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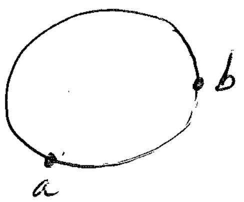
Riemann-Hilbert problem - Japanese viewpoint.

One ~~starts~~ with a representation of $\pi_1(X-S)$. Let's take $X = \mathbb{P}^1$ with coordinate z, and take S to lie on the unit circle. I do this because then I get a natural Hilbert space to work in. I'm trying to construct a vector bundle over $\mathbb{P}^1$. Let's reformulate as follows:

Starting with $X = \mathbb{P}^1$ and a representation of $\pi_1(X-S)$, I want to construct over X a vector bundle E which is trivial and hence has exactly r everywhere independent sections, where r is the rank. Then these r sections will be the solution of the Riemann-Hilbert problem. It's not yet clear where the connection comes from.

Let's consider the simplest example. Take 2 points on S^1



and consider the 1-diml representation which assigns to a counterclockwise circle around a the number $e^{2\pi i L}$ and around b the number $e^{-2\pi i L}$. The solution of the RH problem here is

$$Y(z) = (z-a)^L (z-b)^{-L} = \left(\frac{z-a}{z-b}\right)^L$$

where we normalize so that the arc from a to b is the branch cut and also the function $\to 1$ as $z \to \infty$. The differential equation is

$$\frac{dY}{dz} = \left(\frac{L}{z-a} - \frac{L}{z-b}\right) Y$$

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$$V = \mathbb{C}[z, z^{-1}] = \underbrace{z^{-1}\mathbb{C}[z^{-1}]}_{V_-} \oplus \underbrace{\mathbb{C}[z]}_{V_+}$$

Now equip V with the topology formed by the spaces $z^n\mathbb{C}[z]$ and form

$$\Lambda(V; V_+) \cong \Lambda V_- \otimes \Lambda(V_+)^\vee$$

Here the dual $(V_+)^\vee$ denotes the continuous dual. The natural basis for $\Lambda(V; V_+)$ is given by wedges:

$$z^{n_1} \wedge z^{n_2} \wedge \cdots$$

where $n_1 < n_2 < \cdots$ and $n_k = n_{k-1} + 1$ for large k.

On $\Lambda(V; V_+)$ we have the operators

$$a_n = \text{interior mult. by } \langle z^n|$$
$$a_n^* = \text{exterior mult. by } z^n,$$

satisfying the usual commutation relations. Any operator on V which carries V_+ into a closed subspace commensurable with V_+ extends to a endomorphism of $\Lambda(V; V_+)$ unique up to a scalar.

I want derivations of $\Lambda(V; V_+)$ associated to endos. of V. For example, multiplication by z^n on V has the matrix form

$$z^n = \sum_{k \in \mathbb{Z}} |z^{n+k}\rangle\langle z^k|$$

and should extend as a derivation to $\Lambda(V; V_+)$ by the formula:

$$\rho_n = \sum_k a_{n+k}^* a_k$$

But one must be careful how to interpret the infinite sum. If $n \neq 0$, then we can define the operator ρ_n as

$$\rho_n = \lim_N \sum_{|k| \leq N} a_{n+k}^* a_k$$

because for any element u of $\Lambda(V; V_+)$, only finitely many of the $a^*_{n+k} a_k u$ are $\neq 0$. For example, if $u = z^0 \wedge z^1 \wedge \cdots$ and $n \leq 0$, then

$$a^*_{n+k} a_k u = z^{n+k} \wedge z^0 \wedge \cdots \wedge \hat{z^k} \wedge \cdots \qquad k \geq 0,\ n+k < 0$$
$$= 0 \qquad \text{otherwise}$$

[redacted] On the other hand [redacted] $\sum a^*_k a_k$ does not make sense. So we will define

$$\rho_0 = \sum_{k<0}' a^*_k a_k - \sum_{k \geq 0} a_k a^*_k$$

Then ρ_0 has the value n on $\Lambda^n(V; V_+)$

<u>Question</u>: What conditions on $A: V \to V$ guarantee[redacted] it extends to a derivation of $\Lambda(V; V_+)$?

[redacted] Ask when $\sum A_{mn} a^*_m a_n$ when applied to any element of $\Lambda(V; V_+)$ is a finite sum. We certainly want for each n only finitely many $m < n$ with $A_{mn} \neq 0$, but it is not clear.

Need the commutation relations between the ρ_n. Clearly one has

$$[\rho_n, \rho_0] = 0$$

because ρ_n is an operator of degree 0. Next

$$[\rho_n, \rho_m] = \lim \sum_{|k| \leq N} \sum_{|l| \leq N}' [a^*_{n+k} a_k, a^*_{m+l} a_l]$$
$$a^*_{n+k} \delta_{k, m+l} a_l - a^*_{m+l} \delta_{n+k, l} a_k$$

$$= \lim \sum_{|k| \leq N} \sum_{|l| \leq N} a^*_{n+l} \delta_{l, m+k} a_k - a^*_{m+l} \delta_{n+k, l} a_k$$

$$= \lim \sum_{\substack{|k| \leq N \\ |m+k| \leq N}} [redacted]\, a^*_{n+m+k} a_k - \sum_{\substack{|k| \leq N \\ |n+k| \leq N}} a^*_{m+n+k} a_k$$

If $m+n \neq 0$, then $\sum_k a^*_{n+m+k} a_k$ makes sense as we have seen, so

$$[\rho_n, \rho_m] = 0 \quad , \quad n+m \neq 0$$

If $m = -n$, then $(n>0)$

$$[\rho_n, \rho_{-n}] = \lim \sum_{\substack{-N \le k \le N \\ -N+n \le k \le N+n}} a_k^* a_k - \sum_{\substack{-N \le k \le N \\ -N-n \le k \le N-n}} a_k^* a_k$$

$-N+n \le k \le N$ $\qquad$ $-N \le k \le N-n$

On an element u one has $a_k^* a_k u = 0$ for $k \ll 0$
$= u$ for $k \gg 0$

so $$[\rho_n, \rho_{-n}] = \lim_{N\to\infty} \sum_{N<k\le N+n}' a_k^* a_k - \sum_{-N-n \le k < -N}' a_k^* a_k$$

and we get $[\rho_n, \rho_{-n}] = n$. Thus

<u>Prop</u>: $$[\rho_n, \rho_m] \begin{cases} = 0 & n+m \neq 0 \\ = n & \text{if } n+m=0. \end{cases}$$

In addition we need the operator σ [redacted] $=$ automorphism of $\Lambda(V; V_+)$ corresponding to multiplication by z. Thus $\sigma(z^{n_1} \wedge z^{n_2} \wedge \cdots) = z^{n_1+1} \wedge z^{n_2+1} \wedge \cdots$

and we have
$$\sigma a_n \sigma^{-1} = a_{n+1}$$
$$\sigma a_n^* \sigma^{-1} = a_{n+1}^*$$

e.g. $$\sigma a_n \sigma^{-1}(z^0 \wedge z^1 \wedge z^2 \wedge \cdots) = \sigma a_n (z^{-1} \wedge z^0 \wedge z^1 \wedge \cdots)$$
$$= \sigma(z^{-1} \wedge \cdots \wedge \widehat{z^n} \wedge \cdots) = (z^0 \wedge z^1 \wedge \cdots \wedge \widehat{z^{n+1}} \wedge \cdots).$$

Thus
$$\sigma \rho_n \sigma^{-1} = \sum \sigma a_{n+k}^* a_k \sigma^{-1} = \sum a_{n+k+1}^* a_{k+1} = \rho_n$$

for $n \neq 0$, and
$$\sigma \rho_0 \sigma^{-1} = \sigma\Big(\sum_{k<0}' a_k^* a_k - \sum_{k\ge 0}' a_k a_k^*\Big)\sigma^{-1}$$
$$= \sum_{k\le 0}' a_k^* a_k - \sum_{k>0}' a_k a_k^*$$
$$= 1 + \rho_0$$

[redacted]

~~Remark: Given $A \in \mathrm{End}(V)$, and a subspace $W \subset V$, then~~

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so acting as n on [redacted] σ^{-n}. Hence we get a unique injective map sending 1 to $|0\rangle$ compatible with the operators:

$$(*) \qquad S[b_1^*, b_2^*, \dots\dots] \otimes \mathbb{C}[\sigma, \sigma^{-1}] \hookrightarrow \Lambda(V; V_+)$$

the injectivity coming from irreducibility.

Now the Jacobi triple product identity implies this map is an isomorphism. [redacted] Let's go over this. Introduce the Hamiltonian H on V with $Hz^n = -nz^n$. [redacted] Let's extend H to $\hat{H}$ on $\Lambda(V; V_+)$ so that $|0\rangle = z^0 \wedge z^1 \wedge \dots$ has eigenvalue 0. We have also the number operator $\hat{N} = \rho_0$ with value n on Λ^n. Then form the partition function

$$\mathrm{tr}\left(e^{-\beta(H - \mu \hat{N})}\right)$$ [redacted]

which is a generating function for $\Lambda(V; V_+) = \Lambda(V_-) \otimes \Lambda(V_+)^\vee$. Then we have $H = +n$ on $z^{-n} \in V_-$ so that

$$\text{gen. function of } \Lambda(V_-) = \prod_1^\infty \left(1 + e^{-\beta n} e^{\beta\mu}\right)$$

$$\text{gen. function of } \Lambda(V_+) = \prod_0^\infty \left(1 + e^{-\beta n} e^{-\beta\mu}\right)$$

Now on the side of $S[b_1^*, b_2^*, \dots] \otimes \mathbb{C}[\sigma, \sigma^{-1}]$, we know the operators $b_n^* = \rho_{-n}/\sqrt{n}$ are of exterior degree 0. Also ρ_{-n} corresponds to mult. by z^{-n} so it raises H [redacted] eigenvalues by n. Thus

$$\text{gen. fn. of } S[b_1^*, \dots] = \frac{1}{\prod_1^\infty (1 - e^{-\beta n})}$$

Finally we need the generating fn. of $\mathbb{C}[\sigma, \sigma^{-1}]$ which has the basis $\sigma^n \mapsto \sigma^n|0\rangle = z^n \wedge z^{n+1} \wedge \dots$. If $n > 0$ we have deleted the states [redacted] $z^0, z^1, \dots, z^{n-1}$ which have energies $0, -1, \dots, -n+1$ from $|0\rangle = z^0 \wedge z^1 \wedge \dots$ which has energy 0. Thus $H = 0 + 1 + \dots + n-1 = \frac{n(n-1)}{2}$ on $\sigma^n|0\rangle$. Similarly for $n \le 0$. Thus we get (since σ^n has $\hat{N} = -n$)

gen. function of $\mathbb{C}[\sigma,\sigma^{-1}] = \sum_{n\in\mathbb{Z}} e^{-\beta\frac{n(n-1)}{2}} e^{-\beta\mu n}$

Put $q = e^{-\beta}$, $e^{-\beta\mu} = u$ and then from

$$\frac{1}{\prod_{n\geq 1}(1-q^n)} \sum_{n=-\infty}^{\infty} q^{\frac{n(n-1)}{2}} u^n = \prod_{n\geq 0}(1+q^n u)\prod_{n\geq 1}(1+q^n u^{-1})$$

which is the Jacobi identity, one deduces that ⊛ is an isomorphism.

Now comes an interesting point. If the map ⊛ is an isomorphism, then it should be possible to express the basic operators a_n, a_n^* on $\Lambda(V; V_+)$ in terms of the basic operators ρ_n, σ. The vertex operators on $S[b_1^*, \dots] \otimes \mathbb{C}[\sigma,\sigma^{-1}]$ correspond to the field operators

$$\psi_\zeta = \sum \zeta^n a_n \qquad \psi_\zeta^* = \sum \zeta^{-n} a_n^* .$$

which are simpler to deal with that the a_n, a_n^*.

$$[\rho_n, \psi_\zeta] = \sum_{k,l} \zeta^k [a_{n+k}^* a_k, a_l] = -\sum \zeta^l \delta_{n+k,l}\, a_k$$

$$= -\sum \zeta^l a_{l-n} = -\zeta^n \psi_\zeta$$

Better $[\rho_n, a_l] = a_{l-n}$ $\qquad [\rho_n, a_l^*] = \sum [a_{n+k}^* a_k, a_l^*]$
$$= a_{l+n}^*$$

Thus $[b_n, \psi_\zeta] = \frac{-1}{\sqrt{n}} \zeta^n \psi_\zeta$ $\;n>0$ which means that ψ_ζ should contain the factor $e^{-\sum_1^\infty \frac{1}{\sqrt{n}}\zeta^n b_n^*} = e^{-\sum_1^\infty \frac{\zeta^n}{n}\rho_{-n}}$.

Similarly it should contain the factor) $\qquad [b_n^*, \psi_\zeta] = -\frac{1}{\sqrt{n}}\zeta^{-n}\psi_\zeta$

$$e^{\sum_1^\infty \frac{1}{\sqrt{n}}\zeta^{-n} b_n} = e^{\sum_1^\infty \frac{\zeta^{-n}}{n}\rho_n}$$

Next $[\rho_0, \psi_\zeta] = -\psi_\zeta$, so ψ_ζ contains the factor σ.

In effect $\sigma\rho_0\sigma^{-1} = 1 + \rho_0 \Rightarrow \rho_0\sigma - \sigma\rho_0 = -\sigma$. Finally

$$\sigma\psi_\zeta\sigma^{-1} = \zeta^{-1}\psi_\zeta \qquad\qquad \sigma a_n \sigma^{-1} = a_{n+1}$$

and hence ψ_ζ should contain the factor ζ^{ρ_0} as

In general given a subspace W of a fin. diml V, $\dim W = p$, then the tangent space to the line $\Lambda^p W$ in $\mathbb{P}(\Lambda^p V)$ is

$$\mathrm{Hom}(\Lambda^p W, \Lambda^p V/\Lambda^p W).$$

Now $\Lambda^p V/\Lambda^p W$ contains canonically $\Lambda^{p-1} W \otimes V/W$ and hence we get the subspace

$$(\Lambda^p W)^\vee \otimes \Lambda^{p-1} W \otimes V/W \;=\; W^\vee \otimes V/W = \mathrm{Hom}(W, V/W)$$

embedded in the tangent space to the line $\Lambda^p W$ in $\mathbb{P}(\Lambda^p V)$. This is obviously the image of the tangent space to W in the Grassmannian. So if $A \in \mathrm{End}(V)$, then the induced tangent vector to $\Lambda^p W$ should belong to the map from W to V/W induced by A. And so when we pass to infinite diml situation we will want to know that for every closed W commensurable with V_+ the map $A : W \to V/W$ lies in $W^\vee \otimes V/W$, i.e. is a transformation of finite rank. Next I want to know if this is independent of W. So let W' be of finite index in W. Then from

$$\begin{array}{ccl} W & \longrightarrow & V/W' \\ & \searrow^{A} & \downarrow \text{ finite diml kernel } W/W'. \\ & & V/W \end{array}$$

[redacted] we see that $A(W)$ in V/W fin. diml $\Longleftrightarrow$ in V/W' fin. diml, and so it's clear.

So now on $\Lambda(V; V_+)$ we have these operators ρ_n, σ. Also we have $|0\rangle =$ [redacted] $z^0 \wedge z^1 \wedge \cdots$ which is killed by the ρ_n for $n \geq 0$. Now the operators $b_n = \frac{1}{\sqrt{n}}\rho_n$ [redacted] $b_n^* = \frac{1}{\sqrt{n}}\rho_{-n}$ satisfy boson creation + annih. operator commutation relations. So from $|0\rangle$ these operators generate a symmetric algebra. Also the operators ρ_0, σ applied to $|0\rangle$ will generate an irreducible repn $\cong \mathbb{C}[\sigma, \sigma^{-1}]$ with

$$\sigma \zeta^{-\rho_0} \sigma^{-1} = \zeta^{-\sigma \rho_0 \sigma^{-1}} = \zeta^{-1-\rho_0} = \zeta^{-1} \zeta^{-\rho_0}$$

Thus the candidate for ψ_ζ is

$$\psi_\zeta = e^{-\sum_1^\infty \frac{\zeta^n}{n} \rho_{-n}} \, \zeta^{-\rho_0} \, \sigma \, e^{\sum_1^\infty \frac{\zeta^{-n}}{n} \rho_n} .$$

As a check let's compute the function on the circle belonging to this operator:

$$-\sum_1^\infty \frac{\zeta^n}{n} z^{-n} - \log \zeta + \log z + \sum_1^\infty \frac{\zeta^{-n} z^n}{n}$$

$$= \sum_{n \neq 0} \frac{\zeta^{-n} z^n}{n} + \log(z/\zeta)$$

It seems this is the series expansion of a Heaviside function which jumps $2\pi i$ as z passes thru ζ. Thus ψ_ζ is the operator belonging to a loop of degree 1 which starts at 1 then jumps around the circle as z passes thru ζ.

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$V = L^2(S^1)$, Λ = Fock space, $f : S^1 \longrightarrow \mathbb{C}^*$, $\hat{f}$ = [redacted] and operator on Λ belonging to f. Then we have

$$\hat{f}\, e(v)\, \hat{f}^{-1} = e(fv)$$
$$\hat{f}\, i(\lambda)\, \hat{f}^{-1} = i(\lambda f^{-1})$$

so if we take $v = \delta_\zeta = \sum z^n \langle [redacted]\, z^{+n} | \delta_\zeta \rangle = \sum (z/\zeta)^n$ whence

$$e(\delta_\zeta) = \psi_\zeta^* = \sum_{n\in\mathbb{Z}} \zeta^{-n} a_n^*$$
$$i(\langle \delta_\zeta |) = \psi_\zeta = \sum_{n\in\mathbb{Z}} \zeta^n a_n$$

we get

$$\boxed{\begin{aligned} \hat{f}\,\psi_\zeta\,\hat{f}^{-1} &= f(\zeta)^{-1}\,\psi_\zeta \\ \hat{f}\,\psi_\zeta^*\,\hat{f}^{-1} &= f(\zeta)\,\psi_\zeta^* \end{aligned}}$$

Let's compute

$$z^0 \wedge \cdots \wedge \widehat{z^n} \wedge \cdots \quad n \geq 0$$

$$\langle 0 | \psi_{z'}^* \psi_z | 0 \rangle = \sum (z')^{-m} z^n \langle 0 | \overbrace{a_m^* a_n}\, | 0 \rangle$$

$$= \sum_{n\geq 0} (z/z')^n = \frac{1}{1-(z/z')}$$

This is for $z \neq z'$ on S^1; as $z \to z'$ one defines it to be the distribution <u>such</u> that it is analytic for $|z| < 1$.

Now let's compute

$$\langle 0 | \hat{f}\, \psi_{z'}^* \psi_z | 0 \rangle$$

On one hand it is analytic inside S^1 as $\psi_z |0\rangle = \sum_{n\geq 0} z^n a_n |0\rangle$. However we have

$$\langle 0 | \hat{f}\, \psi_{z'}^* \psi_z | 0 \rangle = \langle 0 | \hat{f} | 0 \rangle\, \delta_{z'}(z) - \langle 0 | \hat{f}\, \psi_z \psi_{z'}^* | 0 \rangle$$

$$= \langle 0 | \hat{f} | 0 \rangle\, \delta_{z'}(z) - f(z)^{-1} \underbrace{\langle 0 | \psi_z \hat{f}\, \psi_{z'}^* | 0 \rangle}_{\substack{\text{analytic outside } S^1 \text{ as} \\ \langle 0 | \psi_z = \sum_{n<0} z^{-n} \langle 0 | a_n}}$$

If we multiply by $(1-z/z')$ to kill $\delta_{z'}(z)$ we get

$$\left(1-\frac{z}{z'}\right)\langle 0|\hat{f}\,\psi^*_{z'}\psi_z|0\rangle = -f(z)^{-1}\left(1-\frac{z}{z'}\right)\langle 0|\psi_z\hat{f}\,\psi^*_{z'}|0\rangle$$

which looks like a factorization of $f(z)$ in the form

$$f_+ = f^{-1}f_- \qquad \text{or} \qquad f f_+ = f_-$$

To see that $\left(1-\frac{z}{z'}\right)\langle 0|\hat{f}\,\psi^*_{z'}\psi_z|0\rangle$ is well-behaved on S', the only way I know is, to assume the factorization $f = f_-(f_+)^{-1}$. Then up to a scalar

$$\begin{aligned}\langle 0|\hat{f}\,\psi^*_{z'}\psi_z|0\rangle &= \underbrace{\langle 0|\hat{f}_-}\,\hat{f}_+^{-1}\psi^*_{z'}\psi_z|0\rangle \\ &= \overbrace{\langle 0|}\,\psi^*_{z'}\psi_z\underbrace{\hat{f}_+^{-1}|0\rangle}_{|0\rangle}\, f_+(z')^{-1}f_+(z)\end{aligned}$$

$$\therefore\ \boxed{\left(1-\frac{z}{z'}\right)\langle 0|\hat{f}\,\psi^*_{z'}\psi_z|0\rangle = f_+(z)/f_+(z')}$$

Let's normalize the scalars as follows: If $f = f_-f_+^{-1}$, then define $\hat{f} = \hat{f}_-\hat{f}_+^{-1}$ where $\hat{f}_+|0\rangle = |0\rangle$ and $\langle 0|\hat{f}_- = \langle 0|$. Then $\langle 0|\hat{f}|0\rangle = 1$. Thus for any choice of $\hat{f}$ one has

$$\boxed{\left(1-\frac{z}{z'}\right)\frac{\langle 0|\hat{f}\,\psi^*_{z'}\psi_z|0\rangle}{\langle 0|\hat{f}|0\rangle} = f_+(z)/f_+(z')}$$

where on the right is the f_+ part of the factorization normalized so as to be 1 at z'.

The field theory doesn't give more information. In order to use the above formula to construct f_+, one needs to know $\langle 0|\hat{f}|0\rangle \neq 0$, which is completely equivalent to fH_+ being complementary to $z^{-1}H_-$, and this is equivalent to there being the factorization.

Deformations: suppose one has an f [redacted] admitting a factorization, i.e. such that $V = z^{-1}H_- \oplus f H_+$ and one changes f infinitesmally to [redacted]

$$f + \delta f = (1 + (\delta f) f^{-1}) f.$$

In general given $V = W_- \oplus W_+$ and $A \in \mathrm{End}(V)$, which we think of as an infinitesimal motion, we [redacted] can look at the tangent vectors to W_+ and W_- in their respective Grassmannians. Thus we look at $A_{-+} : W_+ \to W_-$ and $A_{+-} : W_- \to W_+$ and we can form the intrinsic quantity

$$\mathrm{tr}\,(A_{+-} A_{-+})$$

which is a quadratic function of A. For example in the $V = z^{-1}H_- \oplus H_+$ situation and $g = \sum c_n z^n$, then the matrix of g is

$$\begin{array}{cc|cc}
 & & c_{-2} & \\
 & c_0 & c_{-1} & c_{-2} \\
\hline
c_2 & c_1 & c_0 & c_{-1} \\
 & c_2 & c_1 & c_0
\end{array}$$

and so you get [redacted] $c_1 c_{-1} + 2 c_2 c_{-2} + 3 c_3 c_{-3} + \cdots\cdots$

$$= \mathrm{res}\,(g_- \, dg_+).$$

Thus we have an intrinsic quadratic function on the deformation[redacted] space.

Thus if $g = \sum c_n z^n$ is a infinitesimal change of $f = 1$, then $\hat{g} = \sum c_n \rho_n$ and

$$\langle 0 | \hat{g}^2 | 0 \rangle = \sum c_m c_n \langle 0 | \rho_m \rho_n | 0 \rangle$$

$$= \sum_{m>0} c_m c_{-m} \langle 0 | \rho_m \rho_{-m} | 0 \rangle = \sum_{m>0} m c_m c_{-m}$$

Other possibilities are to use other powers of $\hat{g}$. One possibility is to define

$$(f+\delta f)^{\wedge} = (1+\hat{g})\hat{f}$$

where $g = \delta f f^{-1}$ is lifted to $\hat{g}$ by the explicit choice of ρ_n. If one does this along a path, one gets a candidate for $\hat{f}$ and $\langle 0|\hat{f}|0\rangle$. One has probably

$$\delta \log \langle 0|\hat{f}|0\rangle = \frac{\langle 0|\hat{g}\hat{f}|0\rangle}{\langle 0|\hat{f}|0\rangle}$$

which involves the quantities

$$\frac{\langle 0|\rho_n \hat{f}|0\rangle}{\langle 0|\hat{f}|0\rangle}$$

which can't be too far from the $\langle 0|\psi_z^* \psi_z \hat{f}|0\rangle$ involved in the factorization problem. It's clearly possible to compute all of this using the decomposition $V = z^{-1}H_- \oplus fH_+$

Here is the canonical symplectic structure: suppose $V = W_+ \oplus W_-$. Then à la Kostant–Kirilov one gets a skew-symmetric form on the orbit of $P_+ =$ projection on W_+

$$A, B \longmapsto tr([A,B]P_+) = tr(A[B,P_+])$$

$$\begin{matrix} BP_+ \\ -P_+B \end{matrix} = \begin{pmatrix} B_{++} & 0 \\ B_{-+} & 0 \end{pmatrix} - \begin{pmatrix} B_{++} & B_{+-} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B_{+-} \\ B_{-+} & 0 \end{pmatrix}$$

$$tr(A[B,P_+]) = tr(A_{+-}B_{-+} - A_{-+}B_{+-})$$

So now consider the KdV dictionary and try to see what these quantities are.

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Let $V = W_- \oplus W_+$ and let $A \in \mathrm{End}(V)$. In order to associate to A a derivation $\hat{A}$ of $\Lambda(V; W_+)$, it seems one must assume that $A: W_+ \longrightarrow V/W_+$ is of finite rank. $\hat{A}$ is determined up to a scalar which can be fixed by requiring $\langle 0|\hat{A}|0\rangle = 0$. Another way to see this is to use a basis v_n for V with $v_n \in W_+$ for $n \geq 0$ and $v_n \in W_-$ for $n<0$. Then if $A_{mn} = \langle v_m | A | v_n \rangle$, we want

$$\hat{A} = \sum a_m^* A_{mn} a_n .$$

However then $\langle 0|\hat{A}|0\rangle = \sum_{m\geq 0} A_{mm} = \mathrm{tr}(AP^+)$ which needn't be defined. The normalization $\langle 0|\hat{A}|0\rangle = 0$ amounts to defining

$$\hat{A} = \sum_{m\neq n}' a_m^* A_{mn} a_n + \sum_{m<0} A_{mm} a_m^* a_m - \sum_{m\geq 0} A_{mm} a_m a_m^*$$

Now let's compute

$$\langle 0|\hat{A}\hat{B}|0\rangle = \sum_{\substack{m,n\\ p,q}} A_{mn} B_{pq} \underbrace{\langle 0| a_m^* a_n a_p^* a_q |0\rangle}_{0 \text{ unless } q\geq 0,\ m\geq 0.}$$

Now $a_p^* a_q |0\rangle \neq 0$ implies $q \geq 0$ and either $p<0$ or $p=q$. Thus we get $\langle 0| a_m^* a_n a_p^* a_q|0\rangle \neq 0 \Longrightarrow$ $q\geq 0$, $m\geq 0$ and either $p=n<0$ or $p=q$ and $n=m$

so

$$\langle 0|\hat{A}\hat{B}|0\rangle = \sum_{\substack{m\geq 0\\ n<0}} A_{mn}B_{nm} + \Big(\sum_{m\geq 0} A_{mm}\Big)\Big(\sum_{m\geq 0} B_{mm}\Big)$$

First note that

$$\langle 0|[\hat{A},\hat{B}]|0\rangle = \sum_{\substack{m\geq 0\\ n<0}}' A_{mn}B_{nm} - B_{mn}A_{nm}$$

is a canonical skew-form, i.e. independent of the choice of the scalar in $\hat{A}$. It is at least in the finite-dimensional case the canonical 2-form

$$\mathrm{tr}(P_+[A,B]) = \mathrm{tr}([P_+,A]B)$$

giving the symplectic structure on the orbit of P_+.

So now the problem is to understand how this symplectic structure on the space of splittings $V = W_- \oplus W_+$ might be used to understand flows. It seems I want to take a flow of the form e^{tA} and actually a family of commuting flows $e^{tA+uB+\cdots}$. These things lie in the group of autos. of V, but we are interested in their form when restricted to the orbit of P_+. How does the symplectic structure help? The point perhaps is that it doesn't, if you have already integrated the flow. So what seems to be the case is that one has for each flow a function, and that the functions are constant on the joint orbits of the flows. Thus one has a set of action variables which specify the orbits, and each orbit has angle variables so it appears to be a torus at least locally.

January 20, 1982

Let's consider a Dirac system $\partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \tilde{p} & -ik \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ with $\tilde{p} = \bar{p}$ and $p(x)$ rapidly decreasing. To such a system we associate solutions

$$f_k(x) \sim e^{ikx} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \tilde{f}_k(x) \sim e^{-ikx} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $x \longrightarrow +\infty$ and

$$\phi_k(x) \sim e^{-ikx} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \tilde{\phi}_k(x) \sim e^{ikx} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

as $x \longrightarrow -\infty$. Then f_k, ϕ_k are analytic in UHP and $\tilde{f}, \tilde{\phi}$ in the LHP.

January 24, 1982

The problem: Take a r-diml. vector space V over the fn. field of a curve X and form the modified $\Lambda(V)$ in which one has lines attached to the various [redacted] vector bundles [redacted] embedded in V. Can there exist a [redacted] natural inner product on this $\Lambda(V)$?

Take $X = \mathbb{P}^1(\mathbb{C})$ and $V = \mathbb{C}(z)$. A line bundle embedded in V is simply a divisor on X. We consider $\Lambda(V)$ centered at the trivial line bundle L_0 belonging to the divisor 0. [redacted] Then we have

$$\Lambda(V) = \varinjlim \Lambda(L/L') \otimes \lambda(L_0/L')^*$$

as L, L' run over ~~line~~ bundles with $L \supset L_0 \supset L'$. Now to obtain an inner product on $\Lambda(V)$, the obvious thing to do is to have [redacted] inner products on L/L' which are mutually consistent.

[redacted] Let's consider the case where $L' = L_0$ and [redacted] $\dim(L/L_0) = 1$. Then L/L_0 is supported at a unique point of X, and there is a 1-1 correspondence between such L and points of X. Assuming ~~we~~ have [redacted] an inner product on $\Lambda(V)$, we then have a map

$$X \longrightarrow \mathbb{P}(W)$$

for some Hilbert space W. In fact W is just the subspace of V/L_0 spanned by the L/L_0 of dim. 1. Since $X = \mathbb{P}^1$ its clear there is a minimal canonical such [redacted] map $X \longrightarrow \mathbb{P}(W)$, namely using the line bundle $\mathcal{O}(1)$ and $W = H^0(X, \mathcal{O}(1))$. Further canonical maps of this type $X \longrightarrow \mathbb{P}(W)$ are provided by the line bundles $\mathcal{O}(n)$, but we can't get an infinite-dimensional W in this way.

The space $\prod' V \otimes_K \hat{K}_p$ is simply $V \otimes_K$ (adeles of K) $\mathfrak{a}$.

To simplify take $V = K$. The space $\prod_p E_p \otimes \hat{O}_p$ is the parallelotope $\Lambda(D)$ where D is the divisor corresponding to E embedded in K. So one has Weil's formula for the cohomology:

$$H^0(\mathcal{O}(D)) = \Lambda(D) \cap K \qquad H^1(\mathcal{O}(D)) = \mathfrak{a}/\Lambda(D) + K$$

Finally given a differential form $y\,dx \in \Omega_{K/k}$ one gets a k-linear form on $\mathfrak{a}$

$$(\xi_p) \longmapsto \sum_p \operatorname{res}(\xi_p\, y\,dx)$$

vanishing on $\Lambda(D) + K$ for some D, and this gives an isomorphism of the K-vector space $\Omega_{K/k}$ with the K-vector space of such k-linear forms.

Better

$$\mathfrak{a} = \varinjlim_D \varprojlim_{D'} \underbrace{\Gamma(\mathcal{O}(D)/\mathcal{O}(D'))}_{\Lambda(D)\cap K/\Lambda(D')\cap K}$$

has the top dual

$$\mathfrak{a}^* = \varprojlim_D \varinjlim_{D'} \underbrace{\Gamma(\mathcal{O}(D)/\mathcal{O}(D'))^*}_{\Gamma(\mathcal{O}(D')^\vee \otimes \omega / \mathcal{O}(D)^\vee \otimes \omega)}$$

which is the space of adeles in the K-vector space $\Omega_{K/k}$. So one should really be able to see a duality on the local level between $\hat{K}_p$ and $\Omega_{K/k} \otimes_K \hat{K}_p$. ~~given by an element~~ The candidate is

$$(\xi_p, \eta_p\, y\,dx) \longmapsto \operatorname{res}_p(\xi_p \eta_p\, y\,dx)$$

It's clear we just have to see that $f, g \longmapsto \operatorname{res} fg\,dz$ is a non-singular pairing for formal ~~power~~ Laurent series in z and this is obvious using the basis z^n.

<u>Duality</u>: Can one generalize the duality of ΛV and $\Lambda(V^*)$ in finite-dimensions. We have that $\Lambda(V)$ is essentially the limit of $\Lambda(\Gamma(L/L'))$. But for $L \gg 0$ and $L' \ll 0$ one has

$$0 \to \Gamma(L) \longrightarrow \Gamma(L/L') \longrightarrow H^1(L') \to 0$$

$$\text{dual to} \qquad\qquad\qquad\qquad \text{dual to}$$

$$0 \leftarrow H^1(L^\vee \otimes \omega) \longleftarrow \Gamma(L'^\vee \otimes \omega / L^\vee \otimes \omega) \longleftarrow \Gamma(L'^\vee \otimes \omega) \leftarrow 0$$

which suggests a duality between

$$\Gamma(L/L') \text{ and } \Gamma(L'^\vee \otimes \omega / L^\vee \otimes \omega).$$

So how can one prove this? Note that

$$L^\vee \otimes \omega = \mathcal{H}om_{\mathcal{O}_X}(L, \omega)$$

and so from $0 \to L' \to L \to L/L' \to 0$ one has

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(L, \omega) \to \mathcal{H}om_{\mathcal{O}_X}(L', \omega) \to \mathcal{E}xt^1_{\mathcal{O}_X}(L/L', \omega) \to 0$$

On the other hand

$$\operatorname{Ext}^1(L/L', \omega) = \Gamma(\mathcal{E}xt^1_{\mathcal{O}_X}(L/L', \omega))$$

because a **certain spectral sequence** degenerates. Finally one knows $\Gamma(L/L') \otimes \operatorname{Ext}^1(L/L', \omega) \longrightarrow H^1(\omega) = \mathbb{C}$ is a non-degenerate **pairing**. So what next?

It seems then that we will get a pairing between the exterior algebra for V and the exterior algebra for $\operatorname{Hom}_K(V, \Omega_{K/k})$. To make this clear we will need to understand Weil's proof of Riemann-Roch.

The idea is to compute the cohomology of a vector bundle E embedded in V using the complex

$$V \oplus \prod_P E \otimes_{\mathcal{O}} \hat{\mathcal{O}}_P \longrightarrow \prod_P{}' V \otimes_K \hat{K}_P .$$

In general for any f.d. vector space V over K we have

$$\mathcal{a}(V) \ , \quad \mathcal{a}(V^* \otimes_K \Omega'_{K/k})$$

are naturally dual over k, taking topologies into account. Now V and $V^* \otimes \Omega'_{K/k}$ are orthogonal. If one is given an isomorphism $V \cong V^* \otimes \Omega'_{K/k}$, then $\mathcal{a}(V)$ becomes <u>self-dual</u> and V is an isotropic subspace.

So what seems to be going on is that given a generator ω for $\Omega'_{K/k}$, then one gets a [redacted] duality on $\mathcal{a}(K) = \prod_P' \hat{K}_P$ by

$$(\xi_P) \ (\eta_P) \longmapsto \sum_P{}' \operatorname{Res}(\xi_P \eta_P \, \omega)$$

So if we look at a fixed point P then this pairing

$$\hat{K}_P \times \hat{K}_P \longrightarrow k$$
$$(\xi \, , \eta) \longmapsto \operatorname{res}_P (\xi \eta \omega)$$

is not the obvious pairing [redacted] $\operatorname{res}(\xi \eta \frac{dz}{z})$, but is shifted according to the order of ω at P.

Another feature of this pairing is that, except for this shift due to the canonical divisor, different points P, Q are orthogonal. Thus the pairing is diagonal in P. So it not like the pairing $\frac{1}{2\pi i}\int_{S^1} f g \frac{dz}{z}$ of functions on S^1. ?

January 27, 1982 [redacted]

Fredholm theory: One has an equation of the form
$$(I-\lambda K)f = g$$
The usual Neumann series
$$\frac{1}{I-\lambda K} = I + \lambda K + \lambda^2 K^2 + \cdots$$
converges only for small λ. Fredholm's idea is to use Cramer's formula
$$\frac{1}{I-\lambda K} = \frac{\mathrm{Cof}(1-\lambda K)}{\det(1-\lambda K)}$$
where the numerator & denominator turn out to be entire functions of K.

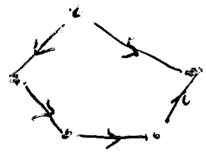
Diagram approach: One has Grassmannian integration fmla:
$$\frac{\int e^{-\psi^* A \psi}\, \psi_y \psi_x^*}{\int e^{-\psi^* A\psi}} = \frac{(\mathrm{Cof}\,A)_{yx}}{\det(A)} = (A^{-1})_{yx}$$

Put in $A = 1-\lambda K$. Then in
$$\int e^{-\psi^*\psi + \psi^*\lambda K\psi}$$
you expand out the second exponential, getting a sum of terms indexed by diagrams with

edges	$x \leftarrow y$	contribution	δ_{xy}
vertex	$\leftarrow x\ y \leftarrow$	"	λK_{xy}
for each loop		"	-1

Thus
$$\int e^{-\psi^*\psi+\psi^*\lambda K\psi} = e^{\text{conn. diag.}}$$
$$= e^{-\sum_1^\infty \frac{1}{n}\operatorname{tr}(\lambda K)^n} = e^{\operatorname{tr}\log(1-\lambda K)}$$
$$= \det(1-\lambda K)$$

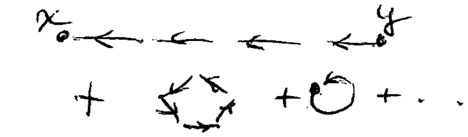
because the connected diagrams are loops



having symmetry factor $\frac{1}{n}$.

For the numerator we get

$$\int e^{-\psi^*\psi + \psi^* \lambda K \psi} \psi_x \psi_y^* = $$ sum over diagrams with in vertex y and out vertex x



From the diagrams results that if

$$\mathrm{Cof}(1-\lambda K) = \sum \lambda^n C_n \qquad \det(1-\lambda K) = \sum \lambda^n D_n$$

then we get the recursion formulas

$$\begin{cases} nD_n = -\mathrm{tr}(KC_{n-1}) \\ C_n = KC_{n-1} + D_n \cdot \mathrm{Id} \end{cases}$$ (break a family of loops at one vertex)

Next we want to [redacted] generalize the formula

$$\det_V(1+A) = \mathrm{tr}(\hat{A} \text{ on } \Lambda V)$$

where $\hat{A}$ denotes the operator $\Lambda(A)$ [redacted] induced by A on ΛV [redacted] which is a ring homomorphism.

So the first question is when does an operator $A: V \to V$ induce an operator $\hat{A}$ on $\Lambda(V; L_0)$. If A is an isomorphism and $A(L_0)$ is commensurable with L_0, then we know $\hat{A}$ exists up to a multiplicative scalar. The reason is that A induces

$$\Lambda(V; L_0) \longrightarrow \Lambda(V; AL_0) = \Lambda(V; L_0) \otimes \lambda(AL_0/L_0)$$

and so one gets $\hat{A}$ by choosing a generator for $\lambda(AL_0/L_0)$.

One can replace [redacted] assuming A is an isomorphism with A injective, and then by shrinking L_0 one sees that it suffices to assume:

$$* \quad \begin{cases} \operatorname{Ker} A \cap L_0 & \text{is finite-dimensional} \\ A(L_0) & \text{is commensurable with } L_0 \end{cases}$$

January 28, 1982.

Let's review $\operatorname{tr}_{\Lambda(V)}(\hat{A}) = \det_V(1+A)$. I want to generalize this $\Lambda(V;L_0)$ and I have already convinced myself that in order to lift A to an operator $\hat{A}$ on $\Lambda(V;L_0)$ I should assume $*$ above. There is uncertainty in the scalar connected with $\hat{A}$ which will somehow affect the $\det(1+A)$.

Consider the basic example. Suppose $A = e^{-\beta H}$ and L_0 is such that $A(L_0) = L_0$, say $L_0 =$ part of V where $H \leq \mu$. In fact suppose ~~has~~ eigenvalues λ_n i.e. discrete spectrum. If I naively compute I get:

$$\det(1+A) = \prod (1+e^{-\beta\lambda_n})$$

and this won't converge if $\{\lambda_n\}$ isn't bounded below. To compensate I formally have to ~~[illegible]~~ multiply $\det(1+A)$ by $\prod_{\lambda_n \leq \mu} e^{\beta\lambda_n}$ and then you get

$$\prod_{\lambda_n \leq \mu} e^{\beta\lambda_n} \det(1+A) = \prod_{\lambda_n \leq \mu} (e^{\beta\lambda_n}+1) \prod_{\lambda_n > \mu} (1+e^{-\beta\lambda_n})$$

which in good cases converges. The first term is

$$\frac{\det_V(1+A)}{\det_{L_0}(A)}$$

where the bottom term $\det_{L_0}(A)$ is the normalization constant required to define $\hat{A}$.

There is something missing, possibly, you want to work with $\det(1+zA)$. Let's simplify by supposing that H has the eigenvalues ε_n arranged in order for $n \in \mathbb{Z}$, and $A = e^{-\beta H}$. Then we break up $V = W_- \oplus W_+$ and form

$$\Lambda(V; W_+) = \Lambda W_- \otimes \Lambda(W_+)^*$$

and define $\hat{A}$ on this somehow. H has eigenvalues $\varepsilon_n \leq \mu$ on W_+ and $\varepsilon_n > \mu$ on W_-. To define $\hat{A}$ we take A_- on W_- and $((A_+)^t)^{-1}$ on $(W_+)^*$. Then

$$\operatorname{tr} \hat{A} = \prod_{\varepsilon_n > \mu} (1 + e^{-\beta\varepsilon_n}) \prod_{\varepsilon_n \leq \mu} (1 + e^{\beta\varepsilon_n})$$

If instead of A you want zA, then

$$\operatorname{tr} \widehat{zA} = \prod_{\varepsilon_n > \mu} (1 + z e^{-\beta\varepsilon_n}) \prod_{\varepsilon_n \leq \mu} (1 + z^{-1} e^{\beta\varepsilon_n})$$

So to define the partition function

$$\operatorname{tr}\left(e^{-\beta(\hat{H} - \mu\hat{N})}\right)$$

we need to specify $\hat{H}, \hat{N}$, I mean, we need to choose the additive constants $\langle 0|\hat{H}|0\rangle$ and $\langle 0|\hat{N}|0\rangle$ giving the energy and number of particles in the ground state.

In the physics one is not only interested in the partition function, but also the Green's functions, e.g.

$$G(t,t')_{xy} = \langle 0| T[\psi_x(t)\, \psi_y(t')^*]|0\rangle$$

which is the Fourier transform of $\left(\frac{1}{\omega - H}\right)_{xy}$ approached from different sides of the real axis for $\omega > \mu$ and $< \mu$.

Fredholm formula:

$$\circledast \qquad \left(\frac{1}{1+K}\right)_{xy} = \frac{tr\,(\hat{K}\psi_x\psi_y^*)}{tr\,(\hat{K})}$$

Here's a Schwinger type proof: Let K undergo an infinitesimal variation $K \mapsto K + \delta K = (1+\delta K.K^{-1})K$. Then $\hat{K} \mapsto (1+\delta K.K^{-1})^\wedge . K^\wedge$ and we know that

$$(1+\delta K.K^{-1})^\wedge = 1 + \sum \psi_x^* A_{xy}\psi_y \qquad A = \delta K.K^{-1}$$

Thus

$$\delta\, tr(\hat{K}) = tr\Big(\sum_{xy} A_{xy}\psi_x^*\psi_y\hat{K}\Big)$$

$$= \sum_{xy} A_{xy}\, tr(\hat{K}\psi_x^*\psi_y)$$

and

$$\delta \log tr(\hat{K}) = \sum_{xy} A_{xy}\frac{tr\,(\hat{K}\psi_x^*\psi_y)}{tr(\hat{K})}$$

$$\|$$

$$\delta_1 \log\det(1+K) = tr\left(\frac{1}{1+K}\delta K\right)$$

$$= tr\left(\frac{1}{1+K}AK\right)$$

$$= tr\left(A\frac{K}{1+K}\right) = tr(A) - tr\left(A\frac{1}{1+K}\right)$$

But

$$\frac{tr\,(\hat{K}\psi_x^*\psi_y)}{tr\,\hat{K}} = \delta_{xy} - \frac{tr\,(\hat{K}\psi_y\psi_x^*)}{tr(\hat{K})}$$

so we conclude that

$$\sum A_{xy}\frac{tr\,(\hat{K}\psi_y\psi_x^*)}{tr(\hat{K})} = tr\left(A\frac{1}{1+K}\right)$$

for all operators, which proves that the difference of the two sides of $\circledast$ is constant. On the other hand the two sides agree for $K=0$. Wait! If $K=1$, then $\hat{K} = id$. If $x=y$, then $\psi_x\psi_x^* = 0$ on a wedge

containing x and $= 1$ on a wedge not containing x.
Thus
$$\frac{tr(\psi_x \psi_x^*)}{tr(1)} = \frac{2^{n-1}}{2^n} = \frac{1}{2} = \left(\frac{1}{1+1}\right)_{xx}$$

So it works: $K = 0 \Rightarrow \hat{K} = 1$ on Λ^0 and 0 elsewhere on $\Lambda(V)$.

Now I still don't understand the role of time. Something else that is funny is the use of the trace instead of the vacuum expectation value. But one can get the vacuum expectation value by letting $\beta \longrightarrow \infty$. If $K = ze^{-\beta H}$, then
$$\frac{1}{1+ze^{-\beta H}} \longrightarrow \text{projection on space where } H > 0$$
assuming 0 is not an eigenvalue. What about $\hat{K}$? As $\beta \longrightarrow \infty$ only the line $|0\rangle$ in ΛV corresponding to the subspace of negative eigenvalue eigenvectors counts. Thus
$$\frac{tr(\hat{K} \psi_x \psi_y^*)}{tr(\hat{K})} \longrightarrow \langle 0 | \psi_x \psi_y^* | 0 \rangle$$
which checks.

January 29, 1982

$$\partial_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ik & p \\ \bar{p} & -ik \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$f \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \qquad \tilde{f} \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \qquad x \to +\infty$$

$$\phi \sim \begin{pmatrix} 0 \\ e^{-ikx} \end{pmatrix} \qquad \tilde{\phi} \sim \begin{pmatrix} e^{ikx} \\ 0 \end{pmatrix} \qquad x \to -\infty$$

$$\phi = A\tilde{f} + Bf \qquad \tilde{A} = \overline{A(\bar{k})} \qquad \tilde{B} = \overline{B(\bar{k})}$$

$$\tilde{\phi} = \tilde{B}\tilde{f} + \tilde{A}f$$

If $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ write $u = \begin{pmatrix} e^{ikx} & \\ & e^{-ikx} \end{pmatrix} \underset{\sim}{u}$, so that

$$\partial_x \underset{\sim}{u} = \begin{pmatrix} 0 & pe^{-2ikx} \\ \bar{p}e^{2ikx} & 0 \end{pmatrix} \underset{\sim}{u}$$

From

$$\begin{pmatrix} \tilde{\phi} & \phi \end{pmatrix} = \begin{pmatrix} f & \tilde{f} \end{pmatrix} \begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix}$$

we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underset{-\infty \leftarrow x}{\sim} \begin{pmatrix} \underset{\sim}{\tilde{\phi}} & \underset{\sim}{\phi} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{f} & \underset{\sim}{\tilde{f}} \end{pmatrix} \begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix} \underset{x \to \infty}{\sim} \begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix}$$

whence $\begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix}$ is the transfer matrix

$$= T\left\{ e^{\int_{-\infty}^{\infty} \begin{pmatrix} 0 & pe^{-2ikx} \\ \bar{p}e^{2ikx} & 0 \end{pmatrix} dx} \right\}$$

in the interaction representation. In particular one has

$$\begin{pmatrix} \tilde{A} & B \\ \tilde{B} & A \end{pmatrix} = I + \int_{-\infty}^{\infty} \begin{pmatrix} 0 & pe^{-2ikx} \\ \bar{p}e^{2ikx} & 0 \end{pmatrix} dx$$

$$+ \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \begin{pmatrix} 0 & pe^{-2ikx_1} \\ \bar{p}e^{2ikx_1} & 0 \end{pmatrix} \begin{pmatrix} 0 & pe^{-2ikx_2} \\ \bar{p}e^{2ikx_2} & 0 \end{pmatrix} + \cdots$$

so

$$B = \int_{-\infty}^{\infty} pe^{-2ikx} dx + O(p^3)$$

$$A = I + \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \, \bar{p}(x_1) p(x_2) e^{2ik(x_1 - x_2)} + O(p^4)$$

and the basic pattern of the higher coefficients is clear. Now notice that $(\tilde{\phi}(x)\,\phi(x))$ is the solution starting at $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ at $-\infty$ and hence

$$\left(\tilde{\phi}(x)\ \ \phi(x)\right) = I + \int_{-\infty}^{x} dx_1 \begin{pmatrix}0 & p e^{-2ikx_1}\\ \bar{p} e^{2ikx_1} & 0\end{pmatrix} + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2 \begin{pmatrix}p & \\ & 0\end{pmatrix}\begin{pmatrix}0 & \\ & 0\end{pmatrix} + \cdots$$

just like for the transfer matrix except we stop at x.

Thus

$$\phi(x) = \begin{pmatrix} \int_{-\infty}^{x} p(x_1)e^{-2ikx_1}dx_1 + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2 \int_{-\infty}^{x_2} dx_3\; p(x_1)e^{-2ikx_1}\,\bar{p}(x_2)e^{2ikx_2}\,p(x_3)e^{-2ikx_3} \\ 1 + \int_{-\infty}^{x} dx_1 \int_{-\infty}^{x_1} dx_2\; \bar{p}(x_1)e^{2ik(x_1-x_2)}p(x_2) + \cdots \end{pmatrix}$$

which shows that $\phi(x) \in \begin{pmatrix} e^{-2ikx}H_+ \\ 1+H_+ \end{pmatrix}$ $\quad \tilde{\phi}(x) \in \begin{pmatrix} 1+H_- \\ e^{2ikx}H_+ \end{pmatrix}$

Similarly

$$\left(f(x)\ \tilde{f}(x)\right) = \begin{pmatrix}1&0\\0&1\end{pmatrix} - \int_{x}^{\infty} dx_1 \begin{pmatrix}0 & p(x_1)e^{-2ikx_1}\\ \bar{p}(x_1)e^{2ikx_1} & 0\end{pmatrix} + \cdots$$

$$\Rightarrow \quad f(x) \in \begin{pmatrix} 1+H_+ \\ e^{2ikx}H_+ \end{pmatrix} \qquad \tilde{f}(x) \in \begin{pmatrix} e^{-2ikx}H_- \\ 1+H_- \end{pmatrix}$$

Hence

$$\phi(x) \in e^{-ikx}\begin{pmatrix} H_+ \\ 1+H_+ \end{pmatrix} \qquad f(x) \in e^{ikx}\begin{pmatrix} 1+H_+ \\ H_+ \end{pmatrix}$$

$$\tilde{\phi}(x) \in e^{ikx}\begin{pmatrix} 1+H_- \\ H_- \end{pmatrix} \qquad \tilde{f}(x) \in e^{-ikx}\begin{pmatrix} H_- \\ 1+H_- \end{pmatrix}$$

But now recall the basic scattering relation

$$\left(\tilde{\phi}\ \tilde{f}\right) = \left(f\ \phi\right)\begin{pmatrix} \frac{1}{A} & -\frac{B}{A} \\ \frac{\tilde{B}}{A} & \frac{1}{A} \end{pmatrix}$$

$$\Longrightarrow \underbrace{\left(e^{-ikx}\tilde{\phi}(x)\ \ e^{ikx}\tilde{f}(x)\right)}_{\in I + H_-} = \underbrace{\left(e^{-ikx}f(x)\ \ e^{ikx}\phi(x)\right)}_{I + H_+}\begin{pmatrix} \frac{1}{A} & -\frac{B}{A}e^{+2ikx} \\ \frac{\tilde{B}}{A}e^{-2ikx} & \frac{1}{A} \end{pmatrix}$$

Now what am I after? The above formulas show that by factoring the S-matrix

$$S_x = \begin{pmatrix} \frac{1}{A} & -\frac{B}{A}e^{2ikx} \\ \frac{\tilde{B}}{A}e^{-2ikx} & \frac{1}{A} \end{pmatrix}$$

we obtain the functions $f(x)$ $\phi(x)$ $\tilde{f}(x)$ $\tilde{\phi}(x)$, whose asymptotic behavior in k determines p. Now I want to go over the symplectic structure connected with the factorization problem, and translate into that for the function p.

<u>January 30, 1982</u>

Question: Can you make ΛV into a functor on the Q-category of f.d. vector spaces V? The idea is that if $V \twoheadrightarrow V/W$ is a surjective map, then we have a [redacted] canonical map

$$\text{①} \qquad \Lambda(V/W) \otimes \lambda(W) \hookrightarrow \Lambda(V),$$

[redacted] which enjoys the following transitivity property, namely given $W' \subset W$, then the composition

$$\text{②} \qquad \Lambda(V/W) \otimes \lambda(W/W') \otimes \lambda(W') \longrightarrow \Lambda(V/W') \otimes \lambda(W') \longrightarrow \Lambda(V)$$

belonging to the surjections $V/W' \longrightarrow V/W$ and $V \longrightarrow V/W'$ coincides with the [redacted] map ① via the canonical isom.

$$\lambda(W/W') \otimes \lambda(W') \xrightarrow{\sim} \lambda(W).$$

So we can now define a kind of oriented Q-category with the same objects as Q, in which a [redacted] morphism is a Q-map

$$\begin{array}{ccc} V_0 & \hookrightarrow & V' \\ \downarrow & & \\ V & & \end{array}$$

together with a generator for $\lambda(\mathrm{Ker}(V_0 \twoheadrightarrow V))$. Then $\Lambda(V)$ becomes a functor from this oriented Q-category to the category of f.d. vector spaces and injections.

<u>Question</u>: What is the homotopy type of this oriented Q-category? And what topological K space is it analogous to?

[redacted] I know Q is analogous to classifying space of $M \times M$ acting on M, where M is the monoid $\coprod_{n \geq 0} BU_n$, hence

$$Q \sim M/_{M \times M} \sim BM.$$

Now the oriented Q-category, denote it $\tilde{Q}$, [redacted] should be analogous [redacted] to $M/M\times\tilde{M} \sim B\tilde{M}$ where $\tilde{M} = \coprod_{n\geq 0}^{\infty} BSU_n$.

It seems that we also [redacted] want to kill the $B\mathbb{Z}_1 = S^1$ in [redacted] $\tilde{Q}$ so that can talk about the degree in the exterior algebra. Let us therefore consider the category consisting of pairs (V, p) where V is a vector space and p is an integer [redacted] restricted by $0 \leq p \leq \dim V$. A map $(V,p) \longrightarrow (V',p')$ is a Q-map $V \twoheadleftarrow V_0 \hookrightarrow V'$ such that $p' = \dim(\operatorname{Ker} V_0 \twoheadrightarrow V) + p$ together with a generator [redacted] for $\lambda(\operatorname{Ker} V_0 \twoheadrightarrow V)$. Then clearly

$$(V,p) \longmapsto \Lambda^p(V)$$

is a functor on this category. I think it is reasonable to expect this category to [redacted] be analogous to $M\times N/M\times\tilde{M} \sim N/\tilde{M} \sim B(BSU)$.

So what is next? [redacted] Ind objects in the Q-category.

$$\begin{tikzcd}
V_{\infty 0} \arrow[r, hook] \arrow[d, two heads] & V_{\infty 1} \arrow[r, hook] & \cdots \arrow[r, hook] & V_{\infty\infty} \arrow[d, two heads] \\
V_{20} \arrow[d, two heads] \arrow[r, hook] & {} & & \\
V_{10} \arrow[r, hook] \arrow[d, two heads] & V_1 \arrow[r, hook] \arrow[d, two heads] & & V_{1\infty} \\
V_0 \arrow[r, hook] & V_{01} \arrow[r, hook] & V_{02} \arrow[r, hook] \cdots & V_{0\infty}
\end{tikzcd}$$

Put $V_{\infty j} = \varprojlim_i V_{ij}$ and $V_{i\infty} = \varinjlim_j V_{ij}$ and finally $V_{\infty\infty} = \varinjlim_j V_{\infty j} = \varprojlim_i V_{i\infty}$. Now $V_{\infty\infty}$ has a natural topology; being the ∞ limit of f.d. vector spaces it is linearly compact. Thus $V_{\infty\infty}$ is locally linearly compact. It seems that ind-objects in the Q-category can be identified with locally linearly compact v.s. In effect given such a L one gets an inductive system in Q by considering the layers of open linearly compact subspaces. Morphisms of Ind objects are the same as Q-morphisms (using closed subspaces).

Given a locally linearly compact V and an open linearly compact subspace W one has defined

$$\Lambda(V;W)$$

containing lines for every open linearly compact subspace. Changing W varies this by a 1-diml space.

A linearly-compact vector space is determined by its dual which is discrete. Hence there is only one linearly compact vector space up to isom. whose dual has infinite countable dimension. Similarly there is up to isomorphism only one locally linearly compact v.s. V such that if W is open lin. compact, then V/W and $W^\vee$ are of dimension $\aleph_0$.

[redacted] Let's consider duality. Given V locally linearly compact, we can write it $V = C \oplus W$ with C discrete and W linear compact. Then $V^* = W^* \oplus \underbrace{C^*}_{W^\perp}$ and

$$\Lambda(V;W) = \Lambda(C) \otimes \Lambda(W^*)$$

$$\Lambda(V^*;C^*) = \Lambda(W^*) \otimes \Lambda(C)$$

so $\Lambda(V;W) \simeq \Lambda(V^*;C^*)$. [redacted]

We have seen that to a continuous map $A: V \longrightarrow V$ preserving W_0 up to commensurability in the sense that $(\operatorname{Ker} A) \cap W_0$ is f.d. and AW_0 comm. with W_0, there is an endom. $\hat{A}: \Lambda(V;W_0) \circlearrowleft$ defined up to a multiplicative scalar. But above we have endos. of $\Lambda(V;W)$ associated to Q-maps $V \to V$. So what is the general class?

In finite dimensions any correspondence between

V and V' induces an operator from $\Lambda(V)$ to $\Lambda(V')$ unique up to a scalar. A correspondence is a subspace $Z \subset V \times V'$ and then we have the line

$$\lambda(Z) \subset \Lambda(V \times V') = \Lambda(V) \otimes \Lambda(V') = \lambda(V) \otimes \Lambda(V)^* \otimes \Lambda(V')$$
$$= \lambda(V) \otimes \mathrm{Hom}(\Lambda V, \Lambda V'),$$

and to get an actual map of ΛV to $\Lambda V'$ one must pick a generator for $\lambda(V)^* \otimes \lambda(Z)$, which agrees with what happens in the $\mathbf{Q}$-case
$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & V' . \\ V & \end{tikzcd}$$

Review from Dec. 1: Let $\dim V = n$. Then

$$\Lambda^n(V \oplus V') = \bigoplus_q \Lambda^{n-q} V \otimes \Lambda^q V'$$
$$\simeq \bigoplus_q \Lambda^q(V^*) \otimes \Lambda^q V' \subset \Lambda^{ev}(V^* \oplus V')$$

A map $A : V \to V'$ has a graph which gives a line in $\Lambda^n(V \oplus V')$, and the corresponding elt of $\Lambda^{ev}(V^* \oplus V')$ is e^A where $A \in V^* \otimes V' \subset \Lambda^2(V^* \oplus V')$.

Clearer is the following

$$\mathrm{Hom}(\Lambda V, \Lambda V') = (\Lambda V)^* \otimes \Lambda V'$$
$$\simeq \Lambda(V^*) \otimes \Lambda V' = \Lambda(V^* \oplus V')$$

The inverse of this isomorphism is called writing things in normal product form. Thus given $A \in V^* \otimes V' \subset \Lambda^2(V^* \oplus V')$ one has

$$:e^A: \in \mathrm{Hom}(\Lambda V, \Lambda V')$$

and this is just $\hat{A}$.

February 4, 1982

If X is a [redacted] curve, one has Bloch's regulator map

$$K_2(X) \longrightarrow H^1(X, \mathbb{C}^*)$$

and I have had difficulty getting at this via the Λ-construction. There are various ways of constructing this map.

Deligne cohomology: Recall this is defined so that [redacted] a Mayer-Vietoris sequence holds:

$$\begin{array}{ccccc} \to & H^\cdot(X, D(p)) & \longrightarrow & H^\cdot(X, F_p\Omega^\cdot) & \to \\ S\downarrow & \downarrow & & \downarrow & \downarrow S \\ \to & H^\cdot(X, \mathbb{Z}) & \longrightarrow & H^\cdot(X, \mathbb{C}) = H^\cdot(X, \Omega^\cdot) & \to \end{array}$$

$D(p)$ is the complex

$$\begin{array}{ccccccccccc} \cdots 0 \longrightarrow 0 \longrightarrow & \mathbb{Z} & \to & \Omega^0 & \to & \Omega^1 & \to \cdots \to & \Omega^{p-1} & \to 0 \longrightarrow \cdots \\ & \text{deg } 0 & & 1 & & 2 & & p & \end{array}$$

So that $D(1) \cong \mathcal{O}^*[1]$ and first Chern classes lie in

$$c_1 \in H^2(X, [redacted] D(1)) = H^1(X, \mathcal{O}^*) = \mathrm{Pic}(X)$$

as they should. The 2nd [redacted] Chern class in Deligne cohomology induces a mapping

$$\begin{array}{rl} K_2(X) \longrightarrow & H^{4-2}(X, D(2)) \\ & \parallel \\ & H^2(X, \mathbb{C}^*[1]) = H^1(X, \mathbb{C}^*) \end{array}$$

which is Bloch's regulator map. Here I have used that for a [redacted] Riemann surface one has

$$D(2): \quad \mathbb{Z} \to \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots\cdots$$

is quasi-equivalent to $\mathbb{Z} \to \mathbb{C} \quad \sim \quad \mathbb{C}^*[1]$.

[redacted] Another approach is based on the diagram

$$K_2(X) \longrightarrow K_2(F) \xrightarrow[\text{symbol}]{\text{tame}} \coprod_{x \in X} \mathbb{C}^* \longrightarrow K_1(X)$$
$$\hspace{6em} \|$$
$$0 \longrightarrow H^1(X, \mathbb{C}^*) \longrightarrow \varinjlim_{S} H^1(X-S, \mathbb{C}^*) \longrightarrow \coprod_{x \in X} \mathbb{C}^* \longrightarrow H^2(X, \mathbb{C}^*) \to 0$$

and the fact that $K_2(F)$ is described by symbols. Thus what you do is to associate to $f, g \in F^*$ an element of $H^1(X-S, \mathbb{C}^*)$, where S contains the zeros + poles of both f, g, such that $f, 1-f$ goes to zero, and such that it is compatible with the tame symbol. This is probably Deligne's approach.

Let's try to work it out. Suppose then that $f, g \in \Gamma(X, \mathcal{O}^*)$ and interpret them as elements of $H^1(X, D(1))$. Then take their cup product, and this lies in $H^2(X, D(2))$ which we have seen is $H^1(X, \mathbb{C}^*)$.

Hence, in particular, when X is an annulus, there is a canonical map which associates to invertible analytic functions f, g an element of $H^1(X, \mathbb{C}^*) = \mathbb{C}^*$. This is interesting because I thought there were two possible ways of doing it.

$D(1)$ fits in a homotopy-cartesian square:

$$\begin{tikzcd}
D(1) \arrow[r] \arrow[d] & F_1\Omega^\cdot \arrow[d] \\
\mathbb{Z}[0] \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \Omega^\cdot
\end{tikzcd}$$

Now $H^1(X, D(1)) = H^0(X, \mathcal{O}^*)$, so that given $f \in \mathcal{O}^*$ we get a section of degree one of $D(1)$ in the derived category. This should be given by $0 \in \mathbb{Z}[0]_1 = 0$, $\frac{1}{2\pi i} d\log f \in \Omega^1 = (F_1\Omega^\cdot)_1$ and the homotopy from 0 to $\frac{1}{2\pi i} d\log f$ given by $\frac{1}{2\pi i}\log f \in \Omega^0$. I am assuming $\log f$ exists globally.

Now the cup $D(1) \underset{\mathbb{Z}}{\otimes} D(1) \longrightarrow D(2)$ is needed.

$$\begin{tikzcd}
D(2) \arrow[r] \arrow[d] & F_2\Omega^\cdot = 0 \arrow[d] \\
\mathbb{Z}[0] \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \Omega^\cdot
\end{tikzcd}$$

so what we have is that the cup product will give us 0, $\frac{1}{2\pi i} d\log f \wedge \frac{1}{2\pi i} d\log g$, and the homotopy between these two given by

$$\frac{1}{2\pi i}\log f \cdot \frac{1}{2\pi i} d\log g \quad \in \Omega^1 = (\Omega^\circ),$$

Let's interpret this as an elt of $H^1(X, \mathbb{C}^*)$, using $D(2) \sim \mathbb{C}^*[1]$.
Quasi-isos:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] & \Omega^0 \arrow[r, "d"] \arrow[d, "e^{2\pi i}"'] & \Omega^1 \arrow[r] \arrow[d, equal] & \Omega^2 \arrow[d, equal] \\
 & \mathcal{O}^* \arrow[r, "\frac{1}{2\pi i} d\log"'] & \Omega^1 \arrow[r] & \Omega^2 \\
 & \mathbb{C}^* \arrow[u, hook] & &
\end{tikzcd}$$

So we have this one-form $\left(\frac{1}{2\pi i}\log f\right)\left(d\frac{1}{2\pi i}\log g\right)$ which is closed because $\dim X = 1$. We get a $\mathbb{C}^*$-torsor by considering all h holomorphic such that

$$\frac{1}{2\pi i} d\log h = \frac{1}{2\pi i}\log f \, \frac{1}{2\pi i} d\log g$$

Thus in some sense

$$h = e^{\frac{1}{2\pi i}\int \log f \, d\log g}.$$

February 5, 1982

X Riemann surface, f g invertible holomorphic fns. on X. I propose to define a $\mathbb{C}^*$-torsor over X associated to (f,g).

First we form the principal $\mathbb{Z}$-bundle $\tilde{X}$ over X given by taking $\frac{1}{2\pi i}\log f$:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\frac{1}{2\pi i}\log f"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "e^{2\pi i}"] \\
X \arrow[r, "f"] & \mathbb{C}^*
\end{tikzcd}$$

Then over $\tilde{X}$ we consider the $\mathbb{C}^*$-torsor $\tilde{P}$ whose local sections are inv. analytic fns. h such that

$$\frac{dh}{h} = \frac{1}{2\pi i}(\log f)\,\pi^*\!\left(\frac{dg}{g}\right)$$

Now we lift the $\mathbb{Z}$ action on $\tilde{X}$ to $\tilde{P}$. It suffice to do this locally over X in a canonical way, hence I must give an isomorphism of the $\mathbb{C}^*$-torsors over X obtained by pulling $\tilde{P}$ back by various choices for $\frac{1}{2\pi i}\log f$. So choose a principal branch $L_0(f)$. Then the other branches are $L_0(f)+n$ $n \in \mathbb{Z}$. And

$$h_n = e^{[redacted]\int (L_0(f)+n)\frac{dg}{g}} = h_0 g^n$$

or more precisely $h_n = h_0 g^n$ is an isom. between solns of

$$\frac{dh_n}{h_n} = L_n(f)\frac{dg}{g} \quad \text{and} \quad \frac{dh_0}{h_0} = L_0(f)\frac{dg}{g}$$

I think the rest is clear, I mean, the fact[redacted] that this is an action and so by descent gives us a $\mathbb{C}^*$-torsor over X. [redacted]

The remaining problems are to see that this torsor is trivial when [redacted] $g = 1-f$, and to see that it gives the tame symbol in the case of meromorphic functions on a punctured disk.

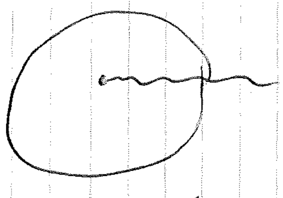
Bilinearity of this construction is clear. Furthermore in the punctured disk case, if f is of degree 0, i.e. extends as an invertible holomorphic function across $z=0$, then taking the loop $\circlearrowleft$ and deforming it down to 0 gives for monodromy

$$(f,g) = e^{\frac{1}{2\pi i}\int \log f \frac{dg}{g}} = e^{\log f(0)\deg(g)} = f(0)^{\deg g}.$$

Next consider (z,z). Put [redacted]

$$h(z) = e^{\frac{1}{2\pi i}\int_1^z \log t \, \frac{dt}{t}}$$

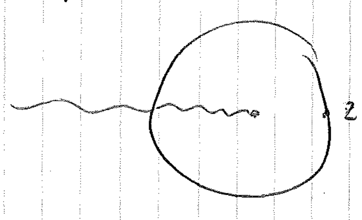
One can use this formula to compute parallel translation in our torsor over any region where $\log t$ is single-valued. If we cross a branch cut for $\log t$ we must multiply by the appropriate value of g. Thus if the cut is



then crossing the cut at $z=1$ doesn't change things. So compute

$$e^{\frac{1}{2\pi i}\oint_1^1 \log t \, \frac{dt}{t}} = e^{\frac{1}{2\pi i}\int_0^{2\pi} i\theta \, i\, d\theta} = e^{\frac{i}{2\pi}\frac{(2\pi)^2}{2}} = -1$$

If I put the cut



then the monodromy is

jump in $\frac{1}{2\pi i}\log t$

$$e^{\frac{1}{2\pi i}\oint_1^{-1} \log t \, \frac{dt}{t}} \cdot (-1)^{-1} \cdot e^{\frac{1}{2\pi i}\oint_{-1}^{1} \log t \, \frac{dt}{t}}$$

value of g

$$= e^{\frac{1}{2\pi i} i^2 \frac{\pi^2}{2}} (-1) \cdot e^{\frac{1}{2\pi i} i^2 \left[\frac{\theta^2}{2}\right]_{-\pi}^{0}} = -1$$

So we see that

$$(z, z) = -1$$

from which we get $(z^m, z^n) = (-1)^{mn}$ using bilinearity.

Assume g invertible holomorphic

$$(z^n, g) = \cancel{e^{\frac{1}{2\pi i}\int_0^{2\pi} i\theta \frac{g'(re^{i\theta})}{g(re^{i\theta})} i\,d\theta} \; g(0)}$$

$$= e^{\frac{1}{2\pi i}\int_0^{2\pi} i\theta \underbrace{\frac{dg}{g}}_{\substack{\downarrow \\ 0 \text{ as circle shrink}}}} \; g(0)^{-1} \quad \leftarrow \text{jump in } \log z$$

$$= g(0)^{-1}$$

So now it's clear we have the tame symbol.

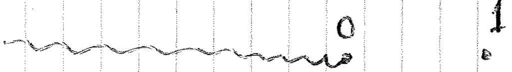
Why $(f, 1-f) = 0$:

$$h(z) = e^{\frac{1}{2\pi i}\int^z \log f(t) \frac{-df(t)}{1-f(t)}} = e^{\frac{1}{2\pi i}\int^{f(z)} \log x \frac{d(1-x)}{1-x}}$$

hence we have only to show that

$$h(z) = e^{\frac{1}{2\pi i}\int^z \log x \frac{d(1-x)}{1-x}}$$

is single-valued on $\mathbb{C} - \{0, 1\}$.



A small circle around 1 gives $e^{\frac{1}{2\pi i}\log 1} = 1$.
A small circle around 0 gives negligible integral since $r \log r \longrightarrow 0$ as $r \to 0$, so one only has to worry about crossing the cut, where one multiplies by $(1-x)^{-1}$ at $x=0$, which is 1.

February 5, 1982 (cont.)

Two ways to obtain $\mathrm{Res}(f\,dg)$. Let A be a d.v.r. with quotient field F, both algebras over residue field k. Then one chooses a projection $P : F \to A$. Given $f, g \in F$ one has the Tate formula

$$\mathrm{res}(f\,dg) = \mathrm{tr}_A([Pf, Pg])$$

See Oct 28, 1982 p. 179. The formula is easy to see when $\mathrm{Ker}\,P$ contains $\{z^n, n<0\}$, z a parameter. It is enough to look at z^n, z^m. The matrix $[Pf, Pg]$ has diagonal elements when $n+m=0$. Take $f = z^{-1}$, $g = z$. Then P_z is the injective shift, so $P_{z^{-1}}P_z = I$, $P_z P_{z^{-1}} = I$ except for killing z^0. Thus

$$\mathrm{tr}\,[P_{z^{-1}}, P_z] = +1 = \mathrm{res}(z^{-1}dz)$$

The other formula goes as follows: The projection P is equivalent to a splitting $F = C \oplus A$ and there is a symplectic structure on the tangent space to such splittings. So give f we associate to it the tangent vector to the splitting which is the pair of operators $Pf(1-P)$, $(1-P)fP$. Then the symplectic pairing is

$$\mathrm{tr}\Big(Pf(1-P)(1-P)gP - Pg(1-P)(1-P)fP\Big)$$

$$= \mathrm{tr}\Big(Pf(1-P)gP - Pg(1-P)fP\Big)$$

$$= \mathrm{tr}\Big(\cancel{PfgP} - PfPgP - \cancel{PgfP} + PgPfP\Big)$$

$$= \mathrm{tr}_A([Pg, Pf]).$$

So the two formulas are essentially the same. Notice that one has used the fact $[f,g]=0$, consequently the symplectic definition is more general.

February 21, 1982

Goal: To understand the Japanese τ function, and how the Baker-Akhieser fn. can be expressed as the quotient of two τ functions.

Put $V = L^2(S^1)$, let H_- be the closed subspace spanned by the z^n, $n \leq 0$ and let H_+ be spanned by z^n, $n \geq 0$. Let W be a closed subspace complementary to $z^{-1}H_-$ such that $z^2 W \subset W$. Then for small x there is a unique

$$\psi(x,z) \in e^{xz}(1 + z^{-1}H_-) \cap W$$

called the Baker-Akhieser function, and this is because for small x, $e^{xz}(z^{-1}H_-)$ remains complementary to W. Also I believe that for small x we get a potential function $q(x)$ such that

$$(\partial_x^2 - q)\psi(x,z) = z^2 \psi(x,z)$$

Now I want to get at this business by means of a factorization problem. So I will regard $L^2(S^1)$ as two copies of $L^2(S^1)$ in the variable z^2, and assume that $W = S(H_+)$ where S is a unitary operator on V which commutes with z^2, and hence I can identify S with a loop in U_2. One gets this loop by choosing an orthonormal basis for $W \ominus z^2 W$ and then writing it out in terms of the natural basis $1, z$ for $H_+ \ominus z^2 H_+$.

It's clear that $\psi(x,z)$ should be simply related to the factorization for $e^{-xz} S$

February 22, 1982

$V = L^2(S^1)$ W closed subspace complementary to $z^{-1}H_- = \text{span}\{z^n \mid n<0\}$, and also W not too far from H_+ so that it determines a unit vector u_W in the Fock space. I want to determine the unique element [redacted] in

[redacted] $\cdot\ (1+z^{-1}H_-) \cap W$

by [redacted] an explicit formula. Let ψ_ζ be the interior multiplication by the linear functional of evaluation [redacted] at ζ [redacted]. Then if ω_n is an orth. basis of W we have $u_W = \omega_1 \wedge \omega_2 \wedge \cdots\cdots$ and so

$$\psi_\zeta(u_W) = \sum (-1)^{n-1}\, \omega_n(\zeta)\ \omega_1 \wedge \cdots \wedge \hat{\omega}_n \wedge \cdots\cdots$$

and hence $\langle \alpha | \psi_\zeta$ [redacted] $u_W \rangle$

is an element of W for any α in the Fock space.

So now take $|\alpha\rangle$ to correspond to the subspace zH_+ which has the same dimension as $\psi_\zeta u_W$. Recall

$$\psi_\zeta = \sum \zeta^n a_n \qquad \langle \alpha | a_n = (a_n^* | \alpha\rangle)^* = 0 \quad n>0$$

hence

$$\langle \alpha | \psi_\zeta = \sum_{n\le 0} \zeta^n \langle \alpha | a_n$$

Thus we see that $\langle \alpha | \psi_\zeta | u_W \rangle$ is a power series in ζ^{-1}. So the unique element in $(1+z^{-1}H_-)\cap W$ is

$$\frac{\langle \alpha | \psi_\zeta | u_W \rangle}{\langle \alpha | a_0 | u_W \rangle}$$

which if we let $|0\rangle$ correspond to H_+ so that $|\alpha\rangle = a_0|0\rangle$ we get the formula

$$\frac{\langle 0 | a_0^* \psi_z | u_W \rangle}{\langle 0 | u_W \rangle} \in (1+z^{-1}H_-)\cap W$$

It will be soon be essential to understand the Θ function (or τ-fn) which gives rise to the Baker-Akhieser function. So let me consider an elliptic curve $X=\mathbb{C}/\Gamma$ where $\Gamma=\mathbb{Z}+\mathbb{Z}\tau$, $\operatorname{Im}\tau>0$, and call ∞ the image of $0\in\mathbb{C}$. I am interested in line bundles L over X with $h^0=h^1=0$, hence in line bundles of degree 0 which are not the trivial line bundle. Fix such a line bundle L_0. Draw a small S^1 around ∞ and describe L_0 in terms of the space V of L^2-sections of L_0 over S^1 and the subspaces of sections holomorphic inside and outside S^1 which are complementary.

The Baker-Akhieser function is a generator for the 1-diml space $\Gamma(X, L_0(\infty))$, so pick such a generator. Then this section vanishes at a unique pt P_0 of X $\neq\infty$, so that I can identify $L_0(\infty)$ with the line bundle $\mathcal{O}(P_0)$ and L_0 with the line bundle $\overbrace{\text{belonging to}}^{\mathcal{O}(P_0-\infty)}$ the divisor $P_0-\infty$. This means that the holomorphic sections of L_0 on $X-\infty$ are those meromorphic functions having at most a simple pole at P_0. This is the space W when closed up in $L^2(S^1)$. In addition, one has the space H_- of functions analytic inside S^1. Finally $W\cap H_-$ contains 1.

Now we take a uniformizer z at ∞, say the square root of the Weierstrass $\wp$-function. Recall

$$\wp(u)=\frac{1}{u^2}+\cdots \qquad u=\text{coor in }\mathbb{C}$$

so it has a square root $z=\sqrt{\wp}=\frac{1}{u}+\cdots$. I look for then an element of

$$W \sim e^{xz}(1+z^{-1}H_-).$$

In other words I want a holomorphic function on the elliptic curve outside of P_0 where it has a simple pole, and outside of ∞ where it has an essential singularity of type $e^{x/u}$.

Clearly this description is effectively useless, which is why one needs somehow the Θ-functions. Let's try integrating the Weierstrass $\wp$-function

$$f(P) = e^{-x\int_{P_0}^{P}\wp(u)\,du}$$

since $\wp(u) = \frac{1}{u^2} + O(1)$ near $u=0$ it follows that near $P=\infty$ or $u=0$ this function has the asymptotic behavior $e^{x/u}$ near $u=0$. And since $\wp$ has only double poles it gives us a single-valued function on the universal covering $\mathbb{C}$. The only problem is that $\int \wp(u)\,du$ might have periods. So how can one adjust?

I guess the idea is to use other differentials. For example I can take a doubly-periodic fn $g(u)$ with simple poles $\frac{1}{u-a} - \frac{1}{u-b}$ averaged over the period lattice.

$$e^{\int\left(\frac{1}{u-a} - \frac{1}{u-b}\right)du} = e^{\log\left(\frac{u-a}{u-b}\right)} = \frac{u-a}{u-b}$$

or simply take a Weierstrass σ function

February 23, 1982

To construct the Baker-Akhieser fn. for an elliptic curve. Suppose the curve is $X = \mathbb{C}/\Gamma$, $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$, $\operatorname{Im}\tau > 0$. Also $\mathbb{C}/\Gamma = \mathbb{C}^*/\{q^n\}$, where $q = e^{2\pi i \tau}$. Denote by u the natural coordinate on $\mathbb{C}$ and by $t = e^{2\pi i u}$ the natural coordinate on $\mathbb{C}^*$. ∞ = image of $u = 0$.

I start a line bundle L_0 on X with $h^0 = h^1 = 0$, hence of degree 0 and not the trivial bundle. $L_0(\infty)$ has a ~~unique se~~ one-diml space of sections, and if I choose a generator, then $L_0(\infty) = \mathcal{O}(P_0)$ for a unique pt P_0 of X not ∞. Sections of $\mathcal{O}(P_0)$ on $X - \{\infty\}$ are meromorphic functions with at most a simple pole at P_0. Now choose a uniformizing parameter around ∞, call it z^{-1}, and then for x small the Baker-Akhieser function will be analytic on X except at P_0, where it has a simple pole, and at ∞ where it has an essential singularity of type $e^{xz}(1 + \frac{a_1}{z} + \cdots)$.

Notice that this is independent of the choice of z in the following sense. If $z' = z + b_0 + b_1/z + b_2/z^2 + \cdots$ has same first order behavior near ∞, then

$$e^{xz'} = e^{xz}\Big(\underbrace{e^{b_0 + b_1/z + \cdots}}_{e^{b_0}(1 + O(\frac{1}{z}))}\Big)$$

Now in general as x varies the Baker-Akhieser function will have a 0, so that the alg. line bundle of which the BA fn. is the only section up to a scalar is going to be $\mathcal{O}(Q)$ where Q is this zero. When $x = 0$, the zero will have to be reinterpreted as the absense of the pole at P_0. Thus we map x to a line bundle of degree 1.

Let's now construct the BA function using the Θ-fn. for the elliptic curve which is Jacobi's function:

$$\theta(t) = \sum' g^{\frac{n(n-1)}{2}}(-t)^n = \prod_{n\geq 0}(1-g^n t)(1-g^n)\prod_{n\geq 1}(1-g^n t^{-1})$$

and which satisfies the identity

$$\theta(gt) = \sum' g^{\frac{(n+1)n}{2}}(-t)^n = \sum g^{\frac{n(n-1)}{2}}(-t)^{n-1}$$

$$= (-t)^{-1}\theta(t)$$

or

$$\boxed{\theta(t) = -t\,\theta(gt)}$$

The function $\theta(t)$ vanishes at $t = g^n$ $n \in \mathbb{Z}$, and has simple zeroes. ~~Hence~~

For various reasons it seems better to use the variable u, instead of t: $e^{2\pi i u} = t$.
Then $\theta(u)$ has simple zeroes at the point of the lattice Γ. Hence

$$\frac{\theta'(u)}{\theta(u)} = \frac{d}{du}\log\theta$$

has simple poles at each lattice point. So

$$e^{x\frac{\theta'(u)}{\theta(u)}}$$

except for the fact that it is not doubly-periodic, looks like the desired BA fn. So let's figure how to correct the non-periodicity.

$$\theta(u+\tau) = \theta\Big(\underbrace{e^{2\pi i\tau}}_{g}\underbrace{e^{2\pi i u}}_{t}\Big) = -\frac{1}{e^{2\pi i u}}\,\theta(u)$$

$$\theta(u+\tau) = -e^{-2\pi i u}\,\theta(u)$$

$$\frac{\theta'(u+\tau)}{\theta(u+\tau)} = \frac{d}{du}\log\theta(u+\tau) = \frac{d}{du}\left(-2\pi i u + \log\theta(u)\right)$$

$$= -2\pi i + \frac{\theta'(u)}{\theta(u)}$$

$$e^{x\frac{\theta'(u+\tau)}{\theta(u+\tau)}} = e^{-2\pi i x}\, e^{x\frac{\theta'(u)}{\theta(u)}}$$

Now $\theta(u \mp y)$ satisfies $\theta(u+\tau \mp y) = -e^{-2\pi i(u\mp y)}\theta(u+y)$

and so $\dfrac{\theta(u\mp y_1)}{\theta(u\mp y_2)}$ satisfies $\dfrac{\theta(u+\tau\mp y_1)}{\theta(u+\tau\mp y_2)} = e^{+2\pi i(y_1-y_2)}\dfrac{\theta(u+y_1)}{\theta(u+y_2)}$

Hence

$$e^{x\frac{\theta'(u)}{\theta(u)}}\ \frac{\theta(u\mp y_1)}{\theta(u\mp y_2)}$$

will be doubly-periodic provided $-x+y_1 \mp y_2 = 0$ or $x = y_1 - y_2$. For the Baker-Akhieser function we want a simple pole at a fixed point P_0, so y_2 is fixed and $y_1 = y_2 + x$. Hence one sees immediately that x is acting like translation on the Jacobian. So the Baker-Akhieser function is simply

$$\cancel{e^{x\frac{\theta'(u)}{\theta(u)}}\ \frac{\theta(u-x-y)}{\theta(u-y)}} \qquad e^{x\frac{\theta'(u)}{\theta(u)}}\ \frac{\theta(u-x-y)}{\theta(u-y)}$$

$$\cancel{e^{x\left(\frac{\theta'(u)}{\theta(u)}-\frac{\theta'(y)}{\theta(y)}\right)}\ \frac{\theta(u-x-y)}{\theta(u-y)}\ \frac{\theta(-y)}{\theta(-x-y)}}$$

where $y \in \mathbb{C}$ maps down to the point P_0 on X, not ∞.

Now the next project will be to compute the potential belonging to this BA function. Put

$$S_x = e^{x\frac{\theta'(u)}{\theta(u)}}\ \frac{\theta(u-x-y)}{\theta(u-y)}$$

It should be the case that $\left[\partial_x^2 - q(x)\right] s_x = z^2 s_x$ where $z^2 = \wp(u)$. Now

$$\frac{\partial^2 s}{s} = \partial\left(\frac{\partial s}{s}\right) + \left(\frac{\partial s}{s}\right)^2$$

$$\frac{\partial s}{s} = \frac{\partial}{\partial x} \log s_x = \frac{\theta'(u)}{\theta(u)} - \frac{\theta'(u-x-y)}{\theta(u-x-y)}$$

Recall that $\dfrac{\theta'(u+\tau)}{\theta(u+\tau)} = -2\pi i + \dfrac{\theta'(u)}{\theta(u)}$.

It follows that $\frac{\partial s}{s}$ is a doubly-periodic function. It is a meromorphic function with simple poles of residues $+1$, -1 at the points $u=0$, $u=x+y$ on X, hence is unique up to an additive constant.

Now $\left(\dfrac{\theta'(u)}{\theta(u)}\right)' = \dfrac{d^2}{du^2}\log\theta$ is a doubly-periodic function with behavior $-\frac{1}{u^2}$ as $u \to 0$, hence has to coincide with $-\wp(u)$ up to an additive constant.

So

$$\partial\left(\frac{\partial s}{s}\right) = +\left(\frac{\theta'(u-x-y)}{\theta(u-x-y)}\right)' = -\wp(u-x-y) + c(x+y)$$

$$\left(\frac{\partial s}{s}\right)^2 = \text{merom. function with double poles at } 0,\ x+y$$
$$= \wp(u) + \wp(u-x-y) + \text{const}\,\frac{\partial s}{s} + \text{const}(x+y)$$

So it should be true that the term ↑ doesn't occur and then we will get a potential depending on $x+y$.

At this point we have to work out the constants carefully. Start with the Weierstrass fn.

$$\wp(u) = \frac{1}{u^2} + \sum{}'\left(\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2}\right).$$

The series converges absolutely by a simple estimate based on $\sum \frac{1}{|\gamma|^3} \sim \int \frac{1}{r^3}\, r\,dr\,d\theta$. ~~Maybe~~

~~simpler proof is to~~ It's also doubly-periodic.

A slicker proof: Start with the convergent series

$$\sum \frac{1}{(u-\gamma)^3}$$

which gives an obvious doubly-periodic function with behavior $\frac{1}{u^3}$ at lattice points. Since the poles have no residues we can integrate

$$(-2)\int_\alpha^u \sum \frac{1}{(u-\gamma)^3}\,du = \sum\left(\frac{1}{(u-\gamma)^2} - \frac{1}{(\alpha-\gamma)^2}\right)$$

to get a doubly-periodic function with behavior $\frac{1}{u^2} + O(1)$ at lattice points. It's doubly-periodic because of the following argument: Let

$$g(u) = \sum\left\{\frac{1}{(u-\gamma)^2} - \frac{1}{(\alpha-\gamma)^2}\right\}$$

The series converges absolutely, so if μ is a period, then I can rearrange:

$$g(u+\mu) - g(u) = \sum\left\{\frac{1}{(u+\mu-\gamma)^2} - \frac{1}{(u-\gamma)^2}\right\}$$

and now one does the sum over a big box, and it comes down to boundary sums which give zero in the limit, probably because of the quadratic exponent.

(Point: The idea that integration should help when there are no residues is no good because one has the periods which can make the integral multiple-valued.)

So now take $g(u)$ and add a scalar so as to kill the constant term in the Laurent expansion at $u=0$; this gives $\wp(u)$.

$$\wp(u) = \frac{1}{u^2} + \sum{}'\left\{\frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2}\right\}$$

$$\frac{1}{(\gamma-u)^2} = \frac{1}{\gamma^2}\left(1-\frac{u}{\gamma}\right)^{-2} = \frac{1}{\gamma^2}\left[1 + (-2)\left(-\frac{u}{\gamma}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{u}{\gamma}\right)^2 + \cdots\right]$$

$$= \frac{1}{\gamma^2} + 2\frac{u}{\gamma^3} + 3\frac{u^2}{\gamma^4} + \cdots$$

hence
$$\wp(u) = \frac{1}{u^2} + 2u \sum' \frac{1}{\gamma^3} + 3u^2 \sum' \frac{1}{\gamma^4} + \cdots$$

February 24, 1982

Yesterday I found the Baker-Akhieser function over an elliptic curve in terms of the Θ fn and I was then trying to compute the potential function. It seems possible to replace $\Theta(u)$ by the Weierstrass σ function, in which case the calculations become easier. One has

$$\wp(u) = \frac{1}{u^2} + \sum' \left\{ \frac{1}{(u-\gamma)^2} - \frac{1}{\gamma^2} \right\}$$

$$\zeta(u) = \frac{1}{u} + \sum' \left\{ \frac{1}{u-\gamma} + \frac{1}{\gamma} + \frac{u^2}{\gamma^2} \right\}$$

$$\sigma(u) = u \prod_\gamma{}' \left(1 - \frac{u}{\gamma}\right) e^{u/\gamma + u^2/2\gamma^2}$$

and hence
$$\zeta'(u) = -\wp(u) \qquad \frac{\sigma'(u)}{\sigma(u)} = \zeta(u)$$

so that $\quad \dfrac{d^2}{du^2} \log \sigma(u) = -\wp(u)$.

Since $\wp$ is doubly-periodic it follows, that $\log \sigma(u+\gamma)$ and $\log \sigma(u)$ differ by a linear function of u. Similarly $\zeta(u+\gamma) - \zeta(u)$ is a constant depending additively on the period γ.

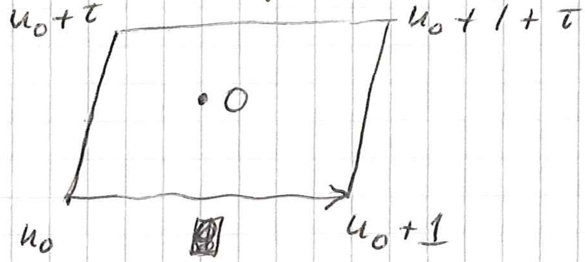
Put $\quad \zeta(u+\gamma) - \zeta(u) = a(\gamma) \in \mathbb{C}$. Then
$$a : \Gamma \longrightarrow \mathbb{C}$$
and it would be nice to know if a were actually given by a complex number times the actual embedding $\Gamma \subset \mathbb{C}$. However if this were the case,

say $a(\gamma) = \alpha\gamma$, then

$$\zeta(u+\gamma) - \zeta(u) = \alpha(u+\gamma) - \alpha(u)$$

and so $\zeta(u) - \alpha u$ [redacted] would be doubly-periodic with only a simple pole in the period parallelogram, contradicting $\sum \text{res} = 0$.

Consider a period parallelogram



Then

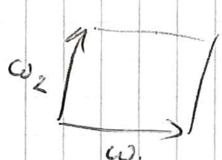
$$2\pi i = \oint \zeta(u)\,du$$

$$= \int_{u_0}^{u_0+1} \zeta + \underbrace{\int_{u_0+1}^{u_0+1+\tau} \zeta(u)\,du}_{\int_{u_0}^{u_0+\tau} \zeta(u+1)\,du} - \underbrace{\int_{u_0+\tau}^{u_0+1+\tau} \zeta}_{\int_{u_0}^{u_0+1} \zeta(u+\tau)\,du} - \int_{u_0}^{u_0+\tau} \zeta$$

$$= \int_{u_0}^{u_0+1} [\underbrace{\zeta(u) - \zeta(u+\tau)}_{-a(\tau)}]\,du + \int_{u_0}^{u_0+\tau} [\underbrace{\zeta(u+1) - \zeta(u)}_{a(1)}]\,du$$

$$\boxed{2\pi i = \tau \cdot a(1) - 1 \cdot a(\tau)}$$

In general given a period parallelogram



we have the periods

$$a(\omega_i) = \int_u^{u+\omega_i} \wp(u)\,du = \zeta(u+\omega_i) - \zeta(u)$$

and one has the formula (Legendre relation)

$$\boxed{2\pi i = \omega_1 \cdot a(\omega_2) - \omega_2 \cdot a(\omega_1)}$$

Recall that $\frac{\theta'(u)}{\theta(u)}$ has simple poles of residue 1 at each of the lattice points, and satisfies

$$\frac{\theta'(u+1)}{\theta(u+1)} = \frac{\theta'(u)}{\theta(u)} \qquad \frac{\theta'(u+\tau)}{\theta(u+\tau)} = -2\pi i$$

i.e. for $\frac{\theta'(u)}{\theta(u)}$ one has $\breve{a}(1)=0$, $\tilde{a}(\tau) = -2\pi i$ in agreement with the Legendre relation.

So now it becomes clear that $\frac{\theta'(u)}{\theta(u)}$ and $\zeta(u)$ differ by a linear function

$$\frac{\theta'(u)}{\theta(u)} = \zeta(u) + \alpha u + \beta$$

where
$$a(1) + \alpha = 0.$$

So let us see if we can compute the potential using the σ-fn instead of the θ-fn.

$$e^{x\frac{\sigma'(u)}{\sigma(u)}} = e^{x\zeta(u)}$$

$$e^{x\zeta(u+\gamma)} = e^{x\zeta(u)}\, e^{x a(\gamma)}$$

$$d\log\frac{\sigma(u+\gamma)}{\sigma(u)} = \zeta(u+\gamma) - \zeta(u) = a(\gamma)$$

$$\therefore \quad \frac{\sigma(u+\gamma)}{\sigma(u)} = e^{a(\gamma)u + b(\gamma)}$$

So
$$\frac{\sigma(u+\gamma-y_1)}{\sigma(u+\gamma-y_2)} = \frac{\sigma(u-y_1)}{\sigma(u-y_2)}\ \underbrace{\frac{e^{a(\gamma)(u-y_1)+b(\gamma)}}{e^{a(\gamma)(u-y_2)+b(\gamma)}}}_{e^{a(\gamma)(y_2-y_1)}}$$

So we want $x + y_2 - y_1 = 0$

or $\quad y_1 = x + y_2$. Thus the BA function becomes

$$s(x,u) = e^{x\frac{\sigma'(u)}{\sigma(u)}} \frac{\sigma(u-x-y)}{\sigma(u-y)} \qquad y \text{ fixed}$$

To simplify I will put $y=0$. Now

$$\partial_x \log s = \zeta(u) + \partial_x \log \sigma(u-x)$$
$$= \zeta(u) - \zeta(u-x)$$

February 25, 1982

Let's compute the Laurent series around $u=0$.

$$\zeta(u) = \frac{1}{u} + \sum{}' \frac{1}{u-\gamma} + \frac{1}{\gamma} + \frac{u}{\gamma^2} \qquad \zeta' = -\wp$$

$$= \frac{1}{u} - u^2 \underbrace{\sum{}' \frac{1}{\gamma^3}}_{=0} - u^3 \sum{}' \frac{1}{\gamma^4} - \cdots$$

$$= \frac{1}{u} - u^3 G_4 - u^5 G_6 - \cdots$$

$$\zeta(u-x) = \zeta(-x) + \zeta'(-x)u + \cdots$$
$$= \zeta(x) - \wp(x)u + \cdots$$

Thus

$$\zeta(u) - \zeta(u-x) - \zeta(x) = \frac{1}{u} + \wp(x)u + O(u^2)$$

is a meromorphic fn on the elliptic curve with simple poles of residue 1, -1 at 0, x. Thus

$$[\zeta(u) - \zeta(u-x) - \zeta(x)]^2 = \frac{1}{u^2} + 2\wp(x) + O(u)$$

has double poles with leading coeff 1.

$$\wp(u) + \wp(u-x) = \frac{1}{u^2} + \wp(x) + \wp'(-x)u + O(u^2)$$

So

$$[\zeta(u) - \zeta(u-x) - \zeta(x)]^2 - [\wp(u) + \wp(u-x) + \wp(x)] = O(u)$$

has no singularity at $u=0$, and at most a simple pole at $u=x$, so it must be constant, hence 0. Thus we get the identity

$$\left[\zeta(u) + \zeta(x-u) + \zeta(-x)\right]^2 = \wp(u) + \wp(x-u) + \wp(-x)$$

which [redacted] can be written more symmetrically using any three points u_1, u_2, u_3 with $u_1+u_2+u_3=0$ instead of $u, x-u, -x$.

The preceding shows that I probably [redacted] have the wrong Baker-Akhiezer fn. [redacted] Recall that I want

$$s(x,z) = e^{xz}\left(1 + \frac{a_1(x)}{z} + \cdots\right)$$

where $z = \sqrt{\wp(u)} = \left(\frac{1}{u^2} + 3u^2 G_4 + \cdots\right)^{1/2} = \frac{1}{u} + O(u^3)$.

Hence

$$\log s(x,z) = xz + \frac{a_1(x)}{z} + O\left(\frac{1}{z^2}\right)$$

$$\partial_x \log s(x,z) = z + \frac{a_1'(x)}{z} + O\left(\frac{1}{z^2}\right)$$

So try

$$\begin{aligned}\partial \log s_x(u) &= \zeta(u) - \zeta(u-x-y) - \zeta(x+y) \\ &= \frac{1}{u} + \wp(x+y)\,u + O(u^2) \\ &= \frac{1}{z} + \wp(x+y)\,z + O(u^2)\end{aligned}$$

Thus it follows that the potential is

$$\boxed{q(x) = 2\wp(x+y)}$$

So the correct formula for the BA function is

$$\boxed{s_x = e^{x\zeta(u)} \underbrace{\frac{\sigma(u-x-y)\,\sigma(-y)}{\sigma(u-y)\,\sigma(-x-y)}}_{\text{rigged to have value 1 at } u=0}}$$

Then $$\partial \log s_x = \zeta(u) - \zeta(u-x-y) + \zeta(-x-y)$$
$$= \zeta(u) - \zeta(u-x-y) - \zeta(x+y)$$

and so
$$\frac{\partial^2 s_x}{s_x} = \partial\left(\frac{\partial s}{s}\right) + \left(\frac{\partial s}{s}\right)^2$$
$$= +\zeta'(u-x-y) - \zeta'(x+y)$$
$$+ \left[\zeta(u) - \zeta(u-x-y) - \zeta(x+y)\right]^2$$
$$= -\cancel{\wp(u-x-y)} + \wp(x+y)$$
$$+ \wp(u) + \cancel{\wp(u-x-y)} + \wp(x+y)$$
$$= \wp(u) + 2\wp(x+y)$$

or $$\left[\partial^2 - 2\wp(x+y)\right] s_x = \wp(u)\, s_x$$

which checks.

February 25, 1982

Differentials of first, 2nd, 3rd kind. (Think of a Riemann surface - but it works more generally.)

Differentials of first kind are elements of $H^0(\Omega^1)$ e.g. $du = \frac{dx}{y}$ $\quad x = \wp(u),\ y = \wp'(u)$ on an elliptic curve. In the complete curve case one has Hodge splitting:

$$0 \to H^0(X,\Omega^1) \to H^1(X,\mathbb{C}) \to H^1(X,\Omega^0) \to 0$$

so only half of $H^1(X,\mathbb{C})$ is represented by dfl's of first kind.

Differentials of 2nd kind are meromorphic differentials whose residues are all 0. Hence if $\mathcal{M}$ = sheaf of meromorphic functions, and $\Omega^1_{(2)}$ = sheaf of merom. dfls. we have

$$0 \to \mathbb{C} \to \mathcal{M} \xrightarrow{d} \Omega^1_{(2)} \to 0$$

and since $H^1(\mathcal{M}) = 0$ quite generally we have

$$0 \to \mathbb{C} \to H^0(\mathcal{M}) \to H^0(\Omega^1_{(2)}) \to H^1(\mathbb{C}) \to 0$$

so that every element of $H^1(X,\mathbb{C})$ is represented by a diff'l of the 2nd kind. e.g. on an elliptic curve $\wp(u)\,du = x\frac{dx}{y}$ gives an independent set of periods from du.

Differentials of 3rd kind are merom. diff'ls having only simple poles whose residues are integers:

$$0 \to \mathbb{C}^* \to \mathcal{M}^* \xrightarrow{d\log} \Omega^1_{(3)} \to 0$$

hence

$$0 \to \mathbb{C}^* \to H^0(\mathcal{M}^*) \to H^0(\Omega^1_{(3)}) \to H^1(X,\mathbb{C}^*) \to H^1(\mathcal{M}^*)$$

Question: Is $H^1(X,\mathcal{M}^*) = 0$? If so, then any element of $H^1(X,\mathbb{C}^*)$ can be represented by a diff'l. of the third kind. In any case one has

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, "d"] \arrow[d, "\exp"] & \Omega^1 \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] & \mathcal{M}^* \arrow[r, "d\log"] \arrow[d] & \Omega^1_{(3)} \arrow[r] & 0 \\
& & \mathcal{D} \arrow[d] & & \\
& & 0 & &
\end{tikzcd}$$

where $\mathcal{D} = \coprod_{p \in X} (i_p)_* \mathbb{Z}$ is the divisor sheaf. Thus

$$0 \longrightarrow \Omega^1 \longrightarrow \Omega^1_{(3)} \xrightarrow{res} \mathcal{D} \longrightarrow 0$$

which yields

$$\cancel{0 \to H^0(X,\Omega^1) \to H^0(X,\Omega^1_{(3)}) \to D \to H^1(X,\Omega^1) \to H^1(X,\Omega^1_{(3)}) \to 0}$$

$$\cancel{0 \to H^0(X,\mathcal{O}) \to H^0(X,\mathcal{M}^*) \to D \to H^1(X,\mathcal{O}) \to H^1(X,\mathcal{M}^*) \to 0}$$

$$0 \to H^0(X,\Omega^1) \to H^0(X,\Omega^1_{(3)}) \longrightarrow D \to \underset{\mathbb{C}}{\underset{\wr}{H^1(X,\Omega^1)}} \to H^1(X,\Omega^1_{(3)}) \to 0$$

Wait. One has

$$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{D} \to 0$$

so
$$0 \to H^0(\mathcal{O}^*) \to H^0(\mathcal{M}^*) \to D \to H^1(\mathcal{O}^*) \to H^1(\mathcal{M}^*) \to 0$$

But one knows every element of $H^1(\mathcal{O}^*)$ comes from a divisor, hence one concludes, $H^1(\mathcal{M}^*) = 0$, and so one has

$$\begin{tikzcd}
& & & 0 \arrow[d] & \\
& & & H^0(X,\Omega^1) \arrow[d] & H^0(X,\Omega^1) \arrow[d] & \\
0 \arrow[r] & \mathbb{C}^* \arrow[r] & H^0(\mathcal{M}^*) \arrow[r] & H^0(\Omega^1_{(3)}) \arrow[r] \arrow[d] & H^1(X,\mathbb{C}^*) \arrow[r] \arrow[d] & 0 \\
& & & D \arrow[r] \arrow[d] & H^1(X,\mathcal{O}^*) \arrow[d] & \\
& & & H^1(X,\Omega^1) & H^1(X,\Omega^1) \arrow[d] & \\
& & & & H^2(X,\mathbb{C}^*) &
\end{tikzcd}$$

It seems to follow that for a complete curve

$$0 \longrightarrow H^0(X,\Omega^1) \longrightarrow H^0(\Omega^1_{(3)}) \longrightarrow \underset{\text{of degree } 0}{\text{Divisors}} \longrightarrow 0$$

which is completely [redacted] clear. In effect just look at the question of producing a [redacted] differential with a first order pole at P.

$$0 \longrightarrow \Omega^1 \longrightarrow \Omega^1(P) \longrightarrow \mathbb{C} \longrightarrow 0$$

$$H^0(\Omega^1) \longrightarrow H^0(\Omega^1(P)) \xrightarrow{\text{res}} \mathbb{C} \longrightarrow \underset{\mathbb{C}}{\overset{\|}{H^1(\Omega^1)}} \longrightarrow H^1(\Omega^1(P)) \to 0$$

and $H^1(\Omega^1(P))$ is dual to $H^0(\mathcal{O}(-P)) = 0$. So you can't find such a differential but clearly we we have

$$H^0(\Omega^1) \longrightarrow H^0(\Omega^1(P+Q)) \xrightarrow{\text{res}} \mathbb{C}\oplus\mathbb{C} \longrightarrow \underset{\mathbb{C}}{\overset{\|}{H^1(\Omega^1)}} \longrightarrow 0$$

etc., etc.

Finally the surjection

$$H^0(\Omega^1_{(3)}) \longrightarrow H^1(X,\mathbb{C}^*) \longrightarrow 0$$

can be interpreted roughly as a kind of solution to the RH problem. Namely, start with $\alpha \in H^1(X,\mathbb{C}^*) = \mathrm{Hom}(\pi_1(X),\mathbb{C}^*)$, then look for a differential equation

$$dy = (A\,dz)\,y$$

over X whose monodromy is α. [redacted] Clearly a differential form of the third kind will give trivial monodromy at each of the poles, and hence will give a map $\pi_1(X) \longrightarrow \mathbb{C}^*$. So representing α by a differential form of the third kind amounts to solving the RH problem.

Perverse sheaves and holonomic modules over a Riemann surface X. The simplest example is the equivalent description of a locally constant sheaf of f.d. $\mathbb{C}$ vector spaces, as a vector bundle with integrable connection. The point is to generalize to constructible sheaves.

$\mathcal{D}_X$ is the sheaf of differential operators of finite order on X. Locally an element of $\mathcal{D}_X$ is of the form $\sum_{n\leq N} a_n(z)\left(\frac{d}{dz}\right)^n$ with the $a_n(z)$ analytic fns. Thus $\Gamma(U,\mathcal{D}_X) = \Gamma(U,\mathcal{O}_X)\left[\frac{d}{dz}\right]$ is a non-commutative polynomial ring. ~~The next point~~ (There is a standard nonsense about $\Gamma(U,\mathcal{O}_X)$ being non-noetherian (?), so to avoid this nonsense let's think of $\Gamma(U,\mathcal{D}_X)$ as being something like $A[\partial]$, where A is a noetherian ring with a derivation ∂. e.g. $A = \mathcal{O}_x = \mathbb{C}\{z\}$.)

Next we let $\mathcal{M}$ be a coherent sheaf of $\mathcal{D}_X$ modules, which in my picture will be a f.g. $A[\partial]$-module M. I know that I can choose a filtration $F_n M \subset F_{n+1} M$ by f.g. A-modules such that $\partial F_n \subset F_{n+1}$, and such that $gr(M)$ is finitely generated over $gr\, A[\partial] =$ poly. ring $A[T]$. One says $\mathcal{M}$ is holonomic when $\dim gr(M) = 1$. This seems to mean that at most of the points of X, M is finite-diml. over A. At some isolated points M will not be finitely generated, but $gr_n M$ will be supported here for large n, and $\partial : gr_n M \xrightarrow{\sim} gr_{n+1} M$.

So the typical example occurs when you try to take $M = A[\frac{1}{z}]$ with the operator

$$Df = \left(\partial + \frac{a}{z}\right) f.$$

Then

$$D(1) = \frac{a}{z}$$

$$D\left(\frac{1}{z}\right) = -\frac{1}{z^2} + \frac{a}{z^2} = \frac{a-1}{z^2}$$

$$D\left(\frac{1}{z^k}\right) = -\frac{k}{z^{k+1}} + \frac{a}{z^{k+1}} = \frac{a-k}{z^{k+1}}$$

and so M is generated over $A[D]$ by the element 1 provided $a \notin \mathbb{N}$. Presumably this is what one expects in general.

Example: $A = \mathbb{C}\{z\}$, $M = \mathbb{C}\{z\}[\frac{1}{z}]$ with D on $M = \partial$. Then M is generated by the element $1/z$. But also M contains the submodule $\mathbb{C}\{z\}$, so that another example is

$$M/A = \sum_{n \geq 1} \mathbb{C} z^{-n}$$

Now from a holonomic $\mathcal{D}_X$-module one obtains a de Rham complex the usual way. So let's take $M = \mathbb{C}\{z\}[z^{-1}]$ with $D = \partial - \frac{a}{z}$ and we want to compute the cohomology of the complex

$$M \xrightarrow{D} M$$

i.e.

$$\left(\partial - \frac{a}{z}\right) f = g \qquad \partial - \frac{a}{z} = z^a \partial z^{-a}$$

If $g = 0$, then $z^a \partial (z^{-a} f) = 0 \implies f = c z^a$ which is not in M for $a \notin \mathbb{Z}$. Similarly for $g = z^n$:

$$z^a \partial (z^{-a} f) = z^n$$

$$\partial (z^{-a} f) = z^{n-a} \implies z^{-a} f = \frac{z^{n-a+1}}{n-a+1} + c$$

$$f = \frac{z^{n+1}}{n-a+1} + Cz^a$$

So D is bijective, when $a \notin \mathbb{Z}$. If $a \in \mathbb{Z}$, then D has a kernel spanned by z^a and a cokernel spanned by z^{a-1}.

However I should be thinking of this complex as a complex of sheaves over a disk, so this kernel and cokernel should represent some kind of constructible sheaves.

Examples over the disk:

$$\mathcal{O} \xrightarrow{d} \Omega^1 \qquad\qquad \underline{H}^0 = \mathbb{C}$$

$$\mathcal{O}[\tfrac{1}{z}] \xrightarrow{d} \Omega^1[\tfrac{1}{z}] \qquad\qquad \underline{H}^0 = \mathbb{C} \qquad \underline{H}^1 = (\iota_0)_* \mathbb{C}$$

$$\mathcal{O}[\tfrac{1}{z}]/\mathcal{O} \longrightarrow \Omega^1[\tfrac{1}{z}]/\Omega^1 \qquad\qquad \underline{H}^1 = (\iota_0)_* \mathbb{C}$$

Consider: $\mathcal{O}[\frac{1}{z}] \xrightarrow{\partial - \frac{a}{z}} \mathcal{O}[\frac{1}{z}]$ where $a \notin \mathbb{Z}$.

The stalk at 0 is acyclic by the above, and away from 0 the complex is the DR complex of the locally constant sheaf with sections $\mathbb{C}z^a$. Hence if $j : \text{Disk} - 0 \hookrightarrow \text{Disk}$ we have

$$\underline{H}^0 = j_*(\mathbb{C}z^a)$$

According to Dale perverse sheaves over a curve are complexes having only a $\underline{H}^{-1}$ and an $\underline{H}^0$, where $\underline{H}^{-1}$ has no finite support sections, and $\underline{H}^0$ has only finite support sections.

February 26, 1982:

Recall the scattering situation for
$$[-\partial^2+q]u = k^2 u$$
where q decays as $x\to\infty$, one has an interpretation of the Jost function as a Fredholm determinant:
$$A(k) = \frac{W(\phi,f)}{W(\phi_0,f_0)} = \det(1-G_0 q).$$
(Here $G_0 = (k^2+\partial)^{-1}$ is the Green's function for the Schroed. op with potential 0, and $\phi \sim e^{-ikx}$ $x\to-\infty$, $f\sim e^{ikx}$ as $x\to+\infty$ and $k\in UHP$.) However, if q leads to a discrete spectrum, then one can define a version of $A(k)$ as follows which is an entire function of k as follows: Consider over the **k-plane** the 2-diml vector bundle E of solutions of the Schroedinger equation. Then E has a natural symplectic structure given by the Wronskian. The subspace of solutions decaying ~~at~~ $x\to\pm\infty$ gives a sub-line-bundle $L_\pm$ of E, hence we have a map
$$L_+\oplus L_- \longrightarrow E$$
$$L_+\otimes L_- \longrightarrow \Lambda^2 E = \mathcal{O}$$
Holomorphic line bundles over $\mathbb{C}$ are trivial, hence picking a generator for $L_+\otimes L_-$ gives an entire function $A(k)$ whose zeroes are the spectrum of the operator. A natural question then is whether this $A(k)$ can be interpreted as a determinant, a possibly generalized determinant in the sense of the wedge space.

February 26, 1982:

Let X be a compact Riemann surface, E fixed C^∞ vector bundle over X of rank r and degree d. I consider a fixed hermitian structure on E, and then obtain a 1-1 correspondence between connections on E preserving the metric and holom. ~~complex~~ structures on E. The gauge group is the group of autos. of E as a C^∞-hermitian vector bundle and it acts on the space of connections. The complex gauge group of all autos. of E acts on the space of complex structures. So we have a familiar situation:

$$\begin{array}{ccc} \{\text{complex gauge gp}\} & & \{\text{holom. } \cancel{\text{complex}} \text{ structures}\} \\ \cup & & \wr\| \\ \{\text{gauge gp}\} & & \{\text{connections}\} \end{array}$$

which is analogous to any of the following:

$$\begin{array}{ccc} GL_n & & GL_n/P \\ \cup & & \wr\uparrow \\ U_n & & U_n/U_n\cap P \end{array}$$

$$\begin{array}{ccc} Maps(S^1, GL_n) & & \{\text{outgoing subspaces}\} \\ \cup & & \wr\uparrow \\ Maps(S^1, U_n) & & \left\{\begin{array}{l} n\text{-dim'l subspaces } W \\ \ni \quad W \perp z^n W \quad \text{all } n \neq 0 \end{array}\right\} \end{array}$$

~~So next what's import~~

Remark: The $\{$holom. ~~complex~~ structures$\}$ over a Riemann surface is some kind of building for the complex gauge group, perhaps. The analogous thing for the loop group is the space of connections somewhere?

I think the correct analogy goes as follows. Consider $GL_n(\mathbb{C})$. Inside this group are the ways of heading toward ∞ which can be identified with rays in the Lie algebra of U_n. Hence the building doesn't appear to have a complex structure, although $GL_n(\mathbb{C})$ acts on it.

In the case of $Map(S^1, GL_n)$, the building consists of connections on the trivial bundle $S^1 \times U_n \longrightarrow S^1$. Thus it is not immediately clear how the complex gauge group acts on it, although presumably in some way this building coincides with the building of parahoric subgps. Hence we seem to again have a building constructed out of holom. orbits but real parameters joining the orbits.

Next in the Riemann surface case one has all holomorphic structures on a fixed C^∞-bundle which looks like a complex manifold, at least it's clear what a holomorphic map into this space is. The complex gauge group acts on it

Wait: Because you think in terms of Laurent polys. you ought to check the loop case carefully.

But go back to the Riemann surface case. Given a ~~complex~~ holomorphic structure one gets finite-diml cohomology and one can take highest exterior power so as to get a line. Thus we get a canonical holomorphic line bundle over the space of holomorphic structures.

You should work this all out in the case of the trivial line bundle. A connection is a purely imaginary 1-form on the curve.

Suppose L is a holom. line bundle with a metric. Fix a gen. hol. section s and let

$$Ds = \theta s \qquad \theta \in \Omega^{10}$$

Then for the connection to preserve the metric means

$$d|s|^2 = \theta s \bar{s} + s\overline{\theta s}$$
$$= |s|^2(\theta + \bar{\theta})$$

so that $\theta = \partial \log |s|^2$. Conversely suppose that t is a section with $|t|^2 = 1$, and we want to know what sections ft are holomorphic. Then we look at

$$L \xrightarrow{D} T^* \otimes L \longrightarrow \Omega^{0,1} \otimes L$$

and take the kernel. The point then is that a holomorphic structure on L is simply an operator

$$L \xrightarrow{\partial} L \otimes \Omega^{0,1}$$

satisfying the derivation property wrt $\partial : \mathcal{O} \to \Omega^{0,1}$. This amounts to a dotted arrow in

$$\begin{tikzcd} L \otimes T^*_{\mathbb{C}} \arrow[r, hook] \arrow[d] & J_1(L) \arrow[dl, dashed] \\ L \otimes \Omega^{01} & \end{tikzcd} \qquad (*)$$

since if f vanishes at x, then

$$\begin{tikzcd} (s \otimes df)(x) \arrow[r, mapsto] \arrow[d, mapsto] & j_1(sf) \arrow[dl] \\ \partial(sf)(x) = s(x) \otimes \partial f(x) & \end{tikzcd}$$

and conversely assuming ⊗ commutes, then

$$\big(\partial\ s[f - f(x)]\big)(x) = s(x)(\partial f)(x)$$

or $$\partial(sf) - (\partial s)f = s\partial f \quad \text{at } x, \quad \text{etc.}$$

So therefore one sees that the set of holomorphic structures on L are simply splittings of an exact sequence of vector bundles

$$0 \longrightarrow L \otimes \Omega^{01} \longrightarrow J_1(L)/L\otimes\Omega^{10} \longrightarrow L \longrightarrow 0$$

and hence this set is an affine space over the complex vector space $\Gamma(\mathrm{Hom}(L, L\otimes\Omega^{01}))$.

So in the case of the trivial bundle a connection preserving the metric is an imaginary 1-form which is the same thing as a form of type 1,0.

February 27, 1982.

X Riemann surface, E fixed C^∞ vector bundle over X, $\mathcal{A}$ is the space of holom. structures on E. Over $\mathcal{A}$ I define a holomorphic line bundle by taking the determinant bundle of the cohomology of E for the given holom. structure. This cohomology is computed using the $\bar\partial$ complex

$$E \xrightarrow{\bar\partial} E \otimes \Omega^{0,1}.$$

As we vary over the space of holomorphic structures, the bundles $E, E\otimes\Omega^{0,1}$ don't change but $\bar\partial$ does. In fact one has

$$\bar\partial = \bar\partial_0 + \omega$$

for a fixed holom. structure, and where ω runs over $\Gamma(\mathrm{Hom}(E, E\otimes\Omega^{0,1}))$.

Now I want to bring in Singer's determinant ideas. So let us suppose that the [redacted] holomorphic structure $\bar\partial_0$, which we have chosen, gives a bundle with $h^0 = h^1 = 0$, in which case we know that the determinant bundle [redacted] of the cohomology of E_0 is canonically trivial. This is so for all structures near to that of E_0.

Better: Two things are happening for holomorphic structures such that $h^0 = h^1 = 0$. On one hand

$$C^\infty(X,E) \xrightarrow{\bar\partial} C^\infty(X, E\otimes\Omega^{0,1})$$

is an isomorphism, hence it perhaps is possible to define the relative determinant

$$\det\left[(\bar\partial_0)^{-1}\bar\partial\right]$$

which will be an actual function, presumably entire, on the space of holomorphic structures. On the other hand the determinant bundle of the cohomology is canonically trivial.

Concrete question: Take an elliptic curve $X = \mathbb{C}/\Gamma$ and consider the family of line bundles of degree 0 parameterized ~~in some convenient way, say by characters of the fundamental group. Better~~ by different 1-forms, i.e. different connections on the trivial line bundle. Compute the above determinant.

On the elliptic curve we have a natural coordinate u, and $\Omega^{0,1}$ has the basis [redacted] $\overline{du} = d\bar u$. Thus we want to look at the differential operator

$$\frac{\partial}{\partial\bar u} : \mathcal{O} \longrightarrow \mathcal{O}$$

for the trivial bundle. The [redacted] different holom. structures are [redacted] described by elements of $\Gamma(\Omega^{0,1})$ i.e. things of the form $f(u)\,d\bar{u}$. The [redacted] complex for computing the cohomology of the holom. line bundle is

$$\mathcal{O} \xrightarrow{\frac{\partial}{\partial \bar{u}} - f} \mathcal{O}$$

and now we are going to try to compute the relative determinant of this operator by varying f. In other words I want to compute

$$\delta \log \det \left(\frac{\partial}{\partial \bar{u}} - f\right) = \log \det \left(\frac{\partial}{\partial \bar{u}} - f\right)^{-1}\left(\frac{\partial}{\partial \bar{u}} - f - \delta f\right)$$

$$= \log \det \left(1 - \left(\frac{\partial}{\partial \bar{u}} - f\right)^{-1} \delta f\right)$$

$$= - \operatorname{tr}\left[\left(\frac{\partial}{\partial \bar{u}} - f\right)^{-1} \delta f\right]$$

Simplest case is where f is constant

$$\left(\frac{\partial}{\partial \bar{u}} - f\right)\psi = 0$$

$$\frac{\partial}{\partial \bar{u}}(\log \psi) = f$$

$$\log \psi = f\bar{u} + g(u) \qquad\qquad g(u) \text{ analytic}$$

$$\psi = e^{f\bar{u} + g(u)}$$

This is the formula for the general solution upstairs on the u-plane. When is it doubly-periodic? [redacted] If it is doubly-periodic, then so is

$$\frac{\partial}{\partial u} \log \psi = \frac{\partial \psi}{\partial u} \Big/ \psi = g'(u)$$

and hence $g'(u)$ has to be a constant, so in fact [redacted]

$$\psi = C e^{f\bar{u} + gu} \qquad \text{where } f, g \text{ are constants}$$

Hence we obtain a doubly-periodic function exactly when

$$u \longmapsto f\bar{u} + gu$$

carries Γ into $2\pi i \mathbb{Z}$. So if $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$ we must have $f + g \in 2\pi i \mathbb{Z} \Rightarrow Re(f) + Re(g) = 0$

First note that the conditions

$$f + g \in i\mathbb{R}$$
$$f\bar{\tau} + g\tau \in i\mathbb{R}$$

uniquely determine g from f, since subtracting two solutions would give $\Delta g, (\Delta g)\tau \in i\mathbb{R}$ which is impossible. Clearly $g = -\bar{f}$ is a solution, so we set

$$\psi = C e^{f\bar{u} - \bar{f}u}$$

This is doubly periodic when f satisfies

$$f - \bar{f} \in 2\pi i \mathbb{Z}$$
$$f\bar{\tau} - \bar{f}\tau \in 2\pi i \mathbb{Z}$$

which means simply that f belongs to some sort of dual lattice to Γ. For example take f to be real and you get $f \in \mathbb{Z}\frac{\pi}{\operatorname{Im}\tau}$; or take $f = a\tau$ with a real and you get $a\tau - a\bar{\tau} = a(2i \operatorname{Im}\tau) \in 2\pi i \mathbb{Z}$ or $a \in \mathbb{Z}\frac{\pi}{\operatorname{Im}\tau}$. Thus ψ is doubly-periodic when

$$f \in \frac{\pi}{\operatorname{Im}\tau}\left[\mathbb{Z} + \mathbb{Z}\tau\right]$$

This checks that I get line bundles parameterized by $f \in \mathbb{C} \bmod \frac{\pi}{\operatorname{Im}\tau}\Gamma$.

Change the constant f to a. We want to compute the relative determinant of

$$\frac{\partial}{\partial \bar{u}} - a \ : \ \Gamma(\mathcal{O}) \longrightarrow \Gamma(\mathcal{O}).$$

For basis of $\Gamma(\mathcal{O})$ we can takes the exponential functions

$$e^{c\bar{u} - \bar{c}u} \qquad c \in \frac{\pi}{\operatorname{Im}\tau}\Gamma$$

The operator $\frac{\partial}{\partial \bar{u}} - a$ has the eigenvalues $c-a$ and so our relative determinant is something like

$$\prod_{c \in \frac{\pi}{\operatorname{Im}\tau}\Gamma} \frac{c-a}{c} \qquad \text{better} \qquad a \prod_{c \in \frac{\pi}{\operatorname{Im}\tau}\Gamma}{}' \left(1 - \frac{a}{c}\right).$$

We know this expression has convergence problems which one can solve by the [redacted] Weierstrass device of [redacted] putting in the <u>convergence</u> factors:

$$\sigma(a) = a \prod_{c \in \frac{\pi}{\operatorname{Im}\tau}\Gamma}{}' \left(1 - \frac{a}{c}\right) e^{\frac{a}{c} + \frac{a^2}{2c^2}}$$

These factors are required to make the resolvant have a trace:

$$\operatorname{tr}\left(\frac{\partial}{\partial \bar{u}} - a\right)^{-1} = \sum \frac{1}{c-a}$$

is essentially the Weierstrass ζ-function before [redacted] the subtractions are made. Hence one consequence is that there are problems with defining this determinant as a function of the line bundle.

~~[illegible]~~

February 27, 1982 (cont.)

Review Singer's business on determinants and analytic torsion.

Suppose I take a complex

$$0 \to C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots\cdots \xrightarrow{d} C^n \to 0$$

of finite dimensional $\mathbb{C}$-vector spaces on which hermitian metrics are given, and suppose the complex is acyclic. Then because it's acyclic one has a canonical isomorphism

$$\lambda(C^\cdot) = \lambda(C^0) \otimes \lambda(C^1)^\vee \otimes \cdots$$

with $\lambda(H^\cdot) = \lambda(0) = \mathbb{C}$. On the other hand from the metrics on the C^δ one inherits a natural metric on $\lambda(C^\cdot)$, and so one gets a metric on $\mathbb{C}$, that is, a definite positive number.

I think what Singer does it to calculate out this number using the Laplacians $\Delta = d\delta + \delta d$ on the above complex. So how does this work in the simplest case, i.e.

$$0 \longrightarrow C^0 \xrightarrow{d} C^1 \longrightarrow 0$$

Choose an orthonormal basis u_i for C^0 consisting of eigenfns for δd: $\delta d u_i = \lambda_i u_i \Longrightarrow$ $(d\delta) du_i = \lambda_i du_i$ so we can choose an orth. basis v_i for C^1 with $du_i = \mu_i v_i$. Then d is effectively diagonal with the eigenvalue μ_i, so clearly by defn. of δ we have $\delta v_i = \mu_i u_i$, so $\mu_i^2 = \lambda_i$.

$$\lambda(C^0) \xrightarrow{\lambda(d)} \lambda(C^1)$$

$$u_1 \wedge u_2 \wedge \cdots \longmapsto \prod \mu_i \, v_1 \wedge \cdots \wedge v_n$$

and hence

$$\lambda(C^0) \otimes \lambda(C^1)^\vee \longrightarrow \mathbb{C}$$

$$(u_1 \wedge \cdots \wedge u_n) \otimes (v_1 \wedge \cdots \wedge v_n)^* \longmapsto \prod \mu_i$$

and so in this case we get the formula

$$\text{torsion} = \prod \mu_i = \det(\delta d)^{1/2}$$

In general one has

$$2\log(\text{torsion}) = \log\det \delta d \text{ on } C^0 - \log\det \delta d \text{ on } (Z^1)^\perp + \cdots$$

and

$$\log\det \delta d \text{ on } (Z^p)^\perp = \log\det \Delta_p - \log\det d\delta \text{ on } Z^p$$

$$= \log\det \Delta_p - \log\det d\delta \text{ on } (Z^{p-1})^\perp$$

$$= \log\det \Delta_p - \log\det \Delta_{p-1} + \log\det \Delta_{p-2} - \cdots$$

So

$$\log\det \delta d \text{ on } (Z^p)^\perp = \sum_{j=0}^{p} (-1)^{p-j} \log\det \Delta_j$$

and

$$2\log\text{torsion} = \sum_p (-1)^p \log\det(\delta d \text{ on } (Z^p)^\perp)$$

$$= \sum_{p=0}^{n-1} (-1)^p \sum_{j=0}^{p} (-1)^{p-j} \log\det \Delta_j$$

$$= \sum_{j=0}^{n-1} (-1)^j \log\det \Delta_j \sum_{p=j}^{n-1} 1$$

$$\therefore \boxed{2\log\text{torsion} = \sum_{j=0}^{n-1} (-1)^j (\log\det(\Delta_j))(n-j)}$$

Notice this is independent of n for large n.

Next we want to understand this for elliptic curves, and the operator $\left(\frac{\partial}{\partial\bar{u}} - a\right) : \mathcal{O} \longrightarrow \mathcal{O}$.

But I forgot to mention the analysis, namely how one proposes to compute $\log\det(\Delta)$. The point is that one has the ζ function

$$\zeta(s) = \sum_\lambda \lambda^{-s} \qquad \text{so } -\zeta'(s) = \sum \lambda^{-s}\log\lambda$$

and so $$-\zeta'(0) = \sum \log\lambda = \log\det(\Delta).$$

So the real idea will be to use some kind of analytic continuation to define the determinants.

So let's return to the operator

$$\frac{\partial}{\partial\bar{u}} + a : \mathcal{O} \longrightarrow \mathcal{O}$$

over an elliptic curve. Here the second $\mathcal{O}$ is really $\Omega^{0,1} = \mathcal{O}\otimes d\bar{u}$. In order to put metrics on $\Gamma(X,\mathcal{O})$ and $\Gamma(X,\Omega^{0,1})$, we of course look at translation invariant metrics, so we have to choose a size for $d\bar{u}$ as well as a volume on X. Different choices lead to rescaling the metrics, hence won't affect the eigenspaces of the Laplacean. So what this means is that the eigenvalues of Δ on $\Gamma(X,\mathcal{O}) = \mathbb{C}^0$ are $|\gamma-a|^2\cdot$const with $\gamma\in\frac{\pi}{\operatorname{Im}\tau}\Gamma$, where the constant >0 depends on our metric choices. The ζ-fn. is then

$$\zeta_c(s) = \underbrace{\sum{}' \frac{1}{|a-\gamma|^{2s}}}_{\text{call this } \zeta(s)}\frac{1}{c^s}$$

and so

$$\zeta_c'(s) = \frac{\zeta'(s)}{c^s} + \zeta(s)\left[c^{-s}(-\log c)\right]$$

$$\zeta_c'(0) = \zeta'(0) + \zeta(0)(-\log c).$$

Hence if it should happen that ζ vanishes at 0, then one will get an answer independent of c. In any case I can fix the metric choices and then study the variation in a.

Change a to z. In order to do the analytic continuation

$$\Gamma(s)\zeta(s) = \sum{}' \frac{1}{|z-\gamma|^{2s}}\int_0^\infty e^{-t}t^s\frac{dt}{t}$$