11/16/83

Course:

G Lie gp. $\quad \mathfrak{g}$ = Lie alg

P principal bundle

G acts on the right of P, hence acts on $\Omega(P)$ by

$$g \cdot \omega = (R_g)^* \omega \qquad R_g = \text{right mult by } g \in G \text{ on } P.$$

If $X \in \mathfrak{g}$ define

$$\cancel{\mathcal{L}_X \omega = \frac{d}{dt} e^{tX} \cdot \omega}$$

a vector field X on P by

$$Xf = \frac{d}{dt} e^{tX} \cdot f \Big|_{t=0}$$

more generally

$$\mathcal{L}_X \omega = \frac{d}{dt} e^{tX} . \omega \Big|_{t=0} .$$

Then one has usual rules

$$[d, \iota_X] = \mathcal{L}_X \qquad [\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}$$

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$$

Connection on P is a $\theta \in \Omega^1(P) \otimes \mathfrak{g}$ $\qquad \theta = \theta^a X_a$

i) $\iota_X \theta = X \iff \iota_{X_a} \theta^b = \delta_a^b$

ii) G-invariant

$$Ad(g) R_g^* \theta = 0$$

$$\Longrightarrow [X, \theta] + \mathcal{L}_X \theta = 0 \qquad \Longleftarrow \text{ if } G \text{ is conn.}$$

Let's take $P = G$. Then to each $X \in \mathfrak{g}$ we have assigned a vector field [redacted] X on G, namely, the ~~vector~~ vector field corresp. to $R_{e^{tX}}$. Thus the vector field is left-invariant. $\therefore$

$$\mathfrak{g} \xrightarrow{\sim} \text{left-invariant v.f. on } G$$

Also [redacted] $\mathfrak{g} \xrightarrow{\sim} T_{G,g}$ for any $g \in G$

so can define $\cancel{\mathfrak{g}^* \to \Omega^1(G)}$

$$\lambda \longmapsto \Theta_\lambda = \text{the unique 1-form with } \iota_X \Theta_\lambda = \lambda(X)$$

$$\therefore \quad \mathfrak{g}^* \xrightarrow{\sim} \text{left invariant 1-forms on } G$$

$$\Lambda \mathfrak{g}^* \xrightarrow{\sim} \text{left-inv. forms on } G.$$

<u>Question</u>: What is d on left-inv. forms

Now [redacted] G has a unique connection $\theta \in \Omega^1(G) \otimes \mathfrak{g}$
(This is not too clear.)

$$\theta = \theta^a X_a$$

(i) $\iota_X \theta = \theta \iff \iota_{X_b}(\theta^a) = \delta^a_b$

($\Rightarrow \theta^a$ are the left-inv. 1-forms belonging to the basis of $\mathfrak{g}^*$ dual to X_a)

(ii) [redacted]

$$\mathrm{Ad}(g) \cdot R_g^* \theta = \theta \implies \mathcal{L}_X \theta + [X, \theta] = 0$$

(iii) $L_g^* \theta = \theta \iff$ each $\theta^a \in \mathfrak{g}^*$.

(I could use a direct proof of (ii).)

Define $[\theta,\theta] \in \Omega^2(P)\otimes\mathfrak{g}$ as follows

$$\theta\otimes\theta \in \left(\Omega^1(P)\otimes\mathfrak{g}\right)\otimes\left(\Omega^1(P)\otimes\mathfrak{g}\right) \longrightarrow \Omega^2(P)\otimes\mathfrak{g}\otimes\mathfrak{g}$$

$$\theta\otimes\theta = \left(\theta^a\otimes X_a\right)\otimes\left(\theta^b\otimes X_b\right) \longmapsto \theta^a\theta^b\otimes X_a\otimes X_b$$

$$\begin{array}{ccc} \theta^a\theta^b\otimes X_a\otimes X_b & & \\ \downarrow & & \downarrow \\ & & \Omega^2(P)\otimes\mathfrak{g} \\ \theta^a\theta^b\otimes[X_a,X_b] & & \end{array}$$

Clearly $\iota_X[\theta,\theta] = [\iota_X\theta,\theta] - [\theta,\iota_X\theta]$

$= [X,\theta] - [\theta,X] = 2[X,\theta]$

$\therefore\ \mathcal{L}_X\theta + [X,\theta]$

$= d\underbrace{\iota_X\theta}_{X\in\Omega^0(P)\otimes\mathfrak{g}} + \iota_X d\theta \qquad \therefore dX = 0$

$$0 = \iota_X d\theta + [X,\theta] = \iota_X\left(d\theta + \tfrac{1}{2}[\theta,\theta]\right)$$

Specialize to $P = G \implies \boxed{d\theta + \tfrac{1}{2}[\theta,\theta] = 0}$

$\therefore$ have

<u>Prop</u>: The MC form $\theta \in \mathfrak{g}^*\otimes\mathfrak{g} \subset \Omega^1(G)\otimes\mathfrak{g}$ satisfies $d\theta + \tfrac{1}{2}[\theta,\theta] = 0$.

Claim this determines d on $\Lambda\mathfrak{g}^* =$ left-inv. forms

$\theta = \theta^a X_a \qquad [X_a,X_b] = f_{ab}^{\ c} X_c$ ← struc. const of $\mathfrak{g}$

Then
$$d\theta^a + \frac{1}{2} f^a_{bc}\, \theta^b \theta^c = 0$$
so you know d = the deriv. on $\Lambda^{\cdot} \mathfrak{g}^*$ = ext. alg gen by θ^a with d given as above.

In the case of matrix valued forms
$$\frac{1}{2}[\theta,\theta] = \frac{1}{2}(\theta\theta + \theta\theta) = \theta^2$$
so I will usually write $\boxed{d\theta + \theta^2 = 0}$ for the structural equations of G