To back over the determinant line bundle and how one computes its first Chern class as an equivariant form. I place myself over the $2^n$-sphere so that the Todd or A-class is 1. What I have is a compact spin manifold $M$ and hence a spinor bundle $S = S^+ \otimes S^-$. Then I take a vector bundle $E$, let $A \in \text{SO}(E)$ a gauge transf. Then in $A \otimes A$ I get $\mathcal{D}: \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$. In this way I get a family of elliptic operators $A \otimes A$ equivariant for $A$.

More precisely over $A \times M$ I have an operator $p_2^*(S^+ \otimes E) \to p_2^*(S^- \otimes E)$ linear with fibres in $A$. Can take bundle over $A$ whose fibre is $\Gamma(S^+ \otimes E)$ at each point.

Anyway what I want to do is quotient out by $\mathcal{D}$. I get over $B^A \times M$ a tangent bundle $\bar{E}$. Then I can form $p_2^*(S^+ \otimes \bar{E})$ and I have a family of elliptic operators on the fibres over $B^A$.

The index theorem for families says

$\text{ch}(\text{index}) \in H^*(B^A)$

is given by $\int_M \text{ch}(\bar{E}) A(M)$. Thus we want to compute $\text{ch}(\bar{E})$. 

$\tilde{E} =$ taut. bundle over $G \times M$, belonging to $E/M$ considered as an $G$-bundle over $M$.

If I had an $G$-invariant connection in $E$, then I could compute the character of $\tilde{E}$ in the alg. of equivariant forms:

\[ [W(\tilde{g}) \otimes \Omega(M)]_{\text{basic}} \]

It is worth doing this calculation separately. So suppose $G$ acts in $M$ and $E$ is an equiv. bundle, $D: \Omega^*(M, E) \leftarrow$ is an invariant connection. Can you explain how to construct $\text{ch}_n(D)$ as an equivariant form?

Introduce the principal bundle $P$ for $E$. $\pi: P \to M$, $G$ acts to the left and $\mathcal{U}_N$ to the right. Now $\pi^*(E)$ is canonically trivial whence $D = d + \theta$ where

$\theta \in \Omega^1(P, \mathcal{U}_N)$

So we get

$W(\mathcal{U}_N) \to \Omega(P)$
G acts on $\mathcal{M}$, $E$ in a $G$-vector bundle, and $D$ is an invariant connection. I want to construct $\text{ch}_n(E)$ as an equivariant form. Geometrically what we do is to pull $E$ back to $PG \times \mathcal{M}$. Over $PG$ is a canonical connection $\Theta$ relative to $BG$; this is what we assume when we think of there being a canonical map

$$\psi : W(g) \longrightarrow \Omega(PG).$$

Now over $PG \times \mathcal{M}$ we have $\text{pr}_2^*(E)$ with the connection $D$.

So let's now look at the situation when $G$ acts freely on $\mathcal{M}$ and one is given a connection form $\theta$ for $\mathcal{M} \rightarrow G/\mathcal{M}$. This means

$$\begin{cases} 
\theta \in \Omega^1(P) \\
ix \theta = \chi 
\end{cases}$$

is $G$-invariant.

We want to descend $D$, but can't because

$$[i_x D] = \mathcal{L}_x \varphi_x \in \Omega^0(M, \text{End} E)$$

so we replace $D$ by $\tilde{D} = D + \varphi_x \Theta^x$. Then

$$\theta = \frac{1}{\varphi} \Theta^x$$

basis for $g$

$$\begin{cases} 
\tilde{D} = D + \varphi_x \Theta^x \\
[i_x \tilde{D}] = [i_x D] + [i_x \varphi \Theta^x] = \varphi_x - \varphi_x = 0
\end{cases}$$
Also \( [\mathcal{L}_x, \overline{D}] = [\mathcal{L}_x, D] - [\mathcal{L}_x, \psi \Theta] \) should be OKAY.

\[
\begin{align*}
\theta &\in g \otimes \Omega^1(M) \\
\psi &\in \Omega^0(M, \text{End} E) \otimes g^* \\
\therefore \psi \theta &\in \Omega^1(M, \text{End} E)
\end{align*}
\]

So now we have \( \overline{D} = D - \psi \Theta \) acting on \( \Omega^*(M, E) \). So we next look at

\[
\begin{align*}
\overline{D}^2 &= (D - \psi \Theta)^2 \\
\overline{D}^2 &= D^2 - [D, \psi \Theta] + \psi \Theta \Psi \Theta
\end{align*}
\]

So what next?

\[
W(g) \otimes \Omega(M^1, E)
\]

\[
\overline{D} = D - \psi \Theta
\]

\[
\text{tr}(\overline{D}^2)^n \in [W(g) \otimes \Omega(M)]^n_{\text{basic}}
\]
Repeat:

I'm looking at the simple question of computing equiv. char. classes for an equivariant bundle $E/M$ when an inv. conn. $D$ is given. The answer is that I define $\overline{D} = D - \varphi \Theta$ operator on $W(\mathcal{g}) \otimes \Omega(M, E)$ degree 1 deriv. over $W(\mathcal{g}) \otimes \Omega(M)$. Then $tr(\overline{D}^2)^n \in \left[W(\mathcal{g}) \otimes \Omega(M)\right]_{\text{basic}}$

The next step is to explain what this becomes under the inv.

$$\left[W(\mathcal{g}) \otimes \Omega(M)\right]_{\text{basic}} \to \left[S(\mathcal{g}^*) \otimes \Omega(M)\right]^G$$

which I know is induced by the map

$$W(\mathcal{g}) \to S(\mathcal{g}^*)$$

$$\Theta^a \mapsto 0$$

$$\omega^a \mapsto \Omega^a$$

$$\overline{D}^2 = D^2 - [D, \varphi \Theta^a] + \varphi \Theta^a \varphi_b \Theta^b$$

$$[D, \varphi \Theta] = [D, \varphi] \Theta + \varphi d\Theta$$

$$\overline{D}^2 \to D^2 - \varphi \Omega$$
Check this computation out with bases.

\[ \Omega = d\theta + \theta^2 = d\theta + \frac{1}{2} [\theta, \theta] \]

\[ \lambda_\alpha \Omega^\alpha = d(\lambda_\alpha \theta^\alpha) + \frac{1}{2} [\lambda_\alpha \theta^b, \lambda_\beta \theta^c] \]

\[ = \lambda_\alpha d\theta^\alpha + \frac{1}{2} f^{\alpha}_{\beta\gamma} \lambda_\alpha \theta^\beta \theta^\gamma \]

\[ \Omega^\alpha = d\theta^\alpha + \frac{1}{2} f^{\alpha}_{\beta\gamma} \theta^\beta \theta^\gamma \]

\[ \tag{6} \]

\[ \omega_\alpha = [\xi_\alpha, D] \]

\[ [\xi_\alpha, D_i] = \left( D_i \xi_\alpha D + D_j \xi_\alpha D_{ij} \right) - (D_i \xi_\alpha D + D_{ij} \xi_\alpha D^{ij}) \]

\[ = D^2 \]

\[ = [D, \xi_\alpha D + D_i \xi_\alpha] = [D^2, \xi_\alpha] \]

\[ D^2 \phi \Omega \rightarrow (1 - \theta^\alpha \xi_\alpha + \frac{1}{2} \theta^\alpha \theta^\beta \xi_\alpha \xi_\beta)(D^2 \phi \Omega) \]

\[ D^2 \phi \Omega = \theta^\alpha \xi_\alpha D^2 + \frac{1}{2} \theta^\alpha \theta^\beta \xi_\alpha \xi_\beta D^2 \]

\[ + \theta^\alpha \xi_\alpha \phi \Omega \]

\[ = \xi_\alpha \phi \Omega \]

\[ \tag{6} \]

\[ 6 \xi_\alpha D^2 = 2 \xi_\alpha (D \phi \Omega) \]
\[ D^2 = (D - \varphi \Theta)^2 = D^2 - [D, \varphi \Theta] + \varphi \Theta \varphi \Theta \]

\[ (D^2 - \varphi \Omega) \quad \longrightarrow \quad D^2 - \varphi \Omega = \Theta^a \varepsilon_a D^2 + \frac{1}{2} \Theta^a \Theta^b \varepsilon^a \varepsilon^b D^2 \]

Mistake back on p.3.
\[ [\lambda_x, D] = \lambda_x + \varphi_x \]

\[ \overline{D} = D - \varphi \Theta \]

\[ [\lambda_x, \overline{D}] = \lambda_x^x + \varphi_x - [\lambda_x, \varphi \Theta] = \lambda_x \]

\[ [\lambda_x, D^2] = \lambda_x \overline{D} - \overline{D} \lambda_x = [\lambda_x, \overline{D}] = 0. \]

So what we have is the map
\[ (W(g) \otimes \Omega(E))_{\text{basic}} \longrightarrow (\Omega(g) \otimes \Omega(M, EndE)) \]
\[ D^2 = (D - \varphi \Theta)^2 \quad \longrightarrow \quad D^2 - \varphi \Omega \]

\[ \text{Chernology...} \]
At this point I understand how to compute the equivariant classes of an equivariant bundle $E$ over $M$ assuming the existence of an invariant connection $D$. One has the equivariant curvature in

$$(W(g) \otimes \Omega(M, \text{End} E))_{\text{basic}} \rightarrow (S(g^*) \otimes \Omega(M, \text{End} E))^g$$

$$D^2 = (D - \Theta)^2 \rightarrow D^2 - \Phi \Omega$$

where

$$\Theta \in g \otimes W^1(g)$$

$$\Omega = d\Theta + \frac{1}{2} [\Theta, \Theta] \in g \otimes W^2(g)$$

are the canonical tensors.

So now if I want the equivariant class, it is

$$\frac{1}{n!} \text{Tr} (D^2 - \Phi \Omega)^n \in [S(g^*) \otimes \Omega(M)]^G \otimes \mathbb{R}$$

and it can be brought back to

$$[W(g) \otimes \Omega(M)]_{\text{basic}}$$

via the operator

$$\Pi_\alpha^a (1 - \Theta^a \Theta_a) = 1 - \Theta^a \Theta_a + \frac{1}{2} \Theta^a \Theta_b \Theta_{ab}$$
Now we go back to the case of $\frac{pr_2^*(E_0)}{\bigwedge} \otimes \Lambda^1_{\text{ext}}$ and $G = \mathbb{R}^+$ the group of gauge transformations. Then $E = \text{pr}_2^*(E_0) = A \times E_0$ over $A \times M$ has a canonical connection. (There is a problem in writing it down.) Suppose $E_0$ is trivial. Then any connection can be compared to $d$. So we have

$$D = \frac{d}{A \times M} + A \mu \text{d}x^\mu$$

Here $A \in \Omega^{0,1}(A \times M, \mathfrak{gl}_n)$.

Now we need the action of $g \in G$. Then $g$ acts on $A \times E_0$ and $D$ should be covariant

$$gDg^{-1} = \frac{d}{A \times M} + (g \text{d}g^{-1} + gAg^{-1})$$

This should be the pull back of $D$ under $g : (A, m) \mapsto (g(a), g(A), m)$

If $v$ is an infinitesimal gauge transformation, then $A, m \mapsto A = dv + [v, A], m$

$$\iota_v D =$$

In any case $\varphi_v \in \Omega^0(A \times M, \mathfrak{gl}_n)$ is linear $\varphi_v : (A, m) \mapsto \varphi_v(m)$
Need curvature as well as \( \Phi \):

\[
D^2 = \delta A_\mu \, dx^\mu + \frac{1}{2} F_{\mu \nu} \, dx^\mu dx^\nu + 0,2
\]

Here \( A_\mu \) is a matrix function on \( A \) and \( \delta A_\mu \) is its differential.

\[\text{\lowercase{(10)}}\]

Now do a Riemann surface. What do you get???

\[
k^2 \int \frac{tr (\frac{1}{2} Q^2)}{M} = \frac{k^2}{2} \int \frac{tr (\delta A_\mu \, dx^\mu)^2}{M}
\]

Next do a 4-manifold to get

\[
\frac{k^3}{3!} \int \frac{tr [(\delta A_\mu \, dx^\mu)^2 F]}{M}
\]

where \( F \) is the curvature.

This is just calculating the curvature. In addition I need to compute the Higgs field

\[
\int \frac{1}{2} \left( D^2 - \Phi \Omega \right)^2 = \int \frac{1}{2} \frac{tr (D^2)^2 - \Phi D^2 \Phi - \Phi \Phi D^2}{M}
\]

So it would appear that I get
\[ e_x = \int \text{tr}(FX) \] for a surface \( R^2 \)

\[
\int_M \frac{1}{3!} \text{tr}\left((D^2 - \varphi \Omega)^3\right) = -\Omega \int \frac{1}{3!} \text{tr}\left(3(D^2)^2 \varphi\right)
\]

so get \( \varphi_x = \frac{1}{2} \int \text{tr}\left(F^2 X\right) \) for \( R^2 \).

Let's compare these formulas.

\[
\begin{align*}
\omega &= \frac{K}{2} \int_M \text{tr}\left(\delta A_\mu dx^\mu\right)^2 \\
\varphi_x &= K \int_M \text{tr}(FX)
\end{align*}
\]

\[
\begin{align*}
\omega &= \frac{K^2}{3!} \int_M \text{tr}\left(\delta A_\mu dx^\mu\right)^2 F \\
\varphi_x &= \frac{K^2}{2} \int_M \text{tr}\left(F^2 X\right)
\end{align*}
\]

Recall that under the isomorphism

\[
[W(g) \otimes \Omega(M)] \overset{\sim}{\longrightarrow} -[S(g^*) \otimes \Omega(M)]
\]

I need the formula for \( \alpha \).

\[
p(\alpha)(1 - \theta^i \dot i_a + \frac{1}{2} \theta^i \Theta^a \epsilon_{i,j} \ldots \Theta^a) \rho(dx) \alpha \\
\overset{\int d}{\sim} \theta \mapsto \rho(\Omega)(dx - \Omega^a \epsilon_{i,a} \alpha)
\]
Hence on \( [S(g^*) \otimes \Omega(M)]^G \) the diff. is
\[ d_m - \Delta^a_i a. \]

Indeed,
\[ (d_m - \Delta^a_i a)^2 = d^2 - \Delta^a_i a - \Delta^b_i a + \Delta^a_i b \]
\[ (d_m - \Delta^a_i a)^2 = 0 \] on the invariants.

Assertion: \( \varphi \omega \) is an \( \Delta \) equivariant 2
form:
\[ \omega - \Delta \varphi \]

\[ (d_m - \Delta^a_i a)(\omega - \Delta \varphi) = 0 \text{ or} \]
\[ d\omega = 0, \quad -\Delta d\varphi - \Delta \omega = 0 \]
\[ d\omega = 0 \text{ and } \Delta d\varphi + \omega = 0 \]

So at this point we reach the following problem. We have described \( c_1(Z) \in H^2(BG) \) by this
in \( \omega, \varphi \) or better \( \omega - \Delta \varphi \), Now I have
a problem of constructing the transgression. This
means that I must use the contractibility of \( A \).
What do I know in the case of a surface? Fix degree 0 case. Then I computed the curvature and found it to be as above! 

But what is the anomaly?

Let's go back to the idea that one has two things: two halves of the Dirac operator. How am I to handle this??

\[ \mathcal{D}_A = \frac{1}{i} \gamma^\mu \left( \partial_\mu + A_\mu \right) = \begin{pmatrix} 0 & D_A^* \\ D_A & 0 \end{pmatrix} \]

\[ \delta \log \det (\mathcal{D}_A) = \text{Tr} \left( \mathcal{D}_A^{-1} \delta \mathcal{D}_A \right) \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \delta \mathcal{A} = \frac{1}{i} \gamma^\mu \delta A_\mu \]

. . . \[ \delta \log \det (\mathcal{D}_A) = \text{Tr} \left( \mathcal{D}_A^{-1} \delta \mathcal{D}_A \right) + \text{Tr} \left( D_A^* \delta D_A^* \right) \]

In the surface case what I do is regularize the trace in a certain way. Namely, I choose a connection extending the \( 3 \)-form. Then I remove it off to get the current \( J \).

\[ D^{-1} - Q = J \]

\[ \text{Tr} (J \delta D) = \int dx dy J(x,y) SD(y,x) \]
19) So what are does is to conclude
that if $\mathbf{J}$ is actually defined as a smooth
kernel, but $\text{Tr}(\mathbf{J}SD) = \int \text{Tr}(\mathbf{JSD}) d^2x$.
In any case the anomaly is furthermore!!!

$$-\text{Tr}(\mathbf{J} [D,X])$$

$$= -\text{Tr} ( \mathbf{J} (DX^+ - X^-) )$$

$$= -\text{Tr} ( \mathbf{JD} X^+) + \text{Tr} (D\mathbf{J} X^-)$$

$$= -\text{Tr} ( (I-QD)X^+ ) + \text{Tr} (I-DQ)X^-$$

$$= -\text{Tr} ( (I-QD)X^+ ) + \text{Tr} (I-DQ)X^-$$

What I find is this. That

$$\text{Tr}(\mathbf{J} [X,D]) = \text{Tr}(\mathbf{e} KX)$$

Canonical section 5