Go back over the determinant line bundle and how one computes its ~~equivariant~~ first Chern class as an equivariant form. I place myself over the $2n$-sphere so that the Todd or $\hat{A}$-class is 1. What I have is a compact spin manifold M and hence a spinor bundle $S = S^+ \oplus S^-$. Then I take a vector bundle E, let $\mathcal{A}$ = conn. on it, $\mathcal{G}$ = gauge transf. Given $A \in \mathcal{A}$ I get $D_A : \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E)$. In this way I get a family of elliptic operators on M param. by $\mathcal{A}$ equivariant for $\mathcal{G}$.

More precisely over $\mathcal{A} \times M$ I have an operator $pr_2^*(S^+ \otimes E) \longrightarrow pr_2^*(S^- \otimes E)$ linear wrt fns. on $\mathcal{A}$. Can take bundle over $\mathcal{A}$ whose fibre is $\Gamma(S^+ \otimes E)$ at each point.

Anyway what I want to do is quotient out by $\mathcal{G}$. I get over $B\mathcal{G} \times M$ a taut. bundle $\tilde{E}$. Then I can form $pr_2^*(S^+) \otimes \tilde{E}$ $\quad pr_2^*(S^-) \otimes \tilde{E}$ and I have a family of elliptic operators on the fibres over $B\mathcal{G}$.

The index thm. for families says

$$ch(\text{index}) \in H^*(B\mathcal{G})$$

is given by $\int_M ch(\tilde{E})\, \hat{A}(M)$. Thus we want to compute $ch(\tilde{E})$.

[redacted]

$\tilde{E}$ = taut. bdle over $B\mathcal{G} \times M$, belonging to E/M considered as an $\mathcal{G}$-bundle over M. If I had a $\mathcal{G}$-invariant connection D in E, then I could compute the character of $\hat{\tilde{E}}$ in the alg. of equivariant forms:

$$\left[W(\tilde{\mathfrak{g}}) \otimes \Omega(M)\right]_{basic}$$

It is worth doing this calculation separately. So suppose G acts on M and E is an equiv. bdle, $D : \Omega^*(M, E) \circlearrowleft$ is an inv. connection. Can you explain [redacted] how to construct $ch_n(D)$ as an equivariant form?

Introduce the principal bundle P for E. $\pi : P \longrightarrow M$, G acts to the left and U_N to the right. Now $\pi^*(E)$ is canonically trivial whence $D = d + \theta$ where

$$\theta \in \Omega^1(P, \mathfrak{gl}_N)$$

So we get

$$W(\mathfrak{gl}_N) \longrightarrow \Omega(P)$$

?

G acts on M, E is a G-vector bundle, and D is an invariant connection. I want to construct $ch_n(E)$ as an equivariant form. Geometrically what we do is to pull E back to $PG \times M$. Over PG is a canonical connection θ relative to BG; this is what we assume when we think of there being a canonical map

$$W(\mathfrak{g}) \longrightarrow \Omega(PG).$$

Now over $PG \times M$ we have $pr_2^*(E)$ with the connection D.

So let's now look at the ~~situation~~ where G acts freely on M and one is given a connection form Θ for $M \longrightarrow G\backslash M$. This means

$$\begin{cases} \Theta \in \mathfrak{g} \otimes \Omega^1(P) \text{ is } G\text{-invariant}, \quad [X,\theta] + \mathcal{L}_X \theta = 0 \\ i_X \Theta = X \quad \text{for } X \in \mathfrak{g}. \end{cases}$$

We want to descend D, but can't because

$$[\iota_X, D] = \varphi_X \in \Omega^0(M, \mathrm{End}\, E)$$

So we replace D by $\bar{D} = D - \varphi_a \Theta^a$. Then

$\theta = \underbrace{\lambda_a}_{\text{basis for } \mathfrak{g}} \Theta^a$

$$[\iota_X, \bar{D}] = [\iota_X, D] - [\iota_X, \varphi\Theta] = \varphi_X - \varphi_X = 0.$$

Also $\quad [\mathcal{L}_X, \bar{D}] = [\mathcal{L}_X, D] - [\underline{\mathcal{L}_X, \varphi\theta}]$

should be OKAY

$\cancel{\mathcal{L}_a)\, \varphi_b \theta^b = [\mathcal{L}_a, \varphi_b]\theta^b + \varphi_b \mathcal{L}_a \theta^b}$

$$\left.\begin{array}{l} \theta \in \mathfrak{g} \otimes \Omega^1(M) \\ \varphi \in \Omega^0(M, \mathrm{End}\, E) \otimes \mathfrak{g}^* \end{array}\right\} \quad G\text{-inv.}$$

$\therefore \quad \varphi\theta \in \Omega^1(M, \mathrm{End}\, E) \qquad G\text{-inv.}$

So now we have $\bar{D} = D - \varphi\theta$ acting on $\Omega^*(M, E)$. So we next look at

$$\bar{D}^2 = (D - \varphi\theta)^2$$

$$\bar{D}^2 = D^2 - [D, \varphi\theta] + \varphi\theta\varphi\theta$$

So what next?

$$W(\mathfrak{g}) \otimes \Omega(M, E)$$

$$\bar{D} = D - \varphi\theta$$

$$\boxed{\mathrm{tr}(\bar{D}^2)^n \in \cancel{\Omega^{2n}(M)}\ [W(\mathfrak{g}) \otimes \Omega(M)]_{\text{basic}}}$$

Repeat:

I'm looking at the simple question of computing equiv. char. classes for an equivariant bundle E/M when an inv. conn. D is given. The answer is that I define $\bar{D} = D - \varphi\theta$ operator on

$$W(\mathfrak{g}) \otimes \Omega(M, E)$$

degree 1 deriv. over $W(\mathfrak{g}) \otimes \Omega(M)$. Then

$$\operatorname{tr}(\bar{D}^2)^n \in [W(\mathfrak{g}) \otimes \Omega(M)]_{basic}$$

The next step is to explain what this becomes under the isom.

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{basic} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(M)]^G$$

which I know is induced by the map

$$\begin{aligned} W(\mathfrak{g}) &\longrightarrow S(\mathfrak{g}^*) \\ \theta^a &\longmapsto 0 \\ \Omega^a &\longmapsto \Omega^a \end{aligned}$$

$$\bar{D}^2 = D^2 - [D, \varphi_a \theta^a] + \varphi_a \theta^a \varphi_b \theta^b$$

$$[D, \varphi\theta] = [D, \varphi]\theta + \varphi\, d\theta$$

$$\boxed{\bar{D}^2 \longrightarrow D^2 - \varphi\Omega}$$

$$\cancel{D^2 \in (W(g)\otimes \Omega(M, \mathrm{End}E))^2_{basic}}$$

Check this computation out with bases.

$$\Omega = d\theta + \theta^2 = d\theta + \tfrac{1}{2}[\theta, \theta]$$

$$\lambda_a \Omega^a = d(\lambda_a \theta^a) + \tfrac{1}{2}[\lambda_b \theta^b, \lambda_c \theta^c]$$

$$= \lambda_a \, d\theta^a + \tfrac{1}{2} f^a_{bc} \lambda^a \theta^b \theta^c$$

$$\boxed{\Omega^a = d\theta^a + \tfrac{1}{2} f^a_{bc} \theta^b \theta^c}$$

$$\varphi_a = [\iota_a, D]$$

$$[D, \varphi_a] = \cancel{-\tfrac{1}{2}(D \iota_a D + D \iota_a D)} \;\cancel{\varphi(\iota_a D + D\iota_a) D}$$

$$\cancel{= D^2}$$

$$= [D, \iota_a D + D \iota_a] = [D^2, \iota_a]$$

$$D^2 - \varphi\Omega \longrightarrow \left(1 - \theta^a \iota_a + \tfrac{1}{2}\theta^a \theta^b \iota_b \iota_a\right)(D^2 - \varphi\Omega)$$

$$D^2 - \varphi\Omega - \theta^a \iota_a D^2 + \tfrac{1}{2}\theta^a\theta^b \iota_b \iota_a D^2$$

$$+ \theta^a \cancel{\iota_a(\varphi\Omega)} \qquad \cancel{= \iota_b \iota_a \varphi\Omega}$$

$$\cancel{\iota_b \iota_a D^2 = \iota_b(-[D, \varphi_a])}$$

$$\bar{D}^2 = (D - \varphi\Theta)^2 = D^2 - [D, \varphi\Theta] + \varphi\Theta\varphi\Theta$$

$$(D^2 - \varphi\Omega) \longmapsto D^2 - \varphi\Omega - \theta^a \iota_a D^2 + \frac{1}{2}\theta^a\theta^b \iota_b \iota_a D^2$$

Mistake back on p. 3.

$$[\iota_X, D] = \mathcal{L}_X + \varphi_X$$

Sign disagrees with earlier convention $\mathcal{L}_X = [\iota_X, D] + \varphi_X$

$$\bar{D} = D - \varphi\Theta$$

$$[\iota_X, \bar{D}] = \mathcal{L}_X + \varphi_X - [\iota_X, \varphi\Theta] = \mathcal{L}_X$$

$$[\iota_X, \bar{D}^2] = \mathcal{L}_X \bar{D} - \bar{D}\mathcal{L}_X = [\mathcal{L}_X, \bar{D}] = 0.$$

So what we have is the map

$$\left(W(\mathfrak{g}) \otimes \Omega(M, \operatorname{End} E)\right)_{basic} \xrightarrow{\sim} \left(S(\mathfrak{g}^*) \otimes \Omega(M, \operatorname{End} E)\right)^G$$

$$\bar{D}^2 = (D - \varphi\Theta)^2 \longmapsto D^2 - \varphi\Omega$$

equiv. cohomology.

~~At this point I have analyzed the point of~~

At this point I understand how to compute the equiv. char. classes of an equivariant bundle E over M assuming the existence of an invariant connection D. One has the equivariant curvature in

$$\left(W(\mathfrak{g}) \otimes \Omega(M, \mathrm{End}\, E)\right)_{basic} \xrightarrow{\sim} \left(S(\mathfrak{g}^*) \otimes \Omega(M, \mathrm{End}\, E)\right)^G$$

$$\bar{D}^2 = (D - \varphi\Theta)^2 \longmapsto D^2 - \varphi\Omega$$

where $\Theta \in \mathfrak{g} \otimes W^1(\mathfrak{g})$

$$\Omega = d\Theta + \frac{1}{2}[\Theta,\Theta] \in \boxed{\cancel{W^2(\mathfrak{g})\ (\mathfrak{g} \otimes \mathfrak{g} W^2(\mathfrak{g}))}}\ \mathfrak{g} \otimes W^2(\mathfrak{g})$$

are the canonical tensors.

So now if I want the equivariant ch_n it is

$$\frac{1}{n!}\, tr\,(D^2 - \varphi\Omega)^n \in \left[S(\mathfrak{g}^*) \otimes \Omega(M)\right]^{G\ 2n}$$

and it can be brought back to

$$\left[W(\mathfrak{g}) \otimes \Omega(M)\right]_{basic}$$

via the operator $\prod_a (1 - \theta^a i_a) = 1 - \theta^a i_a + \frac{1}{2}\theta^a\theta^b i_b i_a - \cdots$

Now we go back to the case of $pr_2^*(E_0)$ over $\mathcal{A}\times M$ and $G = \mathcal{G}$ the group of gauge transf. Then $E = pr_2^*(E_0) = \mathcal{A}\times E_0$ over $\mathcal{A}\times M$ has a canonical connection. (There is a problem in writing it down.) Suppose E_0 is trivial. Then any connection can be compared to d. So we have

$$D = d_{\mathcal{A}\times M} + \underbrace{A_\mu \, dx^\mu}$$

here $A \in \Omega^{0,1}(\mathcal{A}\times M, gl_n)$.

Now we need the action of $g \in \mathcal{G}$. Then $\mathcal{G}$ acts on $\mathcal{A}\times E_0$ and D should be invariant

$$gDg^{-1} = d_{\mathcal{A}\times M} + (g\,dg^{-1} + gAg^{-1})$$

which should be the pull back of D under

$$g : \mathcal{A}\times M \longrightarrow \mathcal{A}\times M$$
$$(A, m) \longmapsto (g\,dg^{-1} + gAg^{-1}, m)$$

If v is an infinitesimal gauge transf., then

$A, m \longmapsto A + dv + [v, A], m$

$i_v D =$

In any case $\varphi_v \in \Omega^0(\mathcal{A}\times M, gl_n)$ is just $\varphi_v : (A, m) \longmapsto v(m)$

need curvature as well as φ:

$$D^2 = \delta A_\mu \, dx^\mu + \tfrac{1}{2} F_{\mu\nu} \, dx^\mu dx^\nu \qquad \begin{matrix} 1,1 \\ +\,0,2 \end{matrix}$$

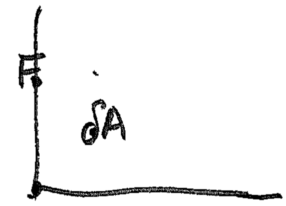
Here A_μ is a matrix function on A and δA_μ is its differential.

K

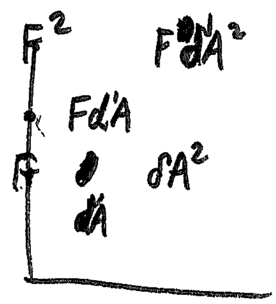
(Now do a Riemann surface. What do you get $\boxed{???}$

$$K^2 \int_M \operatorname{tr}\left(\tfrac{1}{2}(D^2)^2\right)$$

K



$$= \frac{K^2}{2} \int_M \operatorname{tr} \left(\delta A_\mu \, dx^\mu\right)^2$$



2,4

Next do a 4-manifold

Seem to get

2,0
1,1

$$\frac{K^3}{3!} \int_M \operatorname{tr}\left[\left(\delta A_\mu \, dx^\mu\right)^2 F\right]$$

where F is the curvature.

This is just calculating the curvature. In addition I need to compute the Higgs field

$$\int_M \frac{1}{2}\left(D^2 - \varphi\Omega\right)^2 = \int_M \frac{1}{2} \operatorname{tr}\left[(D^2)^2 - D^2\varphi\Omega - \varphi\Omega D^2\right.$$

So it would appear that I get

$$\varphi_X = \int_M \mathrm{tr}(F X) \qquad \text{for } ~~\text{a surface}~~ \mathbb{R}^2$$

$$\int_M \frac{1}{3!} \mathrm{tr}\left((D^2 - \varphi\Omega)^3\right) = -\Omega \int \frac{1}{3!} \mathrm{tr}\left(3 (D^2)^2 \varphi\right)$$

so get $\quad \varphi_X = \frac{1}{2} \int \mathrm{tr}(F^2 X) \quad$ for $\mathbb{R}^4$.

Let's compare then these formulas.

$$\begin{cases} \omega = \dfrac{K}{2} \displaystyle\int_M \mathrm{tr}\,(\delta A_\mu dx^\mu)^2 \\ \varphi_X = K \displaystyle\int_M \mathrm{tr}\,(F X) \end{cases}$$

$$\begin{cases} \omega = \dfrac{K^2}{3!} \displaystyle\int_M \mathrm{tr}\left((\delta A_\mu dx^\mu)^2 F\right) \\ \varphi_X = \dfrac{K^2}{2} \displaystyle\int_M \mathrm{tr}\,(F^2 X) \end{cases}$$

Recall that under the isom

$$[W(\mathfrak{g}) \otimes \Omega(M)]_{\text{basic}} \xrightarrow{\sim} [S(\mathfrak{g}^*) \otimes \Omega(M)]^G$$

I need the formula for d.

$$p(\Omega)\left(1 - \theta^a i_a + \tfrac{1}{2}\theta^a\theta^b i_b i_a \cdots\right) \cancel{p(\Omega)}\alpha \longleftarrow p(\Omega) \otimes \alpha$$

$$\downarrow d \qquad \xrightarrow{\theta \mapsto 0} \qquad p(\Omega)(d\alpha - \Omega^a i_a \alpha).$$

2) Hence on $\left[S(\mathfrak{g}^*)\otimes\Omega(M)\right]^G$ the diffl

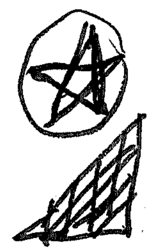
s $\quad d_M - \Omega^a i_a$.

heck $\quad (d-\Omega^a i_a)^2 = d^2 - \cancel{d\Omega\, i_a} - \Omega^a L_a + \cancel{\Omega^a\Omega^b i_a i_b}$

$$(d-\Omega^a \iota_a)^2 = 0 \text{ on the invariants.}$$

Assertion: $\varphi\ \omega$ is an ~~eq~~ equivariant 2 form: $\boxed{\omega - \Omega\varphi}$ i.e.

$$(d-\Omega^a\iota_a)(\omega-\Omega\varphi)=0 \quad\text{or}$$



$$d\omega = 0, \; -\Omega\, d\varphi - \Omega\,\iota\,\omega = 0$$

$$\boxed{d\omega=0} \quad\text{and}\quad \boxed{d\varphi_{*} + \iota_{*}\omega = 0}$$

So at this point we reach the following roblem. We have described $c_1(L)$ $\in H^2(BG)$ ~~I also want a ans~~ by this ~~fact~~ ir ω,φ or better $\omega - \Omega^a\varphi_a$. Now I have e problem of constructing the transgression. This eans that I must use the contractibility of a

13) What do I know in the case of a ~~surface~~ surface? Fix degree 0 case. Then I computed the curvature and found it to be as above!!!

But what is the anomaly?

Let's go back to the idea that one has ~~a simple~~ in physics ~~two things~~. two halves of he Dirac operator. $\boxed{\text{How am I to handle this??}}$

$$\not{D}_A = \frac{1}{i}\gamma^\mu(\partial_\mu + A_\mu) = \left(\begin{array}{c|c} 0 & D_A^* \\ \hline D_A & 0 \end{array}\right)$$

$$\delta \log \det(\not{D}_A) = \mathrm{Tr}(\not{D}_A^{-1}\delta\not{D}_A) \qquad \gamma^5 \equiv \begin{pmatrix} +1 & \\ & -1 \end{pmatrix}$$

$$= \mathrm{Tr}(\not{D}_A^{-1}\delta\not{A})$$

$$\delta\not{A} = \frac{1}{i}\gamma^\mu\delta A_\mu$$

$$\therefore \delta\log\det(\not{D}_A) = \mathrm{Tr}(D_A^{-1}\delta D_A) + \mathrm{Tr}(D_A^{*-1}\delta D_A^*)$$

In the surface case what I do is o regularize the trace in a certain way. lamely I choose a connection extending he $\bar\partial$-operator. Then I remove it off to get he current J

$$D^{-1} - Q = J$$

$$\mathrm{Tr}(J\,\delta D) = \int dx\,dy\ \underline{J(x,y)\,\delta D(y,x)}.$$

14) So what one does is to conclude that J is actually defined as a smooth kernel, but $\mathrm{Tr}(J\delta D) = \int_M \mathrm{tr}(J\delta D)$.

In any case the anomaly is   furthermore!!!!

$$-\mathrm{Tr}(J[D,X])$$

$$= -\mathrm{Tr}(J(DX^+ - X^-D))$$

$$= -\mathrm{Tr}(JDX^+) + \mathrm{Tr}(DJX^-)$$ YES!!

$$= -\mathrm{Tr}((I-QD)X^+) + \mathrm{Tr}((I-DQ)X^-)$$

$$= -\mathrm{Tr}((I-QD)X^+) + \mathrm{Tr}((I-DQ)X^-)$$

---

what I find is this. That

$$\mathrm{Tr}(J[X,D]) = \mathrm{Tr}(\varepsilon K X)$$

Canonical section s