ground energy as a det. 1-21
Graeme's notes on ground energy 21-22
parameters for D: 28-92
flat Hess's fn. 31

Anomaly \( \delta \) + higher order C fn. 43-53

\( \langle 0 | \delta | 0 \rangle \) as a determinant 54-72
The general question is how to interpret the ground energy of an independent fermion gas as a determinant. Single

Let's suppose the particles are described by a Hamiltonian operator $H$ acting on $V$. First suppose $V$ finite-diml. Extend $H$ to a derivation $A$ on $\Lambda(V)$. Assume $H$ doesn't have 0 for an eigenvalue.

Then I know the ground energy of $H$ is the sum of the negative eigenvalues of $H$, and the ground state is non-degenerate. Call this ground energy

$$E(H) = \sum_{\lambda < 0} \lambda.$$ 

We have the variational formula

$$\delta E(H) = \text{Tr} \left( P_+ H \right) \quad P_+ = \text{proj. on } H_{< 0}.$$ 

Next relate this to the operators $\frac{d}{dt} + H$ on $L^2(\mathbb{R}, V)$. The inverse of $\frac{d}{dt} + H$ is given by

$$G(t, t') = \langle t | (\frac{d}{dt} + H)^{-1} | t' \rangle$$

$$= e^{-(t-t')H} \begin{cases} P_+ & t > t' \\ -P_- & t < t' \end{cases}.$$ 

Hence formally we have

$$\delta \log \det \left( \frac{d}{dt} + H \right) = \text{Tr} \left\{ (\frac{d}{dt} + H)^{-1} \delta H \right\}$$

$$= \int dt \text{ tr} \left\{ G(t, t) \delta H \right\}.$$ 

We have to make sense of $G(t, t)$ in some way. Let's do this regularization by taking

$$G(t-, t) = -P_-.$$
Then \[ d \log \det \left( \frac{d}{dt} + H \right) = -\left( \int dt \right) \text{tr} (P \Delta H) \]

Since the factor \( \int dt \) is infinite, it's clear that the determinant is not defined, but rather one wants to work over a finite interval of length \( T \) and divide by \( T \) and let \( T \to \infty \).

**Frequency picture:** \[ f(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} f(\omega). \]

Then \[ \frac{d}{dt} \to -i\omega. \]

\[ \text{Tr} \left\{ \left( \frac{d}{dt} + H \right)^{-1} \Delta H \right\} = \text{Tr} \left\{ (-i\omega + H)^{-1} \Delta H \right\}. \]

Here \( \Delta H \) and \( (-i\omega + H)^{-1} \) are multiplication ops. on \( L^2 \left( \frac{d\omega}{2\pi}; V \right) \), and multiplication operators don't have a trace. The best thing one has is an average. In formulas:

\[ \text{Tr} \left\{ (-i\omega + H)^{-1} \Delta H \right\} = \int d\omega \int d\omega' \text{Tr} \left\{ (-i\omega + H)^{-1} \delta(\omega - \omega') \times \Delta H \delta(\omega - \omega') \right\} \]

\[ = \int d\omega \ \text{tr} \left( \frac{1}{-i\omega + H} \Delta H \right) \delta(\omega) \]

\[ = 2\pi \delta(0) \cdot \int \frac{i d\omega}{2\pi i} \ \text{tr} \left( \frac{1}{i\omega - H} \Delta H \right) \]

\[ = -2\pi \delta(0) \cdot \int_{-i\infty}^{i\infty} \frac{d\lambda}{2\pi i} \ \text{tr} \left( \frac{1}{\lambda - H} \Delta H \right) \]

Some problems occur: The contour integral must be closed off as follows:
so as to get the operator $P_-$. This is the same as choosing the regularization $G(t^{-\beta}t)$. Similarly, we have the infinity $2\pi \delta(0) = \oint dt$.

Possibility: $\det \left( \frac{d}{dt} + H \right)$ makes sense as a von Neumann algebra determinant.

One way to work over a finite interval of length $T$ and let $T \to \infty$ is to use the temperature formalism. In this case we work with

$$e^{-\beta F(\hat{H})} = \text{Tr} \left( e^{-\beta \hat{H}} \right)$$

where $F(\hat{H})$ is the free energy of $\hat{H}$. We have

$$\text{Tr} \left( e^{-\beta \hat{H}} \right) = \det \left( 1 + e^{-\beta H} \right) = \prod_{\alpha} \left( 1 + e^{-\beta \varepsilon_{\alpha}} \right)$$

so that $F(\hat{H}) \to E(\hat{H}) \propto \beta \to \infty$. Variational formula:

$$\delta F(\hat{H}) = -\frac{1}{\beta} \int \delta \log \det \left( 1 + e^{-\beta H} \right)$$

$$= \text{Tr} \left( \frac{e^{-\beta H}}{1 + e^{-\beta H}} \delta H \right)$$

Here $\frac{e^{-\beta H}}{1 + e^{-\beta H}} \left| \alpha \right\rangle = \eta_{\alpha} \left| \alpha \right\rangle$ where $\eta_{\alpha} = \frac{e^{-\beta \varepsilon_{\alpha}}}{1 + e^{-\beta \varepsilon_{\alpha}}}$ is the average no. of particles in the state $\alpha$.

Now relate the above to the operator $\frac{d}{dt} + H$ on $[0, \beta]$ with anti-periodicity boundary conditions. The inverse is given by

$$G(t, t') = e^{-(t-t')H} \begin{cases} \frac{1}{1 + e^{-\beta H}} & t > t' \\ \frac{-e^{-\beta H}}{1 + e^{-\beta H}} & t < t' \end{cases}$$
Hence
\[ G(t^-, t) = -\frac{e^{-\beta H}}{1 + e^{-\beta H}}. \]

Thus starting from
\[ S \log \det \left( \frac{d}{dt} + H \right) = \text{Tr} \left\{ \left( \frac{d}{dt} + H \right)^{-1} \delta H \right\} \]
\[ = \int_0^\beta \text{Tr} \left\{ e^{-\beta H} \delta H \right\} \]
we get
\[ = -\left( \int_0^\beta \text{Tr} \left\{ \frac{e^{-\beta H}}{1 + e^{-\beta H}} \delta H \right\} \right) = S F(H). \]

Which gives
\[ \det \left( \frac{d}{dt} + H \right) \text{ in } L^2([0, \beta], V), \text{ anti-periodic b.c. } \]
\[ G(t^-, t) \text{ reg.} \]
\[ = e^{-\beta F(H)}. \]
The problem is to relate the determinant of $\tilde{\mathcal{D}}$ operators to the determinants of Dirac operators encountered in physics, e.g. Schwinger's work. So let's begin with the Dirac equation which gives the 1-particle Hamiltonian:

$$i \partial_t \psi = H \psi \\
H = \alpha \frac{1}{i} \partial_x + \beta m$$

We want $(\alpha p + \beta m)^2 = p^2 + m^2$ so $\alpha^2 = \beta^2 = 1$, $\alpha \beta + \beta \alpha = 0$.

$$(\frac{1}{i} \partial_t + \alpha \frac{1}{i} \partial_x + \beta m) \psi = 0$$

$$(\beta \frac{1}{i} \partial_t + \beta \alpha \frac{1}{i} \partial_x + m) \psi = 0$$

For imaginary time the eqn. is $(\partial_t + H) \psi = 0$, or

$$\left( \beta \partial_t + \beta \alpha \frac{1}{i} \partial_x + m \right) \psi = 0$$

so that $g^0 = \beta$, $g^1 = \beta \alpha$ are self-adjoint satisfying $g^\mu g^\nu + g^\nu g^\mu = 2\delta_{\mu \nu}$. Standard choices are

$$g^0 = \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so that

$$\frac{1}{i} \beta \alpha = \frac{1}{i} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Now I am interested in the case $m = 0$, in which case

$$\partial_t + H = \begin{pmatrix} \partial_t + \frac{1}{i} \partial_x \\ \partial_t - \frac{1}{i} \partial_x \end{pmatrix}$$

and so the operators $\partial_t + H$ is essentially the direct sum of $\partial \bar{z}$ and $\partial z$. The two helicities...
The massless Dirac equation in 1-space dimension is therefore very simple: The Hamiltonian is just
\[ H = \begin{pmatrix} +\frac{i}{2} \partial_x \\ -\frac{1}{2} \partial_x \end{pmatrix} \]
and if we fix the helicity, then we get
\[ H = +\frac{i}{2} \partial_x . \]

At this point I can try to understand the physics of this Hamiltonian and its gauge transformations. For example, I can put periodic conditions in \( x \) and look at a fermion gas of particles governed by \( H + \text{gauge potential} \) at a given temperature. I can look at the ground energy shift as I vary the potential. These problems involve regularizing the kernel for \( \partial_x + H \) along the diagonal, which seems to involve choices. Somehow the physics makes these choices in a definite way, which I should try to understand.
August 8, 1982

I consider the Hamiltonian $H = \frac{1}{i} \partial_x$ acting on functions of $x$, say periodic with period $\U$. Then I want to form $\hat{H}$ in the Fock space. This already involves a choice of an additive constant for the ground energy. Next I want to consider a perturbation $H = \frac{1}{i} (\partial_x + A)$, where $A$ is a function of $x$, and compute the ground energy shift. To first order it is $\langle 0 | \delta H | 0 \rangle$

where

$$\delta H = \int \psi^*(x) \frac{i}{2} A(x) \psi(x) \, dx$$

so

$$\langle 0 | \delta H | 0 \rangle = \int dx \frac{i}{2} A(x) \langle 0 | \psi^*(x) \psi(x) | 0 \rangle.$$ But this involves $\langle 0 | \psi^*(x) \psi(x) | 0 \rangle$ which is a diagonal part of a Green's function, and therefore has to be defined by a process of regularization.

Formulas: Ortho normal basis, $|k\rangle = \frac{1}{\sqrt{L}} e^{ikx} \frac{k}{2\pi} \mathbb{Z}.$

$$\langle x | = \sum_k \langle x | k \rangle \langle k |$$

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} \psi_k$$

$$\psi^*(x) = \frac{1}{\sqrt{L}} \sum_{k'} e^{-ik'x} \psi^*_k$$

Take the ground state $|0\rangle$ in Fock space to be filled with all $|k\rangle$ with $k < 0$. Then

$$\langle \psi(x) \psi^*(y) \rangle = \frac{1}{L} \sum_k e^{ik(x-y)} \langle \psi_k \psi^*_k \rangle$$

$$\langle x \rangle = \frac{1}{L} \sum_{k>0} e^{ik(x-y)} = \frac{1}{L} \left( \frac{e^{ik(x-y)}}{1 - e^{2\pi i} e^{ik(x-y)}} \right)$$

$$= \frac{1}{L} \left( \frac{1}{e^{-ik(x-y)} - 1} \right) \rightarrow \frac{\alpha}{L \rightarrow \infty} = \frac{1}{2\pi i} \frac{1}{x-y+i0^+} \rightarrow 0$$

We see that there are difficulties with the value of

$$\langle \psi^*(x) \psi(y) \rangle = \frac{1}{L} \sum_{k<0} e^{ik(x-y)} = \frac{1}{L} \left( \frac{1}{1 - e^{2\pi i} e^{ik(x-y)}} \right)$$

$$\frac{\alpha}{L \rightarrow \infty} = \frac{1}{2\pi i} \frac{1}{(x-y)-i0^+}$$
at $x = y$.

Review the 1-particle operator (derivation on Fock space) corresponding to multiplication by f(x).

$$\psi^*(x) \psi(x) = \frac{1}{L} \sum_{k,e} e^{-i k x} \psi^*_k e^{i k y} \psi_e$$

$$= \frac{1}{L} \sum_{k} e^{-i k x} \sum_{e} \psi^*_e \psi_e$$

Hence

$$\int_0^L dx \ f(x) \psi^*(x) \psi(x) = \frac{1}{L} \sum_{k=0}^L \left( \int_0^L dx \ f(x) e^{-i k x} \right) s_0$$

Recall that $s_0$ for $q = 0$ the infinite sum in the definition of $s_0$ makes sense, so we can regard $s_0$ as defined for $q = 0$. For $q = 0$, however, it is the number operator, and we give the special definition

$$s_0 = \sum_{k>0} \psi^*_k \psi_k - \sum_{k<0} \psi_k \psi^*_k$$

so that $s_0 \ket{0} = 0$, where $\ket{0}$ is the state where all $\ket{k}$ with $k < 0$ are filled.

Review the commutation relations.

$$[s_0, s_q] = 0 \quad \text{if} \quad p+q \neq 0$$

For $p > 0$, $$[s_p, s_{-p}] = -\text{the number of } k \text{ with } 0 < k < p$$

$$= -p \frac{1}{2\pi}.$$  

As a check notice that for $p > 0$, $s_p = \sum_k \psi^*_k \psi_k$ kills the ground state $\ket{0}$, in which all $k < 0$ are filled. Hence $s_p$ is a destruction operator.

Now we have made sense of the operators $s_0$ and hence of the operators

$$\hat{f} = \int dx \ f(x) \psi^*(x) \psi(x)$$
with $f$ smooth. Actually it might be more relevant to think of Fock space as a representation of the central extension of the abelian Lie algebra of these functions $f(x)$.

The next thing is to perturb $\hat{H}_0$ by an operator $\hat{f}$ and ask for the ground energy shift. What seems to be the case is the following. For each $f$ we get then a number

$$E(\hat{H}_0 + \hat{f}) = \text{ground energy of } \hat{H}_0 + \hat{f}$$

where $\hat{f}$ is defined using $\otimes$. This will be a kind of determinant for $\frac{\partial}{\partial t} + \frac{1}{2} \partial_x^2 + f$, and hence should correspond to a regularization process on the Green's function. (Perhaps we can even handle time-dependent perturbations ultimately.)
Let's suppose we have a perturbation $H = H_0 + V$ where $H_0$ is time independent and $V = V(t)$ has compact support, say inside $[-T, T]$. Let $U(t, t')$ be the propagator for 
\[ (\partial_t + H) \psi = 0 \]
and let $\langle 0 |$ be the ground state for $H_0$. I will be interested in the quantity
\[
\langle 0 | s | 0 \rangle = \frac{\langle 0 | U(T, -T) | 0 \rangle}{\langle 0 | U_0(T, -T) | 0 \rangle}
\]
which I want to interpret as a determinant in the case of independent fermions. Variational formula:
\[
\delta \log \langle 0 | s | 0 \rangle = -\int_{-T}^{T} dt \frac{\langle 0 | U(T, t) \delta H(t) U(t, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle}
\]
Now I have to switch to Fock space. I suppose $H, H_0, V$ are 1-particle operators in Fock space, e.g.
\[ V(t) = \sum_{\beta, \alpha} \psi_\beta^* \psi_\alpha(t) \psi_\alpha^* \psi_\beta. \]
Then the above integrand is
\[
\sum_{\beta, \alpha} \frac{\langle 0 | U(T, t) \psi_\beta^* \psi_\alpha U(t, -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle} \delta H_{\beta \alpha}(t)
\]
But we have the Greens fn.
\[
G_{\alpha \beta}(t, t') = \begin{cases} 
\frac{\langle 0 | U(T, t) \psi_\alpha U(t, t') \psi_\beta^* U(t', -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle} & t > t' \\
\frac{\langle 0 | U(T, t) \psi_\beta^* U(t', t) \psi_\alpha U(t', -T) | 0 \rangle}{\langle 0 | U(T, -T) | 0 \rangle} & t < t'
\end{cases}
\]
which we know is an inverse for $\mathcal{D} + H$.

Combining the above leads to

$$8 \log \langle 0 | S | 0 \rangle = \int_{-T}^{T} \sum_{\alpha \beta} C_{\alpha \beta} \rho(t, t) \delta H_{\alpha \beta}(t)$$

and from this we see that $\langle 0 | S | 0 \rangle$ has an interpretation as a determinant.

Now I want to apply this to the case where the operator $H_0$ is an extension of the 1-particle operator $\frac{i}{\hbar} \partial_x$ acting on $L^2(\mathbb{R}/2\pi \mathbb{Z})$ and where $V(t)$ comes from a multiplication operator. In this case $V(t)$ does not have the form $\langle x \rangle$, at least as a finite sum.
August 10, 1982

In the case of a constant perturbation \( H = H_\circ - \mu \) the partition \( \rho_\circ \) should be a determinant of \( \frac{d}{dt} + H \) on \([0, \beta]\) with anti-periodic boundary conditions. I have to define \( \hat{H} \) on Fock space which involves the choice of an additive constant. Let's do this by requiring that \( \hat{H} | \Phi_\circ \rangle \equiv 0 \) where \( \Phi_\circ \) is the state where all \( k \leq 0 \) are filled. Then

\[
\hat{H} = \hat{H}_\circ - \mu \hat{N} = \sum (k-\mu) \psi_k^* \psi_k
\]

\[
= \sum_{k>0} (k-\mu) \psi_k^* \psi_k - \sum_{k<0} (k-\mu) \psi_k \psi_k^*
\]

and

\[
\text{Tr} (e^{-\beta \hat{H}}) = \prod_{k>0} (1 + e^{-\beta(k-\mu)}) \prod_{k<0} (1 + e^{\beta(k-\mu)})
\]

Compute variation

\[
\delta \log \text{Tr}(e^{-\beta \hat{H}}) = \sum_{k>0} \frac{e^{-\beta(k-\mu)}}{1 + e^{-\beta(k-\mu)}} \beta \delta \mu
\]

\[
+ \sum_{k<0} \frac{e^{\beta(k-\mu)}}{1 + e^{\beta(k-\mu)}} (-\beta \delta \mu)
\]

Consider

\[
F(\hat{H}) = -\frac{1}{\beta} \log \text{Tr}(e^{-\beta \hat{H}}) \quad \text{as} \ \beta \to \infty.
\]

Suppose \( \mu > 0 \). The second term has \( k - \mu \leq -\mu < 0 \) so gives 0 in the limit. The first gives

\[
\delta F(\hat{H}) \to -\sum \delta \mu + \begin{cases} 
-\frac{1}{2} \delta \mu & \text{if} \ \mu < \frac{2\pi}{\beta} \Xi \\
0 & \text{else} \end{cases}
\]

Do the limit directly for \( F(\hat{H}) \)

\[
F(\hat{H}) = -\frac{1}{\beta} \sum_{k>0} \log (1 + e^{-\beta(k-\mu)}) - \frac{1}{\beta} \sum_{k<0} \ldots
\]
For \( \mu > 0 \)

\[ F(\hat{\mu}) \rightarrow + \sum'_{0 < k < \mu} (k-\mu) = E(\hat{\mu}) \]

Hence the ground energy is discontinuous in \( \mu \),
probably because the particle number changes. This
suggests that I should stick to the finite \( \beta \) situation.

The problem now is to bring in the elliptic
functions. Recall the Jacobi triple product identity
\[
\prod_{n \geq 0} (1 + q^n t) \prod_{n \geq 1} (1 + q^n t^{-1})(1 - q^n) = \sum_{n \in \mathbb{Z}} q^{\frac{n(n+1)}{2}} (qt)^n
\]

We can use this to understand the partition fn.
\( k \epsilon \frac{2\pi}{L} \mathbb{Z} \). Put \( q = e^{-\frac{\beta}{L}} = e^{i(\beta \frac{2\pi i}{L})} \)
\( t = e^{-\beta \mu} \)

Then
\[
\text{Tr} \left( e^{-\beta \hat{H}} \right) = \prod_{n \geq 0} (1 + q^n t) \prod_{n \geq 1} (1 + q^n t^{-1}).
\]

As a function of \( \mu \) this vanishes for
\( t = -q^n \) some \( n \in \mathbb{Z} \)
\(-\beta \mu \in i\pi + \mathbb{Z}(\frac{\beta}{L} - \beta) + 2\pi i \mathbb{Z} \)

\[
\mu \in \frac{2\pi}{L} \mathbb{Z} + \frac{2\pi i}{\beta} (\mathbb{Z} + \frac{1}{2})
\]

So our periodicity lattice is \( \frac{2\pi}{L} \mathbb{Z} + \frac{2\pi i}{\beta} \mathbb{Z} \), and
our \( \Theta \) function is periodic in the \( \frac{2\pi i}{\beta} \) direction.

\[
\frac{S \log \text{Tr}(e^{-\beta \hat{H}})}{-\beta S(\beta \mu)} = \sum_{k \geq 0} \frac{e^{\beta(k-\mu)}}{1 + e^{\beta(k-\mu)}} - \sum_{k \geq 0} \frac{e^{-\beta(k-\mu)}}{1 + e^{-\beta(k-\mu)}}
\]

\[
\frac{\partial}{\partial \mu} F(\hat{\mu}) = \sum_{n \geq 0} \frac{q^n t}{1 + q^n t} - \sum_{n > 0} \frac{q^n t^{-1}}{1 + q^n t^{-1}}
\]
Use the identity
\[ 1 - \frac{q^n t^{-1}}{1 + q^n t^{-1}} = \frac{1}{1 + q^n t} = \frac{q^{-n} t}{1 + q^{-n} t} \]
and one sees that the above \( \frac{\partial}{\partial \mu} F(\hat{H}) \) is formally (up to an inf. constant)
\[ \sum_{h \in \mathbb{Z}} \frac{q^h t}{1 + q^h t} \]
This is going to be a constant times the Weierstrass \( \wp(\mu) \) with linear factor added to make it periodic in \( \frac{2\pi i}{\beta} \) period.

Calculate the residue as a fn. of \( \mu \) at
\[ \mu = \frac{\pi i}{\beta} \] \quad One has the term \[ \frac{e^{-\beta \mu}}{1 + e^{-\beta \mu}} = \frac{1}{e^{-\beta \mu} + 1} \]
\[ \frac{\partial}{\partial \mu} (e^{-\beta \mu} + 1) = \beta e^{-\beta \mu} \bigg|_{\mu = \frac{\pi i}{\beta}} = -\beta \]
Thus
\[ \frac{\partial}{\partial \mu} F(\hat{H}) = -\frac{1}{\beta} \left\{ \sum_{\gamma \in \frac{2\pi i}{\beta} \mathbb{Z} + \frac{2\pi i}{\beta} (2 + 1)} \left( \frac{1}{(\mu - \gamma) \beta + \frac{1}{\gamma} + \frac{\mu}{\gamma^2}} \right)^2 \right\} + \text{linear fn. of } \mu, \bar{\mu} \cdot \]

Review: I want to interpret \( \text{Tr}(e^{-\beta \hat{H}}) \) as a determinant of \( \partial_t + \hat{H} \) over \([0, \beta] \) with anti-periodic b.c. I use the variational formula
\[ 8 \log \det (\partial_t + \hat{H}) = \text{Tr} (\partial_t + \hat{H})^{-1} \delta \hat{H} \]
\[ 8 \log \text{Tr}(e^{-\beta \hat{H}}) = -\beta \delta \mu \left\{ \sum_{k \in \mathbb{Z}} \frac{e^{\beta (k \mu)}}{1 + e^{\beta (k \mu)}} - \sum_{k > 0} \frac{e^{-\beta (k \mu)}}{1 + e^{-\beta (k \mu)}} \right\} \]
In this case \( \delta \hat{H} = -\delta \mu \) and \( -\partial_t + \hat{H} \) has the eigenvalues
\[ \mu - \gamma \quad \gamma \in \frac{2\pi i}{\beta} \mathbb{Z} + \frac{2\pi i}{\beta} (2 + 1) \]
so that

\[- \text{Tr} \ (\partial_t + H)^{-1} = \sum_{\mathbf{p}} \frac{1}{\mu - \mathbf{p}} \]

\[= -\beta \left\{ \sum_{k > 0} \frac{e^{\beta(k - \mu)}}{1 + e^{\beta(k - \mu)}} - \sum_{k < 0} \frac{e^{-\beta(k - \mu)}}{1 + e^{-\beta(k - \mu)}} \right\} \]

so the last expression is the precise regularized version of \(- \text{Tr} \ (\partial_t + H)^{-1}\). What I want now is to understand this regularization process in terms of the actual Schwinger kernel for \((\partial_t + H)^{-1}\).

Since \(\partial_t + \tau(x - \mu)\) is a constant coefficient operator the Schwinger kernel of its inverse is a function of \(x - x', t - t'\), so I can take \(x = x' = 0\). Then as a Fourier transform,

\[G(x, t) = \sum_{k \in \mathbb{Z}^2} \frac{e^{-ikx + i\omega t}}{i\omega + k - \mu} \]

so the sum is over \(k = k + i\omega \in \frac{2\pi}{L^2} \mathbb{Z}^2 + \frac{2\pi i}{\beta} (\frac{1}{2} + \mathbb{Z})\).

What is the regularization process?

One idea is to take \(t = 0^-\). Thus I need to understand

\[G(t) = \sum_{\omega \in \frac{2\pi}{\beta} (\frac{1}{2} + \mathbb{Z})} \frac{e^{i\omega t}}{i\omega + \varepsilon} \]

This is anti-periodic on \([0, \beta]\) and satisfies \((\partial_t + \varepsilon)G = \beta \delta(t)\).

Hence

\[G(t) = Ae^{-\beta t} \quad 0 < t < \beta \]

\[\beta = G(0^+) - G(0^-) = A + G(\beta^-) = A \left(1 + e^{-\beta \varepsilon}\right)\]

\[G(t) = \beta \frac{e^{-\beta t}}{1 + e^{-\beta \varepsilon}} \]

\[G(0^-) = -\beta \frac{e^{-\beta \varepsilon}}{1 + e^{-\beta \varepsilon}} \]

\[\sum \frac{e^{i\omega t}}{i\omega + \varepsilon} = -\beta \frac{e^{-\beta \varepsilon}}{1 + e^{-\beta \varepsilon}} \]
This doesn't seem to do much. I really need a formula for the Green's function so that I can see what the regularization process is.

August 11, 1982

Calculation of Green's function in the elliptic curve case,

$$(\partial_\zeta - \omega) G = \delta(0) = \frac{1}{\text{vol}(\Gamma)} \sum_{\mu \in \Gamma^*} e^{\mu \zeta - \mu^2}$$

$$G = \frac{1}{\text{vol} \Gamma} \sum_{\mu} \frac{e^{\mu \zeta - \mu^2}}{\mu - \omega}$$

But one also has the expression

$$G(\zeta) = \frac{1}{\pi} e^{\frac{\omega}{m} (m \zeta + \zeta^2)} \frac{\sigma(\zeta - \omega)}{\sigma(\zeta) \sigma(-\omega)} \quad m = \frac{n}{\text{vol} \Gamma}$$

I want the Green's fn. for $\partial_t + \frac{i}{2} \partial_x - \mu$:

$$G(x,t) = \frac{1}{L \beta} \sum_{k \in \frac{2\pi}{L}} \frac{e^{ikx + i\omega t}}{i\omega + k - \mu}$$

Let's put $\omega = -\frac{\pi}{L} + \lambda$

$$G(x,t) = e^{-i\frac{i\omega}{L} t} \frac{1}{L \beta} \sum_{k \in \frac{2\pi}{L}} \frac{e^{ikx + i\lambda t}}{i\lambda + k - (\mu + i\frac{\pi}{L})}$$

where $\lambda$ is determined by

$$\left(\partial_t + \frac{i}{2} \partial_x\right) (\alpha \text{m}(x, it)) = \omega$$

$$\alpha = \frac{i\omega}{2m} = \frac{i}{2m} (\mu + i\frac{\pi}{L})$$

$$m = \frac{\pi}{\beta L}$$
Thus the Green's fn. for $\partial_t + \frac{1}{\beta} \partial_x - \mu$ periodic in $x$ of period $L$, anti-periodic in $t$ of period $\beta$ is

$$G(x,t) = \text{const } e^{-i \frac{\pi t}{\beta}} e^{i (m \bar{x} + lx)} \frac{\sigma(z-x)}{\sigma(z) \sigma(-x)}$$

as above.

Expand mean $z = 0$:

$$G(x,t) = \text{const } \frac{1}{z} \left[ 1 - i \frac{\pi t}{\beta} + \alpha(l \bar{x} + m \bar{x}) + \frac{\sigma(-x)}{\sigma(x)} z + O(z^2) \right]$$

Let’s go back to old notation

$$G(z) = \text{const } \frac{1}{z} F(z)$$

and put $t = \frac{z - \bar{z}}{2i}$. Then

$$F(z) = 1 - i \frac{\pi z}{2\beta} + \frac{i \bar{z}}{2\beta} + \alpha(l \bar{x} + m \bar{x}) + \frac{\sigma(-x)}{\sigma(x)} z + \ldots$$

$$F_b(z) = 1 + \left( \frac{i \pi}{2\beta} + \alpha m \right) \bar{z} + R \bar{z} + \ldots$$

Here $R$ is a constant, at least I want to try a constant $\bar{d}$-operator lifting the given $\bar{d}$-operator. In general I could try an $R$ depending analytically on $1/\beta \alpha \bar{x}$. With the above $F_b$ we get

$$F.P. G(z) = \text{const } \left\{ -i \frac{\pi}{2\beta} - R + \alpha \bar{x} - \frac{\sigma(-x)}{\sigma(x)} \right\}$$

Now I have a candidate for this F.P. which is periodic in $\mu$ for the period $\frac{2\pi i}{\beta}$, hence periodic in $x$ with period

$$\Delta \bar{x} = \frac{i}{2m} \Delta \mu = \frac{i}{2m} \frac{2\pi i}{\beta} = -\frac{2\pi}{m \beta} = -\frac{\pi}{\beta \Pi/\beta} = -L$$

Now I know that $\frac{-\sigma(-x)}{\sigma(x)} \alpha \bar{x} - m \bar{x}$ is doubly-periodic with the periods $L$, $i\beta$.

Therefore if I want $\frac{-\sigma(-x)}{\sigma(x)} \alpha \bar{x} + R(x)$ to have the period $L$, I must have $L \alpha + mL = -L \alpha + R(x + L) - R(x) = 0$, or $R(x + L) - R(x) = mL$. The simplest choice
The problem is to calculate the ground energy of $\hat{H}_0 + \hat{f}$ assuming $\hat{H}_0$ chosen so that its ground energy is zero. Here

$$\hat{H}_0 = \sum_{k > 0} k \Psi_k^* \Psi_k - \sum_{k < 0} k \Psi_k^* \Psi_k,$$

and we use normal ordering relative to the ground state $\Omega$ where all $k < 0$ are filled. $\hat{f}$ is defined using the same normal ordering. $f(x)$ is periodic of period $L$ so

$$f(x) = \sum_k e^{ikx} \frac{1}{L} \int_0^L e^{-ik\xi} f(\xi) d\xi,$$

and then

$$\hat{f} = \sum_k c_k \hat{S}_k.$$

The idea will be to use the boson picture in which case $\hat{f}$ is a linear function of creation and annihilation operators, and then we can calculate the ground energy shift by completing the square. For a simple oscillator

$$H = \omega a^*a + ca + \bar{c}a^*$$

$$= \omega (a^* + \frac{c}{\omega})(a + \frac{\bar{c}}{\omega}) - \frac{|c|^2}{\omega},$$

so the ground energy shift is $-\frac{|c|^2}{\omega}$. We have

$$\hat{S}_k = \sum_{\xi} \Psi_k^* \Psi_{k+\xi}$$

in $\Omega$ for $k < 0$, so

$$f_k = \text{const} \ a_k$$

$$\left[ S_{-k} \ S_k \right] = \frac{1}{2\pi} \frac{k}{2\pi}.$$
\[ a_k = \sqrt{\frac{2\pi}{Lk}} s_k \quad a_k^* = \sqrt{\frac{2\pi}{Lk}} \bar{s}_k \quad k > 0 \]

and so
\[ \hat{f} = c_0 \rho_0 + \sum_{k > 0} \left( c_k \sqrt{\frac{Lk}{2\pi}} a_k^* + \bar{c}_k \sqrt{\frac{Lk}{2\pi}} a_k \right) \]

Now \( \mathbf{H}_0 = \sum_{k > 0} k a_k^* a_k \) on the \( N=0 \) piece.

Therefore the ground energy shift on the \( N=0 \) piece of Fock space is
\[ \Delta E = -\sum_{k > 0} |c_k|^2 \frac{Lk}{2\pi} \frac{1}{k} = -\frac{L}{2\pi} \sum_{k > 0} |c_k|^2 \]

From \( f = \sum c_k e^{ikx} \Rightarrow \frac{L}{L} \int_0^L f f^* dx = \sum_k |c_k|^2 \)

So
\[ \Delta E = -\frac{1}{4\pi} \int_0^L f f^* dx = \frac{L}{4\pi} |c_0|^2 \]

\[ \Delta E = -\frac{1}{4\pi} \int_0^L f f^* dx + \frac{L}{4\pi} |c_0|^2 \]

(f is real)

Take \( L = 2\pi \) now.

\[ \hat{\mathbf{H}}_0 = \sum_{n > 0} \psi_n^* \psi_n - \sum_{n > 0} n \psi_n \psi_n^* \]

\[ \hat{\mathbf{N}} = \rho_0 = \sum_{n > 0} \psi_n^* \psi_n - \sum_{n > 0} \psi_n \psi_n^* \]

on \( \Omega = \{0, -1, -2, \ldots\} \) we have \( \hat{\mathbf{H}}_0 = \rho_0 = 0 \)

on \( \{1, -1, -2, \ldots\} \) we have \( \hat{\mathbf{H}}_0 = 0, \quad \rho_0 = -1 \)

On \( \{n, n-1, n-2, \ldots\} \) we have \( \rho_0 = n, \quad \hat{\mathbf{H}}_0 = \frac{n(n+1)}{2} \)

and since this is the vacuum state in the \( \rho_0 = n \) piece
we conclude
\[ \hat{\mathbf{H}}_0 = \sum_{n > 0} \rho_n \hat{s}_n - n + \frac{1}{2}(\rho_0^2 + \rho_0) \]
Suppose \( \phi : S' \to U_n \).

Let \( H = L^2(S'; C^n) = H_+ \oplus H_- \).

and \( \mathfrak{g}_H = \Lambda(H_+ \oplus \overline{H_-}) \).

The operator \( D_0 = -i \frac{d}{dt} \) on \( H \) induces a hermitian operator \( D_0 \) on \( \mathfrak{g}_H \) defined up to an additive constant.

Choose \( D_0 \) on \( \mathfrak{g}_H \) so that \( D_0 \omega = 0 \).

The loop \( \phi \) defines a unitary operator \( U_\phi \) on \( \mathfrak{g}_H \), defined up to a scalar multiple.

Consider \( D_\phi = U_\phi \cdot D_0 \cdot U_\phi^{-1} \). This is an operator on \( \mathfrak{g}_H \) corresponding to \( D_0 + i\phi \phi^{-1} \) on \( H \). Its lowest eigenvalue is \( 0 \). Another operator on \( \mathfrak{g}_H \) corresponding to \( D + i\phi \phi^{-1} \) on \( H \) is \( \tilde{D}_\phi = D_0 + i\phi \phi^{-1} \).

What is the lowest eigenvalue of \( \tilde{D}_\phi \)? In other words, what is \( \tilde{D}_\phi - D_\phi \)?

By assumption \( \langle \Omega, \tilde{D}_\phi \Omega \rangle = 0 \), so we want

\[-\langle \Omega, D_\phi \Omega \rangle = -\langle U_\phi^{-1} \Omega, D_0 \cdot U_\phi^{-1} \Omega \rangle = D_0^{(\phi)} - D_0, \]

where \( D_0^{(\phi)} \) is \( D_0 \) normal ordered with respect to \( U_\phi^{-1} H_+ \oplus U_\phi^{-1} H_- \).

Lemma 99: If \( A : H \to H \) is hermitian, and \( A_0, A_1 \) are the corresponding operators on \( \mathfrak{g}_H \) defined by normal ordering with respect to polarizations \( J_0, J_1 \) of \( H \) (so that \( J_0^2 = 1 \)),
then \[ A_0 - A_1 = \frac{1}{2} \text{ trace } A(\mathcal{J}_0 - \mathcal{J}_1). \]

We apply this with \( A = D_0 = -i \frac{d}{d\Theta} \) and \( \mathcal{J}_0 = U_p^{-1} \mathcal{J} U_p \), \( \mathcal{J}_1 = \mathcal{J} = \text{ standard polarization} \). Then

\[
D_0(\phi) - D_0 = \frac{1}{2} \text{ trace } D_0 \left( U_p^{-1} \mathcal{J} U_p - \mathcal{J} \right) = \frac{1}{2} \text{ trace } \{ D_0 U_p^{-1} \} \left[ \mathcal{J}, U_p \right].
\]

The standard \( \mathcal{J} \) is the \( j \)-integral operator with

kernel \[ \frac{d}{2\pi} \cot \frac{\Theta_1 - \Theta_2}{2}; \] so \( \left[ \mathcal{J}, U_p \right] \) is the integral operator with kernel \[ -\frac{d}{2\pi} \cot \frac{\Theta_1 - \Theta_2}{2} \left( \phi(\Theta_1) - \phi(\Theta_2) \right) \]. Altogether we must calculate the trace of the operator whose kernel is

\[
-\frac{1}{4\pi} \frac{\partial}{\partial \Theta_1} \left\{ \cot \frac{\Theta_1 - \Theta_2}{2} \left( \phi(\Theta_1)^{-1} \left( \phi(\Theta_1) - \phi(\Theta_2) \right) \right) \right\}
\]

\[ = + \frac{1}{8\pi} \csc^2 \frac{\Theta_1 - \Theta_2}{2} \cdot \phi(\Theta_1)^{-1} \left( \phi(\Theta_1) - \phi(\Theta_2) \right) + \frac{1}{4\pi} \cot \frac{\Theta_1 - \Theta_2}{2} \phi(\Theta_1)^{-1} \phi'(\Theta_1) \phi(\Theta_2) \phi''(\Theta_1)
\]

\[ \sim - \frac{1}{2\pi} \frac{1}{(\Theta_1 - \Theta_2)^2} \left\{ (\Theta_2 - \Theta_1) \phi(\Theta_1)^{-1} \phi'(\Theta_1) + \frac{1}{2} (\Theta_2 - \Theta_1)^2 \phi(\Theta_1)^{-1} \phi''(\Theta_1) + \ldots \right\}
\]

\[ - \frac{1}{2\pi} \frac{1}{(\Theta_1 - \Theta_2)} \left\{ \phi(\Theta_1)^{-1} \phi'(\Theta_1) + (\Theta_2 - \Theta_1) \left( \phi(\Theta_1)^{-1} \phi'(\Theta_1) \right)^2 + \ldots \right\}
\]

\[ \rightarrow - \frac{1}{4\pi} \left\{ \phi(\Theta)^{-1} \phi''(\Theta) - \frac{1}{2} \left( \phi(\Theta)^{-1} \phi'(\Theta) \right)^2 \right\} \quad \text{on the obliques}
\]

\[ = - \frac{1}{4\pi} \left\{ \frac{d}{d\Theta} \left( \phi^2 \phi' \right) - \left( \phi^2 \phi'' \right) \right\}.
\]

So that finally

\[ D_0(\phi) - D_0 = + \frac{1}{4\pi} \int_{0}^{2\pi} \text{ trace } (\phi^2 \phi')^2 d\Theta = -\text{ Energy}(\phi). \]
Let's go back to trying to prove a differential form version of GRR when the fibres are curves. Suppose we have $X$ a map $\pi$ of complex manifolds which is proper,  
\[ \begin{array}{c} \pi \\ \downarrow \end{array} \]  
\[ \begin{array}{c} Y \\ \end{array} \]  
also we have a vector bundle $E$ over $X$. Then we have the \( \bar{\partial} \) operator along the fibres  
\[ E \xrightarrow{\bar{\partial}} E \otimes T_{x/y}^{0,1} \]  
which commutes with functions on $Y$. This operator will be used to define a virtual bundle $\pi_!(E)$. I suppose chosen metrics on $E$ and $T_{x/y}^{0,1}$. Then I get an adjoint operator  
\[ E^* \xleftarrow{D^*} E \otimes T_{x/y}^{0,1} \]  
and Laplacians $D^*D$, $DD^*$ which are self-adjoint in each fibre over a point of $Y$.

It seems to be simpler to think of the case of a fixed $C^\infty$ surface $Y$ vector bundle and then to vary the metrics and holomorphic structures. This way as we vary over $Y$ we get a varying operator $D_y : V_1 \rightarrow V_0$ which is Fredholm.
Goal: Fix a $C^\infty$ vector bundle over a Riemann surface $M$ and consider the family of all holom. structures on it. This gives then the picture: GRR says

$$\chi = \alpha \times M \quad \tilde{E}$$

$$\chi \left( f^* \tilde{E}, \text{Todd} (x/y) \right) = \left( f^* \chi \tilde{E}, \text{Todd} (x/y) \right)$$

$$p_1^* (\text{Todd} (x/y))$$

$$1 + \frac{1}{2} c_1 T_M$$

Question: Can $f^* \chi \tilde{E}, \text{Todd} (x/y)$ have interesting components of degree $\geq 2$?

(I suppose $E, M$ equipped with metrics so that these char. classes are given by differential forms, and $f^*$ is just integrating over $M$.)

Let's go over the local calculation of $\chi \tilde{E}$. Locally on $M$ we trivialize $E$ by an orthonormal frame, we choose a local coord. $z$ on $M$ and coordinates $t$ on $\alpha$. Then the holom. structure on $\tilde{E}$ is given by the $\bar{\partial}$-operator

$$(\partial \bar{z} + \alpha) \, dz + \partial_z \, dt$$

where $\alpha$ is holomorphic in $t$. (The notation is lousy—one should think of $\ast$-space as a finite-dim. subspace of $\alpha$; e.g. $\alpha = \alpha_0 + t_1 \alpha_1 + \cdots + t_n \alpha_n$. The hermitian connection corresponding to this $\bar{\partial}$-operator is

$$(\partial \bar{z} - \alpha^*) \, dz + (\partial_z + \alpha) \, d\bar{z} + \partial_t \, dt + \partial_{\bar{t}} \, \bar{dt}$$

and the curvature is

$$K = (\partial \bar{z} \alpha + \partial_z \bar{z} + [\alpha, \alpha^*]) \, dz \, d\bar{z} + (\partial_t \alpha) \, dt \, d\bar{z} - (\partial_{\bar{t}} \alpha^*) \, d\bar{t} \, dz.$$

The character of $\tilde{E}$ is given by the differential form

$$\text{tr} e^{\frac{1}{2\pi} \bar{K}} = \sum_{n} \frac{1}{n!} \left( \frac{j}{2\pi} \right)^n \text{tr} (K^n)$$

In this case $K^n = 0$ for $n \geq 3$ and

$$K^2 = \sum_{i,j} \beta_i \cdot \partial \bar{z} \alpha^* \partial_t \, dt \, d\bar{z}$$
leading to the curvature of the canonical line bundle $c_1(\mathcal{L} \otimes f^*_E)$, as we've seen before.

More generally suppose I consider a holomorphic family $X \to Y$ of Riemann surfaces with holomorphic bundle $E$ over $X$ and suppose metrics chosen on $E$ and $T_{X/Y}$, so that $f_*(\text{ch}(E^{\otimes n} \cdot \text{Todd}(X/Y)))$ is given as a differential form. Locally on $X$ I can choose coordinates $(t, z)$ where the $t_i$ are coords on $Y$. Then the curvature of $E$, or $T_{X/Y}$ will involve the $(1, 1)$-forms

$$dz \wedge \overline{dz}, \quad dt \wedge \overline{dz}, \quad d\overline{z} \wedge dz, \quad dt \wedge d\overline{t}.$$

If none of the $dt \wedge d\overline{t}$ forms occur, then $f_*(\text{ch}(E^{\otimes n} \cdot \text{Todd}(X/Y)))$ will involve no terms of degree $(p, p)$ with $p \geq 2$. One can arrange this by tensoring with a suitable line bundle lifted from $Y$. 


August 17, 1982

I am looking at the situation of a family of invertible operators \( \tilde{D} : E \rightarrow E \otimes T_{x}^{0,1} \) and I suppose metrics chosen, so that I get the function \( \tilde{\kappa} = \text{Tr} (\tilde{D}) \) defined on \( Y \). The torsion \( \tau (\tilde{D}) = \exp (-\tilde{\kappa} (0)) \) gives the norm-squared of the canonical section. Now the problem is that I want to compute the curvature form \( \tilde{D} \log \tilde{\kappa} \) on \( Y \).

Let’s discuss carefully the problems involved with the idea of getting a good invariant viewpoint. We have over each point \( y \in Y \) a Hilbert space \( H_{y} = L^{2}(X, E_{y}) \) and an operator \( (\tilde{D})_{y} \) in \( H_{y} \) and I am looking at the function \( \text{Tr} (\tilde{D}) \). Normally one deals with operators in the same space but here we have a bundle situation.

Suppose \( E \) is a vector bundle over \( X \) and that \( A \) is an endomorphism of \( E \). Then \( \text{Tr}(A) \) is a fn. on \( X \) and we can calculate its differential. Choose a connection \( \nabla : E \rightarrow E \otimes T_{x}^{*} \). Then

\[
\frac{d}{d} \text{Tr}(A) = \text{Tr} \left[ \nabla, A \right]
\]

and this right side is clearly independent of the choice of \( \nabla \). Similarly when \( A \) is invertible

\[
\frac{d}{d} \log \det(A) = \text{tr} (A^{-1} [\nabla, A]).
\]

(To see this at a point \( x \), let \( A_{x} \) be an autom of \( E = A \) at \( x \) and with \( [\nabla, A_{x}] = 0 \) at \( x \). Then we have \( \log \det(A_{x}) = 0 \) at \( x \) because \( A_{x} \) is constant to the first order - actually one can restrict attention to \( X \) a curve. So \( A_{x}^{-1} A = I + B \) with \( B = 0 \) at \( x \), and

\[
\frac{d}{d} \log \det(A) = \frac{d}{d} \log \det (A_{x}^{-1} A) = \frac{d}{d} \text{tr}(B) = \text{tr} ([\nabla, B]) = \text{tr} [\nabla, A_{x}^{-1} A] = \text{tr} A_{x}^{-1} [\nabla, A] = \text{tr} (A^{-1} [\nabla, A])
\]
at \( x \).
Now let's return to my family of operators, or actually to the Laplacian \( D^{x}D \) on \( X \) over \( Y \). I have this Hilbert bundle \( H_y = L^2(X_y, E_y) \) and the operator \( e^{-(D^{x}D)y} \) in it of trace class, and I want to understand how the trace varies in \( y \).

According to the above I need a connection on this bundle. Does there exist a canonical connection obtained from the inner product?

The following example I think is critical. Take \( X \) to be the family of elliptic curves \( C/\mathbb{Z} + \mathbb{Z} \tau \) where \( Y = \{ \tau \mid \text{Im} \tau > 0 \} \), take \( E = \text{trivial line bundle} \), and \( D^{x}D = -(\partial_{\bar{z}} + \bar{w})(\partial_z - \omega) \) where \( \omega \) is a constant chosen so that \( -D^{x}D \) is invertible near the \( \tau \in Y \) one is working. In this example, relative to the local words \( z, \bar{z}, \tau \) the operator \( D^{x}D \) is not changing, however the space \( H_{\tau} = L^2(C/\mathbb{Z} + \mathbb{Z} \tau) \) is changing.

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Problem: Given \( f: X \to Y \) a holomorphic family of curves and a holom. v.b. \( E \) on \( X \), define \( f^*E \) as a virtual holom. vector bundle on \( Y \).

There are several ideas. 1) Use a positive divisor \( o \to O \to L \to F \to o \) with \( F \) torsion such that \( R^1f_*(E \otimes L) = 0 \) on each fibre. Then \( f^*E \) is the difference of \( f_*(E \otimes O) \) and \( f_*(E \otimes F) \) which are holom. v.b. over \( Y \).

2) Grothendieck's general idea that \( f^*E \) is the complex \( Rf_*(E) \), \( E \) sheaf of holom. sections of \( E \), which is a perfect complex of \( O_Y \)-modules. What this means on the \( C^\infty \) level is that one takes \( E \) replaces it by the Dolbeault complex \( 0 \to E \to E \otimes T^0_1 \to E \otimes T^0_2 \to \ldots \) and applies \( f^* \).
Unfortunately the top row\( f_*(E) \otimes T^{0,1}_Y = f_*(E \otimes f^*_Y T^{0,1})\) is not a subcomplex of \( f_*(E \otimes T^{0,1}_X) \), because one could then think of \( f_*(E) \) and \( f_*(E \otimes T^{0,1}_X) \) as being infinite-dimensional holomorphic \( \text{v.b.} \) over \( Y \). Actually something like this happens when \( X = Y \times M \).
August 18, 1982

I want to go over the nature of a parametrix, or Green's function for a \( \overline{\partial} \)-operator on a Riemann surface. This is a local story so I might as well start with \( D = \overline{\partial} \) on \( C \). The standard parametrix is \( \frac{1}{\pi(z-z')} \) relative to the volume form \( \text{d}y \). By a parametrix I mean a kernel \( G(z,z') \) such that \( DG \) and \( GD \) are the identity + smoothing operators on functions with compact support. Now

\[
DG \psi = \overline{\partial} \int dz' G(z,z') \psi(z') = \int dz' \left[ \overline{\partial} G(z,z') \right] \psi(z')
\]

so that we want \( \overline{\partial} G(z,z') = \delta(z-z') + \text{smooth kernel} \) and hence

\[
\overline{\partial} \left[ G(z,z') - \frac{1}{\pi(z-z')} \right] = \text{smooth kernel}.
\]

As \( \overline{\partial} \) is elliptic we have \( G(z,z') - \frac{1}{\pi(z-z')} \) must be smooth.

From past experience it seems useful to write

\[
G(z,z') = \frac{F(z,z')}{\pi(z-z')} \quad \text{with } F \text{ smooth } = 1 \text{ when } z = z'.
\]

If I expand \( F \) in a Taylor series around \( z = z' \):

\[
F(z,z') = 1 + a_{21}(z-z') + b_{21}(z-z')^2 + c_{21}(z-z')^3 + m_{21} z z' + n_{21} \frac{z-z'}{z-z'}^2 + \cdots
\]

the condition that \( \frac{F(z,z')}{\pi(z-z')} = \frac{1}{\pi(z-z')} \) is smooth amounts simply to requiring the coeffs. of \( \frac{z-z'}{z-z'}^k \), \( k \geq 1 \), vanish:

\[
b_{21} = 0, \quad n_{21} = 0, \quad \text{etc.}
\]

For the index thm. I need to know the values on the diagonal of \( DG - \text{id} \), \( GD - \text{id} \):

\[
\left[ \overline{\partial} G(z,z') - \delta(z-z') \right]_{z = z'} = \overline{\partial} \left[ \frac{1}{\pi} \left\{ a_{21} + c_{21} \frac{z-z'}{z-z'} + m_{21} z z' + \cdots \right\} \right]_{z = z'} = \frac{1}{\pi} m_{21}
\]

\[
\left[ - G(z,z') \overline{\partial} - \delta(z-z') \right]_{z = z'} = \overline{\partial} \left[ \frac{F(z,z')}{\pi(z-z')} - \frac{1}{\pi(z-z')} \right]_{z = z'}
\]
\[ -\frac{1}{\pi} \partial \bar{z} \left[ a_z + \bar{z} + \frac{z - \bar{z}}{x^2} + \ldots \right] \]
\[ = -\frac{1}{\pi} \partial \bar{z} a_z + \frac{1}{\pi} m_z \]

Hence we see that the contribution to the index comes from the coeff. \( a_z \), which is somehow to be related to the finite part of the Green's function on the diagonal.

Now I want to take a global viewpoint in which I have a \( D \)-operator on a vector bundle \( E \) over a Riemann surface \( M \). A parametrix for \( D \) will be a kernel \( G(z, z') \) \( dz' \in \text{Hom} (E_z \otimes T_z^{1,0}, E_z) \) with the property that in local holomorphic coordinates
\[ G(z, z') \] dz' = \( \frac{i}{2\pi} \frac{dz'}{z - z'} + \text{smooth} \]
(The \( \frac{i}{2} \) is needed because \( \frac{1}{2} \text{d}z \text{d}z' = \text{d}x \text{d}y \).

At this point I've described what a parametrix for \( D : E \to E \otimes T^{1,0} \) is, and once I have it, I get an integral formula for the index. Now what I want to do is to start with metrics on \( E, M \) and construct a parametrix which will yield the index thus, using the Chern differential forms associated to these metrics. My idea is that there is something called a flat Green's function belonging to the metrics, for at least to the connections.

Here's how to obtain the flat Green's function. Because we have a connection in the tangent bundle to \( M \) we have the notion of a geodesic from \( z' \) to a nearly point \( z \), and can use the connection \( \nabla \) on \( E \) to lift this geodesic to parallel translation giving \( U_z \) an isomorphism
\[ F(z, z') : E_{z'} \to E_z \]

Also the connection in the tangent bundle gives an "exponential map" i.e. a local diffeomorphism of \( T_z \) with a nbhd of \( z' \). Call this exp. map
\[ e_{z'} : T_z \to M \]
and let \( \varphi \) be \( e_{z'}^{-1} \) defined in a nbhd of \( z' \) followed
by an isomorphism $C \cong T_{x_1}$. Then

$$i \frac{F(z, z')}{\psi_z(z')} \in \text{Hom}(E_z, E_z) \otimes T_{x_1}.$$ 

is independent of the choice of $C \cong T_{x_1}$. This expression is defined in a nbhd. of the diagonal and then can be extended by $0$ by first multiplying by a function $=1$ near the diagonal and with support in a nbhd. of the diagonal.

Questions. 1. Should this flat Green's fn. be thought of as a WKB Green's function?

2. Can this flat Green's fn. be used to prove RR?

3. What is the relation between the previous Green's function constructed using orthogonal projection methods and the flat one?

Concerning 3. If $D$ is invertible, then the kernel for $D^{-1}$ is not usually flat — this we know from the anomaly formula. Hence the Hodge-type Green's fn. is not usually flat. In fact the Hodge-type Green's fn. call it $G^H$, is almost an inverse to $D$, more precisely

$$DG^H = I - \text{proj on } H^0$$

$$G^HD = I - \text{proj on } H^1.$$ 

One expects the flat Green's function not to be such a good parametrix.
August 19, 1982

My goal is to understand the flat Green's fn. for a $\Delta$ operator over a Riemann surface. The first thing is to show that it actually is suitable for proving the RR thm. For this purpose I will need to know something about the exponential map for a Riemann surface.

More generally consider a Riemannian manifold and Kinetic Energy Lagrangian: $L = \frac{1}{2} |\dot{g}|^2$. The geodesics are the trajectories. The action is

$$S(g_t, g_t') = \frac{\sqrt{d(g, g')^2}}{2(\Delta t - 1)}$$

where $d(g, g')$ is the distance between $g$ and $g'$. In order to see this recall that the action is the integral over the trajectory going from $g'$ to $g_t$. In this case one sends a particle along the geodesics from $g'$ to $g$ with speed $\frac{d(g, g')}{\Delta t}$ so that it takes the time $\Delta t$. Then

$$S = \int \frac{1}{2} |\dot{g}|^2 dt = \frac{1}{2} \left( \frac{d(g, g')}{\Delta t} \right)^2 \Delta t = \frac{\sqrt{d(g, g')^2}}{2 \Delta t}$$

Now recall the formula that the initial + final momenta are

$$p' = -\frac{\partial S}{\partial \dot{g}_t}, \quad p = \frac{\partial S}{\partial g}$$

for the trajectory joining $g'$ to $g_t$. Put $t-t' = 1$ and let $S(g, g')$ denote the action which is just $\frac{1}{2} |\dot{g}|^2$. Then

$$p' = -\frac{\partial S}{\partial \dot{g}_t} (g, g')$$

gives the initial momentum of the trajectory going from $g'$ to $g$ in unit time. In other words the relation $p' \leftrightarrow g$ expressed by this equation is just the exponential map at the point $g'$. 
Yesterday I realized how small geodesics — in a Riemannian manifold can be described in terms of the “action” function which is just $\frac{1}{2} \Delta \text{d}(g, g')^2$. The
gradient of this function with respect to $g'$ is the unit

tangent vector to the geodesic from $g$ to $g'$. Hence for $g'$ fixed, the
relation:

$$ p' = -\frac{1}{2} \text{d}(g, g')^2 g ' $$

gives the exponential map $p' \mapsto g$
from the tangent space at $g'$ to $M$.

Let's now work on finding the flat Green's fn.
We have a Riemann surface $M$ with metric $ds^2 = g(dx^2 + dy^2)$. Hence $\sqrt{g} \, dx$, $\sqrt{g} \, dy$ is an orthonormal frame in $T_x$, so

$$ \|df\|^2 = \| \partial_x f \, dx + \partial_y f \, dy \|^2 = \frac{1}{g} \left( (\partial_x f)^2 + (\partial_y f)^2 \right) $$

$$ = \frac{4}{g} |\partial_x f|^2 \quad \text{for a real fn.} \, f $$

The Hamilton-Jacobi equation for $H = \frac{1}{2} \|\nabla f\|^2$ is

$$ \partial_t S + \frac{1}{2} \|\nabla S\|^2 = 0, $$

and so if $S = \frac{u(x, y)}{t}$, then we get simply

$$ \frac{1}{2} \|\nabla u\|^2 = u \quad \text{or} \quad \|\nabla \text{grad} u\|^2 = 2u. $$

This has the solution $u = \frac{1}{2} \pi (\text{g}, g')^2$. Clearly,

$$ \frac{4}{g} |\partial_x u|^2 = 2u \quad \text{or} \quad |\partial_x u|^2 = \left( \frac{2}{g} \right) u $$

is the HT equation.

In order to ease the calculations, let's remove the
factors of 2 and put $v = \pi (g, g')^2 \text{ so that the HT eqn.}
\text{is} \quad |\partial_x v|^2 = g v$. 

**Question**: Is $v$ divisible by $|\partial_x|^2$ as a $C^\infty$ fn.?
Let try putting $v = z \overline{z} f(\overline{z})$. Then
\[ \partial_{\overline{z}} v = \overline{z} f + z \overline{z} \partial_{\overline{z}} f = \overline{z} (f + z \partial_{\overline{z}} f) \]
and so we want to solve
\[(f + z \partial_{\overline{z}} f)(f + z \partial_{\overline{z}} f) = pf.\]

I know that $v$ is a $C^\infty$ function of $\overline{z}$, and hence it has a power series expansion $v = \sum c_{mn} z^m \overline{z}^n$. The question of whether it is divisible by $z \overline{z}$ is the same as whether $c_{mn} = 0$ if $m = 0$ or $n = 0$. Hence it should be possible to prove the divisibility of $v$ by $z \overline{z}$ by exhibiting a power series solution of $(\ast)$. Start with the series for $f$

\[ f(z) = f_0 + az + z \overline{z} + \frac{b}{2} z^2 + c z \overline{z} + \frac{d}{2} \overline{z}^2 + \cdots \]

where by rescaling we can suppose that $f_0 = 1$. Put
\[ f(z) = 1 + az + z \overline{z} + \frac{b}{2} z^2 + c z \overline{z} + \frac{d}{2} \overline{z}^2 + \cdots \]

Then
\[ f + z \partial_{\overline{z}} f = 1 + 2az + z \overline{z} + (\frac{b}{2} + b) z^2 + 2c z \overline{z} + \frac{d}{2} \overline{z}^2 + \cdots \]

and so we see that
\[ (f + z \partial_{\overline{z}} f)(f + z \partial_{\overline{z}} f) f^{-1} \]
is a series concocted simply from the coefficients of $f$.

I want to see that there is a unique way to choose the coefficients of $f$ so as to get the series $f$. Assume this has been proved in degrees $< p$. Put $f = \sum c_{mn} z^m \overline{z}^n$. Then the $c_{mn}$ for $m + n < p$ are determined

\[ f + z \partial_{\overline{z}} f = \sum (c_{mn} + m c_{mn}) z^m \overline{z}^n. \]

The new part in the above product of degree $p$ is
\[ \sum' c_{mn} (1 + m) z^m \overline{z}^n + \sum c_{mn} (1 + n) z^m \overline{z}^n - \sum' c_{mn} z^m \overline{z}^n \]

\[ = \sum' c_{mn} (1 + m + n) z^m \overline{z}^n \]

and since $1 + m + n \neq 0$, these $c_{mn}$ exist and
Outline of the proof that \( V = r(z)^2 \) is divisible by \( |z|^2 \) as a \( C^\infty \)-function. \( V \) is \( C^\infty \) because it comes from the exponential, which \( \exp \) is a local diffeom. So \( V \) has a formal power series expansion \( \sum a_n z^n \) at \( z=0 \) satisfying the HT eqn \( \partial_z^2 V \partial_{\bar{z}} V = \rho \cdot V \). But this equation has a unique power series solution starting with the 2nd degree terms \( \sum a_n z^n \) (analogous argument to the above) and it has all terms divisible by \( z\bar{z} \). So we see the power series of \( V \) is divisible by \( z\bar{z} \) which in view of the usual remainder estimates should be enough to guarantee that \( V/|z|^2 \) is \( C^\infty \).
More struggle with the flat Green's fn. Take the case of the trivial line bundle, in which case the flat Green's function is

\[
\frac{\rho(z') \, dz' \, dz}{- \partial_{z'} \, \Lambda(z, z')^2 \, dz'}
\]

It is clear this is well-defined globally near the diagonal. I still have to see that it is a parametrix for \( \tilde{\Delta} \).

Yesterday I saw that

\[
\Lambda(z, z')^2 = |z - z'|^2 \, f(z, z')
\]

where \( f \) is smooth and \( f(z, z) = f(z) \).

\[
- \partial_{z'} \, \Lambda(z, z')^2 = (z - z') \, f(z, z') - |z - z'|^2 \, \partial_{z} \, f(z, z')
\]

\[
= (z - z') \left\{ f - \frac{1}{(z - z')} \partial_{z} \, f \right\}
\]

\[
\text{smooth fn } = f \text{ on } z = z'.
\]

Hence we see that the flat Green's function is of the form

\[
h(z, z') \frac{dz'}{z - z'}
\]

where \( h \) is smooth = 1 on \( z = z' \).

Now let's put \( z' = 0 \) to simplify. A true parametrix for \( \tilde{\Delta} \) with singularity at \( z' = 0 \) satisfies

\[
\pi \, \partial_{z} \, G(z) = \delta(z) + \text{smooth fn}.
\]

\[
\pi \, \partial_{z} \left[ G(z) - \frac{1}{\pi z^2} \right] = \text{smooth}
\]

whence by Weyls lemma

\[
\pi \, G(z) = \frac{1}{\pi z^2} + \text{smooth}
\]

A function like

\[
\frac{h(z)}{z} = 1 + h_1 \frac{z}{2} + h_2 \frac{z^2}{2} + h_{11} \frac{\bar{z}}{2} + h_{12} \frac{z \bar{z}}{2} + \cdots
\]

will be a parametrix only if \( h_{11}, h_{12}, \ldots = 0 \).

For the purposes of the index thing, we want
\[ \partial_{z^2} G - 8 \text{ to have a value at } z = 0 \text{ only so that it is enough to require } h_1 \text{ and } h_2 = 0. \] Perhaps even \( h_2 \) could be \( \pm 0 \) as any symmetric averaging process will assign to \( h_2 \) the value 0.

So the problem arises as to the exact nature of the singularity of the flat Green's fn.

Let's review the equations. We put
\[ n^2(z, z') = |z - z'|^n f(z, z') \]
and want
\[ \partial_{z^2} n^2 \partial_{z^2} n^2 = \rho n^2 \]
or
\[ \left[ f + (z-z') \partial_z f \right] \left[ f + (\bar{z}-\bar{z'}) \partial_{\bar{z}} f \right] = \rho(z) f(z, z') \]
\[ 1 + (z-z') \partial_z \log f = \frac{f(z)}{f + z-z' \partial_z f} \]
\[ \frac{1}{z-z'} + \partial_z \log f = \frac{\rho(z)}{\partial_z n(z, z')^2} \]
Now interchange \( z \) and \( z' \) and use that \( f \) is symmetric and you get
\[ \frac{1}{z-z'} - \partial_z \log f = \frac{f(z')}{-\partial_z n(z, z')^2} \]
From this we see that the flat Green's fn. is a parametrix in the strong sense. In fact one sees immediately the desirability of working directly with \( \log f \) instead of \( n^2 \). \( \log f \) might be related to the Green's function for the Laplacean.

This can be developed as follows. In \( \mathbb{C} \) the Green's function for \( \Delta = 4 \partial_{z} \partial_{\bar{z}} \) is \( \frac{1}{2\pi} \log n = \frac{1}{4\pi} \log n^2 \); hence the Green's fn. for \( \partial_{z} \partial_{\bar{z}} \) is \( \frac{1}{2\pi} \log n^2 \), and so the Green's fn. for \( \partial_{z} \) is \( \frac{1}{2\pi} \log |z|^2 = \frac{1}{\pi} \). The obvious parametrix for \( -(\delta^2) \) is \( \frac{1}{\pi} \log n(z, z')^2 = \frac{1}{\pi} \log |z-z'|^2 + \frac{1}{\pi} \log f(z, z') \), but the order is slightly wrong.
Simpler derivation. We start with the fn.

\[ n(z, z')^2 \text{ intrinsically defined in a nbd. of the diagonal.} \]

We also know that it is of the form locally

\[ n(z, z')^2 = |z-z'|^2 f(z, z') \]

where \( f \) is smooth. Then look at

\[-\partial_{z'} \log (n^2) = -\partial_{z'} \log (|z-z'|^2 f(z, z')) = \left( \frac{1}{z-z'} - \partial_{z'} \log f(z, z') \right) dz' \]

which is intrinsically defined in a nbd. of the diagonal, and is clearly a parametrix for \( \overline{\delta} \).

Now that we have the parametrix it should be possible to derive the index thm. Let's review the calculation on p 28+29.

\[ G(z, z') dz' = \frac{i}{2\pi} \left\{ \frac{1}{z-z'} + a_z + \frac{b_z z - c_z (z-z') + O((z-z')^2)}{2} \right\} dz' \]

Then

\[ \partial_z G(z, z') = \frac{i}{2\pi} \delta(z-z') + \frac{i}{2\pi} a_z \partial_z \delta(z-z') + O(z-z') \]

\[ -\partial_{\overline{z}} G(z, z') = \frac{i}{2\pi} \delta(z-z') - \frac{i}{2\pi} a_{\overline{z}} \partial_{\overline{z}} \delta(z-z') + O(z-z') \]

Hence the index is given by integrating

\[ \frac{i}{2\pi} \partial_{\overline{z}} a_{\overline{z}} \ dz' \partial_{\overline{z}} \]

In the present case

\[ a_{\overline{z}} = -\partial_{z'} \log f(z, z') \bigg|_{z=z'} \]

Because \( f \) is symmetric we have

\[ \partial_{z'} f(z, z') = \boxed{2 \partial_{z'} f(z, z')} \]

\[ \boxed{2 \partial_{z'} f(z, z')} \bigg|_{z=z'} \]
hence \[ a_2 = -\frac{1}{2} \partial \bar{z} \log f(z, \bar{z}) = -\frac{1}{2} \partial \bar{z} \log \rho(z). \]

Now recall that \[ |dz|^2 = |dx|^2 + |dy|^2 = \frac{2}{f} \] and hence the curvature form for the cotangent bundle is
\[ \partial \partial \log |dz|^2 = -\partial^2_{\bar{z}z} \log (\frac{2}{f}) \, dzd\bar{z} = \partial^2_{\bar{z}z} \log (\rho) \, dzd\bar{z}. \]

Thus
\[ \partial^2_{\bar{z}z} \, dzd\bar{z} = -\frac{1}{2} \, \partial^2_{\bar{z}z} \log \rho(z) \, dzd\bar{z} = -\frac{1}{2} \text{ curvature form of } T^{1,0} \]

and so
\[ \int \frac{i}{2\pi} \partial^2_{\bar{z}z} \, dzd\bar{z} = -\frac{1}{2} \deg K = -\frac{1}{2} (2g - 2) = 1 - g \]

which gives RR.

\textbf{Remark:} 0 There is an analogy between constructing a parameter for \( \partial \) and constructing the Weierstrass \( \wp \)-function, or rather a periodic version of it. To do the global construction one must introduce some non-analyticity, which is then detected later as a topological characteristic class.

\textbf{Remark:} 2 The above proof of RR using the flat Green's fn. should be related to the one using the \( \wp \)-fn. or heat kernel. Both seem to use a kind of WKB construction which in the end carries all the information.
Yesterday I constructed a parametrix for $\overline{D}$ over a Riemann surface:

$$\frac{i}{2\pi} \left[ -\frac{1}{2} \frac{t}{i} \log \left( n(z, z')^2 \right) \right] d\zeta'$$

$$\frac{1}{2-\zeta} - \frac{1}{2} \log v(\zeta, z')$$

where $n(z, z')^2 = |z-z'|^2 v(z, z')$.

There seems to be a close connection between this parametrix and the WKB approximation for the heat kernel $e^{-tD^*D}$. Here are some formal ideas:

$$\frac{1}{\Gamma(s)} \int_0^\infty e^{-tD^*D} t^{s-1} dt = (D^*D)^{-s}$$

Take $s=0$:

$$\int_0^\infty e^{-tD^*D} dt = (D^*D)^{-1}$$

Let's substitute the WKB version for $e^{-tD^*D}$

$$\langle z | e^{-tD^*D} | z' \rangle = e^{-\frac{n(z, z')^2}{2t}} \frac{1}{\sqrt{4\pi t}} \theta(z, z') \left[ 1 + O(e) \right]$$

Maybe we get a parametrix for $D^*D$ by using a cutoff

$$\int_0^\infty e^{-tD^*D} dt = \frac{1 - e^{-\epsilon D^*D}}{D^*D}$$

since what we omit is smooth:

$$e^{-\epsilon D^*D} (D^*D)^{-1}$$

So

$$\int_0^\epsilon e^{-\frac{n^2}{2t}} \frac{1}{\sqrt{4\pi t}} dt = \int_0^\infty e^{-\frac{n^2}{2t}} \frac{1}{\sqrt{4\pi t}} dt = \int_{n^2/\epsilon}^\infty \frac{1}{\sqrt{\pi \text{ const.}}} dt$$

$$= - \frac{1}{\sqrt{\pi}} \log n^2 + O(1)$$

Hence to a first approximation the parametrix for $e^{-tD^*D}$ yields the parametrix for $(D^*D)^{-1}$. 

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August 22, 1982
The parametrix is \[ \frac{i}{2\pi} G(z, z') dz' \] where
\[
G(z, z') = \frac{1}{z - z'} - \frac{\partial_{z'} \log V(z, z')}{2 - z'}
\]
\[
= \frac{1}{z - z'} + a_z (z - z') + b_z (\overline{z} - z') + c_z (\overline{z} - z') + O((z - z')^2)
\]
I've already seen that \( a_z = -\frac{i}{2} \partial_z \log \rho(z) \). Formulas for \( b_z \) and \( c_z \) might be useful, especially since \( c_z \) occurs in the diagonal values of \( DP_{-1 \Delta} \) and \( PD_{-1 \Delta} \).

Calculations: First compute \( \partial_z \log V(z, z') \) with \( z' = 0 \).

\[
\lambda^2 = |z|^2 V = |z|^2 \rho_0 e^{\theta}
\]
\[
\partial_z \lambda^2 = \overline{z} \rho_0 e^{\theta} + |z|^2 \rho_0 e^{\theta} \partial_z \theta = \overline{z} \rho_0 e^{\theta} (1 + z \partial_z \theta)
\]
\[
\partial_{\overline{z}} \lambda^2 = z \rho_0 e^{\theta} (1 + \overline{z} \partial_{\overline{z}} \theta)
\]
\[
p = \rho_0 e^{\theta} (1 + z \partial_z \theta)(1 + \overline{z} \partial_{\overline{z}} \theta)
\]
\[
\log p = \log \rho_0 + \theta + z \partial_z \theta - \frac{1}{2} (\overline{z} \partial_{\overline{z}} \theta)^2 \ldots
\]
\[
\quad + \overline{z} \partial_{\overline{z}} \theta - \frac{1}{2} (z \partial_z \theta)^2 \ldots
\]
Put
\[
\theta = \theta_1 \overline{z} + \theta_{\overline{z}} + \theta_2 \overline{z}^2 + \theta_{1\overline{z}} z \overline{z} + \theta_2 z \overline{z}^2 + \ldots
\]
\[
z \partial_z \theta = \theta_1 \overline{z} + 2 \theta_2 \overline{z}^2 + \theta_{1\overline{z}} z \overline{z} + \theta_2 z \overline{z}^2 + \ldots
\]
\[-1/2(z \partial_z \theta)^2 = -1/2 \theta_1 \overline{z}^2
\]
\[
\overline{z} \partial_{\overline{z}} \theta = \theta_{1\overline{z}} \overline{z} + \theta_2 z \overline{z}^2 + \theta_1 z \overline{z} + \theta_2 z \overline{z}^2 + \ldots
\]
\[-1/2(\overline{z} \partial_{\overline{z}} \theta)^2 = -1/2 \theta_1 z^2
\]
\[
\log p = \log \rho_0 + 2 \theta_1 \overline{z} + 2 \theta_{1\overline{z}} z \overline{z} + (2 \theta_2 - \theta_{1\overline{z}}) \overline{z}^2 + 3 \theta_1 z \overline{z} + (2 \theta_2 - \theta_{1\overline{z}}) z \overline{z}^2 + \ldots
\]
\[
\theta_1 = \frac{1}{2} (\partial_z \log \rho)_0 \quad \quad \theta_2 = \frac{1}{3} [ (\partial_{z'} \log \rho)_0 + \frac{1}{4} (\partial_z \log \rho)^2 ]
\]
\[
\theta_{1\overline{z}} = \frac{1}{3} (\partial_{\overline{z}} \log \rho)_0
\]
I want \( \partial_z \log \nu = \partial_{\bar{z}} \tilde{\nu} = g_1 + g_2 z + g_3 \bar{z} + \ldots \)

and get

\[
\partial_z \log \nu(z, \bar{z}') = \frac{1}{2} (\partial_z \log \nu)_{z'} \bar{z}' + \frac{1}{3} \left[(\partial_{zz} \log \nu)_{z'} (z - \bar{z}') + \frac{1}{2} (\partial_z \log \nu)_{z'} (\bar{z} - \bar{z}') \right] + \frac{1}{6} (\partial_{zz} \log \nu)_{z'} (\bar{z} - \bar{z}') \]

Now interchange \( z, \bar{z'} \) and use

\[
\frac{1}{2} \partial_z \log \nu = \frac{1}{2} (\partial_z \log \nu)_{z'} + \frac{1}{2} (\partial_{zz} \log \nu)_{z'} (z - \bar{z}') + \frac{1}{2} (\partial_z \log \nu)_{z'} (\bar{z} - \bar{z}')
\]

and get

\[
\partial_{z'} \log \nu(z, \bar{z}') = \frac{1}{2} (\partial_{z'} \log \nu)_{z'} + \left[ \frac{1}{6} (\partial_{zz} \log \nu)_{z'} - \frac{1}{12} (\partial_z \log \nu)_{z'} \right] (z - \bar{z}') + \frac{1}{6} (\partial_{zz} \log \nu)_{z'} (\bar{z} - \bar{z}')
\]

yielding

\[
\nu(z, \bar{z}') = \frac{1}{z - \bar{z}'} + a_{z'} + b_{z'} (z - \bar{z}') + c_{z'} (\bar{z} - \bar{z}') + \ldots
\]

\[
a_{z'} = -\frac{1}{2} (\partial_z \log \nu)_{z'}, \quad b_{z'} = -\frac{1}{6} (\partial_{zz} \log \nu)_{z'} + \frac{1}{12} (\partial_z \log \nu)_{z'}^2, \quad c_{z'} = -\frac{1}{6} (\partial_{zz} \log \nu)_{z'}
\]

The reason I find this interesting is that

\[
\int \frac{1}{2\pi} c_{z'} \, dz' \bar{d}z' = \frac{1}{6} (2 - 2g) = \frac{1}{3} (1 - g)
\]

which is the constant \( g \) found in the value of \( J(0) \).

I still don't understand very well what a parametrix is. The above construction is pretty computational. When I get to the heat kernel the computation becomes even worse. Therefore I want to analyze directly what it means to construct a
parametrix. These objects exist locally and one has to combine the local gadgets to obtain a global one.

Let's interchange $z, z'$. Then the problem is to construct a singular differential form
\[ \left\{ \frac{1}{z-z'} + \text{smooth}(z, z') \right\} \, dz \]
having residue 1 at $z'$ and depending smoothly on $z'$. This gives a natural fibre space over $M$, namely, to each $z'$ we can associate germs of meromorphic differentials at $z'$ having a simple pole with residue $= 1$ at $z'$. We can work modulo differential forms vanishing at $z'$.

Then we get a fibre bundle consisting of forms
\[ \left\{ \frac{1}{z-z'} + a \right\} \, dz \pmod{z'} \]
This is an affine bundle over $\Omega^1$ and hence determines a class in $H^1(M, \Omega^1)$. This extension
\[ 0 \rightarrow \mathcal{O}' \rightarrow \mathcal{O} \rightarrow O \rightarrow 0 \]
has to be non-trivial holomorphically, as we have seen that $\delta$ gives the topological obstruction.

Try to describe more globally. At a given point $z'$ we have $m_{z'}^{-1}$ = germs of functions with a simple pole at $z'$ and $m_{z'}^{-1} \otimes \Omega^1_z = m_{z'}^{-1} \Omega^1_z = \text{germs of differentials with simple pole at most at } z'$. Since $m_{z'}^{-1} / m_{z'}^{-1} = \Omega^1_{z'}$, it follows that $m_{z'}^{-1} \Omega^1_z / \Omega^1_{z'} = \Omega^{\circ}_{z} / m_{z'}^{-1} = \mathcal{O}$. Thus
\[ \text{gr}_{\mathcal{O}} \left( m_{z'}^{-1} \Omega^1_{z'} \right) = m_{z'}^{8} / m_{z'}^{8+1} \]
As $z'$ varies we get vector bundles over $M$.

These can be described as follows:
\[ \text{gr}_{\mathcal{O}} \left( \Delta^{-1} \otimes \text{pr}_2^* \Omega^1_M \right) = \Delta^8 / \Delta^{8+1} = (\Omega^1_M)^{\otimes \mathcal{O}} \]
Next project: Anomaly formula for denominator-free Greens functions. Suppose given \( D : V_1 \to V_0 \) Fredholm whence we get a line
\[
\mathcal{L}_D \subset \text{Hom}(\rho)(\Lambda V_0, \Lambda V_1) \quad \rho = \text{Ind}(D)
\]
We choose a generator for this line whence we get a linear map \( \Lambda V_0 \to \Lambda V_1 \) of degree \( \rho \), whose matrix elements will be called denominator-free Greens functions. Use notation \( \Lambda D^{-1} \) for this map, since that is what it is when \( D \) is invertible.

\[ \int [d^2x dy] \ e^{-\mathcal{L}D\psi} \ \psi(x_1) \cdots \psi(x_k) \ \bar{\psi}(y_1) \cdots \bar{\psi}(y_l) \]
\[ = \ \langle \psi(x_1) \cdots \psi(x_k) | \Lambda(D^{-1}) | \bar{\psi}(y_1) \cdots \bar{\psi}(y_l) \rangle. \]

Here \( \psi(x_i) \in V_1^* \), \( \bar{\psi}(y_j) \in V_0 \) and
\[ \mathcal{L}D\psi \in V_1^* \otimes V_0 \]
\[ e^{-\mathcal{L}D\psi} \in \Lambda V_1^* \otimes \Lambda V_0. \]

The above formula, which is purely formal, does not specify the diagonal matrix elements, \( \mathcal{L}D\psi \), and in the case where \( D \) is invertible we have a regularization process.
August 23, 1982

Project: Anomaly formula for denominator-free Green's functions. Begin with linear algebra. Suppose given $A: W \to V$. Then we can view $A$ as an element of $W^* \otimes V$ which sits inside $\Lambda^2(W^* \oplus V)$ and form

$$e^A \in \Lambda(W^* \oplus V) = (\Lambda W)^* \otimes \Lambda V.$$ 

Suppose we are given bases of $V, W$. The natural way to write $A$ as a linear transformation is

$$A = \sum \langle j| A| i \rangle \langle i|$$

where $| i \rangle$ is the short notation for $w_i \in W$

$$| j \rangle \quad v_j \in V \quad \text{etc.}$$

So

$$A = \sum v_j A_{ji} w_i^* = \sum a_{ji} v_j w_i^* \in V \otimes W^*.$$ 

I associate to this the element of $\Lambda^2(W^* \oplus V)$

$$A \mapsto \sum a_{ji} w_i^* \wedge v_j.$$ 

This is the convention I have to use, namely, the embedding

$$W^* \otimes V \subset \Lambda^2(W^* \oplus V)$$

$$w^* \otimes v \mapsto w^* \wedge v.$$ 

Why? Because that

$$\delta \log \int e^A = \frac{\int e^{A_{SA}}}{\int e^A} = \sum \frac{\int e^A \omega^*_{ij} s \delta_{ij}}{\int e^A} = \text{Tr}(A^{-1} S A)$$

and so

$$\left( A^{-1} \right)_{ij} = \frac{\int e^A \omega^*_{ij}}{\int e^A}.$$ 

Finally

$$\sum a_{ji} w_i^* \wedge v_j = - \sum v_j a_{ji} w_i^*$$

in the exterior algebra which is why one sees it written

$$\psi^* A \psi$$

where $\psi$ destroys $\psi^*$ creates.
General anomaly picture: suppose we have an elliptic operator $D: E \to F$ and an automorphism $\varphi$ of $E, F$ commuting with the symbol of $D$. We have attached a line $L_D$ to $D$ and similarly a line $L_{\varphi D\varphi^{-1}}$ to $\varphi D\varphi^{-1}$, and we have a canonical isomorphism $L_D \sim L_{\varphi D\varphi^{-1}}$.

For example, if $D$ is invertible then both lines are canonically trivial, and the above isomorphism is compatible with the trivializations. On the other hand using analytic torsion, each line has a metric and the above canonical isomorphism doesn’t preserve the metric, so we get a positive real number attached to $\varphi$.

Let’s examine this carefully in the invertible case, where we know $L_D$ has a canonical generator whose norm squared is the analytic torsion

$$\tau = e^{-\frac{1}{2}\int_{xD}(0)}.$$ 

Replace $\varphi$ by an infinitesimal autom., i.e. an endom. $8\varphi$ of $(E, F)$. Then

$$8\log \tau = -8\int_{xD}(0)$$

and in good cases like $\bar{\varphi}$ I know that

$$-8\int'(0) = \left. \text{Tr} \left( (D^*D)^{-\frac{s}{2}} (D^*D)^{-1} S(D^*D) \right) \right|_{s=0}$$

is a constant term in the $s \to 0$ asymp. exp. for $\text{Tr} \left( e^{-t0^*D}(D^*D)^{-1} S(D^*D) \right)$. 

Now $8D = [S\varphi, D]$, so ultimately we get an expression for $8\log \tau$ as an integral of $S\varphi$ against a quantity depending on $D$. 
For the moment let's work out what happens for a general $\tilde{D}$ operator on a Riemann surface, not just an invertible one where I more or less understand what happens. Review first the steps when $D$ is invertible.

$$\text{Tr} \left( e^{-tD^*D} D^{-1} \delta' D \right) \xrightarrow{t \to 0} \int \text{tr} (JSD)$$

where $J$ is the finite part of the kernel for $D^{-1}$ on the diagonal constructed using the flat Green's $f_w$.

$$J = G - G_b \text{ restricted to } \Delta M.$$ Then when $SD = [\delta \phi, D]$ we get

$$\delta \log \tau = \left[ \int \text{tr} (J [\delta \phi, D]) + \text{c.c.} \right] $$

$$\int \text{tr} \left( [D, J] \delta \phi \right)$$

and

$$[D, J] = -[D, G_b] \big|_\Delta = -(\text{R.R. form for } D)$$

The goal is to understand the anomaly business when $D$ is not invertible. Hence I want to use the line $L_D \subset \text{Hom}(\Lambda V, \Lambda W)$. Because $D$ is usually only densely defined it is nice to think of $D$ in terms of its graph

$$\Gamma_D \subset W \otimes X \times V$$

which is a closed subspace such that the projection

$$p_2 : \Gamma_D \to V$$

is Fredholm. I like to think of $L_D$ as the line in $\Lambda (W \otimes V)$ belonging to the subspace $\Gamma_D$, but this is imprecise. One has

$$L_D \subset \text{Hom}(\Lambda V, \Lambda W) \supset \Lambda (W \otimes V; V) = \Lambda W \otimes \Lambda V^*$$
I thought that the excellent situation occurred when $L_D$ is in the $L^2$-Fock space $\Lambda W \otimes \Lambda V^*$ because then it inherits a metric. But this metric seems to be the wrong one. For example when $D$ is invertible $L_D$ is spanned by $\Lambda (D^{-1}) : \Lambda V \rightarrow \Lambda W$ and the good metric on $L_D$ is given by

$$||\Lambda (D^{-1})||^2 = \det (D^* D)^{-1}.$$ 

On the other hand the natural inner product on $\text{Hom}(\Lambda V, \Lambda W)$ is $\text{Tr}(A^* B)$. Hence

$$||\Lambda (D^{-1})||^2 = \text{Tr} \left( (\Lambda (D^{-1})^* \Lambda (D^{-1})) \right)$$

$$= \det \left[ 1 + (DD^*)^{-1} \right].$$

It might be useful to understand these two functions in the situation Graeme works with. He considers subspaces $\Gamma$ of $W \oplus V$ which are $L^2$-commensurable with $V$ in some sense, and hence which give rise to a line in the Fock space $\Lambda W \otimes \Lambda V^*$. Such subspaces are called polarizations, and each gives then a "vacuum vector" in the Fock space. One should think of these vectors as the Gaussian functions in the metaplectic representation. By associating to $\Gamma$ the corresponding line in the Fock space, we get a canonical line bundle $\mathcal{L}$ over the Grassmannian of these polarizations.

Now restrict attention to those $\Gamma$ such that $\text{pr}_2 : \Gamma \rightarrow V$, whence $\Gamma$ is the graph of a $T : V \rightarrow W$. This is the fat open set in the Grassmannian of $\Gamma$ of index 0 relative to $V$. Choose orthonormal bases for $V, W$ such that $T v_i = \lambda_i \bar{v}_i$ whence $\Gamma$ is spanned by $(T v_i, \bar{v}_i) = (\lambda_i \bar{w}_i, \bar{v}_i)$ and hence
the line \( \mathbb{L} \) in Fock space corres. to \( \Gamma \) has the
generator \( \nu_r = \bigwedge_i (v^i + \lambda_i \omega^i) \).

\( \mathbb{L} \) has a unique element \( \nu_r \) such that \( \langle \Omega | \nu_r \rangle = 1 \)
and one has

\[
\| \nu_r \|^2 = \prod (1 + \lambda_i^2) = \det (1 + T^* T)
\]

It seems that there is no way to obtain a
version of \( \det (D^* D)^{-1} = \det (T^* T) \) from the Fock space
situation. All I have is this line bundle \( \mathbb{L}^* \) over the
Grassmannian with a section \( s \) which is non-vanishing
for the \( \Gamma_E \) with \( \text{pr}_2 : \Gamma \rightarrow V \). The only way to
get a determinant function \( \det (D) \) is by trivializing the
line bundle \( \mathbb{L}^* \) over the \( \Gamma_D \)'s which occur. For
example one can trivialize \( \mathbb{L}^* \) by lifting back to the
central extension of the restricted unitary gp.
In this case \( g \) one gets a spherical fn. \( g \rightarrow \langle \Omega g \Omega \rangle \).
The element \( \Omega \) is a specific generator for \( \mathbb{L} \) belonging
to \( \Gamma = g V \). Hence \( | \langle \Omega | g \Omega \rangle |^2 \) depends only on \( \Gamma \).

\[
g \Omega = \frac{\nu_r}{\| \nu_r \|} \times \text{const of abs. vol. 1}
\]

\[
| \langle \Omega | g \Omega \rangle |^2 = \left| \frac{\langle \Omega | \nu_r \rangle}{\| \nu_r \|} \right|^2 = \frac{1}{\| \nu_r \|^2} = \frac{1}{\det (1 + T^* T)}
\]

But this still doesn't yield anything like \( \det (T^* T) \).
August 24, 1982

\[ D: W \to V \quad \mathcal{L}_D \subset \text{Hom} (\Lambda V, \Lambda W) \]

If \( D \) is invertible, one has formally

\[
\frac{\det (D+B)}{\det (D)} = \det (1+ D^{-1}B) = tr (\Lambda D^{-1} \Lambda B).
\]

Since \( \Lambda D^{-1} \) generates \( \mathcal{L}_D \), one could try to use this formula with \( \Lambda D^{-1} \) replaced by a generator of \( \mathcal{L}_D \). Since \( \Lambda B \) is of degree 0, this won't give anything except in the index 0 case.

Nevertheless, we can try to prove the above formula for \( \delta \) operators. \( B \) is a multiplication operator and \( D^{-1} \) is given by a Greens fn., so \( D^{-1}B \) is represented by a kernel

\[ K(z, z') = G(z, z') b(z'). \]

\( \Lambda ^0 D^{-1} B \) will be represented by the kernel

\[
\det \left[ G(z_i, z'_j) b(z'_j) \right] = \det G(z_i, z'_j) \cdot \prod_j b(z'_j)
\]

To take the trace I need to have \( z'_j = z_j \), and to make sense of \( \det (G(z_i, z'_j)) \).

I know what to do about diagonal elements, \( G(z, z) \).

For \( q = 2 \) I get

\[
\begin{vmatrix}
G(x, x) & G(x, y) \\
G(y, x) & G(y, y)
\end{vmatrix}
\]

and the main problem concerns integrating

\[ G(x, y) G(y, x) b(x) b(y) \quad z = x \quad z' = y. \]

This is an integral of the form

\[
\frac{d^2 \bar{z} \bar{z}'}{(2\pi i)^2} = \frac{d^2 \bar{u} \bar{u}}{u^2}, \quad u = z' - z
\]

and in 2 dims \( \frac{du}{u^2} = r \, dr \, d\theta \) is logarithmically divergent.
although if we do the Θ integral symmetrically there is no problem in getting a good number.

Conclusion: For D invertible we get Green's fun.

$$G(\theta)(z_1, \ldots, z_8) = \det(G(z_i, z_j))$$

which are well-defined distributions on the product $M^8$ such that

$$\frac{\det(D+B)}{\det D} = \sum_{\sigma} \int \prod_{i=1}^8 G^{(\sigma)}(z_{\sigma(i)}, z_{\sigma(8)}) \prod b(z_{\sigma(i)}) \, dz_{\sigma(1)} \cdots dz_{\sigma(8)}$$

$G^{(\sigma)}$ is a sum of terms described by the cycle structure of permutations. The 2 cycles lead to the fact the $G^{(\sigma)}$ are distributions and not functions. There is a problem in showing the above series converges for any $b$.

At this point I understand what the diagonal Green's functions are for $D$ invertible. Now the problem arises how to extend this to the general case. We choose a generator for $\Lambda D$ hence get a map from $\Lambda^1 V$ to $\Lambda^1 W$ whose matrix elements are what we want.

Consider first the case where $D: W \rightarrow V$ is onto and $\dim(\ker D) = 1$. Then the map $\Lambda^1 V \rightarrow \Lambda^1 W$ has degree 1, so we get in particular a map $\Lambda^1 V \rightarrow \Lambda^1 W$ defined up to a scalar. This should amount to choosing a generator $\omega$ for $\ker D$. Then $\Lambda^1 V \rightarrow \Lambda^1 W$ should be $\omega \mapsto \omega \wedge D^{-1} \omega$, or $D^{-1} \omega \wedge \omega$ depending on our conventions. Here $D^{-1}$ is a partial inverse for $D$, but $\omega \wedge D^{-1} \omega$ doesn't depend on the choice of $D^{-1}$. The matrix elements of $\omega \mapsto \omega \wedge D^{-1} \omega$ are

$$\langle z_1, z_2 \mid \omega \wedge D^{-1} \omega \rangle = \begin{vmatrix} w(z_1) & G(z_1, z_2) \\ w(z_2) & G(z_2, z_2) \end{vmatrix}$$

and the matrix elements of $\omega \mapsto \omega \wedge D^{-1} \omega \wedge \omega \wedge D^{-1} \omega \wedge D^{-1} \omega$ are

$$\det\left( \begin{array}{cccc} w(z_1) & G(z_1, z_2) & \cdots & G(z_1, z_8) \\ w(z_2) & G(z_2, z_2) & \cdots & G(z_2, z_8) \\ \vdots & \vdots & \ddots & \vdots \\ w(z_p) & G(z_p, z_2) & \cdots & G(z_p, z_8) \end{array} \right)$$
Interesting situation: In the case of \( \delta \) operators we know how to trivialize \( L_\delta \) as \( D \) varies over \( A \).

Hence, since \( L_\delta \subset \text{Hom}(\Lambda V, \Lambda W) \) we have attached to a finite-dim. subspace \( F \subset V \) and f.d. quotient space \( W \rightarrow Q \) with appropriate dimensions a certain holomorphic \( \text{f.w. on } A \), which is to be thought of as the determinant of a minor of \( D \). On the other hand these minors are also the denominator-free Green's functions.
August 25, 1982

Associated to any 3 operator $D : W \to V$ is the line $L_D = \text{Hom}(W, \Lambda^W)$, so if I have a generator of this line, then I get Green's functions which are the matrix elements of the maps $\Lambda^W \to \Lambda^W$. These $G$-funs. are distributions over products of $W$, and do not come provided with a regularization on the diagonal.

Now the theory I am working on constructs a trivialization of $L_D$ as $D$ varies, and hence the Green's functions obtained from this trivialization will also vary analytically in $D$. We can then differentiate w.r.t. $D$ and this should bring in the regularized diagonal values.

Let's consider this carefully when $D$ is invertible. We can then choose the generator of $L_D$ given by $\Lambda D^{-1}$. In this case, the Green's funs. are the determinants $\det G(z, z')$ where $G$ is the kernel for $D^{-1}$. But now the analytic function $\det(D)$ is constructed so that $(\det D) \cdot \Lambda D^{-1}$ extends to a trivialization of $L_D$ over all the $D$ of index 0. Thus the distributions on $M^8 \times M^8$

$$\det(D) \cdot \det G(z, z')$$

will by extension be defined and analytic in $D$ for all $D$.

But now we have

$$\delta \det(D) = \text{Tr} \left[ \frac{(\det D) G \cdot \delta D}{\text{Cof}(D)} \right]$$

So the definition of $\det(D)$ implies a regularization on the diagonal of $\det(D) G$. Similarly

$$\delta \left[ \det(D) G(z, z') \right] = \delta \det(D) \cdot G(z, z') + \det(D) \cdot \delta G(z, z')$$
\[ = \det(D) \left[ \int dz \ G(z, z) \delta D(z) \ G(z_1, z_1') - \int dz \ G(z, z) \delta D(z) \ G(z_1, z_1') \right] \]

\[ = \int dz \ \delta D(z) \begin{vmatrix} G(z, z) & G(z, z_1') \\ G(z_1, z) & G(z_1, z_1') \end{vmatrix} \frac{\det(D)}{\det(D) \ G^{(2)}(z, z_1 ; z, z_1')} \]

So what is happening is this. Each time we differentiate a denominator-free Green's function, we get a trace with one of higher order on the diagonal. Hence the regularization procedure needed to define the original trivialization of \( L \) should give a meaning to the other Green's functions when the arguments coincide. There might be a consistency problem.
Let's go back to $H_0 = \frac{1}{i} \frac{d}{dx}$ on $L^2(s')$, $s' = \mathbb{R}/L\mathbb{Z}$, and calculating the amplitude $\langle 0|s|0 \rangle$ for the perturbation $H = \frac{1}{i} \frac{d}{dx} + f(x,t)$, where $f$ has compact support.

The operator on Fock space belonging to multiplication by $f(x)$ is

$$\hat{f} = \int dx \, f(x) \varphi^*(x) \varphi(x)$$

$$\varphi(x) = \sum_k \langle x|k \rangle \varphi_k = \sum \frac{1}{\sqrt{L}} e^{ikx} \varphi_k$$

$$\varphi(x)^* = \sum_k \frac{1}{\sqrt{L}} e^{-ik'x} \varphi_k^*$$

$$\therefore \hat{f} = \sum_{k,k'} \left( \frac{1}{L} \int dx \, f(x) e^{-ik'x} \right) \varphi_k^* \varphi_k$$

$$= \sum_g \left( \frac{1}{L} \int dx \, f(x) e^{-igx} \right) \frac{\sum \varphi_k^* \varphi_k}{q \in \frac{2\pi}{L}}$$

Now $H_0|k\rangle = \lambda_k|k\rangle$ and $|0\rangle = \text{all } |k\rangle$, $k \leq 0$, filled, hence $\hat{f} |0\rangle = 0$ for $g < 0$; also for $g = 0$ by the convention about $g_0$. Thus $\hat{f}$ destroys, $\hat{s}_g$ creates for $g > 0$, and the non-trivial commutation relation is

$$[\hat{s}_g, \hat{s}_{g'}] = \frac{L}{2\pi} \delta_{g,g'} \quad g > 0$$

In general for a perturbation such as $\hat{H} = \hat{H}_0 + \hat{f}(t)$, we have

$$S_S(t) = \mathcal{T} \left\{ e^{-\int_0^t \hat{H}_0 dt} \hat{f}(t) e^{-\hat{H}_0 t} \right\}$$

I need $e^{\hat{H}_0 t} \hat{s}_g e^{-\hat{H}_0 t} = e^{g t} \hat{s}_g$. In effect $\hat{H}_0 = \sum_k \lambda_k \hat{p}_k$.

So

$$[\hat{H}_0, \varphi_k] = [k \varphi_k^*, \varphi_k] = -k \{\varphi_k^*, \varphi_k\} = -k \varphi_k$$

$$e^{\hat{H}_0 t} \varphi_k e^{-\hat{H}_0 t} = e^{-\lambda t} \varphi_k$$

$$e^{\hat{H}_0 t} \varphi_k^* e^{-\hat{H}_0 t} = e^{\lambda t} \varphi_k^*$$

$$e^{\hat{H}_0 t} \varphi_k e^{-\hat{H}_0 t} = e^{\lambda t} \varphi_k$$

$$e^{\hat{H}_0 t} \varphi_k^* e^{-\hat{H}_0 t} = e^{\lambda t} \varphi_k^*$$
\[ e^{\hat{H}_0 t} f(t) e^{-\hat{H}_0 t} = \sum_{\delta} f_\delta(t) e^{\delta t} f_{\delta'} \]

\[ = \sum_{\delta > 0} f_\delta(t) e^{\delta t} f_{\delta'} + f_{\delta}(t) e^{-\delta t} f_{-\delta} \]

Recall the formula for a simple oscillator

\[ T \left\{ e^{-\int dt \left( f(t) a^* + f(t)a \right)} \right\} = e^{t \int dt' f(t') f(t')} e^{-\int dt f(t) a^*} \]

\[ \times e^{-(\int dt f(t)) a} \]

Hence in our situation \( \langle 0|s|0 \rangle \) is exp of

\[ \int dtdt' \sum_{\delta > 0} f_{\delta}(t) e^{\delta t} f_{\delta}(t') e^{\delta t'} \left[ f_{-\delta}, f_{\delta'} \right] \]

\[ \text{t > t'} \]

\[ = \int dtdt' dx dx' f(x,t) e^{i\delta x - \delta t} \frac{1}{L} dx' f(x',t') e^{-i\delta x' + \delta t'} \frac{L}{2\pi} \]

\[ = \int dtdt' dx dx' f(x,t)f(x',t') \sum_{\delta > 0} e^{-\delta \left[ \Delta t - i\Delta x \right]} \frac{2\pi}{2\pi} \]

As \( L \rightarrow \infty \), \( \frac{1}{L} \sum_{\delta} \rightarrow \int \frac{d\delta}{2\pi} \) \( \delta \)

\[ \sum_{\delta > 0} e^{-\delta \left[ \Delta t - i\Delta x \right]} \frac{1}{\delta} \frac{\Gamma(2)}{(\Delta t - i\Delta x)^{2}} \]

\[ \text{and so the answer is} \]

\[ \text{log} \langle 0|s|0 \rangle = \int dtdt' dx dx' \frac{f(x,t)f(x',t')}{(2\pi)^2} \frac{\Delta t - i\Delta x}{(\Delta t - i\Delta x)^{2}} \]

Why am I interested in the above calculation? This is a situation, where I can see the generating function \( Z(J) \) and the \( S \)-matrix completely, and where there is some regularization present.
Let's check the answer against previous calculations in the case where $f$ is independent of $t$. When $f$ is time-independent one lets it act for a long time $T$ and then

$$
\langle 0|s|0 \rangle \sim e^{-T \Delta E}
$$

where $\Delta E$ is the energy shift. Thus

$$
\Delta E = - \int_{t>0} \frac{dx dx' f(x)f(x')}{(2\pi)^2 (t-i\Delta x)^2}
$$

$$
= \int \frac{dx dx'}{(2\pi)^2} \frac{f(x)f(x')}{t+i\Delta x}
$$

Actually

$$
\int_0^\infty \frac{dt}{(t-i\Delta x)^2} = \left[ \frac{-1}{t-i\Delta x} \right]_0^\infty = \frac{-1}{\eta-i\Delta x}, \quad \eta=0^+.
$$

So by symmetry

$$
\Delta E = - \int \frac{dx dx'}{(2\pi)^2} f(x)f(x') \left[ \frac{1}{\eta-i\Delta x} + \frac{1}{\eta+i\Delta x} \right] \frac{1}{2}
$$

$$
\frac{\eta}{\Delta x^2 + \eta^2} = \pi \delta(\Delta x)
$$

$$
\Delta E = - \frac{1}{4\pi} \int dx \ f(x)^2
$$

This agrees with the previous calculation on p. 19.

Now I want to take the viewpoint that $\langle 0|s|0 \rangle$ is a determinant for the differential operator $\partial_t + \frac{i}{\hbar} \partial_x + f(x,t)$.

This is Schwinger's idea, but there seems to be problems with the square $(\Delta t - i\Delta x)^2$ in the formula. Let's go through Schwinger's derivation.
\[
\langle 0 | S | 0 \rangle = \langle 0 | e^{\hat{A}_0 T} \hat{U}(T, -T) e^{\hat{A}_0 T} | 0 \rangle \\
= \frac{\langle 0 | \hat{U}(T, -T) | 0 \rangle}{\langle 0 | \hat{U}_0(T, -T) | 0 \rangle}
\]

\[
S \log \langle 0 | S | 0 \rangle = S \log \frac{\langle 0 | \hat{U}(T, -T) | 0 \rangle}{\langle 0 | \hat{U}_0(T, -T) | 0 \rangle}
\]

\[
S \hat{f}(t) = \int dx \delta f(x, t) \psi^*(x) \psi(x)
\]

\[
S \log \langle 0 | S | 0 \rangle = -\int dt dx \delta f(x, t) \frac{\langle 0 | \hat{U}(T, t) \psi^*(x) \psi(x) \hat{U}(t, -T) | 0 \rangle}{\langle 0 | \hat{U}(T, -T) | 0 \rangle}
\]

The Green's function is defined by

\[
G(x, t, x'; t') = \langle \hat{U}(x, t) \psi^*(x', t') \hat{U}(x, t) \rangle
\]

\[
= \left\{ \begin{array}{ll}
\langle 0 | \hat{U}(T, t) \psi^*(x) \hat{U}(t, t') \psi(x') \hat{U}(t, -T) | 0 \rangle \\
\langle 0 | \hat{U}(T, -T) | 0 \rangle
\end{array} \right.
\]

One has

\[
\{ \partial_t + \hat{A}(t) \} \hat{U}(t, t') = 0
\]

hence

\[
\partial_t \{ \hat{U}(T, t) \psi(x) \hat{U}(t, t') \} = \hat{U}(T, t) [\hat{A}(t), \psi(x)] \hat{U}(t, t')
\]

and \( \hat{A}(t) \), being a 1-particle operator, has the form

\[
\hat{A}(t) = \int dy \psi^*(y) \left[ \frac{i}{\hbar} \partial_y + f(y, t) \right] \psi(y)
\]

hence

\[
[\hat{A}(t), \psi(x)] = -\int dy \delta(x-y) \left[ \frac{i}{\hbar} \partial_y + f(y, t) \right] \psi(y)
\]

\[
= - \left[ \frac{1}{i} \partial_x + f(x, t) \right] \psi(x).
\]
Therefore it follows that

\[ \partial_t + \frac{i}{\lambda} \partial_x + f(x, t) \] G(x_t, x_{t'}) = 0 \quad t \neq t' \]

On the other hand the jumps in G as t passes thru t' is \( \delta(x-x') \) because \( \phi(x) \phi(x')^* + \phi(x') \phi(x) = \delta(x-x') \).

Thus

\[ \left[ \partial_t + \frac{i}{\lambda} \partial_x + f(x, t) \right] G(x_t, x_{t'}) = \delta(x-x') \delta(t-t') \]

and so the only remaining information needed is the boundary conditions.

Let \( t \ll 0 \) and let \( t_0 \) be below the support of f. Then \( G(x_t, x_{t'}) \) is an inner product with a vector independent of \( x_t \) with

\[
\hat{U}(t_0, t) \phi(x) \hat{U}^*(t, -T) |0\rangle = e^{-\hat{A}_{t_0}} e^{\frac{it}{\lambda}} \phi(x) e^{-\hat{A}_T} e^{\frac{it}{\lambda}} |0\rangle
\]

\[ \frac{i}{\sqrt{\Lambda}} \sum_k e^{ikx-kt} \phi_k \]

hence

\[ G(x_t, x_{t'}) = \text{lin. comb. of } e^{ikx-kt} \quad \text{for } k \leq 0. \]

Analogously since \( \langle 0 | \phi_k = 0 \) for \( k \leq 0 \), one has

\[ G(x_t, x_{t'}) = \text{linear comb. of } e^{ikx-kt} \quad \text{for } k > 0 \]

if \( t \gg 0 \).

Let's compute \( G \) when \( f = 0 \). Want \( G \) fn.

for \( \partial_t + \frac{i}{\lambda} \partial_x = \frac{2}{\lambda} \left( \partial_x + i \partial_t \right) = \frac{2}{\lambda} \partial_{\tilde{z}} \)

if \( \tilde{z} = x + it \).

Since \( \partial_{\tilde{z}} \) has \( G \) fn. \( \frac{1}{\pi \tilde{z}} \) we get

\[ G = \frac{i}{2\pi \tilde{z}} = \frac{i}{2\pi (x+it)} = \frac{1}{2\pi (t-ix)} \]

assuming the boundary conditions are correct. Directly
\[ G(x,t) = \langle \psi(x,t) \psi(0)^* \rangle = \frac{1}{t} \sum_k e^{ikx-kt} \langle \psi_k^* \psi_k \rangle \quad t > 0 \]

\[ = \int_0^\infty \frac{dk}{2\pi} e^{-k(t-ix)} = \frac{1}{2\pi(t-ix)} \quad t > 0 \]

If \( t < 0 \)

\[ G(x,t) = -\langle \psi(0)^* \psi(x,t) \rangle = \frac{1}{t} \sum_k e^{ikx-kt} \langle \psi_k^* \psi_k \rangle \]

\[ = -\int_\infty^0 \frac{dk}{2\pi} e^{ikx-kt} = \frac{1}{2\pi(t-ix)} \quad t < 0 \]

Next I want the Green's function for a general \( f \) of compact support.

\[ \frac{i}{2} \left( \frac{\partial}{\partial t} + \frac{i}{\hbar} \partial_x + f \right) (G) = \frac{i}{2} \delta \]

\[ \left( \partial_{z'} \frac{i}{2} \right) G(z,z') = \frac{i}{2} \delta(z-z') \]

\[ e^{-\Psi} \partial_z e^{\Psi} i f \partial_z \Psi = \frac{i}{2} f \]

\[ \varphi(z) = \int \frac{i}{2\pi} \frac{1}{z-\omega} f(\omega) \, d\omega \]

\[ \partial_z e^{\varphi} G = \frac{i}{2} e^{\varphi(z')} \delta(z-z') \]

\[ e^{\varphi} G = \frac{i}{2} e^{\varphi(z')} \frac{1}{\Pi(z-z')} \]

\[ G(z,z') = \frac{i}{2\pi} \frac{e^{-\varphi(z')} + \varphi(z)}{z-z'} \]

\[ G(z,z') = \frac{i}{2\pi} \frac{1}{z-z'} e^{\int \frac{d\omega}{2\pi} \left( \frac{1}{\omega-z} - \frac{1}{\omega-z'} \right) f(\omega)} \]

Next you want to extract a finite part when \( z = z' \)

\[ \frac{1}{\omega-z} = \frac{1}{\omega-z'-(z-z')} = \frac{1}{\omega-z'} + \frac{z-z'}{(\omega-z')^2} + \ldots \]
\[ G(z,z') = \frac{i}{2\pi} \frac{1}{z-z'} \left( 1 + (z-z') \int \frac{i \, d^2\omega}{(2\pi)^2} \frac{f(\omega)}{(w-z)^2} + \ldots \right) \]

\[ G(z,z')_{\text{reg}} = \left( \frac{i}{2\pi} \right)^2 \int d^2\omega \frac{f(\omega)}{(w-z)^2} \]

\[ = \left( \frac{i}{2\pi} \right)^2 \int dx \frac{f(x,t)}{(\Delta t-i\Delta x)^2} \quad \Delta t = t-t' \]

From the formula on p. 55 we have

\[ S \log \langle 0 | s | 0 \rangle = \int \frac{dt \, dx \, dx'}{(2\pi)^2} \frac{df(x,t) \, df(x',t) + f(x,t) \, df(x',t)}{(\Delta t-i\Delta x)^2} \]

\[ = \int \frac{dt \, dx \, df(x,t)}{(2\pi)^2} \int \frac{dt' dx'}{(2\pi)^2} \frac{f(x,t')}{(\Delta t-i\Delta x)^2} \]

At this point we have an example of a \( \delta \)-determinant, namely \( \langle 0 | s | 0 \rangle \), a formula for the Green's function of \( \frac{2}{i} \partial \bar{z} + f \), and a way to regularize this Green's function on the diagonal consistent with the determinant. Now there are various things we can do.

1) gauge invariance and trace anomaly.

2) I can rewrite things as energy-momentum integrals.

3) Green's func. and \( S \)-matrix.

For gauge invariance we use

\[ S \log \langle 0 | s | 0 \rangle = \int d^2z \, df(z) \, J(z) \]

in the case where \( df \) arises from an infinitesimal gauge transf. \( df = \hat{\gamma} \left[ \hat{\gamma} \frac{2}{i} \partial \bar{z} + f \right] = -\frac{2}{i} \partial \bar{z} (\hat{\gamma} f) \)
Now \( J(z) = \left(\frac{i}{2\pi}\right)^2 \int d^2 \omega \frac{f(\omega)}{(z-\omega)^2} \).

\[
8 \log \langle 0 | s | 0 \rangle = \int d^2 \bar{z} \left( -\frac{n}{z} \partial_z \delta \varphi \right) J(z) = \frac{2}{i} \int d^2 \bar{z} \delta \varphi \cdot \overline{\partial_{\bar{z}} J(z)}
\]

\[
= \frac{2}{i} \left(\frac{i}{2\pi}\right)^2 \int d^2 \bar{z} \delta \varphi(z) \int d^2 \omega \left[ \partial_{\bar{z}} \left(\frac{1}{(z-\omega)^2}\right) \right] f(\omega)
\]

Hence we must calculate the distribution \( \partial_{\bar{z}} \left(\frac{1}{(z-\omega)^2}\right) \).

\[
\partial_{\bar{z}} \left(\frac{1}{(z-\omega)^2}\right) = \frac{-1}{(z-\omega)^2} = -\partial_\omega \left[ \frac{1}{\pi} \delta(z-\omega) \right]
\]

hence \( \partial_{\bar{z}} \left(\frac{1}{(z-\omega)^2}\right) = \frac{-1}{(z-\omega)^2} \).

\[
\int d^2 \omega \left[ \partial_{\bar{z}} \left(\frac{1}{(z-\omega)^2}\right) \right] f(\omega) = \frac{\pi}{\pi} \int d^2 \omega \partial_\omega \delta(z-\omega) f(\omega) = -\frac{\pi}{\pi} \partial_{\bar{z}} f(z).
\]

Thus \( 8 \log \langle 0 | s | 0 \rangle = \frac{2}{i} \left(\frac{i}{2\pi}\right)^2 (-\pi) \int d^2 \bar{z} \delta \varphi(z) \partial_{\bar{z}} f(z) \)

\[
= \frac{1}{2\pi i} \int d^2 \delta \varphi \partial_{\bar{z}} f
\]

and so there is a trace anomaly, and corresponding lack of \( K \) crude gauge invariance.

Simpler derivation (possibly?)

\[
\langle 0 | s | 0 \rangle = \frac{\det \left(\frac{2}{i} \partial_{\bar{z}} + f\right)}{\det \left(\frac{2}{i} \partial_{\bar{z}}\right)} = \frac{\det \left(1 + \frac{i}{2} (\partial_{\bar{z}})^{-1} f\right)}{K}
\]

\( K \) is given by the kernel \( \frac{1}{2\pi i} \frac{1}{z-z'} f(z') \) and
\[ \log \langle 0 | S | 0 \rangle = \log \det (1 - K) = - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}(K^n) \]

Now \[ -\frac{1}{2} \text{tr} K^2 = \frac{1}{2} \int \frac{d^2 z_1 d^2 z_2}{(2\pi i)^2} \frac{f(z_1) f(z_2)}{(z_1 - z_2)^2}, \]

according to the above formulas, is the only non-trivial term in this series. Let's see this directly:

\[ \text{tr} K^3 = \int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{(2\pi i)^3} \frac{f(z_2) f(z_3) f(z_1)}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)} \]

The numerator is symmetric, the denominator anti-symmetric under \( z_1 \to z_3, z_2 \to z_3, z_3 \to z_1 \), hence \( \text{tr}(K^3) = 0 \). A similar argument shows that \( \text{tr}(K^n) = 0 \) for \( n \) odd, but I don't see why one gets zero for \( n = 4 \).

What matters is the symmetrization of the fn.

\[ \frac{1}{(z_1 - z_2) \cdots (z_n - z_n)(z_n - z_1)} \]

call it \( \psi(z_1, \ldots, z_n) \). If you multiply \( \psi \) by the anti-symmetric fn. \( \omega = \prod (z_i - z_j) \), then you will get an anti-symmetric polynomial of degree \( n(n-1)/2 \), and this is impossible. (This is for \( n \geq 3 \) that \( \omega \psi \) is a polynomial, because then the factors \( z_i - z_j \) are distinct. For \( n = 2 \) \( \omega \psi = (z_1 - z_2) \frac{-1}{(z_1 - z_2)^2} \) is not a poly.)

Therefore we conclude that \( \text{tr}(K^n) = 0 \) \( n \geq 3 \) and we get the same formula for \( \langle 0 | S | 0 \rangle \).

In the non-abelian situation \( f \) is a matrix function of \( z \) and the numerator is \( \text{tr}(f(z_1) \cdots f(z_n)) \) which has cyclic symmetry only, so one can't make a similar vanishing assertion.
August 27, 1982

The central point to be clarified is this. By choosing a regularization process I trivialize the line bundle $L$ and hence for each $D$ give a meaning to the fermion integral

$$\int e^{-\int d^4 \phi \left( \psi(x) \cdot \gamma(x) \psi(x) \right)} \prod \psi(y_i)$$

as a distribution in $M^p \times M^q$. Now if I differentiate with $D$, then I make sense out of diagonal values of these Green's functions. On the other hand, by means of Taylor series, the diagonal Green's functions for one point $D_0$ determine everything for all other values of $D$. I would like to check that this all works.
Let's take the Schwinger viewpoint towards the Hamiltonian $\frac{1}{i} \partial_x + f(x,t)$ extended to Fock space, where $f$ has compact support. By this I mean that we should work in real time and calculate the $S$-matrix which should be a unitary operator on Fock space when $f$ is real.

In the present case $e^{i\hat{H}_0} f(t) e^{-i\hat{H}_0}$ is again a multiplication operator. In fact we have

$$e^{i\hat{H}_0} = e^{t \hat{\partial}_x}$$

and

$$e^{t \hat{\partial}_x} f \ e^{-t \hat{\partial}_x} g = e^{t \hat{\partial}_x} f \ g(-t) = f(+t) g$$

so that

$$(e^{t \hat{\partial}_x} f \ e^{-t \hat{\partial}_x})(x) = f(x+t).$$

Consequently

$$e^{i\hat{H}_0} f \ e^{-i\hat{H}_0} = f(+t).$$

Thus, since these operators commute modulo scalars, we have that modulo a scalar $c$ with $|c|=1$.

$$S = T \{ e^{-i \int dt \ e^{i\hat{H}_0} f(t) e^{-i\hat{H}_0}} \}
\begin{align*}
= c \ e^{-i \int dt \ f(+)^}\n= c \ e^{-i \int dt \ f(x+t,t)}^\n= c' \left[ e^{-i \int dt \ f(x+t,t)} \right]^\n\end{align*}$$

(This last but represents lifting as an autom. not derivative.)

Thus we conclude that the $S$-matrix on Fock space is given by the multiplication operator for the function

$$e^{-i \int dt \ f(x+t,t)}.$$

Next let's check gauge invariance. A gauge
transformation consists of conjugating the Hamiltonian by $e^{ix(st)}$, where $X$ is real-valued and of compact support. Under this gauge transformation, the 1-particle Schrödinger equation becomes

$$0 = e^{-ix} \left( \frac{i}{\hbar} \partial_t + \frac{1}{i} \partial_x + f \right) \psi = \left[ \frac{i}{\hbar} \partial_t + \frac{1}{i} \partial_x + (\partial_x X + \frac{\partial}{\partial x} X + f) \right] (e^{-ix} \psi).$$

Thus the gauge transformation changes $f$ to $f + \partial_x X + \frac{\partial}{\partial x} X$, where $X$ has compact support. Since

$$\int dt \left[ f(x(t)) + \partial_t X(x(t), t) + \partial_x X(x(t), t) \right] = \int dt f(x(t), t)$$

$$\partial_t [X(x(t), t)]$$

it follows that the gauge transformation does not affect the 1-particle S-matrix.

Now we should see what happens to the vacuum-vacuum amplitude $\langle 0 | S | 0 \rangle$. The calculation goes exactly as on p. 59-55 and yields

$$\log \langle 0 | S | 0 \rangle = -\int dt dt' dx dx' \ f(t) f(t') \ \frac{1}{2\pi L} \sum_{g \geq 0} \phi^{i g (\Delta x - \Delta t)} \ e^{ig(Dx - Dc)}$$

$$= \frac{1}{(2\pi)^2} \int dB e^{ig(Dx - Dc + i\gamma)} \ F$$

$$= \frac{1}{(2\pi)^2} \ \frac{1}{(\Delta x - \Delta t + i\gamma)^2}$$

$$\log \langle 0 | S | 0 \rangle = -\frac{1}{2} \int dt dt' dx dx' \ f(t) f(t') \ \frac{1}{(2\pi)^2} \cdot \left[ \frac{\Theta(\Delta t)}{(\Delta x - \Delta t + i\gamma)^2} + \frac{\Theta(-\Delta t)}{(\Delta x - \Delta t - i\gamma)^2} \right]$$

$$= -\frac{1}{2} \int \frac{dt dt' dx dx'}{(2\pi i)^2} \ f(x(t)) f(x(t')) \ \left[ \frac{\Theta(\Delta t)}{(\Delta x + i\gamma)^2} + \frac{\Theta(-\Delta t)}{(\Delta x - i\gamma)^2} \right]$$
Gauge invariance means this expression doesn't change when we make the change
\[ f(x+t, t) \rightarrow f(x+t, t) + \partial_x x(x+t, t) \]
with \( x \) of compact support. This is equivalent to
\[
\partial_t \left[ \frac{\Theta(\Delta t)}{(\Delta x + i \eta)^2} + \frac{\Theta(-\Delta t)}{(\Delta x - i \eta)^2} \right] = 0
\]
which is certainly true in the open set where \( \Delta x \neq 0 \) or \( \Delta t \neq 0 \).

Using \( \partial_t \Theta(\Delta t) = \delta(\Delta t) \) we get
\[
\partial_t \left[ \frac{\Theta(\Delta t)}{(\Delta x + i \eta)^2} + \frac{\Theta(-\Delta t)}{(\Delta x - i \eta)^2} \right] = \delta(\Delta t) \left[ \frac{1}{(\Delta x + i \eta)^2} - \frac{1}{(\Delta x - i \eta)^2} \right]
\]
\[
= \delta(\Delta t) \frac{2}{\Delta x} \left[ \frac{-1}{\Delta x + i \eta} + \frac{1}{\Delta x - i \eta} \right]
\]
\[
= \frac{2i \eta}{(\Delta x)^2 + \eta^2} \rightarrow 2\pi i \delta(\Delta x)
\]

Thus
\[
\partial_t \left[ \frac{\Theta(\Delta t)}{(\Delta x + i \eta)^2} + \frac{\Theta(-\Delta t)}{(\Delta x - i \eta)^2} \right] = 2\pi i \delta(\Delta t) \delta'(\Delta x)
\]
and so we don't have gauge-invariance. The above is clearly quite consistent with the imaginary time calculation on p.69.
Further calculations I could do.

1) Energy-momentum variables
2) S-matrix in imaginary time
3) Schwinger's gauge-invariant determinant
4) Gauge field with non-zero winding number.

Let's consider the problem of the S-matrix in imaginary time. We solve the DE

\[(\partial_t + \hat{H})\psi = 0 \quad H = \frac{1}{i}\partial_x + f\]

in Fock space thereby getting a propagator \(\tilde{A}(t,t')\) which should be a nice operator for \(t > t'\) on the Fock space. Then I can try to compare this operator with the free propagator \(\tilde{A}(t,t')\).

Good idea: There are 2 determinants. One comes from the fact that \(\langle 0 | s | 0 \rangle\) can be interpreted as the determinant of a 3-operator. The other comes from the fact that \(\langle 0 | s | 0 \rangle\) is the determinant of a corner in a block decomposition; this is Graeme's statement that \(\langle 0 | s | 0 \rangle\) is such a determinant.
One of the virtues of having worked thru the above example: \( \partial_t + \frac{i}{\lambda} \partial_x + f \) with \( f \) of compact support, is that it shows we can work over a non-compact Riemann surface provided we deal with compactly supported variations in the \( \bar{\partial} \)-operator. The nice thing about the example is that the Green's function for \( \bar{\partial} \) is so simple.

One possibility now would be to study the case of a small nbh. of a point on a Riemann surface and to allow only changes of the \( \bar{\partial} \)-operator within this small nbh. I place myself in an invertible situation and then ask about \( \tau \)-functions.

Hence it is first necessary to understand what \( \tau \)-functions are. This requires a review of the loop groups representation, etc.

Let's begin with \( V = L^2(S^1) \) with its orthonormal basis \( \{ \phi_n \} \) \( n \in \mathbb{Z} \). We are going to form a Fock space out of \( V \) which will then have operators

\[
\psi_n = \text{int. mult. by } <n|
\]

\[
\psi_n^* = \text{ext. mult. by } |n>
\]

and then we can introduce the operators

\[
\psi(S) = \text{int. mult by } <S| = \sum_n <S\phi_n|\phi_n>
\]

\[
\psi(S)^* = \sum_n S^{-n} \psi_n^*
\]

These are formal operators depending on a complex no. \( S \). These are the same kind of gadgets as \( \psi(x) = e^{\sqrt{-1} \theta(t) x} e^{-\sqrt{-1} \theta(t) x} \) encountered above. Let's work out the relation:

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \sum_k e^{ikx} \psi_k
\]

\[
\psi(x) = \frac{1}{\sqrt{2\pi}} \sum_k e^{ikx - \theta(x)} e^{-\theta(x)}
\]
Hence \( \psi(x,t) = \frac{1}{\sqrt{\lambda}} \sum_k e^{ik(x+it)} \psi_k \)

and so I should really think of \( \mathbb{Z} \) as being \( e^{i(x+it)} \). Hence \( \lambda > 0 \iff \{ 2 \} \).

\[
\langle \psi(x) \psi(x')^* \rangle = \sum_n (S/n)^n \langle \psi_n \psi_n^* \rangle
\]

Hence if we choose the vacuum in Fock space to be filled by the \( \mathbb{Z}^n \) with \( n < 0 \), we get \( \langle \psi_n \psi_n^* \rangle = \{ 1 \quad n > 0 \}
\]

\[
\langle \psi(x) \psi(x')^* \rangle = \frac{S/n'}{1 - (S/n')}
\]

is convergent for \( |S| < |S'| \); this corresponds to \( \lambda > \lambda ' \).

Review next the Japanese factorization formula. The idea here is that we have a subspace \( W \) of \( \mathcal{V} \) which is \( L^2 \)-commeasurable with \( H^- \) and which is complementary to \( H^+ \). Let \( \Omega, \Omega W \) be the unit vectors belonging to \( H^- \) and \( W \) resp.

Because \( H^+ = \text{span} \{ \mathbb{Z}^n, n > 0 \} \) is complementary to \( W \) we know that \( (H^+ + H^-) \cap W \) is 1-dimensional, in fact there is a unique element

\[
s \in (H^+ + H^-) \cap W.
\]

Then \( \psi(x) = 1 + a_1 z + a_2 z^2 + \ldots \) is analytic for \( |z| < 1 \) and we want a formula for it.

Formally \( \Omega W = \omega_0 + \omega_1 + \ldots \) where the \( \omega_n \) form a basis for \( W \). Hence

\[
\psi(x) \Omega W = \sum_{n > 0} \omega_n (s)(-1)^n \omega_0 \ldots \omega_n
\]

and so for any vector \( \Theta \) in Fock space

\[
S \mapsto \langle \Theta \mid \psi(x) \Omega W \rangle = \sum_{n > 0} \omega_n (s)(-1)^n \langle \Theta \mid \omega_0 \ldots \omega_n \rangle
\]

is a function in the space \( W \). On the other
hand if we take $\Theta$ to be the vector belonging to $z^i H^i = \text{span of } z^{-1}, z^{-2}, z^{-3}, \ldots$ then 

$$ \psi_{\Theta}^* \Theta = \sum_{n \geq 0} i^{-n} \psi_n^* \Theta = \sum_{n \geq 0} i^{-n} \psi_n^* \Theta $$

so 

$$ \langle \Theta | \psi_{\Theta} \rangle = i^{-m} \langle \Omega | + j \langle \Theta | \psi_{\Theta} \rangle + j^2 \langle \Theta | \psi_{\Theta} \rangle + \ldots $$

Thus 

$$ \frac{\langle \Theta | \psi_{\Theta} \rangle \Omega_W \rangle}{\langle \Omega | \Omega_W \rangle} = 1 + j \langle \Theta | \psi_{\Theta} \rangle \Omega_W \rangle + \ldots $$

$$ \in (I + H^+)^n W $$

is the desired function $s$. Better:

$$ s(x) = \frac{\langle \Omega | \psi_0^* \psi_{\Theta} \rangle \Omega_W \rangle}{\langle \Omega | \Omega_W \rangle} $$

$$ = 1 - \frac{\langle \Omega | \psi_{\Theta} \rangle \psi_0^* \Omega_W \rangle}{\langle \Omega | \Omega_W \rangle} $$

Baker-Akhiezer function belonging to $W$ is the unique element of $W$ of the form 

$$ s(x, z) = e^{\sum_{n} x_n z^{-n}} \left\{ 1 + \sum_{n} a_n(x) z^n \right\} $$

Thus you want 

$$ e^{\sum_{n} x_n z^{-n}} s(x, z) \in (I + H^+)^n e^{\sum_{n} x_n z^{-n}} W $$

and by the above formula 

$$ e^{\sum_{n} x_n z^{-n}} s(x, z) = \frac{\langle \Omega | \psi_0^* \psi_{\Theta} \rangle \Omega_W \rangle}{\langle \Omega | \Omega_W \rangle} $$

where $T_x$ is an operator on Fock space corresponding to the function $e^{\sum_{n} x_n z^{-n}}$. The denominator is the $T$ function and the numerator can be written as a $T$ function using the vertex operator formula for $\psi_{\Theta}$.
One thing I could try to understand is the relation between the Green's functions and the $S$-matrix. What could you mean by this? The Green's fn.

$$G(x, x') = \langle T [\psi(x) \psi^*(x')] \rangle$$

where

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx - i\lambda t} \psi_k = \frac{1}{\sqrt{2\pi}} \sum_k (e^{-i(x+it)} - k) \psi_k$$

is killed by $\frac{\partial}{\partial z} = \frac{i}{2} (\partial_t + \frac{1}{i} \partial_x)$. Similarly

$$\psi(x', x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} e^{it} \psi_k^* = \frac{1}{\sqrt{2\pi}} \sum_k (e^{-i(x-it)} - k) \psi_k$$

is killed by $\frac{\partial}{\partial \bar{z}}$. Thus the Green's function $G(z, z')$ with interaction will be analytic in both $z, z'$ outside the support of the interaction.

The idea is as follows. Fix $z'$ somehow. Then analytically continue $G(z, z')$ from the regions $t \gg 0$ (or $|z| \ll 1$) back to $|z| = 1$. Similarly continue from $|z| \gg 1$ to $|z| = 1$. Then the ratio will be a function on the circle which is the $S$ matrix.

Of course this doesn't work because as a function of $z$, $G$ either has an analytic continuation or it doesn't, and we know the singularities lie in the support of $f$, as is clear from the formula ($S' \rightarrow \mathbb{R}$)

$$G(z, z') = \frac{i}{2\pi} \int \frac{1}{z - \omega} \frac{e^{i\varphi(\omega)}}{\omega} d^2\omega$$

Nevertheless it should be possible to do the following: Consider the family of $S$ operators of the form $\partial_t + \frac{1}{i} \partial_x + \varphi(x, t)$ with $\varphi$ of compact support. Then it should be possible to map this family to the family of functions $g(x) \in S'$, and to cover this map by a map of line bundles. This is clear, and might form a local basis for calculation of determinants.
Interesting example: Suppose we have a fermion gas described by \( \hat{H}_0 = \sum_k \omega_k \psi^*_k \psi_k \) and we add to it a single fermion state described by the Hamiltonian \( \Omega \phi^* \phi \). Thus we get the unperturbed Hamiltonian

\[
\hat{H}_0 = \sum_k \omega_k \psi^*_k \psi_k + \Omega \phi^* \phi,
\]

to which we add an interaction

\[
\hat{H}_{\text{int}} = \left( \sum_k \frac{v_k}{\sqrt{\omega_k}} \right) \phi + \phi^* \left( \sum_k \frac{v_k'}{\sqrt{\omega_k}} \right) \psi.
\]

The total Hamiltonian is quadratic, so in principle one should be able to see what happens. The problem is that on the level of the operators \( \{ \psi_k, \phi \} \) one should have the possibility of exponential decay, yet somehow this doesn't exist in Fock space.

So let's first understand the Hamiltonian in the 1-particle space which has the basis \( 1k >, 1a > \), where \( \phi^* = \text{ext. mult} by 1a >, \phi = \text{int. mult} by <a1 \).

Question: There is a close connection between Dirac systems in the line \( i \partial \psi = \left( \frac{1}{\hbar} \partial_x - \frac{P}{\hbar^2} \right) \psi \) and wave equations \( \partial_t \phi = (\partial^2_x - \phi) \phi \) given by factorizing the latter. The natural way to quantize the former uses a fermion Fock space, and for the latter a boson Fock space. Are these two quantizations compatible with the correspondence given by factorization?