on a connection.
Does this coincide with Witten's maps

$$\Omega^{2n} U \longrightarrow S^1 \quad ?$$

If so we have succeeded in generalizing his classes from spheres to other manifolds.

Let us summarize what we can do on the K-theory level. The key maps are

$$K_*(M) \otimes K^0(M \times B\mathcal{G}) \longrightarrow K^{-*}(B\mathcal{G})$$

$$K_*(M) \otimes K^{-1}(M \times \mathcal{G}) \longrightarrow K^{-1-*}(\mathcal{G})$$

In $K^0(M\times B\mathcal{G})$, $K^{-1}(M\times\mathcal{G})$ we have the tautological classes, and we have a map

$$K^0(M\times B\mathcal{G}) \longrightarrow K^{-1}(M\times\mathcal{G})$$

corresponding to the map $S^1\mathcal{G}\longrightarrow B\mathcal{G}$. Now

M even, take Dirac $\in K_0(M) \leadsto K^0(B\mathcal{G})$
given by Dirac operators coeffs. in E

M odd, take Dirac $\in K_1(M) \leadsto K^{-1}(B\mathcal{G})$
given by Dirac op with coefficients in E

Note that in the second case one is getting a map $B\mathcal{G}\longrightarrow$ s.a. Fred. ops. instead of $\mathcal{U}$.

By restricting relative to $S\mathcal{G}\longrightarrow B\mathcal{G}$ we get

M even $\leadsto$ class in $K^{-1}(\mathcal{G})$

M odd $\leadsto$ " " $K^0(\mathcal{G})$

The last element $\in K^0(\mathcal{G})$ for M odd can also be realized by Toeplitz operators: The Dirac op. defined a polarization, hence given $g\in\mathcal{G}$, you can take the Toeplitz operator Pg and this gives a map

$$\mathcal{G}\longrightarrow \text{Fred. ops.} \qquad M \text{ odd}$$

It would be nice to have something Toeplitz for M even, i.e. a map $B\mathcal{G}\longrightarrow\mathcal{U}$, but it is not even clear what to do for S^1.

January 2, 1983

[redacted] 1) The setup before operators + K-theory are brought in: a) We have tautological classes

$$\begin{tikzcd} \mathrm{ch}\,\tilde{E} \arrow[d, mapsto] & \in & H^{ev}(M \times B\mathcal{G}) \arrow[d] & = & \mathrm{Hom}^{ev}(H_*(M), H^*(B\mathcal{G})) \arrow[d] \\ \mathrm{ch}\,\tilde{g} & \in & H^{odd}(M \times \mathcal{G}) & = & \mathrm{Hom}^{odd}(H_*(M), H^*(\mathcal{G})) \end{tikzcd}$$

where the vertical map is induced by the canonical map $S(\mathcal{G}) \longrightarrow B\mathcal{G}$. Via cap product with homology classes in M we obtain cohomology classes in $H^*(B\mathcal{G})$ or $H^*(\mathcal{G})$ which are probably always rational generators.

b) If we lift E back to $M \times \mathcal{A}$, then it has a tautological connection so that $\mathrm{ch}\,\tilde{E}$ can be computed using equivariant differential forms on $M \times \mathcal{A}$. Similarly on choosing a connection on E, [redacted] and using a linear connection on [redacted] E pulled back to $M \times S^1 \times \mathcal{G}$, the class $\mathrm{ch}(\tilde{g})$ is represented by a differential form on $\mathcal{G}$ depending on the chosen connection. Integrating over cycles in M then gives differential form representatives for the above classes in $H^*(B\mathcal{G}) = H^*_{\mathcal{G}}(\mathcal{A})$ and $H^*(\mathcal{G})$.

~~c) [redacted] Open problem: [redacted] We have the map~~

$$H^*(\tilde{g}) \longrightarrow H^*(\mathcal{G}) \qquad \tilde{g} = \mathrm{Lie}(\mathcal{G})$$

~~so among the generating classes described in a) or b) for $H^*(\mathcal{G})$ are ones which can be realized by invariant differential forms on $\mathcal{G}$. The problem is to understand which of the classes obtained from $\mathrm{ch}(\tilde{g})$ are Lie-accessible.~~

~~Another problem concerns the formula for $H^*(\tilde{g})$ which results from the Connes theory, and especially its periodicity. Connes theory tells us which classes are Lie accessible, but I want to understand it all much better.~~

Critical example: $M = S^1$ and integrate $\mathrm{ch}_2(\tilde{g})$ over

~~M to obtain a 2-dimensional class for G (see p. 431)~~

c) Open problems: We have the map

$$H^*(\tilde{\mathfrak{g}}) \longrightarrow H^*(\mathcal{G}) \qquad \tilde{\mathfrak{g}} = \mathrm{Lie}(\mathcal{G})$$

so among the classes described above are ones which can be represented by invariant differential forms on $\mathcal{G}$. In other words, which of the forms on $\mathcal{G}$ described in b) are $\mathcal{G}$-invariant.

Alternatively the way we have described classes in $H^*(\mathcal{G})$ and $H^*(B\mathcal{G})$ sheds no light on their Lie accessibility.

However, combining the Loday conjecture with Connes theory we know which classes in $H^*(\mathcal{G})$ are Lie-accessibles. But this has to be understood much better.

First example: $M = S^1$ and cap $ch_2(\tilde{\mathfrak{g}})$ with $[M]$ to get a 2-diml class on $\mathcal{G}$. (See p. 431 and its nbd.) This is connected with c_2 + dilog problem. For the moment I choose not to get involved with diff'l form calculations, which is what I have to do if I wish to pursue things on the level of forms and not bring in operators.

I think ~~that Lie algebra~~ cohomology has a deeper connection with the operators, and this is why Connes found the Liegebra cohomology.

Isolated comment: For the $\mathcal{G}$-invariant connection on E over $M \times \mathcal{A}$, the bundle is flat over $\mathcal{G}$-orbits. This changes when the elliptic operators are brought in.

For a compact gp. one can average over the group to get an invariant connection, then the character is an equivariant form, i.e. in $[\Omega(M) \otimes W(\mathfrak{g})]^G$. Doesn't work for $\mathcal{G}$, so you must lift to $M \times \mathcal{A}$.

2) Next bring in elliptic operators and K-theory.

Let's call 1) above the cohomological picture. Next we want the K-picture: a) [redacted] We have a cap product map

$$K_*(M) \otimes K^*(M \times B\mathcal{G}) \longrightarrow K^*(B\mathcal{G})$$

and use this to generate K^*-classes on $B\mathcal{G}$ from the tautological bundle $\tilde{E}$ over $M \times B\mathcal{G}$ and from K-homology classes on M. K-homology classes on M are [redacted] represented by operators, say elliptic operators on compact manifolds Z mapping to M. The [redacted] above cap product takes an operator on [redacted] $Z \to M$ twists it with the tautological bundle to a get a family of operators on Z parameterized by $B\mathcal{G}$ and then takes the index of this family to get an element of $K^*(B\mathcal{G})$.

[redacted] In the following I will assume that I can represent an element of $K_1(M)$ by a self-adjoint elliptic operator over an odd-diml manifold Z mapping to M. I know I can always represent an elt of $K_0(M)$ by an elliptic operator over an even-diml man. Z mapping to M.

I will also want to restrict to the fundamental cycle of M, i.e. the Dirac operator on M (here $Z=M$).

b) (analogous to 1) b) above). By tensoring the Dirac operator on M with the vector bundle E one gets a family of elliptic operators over M param. by $\mathcal{A}$. This family is equivariant for $\mathcal{G}$. A key problem [redacted] is to make sense of the index of this family as an equivariant virtual bundle on $\mathcal{A}$ with an equivariant connection. In particular I want the character to be an equivariant differential form on $\mathcal{A}$. [redacted]

Repeat: Let us [redacted] consider the Dirac operator on spinors S over M. Suppose given a vector bundle E with a connection (and inner product of course) then we get a Dirac operator on $S \otimes E$. As the connection varies we get a family of elliptic operators on M parameterized by the space $\mathcal{A}$ which is $\mathcal{G}$-equivariant. Notice that all these operators work on the same Hilbert spaces, so that we have a map from $\mathcal{A}$ into Fredholm operators $A \mapsto \not{D}_A : H^+ \longrightarrow H^-$ for $\dim M$ even, and self-adjoint Fredholm operators for $\dim M$ odd. Notice that the gauge group is acting as unitary operators on the Hilbert spaces $H^{\pm}$.

In some sense this map $A \mapsto \not{D}_A$ is [redacted] the index of our family. (It is certainly the same species of gadget as a perfect complex in the sense of Grothendieck.)

The problem is now to [redacted] define the character of this index [redacted] as an equivariant differential form over $\mathcal{A}$. Notice that this is not obvious: The determinant line bundle, or [redacted] determinant of the index is a nice equivariant line bundle over the space of connections. However to put a connection on this line bundle is a non-trivial process involving regularization.

Quite possibly better results are to be obtained with Connes' cycles + cocycles.

January 3, 1982

Atiyah: real part of the dilogarithm is associated to the volume in hyperbolic space and the imaginary part to the first Pontryagin class.

Both of these are related to $SL_2(\mathbb{C})$.

Here I want just to understand characteristic classes for flat bundles with values in $H^{odd}(M, \mathbb{C}^*)$.

Given a flat vector bundle E over M, the associated principal bundle P has a flat connection over M. Hence any assoc. fibre bundle $P \times^G V$ has a flat connection. Given an invariant diffl. form on V it gives rise to a differential form on $P \times^G V$ and the resulting map

$$\Omega^{\cdot}(V)^G \longrightarrow \Omega^{\cdot}(P \times^G V) \qquad\qquad G = GL_n(\mathbb{C})$$

commutes with d.

First case is to take $V =$ symmetric space $GL_n(\mathbb{C})/U_n$. Then the invariant forms on the symmetric space (which one knows are all closed, in fact an exterior algebra with odd-diml generators) give DR cohomology classes in $P \times^G V$. Since V is contractible

$$H^*(M) = H^*(P \times^G V)$$

and so we get odd classes $e \in H^{2i-1}(M, \mathbb{R})$. We get forms representing these classes by chosing a section of $P \times^G V$ over M and pulling back. If there were a hermitian metric in E invariant under the flat connection, then we have an everywhere flat section of $P \times^G V$ and so the e-classes vanish.

Thus the classes $e \in H^{2i-1}(M, \mathbb{R})$ associated to flat bundles measure the non-unitary nature of the representation of the fundamental group. They should give the part of the classes in $H^{odd}(M, \mathbb{C}^*)$ corresponding to the map $\mathbb{C}^* \xrightarrow{\log|\,|} \mathbb{R}$.

At this point I could take $G = SL_2\mathbb{C}$ whence V = hyperbolic space and try to calculate an explicit cocycle in $H^3(G_{disc}, \mathbb{R})$ corresponding to the volume form on V. Then I should probably need to use the boundary $P_1(\mathbb{C})$ of hyperbolic space, and get involved in the Bloch-Dupont calculations.

Let's instead worry about characteristic classes of vector bundles with flat unitary connections. Then I should be able to define classes in $H^{odd}(M, S^1)$. The intuitive reason ~~[illegible] approach~~ for these classes is as follows: Take a characteristic class on vector bundles, i.e. a polynomial ρ in the Chern classes, e.g. c_n. This is an integral cohomology class ~~[illegible]~~ which can be calculated in real DR cohomology using a unitary connection. If the connection is flat, then these ~~[illegible]~~ DR classes vanish ~~[illegible]~~ so because of the sequence

$$H^{2n-1}(M, S^1) \xrightarrow{\delta} H^{2n}(M, \mathbb{Z}) \longrightarrow H^{2n}(M, \mathbb{R})$$

we know $\rho(E)$ can be lifted to $H^{2n-1}(M, S^1)$. Because we have given a reason why $\rho(E)$ vanishes in real cohomology, it should be possible to define canonically a lifting of ρ by analyzing this reason.

There are three approaches to this definition I know:

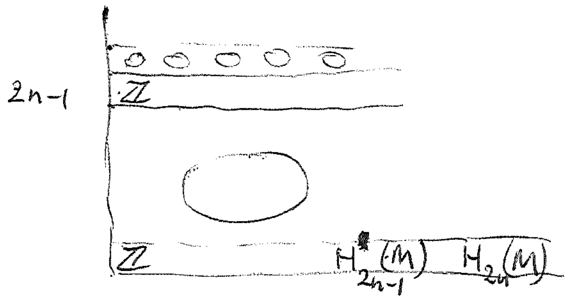
a) Deligne cohomology. I won't go thru this, but want to point out that it does furnish the dilogarithm.

b) Chern-Simons.

c) A direct method which works for the c_n using the fact they are obstructions.

Let's go over c) first to see if it works. ~~[illegible]~~ ~~[illegible]~~ We have a flat $U_N = U$ bundle and consider the fibre bundle $P \times^U (U/U_{n-1})$

where U/U_{n-1} is the Stiefel manifold. The cohomology of U/U_{n-1} is represented by U-invariant differential forms, which we can use to define differential forms on $P \times^{U} U/U_{n-1}$. In particular take the generator for $H^{2n-1}(U/U_{n-1})$ and represent it by a closed U-invariant form; then we get a [redacted] closed $(2n-1)$-form on $P \times^{U} U/U_{n-1}$. Now look at the spectral sequence for this fibre space in integral homology



assume $H_0(M) = \mathbb{Z}$

and we get

$$\to H_{2n}(M) \longrightarrow \underset{\overset{\parallel}{\mathbb{Z}}}{H_{2n-1}(U/U_{n-1})} \longrightarrow H_{2n-1}(P \times^{U} U/U_{n-1}) \longrightarrow H_{2n-1}(M) \to 0$$

Now $\quad H^{2n-1}(M, S^1) = \mathrm{Hom}(H_{2n-1}(M), S^1) \qquad S^1$ injective$/\mathbb{Z}$

and given an elt α of $H_{2n-1}(M)$ we can lift it to $P \times^{U} U/U_{n-1}$ and integrate our closed $(2n-1)$-form over it. Two lifting of α differ by an integral homology class from the fibre which integrates to give an integer, hence we get a well-defined elt. in $\mathbb{R}/\mathbb{Z}$ associated to α.

Disadvantages of this approach are [redacted] a) there might be several classes depending on the choice of invariant form on U/U_{n-1}, b) not clear how to get odd S^1 classes when the bundles are flat along the leaves of a foliation.

Chern-Simons: The key idea here is to use the acyclicity of the universal bundle and ideas of transgression.

Start with a principal G-bundle P with connection over M and an invariant polynomial ρ of degree 2n on $\mathfrak{g}$. Then we have a form $T\rho(\theta)$ on P of degree 2n-1 such that $d(T\rho(\theta)) = \pi^*\rho(\Omega)$ where Θ and Ω are the connection and curvature forms.

(In some sense $T\rho$ exists because P becomes trivial if pulled back over itself. Then choose it once and for all in the universal case.)

Now suppose ρ represents an integral class on BG. We can regard a form as a smooth singular cochain (linear fn. on smooth chains). Then over BG we have that ρ mod $\mathbb{Z}$, i.e. ρ as a smooth singular cochain with values in $\mathbb{R}/\mathbb{Z}$, is a coboundary

$$\boxed{\rho = du \qquad u \in C^{2n-1}(BG, \mathbb{R}/\mathbb{Z}).}$$

Hence $\quad d(T\rho - \pi^* u) = 0 \quad$ in $C^{2n}(PG, \mathbb{R}/\mathbb{Z})$

and since PG is acyclic one has $\boxed{v \in C^{2n-2}(PG, \mathbb{R}/\mathbb{Z}) \ni}$

$$\boxed{T\rho = \pi^*(u) + dv \quad \text{in } C^{2n-1}(PG, \mathbb{R}/\mathbb{Z}).}$$

Now that we have all these universal cochains we take a specific (M, P, Θ) and suppose that $\rho(\Omega) = 0$. Then when we pull u back to M we get a cocycle: $u(M, P, \Theta) \in C^{2n-1}(M, \mathbb{R}/\mathbb{Z})$, $du = 0$.

Actually we didn't use the 2nd box which shows that the u-class in $H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$ lifts to the $T\rho$ class mod $\mathbb{Z}$.

I find this process very mystifying, especially the use of a universal bundle with connection.

Repeat: On the fibred category of principal G-bundles with connection we have a 2n-cocycle $\rho(\Omega)$ with real coefficients. This cocycle represents an integral class, hence we can choose a 2n-1 cocycle u with coeff $\mathbb{R}/\mathbb{Z}$ over a universal bundle such that

$$\rho \bmod \mathbb{Z} = du, \quad u \in C^{2n-1}(BG, \mathbb{R}/\mathbb{Z}).$$

Then if we have an example (M, P, θ) where $\rho = 0$, u gives us an element of $H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$.

Now let us return to yesterday's ideas about maps of the gauge gp. $\mathcal{G}$ to S^1 over even diml. manifolds. Restrict attention to a trivial vector bundle over M. Then $\mathcal{G} = \mathrm{Map}(M, U)$, $U = U_k$, and so for each $g \in \mathcal{G}$ we have a map $M^{2n} \longrightarrow U$.

Idea from Witten: If $M = S^{2n}$, then the map extends to the disk of dim 2n+1. Integrate the e-form of dim 2n+1 over the disk to get a real number, which is well-defined mod $\mathbb{Z}$. Then get a map $\mathcal{G} \to \mathbb{R}/\mathbb{Z}$. Of ~~this has nothing to do with~~ course any closed (2n+1)-form on U with integral values on $H_{2n+1}(U)$ will also give a map.

I'd like to get away from $M = S^{2n}$. I can proceed by using the Stieffel manifold U/U_n and take a closed, integral-period, 2n+1 form on U/U_n and pull-back to U. Then the cycle of U/U_n represented by M will be a boundary, and we can integrate to obtain a number, well-defined mod $\mathbb{Z}$. Here I use that U/U_n has 0 H_{2n}. I don't think the bi-invariant e form on U comes from U/U_n.

Next try elliptic operators: Over M take the Dirac operator with coefficients in the trivial bundle of dim k. Given $g \in \mathcal{G}$ we can transform it to the Dirac operator belonging to a new connection. Then use the linear path ~~of~~ connections from the old to the new connection and look at the determinant line bundle over this path. We can parallel translate along the path using the connection in the determinant line bundle and so get an elt. of S^1.

Note: Any map $\mathcal{G} \xrightarrow{f} S^1$ gives a closed 1-form by pulling back $\frac{d\theta}{2\pi}$ over S^1; the 1-form is $\frac{1}{2\pi i}\frac{df}{f}$. This 1-form determines f up to constant factors on the different components of $\mathcal{G}$. So any of the cohomology schemes just described yield closed 1-forms on $\mathcal{G}$, probably by pulling back a closed $2n+1$ form on $\mathcal{U}$ via the evaluation map

$$M \times \mathcal{G} \longrightarrow \mathcal{U}$$

and then integrating over M.

January 6, 1983

Central problem of interest is that of defining equivariant DR cohomology for the gauge group. Fix a connected manifold with basepoint M, and let E be a unitary vector bundle over it, and let $\mathcal{G}$ be the autom^s of E which are the identity over the basepoint. Then $\mathcal{G}$ acts freely on the space $\mathcal{A}$ of connections. The problem will be to define equivariant de Rham coh. classes for the character of $\tilde{E}$ over $M \times \mathcal{A}$.

But this is too specific. A better idea is to consider $\mathcal{G}$ acting on something like $M \times \mathcal{A}$, and an $\mathcal{G}$-bundle over it, and the problem is to define the character as an equivariant de Rham coh. class.

Example: Take the determinant line bundle $\mathcal{L}$ over $\mathcal{A}$ that would be obtained from a $\bar{\partial}$-operator on a Riemann surface by tensoring with E. This line bdl is equivariant and has an invariant connection that can be described as follows: a) In general $\mathcal{L}$ is holom. for the complex structure on $\mathcal{A}$, so the zeta fn. metric gives a connection. b) In the index 0 case $\mathcal{L}$ has a canonical $\mathcal{G}$-invariant section s, so the connection is given over the open set $s \neq 0$ by a ~~connection~~ 1-form θ. One has

$$i(\delta D)\theta = \operatorname{Tr}\left((D^*D)^{-s} D^{-1}\delta D\right)\Big|_{s=0}$$

$$= \int tr\left(J_D \cdot \delta D\right)$$

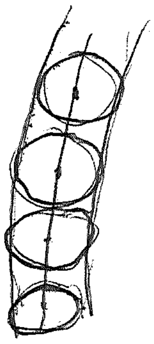
where J_D is the finite part of D^{-1} along the diagonal constructed from the connection corresponding to D and the metric on M.

The idea is that because $\mathcal{G}$ acts freely on $\mathcal{A}$, the bundle $\mathcal{L}$ descends to a line bundle $\bar{\mathcal{L}}$ over $\mathcal{A}/\mathcal{G}$. Then I can choose a connection on $\bar{\mathcal{L}}$ and this will yield a new connection on $\mathcal{L}$ with the property that $\mathcal{G}$ orbits are horizontal. Thus the moment map is zero, and there is no anomaly for the new connection.

Because the canonical section s descends, then the new connection will be given over $s \neq 0$ by a 1-form η killed by $i(X)$ and $\Theta(X)$ for each $X \in \mathrm{Lie}(\mathcal{G}) =$ [redacted] $\tilde{\mathfrak{g}} =$ {skew-adjoint ops. on E}. In general one can say that the new connection will differ from the old by a $\mathcal{G}$-inv. 1-form on $\mathcal{A}$. This will be a purely imaginary 1-form, so it has [redacted] $(1,0)$ and $(0,1)$ components, unlike Θ which has only a $(1,0)$-component.

[redacted] Now there are no obvious invariant 1-forms on $\mathcal{A}$. Analogy: Take $\mathbb{C}^n$ with its Kähler structure and the affine transf. gp. $z \mapsto \alpha z + \beta$ $\alpha \in U_n$. Then there are no invariant 1-forms.

However, one can construct invariant functions by using a tubular nbd. of an orbit.



So we probably want to look at this sort of picture, as it uses the freeness of the $\mathcal{G}$-action.

Notation: We have this invariant connection on $\mathcal{L}$ denoted ∇_Y, Y any tangent vector [redacted] to $\mathcal{A}$. [redacted] We have a moment map $X \mapsto \phi_X$ for $X \in \tilde{\mathfrak{g}}$ such that

$$\Theta(X) = \nabla_{\bar{X}} + \phi_X \qquad \text{on } \Gamma(\mathcal{A}, \mathcal{L})$$

where $\bar{X}$ denotes the vector field on $\mathcal{A}$ produced by X.

Suppose that ϕ is zero; this means gauge orbits in $\mathcal{L}$ are horizontal. Then consider an invariant section s of $\mathcal{L}$, for instance the [redacted] canonical section. We then get a 1-form on $\mathcal{a}$ by pulling back the connection form by the section: hence a 1-form η defined by

$$[redacted] \quad \nabla_Y s = (i(Y)\eta)\, s$$

Now because ϕ is zero [redacted] and s is inv., one has

$$(i(\bar{X})\eta)\, s = \nabla_{\bar{X}} s = \theta(X) s = 0 .$$

Furthermore η is invariant: $\theta(\bar{X})\eta = 0$. So we conclude that η is basic and so descends to the quotient $\mathcal{G}\backslash\mathcal{a}$!

<u>Prop</u>. Let $\mathcal{E}$ be an equivariant vector bundle over $\mathcal{a}$ for $\mathcal{G}$, equipped with an invariant connection. This connection comes from a connection on the induced bundle $\bar{\mathcal{E}}$ over $\mathcal{G}\backslash\mathcal{a}$ iff its Higgs field is zero.

<u>Proof</u>: I am assuming the action is free and [redacted] one has usual slice pictures for tubular nbds of $\mathcal{G}$-orbits. Thus we can find local frames for $\bar{\mathcal{E}}$ of $\mathcal{G}\backslash\mathcal{a}$, and so invariant frames s near any orbit of $\mathcal{G}$ on $\mathcal{a}$. Let the connection be denoted ∇_Y, Y tangent vector fld to $\mathcal{a}$. Then we get a matrix η of 1-forms, by pulling the connection form back using the frame s:

~~$\nabla_Y s = (i(Y)\eta)\, s$~~

~~Because the connection ∇ is $\mathcal{G}$-invariant, and s is $\mathcal{G}$-invariant, the matrix 1-form η is also $\mathcal{G}$-invariant~~

~~$\theta(X)[\nabla_Y s] = \nabla_{[\bar{X},Y]} s = i([\bar{X},Y])\eta\, s$ invariance of ∇, s~~

~~$\theta(X)[(i(Y)\eta)s] = \theta(X)\, i(Y)\eta$~~

A better thing to say is that the invariant frame we have is a trivialization of the vector bdl near our orbit which is equivariant. Thus I can think of sections are vector-valued fns. for A and

$$\theta(X) f = \bar{X} f.$$

The connection is given by a matrix η of 1-forms:

(*) $$\nabla_Y f = (\bar{Y} + \eta(Y)) f.$$

Now $$\theta(X)\left[\nabla_Y f\right] = \nabla_{[\bar{X}, Y]} f + \nabla_Y (\bar{X} f)$$

expresses the invariance of ∇_Y. Now take f to be constant near where we are working with value the vector v

$$\theta(X)\left[\eta(Y) v\right] = \eta([\bar{X}, Y]) v$$

$$\overset{\shortparallel}{}$$

$$\left[\bar{X}\, \eta(Y)\right] v$$

Thus $\bar{X}(i(Y)\, \eta) = i([\bar{X}, Y]\, \eta)$, and in view of the formula $\theta(\bar{X})\, i(Y) - i(Y)\, \theta(\bar{X}) = i([\bar{X}, Y])$

we conclude that $i(Y)\, \theta(\bar{X})\, \eta = 0$ for all Y or that $\theta(\bar{X}) \eta = 0$.

Thus I have checked that the form η is invariant under $\tilde{g}$. But now if I apply (*) above to $Y = \bar{X}$, then I get

$$\nabla_{\bar{X}} = \theta(\bar{X}) + \eta(\bar{X})$$

which shows that η restricted to the field $\bar{X}$ is minus the Higgs field ϕ_X. So $\phi_X = 0 \iff i(\bar{X}) \eta = 0$. But η is basic $\iff \theta(\bar{X}) \eta = i(\bar{X}) \eta = 0$ for all X by defn., so the prop. follows.

Summary: [redacted] I am looking at the determinant line bundle $\mathcal{L}$ in the index-zero Riemann surface case. This has an invariant connection whose moment map is given by the anomaly formula

$$\phi_X(A) = \int tr(X [redacted] F_A) \qquad F_A = [D_A, J_A]$$

where I work with the Dirac operator so as to forget the $\frac{1}{2}(2g-2)$ term.

In this case the tangent space to $\mathcal{A}$ at any point can be identified with $\Omega^1(End_{sk}(E)\otimes T^*) = \Gamma(End E \otimes T^{0,1})$ and the gauge alg. is $\Omega^0(End_{sk}(E)) = \Gamma(End_{sk}(E))$. Put g = the Lie alg. bundle $End_{sk}(E)$, so that $\tilde{g} = Lie\,\mathcal{G} = \Gamma(g) = \Omega^0(g)$. At a point A of $\mathcal{A}$ the map from $\tilde{g}$ to the tangent space at A is

$$\Omega^0(g) \longrightarrow \Omega^1(g) \quad , \qquad X \longmapsto [X, D_A]$$

[redacted] The moment map is $X \longmapsto \phi_X(A) = \int tr(X F_A)$, so the problem [redacted] is to find the dotted arrow

$$\begin{tikzcd}
\Omega^0(g) \arrow[r, "{[\ ,D_A]}"] \arrow[d, "\phi_?(A)"'] & \Omega^1(g) \arrow[dl, dashed, "\tau_A"] \\
\mathbb{R} &
\end{tikzcd}$$

τ invariant under $\mathcal{G}$

Why? Because then τ will be an invariant 1-form on $\mathcal{A}$ which we can use to modify the connection so that the moment map becomes 0, and hence the connection descends to $\mathcal{G}\backslash\mathcal{A}$.

Over the open set where $s \neq 0$ we can take

$$\tau_A([redacted]\, \delta D) = \int tr(J_A\, \delta D)$$

This corresponds to choosing the connection for which the canonical section is flat. In general what we want to do is to find a $\mathcal{G}$-invariant for

$$A \longmapsto L_A \in \Gamma(\mathrm{End}(E) \otimes T^{1,0}_M)$$

such that

$$[D_A, L_A] = F_A$$

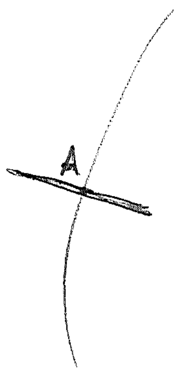
Example: Take the tautological connection on the bundle E pulled back to $M \times \mathcal{A}$. Then we know the Higgs map at $(m, A) \in M \times \mathcal{A}$ assigns to $X \in \Omega^0(\mathfrak{g})$ $= \tilde{\mathfrak{g}}$ the corresponding endo. of the fibre $E|_m$. This time $\tilde{\mathfrak{g}}$ acts trivially on M, so that again we are concerned with the way $\tilde{\mathfrak{g}}$ maps to the tangent space at A to $\mathcal{A}$ which is $\Omega^1(\mathfrak{g})$. So we have at A

$$\begin{tikzcd}
\Omega^0(\mathfrak{g}) \arrow[r, "{[D_A, \ ]}"] \arrow[d] & \Omega^1(\mathfrak{g}) \arrow[dl, dashed, "\tau_{m,A}"] \\
\mathfrak{g}|_m &
\end{tikzcd}$$

where $\tau_{m,A}$ is to be $\mathcal{G}$-invariant. If assume τ doesn't depend on $m \in M$, then we are back to finding an invariant complement for the map

$$\Omega^0(\mathfrak{g}) \xrightarrow{[D_A, \ ]} \Omega^1(\mathfrak{g})$$

assumed to be injective. Geometrically we are looking for a complement to the tangent space to the gauge orbit, i.e. normal spaces to the gauge orbit.



January 7, 1983

The moral from yesterday is that it doesn't seem to help, for the problem of defining $ch(\tilde{E}) \in H^{ev}(M \times B\mathcal{G})$ in DR cohomology, to pull back to $\mathcal{A}$. Although there is an invariant connection there is still the problem of killing the Higgs field.

So perhaps we should use a different model for $B\mathcal{G}$ in order to get a good solution.

Let's go over the fundamental mystery, namely [redacted] why only certain of the generators for $H^*(\mathcal{G})$ are Lie accessible. Firstly we have the diagram

$$\begin{tikzcd}
ch(\tilde{E}) \arrow[d, mapsto] & \in & H^{ev}(M \times B\mathcal{G}) \arrow[d] & = & Hom^{ev}(H_*(M), H^*(B\mathcal{G})) \arrow[d] \\
ch(\tilde{g}) & \in & H^{odd}(M \times \mathcal{G}) & = & Hom^{odd}(H_*(M), H^*(\mathcal{G}))
\end{tikzcd}$$

[redacted] To each $\gamma \in H_*(M)$ belongs then classes

$$\begin{tikzcd}
\gamma \cap ch(\tilde{E}) \arrow[d, mapsto] & \in & H^*(B\mathcal{G}) \arrow[d, "\text{homology susp.}"] \\
\gamma \cap ch(\tilde{g}) & \in & H^*(\mathcal{G}).
\end{tikzcd}$$

At least in the case, $\mathcal{G} = Map(M, U_k)$ rational homotopy theory tells me that the suspension map induces an isomorphism

$$Q\{H^*(B\mathcal{G})\} \xrightarrow{\sim} P\{H^{*-1}(\mathcal{G})\} \qquad * \geq 2.$$

and [redacted] that the classes described above give generators for the cohomology.

What am I after? I need to be able to represent cohomology classes on $B\mathcal{G}$ by something like differential forms. I know how to do this for $\mathcal{G}$. [redacted]

January 8, 1983

Idea: I have a theory of how to regularize the Green's function for a $\bar\partial$ operator on a Riemann surface. The process is very algebraic especially as I did it in the case of a surface with boundary. The idea is to apply this to the case [redacted] where the complex structure on the surface varies and see if you can get out the modular functions. The point is to bypass all calculations involving heat kernels because they are too hard for you to carry out.

Another idea is to work in $K^{1/2}$ where K is the canonical bundle. Somehow I have the feeling that what I need to regularize D^{-1} for $D: E \rightarrow E \otimes T^{0,1}_M$, [redacted] (recall this consists of a connection on E extending the $\bar\partial$ operator D and a metric on M) is roughly the same as a connection on $E \otimes K^{-1/2}$.

Let's go over the regularization of D^{-1}. [redacted]

Idea: D^{-1} blows up along the diagonal. We must [redacted] subtract a parametrix and restrict the difference to to diagonal to get J.

We need to describe the [redacted] singularity of D^{-1} along the diagonal. [redacted] In general the asymptotic behavior of a parametrix for D near the diagonal is determined formally by D. Let's fix a point on the

January 9, 1983

A basic problem seems to be to make sense of the characteristic classes of index virtual bundles. In algebraic geometry it occurs as how to define characteristic classes for perfect complexes. In analysis the problem would be to define the characteristic classes in DR cohomology of the tautological K-element over the space of Fredholm operators.

Here's how I first encountered the nature of the problem: If one has a Dirac operator over M, one can tensor with a vector bundle (unitary) with unitary connection E, and thus obtain a family of elliptic ops. on M parametrized by the space $\mathcal{A}$ of connections, equivariant for the gauge gp. The index is a specific map from $\mathcal{A}$ to Fredholm operators which is $\mathcal{G}$-equivariant. One wants to define the character of the index as an equivariant differential form on $\mathcal{A}$.

The key example for me at the moment of the above occurs when M is a Riemann surface. But the case of a surface with boundary might be simpler. Here the analogue of the index is a map to the Grassmannian of half-spaces at the boundary. We know this Grassmannian is a loop group, so we have a map from connections to the loop group. This leads to:

Problem: Over ΩU_n one has a canonical K-elt. given by periodicity. Describe its character via differential forms on ΩU_n.

Notice that the Grassmannians, being symmetric spaces, have unique invariant differential forms representing each cohomology class. Maybe over ΩU_n one has unique *harmonic* forms which can be described.

Invariant differential forms on the Grassmannian. The Grassmannian is a symmetric space so one knows each cohomology class contains a unique differential form invariant under the unitary group. Actually we will see the only invariant forms are of type (p,p), hence the differentials in the invariant form complex are zero.

At the point in the Grassmannian belonging to the subspace W, the tangent space is canonically $\mathrm{Hom}(W,V)$ where $V = W^\perp$. This is a complex vector space; for the purpose of diff'l forms one wants real multilinear fns. on the tangent space, hence complex multilinear forms on the complexification. As a repn of $U(W) \times U(V)$ the space $\mathrm{Hom}(W,V)$ has the invariant inner product $(T_1 | T_2) = \mathrm{Tr}(T_1^* T_2)$, and hence $\overline{\mathrm{Hom}(W,V)}$ can be identified with $\mathrm{Hom}(W,V)^* = \mathrm{Hom}(V,W)$. So the complexification of $\mathrm{Hom}(W,V)$ ~~can be repre~~ is isomorphic to

$$\mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) = W^* \otimes V \oplus V^* \otimes W.$$

A better thing to say is that

$$\begin{aligned} \mathrm{Hom}(W,V) &\longrightarrow \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W) \\ T &\longmapsto T + T^* \end{aligned}$$

is a complexification map.

Now we want invariant forms. These will be linear fnls on

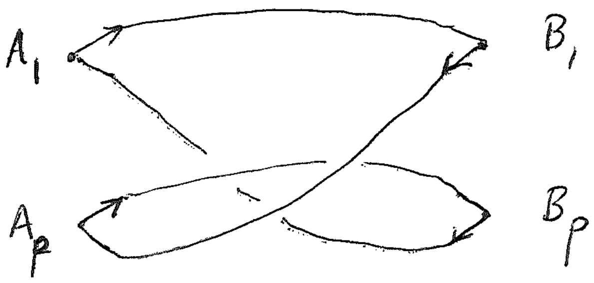
$$\Lambda(W^* \otimes V + V^* \otimes W) = \Lambda(W^* \otimes V) \otimes \Lambda(V^* \otimes W)$$

which are invariant under $U(W) \times U(V)$. Alternatively these are linear fnls on

$$(W^* \otimes V)^{\otimes p} \otimes (V^* \otimes W)^{\otimes q}$$

for various p, q which are invariant and skew-symmetric under Σ_p and Σ_q. Thm of inv. says there are no invariants unless $p=q$, and that the invariants fnls. on the above tensor product are generated by the linear fnls. obtained by contracting the W^*, W factors and the V, V^* factors according to permutations σ, τ.

Suppose given $A_1, \cdot\cdot, A_p \in Hom(W,V)$, $B_1 \cdots B_p \in Hom(V,W)$ and permutations $\sigma, \tau \in \Sigma_p$. Form the diagram



and join the out arrow at A_i to the in arrow at $B_{\sigma(i)}$, while the out arrow at B_j goes to the in arrow at $A_{\tau j}$. Then you get a bunch of loops and for each loop the contraction is given by the trace of the product along the loop. Take prod. over diff loops. Hence for the above loop with $p=2$ we get

$$Tr \left(B_2 A_2 B_1 A_1\right).$$

Thus for each pair of permutations we get an invariant multilinear function.

But now to obtain forms we must anti-symmetrize wrt Σ_p permuting the A_i and Σ_p permuting the B_i. Then the action of σ_1, σ_2 on the pair (σ, τ) is

$$(\sigma_1, \sigma_2) * (\sigma, \tau) = \left(\sigma_2 \sigma \sigma_1^{-1}, \sigma_1 \tau \sigma_2^{-1}\right)$$

so when we anti-symmetrize we will get a form for each orbit for this action. The conjugacy class of $\sigma\tau$ is a complete invariant for the orbit.

It's clear from the anti-symmetrization form of exterior multiplication that we get indecomposable forms by restricting to the case of 1-loop above. This is the same as requiring $\sigma\tau$ to be transitive, i.e. a p-cycle. So we get one indecomposable $2p$ form on the Grassmannian for each p, namely the anti-symmetrization of

$$A_i\,,\,B_i \longmapsto Tr\left(B_pA_p\cdots B_2A_2B_1A_1\right).$$

Next I want to write this as a differential form on the Grassmannian, ~~and~~ make the [redacted] connection with Connes' theory. The point will be to think of a point in the Grassmannian as giving an idempotent matrix e. In general the complexification of the Grassmannian $U_{p+q}/U_p\times U_q$ is $Gl_{p+q}/Gl_p\times Gl_q$ = idempotent matrices of deg $p+q$, rank p. The tangent space to an idempotent e is [redacted] easily found:

$$e^2=e$$

$$\delta e\cdot e+e\cdot\delta e=\delta e \quad \text{or} \quad \begin{array}{l}\delta e\cdot e=(1-e)\,\text{[redacted]}\,\delta e\\ e\cdot\delta e=\delta e\cdot(1-e)\end{array}$$

so that $\delta e=\left(\begin{array}{c|c}0&A\\\hline B&0\end{array}\right)$. [redacted]

[redacted]

Let's consider Connes' differential form on the space of idempotent matrices

$$Tr\left(e\,(de)^{2p}\right)$$

and try to connect it up with the above. This differential form assigns to $2p$ tangent vectors $\delta_1 e$ [redacted] $\cdots\delta_{2p}e$ at e the number

$$\frac{1}{(2p)!}\sum_{\sigma\in\Sigma_{2p}}(-1)^\sigma Tr\left(e\ \delta_{\sigma 1}e\cdots\cdots\delta_{\sigma(2p)}e\right)$$

This form will be determined by its values when $\delta_i e=$ [redacted] $\begin{pmatrix}0&A_i\\0&0\end{pmatrix}$, $1\le i\le p$ and $\delta_{i+p}e=\begin{pmatrix}0&0\\B_i&0\end{pmatrix}$. Then the only way the trace can be $\neq 0$ is for

$$Tr\left(\underset{\begin{pmatrix}1&0\\0&0\end{pmatrix}}{e}\ \underset{A}{\delta_{\sigma 1}}\underset{B}{e}\cdots\cdots\underset{B}{\delta_{\sigma(2p)}}e\right)$$

which restricts σ so the sum is effectively over all permutations of the A and B's respectively. So it is now clear that up to a constant like $\frac{(2p)!}{(p!)^2} = \frac{2^p\,(2p-1)!!}{p!}$ the invariant form obtained by invariant theory agrees with Connes.

Grassmannian connection: Suppose we have a v.b. E which is expressed as a direct summand of a trivial bundle, say $E \underset{i}{\overset{\pi}{\rightleftarrows}} F = M \times \mathbb{C}^{r}$. Then the flat connection on F induces a connection on E, namely $\nabla_E = \pi \nabla_F i$. Express this in terms of the idempotent $e = i\pi$, writing $\nabla_F = d$. Then

$$\nabla_E s = \pi d i s$$

so the curvature is

$$\nabla_E^2 = \pi d i \pi d i = \pi\, d e d\, i.$$

Now

$$de = (de) + ed$$
$$ded = (de)d \qquad \text{as } d^2 = 0.$$

But you haven't realized where the curvature is zeroth order. So put in extra factors of e:

$$\begin{aligned}\nabla_E^2 &= \pi\, e\, d e d e\, i \\ &= \pi\, e\, (de)\, d e\, i \\ &= \pi\, (de)(1-e)\, d e\, i \\ &= \pi\, (de)(1-e)\left[(de) + e d\right] i \\ &= \pi\, (de)(1-e)\,(de)\, i \\ &= \pi\, e\,(de)(de)\, i\end{aligned}$$

where I have used that

$$de = d(e^2) = e(de) + (de)e \implies (de)(1-e) = e(de)$$

Similarly

$$\mathrm{Tr}\left(\nabla_E^{2p}\right) = \mathrm{Tr}\left(\pi\, e(de)^2 \underbrace{i\cdot\pi}_{e}\, e(de)^2 i \cdots\right)$$

$$= \mathrm{Tr}\left(e\,(de)^{2p}\right)$$

since $e(de)^2 = (de)^2 e$. Therefore we see that the differential forms [redacted] $\frac{1}{p!}\mathrm{Tr}\left(e(de)^{2p}\right)$ are the character forms for the Grassmannian connection.

~~January 10, 1983~~

~~Review: Now we have the character differential forms on the finite-dimensional Grassmannians and can extend them formally to infinite dimensions.~~ [redacted] [redacted] ~~But now we can embed the loop group $G = \mathrm{Map}(S^1, U_k)$ into the Grassmannian of half-spaces.~~

~~Formally the free loop group~~

January 10, 1983

Now that we have the character forms on the Grassmannian we should check 2 kinds of behavior: a) suspension map $S(U_n) \longrightarrow BU_n$ b) periodicity map $\Omega U \longrightarrow BU$.

Let's take up periodicity. Here we want to use the explicit map from loops to half-spaces. The free loop group $\mathcal{G} = Map(S^1, U_k)$ acts as unitary transformations on $L^2(S^1; \mathbb{C}^k) = H$. We can then take ~~take~~ the orbit of $H^2(S^1; \mathbb{C}^k) = H^+ \subset H$ and so get ~~an~~ a $\mathcal{G}$-equivariant map

$$\Omega U_k = \mathcal{G}/U_k \hookrightarrow \begin{array}{c}\text{Grassmannian} \\ \text{of "half-spaces" in } L^2(S^1)^k.\end{array}$$

Formally at least, the character forms make sense on the Grassmannian, and so we can pull them back to the $\mathcal{G}/U_k$ to get $\mathcal{G}$-invariant differential forms.

Now this is nice because I have already noted that $\mathcal{G}/U_k$ is the analogue of a flag manifold, and so one has a method to compute the invariant forms on it. The problem was that the differential is non-trivial and one has to do some eigenvalues of Δ, or Hodge theory, to get the cohomology. Also I have Connes cocycle complex for the primitive invariant differential forms on $\mathcal{G}$ which involves the ring $\mathbb{C}[z, z^{-1}] = A$. Both of these complexes are quite inefficient.

I am trying to work with the character of the subbundle over the Grassmannian. The curvature form is $e(de)^2$ and this should associate to two tangent vectors at W an endomorphism of W. Also there should be a Higgs map relative to the action of Unitary gp.

Let $X_i = \left(\begin{array}{c|c} & B_i \\ \hline A_i & \end{array}\right)$ be tangent vectors at the idempotent e giving the splitting $V = W \oplus W'$ so that $A_i : W \longrightarrow W'$. Then

$$i(X_2)\,i(X_1)\quad e\,de\,de = e\left[i(X_1)de \;\; i(X_2)de \; - i(X_2)de\, i(X_1)de\right]$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left[\begin{pmatrix} & B_1 \\ A_1 & \end{pmatrix}\begin{pmatrix} & B_2 \\ A_2 & \end{pmatrix} - \begin{pmatrix} & B_2 \\ A_2 & \end{pmatrix}\begin{pmatrix} & B_1 \\ A_1 & \end{pmatrix}\right]$$

$$= \begin{pmatrix} B_1A_2 - B_2A_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus the curvature sends

$$\begin{pmatrix} & B_1 \\ A_1 & \end{pmatrix}\begin{pmatrix} & B_2 \\ A_2 & \end{pmatrix} \longmapsto B_1A_2 - B_2A_1.$$

Actually it is probably better if I think of the general linear group as acting on the idempotents. Then $X \in \mathfrak{gl}$ and the tangent vector to $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is

$$[X,e] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} - \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}.$$

So we see that if X is skew-hermitian, whence $b = -c^*$ then $[X,e] = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$ is hermitian, as it should for a tangent vector to a hermitian idempotent.

Possible connection with Connes theory:

Let's consider a Toeplitz situation, and to fix the ideas I shall consider an odd-dimensional manifold and a Dirac operator over it. This is a self-adjoint operator, and I propose to concentrate just on the splitting of the Hilbert space into positive + negative eigenspaces. This gives a projection operator e. Now for any vector bundle with connection E I can tensor this Dirac operator and obtain a twisted Dirac operator. Actually I will be concerned only with the case where E is trivial of rank n. Then I can use the gauge transformations on E, i.e. maps $M \longrightarrow U_n$ to act. Specificcally let L be the Hilbert space of the original Dirac operator, i.e. L is the space of L^2 sections of a vector bundle over M. Then the group $\mathcal{G} = \mathrm{Map}(M, U_n)$ acts as unitary transf. on $L^{\oplus n}$. We then get for each element $g \in \mathcal{G}$ a new projector $g e^{\oplus n} g^{-1}$.

Now let us consider the map from $\mathcal{G}$ to the Grassmannian in $L^{\oplus n}$ given by $g \longmapsto g(e^{\oplus n})g^{-1}$. Really I am just looking at the gauge orbit of the Dirac operator $\otimes$ trivial bundle of rank n. In general I have a map from the space of connections A on E to the space of self-adjoint Fredholm operators on $L^2(S \otimes E)$ which is equivariant for the gauge gp. of E. But I am following the idea of the loop group, where to get an invariant 2-form I fix a connection and look at the map to the gauge orbit, and pull-back the 2-form on the gauge orbit.

Since the forms $\mathrm{Tr}(e(de)^{2p})$ on the Grassmannian are invariant, the point will be to see that they are well-defined back on the loop group. If so, then they will be invariant differential forms on

$$\mathcal{G}/U_n$$

since U_n stabilizes $e^{\oplus n}$. Hence we will get Lie algebra cocycles on $\mathfrak{g} = \mathrm{Map}(M, U_n)$ which vanish on the constants and which are U_n-invariant. So they will be given by Connes cocycles, according to the Loday conjecture, assuming that the cocycles are primitive.

Therefore modulo the primitivity problem, it is clear that there should be a Connes cocycle over M involving $2p$ elements of $C^\infty(M)$ which depends only on the idempotent e and commutators $[a,e]$, $a \in C^\infty(M)$ which somehow comes from the form $Tr(e(de)^{2p})$ ~~on~~ the Grassmannian.

Take $p=1$. Try

$$\tau(a,b) = Tr\left(e[a,e][b,e]\right)$$

If this is anti-symmetrized one knows it gives the 2-form $Tr(e(de)^2)$ pulled back to Maps(M,S^1). So is it skew-symmetric?

$$\begin{aligned}\tau(a,b) &= Tr\left([b,e]e[a,e]\right)\\ &= Tr\left((1-e)[b,e][a,e]\right).\end{aligned}$$

Hence it differs from being skew-symmetric by the symmetric form $Tr([a,e][b,e])$. Now instead of idempotents one can use operators F with $F^2=1$, and in this case $F = 2e-1$, so that

$$[a,F] = 2[a,e].$$

So one could try

$$\tau(a,b) = Tr\left(F[a,F][b,F]\right)$$

instead

January 11, 1983

Representations of the symmetric gp. Σ_p. Let V be a vector space of dimension n, let $G = GL(V)$, $\mathfrak{g} = \mathfrak{gl}(V)$. Then $V^{\otimes p}$ is a repn. of $\Sigma_p \times G$. One knows

$$V^{\otimes p} \simeq \sum_j W_j' \otimes W_j''$$

where W_j' (resp W_j'') are non-isom. irreducible repns. of Σ_p (resp. G). Thus we have a bijection between a subset of the spectrum of Σ_p and a subset of the spectrum of G. Also we have

$$\mathbb{C}[\Sigma_p] \longrightarrow \mathrm{End}(V^{\otimes p})^G$$

$$\begin{matrix}\mathbb{C}[G] \\ \text{or} \\ U(\mathfrak{g})\end{matrix} \twoheadrightarrow \mathrm{End}(V^{\otimes p})^{\Sigma_p}$$

(Actually the second is proved first using that $\mathfrak{g}$ spans $\Gamma_p(\mathfrak{g}) = \mathrm{End}(V^{\otimes p})^{\Sigma_p}$. Then one uses complete irred. of repns. of Σ_p over $\mathbb{C}$ + double-commutator thm. to get the rest.)

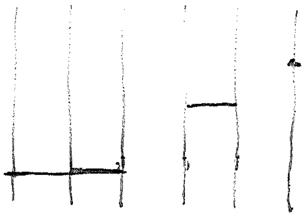
Loday would like to describe the kernel of

$$\mathbb{C}[\Sigma_p] \twoheadrightarrow \mathrm{End}(V^{\otimes p})^G$$

which is a 2-sided, and hence is generated by a central idempotent. This idempotent must be zero on all irreducible repns. of Σ_p occurring in $V^{\otimes p}$, and the identity on the other irred. reps.

Choose a basis S for V. Then $V^{\otimes p}$ has the basis S^p which is stable under Σ_p. Hence $V^{\otimes p}$ can be decomposed according to the orbits of Σ_p on S^p. If S is ordered we can think of S^p graphically as

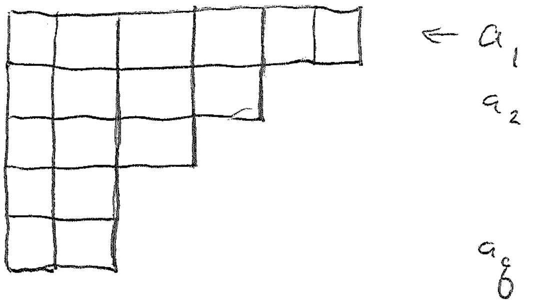
p copies of S arranged vertically over $\{1, \ldots, p\}$, and an element of S^p is a section. Each orbit contain a unique section of the form



i.e. there is a subset $s_1 < \cdots < s_q$ in S and integers $a_1 \geq a_2 \geq \cdots \geq a_q \geq 1$ such that the orbit in S^p contains ~~unique~~ the element

$$(*) \qquad s_1^{\otimes a_1} \otimes \cdots \otimes s_q^{\otimes a_q} \subset V^{\otimes p}.$$

Thus we get a partition $p = a_1 + \cdots + a_q$ $\quad a_1 \geq a_2 \geq \cdots \geq a_q \geq 1$ which we describe graphically by a tableau



so that the a_i's are the lengths of the rows. Note that the number of rows q is $\leq n$. The stabilizer of the elt $(*)$ is

$$\Sigma_{a_1} \times \Sigma_{a_2} \times \cdots \times \Sigma_{a_q}.$$

Notation: $\alpha = (a_1 \geq a_2 \geq \cdots)$ will denote a partition and

$$\Sigma_\alpha = \Sigma_{a_1} \times \cdots$$

Thus for each partition of p α with $q \leq n$ rows we have $\binom{n}{q}$ orbits in S^p of type Σ_p / Σ_α, corresp. to the subsets $s_1 < \cdots < s_q$. Hence

$$V^{\otimes p} = \bigoplus_\alpha \mathbb{C}[\Sigma_p / \Sigma_\alpha]^{\oplus \binom{n}{q}}$$

as a repn. of Σ_p.

Now it is necessary to bring in the description of the irreducible repns. of the symmetric group. The idea is that for each partition α of p we can associate two representations

$$V_\alpha = \mathrm{Ind}_{\Sigma_\alpha \to \Sigma_p}(1) \qquad W_\alpha = \mathrm{Ind}_{\Sigma_\alpha \to \Sigma_p}(\mathrm{sgn})$$

and we can compute the intertwining number of V_α with W_β via the double coset formula. This expresses

$$(W_\beta, V_\alpha) = \dim \operatorname*{Hom}_{\Sigma_\beta}\left(\mathrm{sgn}, \mathrm{Ind}_{\Sigma_\alpha \to \Sigma_p}(1)\right)$$

as sum over the orbits of Σ'_β on Σ_p / Σ_α of the intertwining no. of sgn and 1 pulled back to the stabilizer of a point of each orbit.

$$\Sigma_p / \Sigma_\alpha = \text{partitions of the set } \{1, \ldots, p\} \text{ of type } \alpha$$

not quite correct if the blocks have same size. The correct thing is to write $\alpha = (a_1 \geqslant a_2 \geqslant \cdots \geqslant a_q \geqslant 1)$ and then a point of Σ_p/Σ_α is an ordered decomposition

$$\{1, \ldots, p\} = S_1 \amalg S_2 \amalg \cdots \amalg S_q$$

where S_i has card i. The stabilizer of such an ordered decomposition of type α is the product of the permutation groups of each piece S_i.

Σ'_β is the stabilizer of an ordered decomposition

$$\{1, \ldots, p\} = T_1 \amalg \cdots \amalg T_r$$

where $\beta = (b_1 \geqslant \cdots \geqslant b_r)$. The intersection of Σ_β with the stabilizer of the S_i-decomposition is the stabilizer of the common refinement of the S_i and T_j decomposition. This subgroup of Σ_β will be of the form Σ_γ, where γ

refines β, and the point that as soon γ is not the partition with all blocks of size 1, then Σ_γ acts non-trivially on the sign representation, and so the intertwining no. with the trivial repn. is 0.

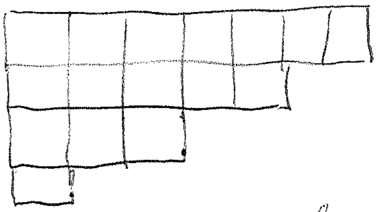
This means that $(W_\beta | V_\alpha) = 0$ unless it is possible for an α type decomposition to be transversal to a β-type decomposition. In fact we have

$(W_\beta | V_\alpha) =$ number of α-type decompositions which are transversal to a given β-type decomposition, modulo Σ_β

$=$ number of β-type decomps. which are transversal to a given α-type decomp., modulo Σ_α.

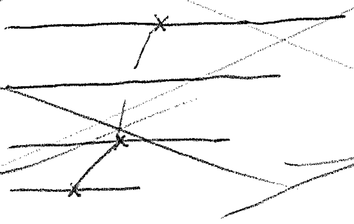
(the equivalence follows because both number are the number of double cosets $\Sigma_\beta g \Sigma_\alpha$ of size $|\Sigma_\beta| \cdot |\Sigma_\alpha|$).

So let's start with the partition α

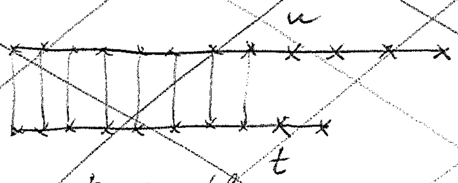


which we think of as the lengths of the rows in the above tableau. Take β ~~to~~ be the partition, the so-called dual partition, by [illegible] the lengths of the columns. Then the decompositions indicated by the tableau are transversal.

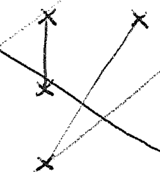
~~But more generally fix an [illegible] α-type decomposition and transversal β-type decomposition where β is now arbitrary. I claim that the β-type decomposition can be "un-refined (collapsed) to a transversal α^*-type partition unique up to Σ_α." Draw the α-~~partition~~ type decomp. as lines.~~



~~and the β-type decomp. as vertical linkings. Under Σ_α we can arrange the first two rows to look as follows~~



~~If some element t of the second row is not linked to an element of the first row, then we can choose a u in the first row not linked to anything in the second row, and we can the blocks of the β-type decomp. containing u and t and form a single block. This unrefines the β-type decomp. without affecting . . .~~ ~~Nope: This doesn't work.~~



One thing is clear: If β is a partition which can be ~~transversal~~ to α, then the number of blocks in β is ≥ the number of columns in α. This is just to get the first row of α separated out. Also it is clear that if β is dual to α, then any β-type decomposition transversal to an α-type decomp. is equivalent under Σ_α to the column decomp. Thus we will get

$$(W_\beta \mid V_\alpha) = 1 \qquad \text{if} \quad \beta = \alpha^*$$

$$(W_\beta \mid V_\alpha) = 0 \qquad \text{if} \quad \text{size } \beta < \text{size } \alpha^*$$

One knows the V_α form a $\mathbb{Z}$-basis for $R(\Sigma_p)$, and also the W_α. It would be nice to have a linear ordering on the partitions so that the matrix

$$(W_{\beta^*} | V_\alpha)$$

[redacted] was triangular with ones on the diagonal.

In any case because $(W_{\alpha^*} | V_\alpha) = 1$ we know that V_α and W_{α^*} have in common a unique irreducible representation.

Consider a [redacted] k.algebra $\bar{A}$ with zero multiplication. Then $gl_n(A)$ is an abelian Lie algebra, hence its Lie alg. homology is just

$$H_p(gl_n \bar{A}) = \Lambda^p(gl_n \bar{A}).$$

Note that this is not the thing related to Connes homology which involves taking coinvariants wrt gl_n first. [redacted] These are not inner automorphisms.

On the other hand, suppose A is a k-algebra with 1. Then I can consider the relative homology

$$H_p(gl(A), gl(k))$$

which one computes [redacted] as follows. One takes the complex

$$C_p = \Lambda^p(gl(A)/gl(k))_{gl(k)}$$

which is a quotient of the Lie alg. complex for $gl(A)$. Now the proof of the Loday conjecture makes it clear that the primitive part of the relative complex = the Connes complex constructed from A/k. As a check let's see that we can form such a quotient of Connes complex

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r] & \underset{a\otimes b}{A\otimes A} \arrow[r, shift left] \arrow[r, mapsto, shift right] & \underset{ab-ba}{A} \\ a\otimes b\otimes c \arrow[r, mapsto] & ab\otimes c - a\otimes bc + ca\otimes b & \end{tikzcd}$$

If $a=1$ then $1\otimes b\otimes c \longmapsto b\otimes c - 1\otimes bc + c\otimes b$, so it will go to zero upon skew-symmetrizing. In general

$$d(x_1 \otimes \cdots \otimes x_n) = \sum_1^{n-1} (-1)^{i-1} x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$
$$+ (-1)^n x_n x_1 \otimes x_2 \otimes \cdots \otimes x_{n-1}$$

Put $x_1 = 1$, and look at the image in $(A/k)^{\otimes(n-1)}$

$$d(1 \otimes x_2 \otimes \cdots \otimes x_n) \equiv \text{[redacted]}\, x_2 \otimes x_3 \otimes \cdots \otimes x_n + (-1)^n x_n \otimes x_2 \otimes \cdots \otimes x_{n-1}$$

and this vanishes by cyclic symmetry.

The conjecture is now that there is a fibration of some sort looking like

⊛ $$gl(k) \longrightarrow gl(A) \longrightarrow gl(A)/gl(k)$$

where the last term is to be interpreted as the relative Lie algebra [redacted]homology.

So now let $\bar{A}$ be an algebra not necessarily with a unit. Then we can add a unit to obtain $A = k + \bar{A}$ and then take relative Lie homology. ~~One has~~

~~Prim $[H_*(gl(A), gl(k))] = HC_*(A/k)$~~

[redacted] ~~we have taken the $gl(k)$-invariants in the Lie algebra complex for $gl(\bar{A})$~~

One has

$$\text{[redacted]} \; \left[\Lambda\, gl(A)/gl(k)\right]_{gl(k)} = \Lambda(gl(\bar{A}))_{gl(k)}$$

and so the relative Lie algebra [redacted]homology will [redacted] have the Connes complex on $\bar{A}$ for its primitive subspace.

Precise interpretation of ⊛ would be an exact sequence

(**) $$\longrightarrow HC(k) \longrightarrow HC(A) \longrightarrow HC(A/k) \xrightarrow{\partial} \cdots .$$

[redacted] Let's check **consistency**: If $\bar{A}$ already had a unit, then $A = k \times \bar{A}$ as rings. One knows from the Lie algebra interpretation that HC adds for direct products, and so the exact sequence (**) we want is true.

Notice that the exact sequence would allow us to compute $HC(A)$ for $A = k \oplus \bar{A}, \bar{A}^2 = 0$ generalized dual numbers, because then $A/k = \bar{A}$ has zero multiplication. Thus its Connes complex is the Connes homology. So we see Connes complex is minimal for computing the relative theory.

Next I want to see if I can prove the exact sequence (**) by means of Lie algebra cohomology. Here I want to use the analogy with geometry. Suppose I have a compact conn. Lie gp. G acting on a manifold M freely. Then we know that the "basic" forms in $\Omega^\cdot(M)$ are the forms on $\Omega^\cdot(M/G)$. On the other hand we can first tensor with $W(\mathfrak{g})$ and then take basic forms

$$\left[\Omega^\cdot(M) \otimes W(\mathfrak{g})\right]_{\text{basic}} = \text{a model for } \Omega^\cdot\left(PG \overset{G}{\times} M\right).$$

One has a spectral sequence for the latter

$$H^*(BG) \otimes H^*(M) \Longrightarrow H^*\left(\left(\Omega^\cdot(M) \otimes W(\mathfrak{g})\right)_{\text{basic}}\right)$$

which one gets by explicit filtration on $\left[\Omega^\cdot(M) \otimes W(\mathfrak{g})\right]_{\text{basic}}$. On the other hand there should also be a map

$$\Omega^\cdot(M/G) \longrightarrow \left[\Omega^\cdot(M) \otimes W(\mathfrak{g})\right]_{\text{basic}}$$

which induces an isom. on cohomology for a free action.

Now I have the cochains $C^\cdot(\tilde{\mathfrak{g}})$ on $\tilde{\mathfrak{g}} = \mathfrak{gl}(A)$ which is acted on by $GL(k)$, which is reductive. So now I take the basic complex for the G-action and I would like the spectral sequence. But it would nicer if I could actually see within $C^\cdot(\tilde{\mathfrak{g}})^G$ the ~~structure~~ that might lead to the desired triangle of primitives.

Bott-Segal thm: Take a principal G-bundle P. Then it is standard in the sense that for a connection $W(\mathfrak{g}) \to \Omega^\cdot(P)$ one gets an isomorphism

$$W(\mathfrak{g}) \otimes_{S(\mathfrak{g}^*)} \Omega^\cdot(P)_{horiz} \xrightarrow{\sim} \Omega^\cdot(P).$$

Also G acts completely reducibly on $\Omega^\cdot(P)$. Now their theorem asserts that the map induced from the above

$$W(\mathfrak{g}) \otimes_{S(\mathfrak{g}^*)^{\mathfrak{g}}} \underset{\Omega^\cdot(P/G)}{\underset{\|}{\Omega^\cdot(P)^{\mathfrak{g}}_{horiz}}} \longrightarrow \Omega^\cdot(P)$$

is a quasi-isomorphism. It will follow by taking G-invariants that

$$\circledast \qquad W(\mathfrak{g})^{\mathfrak{g}} \otimes_{S(\mathfrak{g}^*)^{\mathfrak{g}}} \Omega^\cdot(P/G) \longrightarrow \Omega^\cdot(P)^{\mathfrak{g}}$$

is a quis.

What I want is to see an exact triangle of primitive complexes for

$$\Lambda(\mathfrak{g}^*)^{\mathfrak{g}} \qquad (\Lambda\tilde{\mathfrak{g}}^*)^{\mathfrak{g}} \qquad \left(\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^*\right)^{\mathfrak{g}}$$

Here $\Lambda\tilde{\mathfrak{g}}^*$ "=" $\Omega^\cdot(P)$, and $\left(\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^*\right)^{\mathfrak{g}} = \Omega^\cdot(P/G)$. We will need a connection form & curvature

$$S(\mathfrak{g}^*)^G \longrightarrow \Omega^\cdot(P/G)$$

and that probably results from splitting $\mathfrak{g} \to \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/\mathfrak{g}$.

I want to take primitives in $\circledast$, and I know maybe already the primitives in $W(\mathfrak{g})^{\mathfrak{g}}$ look ugly. A combinatorial proof with Connes complexes is simpler.

January 12, 1983:

Idea last night: Let $A = k \oplus \bar{A}$ where $\bar{A}^2 = 0$. If it is true that Connes exact sequence [redacted] relating Connes homology and Hochschild homology is valid for rings without 1, then this exact sequence would be true for the ring $\bar{A}$. But then all differentials are zero, so the combinatorics would be right on the surface of representations of $GL(V)$.

So we have Hochschild's complex

$$\dots \quad V\otimes V\otimes V \quad V\otimes V \quad V$$

with zero differentials and we have Connes' complex

$$\dots \quad V\otimes V\otimes V/\sim \quad V\otimes V/\sim \quad V$$

again with zero differentials. The exact sequence in general is

$$HC_{p+1}(A) \longrightarrow HC_{p-1}(A) \text{ [redacted] } \longrightarrow H_p(A) \longrightarrow HC_p(A) \longrightarrow HC_{p-2}(A)$$

where I have used the indexing according to the no. of factors in the tensor product. This becomes then

$$V^{\otimes(p+1)}/_{\sim} \longrightarrow V^{\otimes(p-1)}/_{\sim} \longrightarrow V^{\otimes p} \longrightarrow V^{\otimes p}/_{\sim} \longrightarrow V^{\otimes(p-2)}/_{\sim} .$$

So it is obviously not going to work, because there is no invariant way to map $V^{\otimes p}$ to $V^{\otimes q}$ for $p \neq q$.

I conclude that I have to look at the ring with identity $A = k \oplus V$. Think of this as a functor of the vector space V. Then all the functors we are interested in will be [redacted] functors on the category of vector spaces. [redacted] (As

$$\mathrm{Hom}_{k\text{-}alg}(k\oplus V, k\oplus W) = \mathrm{Hom}_k(V, W)$$

[redacted] there is no improvement to an affine group.)

Let's look at the quotient of the first inclusion. This gives a complex of the form.

$$I\otimes I\otimes I \longrightarrow I\otimes I \longrightarrow I \longrightarrow k$$

Let's compute $d(a\otimes b\otimes c)$. We lift $a\otimes b\otimes c \in I^{\otimes 3}$ back to $1\otimes a\otimes b\otimes c$ in $A\otimes I^{\otimes 3}$ and take d.

$$d(1\otimes a\otimes b\otimes c) = a\otimes b\otimes c - 1\otimes ab\otimes c + 1\otimes a\otimes bc - c\otimes a\otimes b$$

$$\equiv -1\otimes ab\otimes c + 1\otimes a\otimes bc \quad \bmod I^{\otimes 3}$$

and so we get

$$d(a\otimes b\otimes c) = -ab\otimes c + a\otimes bc .$$

Similarly

$$\underset{\in I^2}{a\otimes b} \qquad \underset{A\otimes I^{\otimes 2}}{1\otimes a\otimes b} \xrightarrow{d} \underset{A\otimes I^{\text{[illegible]}}}{a\otimes b - 1\otimes ab + b\otimes a} \qquad \underset{I}{-ab}$$

so that $\qquad d(a\otimes b) = -ab.$

Summarizing: If $A = k\oplus I$ is an augmented k-algebra, then the Hochschild cohomology is part of a triangle

$$H_p\left\{\begin{matrix}\text{complex with}\\ I^{\otimes(p+1)} \text{ in degree}\\ p \text{ and cyclic}\\ \text{boundary}\end{matrix}\right\} \longrightarrow \operatorname{Tor}_p^{A\otimes A^{op}}(A,A) \longrightarrow \operatorname{Tor}_p^{A}(k,k) \quad \cdots$$

Now take the case when $I^2 = 0$ and one gets a long exact sequence

$$\xrightarrow{\partial} I^{\otimes(p+1)} \longrightarrow \operatorname{Tor}_p^{A\otimes A^{op}}(A,A) \longrightarrow I^{\otimes p} \xrightarrow{\partial} I^{\otimes p} \longrightarrow \cdots$$

and we want to see what ∂ is. Claim:

$$\boxed{\partial(a_1\otimes\cdots\otimes a_p) = a_1\otimes\cdots\otimes a_p + (-1)^p a_p\otimes a_1\otimes\cdots\otimes a_{p-1}}$$

First we compute the Hochschild cohomology for $A = k \oplus V$. We need an $A \otimes A^{op}$ resolution of A; start with the standard simplicial resolution

$$\cdots \; \overset{2}{A \otimes A \otimes A \otimes A} \; \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \; \overset{1}{A \otimes A \otimes A} \; \substack{\longrightarrow\\\longrightarrow} \; \overset{0}{A \otimes A}$$

for which there are 2 contracting homotopies given by putting 1 at far left or far right. Normalize which means we divide out by the degenerate subcomplex; this involves 1's in the middle factors. Thus we get

$$\longrightarrow A \otimes \bar{A} \otimes \bar{A} \otimes A \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow A \otimes A$$

where $\bar{A} = A/k$. As a check observe that

$$d(a \otimes \bar{b} \otimes c) = ab \otimes c - a \otimes bc$$

is a well-defined function of $\bar{b} = b \pmod k$. Now tensor with A over $A \otimes A^{op}$ and you get

$$\longrightarrow A \otimes \bar{A} \otimes \bar{A} \longrightarrow A \otimes \bar{A} \longrightarrow A$$

$$d(a \otimes \bar{b}) = ab - ba$$

$$d(a \otimes \bar{b} \otimes \bar{c}) = ab \otimes \bar{c} - a \otimes \overline{bc} + ca \otimes \bar{b}$$

Now suppose A is an augmented k-algebra so that we get an ideal I in A with $I \xrightarrow{\sim} \bar{A}$. Then the above complex is isomorphic to the same one with $\bar{A}$ replaced by I. In fact we have subcomplexes

$$\begin{tikzcd}
(1) & \arrow[r] & I \otimes I \otimes I \arrow[r] \arrow[d, phantom, "\cap"] & I \otimes I \arrow[r] \arrow[d, phantom, "\cap"] & I \arrow[d, phantom, "\cap"] \\
(2) & \arrow[r] & A \otimes I \otimes I \arrow[r] \arrow[d, phantom, "\cap"] & A \otimes I \arrow[r] \arrow[d, phantom, "\cap"] & A \arrow[d, phantom, "\cap"] \\
(3) & \arrow[r] & A \otimes A \otimes A \arrow[r] & A \otimes A \arrow[r] & A
\end{tikzcd}$$

and what we have shown is that the second inclusion is a quis.

This seems to demolish the conjecture about there being a long exact sequence NO

$$HC(k) \longrightarrow HC(A) \longrightarrow HC(A,k)$$

because $HC_p(A,k) = I^{\otimes p}/\sim$ is definitely not enough to contribute what occurs in $HC(A)$. NO

So let's go back and try to calculate the Connes homology directly for $A = k \oplus I$, $I^2 = 0$ using what we can about the representations of $\mathrm{End}(I)$.

$$A^{\otimes p} = (k \oplus I)^{\otimes p} = 2^p\text{-fold direct sum.}$$

The idea will be to look carefully at the degree decomposition of the Connes complex. Clearly $A^{\otimes p}$ has components of degree $0, 1, 2, \ldots, p$. We have

$$(A^{\otimes p})_p = I^{\otimes p}$$

$$(A^{\otimes p})_{p-1} = k \otimes I^{\otimes (p-1)} \oplus I \otimes k \otimes I^{\otimes(p-2)} \oplus \cdots\cdots$$

so now we can compute the degree p part of HC_p. We have an exact sequence (since $(A^{\otimes(p-1)}/\sim)_p = 0$)

$$\begin{tikzcd}
1\otimes a_1 \otimes \cdots \otimes a_p & (A^{\otimes(p+1)}/\sim)_p \arrow[r, "d"] & (A^{\otimes p}/\sim)_p \arrow[r] \arrow[d, phantom, sloped, "\cong"] & (HC_p)_p \arrow[r] & 0 \\
a_1 \otimes \cdots \otimes a_p \arrow[u] & I^{\otimes p} \arrow[u, "\cong"] & I^{\otimes p}/\sim & &
\end{tikzcd}$$

Now $d(1 \otimes a_1 \otimes \cdots \otimes a_p) = a_1 \otimes \cdots \otimes a_p + (-1)^p a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}$ which goes to zero in $I^{\otimes p}/\sim$. Thus I seem to get

$$(HC_p)_p = I^{\otimes p}/\sim$$

Notice that from the above we also get it seems

$$\operatorname{Ker} \partial \text{ on } I^{\otimes p} = \operatorname{Coker}\{HC_{p+2} \longrightarrow HC_p\}$$

so that $(HC_p)_p \twoheadrightarrow \operatorname{Ker} \partial$ on $I^{\otimes p}$. Is it possible that $\operatorname{Ker} \partial = \operatorname{Coker} \partial$ on $I^{\otimes p}$. One has

$$0 \to \operatorname{Ker} \partial \longrightarrow I^{\otimes p} \xrightarrow{\partial} I^{\otimes p} \longrightarrow I^{\otimes p}/\sim \to 0$$

and so $\operatorname{Ker} \partial$ and $I^{\otimes p}/\sim$ have the same dimension.

~~Yes. Recall that for a cyclic group $\mathbb{Z}/p$ and module M over it we have~~

~~$0 \to H^0(\mathbb{Z}/p, M) \to M \xrightarrow{1-\sigma} M \to H_0(\mathbb{Z}/p, M) \to 0$~~

Yes. For any G-module over a finite group G we have the norm map $N : H_0(G,M) \longrightarrow H^0(G,M)$, $Nm = \sum_{\sigma \in G} \sigma m$ whose kernel and cokernel are the Tate groups:

$$\hat{H}^{-1}(G,M) = \frac{\operatorname{Ker}\{N : M \to M\}}{\{gm - m\}} \qquad \hat{H}^0(G,M) = \frac{M^G}{\operatorname{Im} N}.$$

The Tate groups vanish if $|G|$ acts invertibly on M, which means that $N : H_0(G,M) \xrightarrow{\sim} H^0(G,M)$. So now take $G = \mathbb{Z}/p$ acting on $I^{\otimes p}$ via $\sigma(a_1 \otimes \cdots \otimes a_p) = (-1)^{p-1} a_p \otimes a_1 \otimes \cdots a_{p-1}$

Actually I am being stupid: As $\partial = 1 - \sigma$ it follows that $\operatorname{Ker} \partial$ is the invariants and $\operatorname{Coker} \partial$ is the coinvariants, and these coincide in char. 0.

But now if my calculation of the Hochschild cohomology is correct we get

$$0 \to I^{\otimes p}/\sim \longrightarrow HC_p \to HC_{p-2} \longrightarrow I^{\otimes(p-2)}/\sim \to 0$$

~~In particular $(HC_p)_{p-1} = 0$, hence also $(HC_p)_{p\text{-odd}} = 0$~~

~~So next let's start the calculation~~

~~$HC_1 = A = \overset{k}{\underset{\oplus}{I}}$~~

$$\begin{tikzcd}
 & I^{\otimes p} \arrow[d] & \\
1\otimes a_1\otimes\cdots\otimes a_p & & a_1\otimes a_2\otimes\cdots\otimes a_p \\
A\otimes I^{\otimes p} \arrow[r, "d"] \arrow[d, "\varepsilon\otimes id"'] & A\otimes I^{\otimes(p-1)} & +(-1)^p a_p\otimes a_1\otimes\cdots\otimes a_{p-1} \\
I^{\otimes p} & & \\
a_1\otimes\cdots\otimes a_p & &
\end{tikzcd}$$

where we have used that $a_i a_{i+1} = 0$ in I. [redacted]

Thus we should think of Cokernel of ∂ as giving the cyclic quotient [redacted] $I^{\otimes p}/(\mathbb{Z}/p)$ that occurs in the Connes complex for I. So this embeds in Tor_{p-1}. Let's consider Connes exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & HC_{p-1} \arrow[r] & H_p \arrow[r] \arrow[d, equal] & HC_p \arrow[r] & HC_{p-2} \arrow[r] & \cdots \\
0 \arrow[r] & (I^{\otimes p})/(\mathbb{Z}/p) \arrow[r] & \mathrm{Tor}_{p-1} \arrow[r] & \mathrm{Ker}\,\partial \text{ on } I^{\otimes(p-1)} \arrow[r] & 0 &
\end{tikzcd}$$

Here HC_p involves $A^{\otimes}$[redacted]$p/\sim$ and similarly H_p. Now we know that the image of $I^{\otimes p}$ in $A^{\otimes p}/\sim$ is a group of cycles, and it seems to be "disjoint" from the boundaries which will involve 1's, so it looks like

$$I^{\otimes p}/\sim \quad \subset \quad \mathrm{Ker}\{HC_p \longrightarrow HC_{p-2}\}$$

Next look at the degrees, by which I mean that we have functors of I as a vector space and we just keep track of the degree of these functors. Clearly HC_p being a sub-quotient of $(k\oplus I)^{\otimes p}$ is of degree $\leq p$, and the degree p part is exactly [redacted] what comes from $I^{\otimes p}$. From Connes exact sequence and the calculation of the Tor it is clear we have

$$I^{\otimes p}/\sim \;=\; \mathrm{Ker}\{HC_p \longrightarrow HC_{p-2}\}$$

$$\mathrm{Ker}\,\partial \text{ on } I^{\otimes(p-1)} \;=\; \mathrm{Coker}\{HC_{p+1} \longrightarrow HC_{p-1}\}.$$

So now let's start off the calculation:

$$\boxed{HC_1 = A = \begin{matrix} k \\ \oplus \\ I \end{matrix}}$$

I need HC_2:

$$\begin{tikzcd} A^{\otimes 3}/\sim \arrow[r] & A^{\otimes 2}/\sim \arrow[r, "0"] & A \end{tikzcd}$$

$$A^{\otimes 2}/\sim \;=\; \Lambda^2(k+I) = k\otimes I + \Lambda^2 I$$

$$d(1\otimes 1\otimes a) = 1\otimes a - 1\otimes a + a\otimes 1 = a\otimes 1$$

so that the $k\otimes I$ part is killed. Conclude from what I know that

$$\boxed{HC_2 = \Lambda^2 I}$$

Now from

$$\begin{tikzcd} 0 \arrow[r] & I^{\otimes 3}/\sim \arrow[r] & HC_3 \arrow[r] & \overset{\begin{matrix} k \\ \oplus \\ I \end{matrix}}{HC_1} \arrow[r] & I \arrow[r] & 0 \end{tikzcd}$$

that

$$\boxed{HC_3 = \begin{matrix} k \\ \oplus \\ I^{\otimes 3}/\sim \end{matrix}}$$

and from

$$\begin{tikzcd} 0 \arrow[r] & I^{\otimes 4}/\sim \arrow[r] & HC_4 \arrow[r] & \overset{I^{\otimes 2}/\sim}{\overset{\parallel}{HC_2}} \arrow[r] & I^{\otimes 2}/\sim \arrow[r] & 0 \end{tikzcd}$$

that

$$\boxed{HC_4 = I^{\otimes 4}/\sim .}$$

Therefore the conjecture about

$$\begin{tikzcd} \cdots \arrow[r] & HC(k) \arrow[r] & HC(A) \arrow[r] & HC(A,k) \arrow[r] & \cdots \end{tikzcd}$$

is checked in this case

So let's try now to prove the conjecture. We define an increasing filtration of $A^{\otimes p}$ by

$F_q(A^{\otimes p})$ = spanned by $a_1\otimes\cdots\otimes a_p$ where at most q of the a_j are not in k

Then [redacted] $F_0(A^{\otimes p}) = k^{\otimes p}$

$$F_1(A^{\otimes p}) = A\otimes k^{\otimes(p-1)} \oplus k\otimes A\otimes k^{\otimes(p-2)} \oplus \cdots .$$

[redacted] Another description is to take a subset $i_1<\cdots<i_q$ of $\{1,\dots,p\}$ and consider the map

$$A^{\otimes q} \longrightarrow A^{\otimes p}$$

$$b_1\otimes\cdots\otimes b_q \longmapsto 1\otimes\cdots\otimes \underset{\substack{\uparrow\\ i_1}}{b_1}\otimes\cdots\otimes \underset{\substack{|\\ i_2}}{b_2}\otimes\cdots \quad \text{position}.$$

Then $F_q(A^{\otimes p}) =$ [redacted] $\displaystyle\sum_{\substack{\{i_1<\cdots<i_q\}\\ \subset\{1,\dots,p\}}} A^{\otimes q}$

The Hochschild boundary takes $a_1\otimes\cdots\otimes a_p$ into a sum of $a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_p$. It [redacted] takes a

$$1\otimes\cdots\otimes b_1\otimes \quad \otimes b_2\otimes\cdots\cdots$$

into something of the same form, and it doesn't increase the number of factors outside of k. Thus $F_q(A^{\otimes p})$ is a subcomplex of the Hochschild complex.

So next we define

$$F_q(A^{\otimes p}/\sim) = F_q(A^{\otimes p})/\sim \;=\; \mathrm{Im}\{F_q(A^{\otimes p})\to A^{\otimes p}/\sim\}.$$

(This is the same because $/\sim$ is exact). Thus we get a filtration of the Connes complex by subcomplexes. Now

$$F_0\,C(A) = F_0\,C(k)$$

and what I would like to show is that

$$F_q C(A)/F_{q-1}C(A) \overset{\text{quis}}{\simeq} (\bar{A}^{\otimes q}/\sim)[q] \qquad q>0.$$

Then taking the spectral sequence should [redacted] show me that

$$C(A)/C(k) \overset{\text{quis}}{\sim} C(\bar{A})$$

[redacted]

Some ideas: 1) Try a homotopy operator which put '1' in cyclically.

$$a\otimes b \longmapsto 1\otimes a\otimes b \pm a\otimes 1\otimes b \pm a\otimes b\otimes 1$$

2) If you take a ring such that $A/[A,A]=0$ then the map $HC_1(k) \longrightarrow HC_1(A)$ will not be injective, so the exact sequence

$$\longrightarrow HC(k) \longrightarrow HC(A) \longrightarrow HC(A,k) \longrightarrow$$

will not split into short exact sequences.

January 13, 1983

Loday's work shows that the theorem of invariants should be viewed as saying one has an injection

$$(\mathfrak{g}^{\otimes p})_{\mathfrak{g}} \hookrightarrow k[\Sigma_p]^* \qquad \mathfrak{g} = \mathrm{End}(V)$$

which is onto for $\dim V \geq p$. Equivalently any invariant linear fnl. on $\mathfrak{g}^{\otimes p}$ is given by a function on permutations. How? An [redacted] invariant linear fnl on $\mathfrak{g}^{\otimes p}$ is a linear combination of contractions:

$$\mathfrak{g}^{\otimes p} = (V \otimes V^*) \otimes (V \otimes V^*) \otimes \cdots \otimes (V \otimes V^*) \longrightarrow k$$

one for each permutation $\sigma \in \Sigma_p$. Let $X^1, \ldots, X^p \in \mathfrak{g}$. In the above example we contract the first index of X^1 with the last index of X^2

$$x^1_{ij} \quad x^2_{ji} \quad \cdots \quad x^p_{kk}$$

$$= x^1_{i_1 i_{\sigma^{-1}1}} \quad x^2_{i_2 i_{\sigma^{-1}2}} \quad x^p_{i_p i_{\sigma^{-1}(p)}}$$

So the formula in general is

$$f(X^1, \ldots, X^p) = \sum_{\sigma, i_1, \ldots, i_p} f_\sigma \quad x^1_{i_1 i_{\sigma^{-1}1}} \; x^2_{i_2 i_{\sigma^{-1}2}} \cdots x^p_{i_p i_{\sigma^{-1}p}}$$

More generally when a vector space A is present, one has an injection (isom for $n \geq p$)

$$((\mathfrak{g} \otimes A)^{\otimes p})_{\mathfrak{g}} = (\mathfrak{g}^{\otimes p})_{\mathfrak{g}} \otimes A^{\otimes p} \hookrightarrow k[\Sigma_p]^* \otimes A^{\otimes p}$$

so that any linear fnl on $(\mathfrak{g} \otimes A)^{\otimes p}$, invariant under $\mathfrak{g}$, is given by a map $\sigma \mapsto f_\sigma$ from Σ_p to $(A^{\otimes p})^*$. How? Let $X^i = (x^i_{\mu\nu}) \in \mathfrak{g} \otimes A$. Given a perm., say the one pictured above, I want a linear [redacted] map

$$(V\otimes V^*\otimes A)\otimes(V\otimes V^*\otimes A)\otimes\cdots \longrightarrow A\otimes A\otimes\cdots$$

which contracts the first index of X^1 with the last index of X^2.

$$x^1_{ij}\otimes x^2_{ij}\otimes\cdot\otimes x^p_{kk}$$

$$= x^1_{i_1 i_{\sigma^{-1}1}}\otimes x^2_{i_2 i_{\sigma^{-1}2}}\otimes\cdots$$

and so the formula is

$$\boxed{f(X^1,\dots,X^p)=\sum_{\sigma, i_1\cdots i_p} f_\sigma\left(x^1_{i_1 i_{\sigma^{-1}(1)}}, x^2_{i_2 i_{\sigma^{-1}2}},\dots, x^p_{i_p i_{\sigma^{-1}(p)}}\right)}$$

Thus ~~one~~ can prove

Proposition: Any $\mathfrak{g}$-invariant linear fnl on $(\mathfrak{g}\otimes A)^{\otimes p}$ can be represented in the above form where $\sigma\longmapsto f_\sigma$ is a map: $\Sigma_p\longrightarrow(A^{\otimes p})^*$. The representation is unique provided $n\geq p$, where $\mathfrak{g}=\mathfrak{gl}_n$.

Next one can how permuting the factors affects the family of f_σ. Take a perm. τ and consider

$$f^\tau(X^1,\dots,X^p)=f(X^{\tau^{-1}1},\dots,X^{\tau^{-1}p})$$

$$=\sum_{\sigma,i} f_\sigma\left(x^{\tau^{-1}1}_{i_1 i_{\sigma^{-1}1}}, x^{\tau^{-1}2}_{i_2 i_{\sigma^{-1}2}},\dots\right)$$

$$=\sum_{\sigma,i} f^\tau_\sigma\left(x^1_{i_{\tau1} i_{\sigma^{-1}\tau1}}, x^2_{i_{\tau2} i_{\sigma^{-1}\tau2}},\dots\right)$$

$j_\mu=i_{\tau\mu}$

$$=\sum_{\sigma,j} f^\tau_\sigma\left(x^1_{j1\, j_{\tau^{-1}\sigma\tau1}},\dots\right.$$

$$=\sum_{\sigma,j} f^\tau_{\tau\sigma\tau^{-1}}\left(x^1_{j1\, j_{\sigma^{-1}1}},\dots\right)$$

which shows that the transformed family is

$$(f^{\tau})_{\sigma}(a_1,\dots,a_p) = f^{\tau}_{\tau\sigma\tau^{-1}}(a_1,\dots,a_p)$$
$$= f_{\tau\sigma\tau^{-1}}(a_{\tau^{-1}1},\dots,a_{\tau^{-1}p})$$

Given f if you want to get f_{τ}, then you take $X^1 = a_1 E_{1,\tau^{-1}1}$, $X^2 = a_2 E_{2,\tau^{-1}2}$, ... etc. Then in the formula for f the only contribution will occur when $i_j = j$, $i_{\sigma^{-1}j} = \tau^{-1}j \Rightarrow \sigma = \tau$ and $i_j = j$. Thus

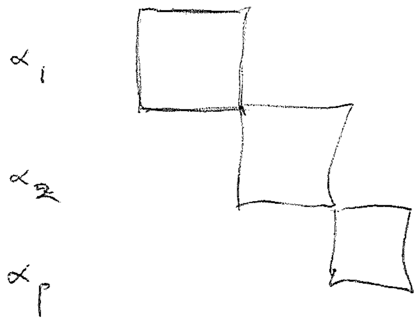
$$\boxed{f_{\sigma}(a^1,\dots,a^p) = f(a^1 E_{1,\sigma^{-1}1}, a^2 E_{2,\sigma^{-1}2}, \dots, a^p E_{p,\sigma^{-1}p})}$$

Let's go back to the transformation law above and ask that f be skew-symmetric. Then f_{σ} determines $f_{\tau\sigma\tau^{-1}}$ which means that the skew-symmetric f decompose according to the conjugacy classes of Σ_p. Therefore to give an skew-symmetric f, it is enough to say what f_{σ} is for some σ in each conjugacy class. Conjugacy classes are described by partitions and to a partition $p = \alpha_1 + \alpha_2 + \dots + \alpha_r$ $\quad \alpha_1 \geq \alpha_2 \geq .. \alpha_r \geq 1$ we get a representative by taking standard cycles of lengths $\alpha_1, \alpha_2, \dots$ etc.

~~The important case is where there is one block, and σ is the standard p-cycle.~~

~~Observation: From the formula for f_{σ} above one sees the following, assuming f is skew-symmetric.~~

Call a skew-symmetric f primitive when $f_\sigma = 0$ unless σ is a p-cycle. How can one tell when f is primitive? It is enough to look at a product of more than one cycle. If $X^i = a^i E_{i,\sigma^{-1}i}$ and σ is a product of standard cycles, then clearly the X^i belong to subspace of block matrices of the form



What are you after? The ultimate idea will be to recognize when invariant differential forms on G are primitive, and so come from Connes cochains.

Let's return to the problem of showing that there is a long exact sequence

$$\cdots \longrightarrow HC(k) \longrightarrow HC(A) \longrightarrow HC(A/k) \longrightarrow \cdots\cdots$$

Now I believe that given a reductive subalgebra $\mathfrak{g}$ of a Lie algebra $\tilde{\mathfrak{g}}$, one has a spectral sequence

$$H^*(\tilde{\mathfrak{g}}, \mathfrak{g}) \otimes H^*(\mathfrak{g}) \Longrightarrow H^*(\tilde{\mathfrak{g}})$$

The idea is to use this spectral sequence or a related one such as

$$H^*(B\mathfrak{g}) \otimes H^*(\tilde{\mathfrak{g}}) \Longrightarrow H^*(\tilde{\mathfrak{g}}, \mathfrak{g})$$

to establish the above exact sequence.

One must show that the map of Connes cxs.

$$C(k) \longrightarrow C(A) \longrightarrow C(A,k)$$

constitute an exact triangle. That means we show that either of the maps

$$C(k) \longrightarrow \mathrm{Ker}\{C(A) \longrightarrow C(A,k)\}$$

$$C(A)/C(k) \longrightarrow C(A,k)$$

is a quis. These complexes occur inside various Hopf algebras, hence we might be able to deduce the result we need by working with the Hopf algebra.

Let's put down what we know:

$$\tilde{S}\,\{C(A)\} \xrightarrow{\sim} \left[\Lambda^{\cdot}(gl(A))\right]_{gl}$$

$$\tilde{S}\,\{C(A,k)\} \xrightarrow{\sim} \left[\Lambda^{\cdot}(gl(A)/gl)\right]_{gl}.$$

It would seem that if I have an exact sequence of complexes

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$$

then I would get a spectral sequence in homology

$$H(\tilde{S}K'') \otimes H(\tilde{S}K') \Longrightarrow H(\tilde{S}K).$$

Let's admit this. Then if I use the known spectral sequence

$$H_{\cdot}(gl(A), gl(k)) \otimes H_{\cdot}(gl(k)) \Longrightarrow H_{\cdot}(gl(A))$$

and the comparison theorem for spectral sequences I should get a quis $C(k) \longrightarrow K' = \mathrm{Ker}\{C(A) \to C(A,k)\}$ which is what I want.

(This whole business is tricky: One would think that by understanding the quis $C(A \times B) \longrightarrow C(A) \times C(B)$, then we could manipulate the formulas to get the triangle $C(k) \longrightarrow C(A) \longrightarrow C(A,k)$. But if one takes the fibre product $A \times_k B$ of two augmented k-algebras, then $C(A \times_k B, k)$ looks like $C(\bar{A} \times \bar{B})$. But $C(\bar{A} \times \bar{B})$ is not quis $C(\bar{A}) \times C(\bar{B})$, otherwise you would contradict formula for $A = k \oplus I$, $I^2 = 0$.)

So it becomes clear that I have to understand various of the arguments in Lie algebra cohomology, in particular the spectral sequences encountered before. So let's consider $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ a reductive subalgebra and derive the spectral sequence relative to

$$\Lambda \mathfrak{g} \qquad \Lambda \tilde{\mathfrak{g}} \qquad (\Lambda \tilde{\mathfrak{g}}/\mathfrak{g})_{\mathfrak{g}} .$$

Perhaps it would be better to work with cohomology and follow the Bott-Segal paper.

They put $A = \Lambda \tilde{\mathfrak{g}}^*$ or $\Omega(P)$ where P is a principal G-bundle and then use an isomorphism

$$A_{hor} \otimes_{S\mathfrak{g}^*} W(\mathfrak{g}) \xrightarrow{\sim} A$$

which in this case is just

$$\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda(\mathfrak{g}^*) \xrightarrow{\sim} \Lambda(\tilde{\mathfrak{g}}^*)$$

and comes from a $\mathfrak{g}$-invariant splitting of $\mathfrak{g} \to \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}/\mathfrak{g}$. Now one uses the decreasing filtration

$$F_p A = \text{ideal generated by } \bigoplus_{q \geq p} A^q_{horiz}$$

$$= \bigoplus_{q \geq p} \Lambda^p(\tilde{\mathfrak{g}}/\mathfrak{g})^* \otimes \Lambda(\mathfrak{g}^*)$$

This is independent of the splitting or connection since $A_{horiz} \longrightarrow A$ is intrinsic.

Bott claims $F_p(A)$ is stable under d, but this isn't obvious since A_{horiz} is not stable under d. (Bott is probably thinking geometrically).

Anyway one knows this will work for foliations, so one can probably justify that $F_p A$ is closed under d and that on

$$gr_p(A) = \Lambda^p(\tilde{\mathfrak{g}}/\mathfrak{g}) \otimes \Lambda(\mathfrak{g}^*)$$

one gets the cochains on $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\Lambda^p(\tilde{\mathfrak{g}}/\mathfrak{g})$. Then to go to E_1 one uses the reductivity of $\mathfrak{g}$, or really the semi-simplicity of $\Lambda(\tilde{\mathfrak{g}}/\mathfrak{g})$, to conclude

$$E_1^p = \left(\Lambda^p(\tilde{\mathfrak{g}}/\mathfrak{g})\right)^{\mathfrak{g}} \otimes H^q(\mathfrak{g}).$$

Then $$E_2 = H(\tilde{\mathfrak{g}}, \mathfrak{g}) \otimes H(\mathfrak{g}).$$

The geometrical origin of this spectral sequence is clear: It results from looking at $\tilde{G}$-invariant forms on $\tilde{G}$, for the left action, and then using the foliation by the right G orbits. It might be simpler if I replaced $\tilde{G}$ by a principal G-bundle P. Then on P I have a foliation by the map $P \longrightarrow P/G = M$. This gives me standard filtration on $\Omega(P)$, or better on the vector bundle $\Lambda^{\cdot} T_P^*$. Namely we have

$$\begin{tikzcd} 0 \arrow[r] & \overbrace{T_{P/M}}^{S} \arrow[r] & T_P \arrow[r] & \overbrace{\pi^* T_M}^{Q} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \pi^* T_M^* \arrow[r] & T_P^* \arrow[r] & T_{P/M}^* \arrow[r] & 0 \end{tikzcd}$$

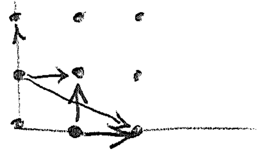
$$\begin{aligned} F_p(\Lambda^{\cdot} T_P^*) &= p\text{th power of ideal gen. by } \pi^* T_M^* \\ &= \pi^* \Lambda^p T_M^* \cdot \Lambda^{\cdot} T_P^* \end{aligned}$$

This is a decreasing filtration with

$$gr_p(\Lambda^{p+q} T_P^*) = \pi^*(\Lambda^p T_M^*) \otimes \Lambda^q T_{P/M}^*.$$

~~The next point is that~~

The next point is that because we have a foliation [illegible] the ideal $F_1(\Lambda^{\cdot} T_P^*)$ is closed under d; here I have to pass to sheaf of sections or all sections. This means that the ~~filtration~~ picture is:



Now let's take global sections, and look at the associated spectral sequence. The $E_1^{p\cdot}$ term is just the DR-complex along the fibres of the bundle $\pi^*\Lambda^p T_M^* = \Lambda^p(Q^*)$, which is flat along the fibres.

In general for a foliation, the bundle Q restricted to a leaf is a flat bundle, and in Connes theory one trivializes this bundle by using the holonomy groupoid for the foliation. In the situation above, where one has a fibration, $Q = \pi^* T_M$ is trivial when restricted to any leaf. Therefore

$$E_1^{p\cdot} = \Lambda^p T_M^* \otimes H^\cdot(G)$$

and the rest is clear. This doesn't use any compactness.

~~[illegible]~~

So far I have worked with a general fibration. The only subtle point perhaps is the Gauss-Manin connection: Why the vector bundles on the base with fibre the $H^\cdot$(fibre) are flat.

Next problem. Look at the situation, i.e. filtration and spectral sequence, after taking G-invariants

14 January 1983

Dear Loday,

I am very pleased with the note you have written. It is very clear and clarifies several points about which I was confused, notably the fact that the map

$$\left(\mathfrak{g}_r^{\otimes n}\right)_{\mathfrak{g}_r} \longrightarrow k[\Sigma_n]^* \simeq k[\Sigma_n]$$

is the good arrow, not the surjection used in my letter.

In your letter you mention applications of the theorems, in particular computation of Lie K-groups for regular algebras in char. 0. This goes from Connes theory to the Lie algebra homology, but there are applications the other way. Two trivial ones are

$$HC(A \times B) = HC(A) \oplus HC(B)$$
$$HC(M_r(A)) = HC(A).$$

(I assume these formulas are proved in the Connes theory using his exact sequence and the corresponding formulas for Hochschild homology. Would you let me know if there are different proofs?)

Another application is based on the fact that one has a spectral sequence using relative Lie algebra homology for the inclusion $\mathfrak{g} \subset \mathfrak{g} \otimes A$. The relative Lie algebra complex is

$$\left[\Lambda(\mathfrak{g} \otimes A/\mathfrak{g})\right]_{\mathfrak{g}}.$$

and for $\mathfrak{g} = \mathfrak{gl}(k)$, the primitive part is the quotient complex of $C(A)$ given by

$$C_p(A, k) = (A/k)^{\otimes p} / \text{cyclic}.$$

Assertion: The inclusion $C(k) \subset C(A)$ induces a quasi-isomorphism $C(k) \longrightarrow \mathrm{Ker}\{C(A) \longrightarrow C(A, k)\}$ and hence there is a long exact sequence

$$\longrightarrow HC(k) \longrightarrow HC(A) \longrightarrow HC(A,k) \xrightarrow{\partial} \cdots$$

According to Koszul's thesis, one has a spectral sequence

$$E^2 = H_p(\tilde{\mathfrak{g}}, \mathfrak{g}) \otimes H_q(\mathfrak{g}) \Longrightarrow H_{p+q}(\tilde{\mathfrak{g}})$$

where $\mathfrak{g}$ is a sub-Lie-algebra of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}$ is semi-simple as a $\mathfrak{g}$-module. Apply this to $\mathfrak{g}_n \subset \mathfrak{g}_n \otimes A$ and let $n \to \infty$, and we get a spectral sequence for the first row of the following diagram of Hopf algebras

$$\begin{tikzcd}
\Lambda(\mathfrak{g})_{\mathfrak{g}} \arrow[r] \arrow[d, hook] & \Lambda(\mathfrak{g}\otimes A)_{\mathfrak{g}} \arrow[r] \arrow[d, equal] & \Lambda(\mathfrak{g}\otimes A/\mathfrak{g})_{\mathfrak{g}} \arrow[d, equal] \\
\tilde{S}[K] \arrow[r] & \tilde{S}[C(A)] \arrow[r] & \tilde{S}[C(A,k)]
\end{tikzcd}$$

In the bottom row $K = \mathrm{Ker}\{C(A) \to C(A,k)\}$ and $\tilde{S}$ = symmetric algebra in the graded anti-commutative sense. It is easy to see by choosing a section of $C(A) \to C(A,k)$ that there is a corresponding spectral sequence for the bottom row. The comparison theorem for spectral sequences shows the left vertical inclusion is a quasi-isomorphism, proving the assertion.

For example take a ring of dual numbers $A = k + I$, $I^2 = 0$. Then $C(A,k)$ has zero differentials, so one gets

$$HC_p(A) = \left(I^{\otimes p}/\sim\right) \oplus \begin{cases} k & p \text{ odd} \\ 0 & p \text{ even} \end{cases}$$

This formula can also be proved using Connes exact sequence.

This example shows that the relative Connes complex $C(A,k)$ is a minimum functor in some sense for computing the Lie K-groups $HC(A,k)$.

If you know another proof of the above exact sequence, say using the combinatorics that go into the proof of Connes exact sequence, I would be very interested

Best regards,
Daniel G. Quillen

Letter to Loday Jan 15 – Summary.

If W is a repn. of a finite gp e.g. Σ_n then the transpose of $k[\Sigma_n] \longrightarrow \mathrm{End}(W)$

is the map

$$\begin{array}{ccc} \mathrm{End}(W) = \mathrm{End}(W)^* & \longrightarrow & k[\Sigma_n]^* \\ A & \longmapsto & \left(\sigma \mapsto \mathrm{Tr}(\rho_W(\sigma)A)\right) \end{array}$$

whose image is the set of "representative" functions.

It seems unlikely that ~~his~~ filtration $F_r k[\Sigma_n]^*$ $=$ Image from $\mathrm{End}(V_r^{\otimes n})$ had a close relation to the periodicity filtration on HC_n. Example: Relative group $HC_n(A,k)$ when $A = k \oplus I$, $I^2 = 0$. Then the periodicity filtration is trivial, since the groups $HC_n(A,k)$ $= I^{\otimes n}/\sim$ have different degrees as functors of I. But the filtration coming from the $gl_r \subset gl$ is non-trivial, the top quotient being $I^{\otimes n}/\sim \twoheadrightarrow \Lambda^n I$.

January 16, 1983

To understand Witten's business about the operator $d + s\, i(X)$. Especially I want to connect up the analogy, pointed out by Bott, of Connes complex with the free loop space. Picture:

$$\begin{tikzcd}
\mathrm{Loop}(M) \arrow[rr, "\text{Witten' version}", "\text{of Duistermaat}"'] \arrow[dr, dash, "\text{Bott}"'] & & \text{Index thm} \\
& (\text{Connes theory}) \arrow[ur, dash, "\text{Connes}"'] &
\end{tikzcd}$$

[illegible]

Let S = circle act on M (possibly infinite-dimensional) with fixpoint set N, and let X denote the corresponding vector field on M. One defines the equivariant cohomology

$$H_S^*(M) = H^*(PS \times^S M).$$

There are two ways to calculate this. The first is by the Gysin sequence of the line bundle obtained from pulling back the standard line bundle $\mathcal{O}(1)$ on $BS = \mathbb{C}P^\infty$.

Notation: $$H_S^* = H_S^*(pt) = H^*(BS) = k[u] \qquad u = c_1(\mathcal{O}(1)).$$

The Gysin sequence is

$$\longrightarrow H^{n-1}(M) \xrightarrow{\delta} H_S^{n-2}(M) \xrightarrow{u} H_S^n(M) \longrightarrow H^n(M) \longrightarrow$$

~~and we have used the fact that PS = total space of the unit circle bundle in O(1), hence the circle bundle~~ [illegible] ~~over $PS\times^S M$~~ I should have said this is the Gysin sequence in equivariant cohomology for the equivariant complex line bundle over M one gets by pulling back the equivariant line bundle over a pt given by the tautological character of S. Then the sphere bundle is $S \times M$ and we have used

$$H_S^*(S\times M) = H^*(M).$$

The other way of calculating the equivariant cohomology of M is via the spectral sequence of the fibration

$$M \longrightarrow PS\times^S M \longrightarrow BS$$

which looks like

$$E_2^{pq} = \underset{\begin{cases} k\cdot u^{\frac{p}{2}} & p \text{ even} \\ 0 & p \text{ odd}\end{cases}}{\overset{}{H_S^p}} \otimes H^q(M) \Longrightarrow H_S^{p+q}(M).$$

Related to this picture is what you get from the complex of equivariant differential forms. Let $\Omega^\cdot(M)$ be the de Rham complex of M. The circle action on M is realized on $\Omega^\cdot(M)$ by the operators $i(X)$, $\theta(X)$. One knows that the inclusion of the invariant differential forms

$$\Omega^\cdot(M)^S \subset \Omega^\cdot(M)$$

is a quis; this is because the group S is ~~compact~~ connected, hence acts trivially on the cohomology of M, hence only the identity representation appears the cohomology, and because S is compact and one can decompose $\Omega^\cdot(M)$ according to the irreducible repns. of S. Because X is S-invariant $i(X)$ acts on $\Omega^\cdot(M)^S$.

Now the equivariant DR complex is defined to be

$$\left[\Omega^\cdot(M) \otimes W(\mathfrak{g})\right]_{\text{basic}}$$

For S we have $W(\mathfrak{g}) = \Lambda[\theta] \otimes k[u]$ where $d\theta = u$, $i(X)\theta = 0$. When we take the forms in

$$\Omega^\cdot(M) \otimes \Lambda[\theta] \otimes S[u]$$

which are killed by $i(X)$ we get something isom. to

$$\Omega^\cdot(M) \otimes S[u].$$

Precisely a form in $\Omega(M) \otimes \Lambda[\theta] \otimes k[u]$ can be written uniquely

$$\sum_p \left(\theta \omega'_p + \omega_p\right) u^p .$$

It is killed by $i(X)$ $\iff$ $\omega'_p + i(X)\omega_p = 0$, hence the horizontal complex consists of forms

$$\sum_p \left[\omega_p - \theta\, i(X)\omega_p\right] u^p .$$

Next take S-invariant forms so that $\theta(X)\omega_p = 0$

One has $d(\omega - \theta\, i(X)\omega] = d\omega - d\theta\; i(X)\omega + \theta\, \underbrace{d\, i(X)}_{\theta(X) - i(X) d}\omega$

$= [1 - \theta\, i(X)]\, d\omega - u\, i(X)\,\omega$

Therefore we conclude that

$$\boxed{\begin{array}{l} \left[\Omega(M) \otimes W(S)\right]_{basic} \cong \Omega(M)^S[u] \\ d\omega = d_M\omega - u\, i(X)\,\omega \ , \quad du = 0 \end{array}}$$

with

Notationally we should write

$$W(g) \otimes \Omega(M) = S(g^*) \otimes \Lambda(g^*) \otimes \Omega(M)$$

and then a form is

$$\sum_p u^p(\omega_p + \theta \omega'_p)$$

and $i(X)[\qquad\qquad] = \sum_p u^p\big(i(X)\omega_p + \omega'_p - \theta\, i(X)\omega'_p\big)$

so that $i(X)[\qquad] = 0 \iff i(X)\omega_p + \omega'_p = i(X)\omega'_p = 0$

$\iff -i(X)\omega_p = \omega''_p$.

So now we reach the following algebraic situation. θ suppose given a complex K and a degree -1 endomorphism $i(X)$ of it such that

$d\,i(X) + i(X)\,d = 0$ and $i(X)^2 = 0$. For example $K = \Omega(M)^S$ in the case of a circle action. Then we can form the complex $K[u] = k[u] \otimes K$, with the diffl.

$$d = 1\otimes d_K + u\otimes i(X). \qquad \text{or simply } \boxed{d = d_K + u\,i(X)}$$

(Note that I have changed the sign of u.) Let's check that we get the algebraic properties of equivariant cohomology.

$$d^2 = (1\otimes d_K)^2 + (1\otimes d_K)(u\otimes i(X)) + (u\otimes i(X))(1\otimes d_K) + (u\otimes i(X))^2$$

$$= \cancel{d_K^2} + u\,[d\,\cancel{i(X) + i(X)\,d}] + u^2\,\cancel{i(X)^2} = 0$$

Also we have an exact sequence of complexes

$$0 \longrightarrow \Sigma^{-2}K[u] \xrightarrow{u} K[u] \longrightarrow K \longrightarrow 0$$

which leads to a Gysin sequence. Finally it's clear that we can filter $K[u]$ by the $u^p K[u]$ and we get the spectral sequence in equivariant cohomology described above.

January 18, 1983

I want to understand Connes cocycles, which he constructs using an operator F of square 1. What I want to do is to go directly from an analytical situation to an invariant differential form on the gauge group.

The situation is the following. Given a vector bundle E over M one gets a Hilbert space which is a module for $A = C^\infty(M)$ by taking the L^2 sections of the vector bundle. Call this Hilbert space $\mathcal{H}$. Also the gauge group $\mathcal{G}$ of E acts as unitary operators on $\mathcal{H}$. Now suppose one takes a projection operator e in $\mathcal{H}$. Then we get a $\mathcal{G}$-map, $g \mapsto geg^{-1}$ from $\mathcal{G}$ into a Grassmannian. On the Grassmannian are the invariant differential forms which are the character of the subbundle for the canonical connection. These are

$$\mathrm{Tr}\left(e(de)^{2p+1}\right)$$

and the trace will be defined for p large when pulled back to $\mathcal{G}$.

But now that I have invariant closed differential forms on $\mathcal{G}$, I ~~can~~ use the Loday-Quillen theorem to express these forms in terms of Connes cocycles. This involves making the formulas work.

Let's review the formulas: According to the thm on invariants for GL_n the invariant maps

$$\mathfrak{g}^{\otimes p} = (V\otimes V^*)\otimes(V\otimes V^*)\otimes\cdots\otimes(V\otimes V^*) \longrightarrow k$$

are linear combinations of the maps given by contracting according to a permutation $\sigma \in \Sigma_p$. Thus corresp. to the permutation $\sigma(1)=2 \quad \sigma(2)=1 \quad \sigma(3)=3$, one has the contraction scheme

$$(V\otimes V^*)\otimes(V\otimes V^*)\otimes \underbrace{(V\otimes V^*)}.$$

January 19, 1983

Connes basic examples of cocycles. The odd case: [redacted] This is where we have a Hilbert space H which is a module over $C^\infty(M) = A$ and a projector e in H such that $[a,e]$ is compact for all $a \in A$. Then for any n [redacted] we consider the projector $e^{\oplus n}$ in $H^{\oplus n}$. $H^{\oplus n}$ is a module for $GL_n(A) = \mathcal{G}$. Then we get a map

$$\mathcal{G} \longrightarrow \text{projectors in } H^{\oplus n}$$
$$g \longmapsto g e^{\oplus n} g^{-1}$$

On the space of projectors we have the character forms which are invariant for conjugation, so if we pull-back these to $\mathcal{G}$ we get left-invariant differential forms which are closed on $\mathcal{G}$.

I recall that [redacted] on the space of projectors, the subbundle has a canonical connection. Think of a vector bundle E over a manifold with an embedding [redacted] as a direct summand of a trivial vector bundle $E \underset{i}{\overset{\pi}{\rightleftarrows}} 1^{\oplus r}$. Then we have the connection $\nabla = \pi d i$ on E. The curvature is

$$\begin{aligned}\nabla^2 &= \pi d i \pi d i = \pi e d e d e i = \pi e (de) de i \\ &= \pi (de)(1-e) de i = \pi (de)(1-e)(de) i \\ &= \pi e (de)(de) i = \pi (de)^2 i\end{aligned}$$

which leads to the character forms

$$\begin{aligned}tr(\nabla^{2p}) &= tr((\pi \, [redacted] dede \, i)^{2p}) \\ &= tr(e de de)^{2p} = tr(e(de)^{2p}).\end{aligned}$$

$\times \frac{1}{p!}$

For reasons which will become clear I want to write these forms in terms of the "polarization" F associated to e. $F = +1$ on $\operatorname{Im} e$, -1 on $\operatorname{Im}(1-e)$.

$$F = e - (1-e) = 2e - 1. \qquad e = \tfrac{1}{2}(F+1)$$

Now cyclic $\mathrm{tr}\,(de)^{2p} = 0$ by symmetry of the trace.
Hence

$$\mathrm{tr}\left(e(de)^{2p}\right) = \mathrm{tr}\left(\tfrac{1}{2}(F+1)\left(d\,\tfrac{1}{2}(F+1)\right)^{2p}\right)$$
$$= 2^{-p-1}\,\mathrm{tr}\left(F\,(dF)^{2p}\right)$$

which gives

$$\boxed{ch_p\,(\nabla) = \frac{1}{p!}\,\frac{1}{2^{p+1}}\,\mathrm{tr}\left(F(dF)^{2p}\right)}$$

Now let's return to the situation of the orbit of the projector $e^{\oplus n}$ under $G = GL_n(A)$. If we pull back the form ch_p on the space of projectors in $H^{\oplus n}$ we get a left-invariant form on G. This can be identified with the skew-symmetric form on the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{gl}_n(A)$ obtained by looking at the value of ch_p on the tangent space to the G-orbit at the point $e^{\oplus n}$. If $X \in \tilde{\mathfrak{g}}$, then the tangent vector to the space of polarizations at the point $F_n = F^{\oplus n}$ is the operator $[X, F_n]$. Hence the pull back of ch_p is

$$X^1, \ldots, X^{2p} \longmapsto \frac{1}{2^{p+1}} \text{ skew-symm. of } \mathrm{tr}\left(F_n[X^1, F_n]\cdots[X^{2p}, F_n]\right)$$

Now $[X, F_n]$ is the operator on $H^{\oplus n}$ with entries $[x_{ij}, F]$. Also

$$\mathrm{tr}\left(F[a^1, F]\cdots[a^{2p}, F]\right)$$

is cyclic skew-symmetric, since $0 = [a, F^2] = [a,F]F + F[a,F]$. Thus it is clear that the above $2p$-form on $\mathfrak{gl}_n(A)$ is the one associated to the Connes cochain:

$$\frac{1}{2^{p+1}}\,\mathrm{tr}\left(F[a^1, F]\cdots[a^{2p}, F]\right)$$

Next let's consider the even case: Here we are given two Hilbert spaces H_+, H_- which are A-modules and an invertible operator $P : H_+ \longrightarrow H_-$ such that

$[a,P]$ is compact for all $a \in A$. Then we form $P_n : H_+^n \longrightarrow H_-^n$ where $\mathcal{G} = GL_n A$ acts on $H_\pm^n$. and we can consider the map

$$\begin{aligned} \mathcal{G} &\longrightarrow \operatorname{Isom}(H_+^n, H_-^n) \\ g &\longmapsto g_- P_n g_+^{-1} \end{aligned}$$

where I have written g_+ and g_- the effects of g on the spases $H_\pm^n$ resp. Now we can take a biinvariant differential form on the space of Isomorphisms e.g.

$$\operatorname{tr} (Q^{-1} dQ)^{2p-1} ,$$

and pull-back by the above map to get a left-invariant differential form on $\mathcal{G}$.

Let's calculate the form we get on $\tilde{\mathcal{g}} = gl_n(A)$ by restricting $\operatorname{tr}(Q^{-1}dQ)^{2p-1}$ on $\operatorname{Isom}(H_+^n, H_-^n)$ to the tangent vectors at $Q = P_n$ obtained from elements of $\tilde{\mathcal{g}}$. The tangent vector at P_n belonging to X is $[X, P_n]$, or more accurately $X_- P_n - P_n X_+$. So at the moment I have the skew-$(2p-1)$ form on $gl_n(A)$ given by

$$X^1, \dots, X^{2p-1} \longmapsto \text{Skew-symm of } \operatorname{tr}_{H_+^n}\left(P_n^{-1}[X^1, P_n] \cdots P_n^{-1}[X^{2p-1}, P_n]\right).$$

I know this has to be given by a Connes cochain; which one? It is obviously

$$\operatorname{tr}_{H_+}\left(P^{-1}[a^1, P] \cdots P^{-1}[a^{2p-1}, P]\right)$$

since this is cyclic skew-symmetric. So now comes the question of comparing it with Connes formula

$$\operatorname{tr}\left(\varepsilon F [a^1, F] \cdots [a^{2p-1}, F]\right)$$

where $\varepsilon = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $F = \begin{pmatrix} 0 & P^{-1} \\ P & 0 \end{pmatrix}$.

Note that $0 = [a, P^{-1}]P + P^{-1}[a, P]$ so that

~~[illegible]~~

$$[a, P^{-1}] = -P^{-1}[a,P]\,P^{-1}$$

which is just the familiar formula for derivatives of X^{-1}. Then

$$tr\left(\varepsilon F[a^1, F]\cdots[a^{2p-1}, F]\right) = tr\left(\begin{pmatrix}1 & \\ & -1\end{pmatrix}\begin{pmatrix}0 & P^{-1}\\ P & 0\end{pmatrix}\begin{pmatrix}0 & [a^1, P^{-1}]\\ [a^1, P] & 0\end{pmatrix}\cdots\cdots\right)$$

$$= tr\left(P^{-1}[a^1, P][a^2, P^{-1}]\cdots\cdots[a^{2p-1}, P]\right)$$

$$-tr\left(P\,[a^1, P^{-1}][a^2, P]\cdots\cdots[a^{2p-1}, P^{-1}]\right)$$

$$= tr\left(P^{-1}[a^1, P]\cdot P^{-1}[a^2, P]\cdot P^{-1}[a^{2p-1}, P]\right)(-1)^{p-1}$$

$$-tr\left(P\;P^{-1}[a^1, P]P^{-1}[a^2, P]\cdots\cdots P^{-1}[a^{2p-1}, P]\;P^{-1}\right)(-1)^{p}$$

$$= 2(-1)^{p-1}\;tr\left(P^{-1}[a^1, P]\cdots\cdots P^{-1}[a^{2p-1}, P]\right)$$

Notice that the way a Connes cochain extends to matrices is as follows. Given $f(a^1, .., a^n)$, one extends it to matrices by

$$\tilde{f}(X^1, .., X^n) = \sum{}' f(x^1_{i_1 i_2}, x^2_{i_2 i_3}, \ldots, x^n_{i_n i_1}).$$

Then if f is cyclically skew-symmetric so is $\tilde{f}$. Then one anti-symmetrizes $\tilde{f}$ so as to get the desired skew-form on matrices.

<u>Example</u>: $f(a^1, .., a^n) = \underbrace{a^1 \ldots a^n.}_{n \text{ odd}}$ Then

$$\tilde{f}(X^1, .., X^n) = tr(X^1 .. X^n)$$

and the skew form is

$$\frac{1}{n!}\sum(-1)^{\sigma} tr\left(X^{\sigma(1)}\ldots X^{\sigma(n)}\right).$$

Idea for tomorrow: Try to deal with self-adjoint Fredholm operators as linear combinations of projection operators. Put another way, the orbits of self-adjoint operators are [redacted] split flag manifolds where the quotients have numbers attached.

Lie theory versus the discrete group. There should be fibration sequences

$$\begin{tikzcd} G^\delta \arrow[r] & G \arrow[r] \arrow[d, equal] & G/\!/G^\delta \arrow[r] \arrow[d] & BG^\delta \arrow[r] \arrow[d] & BG \arrow[d, equal] \\ & G \arrow[r] & \mathfrak{g} \arrow[r] & B_{cc}G \arrow[r] & BG \end{tikzcd}$$

where $G/\!/G^\delta$ denotes the homotopy gadget $(PG^\delta \times G)/G^\delta$ = homotopy fibre of $BG^\delta \longrightarrow BG$. Here $B_{cc}G$ is some sort of continuous cochain gadget.

Specific example: $G = GL_n(\mathbb{C})$. Then

$G/\!/G^\delta$ classifies flat vector bundles of dim n with C^∞ trivialization,

and so we should see how a cohomology class of the Lie algebra gives a characteristic class for such things. This is clear: Given the principal bundle P, the flat connection gives us a map of complexes (really Diff algs.)

$$\Lambda^\bullet \mathfrak{g}^* \longrightarrow \Omega^\bullet(P),$$

but then the trivialization gives a section of P whence a map $\Omega^\bullet(P) \longrightarrow \Omega^\bullet(M)$.

So can we apply this to deduce anything about the K-theory of $\mathbb{C}$? It seems not because $G \longrightarrow \mathfrak{g}$ induces an isomorphism on cohomology (real coeffs.), and so $B_{cc}G = 0$. Thus you get

$$G/\!/G^\delta \sim G \times BG^\delta$$

and the classes obtained from $\mathfrak{g}$ live on the first factor. Put another way the flat connection gives an effective trivialization of the bundle over $\mathbb{C}$, so comparing it with the given trivialization gives you a map to G.

January 20, 1983

Tomorrow's lecture will be on determinant line bundles. So what do I say?

Let $D : \Gamma(E) \longrightarrow \Gamma(F)$ be an elliptic over a compact manifold. It has as canonical line

$$\mathcal{L}_D = \Lambda^{max}(\operatorname{Ker} D)^* \otimes \Lambda^{max}(\operatorname{Cok} D)$$

which we call the determinant line.

To discuss this algebraically first. Suppose we have $A : W \longrightarrow V$ a linear map between vector spaces such that $\operatorname{Ker} A$, $\operatorname{Cok} A$ are finite dimensional, say of dimensions p, q respectively. Can define a canonical embedding

$$\mathcal{L}_A^* = \Lambda^{max}(\operatorname{Cok} A)^* \otimes \Lambda^{max}(\operatorname{Ker} A) \hookrightarrow \operatorname{Hom}^{(p-q)}(\Lambda V, \Lambda W).$$

Idea is to choose F finite-diml $\subset V$ transversal to $\operatorname{Im}(A)$.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} A \arrow[r] & W \arrow[r] & V \arrow[r] & \operatorname{Cok} A \arrow[r] & 0 \\
 & \| & \cup & \cup & \| & \\
0 \arrow[r] & \operatorname{Ker} A \arrow[r] & A^{-1}F \arrow[r] & F \arrow[r] & \operatorname{Cok} A \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A^{-1}F \arrow[r] & F \oplus W \arrow[r] & V \arrow[r] & 0
\end{tikzcd}$$

$$\Lambda^{max}(A^{-1}F) \otimes \Lambda\underbrace{\left(F \oplus W / A^{-1}F\right)}_{V} \hookrightarrow \Lambda(F \oplus W) \twoheadrightarrow \Lambda^{max} F \otimes \Lambda(W)$$

$$(\Lambda^{max} F)^* \otimes \Lambda^{max}(A^{-1}F) \longrightarrow \operatorname{Hom}(\Lambda V, \Lambda W)$$

This doesn't show injectivity.

Better explanation: surjection $\overset{p}{K} \to W \longrightarrow \bar{W}$.

$$\Lambda^p(K) \otimes \Lambda^n \bar{W} \hookrightarrow \Lambda^{p+n} W$$

injection: $\bar{W} \hookrightarrow V$ with $q = \dim V/\bar{W}$.

$$\Lambda^n V \twoheadrightarrow \Lambda^q(V/\bar{W}) \otimes \Lambda^{n-q}\bar{W}.$$

Clearer seems to be the map

$$\Lambda^q(V/\bar{W})^* \otimes \Lambda^n V \twoheadrightarrow \Lambda^{n-q}\bar{W}$$

which one knows because if $0 \neq \lambda \in V^*$, then $i(\lambda)$ on ΛV can be viewed as mapping ΛV onto $\Lambda(\mathrm{Ker}\,\lambda)$.

So now you can combine to get the map

$$\Lambda^p(K) \otimes \Lambda^q(V/W)^* \otimes \Lambda^n V \twoheadrightarrow \Lambda^p K \otimes \Lambda^{n-q}\bar{W} \hookrightarrow \Lambda^{n+p-q} W$$

which gives us the embedding

$$\Lambda^p \mathrm{Ker}\,A \otimes \Lambda^q(\mathrm{Cok}\,A)^* \hookrightarrow \mathrm{Hom}^{(p-q)}(\Lambda V, \Lambda W)$$

The formula is to pick a basis $k_1 \cdots k_p$ for Ker A and $\gamma_1 \cdots \gamma_q$ for (Cok A)* and then the map is given by

$$\omega^n \longmapsto e(k_1)\cdots e(k_p)\Lambda^{n-q}(A^{-1})\big(i(\gamma_q)\cdots i(\gamma_1)\omega\big)$$

Unfortunately this looks ugly when you want to vary A.

So let us suppose that we have defined the embedding of $\mathcal{L}_A^*$ ~~into~~ Hom$(\Lambda V, \Lambda W)$. Then you get to define elements of $\mathcal{L}_A$ associated to elements of $(\Lambda W)^* \otimes \Lambda V$. So for example a finite dim'l subspace F of V and a finite codim subspace $W_1 \subset W$ will give us the divisor of those A such that

$$W_1 \subset W \xrightarrow{A} V \longrightarrow V/F$$

is not invertible.

First lecture: Topics:

[illegible] definition of determinant line bundle, construction of sections of it, especially the canonical section when the index is zero. Holomorphic structure.

Possible goal: To describe the ratio of sections of the determinant line bundle as a finite-dimensional determinant. (In practice when the canonical section vanishes you want to work with a non-vanishing section.) Order of zero of the canonical section.

[illegible] Determinant line bundle over Fredholm correspondences $T: W \longrightarrow V$ and the embedding in $\mathrm{Hom}^{(uid)}(\Lambda V, \Lambda W)$. Relation with the line bundle on the infinite Grassmannian of subspaces W quasi-complementary to H. Problem of the Fock line

January 21, 1983

Prepare lecture: I will limit the first lecture to defining det(D) locally for D invertible. The method is use the formula

$$d \log \det A = \mathrm{Tr}(A^{-1}dA).$$

In other words I will define a (1,0)-form on the space of invertible D by regularizing the formal expression $\mathrm{Tr}(D^{-1}dD)$. Then I will show the form is closed which defines locally an analytic function $\log\det(D)$.

I need to go over as an example the case of line bundles of degree 0 over an elliptic curve $M = \mathbb{C}/\Gamma$. Here E is the trivial line bundle $\mathbb{1}$ and I look at $\bar\partial$ operators of the form

$$\bar\partial + A : \Omega^{0,0} \longrightarrow \Omega^{0,1}$$

where A is a constant (0,1)-form:

$$A = -\omega\, d\bar z.$$

Thus $D = (\partial_{\bar z} - \omega)\, d\bar z$, and I can identify $\Omega^{0,1} = \Omega^{0,0} = C^\infty(M)$ whence $D = \partial_{\bar z} - \omega$. Natural basis of $C^\infty(M)$ given by characters

$$e^{\mu\bar z - \bar\mu z} \qquad \mu \in \underbrace{\{\mu\in\mathbb{C} \mid \mu\bar\gamma - \bar\mu\gamma \in 2\pi i\mathbb{Z} \text{ for all } \gamma\in\Gamma\}}_{\text{call this } \Gamma^*}$$

since

$$(\partial_{\bar z} - \omega)\, e^{\mu\bar z - \bar\mu z} = (\mu - \omega)\, e^{\mu\bar z - \bar\mu z}$$

D is invertible $\Longleftrightarrow$ $\omega \notin \Gamma^*$.

Formally corresponding to $\delta D = -\delta\omega$

$$\mathrm{Tr}(D^{-1}\delta D) = \sum_{\mu\in\Gamma^*}{}' \frac{+1}{\mu-\omega}(-\delta\omega)$$

which doesn't converge.

What is the Green's function for $\partial_{\bar z} - \omega$?

Call it $G(z-z')$. It must satisfy

$$(\partial_{\bar z} - \omega)\, G(z-z') = \delta(z-z').$$

Hence $\pi G(z) = e^{\omega \bar z}\left\{\frac{1}{z} + \text{holom}\right\}$ near 0, and it must be Γ-periodic.

Weierstrass functions:

$$\sigma(z) = z \prod_{\gamma\in\Gamma}\left(1-\frac{z}{\gamma}\right) e^{\frac{z}{\gamma}+\frac{z^2}{2\gamma^2}}$$

$$\zeta(z) = \frac{d}{dz}\log\sigma(z) = \frac{1}{z} + \sum_{\gamma\in\Gamma}{}'\left\{\frac{1}{z-\gamma}+\frac{1}{\gamma}+\frac{z}{\gamma^2}\right\}$$

$$\wp(z) = \zeta'(z) = \frac{1}{z^2} + \sum{}'\left(\frac{1}{(z-\gamma)^2}-\frac{1}{\gamma^2}\right)$$

$\wp$ is Γ-periodic, but ζ is not

$$\zeta(z+\gamma)-\zeta(z) = \mathbb{R}\text{-linear fn. of } \gamma = a_\gamma$$
$$= l\gamma + m\bar\gamma$$

$$\frac{\sigma(z+\gamma)}{\sigma(z)} = e^{a_\gamma z + b_\gamma}$$

$$\pi G(z) = \frac{\sigma(z+\alpha)}{\sigma(z)}\, e^{\omega\bar z + \beta z}$$

For $\pi G(z)$ to be periodic means

$$a_\gamma(\alpha) + \omega\bar\gamma + \beta\gamma = 0$$

$$\left.\begin{aligned} l\alpha + \beta &= 0 \\ m\alpha + \omega &= 0 \end{aligned}\right| \implies \alpha = -\frac{\omega}{m} \qquad \beta = \frac{l\omega}{m}$$

$$\boxed{\pi G(z) = e^{\omega(\bar z + \frac{l}{m}z)}\, \frac{\sigma(z-\frac{\omega}{m})}{\sigma(-\frac{\omega}{m})\,\sigma(z)}}$$

Now for the parametrix

$$\pi G_b(z) = F(z)\left(-\partial_{z'} \log|z-z'|^2\right)_{z'=0}$$

where $F(z)$ is parallel translation for the connection $\nabla = \partial_z dz + (\partial_{\bar{z}} - \omega) d\bar{z}$.

$$\therefore \quad F(z) = e^{\omega\bar{z}} \qquad \text{near} \quad z = 0$$

$$\therefore \quad \boxed{\pi G_b(z) = \frac{e^{\omega\bar{z}}}{z}}.$$

$$\therefore \quad \pi G(z) - \pi G_b(z) = \left[1 + \omega\left(\bar{z} + \frac{\ell}{m}z\right) + \cdots\right] \frac{\cancel{\sigma(-\frac{\omega}{m})}\left[1 + z\,\zeta(-\frac{\omega}{m})\right]}{\cancel{\sigma(-\frac{\omega}{m})}\, z}$$

$$- \left[1 + \omega\bar{z} + \cdots\right]\frac{1}{z}$$

$$= \left(1 + \omega\bar{z} + \cdots\right)\frac{1}{z}\left\{\cancel{1} + \omega\frac{\ell}{m}z + z\,\zeta\left(-\frac{\omega}{m}\right) + \cdots \cancel{-1}\right\}$$

$$\longrightarrow \quad \zeta\left(-\frac{\omega}{m}\right) + \omega\frac{\ell}{m}$$

So the result of regularizing G for $\partial_z - \omega$ is the function

$$J(z) = \text{F.P. } G(z,z) = \frac{1}{\pi}\left[\zeta\left(-\frac{\omega}{m}\right) + \ell\frac{\omega}{m}\right]$$

Hence

$$\delta \log\det(\partial_z - \omega) = \operatorname{Tr}_{\text{reg}}\left((\partial_z - \omega)^{-1}(-\delta\omega)\right)$$

$$= \int_{\mathbb{C}/\Gamma} J(z)(-\delta\omega) = \frac{\operatorname{vol}(\mathbb{C}/\Gamma)}{\pi}\left[\zeta_\Gamma\left(\frac{\omega}{m}\right) - \ell_\Gamma\frac{\omega}{m}\right]_\wedge^{\delta\omega}$$

Next I need to know that

$$m_\Gamma = \left[s\sum{}' \frac{1}{|\gamma|^2|\gamma|^{2s}}\right]\Bigg|_{s=0} = \frac{\pi}{\operatorname{vol}(\mathbb{C}/\Gamma)}$$

$$\ell_\Gamma = \sum{}' \frac{1}{\gamma^2|\gamma|^{2s}}\Bigg|_{s=0}$$

Also $\Gamma^* = m\Gamma$. $\quad \ell_{\Gamma^*} = \frac{1}{m^2}\ell_\Gamma$

$$\delta \log\det(\partial_{\bar z} - w) = \frac{1}{m}\zeta_\Gamma\left(\frac{w}{m}\right) - \frac{1}{m^2}\ell_\Gamma w$$

$$= \zeta_{\Gamma^*}(w) - \ell_{\Gamma^*} w$$

and so

$$\boxed{\det(\partial_{\bar z} - w) = e^{-\ell_{\Gamma^*} w^2/2}\,\sigma_{\Gamma^*}(w)}$$

see p. 649
May 31, 1982

Local picture for $\bar\partial$-operators:

$$\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y) \qquad \text{kills } z = x+iy$$

If the bundle E is trivialized, then

$$Df = (\partial_{\bar z} + \alpha) f\, d\bar z$$

where α is a matrix function. Locally we can always trivialize by a holomorphic frame, then $\alpha = 0$. In general can solve locally with an invertible matrix g the equation

$$(\partial_{\bar z} + \alpha) g = 0 \qquad \text{or} \qquad \alpha = -(\partial_{\bar z} g) g^{-1}$$
$$= g \cdot \partial_{\bar z}(g^{-1})$$

What does a parametrix for

$$D : \Gamma(E) \longrightarrow \Gamma(E \otimes T^{0,1})$$

look like? It is an operator G in the other direction such that DG and $GD \equiv Id$ mod smooth-kernel operators. The Schwarz kernel of G is a distribution-section of the bundle ~~$\mathrm{Hom}(pr_2^* E\otimes T^{0,1}, pr_1^* E)$~~

~~[illegible]~~ $\quad pr_1^* E \otimes pr_2^*(E^* \otimes T^{1,0}_M)$ over $M \times M$

$$G(z, z') \in E_z \otimes E^*_{z'} \otimes T^{1,0}_{z'}$$

This is too involved. Better to understand $\bar\partial : \Omega^{0,0} \longrightarrow \Omega^{0,1}$ first.

$\Delta = 4 \cdot \partial_{\bar{z}} \partial_z$ has fundamental solution $\frac{1}{2\pi} \log r$

$$\Delta \frac{1}{2\pi} \log r = \delta(z) \quad \text{in } \mathbb{C}$$

$$\partial_{\bar{z}} \underbrace{\partial_z 4 \frac{1}{4\pi} \log |z|^2}_{\frac{1}{\pi z}} = \delta(z)$$

well-defined because it is loc. L^1

So an inverse for $\partial_{\bar{z}}$ over $\mathbb{C}$ is

$$f \in C_0^\infty(\mathbb{C}) \longmapsto (G_0 f)(z) = \int \frac{1}{\pi} \frac{1}{z - z'} f(z') \frac{i}{2} dz' d\bar{z}'$$

and an inverse for $\bar{\partial}$ would be

$$a \, d\bar{z} \in C_0^\infty(T^{0,1}) \longmapsto (G_0 a \, d\bar{z})(z) = \int \underbrace{\frac{i}{2\pi} \frac{1}{z - z'} dz'}_{G_0(z,z')} \cdot a(z') d\bar{z}'$$

But now suppose that $G(z, z') =$ ~~$\frac{i}{2\pi} \frac{1}{z - z'} F(z, z')$~~

satisfies

$$\bar{\partial}_z G(z, z') = \delta(z - z')$$

January 23, 1983

Now I have to consider the determinant line bundle and the singular set.

Let's consider the space $\mathcal{F}$ of Fredholm operators $T : W \longrightarrow V$. How do we define the determinant line bundle over $\mathcal{F}$? The fibre at T is the line

$$\mathcal{L}_T = \lambda(\mathrm{Cok}\,T) \otimes \lambda(\mathrm{Ker}\,T)^*.$$

To see these lines vary nicely with T argue as follows. For each finite-dimensional subspace $F \subset V$ we get an open set $U_F \subset \mathcal{F}$ consisting of the T transversal to F. Transversal to F means that $F \oplus W$ maps onto V, and one knows that $T^{-1}F = \mathrm{Ker}(F \oplus W \xrightarrow{(i,T)} V)$ varies nicely in T, that is, that the family of $T^{-1}(F)$ for $T \in U_F$ is the family of fibres of a vector bundle over U_F. One has also

$$0 \longrightarrow \mathrm{Ker}\,T \longrightarrow T^{-1}F \longrightarrow F \longrightarrow \mathrm{Cok}\,T \longrightarrow 0$$

and hence a canonical isom.

$$\lambda(\mathrm{Cok}\,T) \otimes \lambda(\mathrm{Ker}\,T)^* = \lambda(F) \otimes \lambda(T^{-1}F)^*$$

which makes clear that $\mathcal{L}$ over U_F is essentially the highest exterior power of this vector bundle.

Dually I can consider a finite codim'l subspace $W' \subset W$ and the open set $U_{W'}$ consisting of all T such that $W' \cap \mathrm{Ker}\,T = 0$. Over $U_{W'}$ we have an injection $W \xrightarrow{(\pi, T)} W/W' \oplus V$ whose cokernel, which is isom. to V/TW', varies nicely in T.

Better language: Over U_F we have a family of Fredholm operators which are surjective:

$$F \oplus W \longrightarrow V$$

and hence one knows that the family of kernels $T^{-1}F$ forms a vector bundle over U_F. Similarly over $U_{W'}$, we have a family of injective Fred. ops.

$$W \longrightarrow W/W' \oplus V$$

and hence one knows that the family of cokernels $V/T(W')$ forms a vector bundle over $U_{W'}$. From the exact sequence

$$0 \longrightarrow \operatorname{Ker} T \longrightarrow W/W' \longrightarrow V/T(W') \longrightarrow \operatorname{Cok} T \longrightarrow 0$$

one gets a canon. isom.

$$\lambda(\operatorname{Cok} T) \otimes \lambda(\operatorname{Ker} T)^* = \lambda\big(V/T(W')\big) \otimes \lambda(W/W')^*$$

The space of Fredholm operators is an open subset of the Banach space of all bounded operators from W to V, so it has an obvious holomorphic structure. The above vector bundles with fibres $T^{-1}(F)$, $V/T(W')$ are holomorphic.

A better, more elementary, approach seems to be possible as follows. Given W' and F as above suppose that [illegible] the index of T is such that $\dim(T^{-1}F) = \dim(W/W')$. So

$$\operatorname{Ind}(T) = \dim(T^{-1}F) - \dim F = \dim(W/W') - \dim(F)$$

is the condition.

Start again: Given W' and F as above suppose that

$$\operatorname{Ind}(T) = \dim(W/W') - \dim(F).$$

Then for $T \in U_F$ we have $\dim(T^{-1}F) = \dim(W/W')$, and we have a map $T^{-1}F \hookrightarrow W \xrightarrow{\pi} W/W'$.

Hence we get an induced map $\lambda(W/W')^* \longrightarrow \lambda(T^{-1}F)^*$ and so a map

$$\lambda(F) \otimes \lambda(W/W')^* \longrightarrow \lambda(F) \otimes \lambda(T^{-1}F)^* = \mathcal{L}_T$$

which is non-zero $\iff$ $T^{-1}F \xrightarrow{\sim} W/W'$. Thus if I choose generators for $\lambda(F)$ and $\lambda(W/W')^*$ I get a section of $\mathcal{L}_T$ over U_F which is non-zero on the open set

$$U_{(F,W')} = \left\{ T \,\middle|\, \begin{array}{l} W = T^{-1}F \oplus W' \\ V = F \oplus TW' \end{array} \right\}.$$

(In effect, if $T \in U_F$ and $T^{-1}F \xrightarrow{\sim} W/W'$, then $W = T^{-1}F \oplus W'$ is obvious. Also ~~$F \oplus TW = V$~~ $V = F + TW = F + T(T^{-1}F + W') = F + TW'$ and so $V = F \oplus TW'$.)

<u>Conclusion</u>: Given F, W' we get the open set $U_{(F,W')}$ in the space of T of index $= \dim(W/W') - \dim(F)$ and an explicit trivialization of the determinant line bundle

$$\lambda(F) \otimes \lambda(W/W')^* \xrightarrow{\sim} \mathcal{L}_T.$$

In fact we have

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ker} T \arrow[r] & T^{-1}F \arrow[r] \arrow[d, "\wr"'] & F \arrow[r] & \operatorname{Cok} T \arrow[r] & 0 \\
 & & W/W' & & &
\end{tikzcd}$$

and so over $U_{(F,W')}$, we have a trivialization of the whole index bundle.

Next project is to compare these trivializations of $\mathcal{L}$. I will take the simplest case, where we have index 0 and compare the pair $(0,0)$, where $U_{(0,0)} =$ invertible T, with any pair (F, W') of index zero. The intersection $\mathcal{U}_{(0,0)} \cap \mathcal{U}_{(F,W')}$ consists of invertible T such that $W = T^{-1}F \oplus W'$ or equivalently $V = F \oplus TW'$.

The first thing to understand is this: Fix generators for $\lambda(F)$, $\lambda(W/W')^*$. Then we have a section of $\mathcal{L}$ over $\mathcal{U}_F$ namely

$$k \xrightarrow{\sim} \lambda(F)\otimes\lambda(W/W')^* \longrightarrow \lambda(F)\otimes\lambda(T^{-1}F)^* = \mathcal{L}_T$$

Because T is invertible we have $\mathcal{L}_T = \lambda(0)\otimes\lambda(0)^* = k$, hence we have a well-defined fn. of T for T invertible. What is it?

$$\begin{tikzcd}
W/W' & \\
W \arrow[u, "\pi"] \arrow[r, "T", "\sim"'] & V \\
T^{-1}F \arrow[u, hook] \arrow[r, "\sim"'] & F \arrow[u, "i"']
\end{tikzcd}$$

It's clear that we are comparing the volumes on F and on $T^{-1}F$ via the isom $T^{-1}F \rightarrow W/W'$. Thus the function is

$$T \longmapsto \det(\pi T^{-1} i)$$

where the determinant is defined relative to the volumes. Maybe better would be to say that we have

$$\begin{tikzcd}
\lambda(F) \arrow[r] \arrow[d, hook, "i"] & \lambda(T^{-1}F) \arrow[d, hook] & \\
\Lambda(V) \arrow[r, "T^{-1}"] & \Lambda(W) \arrow[r, "\pi"] & \lambda(W/W')
\end{tikzcd}$$

and are then taking the effect of this on the bases.

Still not very clear.

In any case given (F, W') and generators for $\lambda(F)$, $\lambda(W/W')^*$ we have computed over the set of invertible T

the section of $\mathcal{L}$ defined by this data divided by the canonical section. We find the function

$$f : T \longmapsto \det(\pi T^{-1} i).$$

Now we want to compute its differential at a point T_0.

Let us decompose

$$W = \begin{matrix} T_0^{-1}F \\ \oplus \\ W' \end{matrix} \xrightarrow{\;T_0\;} \begin{matrix} F \\ \oplus \\ T_0(W') \end{matrix} = V$$

and denote $\begin{tikzcd} F \arrow[r, shift left, "i"] & V \arrow[l, shift left, "i^*"] \end{tikzcd}$, $\begin{tikzcd} W \arrow[r, shift left, "\pi"] & W/W' \arrow[l, shift left, "\pi^*"] \end{tikzcd}$ the relevant maps relative to this decomposition. Too ugly!

$$\cancel{T_0 = \begin{pmatrix} T_0' & \\ & T_0' \end{pmatrix} = \begin{pmatrix} i^*T_0\pi & \\ & \end{pmatrix}}$$

Better is to compute the logarithmic differential of $f(T) = \det(\pi T^{-1} i)$ at a point T. Hence we want $\log f(T+\delta T) - \log f(T)$ to first order in T. Now

$$\begin{aligned} \delta \log f(T) &= \delta \log \det(\pi T^{-1} i) \\ &= \operatorname{Tr}\left((\pi T^{-1} i)^{-1} (\pi(-T^{-1}\delta T T^{-1}) i)\right). \end{aligned}$$

Let's compute this relative to the decompositions $W = T^{-1}F \oplus W'$, $V = F \oplus T(W')$. Then

$$T = \begin{pmatrix} T_1 & \\ & T_2 \end{pmatrix} \qquad \delta T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We can identify π with the projection onto $T^{-1}F$; let π^* denote the inclusion of $T^{-1}F \subset W$ and i^* the projection of V onto F. Then

$$\pi T^{-1} i = T_1^{-1} \qquad \pi T^{-1} \delta T T^{-1} i = T_1^{-1} a T_1^{-1}$$

so

$$\delta \log f(T) = -\operatorname{Tr}(a T_1^{-1}) = -\operatorname{Tr}_F(i^* \delta T \pi^* (\pi T^{-1} i))$$

$$= -\mathrm{Tr}_{W/W'}\left(\pi T^{-1} i i^* \delta T \pi^*\right).$$

Now let us go back to where we want to use this calculation.

A key point is that over the open set $\mathcal{U}_{(F,W')}$, where $\begin{cases} W = T^{-1}F \oplus W' \\ V = \quad F \oplus TW' \end{cases}$ there is defined a partial inverse $G_T : V \longrightarrow W$ uniquely characterized such that $\mathrm{Ker}\, G_T = F$ and $G_T D w = w$ for all $w \in W'$. Then

$$\mathrm{Tr}_{reg}\left(G_T \,\delta T\right)$$

is the connection form for the (F,W')-trivialization of $\mathcal{L}$ over $\mathcal{U}_{(F,W')}$.

So what I am comparing are

$$\mathrm{Tr}_{reg}\left(T^{-1}\delta T\right) - \mathrm{Tr}_{reg}\left(G_T \,\delta T\right)$$

and $\quad \mathrm{Tr}\left((\pi T^{-1} i)^{-1}(\pi\, \delta T^{-1} i)\right)$. So it becomes clear that I want to do the calculation using the block decomposition.

New idea: Motivation: Consider $M = S^1$ whence $\mathcal{G}$ = free loop group, and we have the family of operators $\frac{1}{i}\left(\frac{d}{dx} + A\right)$ on $L^2(S^1)^n$ parametrized by $\mathcal{a}$ equivariant for $\mathcal{G}$. Take the polarization J_A obtained by the splitting into positive + negative eigenvalues for the operator. Now by mapping $\mathcal{G}$ onto the orbit of A and then onto the orbit of J_A, we can then pull back the standard invariant forms on the space of polarizations. In particular we get a left invariant closed 2-form ω_A on $\mathcal{G}$ associated to any A. The problem is that the polarization J_A isn't well-defined when A has zero eigenvalues. On the other hand there is a way to defined a closed left invariant 2-form ω_A on $\mathcal{G}$ depending smoothly on A.^{1} Because $\mathcal{a}$ is a part of the dual of the Lie algebra of the central extension of the loop group, we know the $\mathcal{G}$-orbits carry natural symplectic structures. Formula:

$$\omega_A(X,Y) = \int (\underbrace{X*A}_{-X'+[X,A]}, Y)\, dt$$

which clearly doesn't jump in A.

One can see the problem even simpler on the space of self-adjoint operators on a finite-dimensional space. This is essentially the dual of $\mathrm{Lie}(U_n)$, and so each orbit or conjugacy class has a natural symplectic form. The symplectic form is not integral, and should be the average of integral classes weighted according to the eigenvalues.

<u>Question</u>: Does there exist a natural way to extend the character forms on the Grassmannian to a family of invariant forms on the orbits of the unitary group on the self-adjoint matrices?

Review the moment map. Given $f : \mathfrak{g} \to k$ we define a 2-form (skew) on $\mathfrak{g}$ by $X, Y \longmapsto f([X,Y])$. This descends to an invariant symplectic 2-form on the G-orbit of f in $\mathfrak{g}^*$.

So take $f(X) = \mathrm{tr}(HX)$ in the case of $\mathfrak{gl}_n$. Then the 2-form is

$$X, Y \longmapsto \mathrm{tr}(H[X,Y]) = \mathrm{tr}([H,X]Y).$$

Let's recall what we did in the case of projection operators E. The curvature form for the sub-bundle is $E(dE)^2$, i.e. it associates to X, Y the operator $E\big([X,E][Y,E] - [Y,E][X,E]\big)$. Then the first Chern form is $i/2\pi$ times

$$\mathrm{tr}\, E\big([X,E][Y,E] - [Y,E][X,E]\big).$$

This is complicated but I believe it is the same up to a constant as

$$\mathrm{tr}\big(F[X,F][Y,F]\big) \qquad F = 2(E - \tfrac{1}{2})$$

$$= \mathrm{tr}\big([F, F[X,F]]\,Y\big)$$

$$F[X,F] = FXF - X$$

$$[F, F[X,F]] = F(FXF - X) - (FXF - X)F = 2[F,X]$$

$$= 2\,\mathrm{tr}\big([F,X]\,Y\big)$$

Let's do this more carefully. Suppose $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $[X,E] = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} - \begin{pmatrix} X_{11} & X_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -X_{12} \\ X_{21} & 0 \end{pmatrix}$, so

$$[[X,E],[Y,E]] = \begin{pmatrix} -X_{12}Y_{21} + Y_{12}X_{21} & \\ & -X_{21}Y_{12} + Y_{21}X_{12} \end{pmatrix}$$

$$\mathrm{tr}\big(E[[X,E],[Y,E]]\big) = +\,\mathrm{tr}\big(-X_{12}Y_{21} + Y_{12}X_{21}\big)$$

$$\operatorname{tr} E[X,Y] = \operatorname{tr}\left(\cancel{X_{11}Y_{11}} + X_{12}Y_{21} - \cancel{Y_{11}X_{11}} - Y_{12}X_{21}\right)$$

so

$$\boxed{\begin{aligned} -\operatorname{tr}\left(E[X,Y]\right) &= \operatorname{tr}\left[E\left([X,E][Y,E] - [Y,E][X,E]\right)\right] \\ &= i(Y)i(X)\operatorname{tr} E(dE)^2 \end{aligned}}$$

Conclude: In general $H = \sum \lambda_i E_i$ where the $\{E_i\}$ are idempotents: $I = \sum E_i$, $E_i E_j = \delta_{ij} E_j$. So the form giving the symplectic structure on the orbit of H, namely

$$\operatorname{tr}(H[X,Y])$$

is

$$\sum \lambda_i \operatorname{tr}\left(E_i(dE_i)^2\right)$$

which is the sum of the first Chern forms for each of the bundles in the flag weighted by the eigenvalue.

January 27, 1983

Bott-Atiyah paper, last section dealing with Chern numbers. If one has a vector bundle over a $2n$-dimensional oriented compact manifold, then one can define Chern numbers

$$\int_M c_\alpha(E) \qquad \text{where} \quad |\alpha| = n.$$

Suppose one has a circle action on M and E is an equivariant bundle. Then we can refine the Chern classes of E to be equivariant classes and apply the fixpoint formula.

Let's review the formula.

$$\begin{tikzcd} M^S \arrow[r, "i"'] \arrow[dr, "j"'] & M \arrow[d, "\pi"] \\ & pt \end{tikzcd}$$

To start with $\alpha \in H_S^\cdot(M)$ and compute $\pi_*\alpha$. If $\alpha = i_*\beta$ $\beta \in H_S^\cdot(M^S)$, then $\pi_*\alpha = j_*\beta$.

Further $i^*\alpha = i^* i_* \beta = e(\nu_i)\beta \implies \beta = \frac{i^*\alpha}{e(\nu_i)}$ in the localized cohomology. Hence

$$\pi_*\alpha = j_*\left\{\frac{i^*\alpha}{e(\nu_i)}\right\} = \sum_P \frac{i_P^*(\alpha)}{e(\nu_P)}$$

Now in the case of a circle action the equivariant forms are $(\Omega_M^\cdot)^S[u] = k[u]\otimes(\Omega_M^\cdot)^S$ with the differential

$$d = d_M - u\, i(X).$$

Hence an equivariant form is a polynomial in u with invariant forms for coefficients. So when we perform π_* we integrate out the $2n$-form. Thus the classes $\pi_* c_\alpha(E)$ with $|\alpha|=n$ should be just the Chern numbers of E.

On the other hand $i_P(c_\alpha(E))$ should be the equivariant Chern classes of the representation on E_P at the fixpoint P. These should be u^n times the appropriate powers of the eigenvalues of the representation. The same holds for $e(\nu_p)$, so the ratio $\dfrac{i_p^*(\mathbf{C}_\alpha(E))}{e(\nu_p)}$ is indeed a number.

January 25, 1983

It's time to do some computations with the free loop group of $U = U_r$. Here we are looking at [redacted] an r-dimensional vector bundle E over S^1.

For each connection A on E I get a skew-2-form on $\tilde{\mathfrak{g}}$ given by

$$\omega_A(X, Y) = \int tr\left(D_A(X)\, Y\right).$$

Here $D_A(X) = [D_A, X]$, and if a [redacted] trivialization of E is given, we can write $D_A = (\partial_t + \alpha)dt$, whence

$$D_A(X) = \left(X' + [\alpha, X]\right) dt$$

Skew-symmetry follows from

$$\int tr\left(D_A(X)Y\right) + \int tr\left(X\, D_A(Y)\right) = \int tr\, D_A(XY)$$

$$= \int d\, tr(XY) = 0$$

Left-invariant forms on $\mathcal{G}$ can be identified with skew-forms on $\mathfrak{g}$, hence ω_A is just the skew-form on $\tilde{\mathfrak{g}}$ corresponding to a left-invariant form which can be defined as follows. We have the map

$$\begin{tikzcd} \tilde{\mathcal{G}} \arrow[r] & \mathcal{A} \\ g \arrow[r, mapsto] & g(A) \end{tikzcd} \qquad D_{g(A)} = g D_A g^{-1} = D_A + g D_A(g^{-1})$$

which is equivariant. At any point A' the form $\omega_{A'}$ on $\tilde{\mathfrak{g}}$ is really the pull-back to $\tilde{\mathfrak{g}}$ of an intrinsic symplectic form on the $\mathcal{G}$-orbit. So over $\mathcal{A}$ we have these two-forms defined [redacted] on vectors tangent to the gauge orbits.

Now I believe that ω_A is closed:

$$\omega_A([X,Y], Z) = \int tr\left(D_A[X,Y]\, Z\right) = \int tr\left([D_A(X), Y]Z\right)$$
$$+ \int tr\left([X, D_A(Y)], Z\right)$$

$$= \int tr\left(D_A(X)[Y,Z]\right) + \int tr\left(D_A(Y)[Z,X]\right)$$

$$= \omega_A(X,[Y,Z]) + \omega_A(Y,[Z,X]).$$

But a 2-cocycle satisfies:

$$(\delta\omega)(X,Y,Z) = \omega([XY],Z) - \omega([X,Z],Y) + \omega([Y,Z],X)$$
$$= 0.$$

So ω_A is a 2-cocycle, i.e. the corresponding left-invariant form on $\mathcal{G}$ is closed.

Thus for each connection $A \in \mathcal{A}$ I have a real 2-dimensional cohomology class defined on $\mathcal{G}$. Does this class vary with A? Now actually the class is defined in the Lie algebra cohomology, so one should look there first:

$$H^2(\tilde{\mathfrak{g}}) = \Lambda^2 \underbrace{H^1(\tilde{\mathfrak{g}})}_{\substack{\text{distributions} \\ \text{on } S^1}} \oplus \text{Prim}\{\underbrace{H^2(\tilde{\mathfrak{g}})}_{H_1(S^1)\otimes\mathbb{R}}\}$$

This formula is a stable $\tilde{\mathfrak{g}} = \mathfrak{gl}_r(C^\infty(S^1))$, r large, formula; hence one must be careful in applying it. For example the 2-cocycle belonging to the generator of $H_1(S^1)$ is ω_A when $A = 0$, i.e.

$$X,Y \longmapsto \int tr\left(\frac{d}{dt}(X)\,Y\right) dt$$

Using a trivialization of E we have $D_A = (\partial_t + \alpha)dt$, so

$$\omega_A(X,Y) = \int tr(\partial_t X \cdot Y)\,dt + \int tr([\alpha,X]Y)\,dt$$

$$= \int tr(\partial_t X\cdot Y)\,dt + \int tr(\alpha[X,Y])\,dt$$

But the last 2-cochain on $\tilde{\mathfrak{g}}$ is a coboundary.

Therefore the forms ω_A are such that the Lie cohomology class, and hence the class in $H^2(\mathcal{G})$ does not change.

One of my ideas is to produce an analytical formula for the invariant form ω_A.

First note $\mathcal{G} = \mathcal{U} \times \Omega\mathcal{U}$

$$
\begin{aligned}
H^2(\mathcal{G}) = {} & \cancel{H^2(\mathcal{U})} \otimes H^0(\Omega\mathcal{U}) \\
& + H^1(\mathcal{U}) \otimes \cancel{H^1(\Omega\mathcal{U})} \\
& + H^0(\mathcal{U}) \otimes H^2(\Omega\mathcal{U}) = \underbrace{\mathbb{Z}[T,T^{-1}]}_{H^0(\Omega\mathcal{U})} \otimes H^2(\underbrace{\Omega B\mathcal{U}}_{B\mathcal{U}})
\end{aligned}
$$

so that $\operatorname{Prim}\{H^2(\mathcal{G})\} = H_1(S^1) = \mathbb{Z}$.

It is natural to ask if there is a function, or 0-form, on $\mathcal{G}$ which might correspond to an equivariant 1-form on $\mathcal{A}$. It should be related to the one-diml. cohomology class on s.a. Fred. ops.

$$H^1(B\mathcal{G}) = H^1_{\mathcal{G}}(\mathcal{A}) = H^1(B\mathbb{Z}) = \mathbb{Z}.$$

January 26, 1983

[redacted] dim $M = 1$. For any $A \in \mathcal{A}$ we get the left-invariant closed 2-form on $\mathcal{G}$ which corresponds to the 2-cocycle on $\tilde{\mathfrak{g}}$ given by

$$\omega_A(X,Y) = \int \operatorname{tr}\left(D_A(X)\,Y\right)$$

All these 2-cocycles are cohomologous, so represent the same class in $P\,H^2(\tilde{\mathfrak{g}})$ or $P\{H^2(\mathcal{G})\} = H_1(M)\otimes\mathbb{R}$.

There should be similarly-defined forms of any even dimension, and these forms should have an analytical expression.

However the 0-form should be special, because one is not in the convergence range. I want now to describe the zero-form. This is just a function f_A on $\mathcal{G}$ possibly depending on the point A. It should be related to an [redacted] equivariant 1-form on $\mathcal{A}$ by suspension. Now

$$H^1_{\mathcal{G}}(\mathcal{A}) = H^1(B\mathcal{G}) = \operatorname{Hom}(\overset{\mathbb{Z}}{\overset{\shortparallel}{\pi_0\mathcal{G}}},\mathbb{R})$$ [redacted]

so the [redacted] 1-diml form is related to the degree map: $f: \mathcal{G} \longrightarrow \mathbb{Z}$. So lets see how reasonable it is for $f_A = f$ to be expected.

The other ingredient should be that on the space of s.a. Fredholm operators there is a canonical class in $H^1(?,\mathbb{Z})$, which can be represented as a homotopy class of maps from $sa\mathcal{F} \longrightarrow S^1$. Concretely this is realized, I think, by $e^{i\pi\eta_A}$ where η_A is the η-invariant.

More precisely and generally, take an odd-diml. manifold M and the family of Dirac operators over M with coeffs. in E, parametrized by $\mathcal{A}$, equivariant for $\mathcal{G}$. Then the η-invariant of any of these operators is defined. The function

$$A \longmapsto e^{i\pi\eta_A}$$

is then a well-defined smooth map to S^1 which is gauge-invariant, hence a map $\mathcal{G}\backslash\mathcal{A} \longrightarrow S^1$.

~~(What does this map look like over the circle?)~~

From this we can define an integer-valued function on $\pi_0\mathcal{G}$ as follows. Pick a point A and then given $g \in \mathcal{G}$ one joins A to gA by a curve and looks at the winding number of $e^{i\pi\eta}$ along this curve.

Now this integer is probably the same as what you would get by regarding the Dirac operator as an elt. of $K_1(M)$, and $\pi_0\mathcal{G} = [M, U] = K^{-1}(M)$, and then capping. This agrees with the isoms

$$e^{i\pi\eta} \in [B\mathcal{G}, S^1] = H^1(B\mathcal{G}, \mathbb{Z}) = \mathrm{Hom}(\underset{\underset{\displaystyle K^{-1}(M)}{\shortparallel}}{\pi_0\mathcal{G}}, \mathbb{Z})$$

Summary of the situation: The central problem is to construct or define character forms for the index of the family of Dirac operators; these forms are supposed to equivariant forms on $\mathcal{A}$ for the action of $\mathcal{G}$. A simpler problem is to define forms on $\mathcal{G}$ which should result from the character forms via the map $S(\mathcal{G}) \rightarrow B\mathcal{G}$.

I have now a list of examples which I think should make part of the general theory.

1) For odd-diml M we want odd equivariant forms on $(\mathcal{G}, \mathcal{A})$, the first case being degree 1. Here we have the map $\mathcal{G}\backslash\mathcal{A} \longrightarrow S^1$, $e^{i\pi\eta}$, and $d\eta$ should be the desired 1-form on $(\mathcal{G}, \mathcal{A})$. The corresponding 0-form on $\mathcal{G}$ is an integer-valued function which seems to be related to the index map

$$\pi_0\mathcal{G} \longrightarrow K^{-1}(M) \xrightarrow[\substack{\text{capping} \\ \text{with class of } \not{D} \text{ in } K_1(M).}]{} \mathbb{Z}$$

[redacted] 2) For even-dimensional M we want even-diml equivariant forms on $\mathcal{A}$. The 0-degree form is the constant $=$ index. The degree 2 form is the curvature of the determinant line bundle. [redacted] What does one get [redacted] on $\mathcal{G}$ corresponding to this? This is a good question: One can define $f_A : \mathcal{G} \to S^1$ using the straight line from A to gA and the connection on $\mathcal{L}$ over this line. One can also pull-back the connection on $\mathcal{L}$ by the map $\mathcal{G} \longrightarrow \mathcal{A}$, $g \longmapsto gA$, trivialize $\mathcal{L} | \mathcal{G}$ equivariantly, and you get a 1-form on $\mathcal{G}$ which is left-invariant. This is the moment map applied to A considered as a linear functional on $\tilde{\mathfrak{g}}$. One gets an invariant forms which is not closed.

3) Analytically we can define closed invariant forms on $\mathcal{G}$ of high degree, by pulling back standard forms [redacted] like $\mathrm{Tr}(D^{-1}dD)^{2p-1}$, $\mathrm{Tr}(E(dE)^2)^{2p}$. I don't yet see how to define these forms for all A, even in the case of S^1 where I have an explicit candidate.

I seem to have two approaches to the problem:

a) work with small degree forms which we already can define over $\mathcal{A}$, such as $d\eta$ in odd-dimensions and the determinant line bundle in even dimensions. These forms require regularization, and will [redacted] not be "Lie-accessible".

b) work with forms of degree $>\dim M$ which are somehow obviously part of the character of the index bundle, and which don't require normalization. These classes ~~don't require regularization~~ might be easier to understand because they are Lie-theoretic.

Related tidbits?

1) Does ∃ a local formula for $d\eta$? Recall (p196)

$$d\,\mathrm{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) = -s\,\mathrm{Tr}\left(\delta A\,\frac{1}{|A|}|A|^{-s}\right) = -s\,\mathrm{Tr}\left(\delta A\,(A^2)^{-\frac{s+1}{2}}\right)$$

$$= \frac{-s}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \underbrace{\mathrm{Tr}\left(e^{-tA^2}\delta A\right)}\, t^{\frac{(s+1)}{2}}\frac{dt}{t}$$

has $ct^{-1/2}$ term in odd degrees as δA is a diffl operator

$$\underset{s\to 0}{\sim} -s\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^1 c\,t^{s/2}\frac{dt}{t} = -\frac{2}{\sqrt{\pi}}\,c$$

2) In dimension 1 for $\frac{1}{i}\frac{d}{dx} - a$ over $\mathbb{R}/2\pi\mathbb{Z}$

$\eta = 2a - 1 \qquad 0 < a < 1$

so that in general $e^{i\pi\eta} = \det(-M)$

where M is the monodromy

New idea: Concentrate on explicit cocycles and calculating them by explicit formulas.

Start with [redacted] a Dirac operator $D\!\!\!/$ on M, and suppose it is invertible to simplify. [redacted] Fix $p > \dim M$ of parity opposite to dim M. Then Connes associates to $D\!\!\!/$ a p-cocycle which we can interpret as [redacted] a left-invariant closed diffl p-form on $\mathcal{G}_n = \mathrm{Map}(M, \mathcal{U}_n)$ for all n. [redacted] Because you [redacted] expect to prove an explicit index theorem, you should be able to produce a formula for the diffl p-form on $\mathcal{G}_n$, hence a formula for Connes cocycle.

Idea ①: If you can prove an explicit [redacted] index thm. for the family of [redacted] Dirac operators, then you can derive an explicit formula for Connes cocycles.

② Once we have the p-form over $\mathcal{G}_n$ corresponding to the flat connection on the trivial bundle, then for any E which is a direct summand of $\mathbb{1}^{\oplus n}$ we get the p-form on the gauge gp. $\mathrm{Aut}(E)$ corresponding to the Grassmannian connection. This suggests the forms are defined for $D\!\!\!/_A$ which are not invertible, and that the cohomology classes do not depend on A.

At this point it appears I should do some calculation in the circle and Riemann surface cases. I should compute explicitly the Connes cocycles belonging to Dirac operators.

Take $M = S^1$ and let F be the standard polarization in $L^2(S^1)$, which is $+1$ on z^n $n \geq 0$ and -1 for z^n, $n<0$. Then for any p we have the Connes cocycle

$$f(a^1, \dots, a^{2p}) = \mathrm{Tr}\left(F[F,a^1]\cdots\cdots[F,a^{2p}]\right)$$

where the $a^i \in C^\infty(S^1)$. What is this exactly? How can I proceed to understand it?

Up to various constant factors, this cocycle corresponds to the closed invariant $2p$-form on $\mathcal{G}$ = free loops in $U = U_k$ which one obtains from the map

$$\begin{tikzcd}
\mathcal{G} \arrow[r] & \mathcal{G}/U \arrow[r, hook] \arrow[d, phantom, "\shortparallel"] & \begin{array}{c}\text{Grassmannian}\\ \text{of half-spaces}\\ \text{in } L^2(S^1)^k\end{array} \arrow[r] & \mathbb{Z} \times BU \\
& \Omega U & &
\end{tikzcd}$$

and the character forms on the Grassmannians. Now you should review some of the things you want to prove about Lie algebra cohomology of loop groups.

So let's recall that the homogeneous space G/U of the free loop group G is to be thought of as a partial flag manifold, [redacted] an "affine" analogue of $U/$Cent of a torus. One can calculate the cohomology of $U/\mathrm{Cent}(T)$ using the complex of U-invariant forms, however this complex has non-zero differentials in general, unlike the Grassmannians.

The same is true for the complex of invariant diffl forms on G/U. In fact we know the primitive forms are given by the relative Connes complex $[C_\cdot(A,k)]^*$ where $A = k[z,z^{-1}]$. Yet the primitive part of the homology of $G/U = \Omega U$ has one generator in each even dimension.

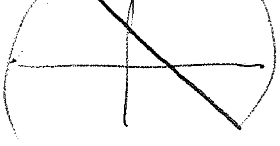
I think that [redacted] Hodge theory on G/U works. More precisely, I think one can understand the Laplacean on G/U as a Casimir operator, and then calculate by root methods what the cohomology is. [redacted]

In any case it is fairly clear that the Connes cocycle is the preferred generator, and perhaps it wouldn't be too hard to see it explicitly in the complex $[C_\cdot(A,k)]^*$.

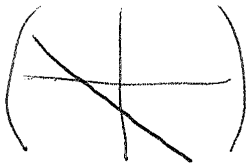
Now
$$F = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$
and

$$\left[\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix}.$$

We are going to apply this to the case of a multiplication operator $a_i =$ mult by z^{n_i}. Then the matrix has the form



or



and hence $[F,a] = 2\begin{pmatrix} & \diagdown \\ & \end{pmatrix}$ or $2\begin{pmatrix} & \\ \diagdown & \end{pmatrix}$.

So the cocycle will be zero unless the signs of $n_1, n_2, \dots, n_{2p}$ alternate and $\sum n_i = 0$.

Let's consider $2p = 4$, with $n_1 > 0 > n_2 < 0 < n_3 > 0 > n_4$ and $n_1 + n_2 + n_3 + n_4 = 0$. Then I want the trace of

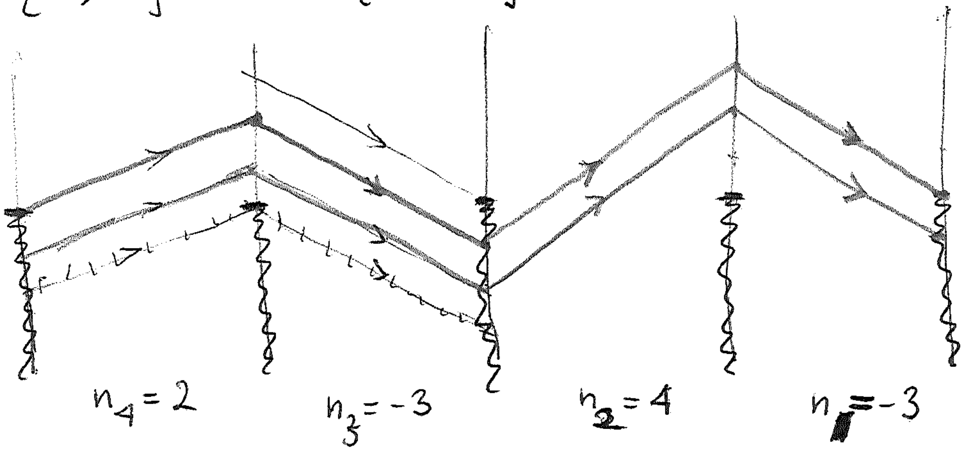
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} & \diagdown \\ & \end{pmatrix}\begin{pmatrix} & \\ \diagdown & \end{pmatrix}\begin{pmatrix} & \diagdown \\ & \end{pmatrix}\begin{pmatrix} & \\ \diagdown & \end{pmatrix}$$

which is the same as the trace of

[redacted]

$$\begin{pmatrix} \diagdown \end{pmatrix}\begin{pmatrix} \diagdown \\ \end{pmatrix}\begin{pmatrix} \\ \diagdown \end{pmatrix}\begin{pmatrix} \diagdown \\ \end{pmatrix},$$

and I want to try to think of these as maps from $\{n \geqslant 0\}$ to $\{n < 0\}$ and back.



So the trace is something like the number of q satisfying

$$q \geqslant 0, \quad q + n_1 < 0, \quad q + n_1 + n_2 \geqslant 0, \quad q + n_1 + n_2 + n_3 < 0, \dots\dots$$

Not too illuminating!

Possibly you have chosen the wrong cocycle: The idea is Bott's map $\mathrm{Grass}_n(\mathbb{C}^{2n}) \subset \Omega SU_{2n}$ uses geodesics from I to $-I$, not circular loops. Only $z^{1/2}, z^{-1/2}$ are used.